LEADERSHIP CARTELS IN INDUSTRIES WITH DIFFERENTIATED PRODUCTS

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No 590

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

Leadership Cartels in Industries with Differentiated Products

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June 5, 2001

Abstract

This article analyses cartels that act as a Stackelberg leader with respect to a competitive fringe in industries supplying differentiated products. The main objectives are to investigate how cartel stability changes with the degree of differentiation and the cartel size, to predict endogenous cartels and to carry out a welfare analysis. Both repeated and static games are considered as well as industries competing in quantities and prices. The results indicate that the degree of stability can be either an increasing, decreasing or non-monotonic function of the degree of product differentiation, depending on the cartel size, the industry size, the competition type and the reaction of cartel loyal members to defection. An endogenous cartel size is also predicted. Other significant results are: some cartels can be sustained under simple static game Nash equilibrium, some cartels may be socially desirable, not all cartels are beneficial for the fringe members and a free riding problem does not necessarily emerge.

JEL classification number: L 13

Keywords: cartel stability, product differentiation, Stakelberg leader

[1] I would like to thank Dr. Jonathan Cave and Dr. Amrita Dhillon for useful comments on this work and the *Conacyt*, Mexico, for financial support.

I. Introduction

One of the most widely accepted structures to characterize industries with collusive behaviour it that of a leadership cartel versus a competitive fringe. Most of the models sharing this view concentrate on the existence of stable cartels in static models of markets without uncertainty. A cartel is defined as stable if neither cartel members, nor fringe members have incentives to move to the fringe or to the cartel, respectively. These models have analysed industries competing in prices (D'Aspremont *et al.* (1983), D'Aspremont and Gabszewick (1986), Donsimoni *et al.* (1986) and Daskin (1989)) as well as industries competing in quantities (Spulber (1989), Martin (1990), Shaffer (1995) and Konishi and Lin (1999)). Nevertheless, all this work has considered industries supplying exclusively homogeneous goods.

Cartels in industries with non-spatial (Chamberlinian) product differentiation have basically been considered under the supergame-theoretic model approach[2], for the particular case of a duopoly (Deneckere (1983), Ross (1992), Rothschild (1992), Lambertini (1995), Rothschild (1997) and; Lambertini and Albaek (1998)) and only a very few cases for industries with more than two firms (Majerus (1988)[3], Eaton and Eswaran (1998) and Posada (2000)). None of these models has considered a leadership role of the cartel.

This paper aims to analyse collusion in industries with differentiated products in which the cartels acts as a Stackelberg leader respect to the competitive fringe. The cartel is assumed to set its output (price) in a sequential movement game, in which it has a first move advantage. Since the products are not homogeneous, in a price competition industry the fringe does not act as a price taker[4] and it only reacts to the price set by the cartel according to its reaction function. The work analyses cartel stability under both a repeated as well as a static game approach. For the former, not only the classical Nash reversion strategy implemented to sustain collusion is considered, but also the strategy of keeping the cartel with one less member. For the latter, the well known D'Aspremont *et al.* (1983) definition of stability is considered. The paper also carries out a welfare analysis.

[2] One exception is Hirth (1999) who considers a static game model for a price competition industry in which the cartel does not play any leader role.

[3] Who considers the particular case of an industry-wide collusion.

[4] Thus, no extra assumptions like that of increasing marginal cost is needed for the fringe to set its output according to the rule price equal marginal cost.

For a quantity competition industry the most significant findings are: a free riding problem does not necessarily emerge, a cartel is not always desirable for the fringe members, some cartels can be sustained under the simple static game Nash equilibrium concept and small cartels are socially desirables. Regarding the stability under the supergame-theoretic model framework, the critical return rate under which collusion is sustained is not a monotonic function. In general, it reaches a maximum, with an infinity value in some cases, for an internal value of the cartel size and the degree of differentiation. Therefore, cartels formed by a subset of the industry and intermediate values of differentiation imply more stability. Under the static game framework, there exists a unique stable cartel whose size goes from 3, for heterogeneous goods, to cartels formed by $40\% - 60\%$ of the industry, for very homogeneous products.

For a price competition industry when the Nash reversion is implemented to sustain collusion, the critical return rate is basically a decreasing function of the degree of homogeneity and of the cartel size. When the cartel remains as a cartel in case a non-loyal member deviates, the stability is almost always negative (no stability at all) with the exception of an industry-wide cartel with homogeneous goods and for cartels of size 3 which, apart from particular cases, also turn out to be the only two stable cartels predicted under the static game approach.

The paper is structured as follows. Section II describes the model. section III analyses the quantity competition case under both a repeated as well as a static game framework. This section also presents a welfare analysis. Section IV extends the analysis of section III for a price competition industry. The final section presents the conclusions and suggests future research.

II. Model

Consider an industry composed of $n \geq 2$ symmetric firms. Assume that the industry produces non-spatial horizontally differentiated products such that the degree of differentiation between the products of any two firms is the same. Hence, the inverse demand function exhibits a Chamberlinian symmetry where the price of product i is given by

$$p_i = a - bq_i - c\sum_{j \neq i} q_j, \qquad a > 0, \qquad b > 0, \qquad 0 < c < b. \tag{1}$$

The value range for c implies that the products are viewed as substitutes rather than complements and that the price of each product is more susceptible to changes on its own demand rather than changes on other product demand.

Let $d \equiv c/b \in (0,1)$ be the parameter to measure the degree of homogeneity between any two products in the industry. Hence, $d = 0$ implies that the products are completely independent and $d = 1$ indicates that they are perfect substitutes. We will usually refer as homogeneous products to a value of d close to 1 and, as heterogeneous, to a value of d close to 0. Although some results are valid for the extreme cases $d = 0, 1$, this is not necessarily true.

Assume that $k \in [0, n]$ of the firms collude to form a leadership cartel. The rest $n - k$ firms, called the fringe, act independently but as Stackelberg followers. Thus, the game basically consist of three stages. In the first one the firms decide independently whether or not to be part of the cartel. In the second stage the cartel collectively sets the Stackelberg leader output (price). Finally, in the third stage, the fringe sets it output (price). For simplicity, and without loss of generality, it is assumed that the total production cost of the firms is equal to zero[5].

III. Quantity competition

To solve the game we can proceed backwards and start the analysis at the third stage. Each fringe member faces the problem of maximising its profits taken as given the output of all other firms in the industry. Hence, the optimization problem confronting one fringe member is

$$\max_{q_i}\ q_i(a - bq_i - c\sum_{j\neq i} q_j) = \max_{q_i}\ q_i(a - bq_i - c\sum_{j\neq i\in F} q_j - c\sum_{j\in C} q_j), \tag{2}$$

where F and C denote the fringe and the cartel, respectively. The first order condition implies

$$a - 2bq_i - c\sum_{j\neq i\in F} q_j - c\sum_{j\in C} q_j = 0. \tag{3}$$

However, by symmetry, all the outputs within the fringe and within the cartel must be the same. Thus, the last expression can be written as

$$a - 2bq_f - c(n - k - 1)q_f - ckq_c = 0, \tag{4}$$

[5]i.e., the model is also valid for cost functions of the type $c(q) = cq + F$.

where q_c and q_f represent the output supplied by each cartel member and by each fringe member, respectively.

On the other hand, the cartel aims to maximise its joint profits which, by symmetry, is k times the profit of each one of its members

$$\max_{q_i} \; kq_i(a - bq_i - c\sum_{j\neq i} q_j) = \max_{q_i} \; kq_i(a - bq_i - c\sum_{j\in F} q_j - c\sum_{j\neq i\in C} q_j) =$$

$$\max_{q_i} \; kq_i(a - bq_i - c(n-k)q_f - c(k-1)q_i).^6 \tag{5}$$

The Stackelberg leader output is obtained by maximising (5), subject to the reaction function of a fringe member (4). Thus, the first order condition implies

$$a - 2bq_c - c(n-k)(q_c\frac{dq_f}{dq_c} + q_f) - 2c(k-1)q_c = 0, \tag{6}$$

where $\frac{dq_f}{dq_c}$ and q_f are directly obtained from (4). Thus, it is directly shown that

$$q_c = \frac{a(d-2)}{2b(-2+3d-d^2-dk-dn+d^2n)}, \tag{7}$$

$$q_f = \frac{a(4-6d+2d^2+d^2k+2dn-2d^2n)}{2b(-2+d+dk-dn)(-2+3d-d^2-dk-dn+d^2n)}, \tag{8}$$

$$\pi_c = \frac{a^2(d-2)^2}{4b(-2+d+dk-dn)(-2+3d-d^2-dk-dn+d^2n)}, \tag{9}$$

and

$$\pi_f = \frac{a^2(4-6d+2d^2+d^2k+2dn-2d^2n)^2}{4b(-2+d+dk-dn)^2(-2+3d-d^2-dk-dn+d^2n)^2}, \tag{10}$$

where π_c and π_f represent the profit of each cartel member and each fringe member, respectively.

It will be assumed that for the cartel to be a leader it must have at least two firms. Therefore, the profit of the firms when the cartel is of the size of zero-one (no cartel) is given exclusively by $\pi_f(k=0)$ and not by $\pi_c(k=1)$. Moreover, any variable referring to a situation where there is no cartel is given by the condition $k=0$ while $k=1$ does not have any interpretation.

[6] When the cartel determines the output set by one of its firms it must not take as given the output of the other cartel members, since these are variable that it also controls and, by symmetry, take the same value.

It can be firstly observed that $\pi_f > \pi_c \Leftrightarrow k > (2 + d(n-1))/2$.[7] Thus, the characteristic free riding problem emerges only in industries with large cartels or heterogeneous products.

As a function of k, the profit of a cartel member increases when this goes from $k = 0$ (no cartel) to $k = 2$ ($\pi_c(k = 2) > \pi_f(k = 0)$)[8]. However, for $k \geq 2$, π_c reaches a local minimum at $k = (2 + d(n-1))/2$ and its global maximum at $k = n$.

It is important to point out that the minimum profit that a cartel member obtains (when $k = (2 + d(n-1))/2$) corresponds to the profit that each firm earns when there is no cartel ($\pi_f(k = 0)$). Therefore, it is always profitable for the cartel members to form a cartel of any size, compared to the situation where there is no cartel. However, small cartels are in general not willing to accept an extra member. The reason is because if the cartel is small an extra member does not mean a considerable increase of profits since the competition level in the industry would be basically the same (the fringe still remains very large) but, on the other hand, the cartel has to share its profits with another member. The cartel is willing to accept an extra member if this means a considerable decrease in the competition level, i.e., when the fringe is small or, in other words, when the cartel is large.

As a function of d, π_c is decreasing for $k > (3 + 5n - \sqrt{9n^2 - 2n - 7})/8$. However, for $k < (3 + 5n - \sqrt{9n^2 - 2n - 7})/8$ it reaches a local minimum at some $d \in (0, 1)$.

Regarding the fringe members profit, one of the most important results is that π_f is a non-monotonic function of k that reaches a global minimum at $k = (2 - d + dn)(2(d-1) + (2 - d)\sqrt{1-d})/d^2$ and its global maximum at $k = n - 1$[9].

Therefore, unlike many models developed in the literature, the profit of one fringe member is not necessarily an increasing function of k. It can be easily shown that a cartel is profitable for the fringe members ($\pi_f(k) > \pi_f(k = 0)$) $\Leftrightarrow k > (2 + d(n-1))/2$. Thus, when the cartel is

[7] As $d \in (0, 1)$ for any cartel of size $k \geq (n+1)/2$ the profit of a fringe member is always larger than the profit of a cartel member.

[8] Except when $d = 2/(n-1)$, in which case both profits are the same.

[9] The global maximum is actually reached at $k = n$. However, the fringe does not exist in this case.

small, the fringe would prefer to be in an industry with no cartel instead of being a Stackelberg follower. The reason is because even though the competition level in the industry has decreased, the fringe now has a follower status, which can lead it into a worse situation. This result might have important implications in explaining the Stackelberg leader position of small cartels since it is not possible for the cartel to threaten the fringe not to form a cartel if it is not allowed to be a Stackelberg leader. However, as it will be shown later in this paper, this is not the case for an endogenous stable cartel.

π_f is a decreasing function of d.

III. 1. Supergame solution

Although we have mentioned that the game consists of three stages, we refer to the process in which the cartel is formed and the supply decisions take place. However, in this section we allow the number of periods in which the firms meet in the market to be infinite. We take k rather as an exogenous variable and analyse stability as a function of k and d.

As Friedman (1971) has shown, it is possible for firms to sustain cooperation in a infinitely repeated game which would not be possible in the corresponding static case. In order to sustain cooperation, every firm in the cartel plays a trigger strategy, i.e., they set an output q_c as long as every other cartel member has done so in previous periods. When one member, called the non-loyal member from now on, deviates to any other output, the remaining members revert to the non-cooperative case or Nash reversion $q_f(k = 0)$ forever, but with one lag period. Cooperation can be sustained if there exists a discount factor in the industry large enough to prevent a firm from deviating. In other words, the extra profits that this non-loyal member earns in the deviating period is offset by the lowered profit the firm gets once every firm has reverted to the non-cooperative case.

The condition of maintaining stability is that the present discounted value of remaining a cartel member must exceed the present discounted value of deviating, i.e.

$$\sum_{t=0}^{\infty} \pi_c \sigma^t \geq \pi_{ch} + \sum_{t=1}^{\infty} \pi_f(k = 0)\sigma^t, \tag{11}$$

where σ is the discount factor of the industry and π_{ch} is the profit of the non-loyal member in the deviating period. Evaluating this condition in terms of the return rate, $r = (1 - \sigma)/\sigma$, results in

$$r \leq r^* \equiv \frac{\pi_c - \pi_f(k = 0)}{\pi_{ch} - \pi_c}. \tag{12}$$

r^* is the critical value below which a cartel member does not have incentives to deviate. A large (low) value of r^* implies that it is more (less) likely that the corresponding return rate of the industry is below this critical value. Therefore, r^* can be seen as a measure of the cartel stability.

Although the trigger strategy ensures a certain degree of stability, the threat of reverting to the non-cooperative case might not be collectively credible since, as is well known, the cartel punishes itself when it punishes the non-loyal member. A further possible reaction is simply to assume that the remaining cartel members will keep acting as a cartel with one less member. Following Eaton and Eswaran (1998) this will be called *stacked reversion* from now on. Hence, the non-loyal member gets a profit equal to $\pi_f(k-1)$ from the second period on. Thus, the stability condition becomes

$$r \leq r^* \equiv \frac{\pi_c - \pi_f(k - 1)}{\pi_{ch} - \pi_c}. \tag{13}$$

It is very important to point out that this strategy is not being considered as a punishment strategy implemented to sustain collusion but for two other different reasons. The first one is that this is the most reasonable reaction we would expect from the loyal members of the cartel: to keep the cartel. Thus, we are interested in finding out if collusion can be endogenously sustained by the market structure itself without considering any kind of punishment from the cartel loyal members. It is also worth mentioning that any punishment strategy from the cartel loyal members must necessarily lie between the Nash reversion (most severe punishment[10]) and the stacked reversion (no punishment at all). The second reason is because this strategy is closely related to the D'Aspremont *et al.* (1983) static definition of internal stability, to be considered ahead. This is because the sign of r^* is given exclusively by the sign of $\pi_c - \pi_f(k-1)$ since, by construction of π_{ch}, the denominator is always non-negative. However, a positive value

[10] It is worth mentioning that there exist in principle more severe punishments than Nash reversion, set the price below the cost (depredation), for instance.

of this amount corresponds to the static internal stability concept.

In the deviating period the non-loyal member maximizes its profits given that the other cartel members have supplied q_c and that the fringe members have each supplied q_f

$$\max_{q_{ch}} \; q_{ch}(a - bq_{ch} - c(n-k)q_f - c(k-1)q_c), \tag{14}$$

which implies

$$q_{ch} = \frac{a(d-2)^2(2-d+dn)}{4b(2-d-dk+dn)(2-3d+d^2+dk+dn-d^2n)}, \tag{15}$$

and

$$\pi_{ch} = \frac{a^2(d-2)^4(2-d+dn)^2}{16b(2-d-dk+dn)^2(-2+3d-d^2-dk-dn+d^2n)^2}. \tag{16}$$

Proposition 1: $q_{ch} = q_c \Leftrightarrow k = (2 + d(n-1))/2$.

Proof. Directly shown by substitution.

The implications of this proposition turn out to be one of the main results of the paper. Given a certain degree of differentiation, there exists a cartel which can be sustained under the simple Nash static equilibrium, from each firm's point of view. In other words, although the Stackelberg leader position of the cartel is not a Nash equilibrium for the cartel as a whole, it is for each individual firm. Therefore, an endogenous cartel size under the classical Nash static equilibrium definition is predicted for this particular market structure. It is worth mentioning that the stable cartel is not in general an integer number however, there always exist certain degrees of differentiation such that any cartel between 1 (for d close to 0) and $(n+1)/2$ (for d close to 1) is stable.

The explanation for this result can be understood as follows. When the products are homogeneous, a non-loyal member deviates by restricting its supply, which is compensated for the increase of prices. Surprisingly, it can be easily shown that the profit of the remaining cartel members as well as the profit of the fringe members also increase in the deviating period. However, when the products are heterogeneous, a non-loyal member deviates by expanding its supply. Unlike the case with homogeneous goods, in this case the cartel loyal member profits as the fringe member profits decrease in the deviating period since, although the products have a rather low substitution level, the consumers switch to buying the much cheaper product of the non-loyal member. Thus, there must be a certain degree of differentiation for which the non-loyal member is willing neither to expand, nor to restrict its supply. Figure 1 helps to understand easily this argument. It shows the difference between q_{ch} and q_c as a function of d for $k = 5$ and $n = 15$. Figure 2 shows the extra profit that the non-loyal member gets in the deviating period which is, of course, non-negative.

That everyone in the industry, the non-loyal cartel member, the cartel loyal members as well as the fringe member, is better off in the deviating period when the products are homogeneous deserves some more attention, since it could be argued that there is no justification for the cartel loyal members and for the fringe to complain about and/or to react to the non-loyal member defection. However, although the fringe members are also better off, their independent nature leads them to react to the non-loyal member movement, which implies an eventual reaction of the cartel as well. Figures 3 and 4 show the extra profits that each loyal member and each fringe member gets in the deviating period for $k = 5$ and $n = 15$, compared with π_c and π_f, respectively. As we can see, the extra profits are positive for homogeneous goods and negative for heterogeneous goods.

III. 1. a. Nash reversion

By direct substitution of (9), (10) and (16), the condition to sustain stability (12) becomes

$$r \le r^* \equiv \frac{4(2-d-dk+dn)(2-3d+d^2+dk+dn-d^2n)}{(d-2)^2(2-d+dn)^2}. \tag{17}$$

Proposition 2: The critical return rate is a non-monotonic function of d that starts at 1 for $d = 0$, it reaches a local minimum at $d = (2(n-1-\sqrt{k(n-k)(n-1)}))/((1+k-n)(n-1))$, a local maximum at $d = 2(k-1)/(n-1)$, where the function takes a value of 1. It finishes at a value of $4k(n+1-k)/(n+1)^2$ at $d = 1$.

Proof: See appendix.

It is directly shown that the minimum is in the valid interval for d (0,1) if and only if $k < (n+1)^2/(n+3)$. Similarly, the maximum is in the valid interval for d if and only if

$k < (n+1)/2$.

The intuition behind this result can be understood as follows. When the products are heterogeneous, there are no incentives to deviate from the cartel since it is not possible to steal other markets and thus the cartel is rather stable. When the products become more homogeneous it gets easier for the non-loyal member to supply other firm's consumers, the incentives to defect increase and the stability decreases. However, as the homogeneity increases the effect described in proposition 1 emerges, the point where the cartel is sustained under the static Nash equilibrium is reached, there are no incentives to deviate and the stability reaches a maximum. As the products become more and more homogeneous the stability decreases again since the non-loyal member, and everyone in the industry, gains by restricting the output. Figure 5 shows the critical return rate as a function of d for $k = 5$ and $n = 15$.

Proposition 3: The critical return rate is a non-monotonic function of k that reaches a global maximum at $k = (2 + d(n-1))/2$ where it takes a value of 1.

Proof. Directly shown by taking $\frac{\partial r^*}{\partial d}$ and $\frac{\partial^2 r^*}{\partial d^2}$.

Figure 6 presents the critical return rate as a function of k for $n = 15$ and $d = 0.9$.

III. 1. b. Stacked reversion

In this case the condition for stability becomes

$$r \leq r^* \equiv \frac{4(2-d-dk+dn)(2-3d+d^2+dk+dn-d^2n)A}{(d-2)^2(2-d-2k+dn)^2(2-dk+dn)^2(-2+4d-d^2-dk-dn+d^2n)^2}, \tag{18}$$

where $A = -48+144d-144d^2+64d^3-13d^4+d^5+64k-192dk+168d^2k-64d^3k+12d^4k-d^5k-16k^2+48dk^2-36d^3k^2+28d^4k^2-9d^5k^2+d^6k^2-24d^2k^3+28d^3k^3-16d^4k^3+3d^5k^3+4d^2k^4-4d^3k^4+2d^4k^4-48dn+112d^2n-80d^3n+22d^4n-2d^5n+64dkn-160d^2kn+136d^3kn-68d^4kn+19d^5kn-2d^6kn-16dk^2n+40d^2k^2n-24d^3k^2n+6d^4k^2n+6d^5k^2n-2d^6k^2n-4d^3k^3n+4d^4k^3n-3d^5k^3n-8d^2n^2+4d^3n^2+11d^4n^2-7d^5n^2+d^6n^2+16d^2kn^2-32d^3kn^2+30d^4kn^2-20d^5kn^2+$

$4d^6kn^2 - 4d^2k^2n^2 + 8d^3k^2n^2 - 7d^4k^2n^2 + 3d^5k^2n^2 + d^6k^2n^2 + 4d^3n^3 - 12d^4n^3 + 10d^5n^3 - 2d^6n^3 + 2d^5kn^3 - 2d^6kn^3 + d^4n^4 - 2d^5n^4 + d^6n^4$.

Due to the complexity of this expression, it becomes very difficult, if not impossible, to prove formally any proposition regarding the behaviour of the critical return rate for this case. However, all the particular cases analysed, informal proofs and mainly some results presented in the next section, closely related to this one, suggest that r^* certainly follows a well defined pattern that can be summarized in the following two conjectures.

Conjecture 1: For cartels of size $k > (5 + 3n - \sqrt{n^2 - 2n - 7})/4$ the stability is negative for every value of d. For cartels of size $k < (5 + 3n - \sqrt{n^2 - 2n - 7})/4$, the stability starts at a negative value $(3-k)/(k-1)$ for $d = 0$[11], it reaches a positive infinity value at $d = 2(k-1)/(n-1)$ and it subsequently decreases, ending up with a positive value.

Figure 7 shows the critical return rate as a function of d for $k = 5$ and $n = 15$. In this case the cartel is rather small and the stability follows the pattern described in the second part of this conjecture.

Regarding the stability as a function of k, this seems to follow a rather complex pattern.

Conjecture 2: For $n = 3$, $r^*(k = 3) < r^*(k = 2)$[12] for every value of d. For $n \geq 4$, $r^*(k = 3) < r^*(k = 2)$ only for low values of d. For $k \in [3, (5 + 3n - \sqrt{n^2 - 2n - 7})/4]$ and $d < 4/(n-1)$, r^* is a decreasing function that starts at a positive value at $k = 3$ and ends with a negative value at $k = (5 + 3n - \sqrt{n^2 - 2n - 7})/4$. For $k \in [3, (5 + 3n - \sqrt{n^2 - 2n - 7})/4]$ and $d > 4/(n-1)$, r^* start at a positive value, it reaches a positive infinity value at $k = (2+d(n-1))/2$ and it subsequently decreases, ending up at a negative value at $k = (5 + 3n - \sqrt{n^2 - 2n - 7})/4$. r^* is always negative for every $k > (5 + 3n - \sqrt{n^2 - 2n - 7})/4$.

Although the pattern followed by the critical return rate seems rather messy, this is only the result of two different effects given by the numerator and denominator of (13). We can firstly

[11] With the exception of $k = 3$ in which case it starts at $r^* = 0$.

[12] $r^*(k = 2)$ is given by (17) and not by (18) since, as has been mentioned before, $k = 1$ makes no sense in this model.

observe that r^* is positive for small cartels and negative for large cartels, which means that a non-loyal member is always willing to leave a large cartel. The reason is because by joining the fringe he gets a larger market size without increasing considerably the competition level in the industry. Moreover, if the products are heterogeneous, even defecting from small cartels does not increase competition too much. On the other hand, if the cartel is small and the products are homogeneous, joining the fringe when this is already large does not mean a considerable gain in the market size but it can bring about high competition levels (the fringe is large and the cartel become even smaller). Therefore, the non-loyal member is likely to remain loyal to the cartel. The fact that the critical return rate becomes infinity is again due to the effect described in proposition 1.

III. 2. Static Stability

In this section we come back to the first stage of the game and try to predict an endogenous cartel size. The section is called static stability in the sense that firms meet in the market only once.

Following D'Aspremont *et al.* (1983) we define a cartel to be stable if it is internally stable, $\pi_c(k) \geq \pi_f(k = 0)$ for $k = 2$ and $\pi_c(k) \geq \pi_f(k-1)$ for $k \geq 3$,[13] and externally stable, $\pi_f(k) \geq \pi_c(k+1)$, for $k \leq n-1$,[14] i.e., no cartel (fringe) member has incentives to join the fringe (cartel).

Proposition 4: A cartel of size $k = 2$ is always internally stable. For $k \neq 2$ a cartel is internally stable $\Leftrightarrow k \in [3, k_o]$, where k_o is the unique root of A within the interval $(3, n)$.

Proof: See appendix.

Taking the extreme values $d = 0, 1$ it can be shown that $k_o \in (3, (5+3n-\sqrt{n^2-2n-7})/4)$ where $\frac{\partial k_o}{\partial d} > 0$. Therefore, for low values of d, apart from $k = 2$, the only internal stable cartel

[13] Although the original definition of stability by D'Aspremont *et al.* (1983) permits the solution $k = 1$, this, as has been mentioned before, makes no sense in our model. Moroever, we would be interested in a non-degenerated solution, i.e., $k \geq 2$.

[14] The definition of external stability makes no sense for $k = n$. Therefore, a cartel of size n is stable as long as it is internally stable only.

is that of size 3. As d increases, larger cartels start to be internally stable up to cartels of size $(5+3n-\sqrt{n^2-2n-7})/4$.

Proposition 5: A cartel is externally stable $\Leftrightarrow k \in (k_o - 1, n-1]$.

Proof: See appendix.

Therefore, for $n > 3$ every cartel of size 2 is externally unstable. On the other hand, for low values of d only cartels of size $k \geq 3$ are externally stable, as d increases, small cartels start to be externally unstable up to cartels of size $k_o(d=1) - 1 = (1+3n-\sqrt{n^2-2n-7})/4$ for homogeneous products. Consequently, every cartel of size $k > (1+3n-\sqrt{n^2-2n-7})/4$ is always externally stable.

Proposition 6: There exists a unique stable cartel. For $n=2$ this is of size 2 for every d, for $n = 3, 4$ this is of size 3 for every d and for $n \geq 5$ its size is defined by the largest integer lower than k_o.

Proof: See appendix.

The last proposition implies that for $n \geq 5$ the unique stable cartel goes from $k=3$, for low values of d, up to a size equal to the largest integer smaller than $(5+3n-\sqrt{n^2-2n-7})/4$ for large values of d.

The next table shows the unique stable cartel for different values of n and d

$d \setminus n$	2	3	4	5	7	10	15	25	50	100
0.1	2	3	3	3	3	3	3	3	4	7
0.3	2	3	3	3	3	3	4	5	9	16
0.5	2	3	3	3	3	4	5	8	14	26
0.7	2	3	3	3	4	5	7	10	19	36
0.9	2	3	3	4	4	6	8	12	24	46

As has been mentioned previously, the fringe is willing to accept a cartel $\Leftrightarrow k > (2+d(n-1))/2$. Numerical calculations show that this amount is always below the stable cartel for the different cases presented in the previous table. Thus, one could conjecture that the unique stable cartel offers larger profits for the fringe members compared to the situation where there is no cartel. Thus, the stable cartel can threaten the fringe not to form a cartel if it is not allowed to lead. Therefore, although an endogenous mechanism to form leadership cartels is not justified for small cartels, it is for the unique stable cartel.

III. 3. Welfare Analysis

A question naturally arises regarding the welfare cost of collusion as a function of the degree of differentiation and the cartel size. The consumer utility function for the particular case of a duopoly has been studied by Dixit (1979) and Singh and Vives (1984). In their model, a representative consumer derives utility from the consumption of two goods and a third numeraire good, sold in a competitive sector. The utility function can be easily extended into an economy with n differentiated products as follows

$$U(q_1, q_2, ..., q_i, ..., q_n) = a\sum_i q_i - \frac{b}{2}\sum_i q_i^2 - \frac{c}{2}\sum_{j\neq i}\sum_i q_i q_j. \tag{19}$$

Thus, U is assumed to be a quadratic and strictly concave function. A representative consumer maximises $U - \sum_i p_i q_i$, which leads directly to the Chamberlinian inverse demand function with n goods (1). Total welfare can be defined as the consumer surplus, $U - \sum_i p_i q_i$, plus the total profit in the industry, $\sum_i p_i q_i$. Considering that there are only two different kind of firms in the industry, k in the cartel and $n-k$ in the fringe, total welfare is given by

$$\begin{aligned} W = U(q_c, q_f) &= akq_c + a(n-k)q_f - \frac{b}{2}kq_c^2 - \frac{b}{2}(n-k)q_f^2 \\ &-\frac{c}{2}k(k-1)q_c^2 - \frac{c}{2}(n-k)(n-k-1)q_f^2 - ck(n-k)q_c q_f, \end{aligned} \tag{20}$$

where the different coefficients stand for the different combinations of pairs of firms within the cartel, within the fringe and between the cartel and the fringe.

The expression for total welfare can be easily calculated by direct substitution of (7), (8) and $c = db$. However, this is not shown here for reasons of space.

The first result we obtain is that, as expected, total welfare is a decreasing function of d, which is explained because consumers value diversity. However, as a function of the size of the cartel we have the surprising result that total welfare is not a non-monotonic function of k that reaches an internal global maximum at some $k \in (0, n)$ and its global minimum at $k = n$.

The implications of this result are outstanding. Total welfare in the economy does not necessarily decrease with the size of a cartel. Therefore, and against antitrust policy principles, the existence of a cartel can be socially desirable. The explanation is related to one of the results we have mentioned before. If a small cartel is formed, the fringe firms sell their product at a lower price and, although the cartel price increases, the effect is not fully compensated and the consumers benefit from the fringe price reduction, increasing thus their surplus. In other words, total welfare in the economy increases at the expenses of the fringe firm's losses. However, it is also very important to mention that total welfare when the cartel is of size $k = (2 + d(n-1))/2$ corresponds to the welfare when there is no cartel ($k = 0$). On the other hand, as has been mentioned before, the unique stable cartel is always of a size $k > (2 + d(n-1))/2$. Therefore, although small cartels can be socially desirable, the endogenous stable cartel is always harmful.

An example of this result can be seen in figure 8, which shows total welfare as a function of the cartel size for $n = 15$ and $d = 0.8$. In this example the most socially desirable cartel is that of size 4, although the unique stable cartel is $k = 7$.

IV. Price competition

The demand function for n differentiated products can be calculated by inverting (1)

$$q_i = \alpha - \beta p_i + \gamma \sum_{j \neq i} p_j, \tag{21}$$

where

$$\alpha = \frac{a}{b - c + nc}, \; \beta = \frac{b + nc - 2c}{(b-c)^2 + nc(b-c)} \text{ and } \gamma = \frac{c}{(b-c)^2 + nc(b-c)}. \tag{22}$$

Following the same argument as in the quantity competition case, it is directly shown that the cartel maximizes its profits

$$\max_{p_c} \; kp_c(\alpha - \beta p_c + \gamma(n-k)p_f + \gamma(k-1)p_c), \tag{23}$$

subject to the reaction function of the fringe members

$$\alpha - 2\beta p_f + \gamma(n-k-1)p_f + \gamma k p_c = 0, \tag{24}$$

which imply

$$p_c = \frac{\alpha(2+\delta)}{2\beta(2+3\delta+\delta^2-\delta k-\delta n-\delta^2 n)}, \tag{25}$$

$$p_f = \frac{\alpha(4+6\delta+2\delta^2+\delta^2 k-2\delta n-2\delta^2 n)}{2\beta(2+\delta+\delta k-\delta n)(2+3\delta+\delta^2-\delta k-\delta n-\delta^2 n)}, \tag{26}$$

$$\pi_c = \frac{\alpha^2(2+\delta)^2}{4\beta(2+\delta+\delta k-\delta n)(2+3\delta+\delta^2-\delta k-\delta n-\delta^2 n)} \tag{27}$$

and

$$\pi_f = \frac{\alpha^2(4+6\delta+2\delta^2+\delta^2 k-2\delta n-2\delta^2 n)^2}{4\beta(2+\delta+\delta k-\delta n)^2(2+3\delta+\delta^2-\delta k-\delta n-\delta^2 n)^2}. \tag{28}$$

where α, and β are given by (22) and $\delta = \gamma/\beta = d/(1+nd-2d)$.

In this case the profit of a fringe member is always larger than the profit of a cartel member. Besides, the profit of a cartel member and the profit of a fringe member are always decreasing functions of d and increasing functions of k.

IV. 1. Supergame solution

As usual, the non-loyal member maximizes its profit given than the other firms have charged prices p_c and p_f, respectively

$$\max_{p_{ch}} \; p_{ch}(\alpha - \beta p_{ch} + \gamma(n-k)p_f + \gamma(k-1)p_c), \tag{29}$$

which implies

$$p_{ch} = \frac{\alpha(2+\delta)^2(2+\delta-\delta n)}{4\beta(2+\delta+\delta k-\delta n)(2+3\delta+\delta^2-\delta k-\delta n-\delta^2 n)} \tag{30}$$

and

$$\pi_{ch} = \frac{\alpha^2(2+\delta)^4(2+\delta-\delta n)^2}{16\beta(2+\delta+\delta k-\delta n)^2(2+3\delta+\delta^2-\delta k-\delta n-\delta^2 n)^2}. \tag{31}$$

IV. 1. a. Nash reversion

The case $k = n$ is an especial case where it makes no sense for the cartel to lead, since there is no fringe. Moreover, this case must be analysed separately since when the non-loyal member deviates to p_{ch}, the remaining demand of the cartel loyal members may become negative for large values of d. Thus, in this case, the non-loyal firm maximizes its profits by reducing its price until the demand of the remaining cartel members is equal to zero

$$\alpha - \beta p_c + \gamma(n-2)p_c + \gamma p_{ch} = 0, \tag{32}$$

which results in

$$p_{ch} = \frac{\alpha(n\delta - 1)}{2\beta\delta(\delta - n\delta + 1)} \tag{33}$$

and

$$\pi_{ch} = \frac{\alpha^2(\delta+1)(n\delta-1)}{4\beta\delta^2(\delta - n\delta + 1)}. \tag{34}$$

The critical value for d, d^*, is found by equalling expressions (30) with (33) evaluated at $k = n$, which implies

$$d^* = \frac{n - 3 + \sqrt{n^2 - 1}}{3n - 5}. \tag{35}$$

Therefore, the non-loyal member deviates according to (30) for $d \in (0, d^*]$ and according to (33) for $d \in [d^*, 1)$.

Hence, the measure of stability, r^*, is given by

$$r^* = \begin{cases} \frac{4(1-d)(1+nd-2d)}{(2+nd-3d)^2} & \text{for } d \in (0, d^*], \quad a) \\ \\ \frac{d^4(n-1)^2}{(2+nd-3d)^2(-1+3d-3d^2-nd+2nd^2)} & \text{for } d \in [d^*, 1). \quad b) \end{cases} \tag{36}$$

Proposition 7: r^* takes a value of 1 at $d = 0$ and a value of $1/(1-n)$ at $d = 1$ for every n. For $n \in \{2, 3, 4\}$, the stability reaches a global minimum at $d = (12 - 4n - 2\sqrt{2(3+n)(n-1)})/(21 - 14n + n^2)$, and it is always a decreasing function of d for $n \geq 5$.

Proof: See appendix.

The economic intuition behind this result is clear. As the products become more homogeneous the incentives to deviate increase, since the non-loyal member can now supply its product to other firm's consumers. When d^* is reached, the profit of the non-loyal member is restricted to avoid the rest of the industry having a negative demand. Therefore, the incentive to deviate diminishes and the stability fall is lessened. Figure 9 shows this pattern for an industry with 7 firms.

For $k < n$ the non-loyal member always deviates according to (30). The condition to sustain stability becomes

$$r \leq r^* \equiv \frac{4(1-2d+dn)(2-3d+dk+dn)(2-5d+3d^2-dk+2d^2k+3dn-4d^2n-d^2kn+d^2n^2)}{(2-3d+dn)^2(2-3d+2dn)^2}. \quad (37)$$

Proposition 8: The critical return rate is a decreasing function of d and k.

Proof: See appendix.

Therefore, the most likely cartels to exist under this scenario are small cartels with heterogeneous products. To understand this we can notice that a large cartel implies a high price in the industry. Therefore, a non-loyal member gets high profits by deviating from large cartels, since the high prices of its *new competitor and* the low numbers of competitors will permit him to get a larger market share and a large gain margin.

IV. 1. b. Stacked reversion

For $k = n$, r^* is given by

$$r^* = \begin{cases} \frac{(1-d)(12-48d+57d^2-17d^3-4n+40dn-77d^2n+33d^3n-8dn^2+32d^2n^2-20d^3n^2-4d^2n^3+4d^3n^3)}{(n-1)(-2+4d-d^2-2dn+d^2n)^2} & \text{for } d \in (0, d^*], \\ \\ \frac{d^4(n-1)(12-48d+57d^2-17d^3-4n+40dn-77d^2n+33d^3n-8dn^2+32d^2n^2-20d^3n^2-4d^2n^3+4d^3n^3)}{4(1-2d+dn)(-2+4d-d^2-2dn+d^2n)^2(-1+3d-3d^2-dn+2d^2n)} & \text{for } d \in [d^*, 1). \end{cases} \quad (38)$$

Proposition 9: The critical return rate is an increasing function of d that takes a negative value of $(3-n)/(n-1)$ at $d = 0$[15] and a positive value of $1/(n-1)$ at $d = 1$.

Proof. See appendix.

[15] For the case $n = 3$ the critical return rate starts at 0.

That the critical return rate takes negative values and no cartel is stable for low values of d is explained because there is no punishment against the non-loyal member. However, the interesting result appears as the products become more homogeneous, the stability increases and can reach positive values. At first glance, this result could seem contradictory since, as was just mentioned, there is no punishment against the non-loyal member. In this case, the punishment comes from the market itself. When the degree of homogeneity is large, so is the degree of competition between the cartel and the non-loyal member, acting now as the fringe. The prices in the industry can fall substantially, nevertheless there are only two entities competing. Although the market share is larger for the non-loyal member, the price fall has reduced its profits. This effect is not observed for heterogeneous goods because competition is not present in this case. The stability is even strengthened when d is close to one, because the profits that the non-loyal member gets in the deviating period is, as is known, restricted for $d \geq d^*$. Figure 10 shows the critical return rate as a function of d for an industry-wide cartel of size 4.

For $k < n$ the critical return rate is given by

$$r \leq r^* \equiv \frac{4(1-2d+dn)(2-3d+dk+dn)BC}{(2-4d+dk+dn)^2(2-3d+2dn)^2(-2+3d+2k-4dk-dn+2dkn)^2D^2}, \quad (39)$$

where $B = 2-5d+3d^2-dk+2d^2k+3dn-4d^2n-d^2kn+d^2n^2$, $C = -48+432d-1584d^2+3008d^3-3085d^4+1587d^5-306d^6+64k-576dk+2088d^2k-3840d^3k+3660d^4k-1583d^5k+174d^6k-16k^2+144dk^2-480d^2k^2+676d^3k^2-188d^4k^2-439d^5k^2+319d^6k^2-24d^2k^3+164d^3k^3-424d^4k^3+493d^5k^3-218d^6k^3+4d^2k^4-28d^3k^4+74d^4k^4-88d^5k^4+40d^6k^4-240dn+1792d^2n-5232d^3n+7414d^4n-5040d^5n+1287d^6n+320dkn-2400d^2kn+6936d^3kn-9492d^4kn+5941d^5kn-1243d^6kn-80dk^2n+600d^2k^2n-1576d^3k^2n+1570d^4k^2n-118d^5k^2n-469d^6k^2n-92d^3k^3n+472d^4k^3n-813d^5k^3n+471d^6k^3n+16d^3k^4n-84d^4k^4n+148d^5k^4n-88d^6k^4n-488d^2n^2+2908d^3n^2-6349d^4n^2+5975d^5n^2-2020d^6n^2+656d^2kn^2-3936d^3kn^2+8502d^4kn^2-7692d^5kn^2+2365d^6kn^2-164d^2k^2n^2+984d^3k^2n^2-1903d^4k^2n^2+1169d^5k^2n^2+49d^6k^2n^2-132d^4k^3n^2+452d^5k^3n^2-389d^6k^3n^2+24d^4k^4n^2-84d^5k^4n^2+74d^6k^4n^2-516d^3n^3+2304d^4n^3-3348d^5n^3+1571d^6n^3+704d^3kn^3-3168d^4kn^3+4546d^5kn^3-2038d^6kn^3-176d^3k^2n^3+792d^4k^2n^3-998d^5k^2n^3+275d^6k^2n^3-84d^5k^3n^3+144d^6k^3n^3+16d^5k^4n^3-28d^6k^4n^3-299d^4n^4+890d^5n^4-646d^6n^4+416d^4kn^4-$

$1248d^5kn^4 + 892d^6kn^4 - 104d^4k^2n^4 + 312d^5k^2n^4 - 191d^6k^2n^4 - 20d^6k^3n^4 + 4d^6k^4n^4 - 90d^5n^5 + 134d^6n^5 + 128d^5kn^5 - 192d^6kn^5 - 32d^5k^2n^5 + 48d^6k^2n^5 - 11d^6n^6 + 16d^6kn^6 - 4d^6k^2n^6$ and $D = -2 + 4d - d^2 + dk - 2d^2k - 3dn + 3d^2n + d^2kn - d^2n^2$.

Again, the complexity of this expression makes it very difficult to prove formally any proposition regarding the behaviour of the r^*. However, all the particular cases analysed, as well as informal proofs, suggest that there exists a well defined pattern for the critical return rate.

Conjecture 3: The critical return rate is always negative for every $k > 3$. For $k = 3$ it is a positive increasing function of d that starts at a value of 0 at $d = 0$.

Therefore, no cartel larger than 3 can exist, with the exception of an industry-wide collusion with homogeneous products. To understand this we can observe that, as long as there is at least one firm in the fringe, a non-loyal member defection does not drastically increase the competition level. This was already present even before he deviates. However, he can now get a larger market share since he does not have to share its profits. Therefore, the only possible stable cartels are those where the profits are not shared among many members, let us say 3. This effect does not take place so drastically in a quantity competition industry since competition is not very severe in that case.

IV. 2. Static stability

Conjecture 4: A cartel of size $k = 2$ is internally stable for every n. For $n = 3$, the cartel $k = 3$ is also internally stable. For $n = 4$, $k = 3$ is internally stable and $k = 4$ is internally stable only for homogeneous products. For $n = 5$, $k = 3$ is internally stable and $k = 4, 5$ are internally stable only for homogeneous products. For $n \geq 6$, apart from $k = 2$, the only two other internally stable cartel are those of size $k = 3$ for every d and $k = n$ for homogeneous products.

Conjecture 5: Every cartel is externally stable with three exceptions: a cartel of size $k = 2$ for every d and n, a cartel of size $k = n - 1$ for homogeneous products and a cartel of size $k = 3$ for $n = 5$ for homogeneous products.

Conjecture 6: For $n = 2$, $k = 2$ is stable. For $n = 3$, $k = 3$ is stable. For $n = 4$ the cartel $k = 3$ is stable only for heterogeneous products and the cartel $k = 4$ is so only for homogeneous products. For $n = 5$, the cartel $k = 3$ is stable only for heterogeneous products and the cartel $k = 5$ is stable for very homogeneous products. For $n \geq 6$ there exist only two stable cartels, that of size 3 for every d and that of size n for homogeneous products.

IV. 3. Welfare analysis

The explicit expression for the total welfare can be easily obtained by calculating $q_c = \pi_c/p_c$ and $q_f = \pi_c/p_f$. However, it is not presented here for reasons of space. Anyway, in this case we have the rather reasonable result that total welfare in the economy is a decreasing function of d and k.

V. Conclusion

The most significant finding of this paper regarding an industry competing in quantities are: fringe members can have lower profits than cartel members, a cartel is not always desirable for the fringe members, some cartels can be sustained under the simple static game Nash equilibrium and small cartels are socially desirable, although the unique stable cartel is not.

Under the repeated game framework for a Nash reversion, the stability of the cartel is not a monotonic function of d. In general, it reaches two maximums, at $d = 0$ and at $d = 2(k-1)/(n-1)$ and a global minimum at $d = 1$. As a function of k the stability reaches a global maximum at $k = (2 + d(n-1))/2$. For the staked reversion, the stability is negative for every $k > (5+3n-\sqrt{n^2-2n-7})/4$. For smaller cartels the stability starts at a negative value for low values of d and it reaches a positive infinity value at $d = 2(k-1)/(n-1)$. Under the static game framework, there exists a unique stable cartel whose size goes from 3, for heterogeneous products up to $k = (5 + 3n - \sqrt{n^2 - 2n - 7})/4$ for very homogeneous products. Total welfare is a decreasing function of d but it reaches an internal global maximum as a function of k.

For a price competition industry, the critical return rate is basically a decreasing function of d and k when Nash reversion is implemented to sustain collusion. For the stacked reversion however, the cartel is almost always unstable with two exceptions: an industry-wide collusion for homogeneous products and cartels of size 3, which also turn out to be the only two stable cartels under the static definition of stability. Total welfare is a decreasing function of d and k.

Future research might be focused on more general forms of the demand and cost function. It is also worth finding out how the predictions of the model would change considering asymmetric firms, i.e., by relaxing the symmetries regarding the size of the market, the cost function and the own and cross demand elasticities. Industries with spacial differentiation products could also be considered.

VI. Appendix

VI. 1. Proof proposition 2

By taking $\frac{\partial r^*}{\partial d}$ it is directly shown that r^* has 4 critical points, at $d = 0$, $d = 2(k-1)/(n-1)$, $d = (2(n-1-\sqrt{k(n-k)(n-1)}))/((1+k-n)(n-1))$ and $d = (2(n-1+\sqrt{k(n-k)(n-1)}))/((1+k-n)(n-1))$. The last critical point is always out of the valid interval for d since for $k = n$ it takes a value of 2, for $k = n-1$ it goes to infinity and it is always negative for $k < n-1$ since $(1+k-n)(n-1) < 0$ and $n-1+\sqrt{k(n-k)(n-1)} > 0$. $\frac{\partial^2 r^*}{\partial d^2}(d = 2(k-1)/(n-1)) = -(n-1)^2(k-1)^2/(2k^2(n-k)^2) < 0$, hence $d = 2(k-1)/(n-1)$ is a maximum. $r^*(d = 2(k-1)/(n-1)) = 1$ is directly shown by substitution. $\frac{\partial^2 r^*}{\partial d^2}(d = 0) = -(k-1)^2/2 < 0$, hence $d = 0$ is also a maximum. Since $d = 0$ and $d = 2(k-1)/(n-1)$ are maximums and there exists only another critical point in the interval (0,1), this must be $d = (2(n-1-\sqrt{k(n-k)(n-1)}))/((1+k-n)(n-1))$ and it must be a minimum.

VI. 2. Proof proposition 4

$\pi_c(k = 2) > \pi_f(k = 0) \Leftrightarrow 2 - d + d^2 + dn - d^2n > 0$, which is true since this is a concave parabola (its second derivative is $-2(n-1)$) which takes a positive value at $d = 0$ (2) and a positive value at $d = 1$ (2). Thus a cartel of size 2 is internally stable. For $k \geq 3$ we can observe that $\pi_c(k) - \pi_f(k-1) \equiv IS = (a^2d^2A)/(4b(2+d(n-k))^2(2-d+d(n-k))(2-3d+d^2+dk+dn-d^2n)(-2+4d-d^2-dk-dn+d^2n)^2)$. Since $2-3d+d^2+dk+dn-d^2n > 0$ (this is concave parabola that takes a positive value at $d = 0$ (2) and a positive value at $d = 1$ (k)), and $2-d+d(n-k) > 0$, $\pi_c(k) > \pi_f(k-1) \Leftrightarrow IS > 0 \Leftrightarrow A > 0$. Therefore, it is enough to prove that $A(k = 3) > 0$, $A(k = n) < 0$ and A has only one root in the interval $(3, n)$. $A(k = 3) > 0 \Leftrightarrow 36 - 20d + 5d^2 - 2d^3 + 9d^4 - 8n + 4dn - 20d^2n + 28d^3n - 24d^4n + 4n^2 - 20dn^2 + 38d^2n^2 - 40d^3n^2 + 22d^4n^2 + 4dn^3 - 12d^2n^3 + 16d^3n^3 - 8d^4n^3 + d^2n^4 - 2d^3n^4 + d^4n^4 \equiv E > 0$. To show that $E > 0$ the particular cases $n = 3$, 4 and 5 can be directly verified. For $n > 5$

we can easily check that $E(d=0)>0$, $\frac{\partial E}{\partial d}(d=0)>0$, $E(d=1)>0$, $\frac{\partial E}{\partial d}(d=1)<0$ and $\frac{\partial^2 E}{\partial d^2}(d=1)>0$. Thus E is a positive increasing function at $d=0$ and a positive decreasing convex function at $d=1$. Therefore, E cannot take negative values because this would require at least three concavity changes, which is not possible for a fourth degree polynomial. $A(k=n)<0 \Leftrightarrow -12+24d-9d^2+d^3+4n-20dn+13d^2n-2d^3n+4dn^2-4d^2n^2+d^3n^2 \equiv F>0$. It is directly shown that $F(d=0)>0$, $\frac{\partial F}{\partial d}(d=0)>0$, $F(d=1)>0$, $\frac{\partial F}{\partial d}(d=1)<0$ for $n\geq 4$. Therefore, F must be positive in all the interval $d\in(0,1)$ since otherwise F would have to reach two maximums and one minimum, which is not possible for a third degree polynomial. At this moment it is worth noticing that the dominant terms of A for large values of k are the terms containing k^4. It can be easily seen that these terms, once factorized, are $2d^2(2-2d+d^2)k^4$. Since this coefficient is always positive, A must take positive values for very large positive values of k and positive values (with a negative slope) for very large negative values of k. On the other hand, $\frac{\partial A}{\partial k}(k=1)>0 \Leftrightarrow 8-12d+6d^2-2d^3+d^4+4dn-4d^2n+2d^3n-2d^4n+d^4n^2>0$, which is true since this function at $n=1$ is positive $(2(2-d)^2)$ its first derivative respect to n at $n=1$ is positive $(2d(2-2d+d^2))$ and its second derivative respect to n is positive$(2d^4)$. Therefore, A necessarily reaches one minimum at some $k<1$ since A has a negative slope for very large negative values of k. For A to have more than one root in the interval $(3,n)$ it would require A to reach at least two other minimums and two maximums, since $A(k=n)<0$ and A finishes up with positive values for large values of k. However, A has at most three critical points since it is a fourth degree polynomial. Therefore, A has only one root in the interval $(3,n)$.

VI. 3. Proof proposition 5

Let $ES \equiv \pi_f(k)-\pi_c(k+1)$. Thus, as long as ES is positive (negative) a cartel of size k is internally stable (unstable). It can be observed firstly that if IS has a root at k_o, then the function ES necessarily has a root at k_o-1. On the other hand, if IS (defined in the previous proof) is a decreasing function at k_o, ES is necessarily an increasing function at k_o-1. Therefore, based on these results and on the proof of proposition 4, proposition 5 is directly implied.

VI. 4. Proof proposition 6

The particular cases can be directly verified by substitution. The general results are a direct implication from propositions 4 and 5 and some of the properties mentioned in the proofs of these two propositions. However, an alternative formal proof of existence would be as follows: We have proved that $\pi_c(k=2) > \pi_f(k=0)$. This implies that $k=2$ is internally stable. $\pi_f(n-1) > \pi_c(k=n) \Leftrightarrow -12+24d-9d^2+d^3+4n-20dn+13d^2n-2d^3n+4dn^2-4d^2n^2+d^3n^2 \equiv G > 0$. It can be easily shown that G takes positive values at $d=0$ and $d=1$ for $n \geq 5$ and its derivative at $d=0(1)$ is positive (negative). Thus, G must be positive for any $d \in (0,1)$ since G could only become negative by reaching at least two maximums and one minimum within the interval $d \in (0,1)$, which is not possible for a cubic function. Therefore, $k=n-1$ is externally stable. We proceed to compare $\pi_f(k=2)$ with $\pi_c(k=3)$. There exist only two possibilities, if $\pi_f(k=2) > \pi_c(k=3) \Rightarrow k=2$ is externally stable and then $k=2$ is a stable cartel. If $\pi_f(k=2) < \pi_c(k=3) \Rightarrow k=3$ is internally stable and we proceed to compare $\pi_f(k=3)$ with $\pi_c(k=4)$. There exist only two possibilities, if $\pi_f(k=3) > \pi_c(k=4) \Rightarrow k=3$ is externally stable and then $k=3$ is a stable cartel. If $\pi_f(k=3) < \pi_c(k=4) \Rightarrow k=4$ is internally stable. If we keep this procedure there will be only two options. If we stop at some point it is because a stable cartel has been found. If not, we will reach a stage where the cartel $k=n-1$ is internally stable. However, it is known that a cartel of size $n-1$ is externally stable and therefore this cartel of size $k=n-1$ would be a stable cartel.

VI. 5. Proof proposition 7

By substitution, it is directly shown that the critical return rate takes a value of 1 at $d=0$ and a value of $1/(1-n)$ at $d=1$ for every n. For the interval $d \in (0, d^*]$, $\frac{\partial r^*}{\partial d} = -4d(n-1)^2/(2+d(n-3))^3$, which is clearly negative for every $n \geq 2$. At this point it is useful to consider that, by construction, the critical return rate is a continuous function. It is also directly shown, by evaluating the first derivative of $36a$ and $36b$ at d^*, that continuity is also a property of the first derivative. Therefore, $36b$ is also a decreasing function at d^*. By calculating $\frac{\partial r^*}{\partial d}$ for the interval $d \in [d^*, 1)$ it is directly shown that r^* has a minimum at $d = (12-4n-2\sqrt{2(3+n)(n-1)})/(21-14n+n^2) \in (d^*, 1)$ only for $n \in \{2,3,4\}$. Since r^* has a negative derivative at d^* and it does not have any minimum for $n \geq 5$, in this case the function is always decreasing in d.

VI. 6. Proof proposition 8

$\frac{\partial r^*}{\partial d} < 0 \Leftrightarrow -4+12d-9d^2+4k-16dk+15d^2k-4dn+6d^2n+8dkn-15d^2kn-d^2n^2+4d^2kn^2 \equiv H > 0$. $H(k=1) = d(n-1)(4+3d(n-2)) > 0$, $H(k=n) = (n-1)(2+d(2n-3))^2 > 0 \Rightarrow H > 0$ for every $k \in [1,n]$ since H is a straight line in k. $\frac{\partial r^*}{\partial k} < 0 \Leftrightarrow -2+3d+2k-4dk-dn+2dkn > 0$, easily shown to be true by evaluating at the extreme values $d = 0,1$ or $k = 1,n$.

VI. 7. Proof proposition 9

The particular case $n = 3$ can be directly verified. For $n \geq 4$ we proceed as follows: For $d \in (0, d^*]$, $\frac{\partial r^*}{\partial d} > 0 \Leftrightarrow 12-66d+132d^2-111d^3+31d^4-4n+54dn-174d^2n+204d^3n-74d^4n-12dn^2+78d^2n^2-137d^3n^2+65d^4n^2-12d^2n^3+40d^3n^3-26d^4n^3-4d^3n^4+4d^4n^4 \equiv I < 0$, provided $-2+4d-d^2-2dn+d^2n < 0$. The second condition is clearly true since this is a convex parabola (its second derivative $(2(n-1))$ is positive) that takes negative values at $d = 0(-2)$ and $d = 1(1-n)$. Since $d^* < \sqrt{3}-1$ for every $n > 2$. It is sufficient to show that $I < 0$ for $d \in (0, \sqrt{3}-1)$. It can be directly checked that I is a negative decreasing concave function at $d = 0$ and a negative increasing convex function at $d = \sqrt{3}-1$ for every $n \geq 3$. Therefore, I is always negative $\in (0, \sqrt{3}-1)$ since otherwise for I to become positive would require it to reach two minimums and one maximum within this interval, i.e., three concavity changes, impossible for a third degree polynomial. For $d \in [d^*, 1]$, it is enough to show that it $\frac{\partial r^*}{\partial d} > 0$ for $d > 2/3$ since $d^* > 2/3$ for every n. $\frac{\partial r^*}{\partial d} > 0 \Leftrightarrow -96+936d-3852d^2+8644d^3-11374d^4+8734d^5-3614d^6+627d^7+32n-744dn+4836d^2n-14844d^3n+24840d^4n-23244d^5n+11414d^6n-2310d^7n+144dn^2-1960d^2n^2+9192d^3n^2-20718d^4n^2+24434d^5n^2-14544d^6n^2+3486d^7n^2+256d^2n^3-2416d^3n^3+8164d^4n^3-12816d^5n^3+9546d^6n^3-2770d^7n^3+224d^3n^4-1488d^4n^4+3444d^5n^4-3378d^6n^4+1227d^7n^4+96d^4n^5-424d^5n^5+600d^6n^5-288d^7n^5+16d^5n^6-40d^6n^6+28d^7n^6 \equiv J > 0$, provided $-2+4d-d^2-2dn+d^2n < 0$, which has been shown to be true previously. It can easily be verified that J, its first, second, third, fourth, fifth and sixth derivative at $d = 2/3$ as well as its seventh derivative are positive for every $n \geq 4$. Therefore $J > 0$ for every $d > 2/3$.

Figures

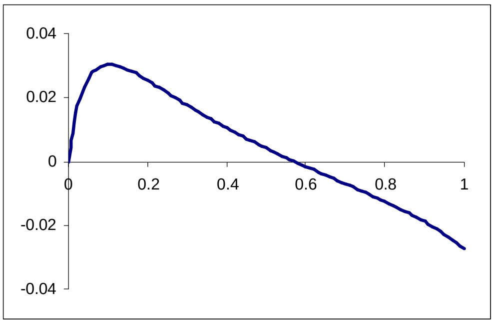

Figure 1. $(q_{ch} - q_c)$ as a function of d for $k = 5$ and $n = 15$.

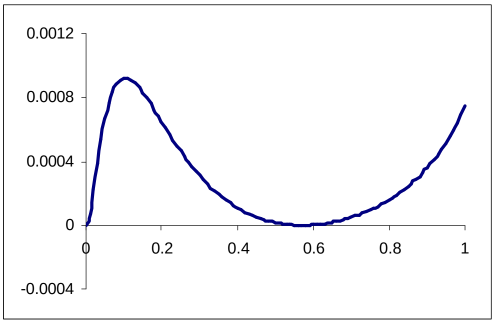

Figure 2. $(\pi_{ch} - \pi_c)$ as a function of d for $k = 5$ and $n = 15$.

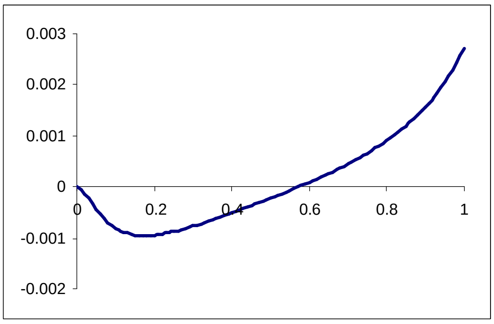

Figure 3. Extra profits of a cartel loyal member in the deviating period as a function of d for $k = 5$ and $n = 15$.

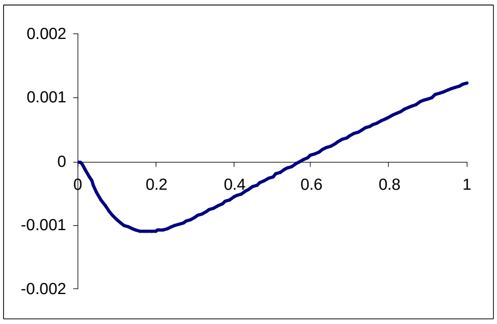

Figure 4. Extra profits of a fringe member in the deviating period as a function of d for $k = 5$ and $n = 15$.

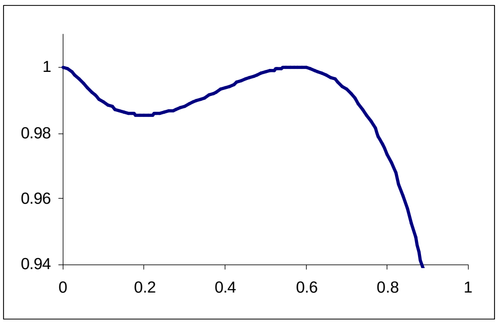

Figure 5. Critical return rate as a function of d for a Nash reversion for $k = 5$ and $n = 15$.

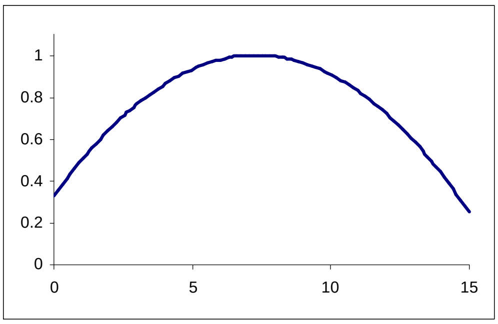

Figure 6. Critical return rate as a function of k for a Nash reversion for $n = 15$ and $d = 0.9$.

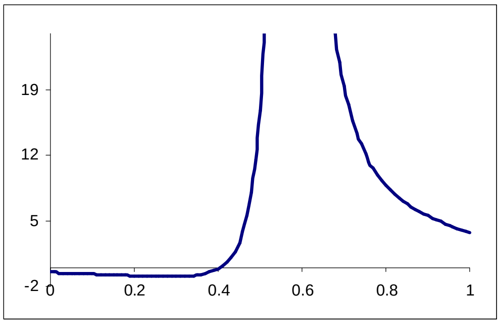

Figure 7. Critical return rate as a function of d for a stacked reversion for $k = 5$ and $n = 15$.

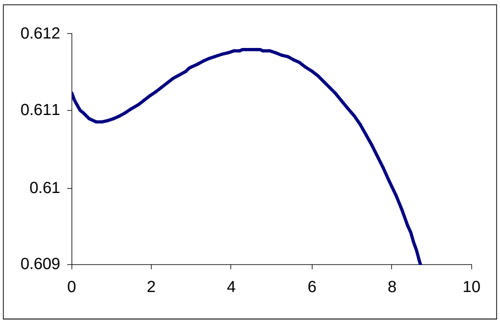

Figure 8. Total welfare as a function of k for $n = 15$ and $d = 0.8$.

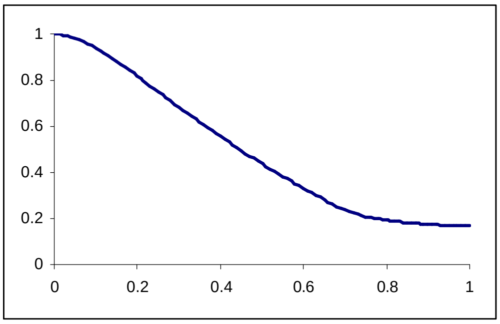

Figure 9. Critical return rate as a function of d for a Nash reversion for $k = n = 7$ in a price competition industry.

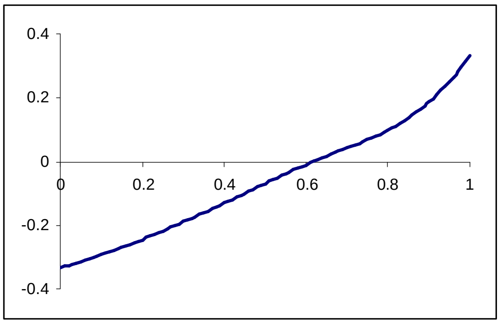

Figure 10. Critical return rate as a function of d for a stacked reversion for $k = n = 4$ in a price competition industry.

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