Early Stage Variation Simulation and Visualization of Compliant Part Based on Parametric Space Envelope

Chen Luo, Pasquale Franciosa, Darek Ceglarek, Zhonghua Ni, and Zhijie Mo

***Abstract*— Compliant, non-rigid parts are widely used in many industries today. Existing variation simulation analysis on manufacturing part focuses on orientation and position deviation with part shape errors being largely omitted. This is valid approach for rigid part, but unrealistic and can be problematic for compliant part. In this study, a new methodology has been introduced to compliant assembly early-phase design to generate various probable variated manufacturing parts that conform to pre-defined tolerance specification or meet certain industrial requirement. The proposed method is based upon novel idea of parametric space envelope, a purpose designed variation tool constructed from parametric curves. Variation of embedded manufacturing part is linked to and controlled by a compact set of envelope's boundary control points. Part variation instances are generated through simulating control points' movement in a systematic and efficient way. Importantly, simulated variations can be visualized in a three-dimensional (3D) virtual space to provide user insight into part variation. The proposed methodology can help identify assembly high-risk region, select proper fixture, guide assembly engineering changes, and optimize assembly operations. An industrial case study on deformable vehicle door hinge plate is presented to illustrate the methodology.**

***Note to Practitioners*— This paper is motivated by two acute problems encountered in geometric variation simulation analysis for compliant parts in early design stage. Firstly, form errors of non-rigid part are not fully captured in existing methods. Secondly, variation visualization can significantly enhance understanding of the geometric variation effects but it is difficult to achieve under existing approaches. Inspired by the idea of parametric space envelope, this paper proposes a new methodology by building a variation tool to aid the task. Under the proposal, geometric variation of compliant part is indirectly modeled through the constructed variation tool. This indirect modeling enables capturing intra-part interactions which are a major source of inaccuracy of existing methods. Simulated geometric variation can be visualized through the variation tool. In addition, the developed method doesn't rely on historical manufacturing experience. This can be valuable for early stage designer when such production information is not available or expensive to get. Furthermore, the proposed method can be integrated into existing Computer-aided Design (CAD), Manufacturing (CAM), and Engineering (CAE) systems to improve overall design quality and reduce reliance on multiple physical prototyping.**

***Index Terms*—Compliant assembly, geometric form tolerance, shape variation modeling, shape variation simulation, geometric form defects visualization.**

I. Introduction

TODAY compliant, non-rigid parts are widely used in industries to meet ever increasing customer demands [1]-[3]. Geometric variation of these deformable parts during manufacturing and assembly can have significant impact on product quality, functionality, and time to market. Variation analysis is required in design stage to predict such uncertainties induced by geometric variation [4]. Failures not predicted during the design phase can appear during prototyping and production ramp-up (or preproduction stage) shown in Fig. 1. And these failures, in turn, require design modification, assembly engineering changes, or even large-scale redesign. This trial-and-error type of fine tuning of new product can add significant assembly costs, and repeated test trial can lead to lengthy delays in launching new products.

Hence, a comprehensive variation simulation analysis in design phase plays a vital role to pre-empt these failures and deliver *right first time* (RFT). Key efforts of this involve generating various probable part variation representatives to mimic production variations. This aids design engineers to make informed decision in earliest design phase. Currently this variation simulation analysis activity is performed with support from computer-aided tolerance (CAT) tools [5], which focus on part orientation and position deviation with part shape errors (or form deviation) largely omitted. This may be valid approach for rigid part but is unrealistic and can be problematic for compliant part assembly [6], [7].

Major difficulties in developing a solid geometric variation simulation method for early design phase compliant assembly lie in two aspects. Firstly, most value-added compliant parts have complex shapes, i.e. with free-form surfaces. Accurately

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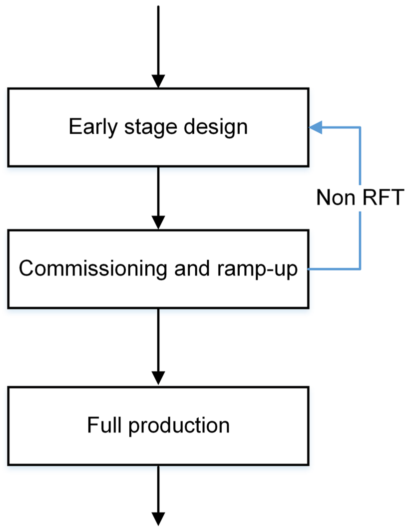

Fig. 1. Illustration of typical industrial production flows with non-RFT (right first time) being handled in the process.

modeling intra-part interactions induced by geometric variation for complex part is a recognized challenge for all modelers and researchers. Secondly, production variation information may not be readily available in early design phase as production has not begun yet. Many existing methods, which rely on historical variation statistics as model inputs, cannot be applied in this stage. For example, principal component analysis (PCA) on a large sample of manufactured surfaces can provide useful information on analyzing and categorizing geometric deviation [8]. But collecting sample variation data from a trial set of testing run is costly, time-consuming and insufficient in many cases.

Recently, digital twins are introduced into manufacturing to improve production efficiency. However, the limited accuracy of digital twins of assembly systems prohibits their effective use for simulating compliant parts' variation. e.g., they are not able to support producing near-zero-defect products and ensure high rate of RFT [9].

In view of these challenges, this paper introduces a new methodology into this area to provide design support through generating probable product variation instances including expected as well as unexpected variations. The new method is inspired by novel idea of parametric space envelope (PSE), a purpose-designed variation tool constructed from a base parametric curve. Compliant part is subsequently embedded in this tool. Variation of target part is linked to and controlled by a compact set of boundary control points of the envelope. Solving a system of inequalities constrained by geometric tolerance, allowed control points' movement can be determined. Simulating control points' movement within their respective influence zone, various tolerance-conforming variation instances can be generated.

Compared with existing methods that are applied in the design phase, the proposed methodology has following distinctive advantages: (i) It is capable of handling complex compliant part. The constructed variation tool is independent of embedded manufacturing part. As such, this indirect modeling method enables handling complex compliant part. (ii) Generated variation instances can conform to specified tolerance or meet certain industry's production requirement, i.e. satisfying 99.74% assembly rate in six sigma (±3σ) manufacturing. (iii) Required model inputs are geometries of target compliant part, which is available from part's CAD model (or file), and related tolerance specification either pre-defined or to be assigned by product designer. As such, variation simulation analysis under the proposed methodology does not rely on historical manufacturing experience. (iv) It allows manufacturing and assembly deformation and deviation to be visualized with high level of realism long before any physical prototypes are being made.

Although the proposed methodology and framework is versatile and can be used throughout the product life-cycle, the scope of this paper is to deal with geometric variation simulation and visualization of compliant part in early design phase. This paper is organized as follows. Section II provides a literature review on compliant part variation modeling and simulation. The concept of parametric space envelope is explained and geometric variation modeling is presented in Section III. The influences of involved modeling parameters are studied in Section IV. Section V presents variation simulation and detailed implementation of proposed methodology. Validity and intuitiveness of the proposed method is demonstrated through an industrial case example in Section VI. Method discussions are provided in Section VII. Validation and use of the simulation results are given in Section VIII. Section IX concludes the analysis and provides direction for future research in this area.

II. Literature Review

In compliant assembly, manufacturing costs and product quality are heavily influenced by compliant part deviation and deformation during production. Variation is controlled through specified tolerance guided by Geometric Dimensioning and Tolerancing (GD&T) standards, such as ASME Y14.5M [10] and ISO 1101[11]. Tolerance related variation analysis is routinely carried out in practice to evaluate geometric deviation impact on quality and functional requirement. Two commonly used methods are worst case method and statistical analysis method. Worst case method works on assembly deviation assumption in worst possible assembly circumstances [12], [13]. It ensures complete interchangeability of parts and 100% acceptability of assemblies [14]. On the other hand, statistical analysis methods are based on theory of probability and statistics [15], [16]. It can ensure acceptability of a certain large number of assemblies while achieve considerable reduction on assembly and manufacturing costs [17].

To carry out a comprehensive variation analysis, a mathematical model to represent and process part defects, and link to geometric tolerance is required [18]. However, most of the established methods under existing CAT tools do not consider form deviation [19]. They treat manufacturing part as rigid body and model only position and orientation errors along six degrees of freedom (DOFs), with three transitional movements and three rotational movements in 3D space. This is not a sufficient method to handle form errors, which in principle involve infinite number of DOF.

In early efforts to address issues related to compliant assembly, the Finite Element Method (FEM) [20] and variation sensitivity matrix were introduced with deformation and spring back effects being considered [21], [22]. However, part shape errors are not explicitly modeled in these linear approaches even it was recognized that these shape errors can have a significant impact on compliant assembly [23], [24].

Limited progress has been made so far on shape variation modeling and simulation. Huang *et al*. proposed Statistical Modal Analysis for variation modeling [25], [26]. The method aims to decompose shape error into several key orthogonal error modes using Discrete Cosine Transformation (DCT) technique, which is widely used in image compression. Under DCT method, a surface is described as a sum of cosine functions sorted by frequencies. This way of surface description allows dividing surface information (form, waviness, roughness, etc.) and enables relatively efficient computation. The first few modes contain key information on shape variation. Modeling accuracy increases when further modes were taken into account. Then surface variation is a linear combination of these modes based defects. Related simulation can be carried out along these basis defects. Summation of them generates the shape variation [27]. The initial 2D method was later on extended to 3D geometries [28].

Modes can also be extracted through Fourier series method [29], [30] and take the form of natural vibration modes [31], [32]. Modes based methods require manufacturing part surface measurement data (i.e., a large amount of points data gathered from tactile or optical measurement machines) as model input to extract the modes. As such, the effectiveness of this type of method is subject to data measurement uncertainties. In addition, modes related methods are less effective to handle random noisy local variation. Further, required measurement data may not be readily available in early design stage. All these impose limitations on the application of modes based methods.

Another method developed to handle geometric form deviation is Skin model [33], [34]. Under which, discrete geometry scheme (such as 3D point clouds) is used to represent variational parts and generated instances are called “Skin model shapes”. Shape deviation is split into two components, i.e., systematic deviation and random derivation. Systematic part deviation is usually modeled using second order shapes (quadrics) [33]. And random geometric deviations, which are added to the systematic part deviations, are simulated by Gaussian random fields [33], [35] or similar approaches. It is worth noting that generated Skin model shapes under this approach do not necessary conform to pre-defined tolerance. Subsequently, scaling method [36] was proposed to enable generating discrete Skin model shapes in conformance to tolerance. One drawback of Skin model is its discrete variation modeling. Point clouds only approximate the surface of target object and they do not encode detailed geometric information about relationships between points. It requires additional modeling efforts to handle surface continuities at edges [37]. By and large, Skin model is still a method-in-developing with latest development provided by [38].

Every manufacturing process leaves on the surface a signature, i.e., a systematic pattern that characterizes all the features produced with that process. This has been seized on by researchers to take into account form errors through manufacturing signature. Manufacturing signature is represented by means of Simultaneous Autoregressive Model of first order [39] or ARMAX model [40]. These models aim to represent and capture correlation of points on the target surface. And developed method has been applied in combination with Skin model to tackle geometric variation related problem [41].

One recent approach for early stage shape variation modeling and simulation is morphing mesh method [42], [43]. Modeled deformation can follow geometric consideration or a physical law including elastic deformation, mass-spring, and particle systems. The morphing method allows for fast generating shape errors with continuous deformation and can be applied to various compliant assemblies. However, generated deformed part instances under morphing mesh method are not guaranteed to be within the tolerance zone, in other words, it does not necessary conform to pre-defined tolerance specification. Further, user needs to identify the region of influence of deformation when applies morphing mesh method, which is a non-trivial task.

More details about existing methods in this area can be found in review paper [44]. Clearly, there is a strong lack of solid approaches to represent and generate probable shape variation instances for compliant assembly. In view of this, a novel methodology has been introduced in this paper to take on this challenge. It can be applied at early design stage to generate part variation representatives, which can be visualized in 3D virtual space to aid modification, analysis and optimization of compliant assembly. A brief comparison of proposed methodology with major existing methods is provided in Table I.

TABLE I
COMPARISON OF METHODS FOR COMPLIANT PART VARIATION MODELING AND SIMULATION

	Not relying on measurement data	Continuous deformation	Tolerance conformance
Modes based methods	✗	✓	✓
Skin model shapes	✓	✗	✓
Morphing mesh method	✓	✓	✗
Proposed approach	✓	✓	✓

III. Basic Concepts and Approach Overview

The proposed methodology is based upon a purpose-built variation tool which circumvents direct modeling geometric variation and brings considerable benefits. The variation tool is constructed from base parametric curves, i.e. Bezier curves, B-Splines or Non-uniform rational basis spline (NURBS) among others. The variation tool allows user to deform a 3D manufacturing part by repositioning a compact set of predefined control points. The developed method utilizes free-form deformation (FFD) technique [45], [46] which is widely used in computer graphics [47], medical image processing [48], and computer animation [49]. For ease of illustrating the method, classic Bezier curves are chosen below to construct the variation tool.

A. Variation tool and involved technique

A parametric space envelope is a variation tool that can be constructed by Bezier curves with extension from one-dimensional curve to three-dimensional Bezier tensor product volume. Let Bezier volume of degree l, m, and n be defined by a set of $(l+1)\times(m+1)\times(n+1)$ control points $\mathbf{P}_{ijk}$ ($i = 0, 1, \ldots, l$; $j = 0, 1, \ldots, m$; $k = 0, 1, \ldots, n$). A three-dimensional Bezier volume is a parametric volume where the position of a point $\mathbf{S}$ as a function of the parametric coordinates u, v, w is given by:

$$\mathbf{S}(u,v,w)\big|_{XYZ} = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} B_i^l(u)\,B_j^m(v)\,B_k^n(w)\cdot \mathbf{P}_{ijk}\big|_{XYZ} \qquad (1)$$

where $B_i^l(u)$, $B_j^m(v)$ and $B_k^n(w)$ are Bernstein polynomials of degree l, m and n, respectively. When manufacturing part is enclosed into the Bezier volume, it gets deformed under the influence of control points and the deviation is as follows:

$$\Delta\mathbf{S}(u,v,w)\big|_{XYZ} = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} B_i^l(u)\,B_j^m(v)\,B_k^n(w)\cdot \Delta\mathbf{P}_{ijk}\big|_{XYZ} \qquad (2)$$

where $\Delta\mathbf{P}_{ijk}$ denotes control points' displacement. The deformed surface $\mathbf{S'}$ of target compliant part can be calculated as:

$$\mathbf{S'} = \mathbf{S} + \Delta\mathbf{S} \qquad (3)$$

The above variation tool applies free-form deformation technique, and the variation is carried out in three steps: (i) Mapping the target object from physical domain (the global coordinate system or the *xyz*-space) to the reference one (the local coordinate system or the *uvw*-space). This is illustrated in Fig. 2(b) where a plain metal sheet is placed into the deformation tool. Local coordinates (u, v, w) is assigned to target object. (ii) Moving some control points to deform the lattice. This is shown in Fig. 2(c) where middle control points were perturbed and the enclosed metal sheet gets deformed accordingly. (iii) Mapping back to the physical domain. The deformed metal sheet (Fig. 2(d)) can be revealed by mapping back to physical domain. The mapping is done through equations (1-3).

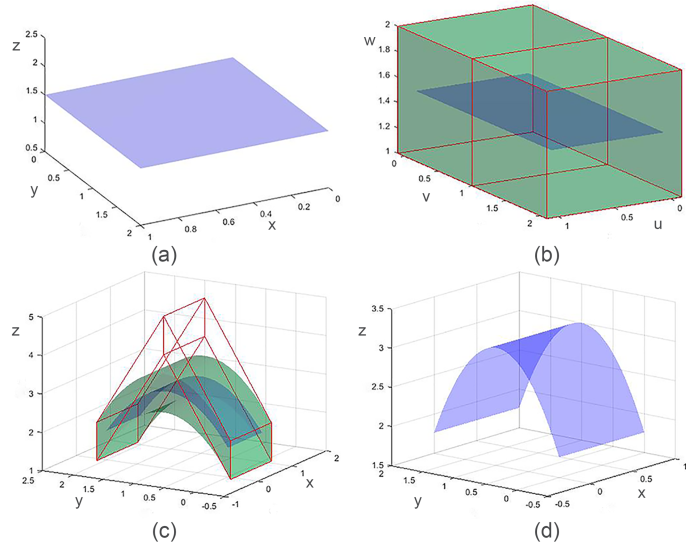

Fig. 2. Variation of a plain metal sheet under constructed variation tool. (a) A plain metal sheet. (b) Metal sheet is placed into a variation tool constructed from a Bezier curve. (c) Control points were moved and embedded metal sheet gets deformed accordingly. (d) Deformed metal sheet.

B. Benefits of constructed variation tool

The constructed variation tool can provide smooth continuous deformation observed in compliant assembly production line, i.e., without fracture[1]. This is due to the properties of the underlying base parametric curve [50] that modeled deformation can maintain C^k ($k \geq 2$) continuity which practically means smooth deformed surface. The involved technique is termed "free-form" because it is independent of the underlying object to be deformed. This enables the variation tool to handle complex compliant part.

The constructed variation tool is versatile and efficient in terms of representing and modeling non-rigid manufacturing part. In Fig. 3(b), part's dimension was changed following top right and top middle control points' upward movement, i.e. due to stretching force applied to the part during assembly. If there were an external drag force acting along the middle line during assembly, the flatness of the part's top surface will change accordingly and this can be modeled by moving top middle control points upwards (Fig. 3(c)). More importantly, the dimensional deviation and surface flatness deformation can be stacked up under the variation tool through combining the control points' movement with results shown in Fig. 3(d). This simplified example illustrates the modeling efficiency offered by the proposed method. For example, dimensional deviation and surface deformation can be handled

[1] Fracture, or rupture, is the phenomenon wherein a structural component or feature breaks into two or more pieces. This occurs only in extreme cases and is a clear failure, which is defined as non-conformance to a specification in industry.

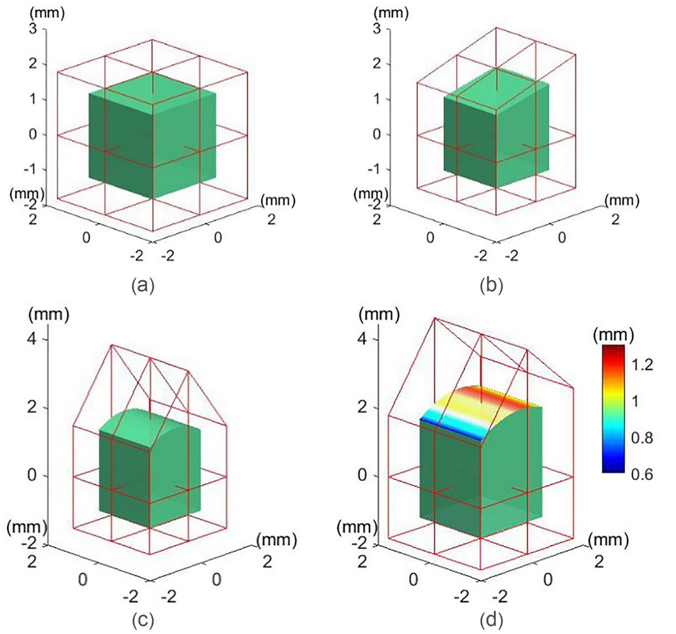

Fig. 3. Dimension and geometric shape variation of a manufacturing part under constructed variation tool. (a) A manufacturing part is enclosed into constructed variation tool. (b) Dimensional deviation under control points' movement. (c) Shape deformation under control points' movement with top plane deformed into a curved surface. (d)Stack-up of dimensional deviation and shape deformation.

simultaneously. User doesn't need to model dimensional deviation and surface flatness deformation separately, and then combine them by taking extra calculation of the interaction between these two.

C. Control point influence zone

To meet product quality and functional requirement, compliant part's deviation **d** during manufacturing and assembly has to be within tolerance:

$$\mathbf{d}=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}B_i^l(u)B_j^m(v)B_k^n(w)\cdot\mathbf{\Delta P}\cdot\mathbf{N}_{(u,v,w)}\leq Tol \qquad (4)$$

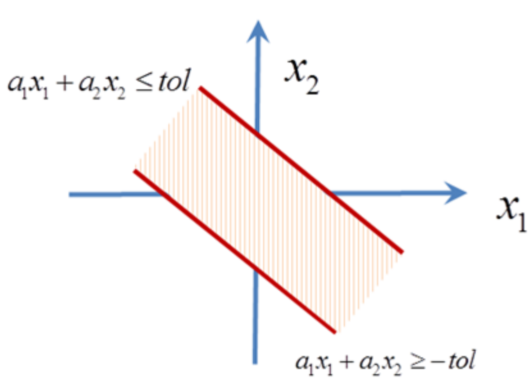

Fig. 4. Illustration of a feasible region for one linear inequality constraint in two-dimensional (2D) space.

where $\mathbf{N}_{(u,\,v,\,w)}$ is the normal at the sampled point and *Tol* is short for tolerance. For *h* sampled points from target compliant part, this is

$$\begin{cases} d_1=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}B_i^l(u)B_j^m(v)B_k^n(w)\cdot N_1\cdot\Delta P_1\leq Tol_1 \\ d_2=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}B_i^l(u)B_j^m(v)B_k^n(w)\cdot N_2\cdot\Delta P_2\leq Tol_2 \\ . \\ . \\ . \\ d_h=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}B_i^l(u)B_j^m(v)B_k^n(w)\cdot N_h\cdot\Delta P_h\leq Tol_h \end{cases} \qquad (5)$$

or written in matrix:

$$\left|\mathbf{A}\cdot\mathbf{\Delta P}\right|\leq\mathbf{t} \qquad (6)$$

where $\mathbf{A}=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}B_i^l(u)B_j^m(v)B_k^n(w)\mathbf{N}_{(u,v,w)}$ is deformation matrix and $\mathbf{t}=\left|Tol_1,Tol_{2,}...,Tol_h\right|^T$. Let r_j (j =1, 2, … , g) be the radius of control point j's local sphere and g is the total number of control points. When control points are perturbed and moved position in their respective local spheres, the embedded manufacturing part will deviate away from its design intent. The problem to find the maximum allowable control point movement r_j can be converted into a *nonlinear programming problem* (NLP):

$$\begin{aligned} &\text{maximize} && r_j \\ &\text{subject to} && \left|\mathbf{A}\cdot\mathbf{\Delta P}\right|\leq\mathbf{t} \\ & && \left\|\mathbf{\Delta P}_j\right\|\leq r_j \end{aligned} \qquad (7)$$

Geometrically, each linear inequality constraint $\left|\mathbf{A}_i\cdot\mathbf{\Delta P}\right|\leq\mathbf{t}\quad i=1,2,...,h$ defines a strip which is the intersection of two half-spaces defined by the two individual inequalities. Fig. 4 provides an illustration in $\mathbb{R}^2$. $\left|\mathbf{A}\cdot\mathbf{\Delta P}\right|\leq\mathbf{t}$ forms a convex hull that defines a minimum enclosed zone which is essentially the solution to the system of linear inequalities. There are well-developed numerical methods (i.e., quasi-Newton method [51]), algorithms (i.e., genetic algorithm [52], [53]), and technique [54] to solve this non-linear programming problem. By solving NLP in (7), we can get allowable control point deviation r_j, j=1, 2 …, g. As such, the geometric form tolerance (*Tol*) of complex compliant part is converted to control point influence zones. It should be noted that each control point, in general, has different influence zone [55], i.e., $r_1\neq r_2$. Other tolerances under GD&T standards can also be modeled under the proposed modeling framework [56].

In above modeling and analysis, all geometric variations are contained within specified tolerance zone. In practice, statistical analysis is widely applied. User can adjust the condition in (7) with $\eta\|\Delta\mathbf{P}_j\| \le r_j$, where η is reliability coefficient. A table of η and the corresponding safety assembly rate can be produced [55]. From the table, user can choose specific η to meet their production needs, i.e. satisfying industry's six sigma (or $\pm 3\sigma$) production requirement to achieve 99.74% successful assembly rate.

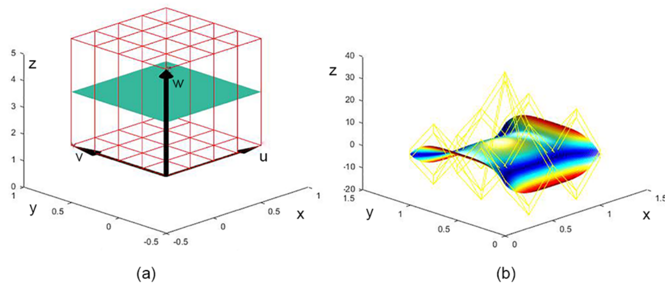

Fig. 5. Deformation of a plain metal sheet with multiple control points. (a) A plain metal sheet is enclosed into a deformation tool. (b) Plain metal sheet gets deformed following multiple control points' movement.

IV. Modeling parameters and Sensitivity analysis

In above developed modeling methodology, geometric variation of non-rigid workpiece is modeled through a purpose-built variation tool which, in turn, is constructed from a base parametric curve. Workpiece variation, the modeling output, gets impacted from various modeling parameters. The impact from these modeling inputs can be assessed with qualitative as well as quantitative analysis.

A. Qualitative analysis

The variation tool can be constructed from Bezier curves, B-splines or NURBS. As the base curve becomes more mathematically complex, the modeling precision increases but with higher computation costs.

B-splines are a more general type of curve than Bezier curve (a B-spline with no 'interior knots' is a Bezier curve) [57]. There are two interesting properties of B-splines that are not part of the Bezier basis functions, namely: (a) the domain is subdivided by knots, and (b) basis functions are not non-zero on the entire interval. In fact, each B-spline basis function is non-zero on a few adjacent subintervals and, as a result, B-spline basis functions are quite "local". The local shape can be controlled by the independent adjustment of local control points. On the other hand, the global definition of the workpiece can be controlled by moving groups of control points simultaneously in a certain fashion. As such, B-spline based method provides user the ability to model global as well as local geometric variation under one modeling variation tool.

Compared with uniform B-splines, non-uniform Rational B-spline (NURBS) offers further modeling flexibility and precision. NURBS are essentially B-splines in homogeneous coordinates [58]. Like B-splines, they are defined by their order, and a knot vector, and a set of control points, but unlike simple B-splines, the control points each have a weight. When the weight is equal to 1, a NURBS is reduced to a B-spline and as such NURBS generalizes both B-splines and Bezier curves. NURBS offers further modeling accuracy in handling complex geometric variation as user can adjust the knots to provide refined geometric variation modeling in critical region.

Geometric variation of enclosed compliant part is linked to control points. From equations (1-3), modeled variation is linked to the degree of Bernstein polynomial, the number of control points selected, as well as the positioning of control points. Cubic Bezier curve with degree of 3 is a good choice for many applications as it can meet various application requirements. In particular, the modeled deformation can maintain C^2 continuity which practically means smooth deformed surface. Higher degree curves (n >3) are more computationally expensive to evaluate while achieving superior results.

Choosing more control points can allow for more variety of deviation and deformation of compliant part. Fig. 5(b) shows a variation pattern generated by applying more control points to the plain metal sheet in Fig. 2. Choosing more control points, i.e. more degrees of freedom, various geometric variation types can be realized. However, these increased benefits will be accompanied by rising computation costs.

Control points' influence on manufacturing part deviation is acted through the Bernstein polynomial weights in equations (1) and (2). The more a control point is close to target part, the more it weighs in local coordinates expression of the target part, and the more its displacement acts on the target part. If key deformation region can be identified in advance, control points can be arranged to concentrate in that area (instead of being located evenly) to enable refined variation modeling and simulation.

B. Quantitative analysis

Above qualitative analysis provides user the insight and the direction of impact from these modeling parameters. For a given application, user can further carry out a quantitative sensitivity analysis to quantify the impact of these modeling inputs. The analytical sensitivity of non-rigid manufacturing part to control points' movement can be calculated as follows:

$$\frac{\partial S}{\partial p} = \begin{bmatrix} \frac{\partial S_x}{\partial p_x} & \frac{\partial S_x}{\partial p_y} & \frac{\partial S_x}{\partial p_z} \\ \frac{\partial S_y}{\partial p_x} & \frac{\partial S_y}{\partial p_y} & \frac{\partial S_y}{\partial p_z} \\ \frac{\partial S_z}{\partial p_x} & \frac{\partial S_z}{\partial p_y} & \frac{\partial S_z}{\partial p_z} \end{bmatrix} \quad (8)$$

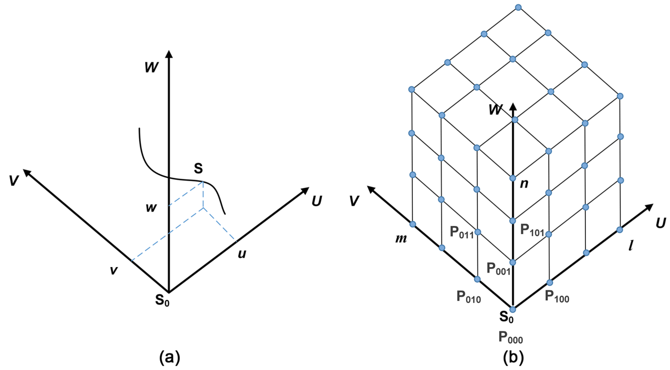

Fig. 6. Target compliant part is mapped from physical domain to reference domain. (a) Point S (sampled from target part) is assigned with local coordinate (*u*, *v*, *w*). (b) Positioning of control points.

This is the first partial derivative of deformable workpiece with respect to control point in (1). This computation examines workpiece deviation from small control points' perturbation. Calculated sensitivities can be checked against specific application requirements including related geometric tolerance. Then user can assess whether further adjustments on modeling parameters are needed. Based on above qualitative as well as quantitative analysis, user can choose suitable modeling parameters to construct the geometric variation tool to meet their application needs.

V. Compliant Part Variation Simulation

When target compliant part is enclosed into a variation tool, it will deviate away from its nominal position whenever boundary control points get perturbed. As such sampling control points' movement within their respective influence zone, user can generate various compliant part variation instances to enable a comprehensive variation analysis.

In earliest design phase, production has not started yet and real assembly deviation information is not available. The manufacturing process may not even be known or decided yet. Deformation type is unknown or may not be reliably inferred from similar other assemblies. Furthermore, part deformation and deviation along production assembly lines can be sporadic due to multiple factors including thermal induced variation, operating conditions caused deviation, and fixturing related variation. In these circumstances, Monte Carlo simulation can be applied to generate part variation instances which include both expected variation as well as unexpected ones.

To apply the proposed methodology, compliant part variation simulation can be conducted broadly in two stages. Stage I involves converting related GD&T tolerance zone to control point influence zone. This stage includes selecting relevant base parametric curve to construct the variation tool, deciding number of control points needed and determining the location of these control points through a sensitivity analysis. Once the modeling parameters are chosen based on application requirement and variation tool has been built, target compliant part is enclosed into the tool. A mapping of the target part from physical domain to reference domain

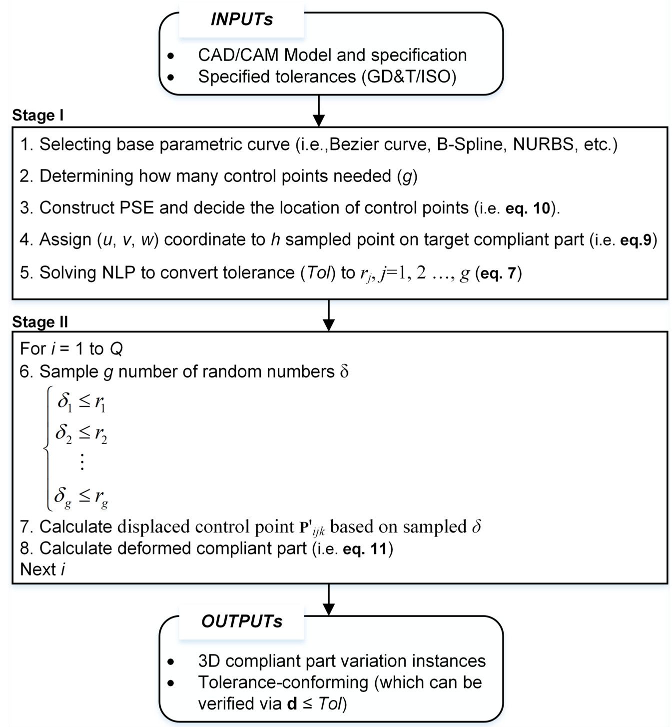

Fig. 7. Implementation of proposed methodology for variation simulation for compliant assembly.

needs to be done. This involves assigning local coordinates to the target compliant part. If Bezier curve is selected as base parametric curve to construct the tool with control points being evenly distributed, the constructed Bezier volume is denoted as $[X_{min}, X_{max}] \times [Y_{min}, Y_{max}] \times [Z_{min}, Z_{max}] \subset \mathbb{R}^3$. An arbitrary point S (sampled from the target part) in Fig. 6 is assigned with local reference frame coordinate (u, v, w) which can be calculated as follows:

$$\begin{cases} u = \dfrac{x' - X_{min}}{X_{max} - X_{min}} & u \in [0 \;\; 1] \\ v = \dfrac{y' - Y_{min}}{Y_{max} - Y_{min}} & v \in [0 \;\; 1] \\ w = \dfrac{z' - Z_{min}}{Z_{max} - Z_{min}} & w \in [0 \;\; 1] \end{cases} \tag{9}$$

where (X_{min}, Y_{min}, Z_{min}) is the coordinate of local frame origin S_0 in the global reference frame. Let l+1, m+1 and n+1 control points (down u, v and w direction, respectively) be selected. Total number of control points is g with $g = (l+1) \times (m+1) \times (n+1)$. The position of these evenly located control points $\mathbf{P}_{ijk}$ can be calculated as:

$$\mathbf{P}_{ijk} = \frac{i}{l}(X_{max} - X_{min}) + \frac{j}{m}(Y_{max} - Y_{min}) + \frac{k}{n}(Z_{max} - Z_{min}), \quad i \in [0,..., l], j \in [0,..., m], k \in [0,..., n] \tag{10}$$

The location of these control points is illustrated in Fig. 6(b).

The influence zone of control point $\mathbf{P}_{ijk}$ can be calculated by solving NLP in (7). As such, specified GD&T tolerance is converted to allowable control points' movement in this stage. This concludes the stage I implementation which is shown in the upper part in Fig. 7.

Stage II of the implementation involves simulating control points' movement to generate part variation instances. Assuming g control points being selected in stage I to construct the variation tool, variation simulation involves sampling g random numbers $\delta_1, \delta_2, \ldots, \delta_g$ independently using multivariate Gaussian methods [59 - 61]. The procedure is as follows:

- Sampling g random numbers $\delta_1, \delta_2, \ldots, \delta_g$ for each simulation trial with $|\delta_1| \leq r_1$, $|\delta_2| \leq r_2$, …, $|\delta_g| \leq r_g$.
- Compute displaced control point $\mathbf{P}'_{ijk}$ based on sampled δ.
- Generate deviated compliant part by combining (1) and (3) with following:

$$\mathbf{S}'(u,v,w)\big|_{XYZ} = \sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n} B_i^l(u)\, B_j^m(v)\, B_k^n(w)\cdot(\mathbf{P}'_{ijk})\big|_{XYZ} \quad (11)$$

- Repeat this process to generate more compliant part variation instances with Q, the number of simulation runs, determined by application needs.

The second stage of implementation is shown in the lower part of the panel in Fig. 7. Multi-Gaussian methods [59, 60] sample each control point movement following a spatial random vector which is generated by a multivariate Gaussian distribution. Conducting large number of simulation runs, different geometric variation representatives can be generated including expected as well as unexpected deviation and deformation cases. The output from Stage II implementation is a large pool of compliant part variation instances. Under proposed method, these generated variation representatives can be visualized in a 3D virtual space through the constructed variation tool. These vivid variation instances can aid designer to identify high-risk assembly region, assess assembly capacity, and optimize assembly operations.

VI. Industrial Case Study

To illustrate the proposed methodology, an industrial case study on a vehicle compliant part has been carried out. In Fig. 8(a), deformable hinge reinforcement plate and door inner panel of a vehicle door are shown. Related geometry including dimension of the target manufacturing part is shown in Fig. 8(b). The hinge reinforcement is joined with door inner panel through welding to provide sufficient strength to hold the door with main automotive body frame. It can be seen that the hinge reinforcement part has smooth complex surface and unique geometric topology. And this compliant part is liable to deform during assembly. In real production, the gap between the hinge reinforcement plate and door inner panel needs to be within a tight range to ensure satisfactory joining quality. Therefore, predicting and assessing this compliant part variation is crucial on production quality as it directly impacts the gap between the two joining parts and the subsequent

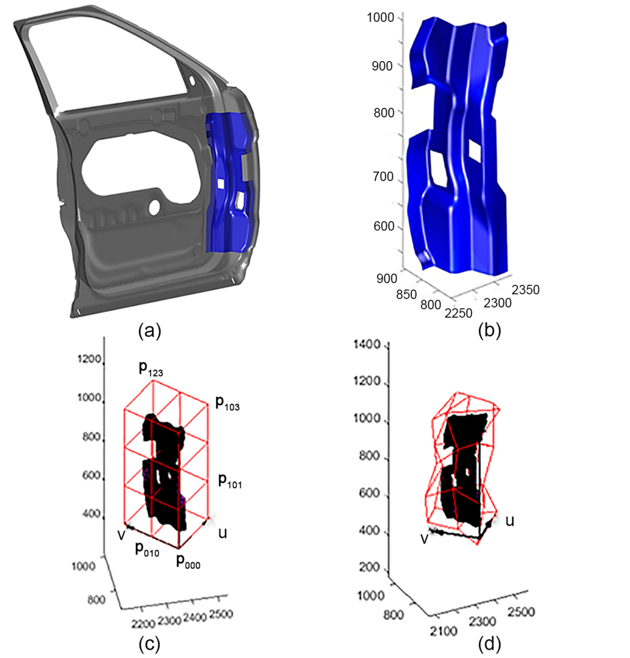

Fig. 8. Variation simulation case study on a compliant industrial automotive part. (a) Vehicle door inner panel and hinge reinforcement plate. (b) Geometry of hinge reinforcement plate. (c) Target deformable part is placed into a variation tool. (d) Target part gets deformed when control points being perturbed.

welding. The specified GD&T form tolerance for the hinge plate is 1 mm.

In this industrial case example, the proposed methodology is applied to simulate geometric variation of the hinge plate. Cubic Bezier curve is selected as base parametric curve to construct the variation tool. 24 control points are selected (2 alone u axis, 3 along v axis and 4 along w axis) and were evenly located. The constructed variation tool was shown in Fig. 8(c). The six sigma method [62] is widely adopted in car manufacturing industry to ensure at least 99.74% of manufactured parts to be within specified tolerance. Generic algorithm method is applied to solve the non-linear optimization problem in (7). The computed influence radii for the 24 control points satisfying six sigma industrial requirements are shown in Table II.

Based on Table II, Monte Carlo simulation is then carried out to generate a large sample of variated hinge plate instances. Some of the simulated representatives are highlighted and shown in Fig. 9. These vivid deformation instances can provide useful information to designer as well as assembly engineers. Based on these visualized deformation cases, designer can identify high-risk assembly region, select proper fixtures, adjust fixture location and fixture layout, and modify assembly operations accordingly.

VII. Method Discussions

The proposed methodology provides a systematic approach for generating compliant part variation instances in a fast and efficient way. One insight of the constructed variation tool is that the assigned local coordinate (u, v, w) of target compliant

TABLE II
RADDII (MM) OF CONTROL POINT (CP) BY SOLVING NLP SATISFYING SIX SIGMA

CP	Radius	CP	Radius	CP	Radius
P_{000}	0.784	P_{010}	0.962	P_{020}	1.324
P_{001}	2.076	P_{011}	1.048	P_{021}	0.898
P_{002}	0.738	P_{012}	1.285	P_{022}	1.175
P_{003}	1.138	P_{013}	0.807	P_{023}	0.791
P_{100}	0.773	P_{110}	0.725	P_{120}	0.776
P_{101}	1.727	P_{111}	1.036	P_{121}	0.896
P_{102}	1.106	P_{112}	1.142	P_{122}	1.225
P_{103}	1.048	P_{113}	0.926	P_{123}	1.763

part does not change throughout the deformation and deviation process. It is the corresponding global coordinate gets changed. Any part embedded in the variation tool will get deformed. If just some portion of the compliant part enclosed in the variation tool, then only that portion gets deformed. And the continuity between the deformed portion and un-deformed portion of target part can be maintained by "freezing" certain rows of control points [63]. As such, local variation simulation can be conducted by the proposed method. Shape variation in real assembly can come from global effect (e.g., spring back effect that affects large or entire part) or local effect (e.g., surface defects that affect a small localized region of a part) [64]. Many existing methods, i.e., Skin model shapes and modes based method in Table I have difficulty to simulate local variation [65].

The constructed variation tool involves a compact set of control points with relatively efficient calculation. Unlike boundary parameterizations, the tool can be applied to arbitrary complex manufacturing part while still keeping a small number of degrees of freedom. More degrees of freedom mean more memory requirement and larger computation costs.

In free-form surfaces, functionality may vary along the surface raising the requirement for non-uniform tolerancing. Latest GD&T standards start to introduce the non-uniform tolerance zone, i.e. in standard ASME Y14.5-2009. This non-uniform tolerance can be handled by the proposed method with vector $\boldsymbol{t}$ to be replaced by $\boldsymbol{t} = [Tol_{1,}Tol_{2,} \dots, Tol_h]^T$, where Tol_1 and Tol_2 can take different value. In early phase design, i.e., before prototyping, nominal product design data is available and typically embedded in part's CAD model (or file). Related GD&T tolerance will also be attached or to be assigned by product designer. These provide required information on conducting variation simulation analysis under our proposed methodology. It does not require historical variation statistics, nor does it need a sample of real deformation cases. This can be valuable to early stage designer when such information is not available or costly to get, i.e., from a set of trial run.

VIII. VALIDATION AND USE OF SIMULATION RESULTS

Geometric variation simulation is an important step of the manufacturing process because it focuses on all sources of variations which may affect the quality of the product and tries to simulate them to predict failures and faults.

In general, geometric shape error correlates with

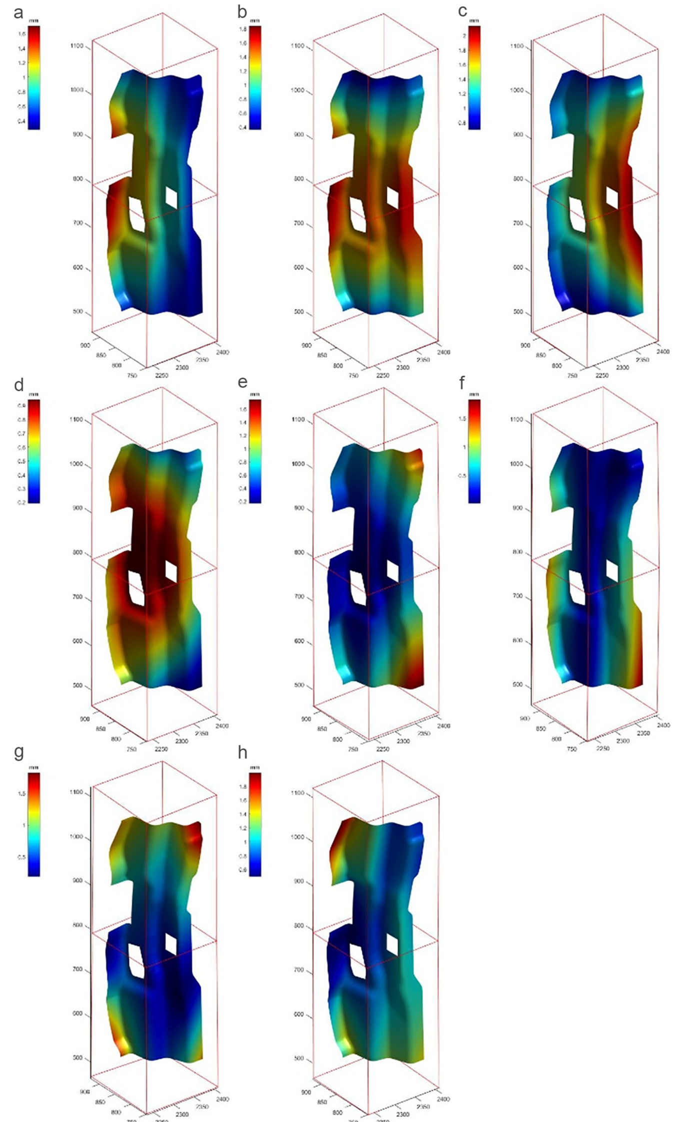

Fig. 9. Simulated variation instances of hinge reinforcing plate.

manufacturing operation. In real production, similar geometric deviations can be observed on nearly every generated part. These deviations are predictable and reproducible, and driven by manufacturing process to a large extent. Under proposed methodology, geometric variation is contained within the convex hull of boundary control points and "follows" the control points' movement [50]. As such, it can model geometric variation from bending, twisting, punching or stretching manufacturing operations that are typically applied to compliant parts during assembly [66]. For example, variation of a deformable part from a punching operation can be modeled through moving control points along the direction of that external load. During a typical sheet metal forming process, bend deformation is occurred to a piece of sheet metal when an external force is applied (Fig. 2(d)). This bending deformation can be modeled by moving certain control points in the external force direction as shown in Fig. 2(c). If certain manufacturing process has been selected in design phase, user can focus on variation instances, in the simulation run output, related to operations (stamping, twisting, clamping, etc.) in that specific manufacturing process.

In early design stage, production information is scarce. However, if certain real-deformation case is available through physical prototype run or collected from similar other assembly line, user can validate the simulation results by checking whether the real-case variation is captured in the simulation run output. On this, various methods are available [67], [68]. Under proposed framework and methodology, it involves fitting control points' movement to reproduce the real deformed manufacturing part. Technically, this involves solving an optimization problem (i.e., minimizing distance between real deformed case and the reproduced part through repositioning control points within their influence zone). If more real deformation cases are available, user can focus on similar type of variation instances from simulation run. And related assembly engineering changes can then be guided by these more likely variation scenarios in production.

In the field of product design, the need to view an object in three dimensions is imperative, as it ensures that the design meets various needs and specifications. The ability to see the deformed part in a virtual space makes it easier for designers to detect and fix any flaws that may exist in their design. The output from the proposed method is vivid three-dimensional part variation instances that resemble what the deformed part will look like in real assembly production. It enables user to do a type of virtual prototyping by viewing a virtual deformed part from different angles. In this way, engineers can quickly explore the performance of design alternatives without investing the time and money required to build physical prototypes. The ability to explore a wide range of design alternatives leads to improvements in performance and design quality.

IX. Conclusion And Future Research

We have presented a new methodology for geometric variation simulation and visualization for compliant part in early design stage. The proposed approach provides a systematic and efficient way to (early stage) designer to conduct variation simulation analysis on complex compliant part. The method also enables user to see deformed part through 3D visualization long before the prototyping stage. The developed methodology is versatile and can be integrated into existing CAD/CAM systems. It augments product designers' capability, provides valuable support for industrial design, reduces the need for multiple prototypes, and speeds up new product delivery by shortening the time period from early stage design to full production. It brings designers one step closer to the point of totally production-ready designs with increased accuracy and efficiency.

Principles of physics govern the dynamic behaviors of objects in the physical world. In our developed geometric method, material properties of compliant part are not explicitly considered. Add physical constraints [69] to the developed method, and take into account varying local rigidity (e.g. change of thickness, stiffeners, etc.) point to the direction of further enhancing the developed methodology in our future research.

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