Generalisations of groups and Cayley graphs

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Mathematics

December 2019

WARWICK

THE UNIVERSITY OF WARWICK

Contents

Acknowledgments

First and foremost I would like to thank my supervisor Agelos Georgakopoulos for taking me as a student and guiding me through this time in my life. I would also like to thank my previous undergraduate supervisors who got me interested in research, Dan Kral, Krzysztof Latuszyński, and Dmitriy Rumynin. During the PhD I did a 3 month internship at the Center for Science and Policy, I would like to thank everyone there for making that experience full of fun and insight.

During my PhD I was lucky enough to go on Academic visits to Iceland and Glasgow, I would like to thank Rögnvaldur Möller and Brendan Owens for supporting me in this. Though I also got the opportunity to work with many excellent academics at Warwick, I would like to thank Katerina Hristova (who I also want to thank for doing a heroic job proof reading my thesis), Giulio Morina, and Stephen Spooner. I would like to generally thank everyone in the Geometry and Topology group, for providing me with a great family to work with and for many eventful evenings!

I would never have made it through without the support of countless people, some of which I have inevitably forgotten to mention by name. I would like to thank my long term house mates and general mathematical crutches James Plowman, and Bogdan Alecu, you had me covered when it comes to Homological Algebra or Graph theory. I would like to thank Ronja Kuhne, Esmee Te Winkle, and Alex Torzewski for always being there to answer questions about topology, algebra or the meaning of life. Lastly I would like to thank my family for providing a safe place to hibernate from the PhD and for always being my fall back proof readers.

I used many computer languages and packages when doing my thesis. Which all too commonly go unacknowledged, I have used Sage, Magma and MatLab. The continual development of which help many academics all the time. Lastly, as someone with Dyslexia, I couldn't have done even the undergraduate degree without Latex. This thesis was typeset with LaTeX 2ε[1] by the author.

Lastly I would like to thank my examiners Derek Holt and Paul Martin, for an interesting conversation and useful contributions to this thesis.

[1] LaTeX 2ε is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The style package *warwickthesis* was used.

Declarations

Chapter 1 is mainly back ground material and introduces the basic notation, this is all widely available in the literature. Chapter 2 is work of my supervisor, Agelos Georgakopolous, and myself with the appendix co-authored by Matthias Hamann. This work will appear as a paper in its own right. Chapter 3 is a combination of work as follows; section 1 is widely available background material, Section 2 is joint work with Katerina Hristova and myself that is currently work in progress, and Section 3 is joint work with Dmitriy Rumynin and myself which is published [48]. Some of the proofs included in chapter 3 are new, taking a more elementary angle. Results from [48] that have proofs are considered work for the thesis, statements with no proof from [48] are considered to be from the paper. Within subsection 3.3.3, a result from the author's masters thesis is used and the updated proof is included for completeness sake.

Abstract

Groups are one of the most fundamental objects in Mathematics and have been generalised in many fashions. This thesis focuses on two generalisations.

Within the study of groups by geometric and combinatorial group theorists, instead of thinking about groups as an algebraic object they choose to study them through their Cayley graph. This has paved the way to many simplified proofs of properties about groups. Cayley graphs are vertex transitive graphs with a regular action by a group. However, not all vertex transitive graphs have a regular action and so cannot be Cayley graphs. This is reflected in the comparable levels of knowledge about them. The first chapter in this thesis generalises the concept of a group presentation and their associated Cayley graph. We hope this will open the door for techniques from combinatorial and geometric group theory to be applied to the study of vertex transitive groups.

The second is the study of groups in higher categories. Cayley pointed out that the study of groups is really just the study of symmetries. When we categorify groups into the setting of 2-categories, we study the symmetries between the symmetries given by a classical group. That is we allow the group axioms to hold only up to natural isomorphism. From this point of view we study 2-groups in the same way people studied classical groups, namely through their actions on vectors spaces. In this setting the 2-Vector spaces. The work provides an explicit formula for their characters.

Chapter 1

Introduction

This thesis combines three projects in two genres. Each genre has its own introduction. However, here we provide an overview of what is included in the thesis. The thesis focuses on generalisations of groups and Cayley graphs.

First, we generalise the concept of group presentations and the associated Cayley graph, so that the definition can generate all vertex transitive graphs. We call these split presentations and split graphs, and show that this generalisation does indeed capture all vertex transitive graphs.

Theorem 2.3.12 Every connected vertex transitive graph has a split presentation.

To do this we prove a result within vertex transitive graphs. This can be read by itself, so we include it in the appendix to this chapter.

Theorem 2.7.1 Every infinite, connected, vertex transitive graph has a perfect matching.

Lastly, we use these ideas to solve an open problem of Watkins [53], and Grimmett and Li [20].

Theorem 2.5.1 There exists a cubic 2-ended vertex transitive graph which is not a Cayley graph.

In the third chapter we look at the generalisation of groups to higher categories, namely 2-groups. However, these generalisations can be done in a number of ways. The first result of this chapter is a new way to get from a skeletal 2-group to a 2-group given by a crossed module.

Theorem 3.2.3 A skeletal 2-group given by (G, H, α) is equivalent to a crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ given by:

- group $B = G \times \mathrm{Mor}_{Set}(G, H)/H$ where $(X, [\theta^1]) \otimes (Y, [\theta^2]) = (XY, [\theta])$ with

$$\theta(z) = \alpha(X, Y, z) + {}^X\theta^2(z) + \theta^1(Yz),$$

- group $A = \mathrm{Mor}_{Set}(G, H)/H \times H$ where $([\theta_1], a) \otimes ([\theta_2], b) = ([\theta_1 + \theta_2], a + b)$,
- where $\partial : A \to B$ being $\partial([\Phi], h) = (1_G, [\Phi])$, and
- with the group action $B \hookrightarrow A$ given by $(X, [\theta]) : ([\Phi], h) \mapsto ([{}^X\Phi(X^{-1}-)], {}^Xh)$.

We study 2-groups, just like in classical group theory, by their representations. Associated to these are characters. We show that the space of 2-representations is the same as a certain Burnside ring and that these characters are a specific mark homomorphism.

Theorem 3.3.5 Let $\mathcal{K} = (A \xrightarrow{\partial} B)$ be a crossed module, P be the subgroup of $\pi_1(\mathcal{K})$ generated by $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$. Let $\alpha := \mathfrak{X}(\mathbf{b}, \mathbf{a}, h)$ considered as a group homomorphism $2\text{-Rep}^1(\mathcal{K}_P) \to \mathbb{K}^\times$. If the order of $\pi_1(\mathcal{K})$ is finite and invertible in the field $\mathbb{K}$, then

$$\mathfrak{X}(\mathbf{b}, \mathbf{a}, h) = f_P^\alpha.$$

We then use this understanding to write down a formula for the character in the case of the 2-group being a finite group.

Theorem 3.3.8 Let $\mathbf{a}, \mathbf{b} \in B$ be commuting elements, Θ a degree one 2-representation of B, $\mu \in Z^2(B, \mathbb{K}^\times)$ a cocycle such that $[\mu] = \{\Theta\}$. Then

$$\mathfrak{X}(\mathbf{b}, \mathbf{a})(\langle \Theta, B \rangle) = \mu(\mathbf{b}, \mathbf{a}^{-1})\mu(\mathbf{a}^{-1}, \mathbf{b})^{-1}.$$

Which we then use to rederive the formula originally given by Orsorno.

Theorem 3.3.11 ([45, Theorem 1]) Let Θ be a 2-representation of B that corresponds to a B-set X and a cohomology class $[\theta]$ for some cochain $\theta \in Z^2(B, (\mathbb{K}^\times)^X)$. Then

$$\mathfrak{X}_\Theta(\mathbf{b}, \mathbf{a}) = \sum_{x \in X,\ x = \mathbf{a}\cdot x = \mathbf{b}\cdot x} \frac{\theta^x(\mathbf{b}, \mathbf{a}^{-1})}{\theta^x(\mathbf{a}^{-1}, \mathbf{b})} = \sum_{x \in X,\ x = \mathbf{a}\cdot x = \mathbf{b}\cdot x} \frac{\theta^x(\mathbf{b}, \mathbf{a}^{-1})\theta^x(\mathbf{a}, \mathbf{b}\mathbf{a}^{-1})}{\theta^x(\mathbf{a}, \mathbf{a}^{-1})\theta^x(1, 1)}$$

for any commuting $\mathbf{a}, \mathbf{b} \in B$.
What follows is some introductory material which will be used throughout all the projects. We also set our notation that we use in the following work.

1.1 Preliminaries

In this section, we review some commonly known material for use in the later sections. We start by fixing our notation for some basic concepts.

1.1.1 Groups

When groups are referred to in this thesis they will in general use the notation G, H, N, A and B with appropriate super- and sub-scripts. We use the short hand $H \leq G$ to denote that H is a subgroup of G, and we use $H \lhd G$ to say it is a normal subgroup.

We define the *centraliser* of a subgroup $H \leq G$ by $C_G(H) = \{g \in G | gh = hg$ for all $h \in H\}$. If a group G is acting on a set X then we use $G_x := \{g \in G | gx = x\}$ to denote the point *stabiliser* of $x \in X$. Similarly, we define $G_Y := \{g \in G | gY = Y\}$ setwise stabiliser for $Y \subset X$. We define the *orbit* of $x \in X$ as $Gx = \{y \in X | y = gx$ for some $g \in G\}$ and similarly for sets with the set of orbits to be $\mathrm{Orb}_X(G)$.

A group action is *regular* if it acts transitively and with trivial point stabilisers. It is *semi-regular* if it acts only with trivial point stabilisers.

Given a set $\mathcal{S}$ we use $F_{\mathcal{S}}$ to denote the *free group* generated by $\mathcal{S}$. Given some words $\mathcal{R} \subset F_{\mathcal{S}}$ we use $\langle \mathcal{R} \rangle \leq F_{\mathcal{S}}$ to denote the subgroup generated by $\mathcal{R}$. We use $\langle\langle \mathcal{R} \rangle\rangle_{F_{\mathcal{S}}} \lhd F_{\mathcal{S}}$ to denote the normal closure of $\mathcal{R}$ in $F_{\mathcal{S}}$, the subscript will be dropped if the secondary group is clear from context. Some groups will be given by *group presentations* $G = \langle \mathcal{S} | \mathcal{R} \rangle$ where $\mathcal{R} \subset F_{\mathcal{S}}$ which is the group $G = F_{\mathcal{S}} / \langle\langle \mathcal{R} \rangle\rangle$. Some standard groups we use through out are the *cyclic group* of order n $\mathbb{Z}/n\mathbb{Z}$ and the *dihedral group* of order $2n$ D_{2n}.

1.1.2 Graphs

Graphs are refered to by the notation Γ, Δ and Λ with appropriate decoration. For the majority of the thesis we use the structure of graphs as in [50], that is a *graph* is:

- a set $V(\Gamma)$ (vertices),
- a set $\overrightarrow{E}(\Gamma)$ (directed edges),
- a fixed point free involution $^{-1} : \overrightarrow{E}(\Gamma) \to \overrightarrow{E}(\Gamma)$ (mapping edges to their 'opposite edge'), and
- a map $\tau : \overrightarrow{E}(\Gamma) \to V(\Gamma)$ (terminus map).

Sometimes elements of $\overrightarrow{E}(\Gamma)$ are expressed as directed pairs (v, w) with $v, w \in V(\Gamma)$, in which case we mean that $\tau((v, w)) = w$ and $(v, w)^{-1} = (w, v)$. We define the set of *undirected edges* $E(\Gamma) = \overrightarrow{E}(\Gamma)/e \sim e^{-1}$. An undirected edge uses notation $[e]$ or $[(u, v)]$ if we use the notation as above.

Note that although we are talking about 'directed edges', we are not talking about 'directed graphs' in the sense of [8]. Our edges can be thought of as undirected pairs of vertices, but our formalism allows us to distinguish between two orientations for each of them. Moreover, our formalism allows for multiple edges between the same pair of vertices, and multiple loops at a single vertex. Thus the pair $(V(\Gamma), E(\Gamma))$ is a multigraph in the sense of [8].

A *path* in a graph is an ordered set of directed edges $e_1, e_2, \ldots, e_n$ such that $\tau(e_i) = \tau(e_{i+1}^{-1})$, we would say this path has length n. A path is a *loop* if $\tau(e_n) = \tau(e_1^{-1})$. An *infinite ray* is an ordered set of directed edges $e_1, e_2, \ldots$ indexed by $\mathbb{N}$ such that $\tau(e_i) = \tau(e_{i+1}^{-1})$. A *bi-infinite ray* is an ordered set of directed edges $\ldots, e_{-1}, e_0, e_1, \ldots$ indexed by $\mathbb{Z}$ such that $\tau(e_i) = \tau(e_{i+1}^{-1})$.

The *neighbourhood* of a vertex $v \in V(\Gamma)$ is the subset of the vertices $\mathrm{Nb}(v) = \{x \in V(\Gamma) | (v, x) \in \overrightarrow{E}(\Gamma)\}$. This comes with a similar notion the *star* of a vertex $\mathrm{St}(v) = \{e \in \overrightarrow{E}(\Gamma) | \tau(e^{-1}) = v\}$. A directed edge $e \in \overrightarrow{E}(\Gamma)$ is a *loop* if $\tau(e) = \tau(e^{-1})$. The *degree* $d(v)$ of a vertex $v \in V(\Gamma)$ is the cardinality of $|St(v)|$. Our graphs can contain loops and double edges, normally when dealing with them topologically. A graph is called *n-regular* (or just *regular* if we don't want to specify n) if for all $v \in V(\Gamma)$ we have $d(v) = n$. We call 3-regular graphs *cubic*. A graph is *locally finite* if each vertex has finite degree.

A *map of graphs* $f : \Gamma \to \Delta$ consists of two maps $f_V : V(\Gamma) \to V(\Delta)$ and $f_E : \overrightarrow{E}(\Gamma) \to \overrightarrow{E}(\Delta)$ such that $f_V \circ \tau = \tau \circ f_E$ and $f_E \circ {}^{-1} = {}^{-1} \circ f_E$. Maps of graphs are considered to be rigid, therefore no collapsing of edges or mapping edges to paths. Define $\mathrm{Aut}(\Gamma) = \{f | f_V \text{ and } f_E \text{ are bijections}\}$ to be the group of *automorphisms* of a graph Γ.

We say that Γ is *vertex transitive* if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$, and *edge transitive* if $Aut(\Gamma)$ acts transitively on $E(\Gamma)$. We say that Γ is *arc-transitive*, or *symmetric*, if $Aut(\Gamma)$ acts transitively on $\overrightarrow{E}(\Gamma)$. We say Γ is *semi-symmetric* if it is edge transitive and regular but not vertex transitive.

For a graph Γ we say that it has a *vertex (edge or directed edge) colouring* by some set X if there is a mapping $c : V(\Gamma) \to X$ ($c : E(\Gamma) \to X$ or $c : \overrightarrow{E}(\Gamma) \to X$). An *orientation* of Γ is a subset $\mathcal{O} \subset \overrightarrow{E}(\Gamma)$ such that $|\mathcal{O} \cap \{e, e^{-1}\}| = 1$ for all $e \in \overrightarrow{E}(\Gamma)$.

We say that Γ is a *directed graph* if it comes with an orientation $\mathcal{O}$. We say that $\phi \in Aut(\Gamma)$ is *colour preserving* if $c = c \circ \phi$ and *direction preserving* if $\phi(\mathcal{O}) = \mathcal{O}$. Some standard graphs used throughout will be the *cyclic graph* on n vertices C_n, the *complete graph* on n vertices K_n and the *rose* on $\mathcal{S}$ $Ro_{\mathcal{S}}$. The rose $Ro_{\mathcal{S}}$ is the one vertex graph with a loop for every element in the set $\mathcal{S}$ which comes with a colouring by $\mathcal{S}$ and an orientation.

Given a graph Γ, its *line graph* $L(\Gamma)$ is the graph where $V(L(\Gamma)) = E(\Gamma)$ with an edge between any two edges that share a end vertex $\overrightarrow{E}(L(\Gamma)) = \{([e_1], [e_2]) | e_1, e_2 \in \overrightarrow{E}(\Gamma)$ and $\tau(e_1) = \tau(e_2)\}$.

Sometimes it is useful to pick out certain subgraphs, throughout we will use induced subgraphs. Given a subset of the edges $X \subset E(\Gamma)$, the *induced graph* Γ_X is defined by $V(\Gamma_X) = \{\tau(e) \in V(\Gamma) | e \in x \in X\}$ and $\overrightarrow{E}(\Gamma_X) = \{e | e \in x \in X\}$. Any subset of the edges $M \subset E(\Gamma)$ such that Γ_M has maximum degree 1 is called a *matching.* Any matching M such that $V(\Gamma_M) = V(\Gamma)$ is a *perfect matching.* A subset of the edges $H \subset E(\Gamma)$ is called a *Hamiltonian cycle* if the induced graph is 2-regular connected with $V(\Gamma_H) = V(\Gamma)$. A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

1.1.3 Group presentations and Cayley graphs

For an in detail look at these definitions please see [32]. To a group G with a subset $\mathcal{S}$ we can associate a directed edge coloured graph $\Gamma := \mathrm{Cay}(G, \mathcal{S})$ called its *Cayley graph.* The vertex set $V(\Gamma) = G$ are the elements of the group G and we connected $(g, gs) \in \overrightarrow{E}(\Gamma)$ by a directed edge coloured with $s \in \mathcal{S}$.

We have a natural left G-action on Γ by mapping $h \cdot g \mapsto hg$ which is a colour and direction preserving automorphism.

For ease of notation we may refer to these as $\mathrm{Cay}(G)$ if the set $\mathcal{S}$ is clear or $\mathrm{Cay}\langle \mathcal{S} | \mathcal{R} \rangle$ for a group given by a presentation with the generating set being $\mathcal{S}$. Unless otherwise stated, we are not assuming that $\mathcal{S}$ generates G. This implies that the Cayley graphs in this thesis are not all connected. In Figure 1.1 is some examples of such graphs.

In Figure 1.1 we have contracted involutions to a single edge, however formally this is not correct. This will be further explained in Chapter 2.

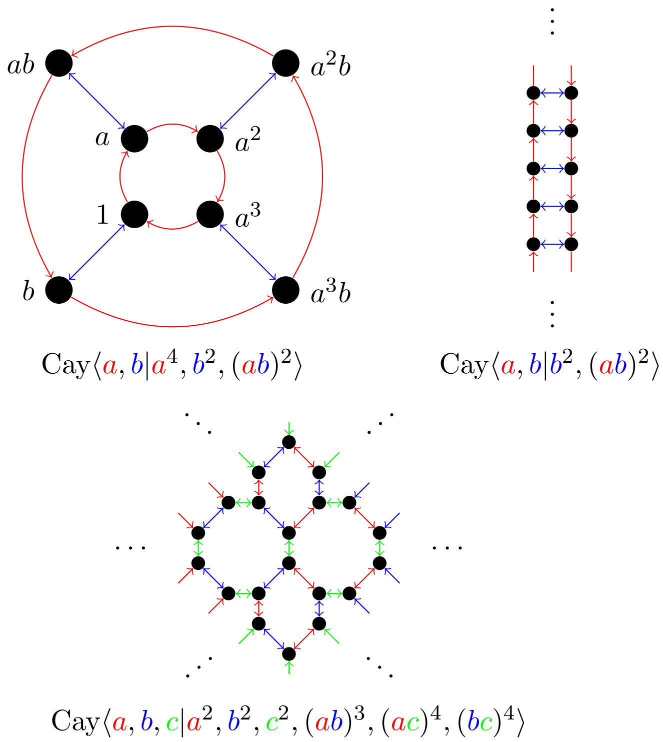

Figure 1.1: Examples of Cayley Graphs

1.1.4 Topology

For an in detail look at these definition, please see [23]. A 0-*dimensional simplicial complex* is a set of points with the discrete topology. A *simplicial complex* of dimension n will be a simplical complex of dimension $n-1$ called C, a set of n-discs $\mathbb{S}$ and a (attaching) map from the boundary of each n-disc to the space $a : \partial\mathbb{S} \to C$ which defines a topological space by $(C \bigcup \mathbb{S})/x \sim a(x)$.

Given a graph Γ with vertex set V, and any orientation on its edges $O \subset \overrightarrow{E}(\Gamma)$, we define a topological space as follows. Associate a point to each vertex, and a closed interval $I_e = [0, 1]$ to each edge $e \in O$. Then define the quotient $I_e(0) \sim \tau(e^{-1})$ and $T_e(1) \sim \tau(e)$ to obtain the topological space

$$\Gamma = (V \cup \bigcup_{e \in O} I_e)/\sim .$$

It is not hard to see that when Γ is connected this topological space is path-connected, locally path-connected and semilocally simply-connected (every point has a neighbourhood which is simply connected). Moreover, different choices of O define homeomorphic topological spaces. Lastly as all closed intervals I_e have a metric given by the subspace topology, we can take the path metric of Γ.

A *path* in a topological space X is a continuous mapping $p : [0,1] \to X$ with end points $p(0)$ and $p(1)$. If p_1 and p_2 are paths in x such that $p_1(1) = p_2(0)$ we can *concatenate* the two paths to get path $p_3 := p_1 \circ p_2 : [0,1] \to X$ so that $p_3(x) = p_1(2x)$ for $x \in [0,1/2]$ and $p_3(1/2 + x) = p_2(2x)$ for $x \in (0,1/2]$. A *homotopy* between paths p_1 and p_2 is a continuous map $h : [0,1] \times [0,1] \to X$ such that $h(0,x) = p_1(x)$ and $h(1,x) = p_2(x)$. A homotopy is *end point preserving* if $p(x,0) = p(y,0)$ and $p(x,1) = p(y,1)$ for all $x, y \in [0,1]$. A path is a *loop* if $p(0) = p(1)$. The *fundamental group* $\pi_1(X,x)$ is the set of loops with end point x up to end point preserving homotopy, this forms a group under the operation of concatenation. In a simplical complex (or graph) every element in the fundamental group has a representative whose image lies in the 0/1-cells (1-skeleton) of the simplical complex. In graphs it is common to think of elements of the fundamental group as loops in the graph theory sense.

A *covering space* (or cover) of a topological space X is a topological space C endowed with a continuous surjective map $\psi : C \to X$ such that for every $x \in X$, there exists an open neighbourhood U of x such that $\psi^{-1}(U)$ is the union of disjoint open sets in C, each of which is mapped homeomorphically onto U by ψ.

Given a map of spaces $\phi : Y \to X$, and a point $y \in Y$ such that $\phi(y) = x$, we obtain an induced map on the level of fundamental groups $\phi_* : \pi_1(Y,y) \to \pi_1(X,x)$ by composition. For a covering map ϕ Hatcher shows that ϕ_* is injective [23, Proposition 1.31]. If Y is arc-connected, and $\pi_1(Y,y) = 1$, that Y is simply connected, we call Y the *universal cover.*

Given a cover $\psi : C \to X$ and a map $\phi : Y \to X$ (with Y path connected, and locally path connected) we obtain a *lift* $\tilde{\phi} : Y \to C$ (where $\phi = \psi \circ \tilde{\phi}$) of ϕ if and only if $\phi_*(\pi_1(Y,y)) \subset \psi_*(\pi_1(C,c))$ [23, Proposition 1.33]. Moreover, for any preimage $c \in \psi^{-1}(x)$ we can choose $\tilde{\phi}(y) = c$.

Lastly we recall the classification of covering spaces:

Theorem 1.1.1. *(Hatcher [23, Theorem 1.38]) Let X be a path-connected, locally path-connected, and semilocally simply-connected topological space. Then there is a bijection between the set of isomorphisms classes of path-connected covering spaces $\psi : C \to X$ and the set of subgroups (up to conjugation) of $\pi_1(X)$, obtained by associating the subgroup $\psi_*(\pi_1(C))$ to the covering space C.*

1.1.5 Geometric group theory

Let X be a topological space, and suppose that

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \dots$$

is an ascending sequence of compact subsets of X whose interiors cover X. Then X has one *end* for every sequence,

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \dots$$

where each U_n is a connected component of $X \backslash K_n$. Then number of ends doesn't depend on the specific sequence K_i.

A map $f : X \to Y$ of two metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometry* if there exists constants $A \geq 1$, $B \geq 0$ and $C \geq 0$ such that the following two conditions hold:

- for all points $x, x' \in X$

$$\frac{1}{A} d_X(x, x') - B \leq d_Y(f(x), f(x')) \leq A d_X(x, x') + B,$$

- and for every $y \in Y$ there exists $x \in X$ such that

$$d(y, f(x)) \leq C.$$

We say that two spaces are *quasi-isometric* if there exists a quasi-isometry between them. This forms an equivalence relations. Moreover, when we say something is quasi-isometric to a group, we mean to a Cayley graph of that group.

It was shown by Brick [5] that if two locally finite graphs are quasi-isometric then they have the same number of ends. Another useful result of geometric group theory is the Šarc-Milnor lemma as stated below.

Theorem 1.1.2. *(Milnor [41]) If a group G acts on a length space X properly discontinuously and cocompactly then G is quasi-isometric to X.*

We note that for this thesis graphs are length spaces with the path metric. An action of G on X is *cocompact* if the quotient of X by G is compact. An action of G on X is *properly discontinuous* if for all compact sets $K \subset X$ the set $\{g \in G | K \cap g \cdot K \neq \emptyset\}$ is finite.

Chapter 2

Split Presentations

Every Cayley graph is (vertex-)transitive but the converse is not true, with the Petersen graph being a well-known example. A lot of research focuses on understanding how much larger the class of transitive graphs is or, what is essentially the same, on extending results from Cayley graphs to transitive graphs. Since the algebraic machinery is helpful in studying Cayley graphs, some of this work concentrates on algebraic descriptions of transitive graphs [42]. This thesis offers a new algebraic way of defining graphs, which we will prove to have the power to present all transitive graphs.

The idea is to still define our graphs by means of generators and relators similarly to Cayley graphs defined via group presentations, but we now allow different vertices to obey different sets of relators. The fewer 'types' of vertices we have the closer our graph is to being a Cayley graph. This is perhaps best explained with an example: in Figure 2.1 we have directed and labelled the Petersen graph with two letters r and b, represented by red and blue edges respectively, that make it look almost like a Cayley graph. But a closer look shows that if we start at any exterior vertex v and follow a sequence of edges labelled $brbrr$ then we return to v, while this is not true if v is one of the interior vertices. In that case, $brrbr$ is an example of a word that gives rise to a cycle.

This example motivates our definition of a *split presentation*, which prescribes a number of types of vertices, and a set of relators for each type. Moreover, it entails a set of generators, and for each generator s it prescribes the type of end-vertex of an edge labelled s for each type of starting vertex. The precise definition of split presentations in the case where there are only two types of vertices, which we call special split presentations, is given in Section 2.1. The case with more classes is more involved, and it is given in Section 2.3.

We show how each split presentation defines a regular graph, by imitating the stan-

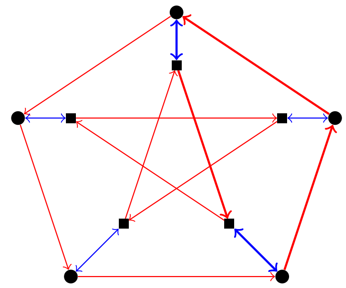

Figure 2.1: The Petersen graph $P(5,2)$. The relation $rbrrb$ is highlighted for the top square vertex.

dard definitions of a Cayley graph via a group presentation: either as a quotient of a free group by the normal subgroup generated by the relators (Definition 2.1.2), or as the 1-skeleton of the universal cover of the presentation complex (Definition 2.1.7). The resulting *split graph* is always regular, with vertex-degree determined by the generating set, and it admits a group of automorphisms acting on it semi-regularly and with as many orbits as the number of types of vertices prescribed by its presentation (Proposition 2.3.13). In particular, special split presentations always give rise to bi-Cayley graphs. We prove this, as well as a converse statement, in Section 2.2.

Our first main result says that our formalism of split graphs is general enough to describe all vertex transitive graph:

Theorem 2.3.12 Every connected vertex transitive graph has a split presentation.

In general, for the proof of this we allow for the vertex types to be in bijection with the vertex set of the graph in question. It would be interesting to study how much the number of vertex types can be reduced, see Section 2.6. In this spirit, we show, in Section 2.4, that every line graph of a Cayley graph Γ admits a split presentation with at most as many vertex types as the number of generators of Γ.

The proof of Theorem 2.3.12 involves decomposing the edge-set into cycles. This decomposition is not obvious though, and it is related to a conjecture of Leighton [28] disproved by Marušič [33]; see Section 2.3.2 for more. To find such decomposition we had to generalise a result of [19, Theorem 3.5.1], saying that every connected finite vertex transitive graph has a matching that misses at most one vertex, to infinite vertex transitive graphs, which might be of independent interest:

Theorem 2.7.1 Let Γ be a connected infinite vertex transitive graph which is locally finite. Then Γ has a perfect matching.

This is proved in the appendix, which can be read independently.

Incidentally, we find a cubic 2-ended vertex transitive graph which is not a Cayley graph, answering a question of Watkins [53], recently revived by Grimmett and Li [20]. Although this construction does not explicitly use the theory developed in this thesis, our study of split presentations helped us understand where to look for such examples.

Theorem 2.5.1 There exists a cubic 2-ended vertex transitive graph which is not a Cayley graph.

This is proved in section 2.5, which can again be read independently.

2.1 Special Split presentations

2.1.1 Algebraic definition

Let G be a group. A *presentation* $\langle \mathcal{S} | \mathcal{R} \rangle$ of G consists of a generating set $\mathcal{S} \subset G$ and a *relator set* $\mathcal{R} \subset F_{\mathcal{S}}$, where $F_{\mathcal{S}}$ denotes the free group with free generating set $\mathcal{S}$, such that $G = F_{\mathcal{S}} / \langle\langle \mathcal{R} \rangle\rangle$. For a group presentation $\langle \mathcal{S} | \mathcal{R} \rangle$, we can construct the *Cayley graph* $\mathrm{Cay}\langle \mathcal{S} | \mathcal{R} \rangle$ in the following manner. Let $T_{\mathcal{S}}$ be the $2|\mathcal{S}|$-regular tree defined by

$$\begin{aligned} &V(T_{\mathcal{S}}) := F_{\mathcal{S}}, \text{and} \\ &\overrightarrow{E}(T_{\mathcal{S}}) := \{(w, ws) | w \in F_{\mathcal{S}}, s \in \mathcal{S} \cup \mathcal{S}^{-1}\}. \end{aligned}$$

We endow $T_{\mathcal{S}}$ with a colouring $c : \overrightarrow{E}(T_{\mathcal{S}}) \to \mathcal{S} \cup \mathcal{S}^{-1}$ defined by $c(w, ws) = s$ and $c(ws, w) = s^{-1}$. Let $R := \langle\langle \mathcal{R} \rangle\rangle$ be the normal closure of $\mathcal{R}$ in $F_{\mathcal{S}}$. Define an equivalence relation $\sim$ on $V(T_{\mathcal{S}}) = F_{\mathcal{S}}$ by letting $v \sim w$ whenever $v^{-1}w \in R$. Extend $\sim$ to $\overrightarrow{E}(T_{\mathcal{S}})$ by demanding $e \sim d$ whenever $c(e) = c(d)$ and $\tau(e) \sim \tau(d)$ and $\tau(e^{-1}) \sim \tau(d^{-1})$. Then $\mathrm{Cay}\langle \mathcal{S} | \mathcal{R} \rangle$ can be defined as the quotient $T_{\mathcal{S}} / \sim$. The corresponding covering map is denoted by $\eta : T_{\mathcal{S}} \to \mathrm{Cay}\langle \mathcal{S} | \mathcal{R} \rangle$. Note that as $\sim$ preserves c, we obtain a unique colouring $c' : \overrightarrow{E}(\mathrm{Cay}\langle \mathcal{S} | \mathcal{R} \rangle) \to \mathcal{S} \cup \mathcal{S}^{-1}$ satisfying $c = c' \circ \eta$.

This definition of the Cayley graph is standard. All Cayley graphs defined this way have an even degree: involutions in $\mathcal{S}$ give rise to pairs of 'parallel' edges with the same endvertices. However, in certain contexts it is desirable to replace such pairs of parallel edges by single edges. To accommodate for this modification —which is important for us later as we want to capture odd-degree graphs such as the Petersen graph with our presentations— we now introduce *modified presentations* and *modified Cayley graphs*.

Let $\langle \mathcal{S}|\mathcal{R}\rangle$ be a group presentation such that $\mathcal{S}^{-1} = \mathcal{S}$ (we allow repeats of elements but these elements should have distinct inverses in $\mathcal{S}$). Define a bijective map $\mathbf{^{-1}} : \mathcal{S} \to \mathcal{S}$ such that $\mathbf{^{-1}} \circ \mathbf{^{-1}}$ is the identity and $s^{\mathbf{-1}} = s^{-1}$. Define the *modified presentation* $P = \langle \mathcal{S}, \mathbf{^{-1}}|\mathcal{R}\rangle$, note $\mathcal{S}$ divides into two sets $\mathcal{I} := \{s = s^{\mathbf{-1}} \in \mathcal{S}\}$ and $\mathcal{U} := \mathcal{S}\backslash\mathcal{I}$. Define the *modified free group* for such a pair $(\mathcal{S}, \mathbf{^{-1}})$ to be $F^{\text{Mod}}_{\mathcal{S},\mathbf{^{-1}}} = \langle \mathcal{S}|ss^{\mathbf{-1}}, s \in \mathcal{S}\rangle =: F^{\text{Mod}}_{P}$. (Thus F^{Mod}_{P} is a free product of infinite cyclic groups, half of one for each $s \in \mathcal{U}$, and cyclic groups of order 2, one for each $s \in \mathcal{I}$.) Let $\phi : F_{\mathcal{S}} \to F^{\text{Mod}}_{P}$ be the unique homomorphism extending the identity on $F_{\mathcal{S}}$, as provided by the universal property of free groups. Define the $|\mathcal{S} \cup \mathcal{S}^{-1}|$-regular tree T_P by

$$
\begin{aligned}
V(T_P) &:= F^{\text{Mod}}_{P}\\
\overrightarrow{E}(T_P) &:= \{(w, ws)|w \in F_P, s \in \mathcal{S} \cup \mathcal{S}^{-1}\}.
\end{aligned}
$$

We proceed as above to define the colouring c and the relation $\sim$, and obtain the *modified Cayley graph* as the quotient $T_P/\sim$.

We now modify the above construction of the Cayley graph, to obtain our split graphs. The basic idea is to partition the vertex set into two (and later more than two) classes V_0, V_1, obeying different sets of relators $\mathcal{R}_0, \mathcal{R}_1$. This bipartition creates the need to partition our generators too into two classes $\mathcal{S}_1, \mathcal{S}_2$, the former corresponding to edges staying in the same partition class, and the latter corresponding to edges incident with both classes V_0, V_1.

We will formally define a *special split presentation* as a 4-tuple $P = \langle \mathcal{S}_1, \mathcal{S}_2|\mathcal{R}_0, \mathcal{R}_1\rangle$, and explain how this data is used to define a split graph, in analogy with the above definition of a Cayley graph $\text{Cay}\langle \mathcal{S}|\mathcal{R}\rangle$ corresponding to a group presentation $P = \langle \mathcal{S}|\mathcal{R}\rangle$. The set $\mathcal{S}_1$ is an arbitrary set of 'generators'. The set $\mathcal{S}_2$ is partitioned into two disjoint sets $\mathcal{S}_2 = \mathcal{U} \cup \mathcal{I}$, so that $\mathcal{S}_1, \mathcal{U}, \mathcal{I}$ are pairwise disjoint. Their union $\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_1^{-1} \cup \mathcal{U} \cup \mathcal{U}^{-1} \cup \mathcal{I}$ will be our set of *generators*. Define $\mathbf{^{-1}}(s) = s^{-1}$ for $s \in \mathcal{S}_1 \cup \mathcal{U}$ and $\mathbf{^{-1}}(s) = s$ for $s \in \mathcal{I}$. The necessity of distinguishing $\mathcal{S}_2$ into $\mathcal{U}, \mathcal{I}$ is to allow for some involutions, namely the elements of $\mathcal{I}$, to give rise to single edges in our graphs, just like in the above definition of modified Cayley graph. This can't be done to $\mathcal{S}_1$ for the same reason the topological Cayley graph can't have odd degree, this will become apparent later.

Just as in our definition of modified Cayley graph, we let $F_P^{\mathrm{Mod}} := F_{\mathcal{S},\text{-1}}^{\mathrm{Mod}}$. Let $|\cdot|_{\mathcal{S}_2}$ be the unique homomorphism from F_P^{Mod} to $\mathbb{Z}/2\mathbb{Z}$ extending

$$|s|_{\mathcal{S}_2} = \begin{cases} 0 & \text{if } s \in \mathcal{S}_1 \\ 1 & \text{if } s \in \mathcal{S}_2 \end{cases}.$$

We have that $K := Ker(|\cdot|_{\mathcal{S}_2})$ is an index-two subgroup of F_P^{Mod}, and so its cosets $\tilde{V_0} := K$ and $\tilde{V_1} := \mathcal{S}_2 K$ bipartition F_P^{Mod}.

Definition 2.1.1. For any two sets $\mathcal{R}_0, \mathcal{R}_1 \subset K$, called *relator sets*, we call the tuple $\langle \mathcal{S}_1, \mathcal{S}_2 | \mathcal{R}_0, \mathcal{R}_1 \rangle$ a *special split presentation.*

(The restriction $\mathcal{R}_i \subset K$ does not have an analogue in the definition of Cayley graph; the intuition is that relators should start and finish at the same side of the bipartition V_0, V_1 because they are supposed to yield cycles in the graph.)

Given a special split presentation $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$, define the ($|\mathcal{S} \cup \mathcal{S}^{-1}|$-regular) tree T_P by

$$\begin{aligned} &V(T_P) := F_P^{\mathrm{Mod}} \\ &\overrightarrow{E}(T_P) := \{(w, ws) | w \in F_P^{\mathrm{Mod}}, s \in \mathcal{S} \cup \mathcal{S}^{-1}\}. \end{aligned}$$

We have a natural colouring $c : \overrightarrow{E}(T_P) \to \mathcal{S} \cup \mathcal{S}^{-1}$ defined by $c(w, ws) = s$. Define the subgroups

R_0 to be the normal closure of $\mathcal{R}_0 \cup \{srs^{-1} : r \in \mathcal{R}_1,\ s \in \mathcal{S}_2\}$ in K, and

R_1 to be the normal closure of $\mathcal{R}_1 \cup \{srs^{-1} : r \in \mathcal{R}_0,\ s \in \mathcal{S}_2\}$ in K.

Here $R_i \le K \le F_P^{\mathrm{Mod}}$ is the analogue of the normal subgroup R of F_P^{Mod} in the definition of $\mathrm{Cay}(P)$, but now having two versions corresponding to our two classes of elements of F_P^{Mod}, namely $\{\tilde{V_0}, \tilde{V_1}\} := \{K, \mathcal{S}_2 K\}$. In analogy with the relation $\sim$ above, we now write $v \sim w$ whenever $v^{-1}w \in R_i$ for $v, w \in \tilde{V_i}$. We extend $\sim$ to the edges of T_P via $e \sim d$ if $c(e) = c(d)$, $\tau(e) \sim \tau(d)$, and $\tau(e^{-1}) \sim \tau(d^{-1})$.

Definition 2.1.2. The *special split graph* $\mathrm{Spl}\langle \mathcal{S}_1, \mathcal{S}_2 | \mathcal{R}_1, \mathcal{R}_2 \rangle = \mathrm{Spl}(P) =: \Gamma$ is the quotient $T_P / \sim$.

The edge set of Γ can thus be written as $\overrightarrow{E}(\Gamma) = \overrightarrow{E}(T_P)/\sim$.

As before, we have a natural colouring $c : \overrightarrow{E}(T_P) \to \mathcal{S} \cup \mathcal{S}^{-1}$ defined by $c(w, ws) = s$, and as $\sim$ preserves c, the latter factors into $c' : \overrightarrow{E}(\Gamma) \to \mathcal{S} \cup \mathcal{S}^{-1}$, i.e. the unique colouring satisfying $c = c' \circ \eta$ where again η denotes the projection map corresponding to $\sim$.

Note that this is a generalisation of the modified Cayley graph. When $\mathcal{I} = \emptyset$ we have a generalisation of the standard Cayley graph.

Borrowing terminology from groupoids [34], we define the *vertex groups* of our split presentation to be $G_i := K/R_i$ for $i \in \{0, 1\}$.

The condition $\mathcal{R}_i \subset K$ implies that if $v \sim w$ then v and w belong to the same coset $\tilde{V_0}$ or $\tilde{V_1}$ of K in F_P^{Mod} by the definitions. Thus factoring by $\sim$ projects the bipartition $\{\tilde{V_0}, \tilde{V_1}\}$ of F_P^{Mod} into a bipartition $\{V_0, V_1\}$ of $V(\Gamma)$, with $V_i := \tilde{V_i}/\sim$. It follows from these definitions that G_i is in canonical bijection with V_i.

As in the case of Cayley graphs, relators in the presentation yield closed walks in Γ, but now we need to start reading our relators at the right side of the bipartition for this to be true: for every $i \in \{0, 1\}$ and each $r \in \mathcal{R}_i$ and $v \in V_i$, if we start at v and follow the directed edges of Γ with colours dictated by r one-by-one, we finish our walk at v.

We now explain how the Petersen graph can be obtained as a special split graph:

Example 2.1.3. Theorem 2.1.14 below asserts that the Petersen graph $P(5, 2)$ is isomorphic to $\mathrm{Spl}(\langle \mathcal{S}_1 = \{a\}, \mathcal{U} = \emptyset, \mathcal{I} = \{b\} \mid \mathcal{R}_0 = \{a^5, aba^2b, b^2\}, \mathcal{R}_1 = \{a^5\}\rangle) = \mathrm{Spl}\langle\ \{a\},\ \emptyset,\ \{b\} \mid \{a^5, aba^2b\},\ \{a^5\}\ \rangle$. For this presentation we have

- $F_P^{\mathrm{Mod}} = \langle a, b | b^2 \rangle$, so that T_P is the 3-regular tree;
- $K = \langle a, bab \rangle \leq F_P^{\mathrm{Mod}}$,
- $R_0 = \langle\langle a^5, aba^2b, ba^5b \rangle\rangle_K$, and
- $R_1 = \langle\langle ba^5b, baba^2, a^5 \rangle\rangle_K$.

The vertex groups $G_i = K/R_i$ are generated by any generating set of K, in particular by $\{a, bab\}$. They abide by the relations that generate R_i so in the case of R_0 these are a^5, $aba^2b = a(bab)^2$ and $ba^5b = (bab)^5$ (when we write them in terms of the generators of K). So we have

$$\begin{aligned} G_0 =& \langle a, bab | a^5, a(bab)^2, (bab)^5 \rangle \\ =& \langle bab | (bab)^{-10}, (bab)^5 \rangle && \text{as } a = (bab)^{-2} \\ =& \mathbb{Z}/5\mathbb{Z} = \langle bab \rangle \end{aligned}$$

and similarly

$$\begin{aligned} G_1 =& \langle a, bab | (bab)^5, (bab)a^2, a^5 \rangle \\ =& \langle a | a^{-10}, a^5 \rangle && \text{as } (bab) = a^{-2} \\ =& \mathbb{Z}/5\mathbb{Z} = \langle a \rangle. \end{aligned}$$

The fact that G_0 is isomorphic to G_1 is not a coincidence as we remark at the end of this section.

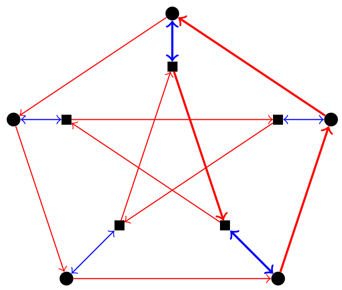

Figure 2.2: The Petersen graph $P(5,2)$ is isomorphic to $\mathrm{Spl}\langle\{a\}, \emptyset, \{b\} | \{a^5, aba^2b\}, \{a^5\}\rangle$. The square vertices are in $V_0 = \tilde{V}_0/\sim$ and circles are in $V_1 = \tilde{V}_1/\sim$. The relation aba^2b is highlighted for the top square vertex.

Note that we have made $\mathcal{S}$ a subset of the group F_P^{Mod}, and so each $s \in \mathcal{S}$ has an inverse s^{-1} in F_P^{Mod}. With these inverses in mind we define $\mathcal{S}^{-1} := \{s^{-1} : s \in \mathcal{S}\}$. Note that $s = s^{-1}$ exactly when $s \in \mathcal{I}$. Moreover, as $\mathcal{S}_1 \subset K$ and $G_i = K/R_i$, we can think of $\mathcal{S}_1$ as a subset of G_i in the following proposition:

Proposition 2.1.4. *For every special split presentation $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$, the subgraph of $\Gamma := Spl(P)$ with edges coloured by $\mathcal{S}_1 \cup \mathcal{S}_1^{-1}$ is isomorphic to the disjoint union of $Cay(G_0, \mathcal{S}_1)$ and $Cay(G_1, \mathcal{S}_1)$.*

Proof. Let T_i be the subgraph of T_P induced by the vertices of $\tilde{V}_i$, and Γ_i be the subgraph of Γ induced by $V_i = \tilde{V}_i/\sim$. We will show that Γ_i is isomorphic to $\mathrm{Cay}(G_i, \mathcal{S}_1)$.

To begin with, recall that $\tilde{V}_0 = K$ and $G_0 = K/R_0$, and so V_0 is canonically identified with G_0. Thus to show that Γ_0 is isomorphic to $\mathrm{Cay}(G_0, \mathcal{S}_1)$, we need to check that (v, w) is a directed edge of Γ_0 coloured s whenever $w = vs$. The latter holds whenever $v's \in \eta^{-1}(w)$ for every $v' \in \eta^{-1}(v)$, which is exactly when $(v', v's)$ is a directed edge of T_P coloured s. This in turn is equivalent to (v, w) being a directed edge of Γ_0 coloured s because $c = c' \circ \eta$.

This proves that Γ_0 is isomorphic to $\mathrm{Cay}(G_0, \mathcal{S}_1)$. To prove that Γ_1 is isomorphic to $\mathrm{Cay}(G_1, \mathcal{S}_1)$ we repeat the same argument multiplying on the left with a fixed element of $\mathcal{S}_2$ throughout. Since $V(\Gamma)$ is the disjoint union of V_0 and V_1, our statement follows. $\square$

Proposition 2.1.5. *For every special split presentation $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$, the graph $\Gamma := Spl(P)$ is $|\mathcal{S} \cup \mathcal{S}^{-1}|$-regular.*

Proof. By Proposition 2.1.4, the subgraph with edges coloured by $\mathcal{S}_1 \cup \mathcal{S}_1^{-1}$ is $2|\mathcal{S}_1|$-regular. It therefore suffices to prove that every vertex in Γ has a unique outgoing edge coloured s for every $s \in \mathcal{S}_2 \cup \mathcal{S}_2^{-1}$. Existence is easy by the definition of T_P. To prove uniqueness, suppose in T_P we have two edges $(v_0, u_0), (v_1, u_1) \in \overrightarrow{E}(T_P)$ where $c(v_0, u_0) = s = c(v_1, u_1)$ and $v_0 \sim v_1$. So by definition $u_i = v_i s$ and $v_0^{-1} v_1 \in R_i$ for $i \in \{0, 1\}$. Note that

$$u_0^{-1} u_1 = s^{-1} v_0^{-1} v_1 s = s^{-1}(v_0^{-1} v_1) s \in s^{-1} R_i s \subset R_{i+1},$$

which means that $u_0 \sim u_1$ and hence $(v_0, u_0) \sim (v_1, u_1)$ proving our uniqueness statement. $\square$

Corollary 2.1.6. *For a special split presentation $P = Spl\langle \mathcal{S}_1, \mathcal{S}_2 | \mathcal{R}_1, \mathcal{R}_2 \rangle$ the universal cover of $\Gamma := Spl(P)$ is T_P. Moreover, every edge with a colour in $\mathcal{S}_1$ connects two vertices in V_i for some $i \in \{0, 1\}$, and every edge with a colour in $\mathcal{S}_2$ connects a vertex in V_i to a vertex in V_{i+1}.*

Proof. Recall that $\sim$ defines a map of graphs $\eta : T_P \to \mathrm{Spl}(P)$, by $\eta(x) = [x]$. As both T_P and $T_P/\sim$ are $|\mathcal{S} \cup \mathcal{S}^{-1}|$-regular by Proposition 2.1.5, and η is locally injective, η is a cover. As the fundamental group of a tree is trivial we deduce that η is in fact the universal cover.

By Proposition 2.1.4, edges labelled $\mathcal{S}_1$ connect vertices in G_i to vertices in G_i, which are exactly the vertices in V_i. Moreover, in T_P edges labelled $\mathcal{S}_2$ connect vertices in $\tilde{V}_i$ to $\tilde{V}_{i+1}$. Therefore, edges labelled $\mathcal{S}_2$ in Γ connect vertices in $\tilde{V}_i/\sim = V_i$ to vertices in $\tilde{V}_{i+1}/\sim = V_{i+1}$. $\square$

2.1.2 Topological definition

We now give an alternative definition of $\Gamma = \mathrm{Spl}(P)$ following the standard topological approach of defining a Cayley graph.

Let X be a set. Define the *rose* Ro_X to be a graph with a single vertex v and edge set $E(\mathrm{Ro}_X) = X$, where each $x \in X = E(\mathrm{Ro}_X)$ signifies a loop at v. To be more precise, we let X^{-1} denote an abstract set disjoint from X and in bijection (denoted $^{-1}$) with X, and let $X \cup X^{-1}$ be the set of directed edges of Ro_X. The terminus map τ of Ro_X maps all edges to v. We colour this rose by $c : \overrightarrow{E}(\mathrm{Ro}_X) \to X \cup X^{-1}$ by an arbitrary choice of orientation; in other words, c is a bijection from $\overrightarrow{E}(\mathrm{Ro}_X)$ to $X \cup X^{-1}$ satisfying $c(e^{-1}) = c(e)^{-1}$ for every $e \in X$.

For a presentation $P = \langle \mathcal{S} | \mathcal{R} \rangle$ of a group one often alternatively defines the Cayley graph in the following more topological way. We start by constructing the *presentation complex* $\mathcal{C}(P)$ as follows. The 1-skeleton of $\mathcal{C}(P)$ is $\mathrm{Ro}_{\mathcal{S}}$ with vertex v. For each relator $r \in \mathcal{R}$, we introduce a 2-cell D_r and identify its boundary with the closed walk of $\mathrm{Ro}_{\mathcal{S}}$ dictated by r (see Definition 2.1.7 below). This completes the definition of $\mathcal{C}(P)$. The Cayley graph $\mathrm{Cay}\langle \mathcal{S} | \mathcal{R} \rangle$ is the 1-skeleton of the universal cover of $\mathcal{C}(P)$.

We now generalise this construction to the context of our special split presentations. We remark that it is not so easy to obtain the modified Cayley graphs using this construction because $\mathrm{Ro}_{\mathcal{S}}$ has even degree, so any cover will also have even degree. But treating $\mathcal{I}$ appropriately we will in fact be able to obtain graphs of odd degree.

Definition 2.1.7. Let $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$ be a special split presentation. We construct the *presentation complex* $\mathcal{C}(P)$ of P as follows. Start with two copies of $\mathrm{Ro}_{\mathcal{S}_1}$, with vertices v_0 and v_1 respectively, and connect v_0 and v_1 with an edge for each element of $\mathcal{S}_2 \cup \mathcal{S}_2^{-1} \subset F_P^{\mathrm{Mod}}$. We will refer to this 1-complex $C(P)$ as the *presentation graph* of P. We can extend the colouring of the two copies of $\mathrm{Ro}_{\mathcal{S}_1}$ to a colouring $c : \overrightarrow{E}(C(P)) \to \mathcal{S} \cup \mathcal{S}^{-1}$ where $c(e)^{-1} = c(e^{-1})$.

To define the 2-cells of $\mathcal{C}(P)$, for each relator $r = s_1 s_2 \ldots s_n \in \mathcal{R}_i$, we start a walk p_r at v_i and extend this walk inductively with the edge labelled $s_i, i = 1, \ldots, n$. The path p_r starts and ends at v_i as $\mathcal{R}_i \subset K$. Attach a 2-cell along each such closed walk p_r to obtain $\mathcal{C}(P)$ from $C(P)$. Finally, we define the *special split complex* to be the universal cover of $\mathcal{C}(P)$, and we define the *(topological) special split graph* $\mathrm{Spl}'\langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$ to be its 1-skeleton.

Our next result, Theorem 2.1.11, says that this gives rise to the same graph as in Definition 2.1.2. To prove it, we will use the theory of covering spaces (Section 1.1.4). For this we need to turn our graphs into topological spaces, and we now recall the standard way to do so.

Given a graph Γ with vertex set V, and any orientation on its edges $O \subset \overrightarrow{E}(\Gamma)$, we define a topological space as follows. Associate a point to each vertex, and a closed interval $I_e = [0, 1]$ to each edge $e \in O$. Then define the quotient $I_e(0) \sim \tau(e^{-1})$ and

$T_e(1) \sim \tau(e)$ to obtain the topological space

$$\Gamma = (V \cup \bigcup_{e \in O} I_e)/\sim .$$

It is not hard to see that when Γ is connected this topological space is path-connected, locally path-connected and semilocally simply-connected. Moreover, different choices of O define homeomorphic topological spaces.

Next, we define a type of colouring that will be useful to establish that certain maps of graphs are covers.

Definition 2.1.8. Let Γ be a graph with a colouring $c : \overrightarrow{E}(\Gamma) \to X$. We say that c is Cayley-like if

1. Γ is $|X|$-regular,
2. for all $e, e' \in \overrightarrow{E}(\Gamma)$, if $c(e) = c(e')$ and $\tau(e) = \tau(e')$ then $e = e'$, and
3. there is an involution $^{\mathbf{-1}} : X \to X$ such that $c(e)^{\mathbf{-1}} = c(e^{-1})$.

Suppose we have two graphs Γ and Δ with Cayley-like colourings $c_\Gamma : \overrightarrow{E}(\Gamma) \to X$ and $c_\Delta : \overrightarrow{E}(\Delta) \to X$. Then any surjective map of graphs $\phi : \Gamma \to \Delta$ which respects these colourings, that is, $c_\Gamma = c_\Delta \circ \phi$, is a covering map of the associated topological spaces. Indeed, ϕ can't map any two edges that share an end vertex to the same edge, as this cannot respect the colourings.

Let $\mathcal{P}_v(\Gamma)$ be the set of walks in Γ starting at a vertex v, and define the modified group $F_{X,\mathbf{^{-1}}} =: F_X$ by the presentation $\langle X | \{x x^{\mathbf{-1}} : x \in X\}\rangle$. Then any Cayley-like colouring $c : \overrightarrow{E}(\Gamma) \to X$ defines a map $\mathcal{W}_v : \mathcal{P}_v(\Gamma) \to F_X$ by $p = v e_1 v_1 \dots e_n v_n \mapsto c(e_1)c(e_2)\dots c(e_n)$. Note that there is a well defined inverse $\mathcal{W}_v^{-1} : F_X \to \mathcal{P}_v(\Gamma)$ as at every vertex $v' \in V(\Gamma)$ there is a unique edge $e \in \overrightarrow{E}(\Gamma)$ with colour $c(e)$ and $\tau(e^{-1}) = v'$. Moreover, $\mathcal{W}_v^{-1}$ is a double sided inverse to $\mathcal{W}_v$, so both these maps are bijections.

Definition 2.1.9. For any $g \in F_X$, we say that $\mathcal{W}_v^{-1}(g)$ is the walk (in Γ) *dictated* by the word g starting at v.

This is a natural definition since we can express g as a word $s_1 \dots s_n$ with $s_i \in X \cup X^{-1}$, and obtain $\mathcal{W}_v^{-1}(g)$ by starting at v and following the directed edges with colours $c(s_1)\dots c(s_n)$; this is well-defined when c is Cayley-like.

It is straightforward to check that if p is homotopic to p', then $\mathcal{W}_v(p) = \mathcal{W}_v(p')$. Thus by restricting to the closed walks we can think of $\mathcal{W}_v$ as a map from $\pi_1(\Gamma, v)$ to F_X, and so the above remarks imply that

Proposition 2.1.10. *$\mathcal{W}_v$ is a group isomorphism from $\pi_1(\Gamma, v)$ to a subgroup of F_X.*

Suppose we have a covering map of graphs $\psi : \Delta \to \Gamma$ both of which have Cayley-like colourings $c_\Delta : \overrightarrow{E}(\Delta) \to X$ and $c_\Gamma : \overrightarrow{E}(\Gamma) \to X$ such that $c_\Delta = c_\Gamma \circ \psi$. For a path $p : [0,1] \to \Gamma$ with $p(0), p(1) \in V(\Gamma)$ and a lift $\tilde{p} : [0,1] \to \Delta$ of p by ψ, it is straightforward to check that

$$\mathcal{W}_{p(0)}(p) = \mathcal{W}_{\tilde{p}(0)}(\tilde{p}) \tag{2.1}$$

where with a slight abuse, we interpreted p as a walk in Γ in the obvious way.

Theorem 2.1.11. *For every special split presentation $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$, the special split graphs $\Gamma = Spl(P)$ and $\Delta = Spl\,'(P)$ are isomorphic.*

Proof. Our presentation graph $C = C(P)$ is $|\mathcal{S} \cup \mathcal{S}^{-1}|$-regular by definition. Therefore, the universal cover of C is the $|\mathcal{S} \cup \mathcal{S}^{-1}|$-regular tree T, and we can let $\theta : T \to C$ be the corresponding covering map. Let $c_C : \overrightarrow{E}(C) \to \mathcal{S} \cup \mathcal{S}^{-1}$ be the colouring of C as above. This lifts to a colouring $c_T : \overrightarrow{E}(T) \to \mathcal{S} \cup \mathcal{S}^{-1}$ of T, by letting $c_T(e) := c_C(\theta(e))$. This colouring allows us to identify T with T_P.

Let $p \in \pi_1(C, v_i)$. As c_C is a Cayley-like colouring of C, we can consider $\mathcal{W}_{v_i}(p) \in F_P$ by Definition 2.1.8 and the discussion thereafter. Any closed walk representing p must use an even number of edges coloured $\mathcal{S}_2 \cup \mathcal{S}_2^{-1}$ by the definition of C, so $\mathcal{W}_{v_i}(p) \in K \subset F_P^{\mathrm{Mod}}$. Moreover, each $k \in K$ gives rise to a closed walk $\mathcal{W}_{v_i}^{-1}(k)$ representing some element of $\pi_1(C, v_i)$. Thus by Proposition 2.1.10,

$$\mathcal{W}_{v_i} \text{ is an isomorphism from } \pi_1(C, v_i) \text{ onto } K. \tag{2.2}$$

Recall that we can identify T with T_P. If in doing so we identify the identity $1_{F_P^{\mathrm{Mod}}} \in V(T_P)$ of F_P^{Mod} with some vertex in $\theta^{-1}(v_0)$ (which we easily can) then (2.2) implies

$$\theta(\tilde{V}_i) = v_i, \tag{2.3}$$

because $\tilde{V}_0 = K$ and $\tilde{V}_1 = \mathcal{S}_2 K$.
Let $\eta : T_P \to \Gamma$ be the covering map found in Corollary 2.1.6. Let $c_\Gamma : \overrightarrow{E}(\Gamma) \to \mathcal{S} \cup \mathcal{S}^{-1}$ be the colouring of Γ as in its definition. Now define a map $\nu : \Gamma \to C$ by letting $\nu(v) = v_i$ whenever $v \in V_i = \eta(\tilde{V}_i)$. If $c_\Gamma(e) = s$ for some $e \in \overrightarrow{E}(\Gamma)$ then ν maps e to the unique edge $e' \in \overrightarrow{E}(C)$ with $c_C(e') = s$ and $\tau(e') = \nu(\tau(e))$. Since for every $v \in \tilde{V}_i$ we have $\eta(v) \in V_i$, we have $\nu(\eta(v)) = v_i$ and hence $\theta = \nu \circ \eta$ by (2.3).

$$
\begin{tikzcd}
 & T_P \arrow[dl, "\eta"'] \arrow[d, "\theta"] \arrow[dr, "\Phi"] \arrow[drr, "\widehat{\Phi}"] & & \\
\Gamma \arrow[dr, "\nu"'] & & \Delta \arrow[r, hook, "i"'] \arrow[dl, "\epsilon"] & \widehat{\Delta} \arrow[dl, "\widehat{\epsilon}"] \\
 & C \arrow[r, hook, "i"] & \mathcal{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 1 \arrow[dl, hook, "\eta_*"'] \arrow[d, "\theta_*"] \arrow[dr, "\Phi_*"] \arrow[drr, hook, "\widehat{\Phi}_*"] & & \\
\pi_1(\Gamma) \arrow[dr, hook, "\nu_*"'] & & \pi_1(\Delta) \arrow[r, two heads, "i_*"'] \arrow[dl, hook, "\epsilon_*"] & 1 \arrow[dl, hook, "\widehat{\epsilon}_*"] \\
 & K \arrow[r, two heads, "i_*"] & K/R_i &
\end{tikzcd}
$$

Figure 2.3: Maps used in Proposition 2.1.11

Let $\widehat{\epsilon} : \widehat{\Delta} \to \mathcal{C}$ be the universal cover of $\mathcal{C} := \mathcal{C}(P)$. We know that Δ and C are the 1-skeletons of $\widehat{\Delta}$ and $\mathcal{C}$ respectively, so we obtain the inclusion maps $i : \Delta \to \widehat{\Delta}$ and $i : C \to \mathcal{C}$. Furthermore, by restricting $\widehat{\epsilon}$ to the 1-skeleton we obtain a covering map $\epsilon : \Delta \to C$. As $\theta : T_P \to C$ is the universal cover of C, it can be lifted through $\epsilon : \Delta \to C$ to a map $\Phi : T_P \to \Delta$ so that $\epsilon \circ \Phi = \theta$ by the definition of a universal cover. This gives us a map $\widehat{\Phi} : T_P \to \widehat{\Delta}$ defined by $\widehat{\Phi} := i \circ \Phi$. Note that all these maps respect the colourings of the edges as θ and $\widehat{\epsilon}$ do.

By Theorem 1.1.1, to show $\Gamma \cong \Delta$ it suffices to show that $\nu_*(\pi_1(\Gamma)) = \epsilon_*(\pi_1(\Delta))$, or equivalently $\mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma))) = \mathcal{W}_{v_i}(\epsilon_*(\pi_1(\Delta)))$ as $\mathcal{W}_{v_i}$ is a bijection. To do so, we will prove that the latter groups are both equal to R_i, where R_i is as defined after Definition 2.1.1.

To show that $\mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma))) = R_i$, let p be a closed walk representing some element of $\pi_1(\Gamma, v)$ with $v \in V_i$. Choose a lift of p to a walk $\tilde{p} : [0,1] \to T_P$ (so $\eta \circ \tilde{p} = p$). We know that $\eta(\tilde{p}(0)) = \eta(\tilde{p}(1)) = v$, so $\tilde{p}(0), \tilde{p}(1) \in \eta^{-1}(v)$ implying $\tilde{p}(0)^{-1}\tilde{p}(1) \in R_i$. So $\mathcal{W}_{v_i}(\nu_*(p)) = \mathcal{W}_{v_i}(\theta(\tilde{p})) \in R_i$, which proves that $\mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma))) \subseteq R_i$.

We would like to use Proposition 2.1.10 to deduce $\mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma))) = R_i$, and for this it now only remains to prove that the former is surjective onto R_i. To show this, pick any $r \in R_i$. As $R_i \subset K \cong \mathcal{W}_{v_i}(\pi_1(C, v_i))$ by (2.2), there is a representative q of an element of $\pi_1(C, v_i)$ such that $\mathcal{W}_{v_i}(q) = r$. Choose a lift $\tilde{q} : [0,1] \to T_P$ of q through $\nu \circ \eta = \theta$, such that $\eta(\tilde{q}(0)) = v$ (and so $\nu \circ \eta \circ \tilde{q} = \theta \circ \tilde{q} = q$). Then as $\mathcal{W}_v(\tilde{q}) = \mathcal{W}_{v_0}(q) = r \in R_i$ we have $\tilde{q}(0)^{-1}\tilde{q}(1) \in R_i$, and so $\tilde{q}(0) \sim \tilde{q}(1)$, with $\sim$ as in the definition of Γ as a quotient of T_P. This means that $\eta(\tilde{q}(1)) = \eta(\tilde{q}(0)) = v$, and so $\eta \circ \tilde{q}$ is a loop representing an element of $\pi_1(\Gamma, v)$. Since $\nu_*(\eta \circ \tilde{q}) = \theta \circ \tilde{q} = q$ represents an element of $\nu_*(\pi_1(\Gamma))$ we deduce that $r = \mathcal{W}_{v_i}(q) \in \mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma, v)))$, proving that $\mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma, v)))$ surjects onto R_i as desired.

Next, we prove $\mathcal{W}_v(\epsilon_*(\pi_1(\Delta, v))) \subseteq R_i$ for every $v \in V(\Delta)$ with $\epsilon(v) = v_i$. It is well-known [23, Proposition 1.26] that the inclusion of the one skeleton into a 2-simplex induces a surjection on the level of fundamental groups, and the kernel is exactly the normal closure of the words bounding the 2-cells. Thus $i_* : \pi_1(C, v_i) \to \pi_1(\mathcal{C}, v_i)$ is a surjection. Combining these remarks with (2.2), it follows that $i_* \circ \mathcal{W}_{v_i}^{-1} : K \to \pi_1(\mathcal{C}, v_i)$ is a surjection, with kernel R_i, since R_i is the normal closure in K of the words onto which $\mathcal{W}_{v_i}^{-1}$ maps the closed walks bounding 2-cells of $\mathcal{C}$ by the definition of $\mathcal{C}$. Thus $\pi_1(\mathcal{C}, v_i) = K/R_i = G_i$. Now pick $v \in V(\Delta)$ with $\epsilon(v) = v_i$. As $i \circ \epsilon = \widehat{\epsilon} \circ i$ and $\pi_1(\widehat{\Delta}) = 1$, we have $(i_* \circ \epsilon_*)(\pi_1(\Delta, v)) = (\widehat{\epsilon}_* \circ i_*)(\pi_1(\Delta, v)) = 1$, and so $\mathcal{W}_v(\epsilon_*(\pi_1(\Delta, v))) \leq ker(i_*) = R_i$ as desired.

Finally, we claim that $R_i \subset \mathcal{W}_v(\epsilon_*(\pi_1(\Delta, v)))$ for every $v \in V(\Delta)$ with $\epsilon(v) = v_i$. For this, pick $r \in R_i$, and note that as $R_i \subset K$ and $K \cong \mathcal{W}_{v_i}(\pi_1(C, v_i))$ by (2.2), there is a representative t of an element of $\pi_1(C, v_i)$ such that $\mathcal{W}_{v_i}(t) = r$. We can write $\mathcal{W}_{v_i}(t) = r = \prod_{j=1}^{n} w_j r_j w_j^{-1} \in F_P^{\text{Mod}}$ for $w_j \in K$ and $r_j \in \mathcal{R}_i \cup s\mathcal{R}_{i+1}s^{-1}$ with $s \in \mathcal{S}_2 \cup \mathcal{S}_2^{-1}$ by the definition of R_i. Choose a lift $t' : [0,1] \to \Delta$ of t through ϵ so that $t'(0) = v$. By (2.1) we have $\mathcal{W}_{v_i}(t) = \mathcal{W}_v(t')$. Note that $\mathcal{W}_v^{-1}(w_j r_j w_j^{-1})$ is a loop of Δ as $\mathcal{W}^{-1}(r_j)$ is contractable in $\widehat{\Delta}$, and so it represents some element of $\pi_1(\Delta, v)$. Applying this to each factor of our above expression $r = \prod_{j=1}^{n} w_j r_j w_j^{-1}$ implies that t' represents some element of $\pi_1(\Delta, v)$. Thus $\mathcal{W}_{v_i}(\epsilon_*(t')) = \mathcal{W}_{v_i}(t) = r$, which means that $R_i \subset \mathcal{W}_v(\epsilon_*(\pi_1(\Delta, v)))$ as claimed.

To summarize, we have proved that $\mathcal{W}_{v_i}(\nu_*(\pi_1(\Gamma))) = R_i = \mathcal{W}_{v_i}(\epsilon_*(\pi_1(\Delta)))$, implying that $\Gamma \cong \Delta$. Moreover, it is straightforward to check that as all the maps above respect the edge colourings, so does this isomorphisms of graphs. $\square$

From now on we just use the notation $\mathrm{Spl}(P)$ for the special split graph obtained in either Definition 2.1.2 or 2.1.7.

As a corollary of the above proof, we deduce that the covers ν, ϵ are equal, and so

$$V_i = \nu^{-1}(v_i) = \epsilon^{-1}(v_i) \tag{2.4}$$

and similarly $V_i = \eta(\tilde{V}_i) = \Phi(\tilde{V}_i)$, so V_i is well defined for either the topological or graph definition, as in the notation of Figure 2.3. From now on we will only use ϵ to denote this covering map.

The following corollary gathers some further facts that we obtained in the proof of Theorem 2.1.11 for future reference.

Corollary 2.1.12. *Let $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$ be a special split presentation with split graph $\Gamma := Spl(P)$. For all $i = 0, 1,$ We have*

1. *$\pi_1(\mathcal{C}(P), v_i)$ is isomorphic to G_i;*

2. *$\mathcal{W}_{v_i}$ is an isomorphism from $\pi_1(C(P), v_i)$ onto K;*

3. *$\mathcal{W}_{v_i}$ is an isomorphism from $\pi_1(\Gamma, v)$ onto R_i for every $v \in V_i$; and*

4. *the sequence $0 \to \pi_1(\Gamma, v) \xrightarrow{\epsilon_*} \pi_1(C(P), v_i) \xrightarrow{i_*} \pi_1(\mathcal{C}(P), v_i) \to 0$ is exact, where $\epsilon : \Gamma \to C(P)$ is the cover in Definition 2.1.7, and $i : C(P) \to \mathcal{C}(P)$ the inclusion.*

Note that from the definition of R_i we have $R_0 = sR_1s^{-1}$ for any $s \in \mathcal{S}_2$. Therefore, we deduce that $G_i := R_i \backslash K \cong R_{i+1} \backslash K$, where an isomorphism $\phi_{s,i} : G_i \to G_{i+1}$ is given by conjugation by any $s \in \mathcal{S}_2$. This follows also from the fact that $\pi_1(C(P))$ is base point invariant, and $\mathcal{W}_{v_i}^{-1}(s)$ is a path from v_i to v_{i+1}. This property isn't enough to guarantee vertex transitivity of Γ, with a counter example given by $P(4,2)$, which will be shown to have a split presentation in Theorem 2.1.14. This invites the following question.

Question 2.1.13. *For which special split presentations P is Spl(P) vertex transitive?*

The *generalised Petersen graph* is denoted by $P(n,k)$ and defined as follows. Let

$$\begin{aligned} V(P(n,k)) &:= \{x_i, y_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}, \text{ and} \\ E(P(n,k)) &:= \{(x_i, x_{i+1}), (x_i, y_i), (y_i, y_{i+k}) \mid i \in \mathbb{Z}/n\mathbb{Z}\}. \end{aligned}$$

The classical example is the Petersen graph, $P(5,2)$, the smallest non-Cayley vertex transitive graph.

Theorem 2.1.14. *The generalised Petersen graph $P(n,k)$ is isomorphic to $\Gamma :=$ Spl$\langle\{a\}, \emptyset, \{b\}|\{a^n, aba^kb\}, \{a^n\}\rangle$.*

Proof. Let $C(P) =: C$ be the presentation graph of $P := \langle\{a\}, \emptyset, \{b\}|\{a^n, aba^kb\}, \{a^n\}\rangle$. Define map $\eta : P(n,k) \to C$ by

$$\eta : \begin{matrix} (x_i, x_{i+1}) \\ (x_i, y_i) \\ (y_i, y_{i+1}) \end{matrix} \quad \mapsto \quad \begin{matrix} a^{-1}(v_1, v_1) \\ b(v_1, v_0) \\ a(v_0, v_0) \end{matrix} .$$

The relations a^n starting at v_0 and v_1 hold in $P(n,k)$ as we have closed cycles y_i y_{i+k} y_{i+2k} $\dots$ y_{i+nk} and x_i x_{i-1} x_{i-2} $\dots$ x_{i-n} with the subscripts taken in $\mathbb{Z}/n\mathbb{Z}$. Next examine the relation aba^kb starting at v_0. This pulls up to the walk y_i y_{i+k} x_{i+k} $x_{i+(k-1)}$ $x_{i+(k-2)}$ $\dots$ x_i y_i which is clearly closed.

Therefore we have covers $\eta : P(n,k) \to C$ and $\epsilon : \Gamma \to C$ such that ϵ lifts to a cover $\widehat{\epsilon} : \Gamma \to P(n,k)$. However look at the vertex group G_1 which acts regularly on $\epsilon^{-1}(v_1) = V_1$. We have

$$\begin{aligned} G_1 &= \langle a, bab | a^n, (bab)a^k, (bab)^n \rangle \\ &= \langle a | a^n, a^{-nk} \rangle && \text{as } bab = a^{-k} \\ &= \langle a | a^n \rangle \cong C_n \end{aligned}$$

This gives $|G_1| = n$, so $|\eta^{-1}(v_1)| = n = |G_1| = |\epsilon^{-1}(v_1)|$, making $\widehat{\epsilon}$ a graph isomorphism. □

2.2 Relationships to Bi-Cayley and Haar graphs

We recall that an action on a graph Γ is *semi-regular* (or *free*) if $g \cdot x = h \cdot x$ implies $g = h$ for every $g, h \in G$ and $x \in V(\Gamma)$. A vertex transitive graph Γ is said to be *n-Cayley over G* if G is a semi-regular subgroup of $\mathrm{Aut}(\Gamma)$ with n orbits of vertices. If $n = 2$ we say that Γ is *bi-Cayley*.

Suppose Γ is bi-Cayley over G. Pick two vertices $e_0, e_1 \in V(\Gamma)$ from different orbits of G. As G has exactly two orbits in $V(\Gamma)$, and it acts regularly on each of them, for any $x \in V(\Gamma)$ there exists a unique $i \in \{0,1\}$ and $g \in G$ such that $g \cdot e_i = x$, so we define $x =: (g)_i$. Each of the two orbits $O_i := \{(g)_i : g \in G\}$ forms a (possibly disconnected) Cayley graph of G with respect to the sets $R = R^{-1} = \{g \in G | [(e_0, (g)_0)] \in E(\Gamma)\}$ and $L = L^{-1} = \{g \in G | [(e_1, (g)_1)] \in E(\Gamma)\}$, respectively. To capture the set E_{01} of edges of the form $[((g)_0, (h)_1)] \in E(\Gamma)$, we introduce the set $S = \{g \in G | (e_0, (g)_1) \in \overrightarrow{E}(\Gamma)\}$, and note that S uniquely determines E_{01} as any $e \in E_{01}$ coincides with $[((g)_0, (h)_1)] = g \cdot [(e_0, (g^{-1}h)_1)]$ for some $g^{-1}h \in S$ and $g \in G$.

To summarize, we can represent any bi-Cayley graph Γ over G as $\mathrm{BiCay}(G, R, L, S)$ where $R, L, S \subset G$ with $R = R^{-1}$ and $L = L^{-1}$. Then the set of directed edges of $\Gamma =: \mathrm{BiCay}(G, R, L, S)$ is

$$\overrightarrow{E}(\Gamma) = \{((g)_0, (gr)_0) | g \in G, r \in R\} \cup \{((g)_1, (gl)_1) | g \in G, l \in L\} \cup$$

$$\{((g)_0, (gs)_1) | g \in G, s \in S\} \cup \{((g)_1, (gs^{-1})_0) | g \in G, s \in S\}.$$

This representation isn't unique: if we choose different vertices for e_0, e_1 or a different action of G we potentially obtain different sets R, S and L. Note that $\mathrm{BiCay}(G, R, L, S)$ is a regular graph if and only if $|R| = |L|$.

Example 2.2.1. Consider again the Petersen graph $\Gamma = P(5,2)$ as in Example 2.1.3

(Figure 2.4). This has a natural action of $G := \mathbb{Z}/5\mathbb{Z} =< a >$ where

$$a^j : \begin{array}{c} x_i \\ y_i \end{array} \mapsto \begin{array}{c} x_{i+j} \\ y_{i+j} \end{array}.$$

To represent this as a bi-Cayley graph with above notation, we could choose $(a^0)_0 := x_0$ and $(a^0)_1 := y_0$. Then we obtain $R = \{a, a^4\}$, $L = \{a^2, a^3\}$ and $S = \{a^0\}$. If instead we chose $(a^0)_1 := y_1$ we would obtain $R = \{a, a^4\}$, $L = \{a^2, a^3\}$ and $S = \{a^4\}$.

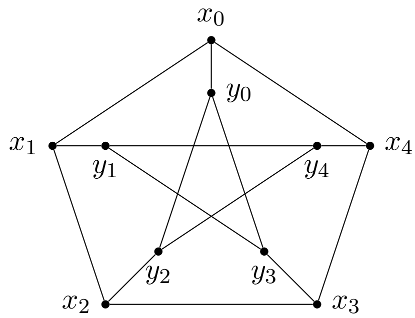

Figure 2.4: The labelling of the Petersen graph used in Example 2.2.1.

Recall that we have endowed $\Gamma := \mathrm{Spl}\langle \mathcal{S}_1, \mathcal{S}_2 | \mathcal{R}_1, \mathcal{R}_2 \rangle$ with a colouring $c : \overrightarrow{E}(\Gamma) \to \mathcal{S} \cup \mathcal{S}^{-1}$. We want to talk about automorphisms that preserve this colouring. The following definition distinguishes between preserving these colours globally or locally.

Definition 2.2.2. Let Γ be a graph with a colouring $c : \overrightarrow{E}(\Gamma) \to X$. We define the following two subgroups of $\mathrm{Aut}(\Gamma)$:

$$\begin{aligned} \mathrm{Aut}_c(\Gamma) =& \{\phi \in \mathrm{Aut}(\Gamma) | c(e) = c(\phi(e)) \text{ for every } e \in \overrightarrow{E}(\Gamma)\}, \text{ and} \\ \mathrm{Aut}_{c-loc}(\Gamma) =& \{\phi \in \mathrm{Aut}(\Gamma) | c(x,y) = c(y,z) \Leftrightarrow c(\phi(x,y)) = c(\phi(y,z)) \\ & \qquad\qquad\qquad \text{for all } (x,y),(y,z) \in \overrightarrow{E}(\Gamma)\}. \end{aligned}$$

Example 2.2.3. Recall the split presentation $P = \langle \{a\}, \emptyset, \{b\} | \{a^5, aba^2b\}, \{a^5\} \rangle$ of $P(5,2)$ as in Example 2.1.3. The corresponding colouring $c : \overrightarrow{E}(P(5,2)) \to \{a, a^{-1}, b\}$ is given by (Figure 2.5)

$$c : \begin{array}{c} (x_i, x_{i+1}) \\ (x_i, y_i) \\ (y_i, y_{i+2}) \end{array} \mapsto \begin{array}{c} a^{-1} \\ b \\ a \end{array} \quad \text{and} \quad c : \begin{array}{c} (x_i, x_{i-1}) \\ (y_i, x_i) \\ (y_i, y_{i-2}) \end{array} \mapsto \begin{array}{c} a \\ b \\ a^{-1} \end{array}.$$

To describe $\mathrm{Aut}_c(P(5,2))$ and $\mathrm{Aut}_{c-loc}(P(5,2))$ we look at edges coloured b. As b edges are self inverses, both $\mathrm{Aut}_c(P(5,2))$ and $\mathrm{Aut}_{c-loc}(P(5,2))$ can be represented

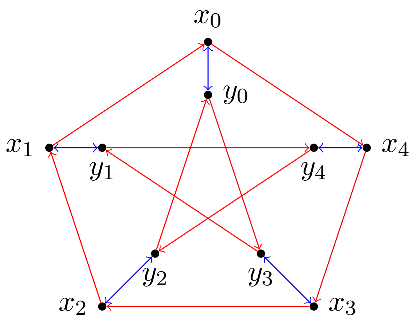

Figure 2.5: Our colouring of the Petersen graph corresponding to the split presentation $P = \langle \{a\}, \emptyset, \{b\} | \{a^5, aba^2b\}, \{a^5\} \rangle$ of $P(5,2)$.

as permutations of the set of b edges

$$B = \{(x_i, y_i) | i \in \mathbb{Z}/5\mathbb{Z}\} = \mathbb{Z}/5\mathbb{Z}.$$

Note $\mathrm{Aut}_c(P(5,2))$ and $\mathrm{Aut}_{c-loc}(P(5,2))$ faithfully sit inside $Sym(B)$ as no non-trivial automorphisms fixes the edges in B setwise giving $\mathrm{Aut}_c(P(5,2)), \mathrm{Aut}_{c-loc}(P(5,2)) \leq Sym(B)$. One can show $\mathrm{Aut}_c(P(5,2)) = \langle (0,1,2,3,4) \rangle = C_5$ and $\mathrm{Aut}_{c-loc}(P(5,2)) = \langle (0,1,2,3,4), (1,2,4,3) \rangle = G(1,5) = \langle a, b \mid a^5, b^4, bab^{-1}a^{-2} \rangle$. We have that Aut_{c-loc} is larger as it is allowed to invert the directions of the a cycles. Note that $\mathrm{Aut}_c(P(5,2)) < \mathrm{Aut}_{c-loc}(P(5,2)) < \mathrm{Aut}(P(5,2))$, so it is useful in some contexts to look at different colour preserving groups.

The action of $\mathrm{Aut}_c(P(5,2))$ makes $P(5,2)$ a bi-Cayley graph. In fact for any special split presentation P there is always a subgroup of $\mathrm{Aut}_c(\mathrm{Spl}(P))$ where c is the colouring coming from P that makes $\mathrm{Spl}(P)$ a bi-Cayley graph.

We remark that for any special split presentation P, there is a subgroup of $\mathrm{Aut}_c(\mathrm{Spl}(P))$ witnessing that $\mathrm{Spl}(P)$ is a bi-Cayley graph:

Proposition 2.2.4. *For every special split presentation $P = \langle \mathcal{S}_1, \mathcal{U}, \mathcal{I} | \mathcal{R}_0, \mathcal{R}_1 \rangle$ the vertex group G_i is a subgroup of $Aut_c(Spl(P))$. Moreover G_i acts regularly on V_i (and on V_{i+1}) for $i \in \mathbb{Z}/2\mathbb{Z}$, and so $Spl(P)$ is bi-Cayley over $G_0 \cong G_1$.*

Proof. Recall that for a covering map $\epsilon : X \to Y$, the group of automorphisms $f : X \to X$ such that $\epsilon \circ f = \epsilon$ is called the *deck group* of ϵ and is denoted by $\mathrm{Aut}(\epsilon)$. It is known that if ϵ is a universal cover $\mathrm{Aut}(\epsilon) = \pi_1(Y)$, and if X is connected and locally path connected then $\mathrm{Aut}(\epsilon)$ acts freely on $\epsilon^{-1}(y)$ for any $y \in Y$ [23].

Let $\Gamma := \mathrm{Spl}(P)$ and let $\widehat{\epsilon} : \widehat{\Gamma} \to \mathcal{C}$ be the universal cover of the presentation complex $\mathcal{C}(P)$ of P. Thus $\mathrm{Aut}(\widehat{\epsilon}) \cong \pi_1(\mathcal{C}(P)) \cong G_i$ by the above remark and Corrolary 2.1.12 (1). As Γ is the 1-skeleton of $\widehat{\Gamma}$ by Definition 2.1.7, we can think of

$\mathrm{Aut}(\widehat{\epsilon}) \cong G_i$ as a subgroup of $\mathrm{Aut}(\Gamma)$. Moreover as elements of $\mathrm{Aut}(\widehat{\epsilon}) \cong G_i$ preserve the cover, they preserve the colouring $c : \overrightarrow{E}(\Gamma) \to \mathcal{S} \cup \mathcal{S}^{-1}$ obtained by lifting our colouring of $\mathcal{C}(P)$ via $\widehat{\epsilon}$, and so we have realised G_i as a subgroup of $\mathrm{Aut}_c(\Gamma)$. As $\mathcal{C}(P)$ is a connected 2-complex it is locally path connected, therefore G_i acts freely on $\widehat{\epsilon}^{-1}(v_i) = V_i$ by the above remarks. $\square$

Proposition 2.2.5. *Every regular connected bi-Cayley graph BiCay(G, R, L, S) where $R \cap R^{-1} = L \cap L^{-1} = \emptyset$ and $|R| = |L|$ can be constructed as a special split presentation.*

Proof. Let $\Gamma := \mathrm{BiCay}(G, R, L, S)$ be a a bi-Cayley graph, and recall our representation of its vertex set as $V(\Gamma) = \{(g)_i | g \in G,\ i \in \{0, 1\}\}$. Choose $\mathcal{S}_1 \subset R$ such that $\mathcal{S}_1 \cap \mathcal{S}_1^{-1} = \emptyset$ and yet $\mathcal{S}_1 \cup \mathcal{S}_1^{-1} = R$. Choose a bijection $f : L \to R$ such that $f(s^{-1}) = f(s)^{-1}$. We use f to define the colouring $c : \overrightarrow{E}(\Gamma) \to \mathcal{S}_1 \cup \mathcal{S}_1^{-1} \cup S$ as follows:

$$c : \begin{array}{ccccc} ((g)_0, (rg)_0) & & r & & r \in R \\ ((g)_1, (lg)_1) & \mapsto & f(l) & \text{for} & l \in L \\ ((g)_0, (sg)_1)^{\pm 1} & & s & & s \in S. \end{array}$$

Note that this colouring is Cayley-like, as there is a unique edge of each colour incident with each vertex. Let $\mathcal{I} := S$, and set $\mathcal{R}_0 := \mathcal{W}_{(e)_0}(\pi_1(\Gamma, (e)_0))$. We have thus constructed a special split presentation $P := \langle \mathcal{S}_1, \emptyset, \mathcal{I} | \mathcal{R}_0, \emptyset \rangle$. We claim that $\Gamma \cong \mathrm{Spl}(P)$.

To see this, let as usual $\mathcal{C}(P) =: \mathcal{C}$ be the presentation complex and $C(P) =: C$ the presentation graph with vertices $v_i, i \in \{0, 1\}$ and edges $\overrightarrow{E}(C) = \{r(v_i, v_i), s(v_i, v_{i+1}) | r \in \mathcal{S}_1 \cup \mathcal{S}_1^{-1},\ s \in S = \mathcal{I}\}$ where $c_C(x(v_i, v_j)) = x$. We will prove $\Gamma \cong \mathrm{Spl}(P)$ by applying Theorem 1.1.1 to a cover $\epsilon : \Gamma \to C$ defined by $\epsilon : (g)_i \mapsto v_i$, and

$$\epsilon : \begin{array}{ccccc} ((g)_0, (rg)_0) & & r(v_0, v_0) & & r \in R \\ ((g)_1, (lg)_1) & \mapsto & f(l)(v_1, v_1) & \text{for} & l \in L \\ ((g)_0, (sg)_1)^{\pm 1} & & (s(v_0, v_1))^{\pm 1} & & s \in S \end{array}.$$

As ϵ is a map of graphs with Cayley-like colourings, and ϵ respects these colourings by definition, it is indeed a cover. We have $\mathcal{W}_{v_0}(\epsilon_*(\pi_1(\Gamma, (e)_0))) = \mathcal{R}_0$ by the choice of $\mathcal{R}_0$. Let $\epsilon : \mathrm{Spl}(P) \to C$ represent the cover given in definition 2.1.7 of $\mathrm{Spl}(P)$ (as in Figure 2.3). By Corollary 2.1.12 (3) we have that $\mathcal{W}_{v_0}(\epsilon_*(\pi_1(\mathrm{Spl}(P)), v)) = R_0 := \langle\langle \mathcal{R}_0 \rangle\rangle_K$ for some $v \in V(\mathrm{Spl}(P))$ such that $\epsilon(v) = v_0$. Note that for any $k \in K$ the path $\mathcal{W}_{(e)_0}^{-1}(k)$ connects $(e)_0$ to $(g)_0$ for some $g \in G$ because it uses an even number of edges with their colour in S. This implies

$$\mathcal{W}_{(e)_0}^{-1}(k)\pi_1(\Gamma, (g)_0)\mathcal{W}_{(e)_0}^{-1}(k)^{-1} = \pi_1(\Gamma, (e)_0). \tag{2.5}$$

As there exists a colour preserving automorphism of Γ mapping $(e)_0$ to $(g)_0$ namely g. Moreover, we have

$$\mathcal{W}_{(g)_0}(\pi_1(\Gamma,(g)_0)) = \mathcal{W}_{(e)_0}(\pi_1(\Gamma,(e)_0)). \tag{2.6}$$

Therefore $\langle\langle \mathcal{R}_0 \rangle\rangle_K = \mathcal{R}_0$ by (2.5), (2.6) and the definition of $\mathcal{R}_0$. Using this we have $\mathcal{W}_{v_0}(\epsilon_*(\pi_1(\mathrm{Spl}(P),v))) = \mathcal{R}_0$. Moreover, we have $\mathcal{W}_{v_0}(\epsilon_*(\pi_1(\Gamma,(e)_0))) = \mathcal{R}_0$ by the definition of $\mathcal{R}_0$ and (2.1). Therefore $\mathcal{W}_{v_0}(\epsilon_*(\pi_1(\mathrm{Spl}(P),v))) = \mathcal{W}_{v_0}(\epsilon_*(\pi_1(\Gamma,(e)_0)))$, and so by Theorem 1.1.1 we have $\Gamma \cong \mathrm{Spl}(P)$. □

A *Haar graph* is a bi-Cayley graph of the form $\mathrm{BiCay}(G,\emptyset,\emptyset,S)$. The following is an immediate consequence of the last two propositions.

Corollary 2.2.6. *Any Haar graph can be represented by a special split presentation and any special split graph with $\mathcal{S}_1 = \mathcal{U} = \emptyset$ is a Haar graph.*

Most of our motivation for introducing split presentations came from studying vertex transitive graphs. Our next proposition gives a sufficient condition for $\mathrm{Spl}(P)$ to be vertex transitive in terms of the 'symmetry' of $\mathcal{C}(P)$. Given two CW complexes $\mathcal{C}_i$ for $i \in \{0,1\}$, recall that a *simplicial map* $\phi : \mathcal{C}_0 \to \mathcal{C}_1$ is a continuous map that maps each n-simplex to an n-simplex for every n. For a CW complex $\mathcal{C}$, the group of bijective simplicial maps from $\mathcal{C}$ to itself is denoted by $\mathrm{Aut}(\mathcal{C})$.

Proposition 2.2.7. *Let P be a special split presentation. As above, the two vertices of the presentation complex $\mathcal{C}$ are denoted by v_0 and v_1. If there exists a simplicial map $\phi : \mathcal{C} \to \mathcal{C}$ such that $\phi(v_0) = v_1$, then Spl(P) is vertex transitive.*

Proof. Set $\Gamma := \mathrm{Spl}(P)$. Lemma 2.2.4 says that G_i acts transitively on V_j for $j \in \{0,1\}$. Thus it only remains to find an automorphism which maps a vertex in V_0 to a vertex in V_1. We have a covering map $\epsilon : \widehat{\Gamma} \to \mathcal{C}$, where $\widehat{\Gamma}$ is the universal cover of $\mathcal{C}$ with 1-skeleton Γ. By the lifting property $\phi \circ \epsilon : \widehat{\Gamma} \to \mathcal{C}$ lifts to an automorphism $\widehat{\phi} \in \mathrm{Aut}(\widehat{\Gamma})$ such that $\phi \circ \epsilon = \epsilon \circ \widehat{\phi}$. For any $v \in V_i$ we have $\epsilon(v) = v_i$ by (2.4). So for $v \in V_0$ we have $\epsilon \circ \widehat{\phi}(v) = \phi \circ \epsilon(v) = \phi(v_0) = v_1$ giving that $\widehat{\phi}(v) \in V_1$. Thus when restricting $\widehat{\phi}$ to the 1-skeleton, Γ, we obtain the required automorphism. □

We remark that this sufficient condition is not necessary for $\mathrm{Spl}(P)$ to be vertex transitive. For example, there is never such an automorphism for the split presentations $\langle \{a\},\{\},\{b\} | \{a^n, aba^kb\},\{a^n\}\rangle$ of Theorem 2.1.14 unless $k = \pm 1$. However, we know that $P(n,k)$ is transitive for many other choices of n and k (such as the case of the Petersen graph $n = 5, k = 2$), see [15].

2.3 General Split Presentations

2.3.1 Definition of General Split Presentations

In this section we generalise our notion of split presentation by allowing for more than two classes of vertices V_i. This will allow us to describe vertex transitive graphs such as the Coxeter graph or the triangulated Petersen graph $TriP(5,2)$ cannot be expressed as a bi-Cayley graph.

In Definition 2.1.1 of a special split presentation we did not explicitly talk about the two vertex classes, but they were implicit in that definition: we had two sets of relators $\mathcal{R}_0, \mathcal{R}_1$, and the definition of K implicitly distinguished our generators into those staying in the same vertex, namely $\mathcal{S}_1$, from those swapping between the two vertex classes, namely $\mathcal{S}_2$. The two vertex classes V_i were defined a-posteriori, and Corollary 2.1.6 confirms that the generators gave rise to edges of the split graph behaving this way.

The following definition is a direct generalisation of Definition 2.1.1, although it is formulated a bit differently. We now make the vertex classes more explicit. The main complication arises from the fact that we have to specify, for each generator s, which vertex class any edge coloured by s will lead to if it starts at a given vertex class. This information is encoded as a permutation $\phi(s)$ of the set of vertex classes. As before, we distinguish our generators into two subsets $\mathcal{U}$ and $\mathcal{I}$ to allow for 'involutions' that make split graphs with odd degrees possible.

We now give the formal definition:

Definition 2.3.1. A split presentation $\langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$ consists of the following data:

1. a set of vertex classes X;
2. two generator sets $\mathcal{U}$ and $\mathcal{I}$; define $\mathcal{S} := \mathcal{U} \cup \mathcal{U}^{-1} \cup \mathcal{I}$ and $^{\mathbf{-1}} : \mathcal{S} \to \mathcal{S}$ by $s^{\mathbf{-1}} = s^{-1}$ for $s \in \mathcal{U}$ and $s^{\mathbf{-1}} = s$ for $s \in \mathcal{I}$, then $F_P^{\mathrm{Mod}} := F_{\mathcal{S},\mathbf{-1}}^{\mathrm{Mod}}$ (a free product of cyclic groups each of order 2 or ∞);
3. a map $\phi : \mathcal{S} \to \mathrm{Sym}_X$ from the generator set to the group Sym_X of permutations of X;
   We remark that any such map defines a right action of $s_1 \dots s_n \in F_P^{\mathrm{Mod}}$ on $x \in X$ via $x \cdot s_1 \dots s_n := \phi(s_n) \circ \dots \circ \phi(s_1)(x)$, where $s_i \in \mathcal{S} \cup \mathcal{S}^{-1}$, and $\phi(s^{-1}) := \phi(s)^{-1}$. We require that
   (a) this action of F_P^{Mod} on X is transitive, and
   (b) for all $s \in \mathcal{I}$ the permutation $\phi(s)$ is fixed point free of order 2;

4. a *relator set* $\mathcal{R}_x \subset Stab(F_P^{\text{Mod}}, x)$ for each $x \in X$, where $Stab(F_P^{\text{Mod}}, x)$ denotes the stabiliser of x with respect to the aforementioned action of F_P^{Mod}. (This is a natural condition, as we want to return to our starting vertex when following a walk labelled by a relator, and in particular we want to return to the same vertex class.)
   The set $\{\mathcal{R}_x : x \in X\}$ of these relator sets is denoted by $\mathcal{R}$.

We now use such a presentation $P = \langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$ to define the split graph $\text{Spl}(P)$, in analogy with Definition 2.1.7. We start by defining the *presentation graph* $C(P)$. This has vertex set X, and directed edge set $\{(x, \phi(s)(x))|$ for all $x \in X$ and $s \in \mathcal{U} \cup (\mathcal{U})^{-1} \cup \mathcal{I}\}$ where $\phi(s^{-1}) = \phi(s)^{-1}$. We colour it by $c : \overrightarrow{E}(C(P)) \to \mathcal{S} \cup \mathcal{S}^{-1}$ defined by $c(x, \phi(s)x) := s$, and note that this is a Cayley-like colouring as in Definition 2.1.8.

The *split presentation complex* $\mathcal{C}(P)$ is the 2-complex obtained from $C(P)$ as follows. For each $x \in X$ and each $r \in \mathcal{R}_x$, we introducing a 2-cell and glue its boundary along the walk of $C(P)$ starting at x and dictated by r (as in Definition 2.1.9). It is straightforward to check that this is a closed walk using (4).

Note that $\mathcal{C}(P)$ is connected by condition (3a). Finally,

Definition 2.3.2. We define the *split graph* $\text{Spl}(P) = \text{Spl}\langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$ to be the 1-skeleton of the universal cover of $\mathcal{C}(P)$.

Letting $\epsilon : \text{Spl}(P) \to C(P)$ be the covering map, we can lift c to the edge-colouring $\tilde{c} = c \circ \epsilon$ of $\text{Spl}(P)$.

Note that if X is a singleton, then we recover the usual group presentations and Cayley graphs by the above definitions. Our special split presentations $\langle \mathcal{S}_1, \mathcal{U}', \mathcal{I}'|\mathcal{R}_0, \mathcal{R}_1\rangle$ of Section 2.1 are tantamount to split presentations as in Definition 2.3.1 with $X = \{0, 1\}$, where $\phi(s_1) = (0)(1)$ for $s_1 \in \mathcal{S}_1$ and $\phi(s_2) = (0, 1)$ for $s_2 \in \mathcal{S}_2 := \mathcal{U}' \cup \mathcal{I}'$, with $\mathcal{U} = \mathcal{S}_1 \cup \mathcal{U}'$ and $\mathcal{I} = \mathcal{I}'$. Then $\pi_1(C(P), x) = Stab(F_P^{\text{Mod}}, x)$ using the colouring c, therefore $\mathcal{R}_x \subset Stab(F_P^{\text{Mod}}, x) = \pi_1(C(P), x)$.

As in Section 2.1, we can alternatively define $\text{Spl}(P)$ as a graph quotient, following the lines of Definition 2.1.2, as follows:

1. Let $\mathcal{S} := \mathcal{U} \cup \mathcal{I}$ and define the group F_P^{Mod} by the presentation $\langle \mathcal{S}|\{s^2 : s \in \mathcal{I}\}\rangle$; this is a free product of infinite cyclic groups, one for each $s \in \mathcal{U}$, and cyclic

groups of order 2, one for each $s \in \mathcal{I}$. Define the tree T_P by

$$V(T_P) := F_P^{\text{Mod}}, \text{ and}$$
$$\overrightarrow{E}(T_P) := \{(w, ws) | w \in F_P^{\text{Mod}}, s \in \mathcal{S} \cup \mathcal{S}^{-1}\}.$$

This is a $(2|\mathcal{U}| + |\mathcal{I}|)$-regular tree, and it comes with a colouring $c : \overrightarrow{E}(T_P) \to \mathcal{S} \cup \mathcal{S}^{-1}$ by $c(w, ws) = s$.

2. We can extend the map ϕ of (3) from $\mathcal{S}$ to a right action of F_P^{Mod} by composition: we let $x \cdot s_1 \dots s_n := \phi(s_n) \circ \dots \circ \phi(s_1)(x)$ for all $x \in X$ and $s_i \in \mathcal{S}$. Let $W_{x,y} = \{w \in F_P^{\text{Mod}} | x \cdot w = y\}$ for $x, y \in X$. Fixing any 'base' vertex class $b \in X$ leads to a partition of $V(T_P) = F_P^{\text{Mod}}$, namely $\tilde{V}_x = W_{b,x}$. Note that two vertices in $u, v \in \tilde{V}_x \subset F_P^{\text{Mod}}$ differ by a word $u^{-1}v \in W_{x,x} = Stab(F_P^{\text{Mod}}, x)$.

3. Let $R_x = \langle wrw^{-1} | r \in \mathcal{R}_y, w \in W_{x,y}, y \in X \rangle \subset W_{x,x}$. Then we say that two vertices in $u, v \in \tilde{V}_x$ are equivalent, and write $u \sim v$, if $u^{-1}v \in R_x$. Similarly, for edges $e, f \in \overrightarrow{E}(T_P)$ we write $e \sim f$ if $c(e) = c(f)$ and $\tau(e) \sim \tau(f)$ and $\tau(e^{-1}) \sim \tau(f^{-1})$.

4. We define $\text{Spl}(P)$ to be the corresponding quotient $T_P / \sim$.

As in Corollary 2.1.6, it is not hard to see that T_P is the universal cover of $\text{Spl}(P)$. Define V_x, $x \in X$ as the image of $\tilde{V}_x$ under the quotient of $\sim$. We have $W_{x,x} = \pi_1(C(P), x)$ and $\pi_1(\mathcal{C}(P), x) = R_x \backslash W_{x,x} =: G_x$, analogously to the special split presentation case. We call G_x, $x \in X$ the *vertex groups*.

We remark that the vertex set of $\text{Spl}(P)$ can be given the structure of a groupoid $\mathcal{G}_{\text{Spl}(P)}$. Indeed, we can think of $\bigcup_{x,y \in X} W_{x,y}$ as the ground set, and define the groupoid operation $W_{x,y} \times W_{y,z} \to W_{x,z}$ by concatenation. Another way to think of this groupoid is $\mathcal{G}_{\text{Spl}(P)} \cong \pi_1(\mathcal{C}(P), X)$, the universal groupoid of the presentation complex $\mathcal{C}(P)$, with paths starting and ending in $V(C)$.

The main result of this section is that every vertex transitive graph Γ is isomorphic to $\text{Spl}(P)$ for some split presentation P. For the proof of this we will need to decompose the edges of Γ into cycles. The next section discusses such decompositions.

2.3.2 Multicycle colourings

Leighton [28] asked whether vertex transitive graphs have similar colouring structures to Cayley graphs of groups. For a Cayley graph $\Gamma = \text{Cay}(G, \mathcal{S})$, the generators canonically induce a colouring $c : E(\Gamma) \to \mathcal{S}$ as above, so that $c^{-1}(s)$ is a disjoint union of cycles of the same length for every $s \in \mathcal{S}$. Leighton calls this a multicycle:

Definition 2.3.3. A *multicycle* is a graph which is either the disjoint union of cycles of the same length or a perfect matching. A *multicycle colouring* of a graph Γ is a colouring $c : E(\Gamma) \to \Omega$ such that the graph with vertex set $V(\Gamma)$ and edge set $c^{-1}(x)$ is a multicycle for each $x \in \Omega$.

Thus every Cayley graph has a multicycle colouring. Leighton [28] conjectured that all vertex transitive graphs have a multicycle colouring [28], but this was shown to be false by Marušič [33], a counter-example being the line graph of the Petersen graph:

Example 2.3.4. Given a graph Δ set $\Gamma := L(\Delta)$ to be the line graph. So $V(\Gamma) := E(\Delta)$ and $\overrightarrow{E}(\Gamma) = \{(e, e')|\tau(e^{\pm 1}) = \tau(e'^{\pm 1})\}$. To see there is no multicycle colouring of $L(P(5, 2))$, note that it has $|V(L(P(5, 2)))| = |E(P(5, 2))| = 15$ vertices, so any mutlicycle will have to consist of triangles or pentagons or a 15-cycle. A 15-cycle in the line graph would correspond to a Hamiltonian cycle in the Petersen graph. The Petersen graph is not Hamiltonian so, there are no such 15-cycle. The only triangles in $L(P(5, 2))$ are formed by edges incident with a single vertex of $P(5, 2)$. As $P(5, 2)$ is not bipartite, there is no way to partition the triangles into disjoint sets that pass through all vertices. So we can only use sets of five cycles, which correspond to sets of edge disjoint pentagons in $P(5, 2)$. As $P(5, 2)$ is cubic, there is no set of pentagons that visits every edge exactly once.

Still, it is possible to express $L(P(5, 2))$ as a split graph:

$\mathrm{Spl}\langle\{1, 2, 3\}|\mathcal{U} := a \mapsto (12)(3), b \mapsto (1)(23)|\mathcal{I} := \emptyset|\{b^5, a^{10}, a^2b\}, \{a^{-2}b^4\}, \{a^5, b^{10}, b^2a\}\rangle.$

This is shown in Figure 2.6.

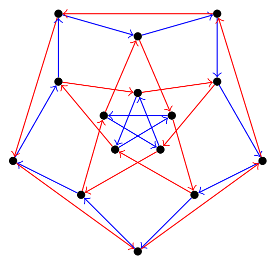

Figure 2.6: The line graph $L(P(5, 2))$ of the Petersen graph.

Our aim now is to weaken the notion of a multicycle colouring enough that every vertex transitive graph will admit one, so that the weakened notion will allow us to find split presentations. This is the essence of Theorem 2.3.7 below.

Definition 2.3.5. A graph Γ is a *weak multicycle*, if it is a vertex-disjoint union of cycles and edges. A *weak multicycle colouring* of a graph Γ is a colouring $c : E(\Gamma) \to \Omega$

such that the graph with vertex set $V(\Gamma)$ and edge set $c^{-1}(x)$ is a weak multicycle for each $x \in \Omega$.

We say that a weak multicycle colouring c is *split-friendly* if $c^{-1}(x)$ is regular for all $x \in \Omega$. In other words, $c^{-1}(x)$ is either a disjoint union of cycles or a perfect matching for all x.

As we will see in the following section, every vertex transitive graph has a split-friendly weak multicycle colouring. The condition of vertex transitivity cannot be relaxed to just regularity. Indeed, let Γ be the 3-regular graph in Figure 2.7. Since its vertex degrees are odd, one of the colours in any weak multicycle colouring must induce a perfect matching. But Γ does not have a perfect matching M, because removing v and the vertex matched to v by M results in at least one component with an odd number of vertices.

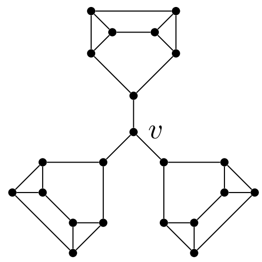

Figure 2.7: A regular graph with no split-friendly weak multicycle colouring.

2.3.3 Multicycle colourings and split presentations

We say a split presentation $P = \langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$ is *uniform*, if for every $s \in \mathcal{S}$, all orbits of $\phi(s)$ have the same size. In other words, if c is a multicycle colouring on $C(P)$. In light of Leighton's aforementioned conjecture, one can ask the following:

Question 2.3.6. *Let Γ be a vertex transitive graph. Does Γ have a multi-cycle colouring if and only if it is the split graph of a uniform split presentation?*

The forward direction is true: if Γ has a multicycle colouring then it has a uniform split presentation given in the proof of Theorem 2.3.7. But the backward direction could be false, as shown by the following example. Consider the special spit presentation $P = \langle\{a\},\{b\},\emptyset|\{a\},\{a^2\}\rangle$. This is trivially uniform, like every special split presentation. However, Spl(P), shown in Figure 2.8, does not have a multicycle colouring.
The following result will be used later to show that every vertex transitive graph admits a split presentation.

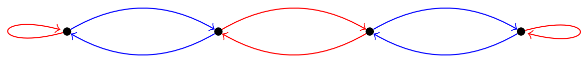

Figure 2.8: $\mathrm{Spl}\langle\{a\},\{b\},\emptyset|\{a\},\{a^2\}\rangle$

Theorem 2.3.7. *A connected graph has a split-friendly weak multicycle colouring if and only if it has a split presentation.*

Proof. Recall that a graph is given by a directed edge set $\overrightarrow{E}(\Gamma)$, but we can also consider the undirected edge set $E(\Gamma) = \overrightarrow{E}(\Gamma)/^{-1}$, so that an undirected edge is a pair $\{e, d\}$ such that $e^{-1} = d$ and $d^{-1} = e$. In the following proof we have to transition between colourings of the directed edges and colourings of the undirected edges. Apart from this, the proof boils down to a straightforward checking of the conditions of the corresponding definitions.

For the forward direction, suppose Γ is connected and it has a split-friendly weak multicycle colouring $c : E(\Gamma) \to \Omega$. To define the desired split presentation P, we start with

- $X = V(\Gamma)$,
- $\mathcal{U} = \{\omega \in \Omega | c^{-1}(\omega)$ is of degree $2\}$, and
- $\mathcal{I} = \{\omega \in \Omega | c^{-1}(\omega)$ is of degree $1\}$.

Since c is split-friendly, we have $\mathcal{U} \cup \mathcal{I} = \Omega$. We want to refine c into a colouring c' of the directed edges of Γ. To do this, for each $\omega \in \mathcal{U}$ we choose an orientation $O_\omega \subset \overrightarrow{E}(\Gamma)$ of $c^{-1}(\omega)\} \subset E(\Gamma)$ (recall this means that $(O_\omega \cup O_\omega^{-1})/^{-1} = c^{-1}(\omega)$ and $O_\omega \cap O_\omega^{-1} = \emptyset$). Since $c^{-1}(\omega)$ is a multicycle, we can choose O_ω so that each of its cycles is oriented, that is, for each vertex $v \in V(\Gamma)$ there is exactly one $e \in O_\omega$ with $\tau(e) = x$. Thus O_ω defines a permutation $\phi(\omega)$ of $X = V(\Gamma)$, by letting $\phi(\omega)(x)$ be the unique $y \in X$ such that $(x, y) \in O_\omega$. Moreover, for each $\omega \in \mathcal{I}$, let $O_\omega = \{e \in \overrightarrow{E}(\Gamma) | [e] \in c^{-1}(\omega)\}$, and let $\phi(\omega)$ be the involution of $V(\Gamma)$ exchanging the endvertices of each edge in $c^{-1}(\omega)$. Thus ϕ satisfies (3b) of Definition 2.3.1 by construction (we will check (3a) below).

We now define c' by

$$c'(e) = \begin{cases} c([e]) & \text{if } e \in O_{c([e])} \\ c([e])^{-1} & \text{otherwise.} \end{cases}$$

This maps $\overrightarrow{E}(\Gamma)$ to $\mathcal{U} \cup \mathcal{U}^{-1} \cup \mathcal{I}$, because for $e \in \overrightarrow{E}(\Gamma)$ such that $c([e]) \in \mathcal{I}$ we have $e, e^{-1} \in O_{c([e^{-1}])}$ by definition. Easily, c' is a Cayley-like colouring. This allows us to define $\mathcal{W}_v$ on Γ as described after Definition 2.1.8. Note that as Γ is connected, for any two $x, y \in V(\Gamma)$ there is a path p connecting x and y. Then the path p

corresponds to a word $\mathcal{W}_x(p) \in F_P^{\mathrm{Mod}}$ such that $\phi(\mathcal{W}_x(p))(x) = y$. Therefore the action of F_P^{Mod} on $X = V(\Gamma)$ is transitive as required by (3a) of Definition 2.3.1.

To complete the definition of our split presentation P, we choose the relators

- $\mathcal{R}_v = \mathcal{W}_v(\pi_1(\Gamma, v)) \subset F_P^{\mathrm{Mod}}$.

We claim that Γ coincides with the presentation graph $C(P)$. To begin with, they have the same vertex set $V(C) = X = V(\Gamma)$. Moreover,

$$\begin{aligned}\overrightarrow{E}(C(P)) &= \{(x, \phi(\omega)(x)) | x \in V(\Gamma),\ \omega \in \mathcal{U} \cup \mathcal{U}^{-1} \cup \mathcal{I}\} \\ &= \cup_{\omega \in \Omega}\{(x, y) | (x, y) \in O_\omega \text{ or } (y, x) \in O_\omega\} \\ &= \cup_{\omega \in \Omega}(c')^{-1}(\omega) = \overrightarrow{E}(\Gamma)\end{aligned}$$

and so our claim is proved.

As we defined $\mathcal{C}(P)$ by glueing in a 2-cell along each closed walk dictated by an element of $\mathcal{R}_v, v \in V(C(P))$, where we have chosen $\mathcal{R}_v = \mathcal{W}_v(\pi_1(\Gamma, v))$, we have forced $\pi_1(\mathcal{C}(P), v)$ to be trivial. Therefore, $\mathcal{C}(P)$ coincides with its own universal cover $\widehat{\mathcal{C}(P)}$. Thus $\mathrm{Spl}(P)$, defined as the 1-skeleton of $\widehat{\mathcal{C}(P)}$, is $C(P) = \Gamma$. Therefore P is a split presentation for Γ.

For the converse direction, let $\Gamma = \mathrm{Spl}(P)$ for some split presentation P. Let $\epsilon : \Gamma \to C(P)$ be the covering map, and $c'_C : \overrightarrow{E}(C) \to \mathcal{U} \cup \mathcal{U}^{-1} \cup \mathcal{I}$ the colouring induced by the generators of P, as in the definition of $\mathrm{Spl}(P)$. We collapse c'_C into a colouring c_C of the undirected edges of C defined by

$$c_C([e]) = \begin{cases} u \in \mathcal{U} & \text{if } c(e) \in \{u, u^{-1}\} \\ i \in \mathcal{I} & \text{if } c(e) = i. \end{cases}$$

We can collapse $c'_\Gamma : \overrightarrow{E}(\Gamma) \to \mathcal{U} \cup \mathcal{U}^{-1} \cup \mathcal{I}$ similarly to obtain an undirected colouring $c_\Gamma : E(\Gamma) \to \mathcal{U} \cup \mathcal{I}$. Note that c_C is a split-friendly weak multicycle colouring, with $c^{-1}(i)$ being of degree 1 for $i \in \mathcal{I}$ and $c^{-1}(u)$ being of degree 2 for $u \in \mathcal{U}$, by the definitions. As $c'_C \circ \epsilon = c'_\Gamma$, it is easy to verify that $c_C \circ \epsilon = c_\Gamma$. This implies that $c_\Gamma^{-1}(x)$ has the same degree as $c_C^{-1}(x)$, and that every vertex has at least one incident edge coloured s for each $s \in \mathcal{I} \cup \mathcal{U}$. This means that c_Γ is a split-friendly weak multicycle colouring of Γ as claimed. □

2.3.4 Weak multicycle colourings of vertex transitive graphs

The aim of this section is to show that every vertex transitive graph Γ has a split-friendly weak multicycle colouring, hence it admits a split presentation by Theorem 2.3.7.

For this, we will use the following result of Godsil and Royle [19, Theorem 3.5.1]:

Theorem 2.3.8 (Godsil & Royle [19, Theorem 3.5.1]). *Let Γ be a connected finite vertex transitive graph. Then Γ has a matching that misses at most one vertex.*

In the Appendix we generalise this to infinite vertex transitive graphs as follows

Theorem 2.7.1 Let Γ be a connected infinite vertex transitive graph which is locally finite. Then Γ has a perfect matching.

In passing, let us mention the following still open conjectue. If true, it would imply that all finite vertex transitive cubic graphs have a uniform split presentation.

Conjecture 2.3.9 (Lovasz [29, Problem 11]). *Let Γ be a finite cubic vertex transitive graph. Then there exists a perfect matching M in Γ such that $\Gamma \setminus M$ consists of either one cycle, and Γ is Hamiltonian, or of two disjoint cycles of the same length.*

The following theorem of Petersen is a rather straightforward application of Hall's Marriage theorem [22]. Although this is well-known, we include a proof for convenience.

Theorem 2.3.10 (Julius Petersen). *Every regular graph of finite positive even degree has a spanning 2-regular subgraph.*

Proof. Let Γ be a $2k$-regular graph. If Γ is finite then it contains an Euler tour C (i.e. a closed walk that uses each edge exactly once) by Euler's theorem [8]. Pick an orientation of $O_C \subset \overrightarrow{E}(\Gamma)$ of C. If Γ is infinite then just choose an orientation with equal in and out degree, which can be constructed greedily. Then construct an auxiliary graph Δ with

$$\begin{aligned} V(\Delta) =& \{v^+, v^- | v \in V(\Gamma)\}, \text{ and} \\ E(\Delta) =& \{(v^+, u^-) | (v, u) \in O_C\}. \end{aligned}$$

By definition, Δ is k-regular and bipartite, with bipartition $V^+ = \{v^+ | v \in V(\Gamma)\}$ and $V^- = \{v^- | v \in V(\Gamma)\}$. For any finite $A \subset V^+$, as Δ is k-regular, the neighbourhood $\mathrm{Nb}(A) = \{u^- | (v^+, u^-) \in E(\Delta) \text{ with } v^+ \in A\}$ of A has size at least $k \times |A|/k = |A|$. So by Hall's Marriage theorem [22], Δ contains a perfect matching $M \subset E(\Delta)$. Then the spanning subgraph $S \subset \Gamma$ given by $\{(v,u)|(v^+,u^-) \in M\} \subset E(\Gamma)$ is 2-regular by construction. $\square$

Combining this with Theorem 2.3.8 and Theorem 2.7.1, we now obtain.

Lemma 2.3.11. *Every vertex transitive graph which is locally finite Γ has a split-friendly weak multicycle colouring.*

Proof. As Γ is vertex transitive it is n-regular for some $n \in \mathbb{N}$. If n is even, then we can apply Theorem 2.3.10 recursively to decompose $E(\Gamma)$ into 2-regular spanning subgraphs, and attributing a distinct colour to the edges of each of those subgraphs yields a split-friendly weak multicycle colouring.

If n is odd, then we first find a perfect matching M, colour its edges with the same colour, and treat $\Gamma \backslash M$ as above to obtain a split-friendly weak multicycle colouring. To obtain M, note that if Γ is finite, then $|V(G)|$ is even since $|E(\Gamma)| = n|V(G)|/2$. Therefore Γ has a perfect matching by Theorem 2.3.8 as no matching can miss exactly 1 vertex in this case. If Γ is infinite, then Theorem 2.7.1 provides a perfect matching. $\square$

This combined with Theorem 2.3.7 yields one of our main results:

Theorem 2.3.12. *Every locally finite vertex transitive graph has a split presentation.*

2.3.5 Generalised results

Here we extend some of our earlier results from special to general split presentations. Where the same arguments apply directly the proofs will be omitted. First we generalise Lemma 2.2.4:

Proposition 2.3.13. *For a split presentation $P = \langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$ there is a natural inclusion of the vertex group $G_x \leq Aut_c(Spl(P))$ for each $x \in X$. Moreover G_x acts regularly on V_x, and so $Spl(P)$ is $|X|$-Cayley.*

The vertex groups are still isomorphic due to the fact that π_1 does not depend on the choice of a base point:

Proposition 2.3.14. *For every split presentation $P = \langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$, and every $x, y \in X$, the vertex groups G_x, G_y are isomorphic.*

Proof. As above, let $C(P) =: C$ be the presentation graph of P. Let $x, y \in X = V(C)$. Recall that $G_x := R_x \backslash W_{x,x}$ is the left quotient of $W_{x,x}$ by R_x, where $W_{x,z}$ is the set of paths in C from x to z up to homotopy (in particular, $W_{x,x} = \pi_1(C,x)$), and

$$R_x := \langle \{w\mathcal{W}_z^{-1}(r)w^{-1} | r \in \mathcal{R}_z, w \in W_{x,z}, z \in X\}\rangle$$

with $\mathcal{W}_z^{-1}$ the map from words in F_P^{Mod} to paths in C defined in section 2. Let $p \in W_{x,y}$ be a path from x to y in C. As π_1 is base point preserving, we have

$$W_{x,x} = pW_{y,y}p^{-1}. \tag{2.7}$$

Moreover, note that

$$\begin{aligned} pR_yp^{-1} =& \langle \{(pw)\mathcal{W}_z^{-1}(r)(pw)^{-1} | r \in \mathcal{R}_z, w \in W_{y,z}, z \in X\}\rangle \\ =& \langle \{w'\mathcal{W}_z^{-1}(r)(w')^{-1} | r \in \mathcal{R}_z, w' \in W_{x,z}, z \in X\}\rangle \\ =& R_x. \end{aligned}$$

This defines a homomorphism $\phi : G_y \to G_x$ by $\phi : R_yw \mapsto R_xpwp^{-1}$ for every $w \in W_{y,y}$. It is surjective by (2.7) and injective as $pR_yp^{-1} = R_x$. Thus it is an isomorphism proving our claim. □

We generalise Proposition 2.2.7 to obtain a sufficient condition for vertex transitivity.

Proposition 2.3.15. *Let $P = \langle X|\mathcal{U}|\mathcal{I}|\phi|\mathcal{R}\rangle$ be a split presentation. If the presentation complex $\mathcal{C}(P)$ is vertex transitive, then so is Spl(P).*

Lastly we would like to talk about what kind of graphs we get up to quasi-isometry.

Proposition 2.3.16. *Given a split presentation P with finite vertex set X. Then $\Gamma :=$ Spl(P) is quasi-isometric to G_x for every $x \in X$.*

Proof. Let $\mathcal{C} := \mathcal{C}(P)$ be the presentation complex associated to P and $C := C(P)$ be the presentation graph. Consider the inclusion map $i : C \to \mathcal{C}$. Hatcher [23, Proposition 1.26] tells us that the inclusion of the one skeleton into a 2-simplex induces a surjection on the level of fundamental groups and the kernal is exactly the normal closure of the words bounding the 2-cells inserted. So we have that $i_* : \pi_1(C, x) \to \pi_1(\mathcal{C}, x)$ is a surjection with kernal exactly R_x, so $\pi_1(\mathcal{C}, x) = R_x \backslash \pi_1(C, x) = R_x \backslash W_{x,x} = G_x$.

Let $\widehat{\Gamma}$ be the universal cover of $\mathcal{C}$, with covering map $\widehat{\eta} : \widehat{\Gamma} \to \mathcal{C}$. As $\pi_1(\mathcal{C}, x) = G_x$ we have an action of G_x on $\widehat{\Gamma}$ (and it's 1-skeleton Spl$(P) =: \Gamma$) by deck transformations.

From Hatcher [23, p 70] we know the quotient of a universal cover by the group of deck transformations gives the space itself. So the quotient of $\widehat{\Gamma}$ by G_x is exactly $\mathcal{C}$, so similarly the quotient of Γ by G_x is exactly C. Lastly we want to show that the action of G_x by deck transformations on Γ is properly discontinuous. Take a compact subset $K \subset \Gamma$, this is bounded in the graph metric. For a fixed integer there are only finitely many classes of paths of length less that this integer, in $\pi_1(\mathcal{C}, x)$. So $|\{g \in G_x | (g \cdot K) \cap K \neq \emptyset\}|$ is finite as any such g would have to come from a path of finite length.

So G_x acts on Γ in a properly discontinuous and co-compact fashion, as X is finite giving that C is compact. By the Švarc–Milnor lemma [41] we have that Γ is quasi-isometric to G_x, giving what is required. $\square$

2.4 Line graphs of Cayley graphs admit split presentations

In this section we show that every line graph of a Cayley graphs can be represented as a split presentation graph. To do this we need to analyse the complete graph K_n. By Lemma 2.3.11, we have found a split-friendly multicycle colouring $c : E(K_n) \to \Omega$ of K_n. Next, we want to associate each colour $\omega \in \Omega$ with a permutation $\pi_\omega \in Sym_n$ of the vertices of K_n. To do so, for each $\omega \in \Omega$ such that $c^{-1}(\omega)$ is 2-regular, we pick an orientation $O_\omega \subset c^{-1}(\omega)$, (such that $O_\omega \cap O_\omega^{-1} = \emptyset$ and $O_\omega \cup O_\omega^{-1} = c^{-1}(\omega)$), and let π_ω be the corresponding permutation (sending each vertex to its successor in O_ω. For each $\omega \in \Omega$ such that $c^{-1}(\omega)$ is 1-regular, we let π_ω be the permutation that exchanges the two end vertices of each edge in $c^{-1}(\omega)$.

Proposition 2.4.1. *Let $\Gamma = Cay\langle\mathcal{S}|\mathcal{R}\rangle$ be a Cayley graph. Then the line graph $L(\Gamma)$ can be represented as $Spl(P)$ for a split presentation P with at most $|\mathcal{S}|$ vertex classes.*

Proof. The split presentation P we will construct will have one vertex class for each generator in $\mathcal{S}$. Since the edges of $L(\Gamma)$ are precisely the pairs of incident edges of Γ, we will identify the generators of P with pairs of generators $s, t \in \mathcal{S}$. Since we need to pay attention to the directions of the edges of Γ, each such pair s, t will give rise to four generators of P, indexed by the elements of $\{-1, 1\}^2$. Similarly, each $s \in \mathcal{S}$ will give rise to two generators of P, since there are pairs of incident edges of Γ labelled by s, and there are two choices for their directions. The relators of P will be of two kinds. The first kind is just obtained by rewriting the elements of $\mathcal{R}$ in terms of the new generators. The second kind will correspond to closed walks in $L(\Gamma)$ contained in the star of a vertex of Γ.

We proceed with the formal definition of P. The vertex classes of P will be identified with the generating set $\mathcal{S}$ of Γ. Let $K_\mathcal{S}$ denote the complete graph with $V(K_\mathcal{S}) = \mathcal{S}$. From the above discussion we obtain a multicycle colouring $M \subset \mathrm{Sym}_\mathcal{S}$ of $K_\mathcal{S}$ where each colour is identified with a permutation of $\mathcal{S}$. The generating set of our split presentation P comprises the formal symbols $\mathcal{U} = \{e, e^{-1}\} \cup \{m_{i,j} | m \in M, i, j \in \{-1, 1\} \text{ where } m^2 \neq 1\}$ and $\mathcal{I} = \{m_{i,j} | m \in M, i, j \in \{-1, 1\} \text{ where } m^2 = 1\}$. Set

$\mathcal{S}' = \mathcal{U} \cup \mathcal{I}$, the generators of P. We need to associate a permutation $\phi(s)$ of the vertex classes with each $s \in \mathcal{S}'$, and we do so by

$$\phi : \begin{array}{c} m_i \\ e, e^{-1} \end{array} \mapsto \begin{array}{c} m \\ 1_{\mathcal{S}} \end{array}.$$

Let $\theta : \overrightarrow{E}(K_{\mathcal{S}}) \to M \cup M^{-1}$ be the colouring of $K_{\mathcal{S}}$ by $M \cup M^{-1}$. We can think of θ as a map from $\mathcal{S} \times \mathcal{S} \backslash \{(s,s) | s \in \mathcal{S}\}$ to $M \cup M^{-1}$ where $\theta(a,b)(a) = b$. Let $S = \{s, s^{-1} | s \in \mathcal{S}\}$ be $\mathcal{S}$ with formal inverses. Define a map $\chi : S \times S \backslash \{(s, s^{-1}) | s \in \mathcal{S}\} \to \mathcal{S}'$ where

$$\chi(a,b) = \begin{cases} e & \text{if } a = b \in \mathcal{S} \\ e^{-1} & \text{if } a = b \notin \mathcal{S} \\ m_{i,j} & \text{if } \theta(a,b) = m \text{ where } a^i, b^j \in \mathcal{S}. \end{cases}$$

Here we make the identification that $(m_{i,j})^{-1} = (m^{-1})_{-j,-i}$.

We now define the sets of relators $\mathcal{R}_a, a \in \mathcal{S}$ of P. For each relator $r := a_1 a_2 \dots a_k \in \mathcal{R}$ we add $\chi(r) := \chi(a_1, a_2)\chi(a_2, a_3) \dots \chi(a_{k-1}, a_k)\chi(a_k, a_1)$ to $\mathcal{R}_{a_1^{\pm 1}}$. (These are the relators of the first kind as explained at the beginning of the proof.) Lastly, we add relations (of the second kind) corresponding to the star of each vertex of Γ as follows. Let $a_1 \dots a_k \in W_{\mathcal{S}}$ be any word equaling the identity in F_P^{Mod}, and add $\chi(a_1, a_2) \dots \chi(a_k, a_1)$ to $\mathcal{R}_{a_1^{\pm 1}}$, where $\chi(s, s^{-1})$ is the empty word. Let $\mathcal{R}' := \{\mathcal{R}_a, a \in \mathcal{S}\}$. We have now constructed our presentation $P := \langle \mathcal{S} | \mathcal{U} | \mathcal{I} | \phi | \mathcal{R}' \rangle$.

Next, we prove that $\mathrm{Spl}(P)$ is isomorphic to $L(\Gamma)$. First label

$$V(L(\Gamma)) = \{[(g, gs)] | g \in G,\ s \in \mathcal{S}\} \text{ and}$$

$$\overrightarrow{E}(L(\Gamma)) = \{(g, s_1, s_2) | g \in G,\ s_1, s_2 \in \mathcal{S} \cup \mathcal{S}^{-1},\ s_1 \neq s_2^{-1}\}$$

so that the edge (g, s_1, s_2) connects $[(g, gs_1)]$ and $[(gs_1, gs_1s_2)]$. Let $C := C(P)$ be the presentation graph of P. Then we can define a map $\epsilon : L(\Gamma) \to C$ by letting $\epsilon([(g, gs)]) = s$ and letting $\epsilon((g, s_1, s_2))$ be the edge of colour $\chi(s_1, s_2)$ coming from $s_1^{\pm 1} \in \mathcal{S}$. One can show that the relations in $\mathcal{R}_x$ hold in $L(\Gamma)$ for all $x \in \mathcal{S}$. It remains to show that these relations suffice.

Intuitively we are going to argue that any closed walk p in $L(\Gamma)$ is labelled by some $r \in \langle\langle \mathcal{R} \rangle\rangle_{F_{\mathcal{S}}}$ interwoven with relations coming from the stars at the vertices. One can observe this by just projecting p to a closed walk in Γ, where after some cancelations happening within the stars of vertices, we are left with a closed walk labelled by a word r than can be expressed in terms of the relators in $\mathcal{R}$. We proceed with this formally.

Define a topological map $\Phi : L(\Gamma) \to \Gamma$ by mapping $[(g, gs)] \in V(L(\Gamma))$ to the midpoint of the edge (g, gs), and $(g, s_1, s_2) \in E(L(\Gamma))$ to the arc in the star of gs_1 connecting the midpoints of $[(g, gs_1)]$ and $[(gs_1, gs_1s_2)]$. Consider a closed walk p in $L(\Gamma)$. We can write $p = \prod_{i=0}^{n-1}(g^i, s_1^i, s_2^i)$. As $\Phi(p)$ is a closed walk in Γ we know it can be contracted to a path given by $g_0g_1 \ldots g_{m-1}$ for $g_i \in G$. Now we want to group the edges of p by the stars of vertices of Γ they lie in. For this, we subdivide the interval $\{0, \ldots, n-1\}$ into disjoint subintervals $\{I_j\}_{j=0}^{m-1}$ such that $\Phi(g^i, s_1^i, s_2^i)$ lies in the star of g_j for all $i \in I_j$ and $0 \le j \le m-1$ (we can assume without loss of generality that no I_j has to be the union of an initial and a final subinterval of $\{0, \ldots, n-1\}$ by rotating p appropriately). Thus $p = \prod_{j=0}^{m-1}\left(\prod_{i \in I_j}(g^i, s_1^i, s_2^i)\right)$.

To each j we can also associate $s_j \in \mathcal{S} \cup \mathcal{S}^{-1}$ so that $g_js_j = g_{j+1}$; these are the generators that p uses in order to move from one star to the next.

We modify p into a closed walk p' by inserting pairs of edges that have the same endvertices and opposite directions each time that p moves from one star to the next. More formally, we define

$$p' := \prod_{j=0}^{m-1}\left(\left(\prod_{i \in I_j}(g^i, s_1^i, s_2^i)\right)(g_{j+1}, s_j^{-1}, s_{j-1}^{-1})(g_{j-1}, s_{j-1}, s_j)\right).$$

Notice that by contracting these pairs of opposite edges $(g_{j+1}, s_j^{-1}, s_{j-1}^{-1})(g_{j-1}, s_{j-1}, s_j)$ we obtain p. Moreover, the sub-walk

$$\left(\prod_{i \in I_j}(g^i, s_1^i, s_2^i)\right)(g_{j+1}, s_j^{-1}, s_{j-1}^{-1})$$

of p' stays within the star of g_j by definition, and it is a closed walk starting and ending at $[(g_{j-1}, g_j)]$. Therefore, it is labelled by one of our relators of the second kind. Easily, $\Phi(p)$ is homotopic to $\Phi(p')$. Moreover,

$$\begin{aligned}\Phi(p') &= \prod_{j=0}^{m-1}\Phi\left(\left(\prod_{i \in I_j}(g^i, s_1^i, s_2^i)\right)(g_{j+1}, s_j^{-1}, s_{j-1}^{-1})\right)\Phi(g_{j-1}, s_{j-1}, s_j)\\ &= \prod_{j=0}^{m-1}\Phi(g_{j-1}, s_{j-1}, s_j)\end{aligned}$$

since a closed walk contained in a star is 0-homotopic.

Now $\prod_{j=0}^{m-1}(g_{j-1}, s_{j-1}, s_j)$ is a closed walk in $L(\Gamma)$ no two consecutive edges of which are contained in the star of a vertex of Γ because of the way we chose the I_j. This implies that the word $s_0 \dots s_{m-1}$ labelling this walk is a relation of Γ, and so it can be written as a product of conjugates of relators $\mathcal{R}$. Recalling that each such relator was admitted as a relator (of the first kind) in $\mathcal{R}'$, we conclude that the word labelling p can be written as products of conjugates of words in $\mathcal{R}'$. □

We explicate an example of this below.

Example 2.4.2. Consider $D_{10} = \langle a, b | a^5, b^2, aba^{-1}b^{-1}\rangle$, which has the Cayley graph and line graph thereof shown in Figure 2.9. As $K_{\{a,b\}}$ is a single edge, we have

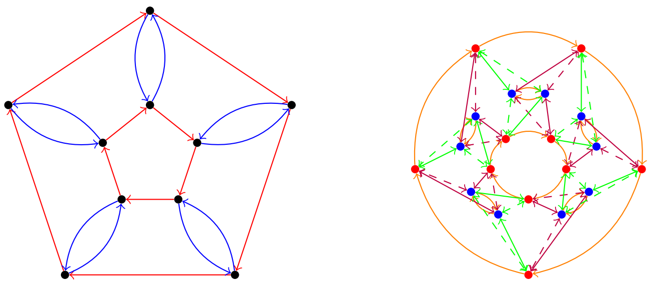

Figure 2.9: $\mathrm{Cay}\langle a, b | a^5, b^2, aba^{-1}b^{-1}\rangle$ and its line graph

$M = \{(1,2)\} =: \{m\}$ with $m_{i,j} \mapsto (1,2)$ for $i, j \in \{1, -1\}$ and $e, e^{-1} \mapsto (1)(2)$ as generators. Define the following function

$$\chi : \begin{array}{rcl} aa & \to & e \\ a^{-1}a^{-1} & \to & e^{-1} \\ bb & \to & e \\ b^{-1}b^{-1} & \to & e^{-1} \end{array} \qquad \begin{array}{rcl} ab & \to & m_{1,1} \\ ab^{-1} & \to & m_{1,-1} \\ a^{-1}b & \to & m_{-1,1} \\ a^{-1}b^{-1} & \to & m_{-1,-1} \end{array} \qquad \begin{array}{rcl} b^{-1}a^{-1} & \to & m_{-1,-1} \\ ba^{-1} & \to & m_{1,-1} \\ b^{-1}a & \to & m_{-1,1} \\ ba & \to & m_{1,1}. \end{array}$$

The original relators $a^5, b^2, aba^{-1}b^{-1}$ are thus translated into relators in the resulting split presentation as follows: $a^5 \to e^5 \in \mathcal{R}_a$, $b^2 \to e^2 \in \mathcal{R}_b$ and $aba^{-1}b^{-1} \to m_{1,1}m_{1,-1}m_{-1,-1}m_{-1,1} \in \mathcal{R}_a$. Lastly, we add relations of the second kind shown in Figure 2.10, which are enough to generate the rest of the relations. The resulting split presentation is

$$\left\langle a, b \,\middle|\, \begin{array}{c} \mathcal{I} = m_{1,1}, m_{1,-1}, m_{-1,1}, m_{-1,-1} \\ \mathcal{U} = e, e^{-1} \end{array} \to \begin{array}{c} (12) \\ (1)(2) \end{array} \right|$$

$$\left| \begin{array}{c} \{e^5, m_{1,1}m_{1,-1}m_{-1,-1}m_{-1,1}, em_{-1,1}m_{-1,-1}, \\ em_{-1,-1}m_{1,-1}, m_{1,1}e^{-1}m_{1,-1}, m_{-1,1}e^{-1}m_{1,1}\}, \{e^2\} \end{array} \right\rangle.$$

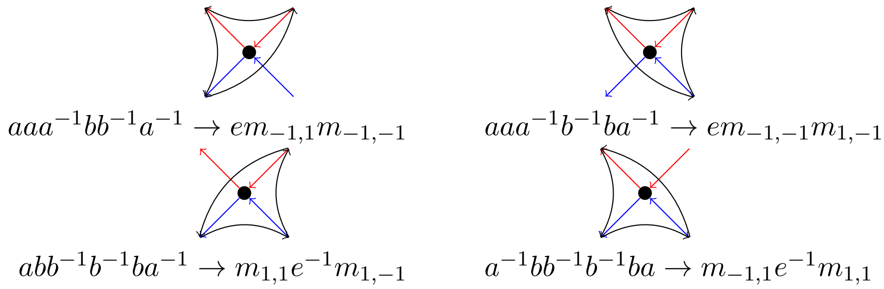

Figure 2.10: Example of relations of the second kind

2.5 A Cubic 2-ended vertex transitive graph which is not Cayley

In this section we construct an example of a 2-ended cubic vertex transitive graph which is not a Cayley graph. This answers a question of Watkins [53] also appearing in [20].

Let Γ be the graph with

$$V(\Gamma) = \{v_{n,k} | n \in \mathbb{Z}, k \in \mathbb{Z}/10\mathbb{Z}\}, \text{ and}$$

$$E(\Gamma) = \{[(v_{n,k}, v_{n,k+1})], [(v_{n,2k+1}, v_{n+1,4k+2})] \mid n \in \mathbb{Z},\ k \in \mathbb{Z}/10\mathbb{Z}\}.$$

By construction Γ is a cubic graph. A useful way to think of this graph is as a 2-way infinite stack of layers $L_n := \{v_{n,k} | k \in \mathbb{Z}/10\mathbb{Z}\}$. Each L_n spans a 10-cycle, and between any two layers L_n and L_{n+1} there is a Petersen graph like structure.

Claim 1. Γ is 2-ended.

Proof. We will show that Γ is quasi-isometric to $\Delta := \mathrm{Cay}(\mathbb{Z}, \{1\})$, hence 2-ended [5]. Let d_Γ and d_Δ be the path metric in Γ and Δ respectively. Our quasi-isometry is the map $f : \Gamma \to \Delta$ defined by $f(v_{n,k}) := n$.

It is straightforward to check that $d_\Gamma(v_{n,0}, v_{n+1,0}) = 2$ and $d_\Gamma(v_{n,k}, v_{n,0}) \leq k$ by the definition. Now for any two vertices $v_{n,k}, v_{n',k'} \in V(\Gamma)$ where $n \leq n'$, we have

$$\begin{aligned} d_\Gamma(v_{n,k}, v_{n',k'}) \leq & d_\Gamma(v_{n,k}, v_{n,0}) \\ & + \sum_{n \leq i < n'} d_\Gamma(v_{i,0}, v_{i+1,0}) + d_\Gamma(v_{n',0}, v_{n',k}) && \text{by the triangle inequality} \\ \leq & 2(n - n') + (k + k') && \text{by the two facts above} \\ \leq & 2d_\Delta(f(v_{n,k}), f(v_{n',k'})) + 20 && \text{by the definition of } f. \end{aligned}$$

Another straightforward consequence of the definition of Γ is that $d_\Gamma(v_{n,k}, v_{n',k'}) \geq$

$n - n'$, which combined with the above inequality yields

$$d_\Gamma(v_{n,k}, v_{n',k'})/2 - 20 \le d_\Delta(f(v_{n,k}), f(v_{n',k'})) = n - n' \le d_\Gamma(v_{n,k}, v_{n',k'}).$$

As f is surjective, this means that it is a quasi-isometry. Since the number of ends of a graph is invariant under quasi-isometry, Γ has 2-ends as $\mathbb{Z}$ does. $\square$

Claim 2. Γ is vertex transitive.

Proof. We introduce the following two maps $\sigma, \tau : V(\Gamma) \to V(\Gamma)$, which will allow us to map any vertex of Γ to any other:

$$\sigma(v_{n,k}) = v_{n+1,k} \quad \text{and} \quad \tau(v_{n,k}) = \begin{cases} v_{-n,k+1} & \text{if } n \equiv 0 \pmod 4 \\ v_{-n,3-k} & \text{if } n \equiv 1 \pmod 4 \\ v_{-n,k+9} & \text{if } n \equiv 2 \pmod 4 \\ v_{-n,7-k} & \text{if } n \equiv 3 \pmod 4 \end{cases}.$$

Intuitively σ just shifts all the layers up by 1, so L_n is mapped to L_{n+1} keeping the positions and orientations of the 10-cycles the same. Whereas τ rotates the 10-cycle on L_0 by one position, which flips the stack of layers by mapping L_n to L_{-n}, and inverts the orientation of the 10-cycles at odd numbered layers.
It is easy to see that τ and σ preserve edges of the form $(v_{n,k}, v_{n,k+1})$. Moreover, it is easy to see that σ preserves edges of the form $(v_{n,2k+1}, v_{n+1,4k+2})$, we now check that τ also preserves these edges $\tau : (v_{n,2k+1}, v_{n+1,4k+2}) \mapsto$

$$\begin{array}{rcll} (v_{-n,2k+2}, v_{-(n+1),1-4k}) & = & (v_{-(n+1)+1,4(-2k)+2}, v_{-(n+1),2(-2k)+1}) & \text{if } n \equiv 0 \pmod 4 \\ (v_{-n,2-2k}, v_{-(n+1),4k+1}) & = & (v_{-(n+1)+1,4(2k)+2}, v_{-(n+1),2(2k)+1}) & \text{if } n \equiv 1 \pmod 4 \\ (v_{-n,2k}, v_{-(n+1),5-4k}) & = & (v_{-(n+1)+1,4(2-2k)+2}, v_{-(n+1),2(2-2k)+1}) & \text{if } n \equiv 2 \pmod 4 \\ (v_{-n,6-2k}, v_{-(n+1),4k+3}) & = & (v_{-(n+1)+1,4(2k+1)+2}, v_{-(n+1),2(2k+1)+1}) & \text{if } n \equiv 3 \pmod 4, \end{array}$$

where we used that $8 \equiv -2 \pmod{10}$. By changing the order of the vertices in the right hand side we see that these are indeed edges of Γ of the form $(v_{n,2k+1}, v_{n+1,4k+2})$. Thus we have checked that $\sigma, \tau \in \mathrm{Aut}(\Gamma)$. Let $G := \langle \sigma, \tau \rangle \le \mathrm{Aut}(\Gamma)$ be the group of automorphisms of Γ generated by σ and τ. For any two vertices $v_{n,k}, v_{n',k'} \in V(\Gamma)$, we have

$$\begin{aligned} \sigma^{n'}\tau^{k'-k}\sigma^{-n}(v_{n,k}) &= \sigma^{n'}\tau^{k'-k}(v_{0,k}) \\ &= \sigma^{n'}(v_{0,k'}) \\ &= v_{n',k'} \end{aligned}$$

proving that G acts transitively on $V(\Gamma)$. $\square$

Claim 3. G is not a regular subgroup of $\mathrm{Aut}(\Gamma)$.

Proof. Observe

$$\begin{aligned}\tau^{-3}\sigma\tau\sigma(v_{0,k}) &= \tau^{-3}\sigma\tau(v_{1,k})\\ &= \tau^{-3}\sigma(v_{-1,3-k})\\ &= \tau^{-3}(v_{0,3-k})\\ &= v_{0,-k}.\end{aligned}$$

This implies $\tau^{-3}\sigma\tau\sigma(v_{0,0}) = v_{0,0}$, yet $\tau^{-3}\sigma\tau\sigma \neq 1_\Gamma$ as $\tau^{-3}\sigma\tau\sigma(v_{0,1}) = v_{0,9}$. Thus the action of G on Γ is not semi-regular. □

We remind the reader the nth layer is denoted $L_n = \{v_{n,k} | \; k \in \mathbb{Z}/10\mathbb{Z}\}$ for $n \in \mathbb{Z}$ and we define the partition $\mathcal{C} := \{L_n \mid n \in \mathbb{Z}\}$ of $V(\Gamma)$.

Claim 4. Let $\phi \in \mathrm{Aut}(\Gamma)$ satisfy $\phi(L_a) = L_b$. If for some $\chi \in \mathrm{Aut}(\Gamma)$ we have $\phi(x) = \chi(x)$ for every $x \in L_a$, then $\phi = \chi$. Moreover, ϕ preserves the partition $\mathcal{C} := \{L_n \mid n \in \mathbb{Z}\}$.

Proof. Observe that $\phi(v_{a+1,2k})$ and $\phi(v_{a-1,2k+1})$ are uniquely determined by $\phi(x), x \in L_a$ because each vertex in L_b has exactly one neighbour outside L_b. This in turn uniquely determines $\phi(v_{a+1,2k+1})$ and $\phi(v_{a-1,2k})$ by a similar argument. Continuing like this, we see that $\phi(\{v_{a+\epsilon,k}|k \in \mathbb{Z}/10\mathbb{Z}\}) = \{v_{b\pm\epsilon,k}|k \in \mathbb{Z}/10\mathbb{Z}\}$ for $\epsilon \in \{-1, 1\}$. By an inductive argument, this uniquely determines ϕ, and moreover ϕ preserves $\mathcal{C}$. □

Claim 5. Any $\phi \in \mathrm{Aut}(\Gamma)$ preserves the partition $\mathcal{C} = \{L_n|n \in \mathbb{Z}\}$.

Proof. Suppose ϕ does not fix $\mathcal{C}$. By Claim 4 we have $\phi(L_0) \neq L_a$ for every $a \in \mathbb{Z}$. Let

$$n := \min\{n' \in \mathbb{Z}|v_{n',k} \in \phi(L_0)\}.$$

Thus there exist $l, k \in \mathbb{Z}/10\mathbb{Z}$ such that

$$\phi : \begin{matrix} v_{0,l} \\ v_{0,l\pm1} \\ v_{0,l\pm2} \\ v_{0,l\pm3} \end{matrix} \mapsto \begin{matrix} v_{n,2k-1} \\ v_{n,2k} \\ v_{n,2k+1} \\ v_{n+1,4k+2} \end{matrix}.$$

Let $\mathrm{Nb}(v_{n,k})$ denote the neigbourhood of $v_{n,k}$ in Γ. Let $v_{\epsilon,a} \in \mathrm{Nb}(v_{0,l\pm1})$ and

$v_{-\epsilon,b} \in \mathrm{Nb}(v_{0,l\pm 2})$ for $\epsilon \in \{-1,1\}$. Then

$$\begin{aligned}\phi(v_{\epsilon,a}) &\in \mathrm{Nb}(\phi(v_{0,l\pm1}))\backslash\{\phi(v_{0,l}),\phi(v_{0,l\pm2})\}\\ &= \{v_{n-1,a'}\}\\ \phi(v_{-\epsilon,b}) &\in \mathrm{Nb}(\phi(v_{0,l\pm2}))\backslash\{\phi(v_{0,l\pm1}),\phi(v_{0,l\pm3})\}\\ &= \{v_{n,2k+2}\}.\end{aligned}$$

Note that $\phi(v_{\epsilon,a}) = v_{n-1,a'}$ and $\phi(v_{-\epsilon,b}) = v_{n,2k+2}$ lie in the same connected component of $\Gamma\backslash\phi(L_0)$ by the definition of n. However, $v_{\epsilon,a}$ and $v_{-\epsilon,b}$ lie in different connected components of $\Gamma\backslash L_0$. This contradicts that ϕ is an automorphism of Γ, and so our claim is proved. $\square$

Claim 6. Any $\phi \in \mathrm{Aut}(\Gamma)$ is uniquely determined by $\phi(v_{0,1})$ and $\phi(v_{0,2})$.

Proof. Assume $\phi(v_{0,1}) = v_{a,b}$. Then by Claim 5, $\phi(v_{0,2}) \in \mathrm{Nb}(v_{a,b}) \cap \{v_{a,k} | k \in \mathbb{Z}/10\mathbb{Z}\} = \{v_{a,b+1}, v_{a,b-1}\}$. In either case, by Claim 5 this uniquely determines $\phi(v_{0,k})$ for $k \in \mathbb{Z}/10\mathbb{Z}$. By Claim 4, this uniquely determines ϕ. $\square$

We remark that this implies $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(v_{0,0}) = \langle \tau^{-3}\sigma\tau\sigma\rangle = \mathbb{Z}/2\mathbb{Z}$. So since $G = \langle \tau, \sigma\rangle$ acts transitively then $G = \mathrm{Aut}(\Gamma)$, in other words τ and σ generate the automorphism group.

Claim 7. Γ is not a Cayley graph.

Proof. Let $T \leq \mathrm{Aut}(\Gamma)$ be a transitive subgroup. Thus we can find automorphisms $\sigma', \tau' \in T$ such that $\sigma'(v_{0,1}) = v_{1,1}$ and $\tau'(v_{0,1}) = v_{0,2}$. By Claim 5, T preserves the partition $\mathcal{C}$. So either $\tau'(v_{0,2}) = v_{0,3}$ and $\tau' = \tau$ or $\tau'(v_{0,2}) = v_{0,1}$ and

$$\tau'(v_{n,k}) = \sigma\tau\sigma(v_{n,k}) = \tilde{\tau}(v_{n,k}) := \begin{cases} v_{-n,3-k} & \text{if } n \equiv 0 \pmod 4\\ v_{-n,k-1} & \text{if } n \equiv 1 \pmod 4\\ v_{-n,7-k} & \text{if } n \equiv 2 \pmod 4\\ v_{-n,k+1} & \text{if } n \equiv 3 \pmod 4\end{cases}$$

by Claim 6. Similarly, either $\sigma'(v_{0,2}) = v_{1,2}$ and $\sigma' = \sigma$ or $\sigma'(v_{0,2}) = v_{1,0}$ and

$$\sigma'(v_{n,k}) = \sigma\tau^{-1}\sigma\tau\sigma(v_{n,k}) = \tilde{\sigma}(v_{n,k}) := \begin{cases} v_{n+1,2-k} & \text{if } n \equiv 0 \pmod 4\\ v_{n+1,4-k} & \text{if } n \equiv 1 \pmod 4\\ v_{n+1,8-k} & \text{if } n \equiv 2 \pmod 4\\ v_{n+1,6-k} & \text{if } n \equiv 3 \pmod 4\end{cases}.$$

by Claim 6. By Claim 3, if $\{\sigma', \tau'\} = \{\sigma, \tau\}$ then T is not regular. If $\{\sigma', \tau'\} = \{\tilde{\sigma}, \tau\}$ then

$$\begin{aligned} \tau\tilde{\sigma}\tau\tilde{\sigma}(v_{0,k}) &= \tau\tilde{\sigma}\tau(v_{1,2-k}) \\ &= \tau\tilde{\sigma}(v_{-1,k+1}) \\ &= \tau(v_{0,5-k}) \\ &= v_{0,6-k} \end{aligned}$$

giving $\tau\tilde{\sigma}\tau\tilde{\sigma}(v_{0,3}) = v_{0,3}$ yet $\tau\tilde{\sigma}\tau\tilde{\sigma} \neq 1_\Gamma$. Similarly, if $\{\sigma', \tau'\} = \{\sigma, \tilde{\tau}\}$ then

$$\begin{aligned} \tilde{\tau}\sigma\tilde{\tau}\sigma(v_{0,k}) &= \tilde{\tau}\sigma\tilde{\tau}(v_{1,k}) \\ &= \tilde{\tau}\sigma(v_{-1,k-1}) \\ &= \tilde{\tau}(v_{0,k-1}) \\ &= v_{0,4-k} \end{aligned}$$

giving $\tilde{\tau}\sigma\tilde{\tau}\sigma(v_{0,2}) = v_{0,2}$ yet $\tilde{\tau}\sigma\tilde{\tau}\sigma \neq 1_\Gamma$. Lastly, if $\{\sigma', \tau'\} = \{\tilde{\sigma}, \tilde{\tau}\}$ then

$$\begin{aligned} \tilde{\tau}\tilde{\sigma}\tilde{\tau}\tilde{\sigma}(v_{0,k}) &= \tilde{\tau}\tilde{\sigma}\tilde{\tau}(v_{1,2-k}) \\ &= \tilde{\tau}\tilde{\sigma}(v_{-1,1-k}) \\ &= \tilde{\tau}(v_{0,5+k}) \\ &= v_{0,-2-k} \end{aligned}$$

giving $\tilde{\tau}\tilde{\sigma}\tilde{\tau}\tilde{\sigma}(v_{0,9}) = v_{0,9}$ yet $\tilde{\tau}\tilde{\sigma}\tilde{\tau}\tilde{\sigma} \neq 1_\Gamma$. Therefore T is never regular, and so Γ is not a Cayley graph. □

Combining the above claims we deduce the following.

Theorem 2.5.1. Γ *is a cubic 2-ended vertex transitive graph which is not a Cayley graph.*

Remark. For an interested reader, note that

$$Aut(\Gamma) = \langle \tau, \sigma | \tau^{10}, (\tau^{-1}\sigma\tau\sigma)^2, \sigma^{-1}\tau^2\sigma\tau^{-4}, (\sigma^{-2}\tau)^2 \rangle.$$

What follows is a discussion about the split presentation of the graph Γ.

Define presentation $P = \langle\ \{0,1\} \mid \{a\} \mid \{b\} \mid a, b \mapsto (01) \mid \{a^{10}, a^2ba^4b\}, \{a^2ba^{-4}b\}\ \rangle$, we will sketch that $\Gamma = \mathrm{Spl}(P)$. Let $\mathcal{C}(P)$ be the presentation complex, label the directed edges

$$\overrightarrow{E}(\mathcal{C}(P)) = \{a(0,1), a^{-1}(0,1), b(0,1), a(1,0), a^{-1}(1,0), b(1,0)\}$$

where $\epsilon(i,j)$ is the edge coloured ϵ going from i to j, these indices will be taken modulo 2. Define cover $\zeta : \Gamma \to \mathcal{C}(P)$, where

$$\zeta : \begin{array}{c} v_{n,k} \\ (v_{n,k}, v_{n,k+1}) \\ (v_{n,2k+1}, v_{n+1,4k+2}) \end{array} \mapsto \begin{array}{c} k \pmod 2 \\ a(k,k+1) \\ b(1,0) \end{array} \quad \text{where}$$

$$\zeta : \begin{array}{c} (v_{n,k+1}, v_{n,k}) \\ (v_{n+1,4k+2}, v_{n,2k+1}) \end{array} \mapsto \begin{array}{c} a^{-1}(k+1,k) \\ b(0,1) \end{array}.$$

Observe the relations hold as the paths

$$\zeta : \begin{array}{c} v_{n,k}, v_{n,k+1}, \dots v_{n,k+9}, v_{n,k} \\ v_{n+1,2k}, v_{n+1,2k+1}, v_{n+1,2k+2}, v_{n,6k+1}, v_{n,6k+2}, \dots v_{n,6k+5}, v_{n,2k} \\ v_{n,2k+1}, v_{n,2k+2}, v_{n,2k+3}, v_{n+1,4k+6}, v_{n+1,2k+5}, \dots, v_{n+1,4k+2}, v_{n,2k+1} \end{array}$$

$$\mapsto \begin{array}{c} (a(k,k+1)a(k+1,k))^5 \\ a(0,1)a(1,0)b(0,1)(a(1,0)a(0,1))^2 b(1,0) \\ a(1,0)a(0,1)b(1,0)(a^{-1}(0,1)a^{-1}(1,0))^2 b(0,1) \end{array}$$

which we leave for the interested reader to deduce that by inserting these 2-cells Γ becomes simply connected (note here we don't need the $a^2ba^{-4}b$ relation). However, there is no simplicial map $\phi : \mathcal{C}(P) \to \mathcal{C}(P)$ with $\phi(0) = 1$. However this changes when we take the 2-fold cover of the split presentation complex $\mathcal{C}(P)$ to give us split presentation

$$P' = \langle\ \{0,1,2,3\} \mid \{a\} \mid \{b\} \mid a \mapsto (01)(23),\ b \mapsto (03)(12) \mid$$

$$|\{\{a^{10}, a^2ba^4b\}, \{a^2ba^{-4}b\}, \{a^{10}, a^2ba^4b\}, \{a^2ba^{-4}b\}\}\rangle.$$

We have cover $\epsilon : \mathcal{C}(P') \to \mathcal{C}(P)$ by $\epsilon : i \mapsto i \pmod 2$. Where the cover $\zeta : \Gamma \to \mathcal{C}(P)$ factors through ϵ by the map $\mu : \Gamma \to \mathcal{C}(P')$ where $\mu(v_{n,k}) = 2n' + k'$ where n' and k' are n and k reduced modulo 2. Where μ maps edges similarly to ζ. The $\mathcal{C}(P')$ has two interesting automorphisms, $\widehat{\sigma} : \mathcal{C}(P') \to \mathcal{C}(P')$ which swaps $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$ but preserves edge labels. The less trivial map $\widehat{\tau} : \mathcal{C}(P') \to \mathcal{C}(P')$ which swaps $0 \leftrightarrow 1$ and $2 \leftrightarrow 3$ and preserves labels of edges between 0 and 1 ($\widehat{\tau} : a(0,1) \mapsto a(1,0)$) though reverse labels of 2 and 3 ($\widehat{\tau} : a(2,3) \mapsto a^{-1}(3,2)$). Then one can observe $\widehat{\tau}$'s action on the 2-cells, both a^{10} 2-cells stay fixed but between 2 and 3 the direction is reversed. We obtain a slightly less trivial action on the other 2-cells:

$$\begin{array}{c} a(0,1)a(1,0)b(0,3)(a(3,2)a(2,3))^2b(3,0) \\ a(1,0)a(0,1)b(1,2)(a^{-1}(2,3)a^{-1}(3,2))^2b(2,1) \\ a(2,3)a(3,2)b(2,1)(a(1,0)a(0,1))^2b(1,2) \\ a(3,2)a(2,3)b(3,0)(a^{-1}(0,1)a^{-1}(1,0))b(0,3) \end{array}$$

$$\mapsto \begin{array}{c} a(1,0)a(0,1)b(1,2)(a^{-1}(2,3)a^{-1}(3,2))^2b(2,1) \\ a(0,1)a(1,0)b(0,3)(a(3,2)a(2,3))^2b(3,0) \\ a^{-1}(3,2)a^{-1}(2,3)b(3,0)(a(0,1)a(1,0))^2b(0,3) \\ a^{-1}(2,3)a^{-1}(3,2)b(2,1)(a^{-1}(1,0)a^{-1}(0,1))^2b(1,2) \end{array}$$

where $a^{-1}(3,2)\ a^{-1}(2,3)\ b(3,0)\ (a(0,1)a(1,0))^2\ b(0,3)$ and $a^{-1}(2,3)\ a^{-1}(3,2)\ b(2,1)$ $(a^{-1}(1,0)a^{-1}(0,1))^2\ b(1,2)$ are the 2-cells $a(3,2)\ a(2,3)\ b(3,0)\ (a^{-1}(0,1)a^{-1}(1,0))^2$ $b(0,3)$ and $a(2,3)\ a(3,2)\ b(2,1)\ (a(1,0)a(0,1))^2\ b(1,2)$ ran in reverse. Using Proposition 2.3.15 we obtain that Γ is vertex transitive. The suggestive notation being correct, $\widehat{\sigma}, \widehat{\tau} \in \mathrm{Aut}(\mathcal{C}(P'))$ lifting to $\sigma, \tau \in \mathrm{Aut}(\Gamma)$.

2.6 Conclusion

In this thesis we showed that every vertex transitive graph admits a split presentation, but we were not able to limit the number of vertex classes required. This suggests

Problem 2.6.1. *Can every connected vertex transitive graph other than K_1 and K_2 be represented as a split graph so that each vertex class V_x for $x \in X$ contains at least two vertices?*

From Proposition 2.3.16 and the result of Diestel-Leader [9] saying there exists a vertex-transitive graph not quasi-isometric to any Cayley graph. We know that there exists a vertex transitive graph that can't be represented using a presentation P with finite X. However this raises the following question.

Problem 2.6.2. *Does a vertex transtive graph Γ have a split presentation with finite X if and only if Γ is quasi-isometric to a Cayley graph?*

Define the *Cayleyness* of a (vertex transitive) graph Γ as the minimum number of vertex classes in any split presentation of Γ. Thus Γ is a Cayley graph if and only if it has Cayleyness 1.

Problem 2.6.3. *Is there a vertex transitive graph of Cayleyness at least n for every $n \in \mathbb{N}$?*

A simple observation gives that the Cayleyness of a graph Γ divides $|V(\Gamma)|$. Therefore this question could be answered if for every prime $p \in \mathbb{N}$, there is a non-Cayley graph on p^n vertices. Then its Cayleyness has to divide p^n but could not be 1, as it is not Cayley.

We say that a (vertex transitive) graph Γ is *finitely presented* if it has a split presentation with finitely many vertex classes and finitely many relators. Is this equivalent to $\pi_1(\Gamma)$ being generated by walks of bounded length?

When comparing knowledge about groups and vertex transitive graphs, a lot more is known about the former. For example, it is an easy exercise in group theory to show there are finitely many finite extensions of finitely presented groups. When it comes to vertex transitive graphs, the analogous question is still open and has been extensively studied in the literature by Gardiner and Praeger, and Neganova and Trofimov amongst other authors [18, 43, 52]. Using split presentations we might be able to develop an analogous proof.

2.7 Appendix

In this appendix we generalise Theorem 2.3.8 of Godsil and Royle [19] to infinite graphs as follows.

Theorem 2.7.1. *Let Γ be a connected infinite vertex transitive graph which is locally finite. Then Γ has a perfect matching.*

Formally a matching $M \subset E(\Gamma)$, is a subgraph of Γ, which we say misses $x \in V(\Gamma)$ if x is not in the induced subgraph of M. Let d be the distance function on vertices, where the distance between any two vertices is the minimum size path containing both vertices, it is infinite if no such path exists. For $x \in V(\Gamma)$ and $n \in \mathbb{N}$ define $B(x,n) = \{y \in V(\Gamma) | d(x,y) \leq n\}$ to be the set of vertices distance at most n from x, we will call this the ball of radius n around x.

To this end we define a metric on the space of matchings for an infinite graph Γ.

Definition 2.7.2. Let Γ be an infinite graph which is locally finite and $x \in V(\Gamma)$ with matching $M \subset E(\Gamma)$. Define the *miss cardinality* of M at x to be the sequence $(m_n)_{n\in\mathbb{N}}$. Where m_n is the number of missed vertices of M in $B(x,n)$. For two matchings M_1 and M_2 with miss cardinality at x to be (a_n) and (b_n) respectively, we say $M_1 < M_2$ if there exists $N \in \mathbb{N}$ such that $a_n = b_n$ for $n < N$ with $a_N > b_N$.

We extend the miss cardinality to a partial order by setting $M < M$ for any matching M. A chain is a sequence of matchings $(M_i)_{i\in\mathbb{N}}$ such that $M_i < M_{i+1}$ for all $i \in \mathbb{N}$. We remind the reader of Zorn's lemma.

Theorem 2.7.3. *(Zorn's lemma, Kuratowski [27]) Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.*

We use this to prove that a maximum matching exists with respect to this partial order.

Lemma 2.7.4. *For a locally finite connected graph Γ with vertex $x \in V(\Gamma)$ there exists a maximal matching with respect to the miss cardinality.*

Proof. Suppose we have a chain of matchings M_i, $i \in \mathbb{N}$ in Γ with respect to the miss cardinality at x. Let M_i have miss cardinality $(m_n^i)_{n\in\mathbb{N}}$, so we know $m_n^{i+1} \leq m_n^i$ as this is a chain.

We will inductively define subsequences $\mathcal{I}_{n+1} \subset \mathcal{I}_n \subset \mathbb{N}$. Let $\mathcal{I}_1 = \mathbb{N}$. Suppose we have defined $\mathcal{I}_n$, as the induced graph on $B(x, n)$ is a finite there are finitely many matchings on it. Therefore infinitely many $i, j \in \mathcal{I}_n$ have $M_i \cap B(x, n) = M_j \cap B(x, n) =: \mathcal{M}_n$, let $\mathcal{I}_{n+1}$ be such an infinite subset.

As $\mathcal{I}_{n+1}$ is infinite with $M_i \cap B(x, n) = M_{i'} \cap B(x, n)$ for $i, i' \in \mathcal{I}_n$ we obtain that $m_n^i \leq m_n^j$ for $i \in \mathcal{I}_{n+1}$ and $j \in \mathbb{N}$. Let $M = \bigcup_{i\in\mathbb{N}} \mathcal{M}_i$, which is a matching as $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ for each $i \in \mathbb{N}$ with $\mathcal{M}_i$ being a matching. Then if $(m_n)_{n\in\mathbb{N}}$ is the miss cardinality of M then $m_n = m_n^i \leq m_n^j$ for $i \in \mathcal{I}_n$ and $j \in \mathbb{N}$ by the definition of M. If $m_n = m_n^i$ for all $n \in \mathbb{N}$, then $m_n^i = m_n^{i+j}$ for all $j \in \mathbb{N}$, however as this is a chain $M_i < M_{i+j}$ which can only happen if $M_i = M_{i+j} = M$ still giving $M_i < M$. Therefore M is an upper bound for the chain $(M_i)_{i\in\mathbb{N}}$.

By Zorn's lemma we obtain the existence of a maximum matching. □

For a matching M a path, ray, or cycle P is alternating with respect to M if every other edge contained in P is also contained in M. The number of edges in a path is called its length. We remind the reader of the symmetric difference $S \oplus T = S \cup T \backslash S \cap T$.

Lemma 2.7.5. *Let Γ be an infinite graph which is locally finite with vertex $x \in V(\Gamma)$. Let M_1 and M_2 be two maximum matchings with respect to the miss cardinality. Then the symmetric difference $M_1 \oplus M_2$ can only contain even length cycles, bi-infinite rays, and even length paths P. Furthermore, for any such path P if it has end vertices $u, v \in V(\Gamma)$ then $d(u, x) = d(v, x)$. Every component of $M_1 \oplus M_2$ is alternating with respect to M_1 or M_2.*

Proof. As M_1 and M_2 are matchings, vertices in $M_1 \oplus M_2$ are degree 0,1 or 2. So $M_1 \oplus M_2$ contains paths, cycles, one way infinite rays and bi-infinite rays, which are alternating with respect to M_1 and M_2. As the components are alternating with respect to M_1 all cycles must be of even length.

Suppose $M_1 \oplus M_2$ contains a one way infinite ray R and let $y \in V(\Gamma)$ be its end point. Without loss of generality M_1 misses y, however $M_1 \oplus R$ does not miss y moreover it does not miss any vertex M_1 does not miss. This contradicts the maximality of M_1 as a matching. So $M_1 \oplus M_2$ doesn't contain any one way infinite rays.

Let P be a path in $M_1 \oplus M_2$. If P has odd length then the end vertices $u, v \in V(\Gamma)$ are without loss of generality missed by M_1. Therefore $M_1 \oplus P$ doesn't miss any vertex M_1 missed but in addition doesn't miss u and v which contradicts maximality of M_1. Therefore assume P has even length.

Suppose P a path in $M_1 \oplus M_2$ connects its end vertices $u, v \in V(\Gamma)$. Suppose that $N := d(u, x) < d(v, x)$. Without loss of generality let M_1 miss u. Let M_1 and $M_1 \oplus P$ have miss cardinality with respect to x, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ respectively. Then for $n < N$ $a_n = b_n$ however in $B(x, N)$ M_1 misses u whereas $M_1 \oplus P$ doesn't not, therefore $a_N > b_N$. This contradicts the maximality of M_1, therefore $d(u, x) = d(v, x)$. □

Lemma 2.7.6. *Let Γ be an infinite connected graph which is locally finite with vertex $x \in V(\Gamma)$. Let $u, v \in V(\Gamma)$ be such that no maximum matching with respect to the miss cardinality at x misses both of them. Suppose M_u and M_v are maximum matchings that miss u and v respectively. Then there is a path of even length in $M_u \oplus M_v$ with end vertices u and v.*

Proof. In $M_u \oplus M_v$, u and v have degree 1. So u and v are the end points of paths in $M_u \oplus M_v$. If they are the end points of the same path we are done. So suppose not, and let P be the path with end points u and $y \neq v$ which by Lemma 2.7.5 $d(u, x) = d(y, x)$. Then $M_v \oplus P$ is a matching that misses u and v, moreover as $d(u, x) = d(y, x)$ it has the same miss cardinality as M_v. However this contradicts the assumption that no maximum matching misses u and v. So the path must connect u and v. □

We call a vertex $v \in V(\Gamma)$ critical if every maximum matching covers it.

Lemma 2.7.7. *Let Γ be an infinite connected graph which is locally finite with vertex $x \in V(\Gamma)$. Let $u, v \in V(\Gamma)$ be distinct and P a path with end points u and v. If no vertex in $V(P)\backslash\{u, v\}$ is critical, then no maximum mathching misses both u and v.*

Proof. We apply induction to the length of the path P. If P is simply an edge, then if a matching missed u and v the addition of this edge would increase the size of the matching.

Suppose $|P| \geq 2$ and let $y \in V(P)\backslash\{u, v\}$. As y is not critical let M_y be a matching missing y. However by induction we know no matching misses both u and y as well as no matching misses v and y. Suppose N was a matching missing both u and v. By Lemma 2.7.6, we know that $N \oplus M_y$ contains a path with end vertices u and y, however it also contains a path with end vertices v and y. However this contradicts $u \neq v$, so no such mathcing N exists. $\square$

Note that for vertex transitive graphs either all vertices are critical or none are, therefore using Lemma 2.7.7 we can deduce the following.

Corollary 2.7.8. *A connected vertex transitive graphs which is locally finite has a maximal matching missing at most one vertex.*

Next we want to improve this to show a perfect matching exists in infinite vertex transitive graphs.

Lemma 2.7.9. *Let Γ be an infinite connected graph which is locally finite with $x \in V(\Gamma)$. If there exists a sequence of matchings $(M_i)_{i \in \mathbb{N}}$ of Γ such that M_n misses no vertex in $B(x, n)$, then Γ has a matching missing no vertex.*

Proof. We inductively build subsequences $(M_i)_{i \in \mathcal{I}_n}$ where $\mathcal{I}_{n+1} \subset \mathcal{I}_n \subset \mathbb{N}$ are infinite subsets. Let $\mathcal{I}_1 = \mathbb{N}$. For $n \in \mathbb{N}$ suppose we have constructed $\mathcal{I}_n$. We know that there are finitely many matchings in $B(x, n)$. Therefore for a sequence of matchings $(M_i)_{i \in \mathcal{I}_n}$ we know for infinitely many $i \in \mathcal{I}_n$ the matchings $M_i \cap B(x, n)$ are equal. Therefore construct infinite subsequence $(M_i)_{i \in \mathcal{I}_{n+1}}$ where $M_i \cap B(x, n) = M_j \cap B(x, n) = \mathcal{M}_n$ for all $i, j \in \mathcal{I}_{n+1}$. Note as only finitely many M_i can miss a vertex in $B(x, n)$, $\mathcal{M}_n$ does not miss any vertex in $B(x, n)$.

Let $M = \cup_{n \in \mathbb{N}} \mathcal{M}_n$ then as $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ we have that M is a matching in Γ. Moreover let $y \in V(\Gamma)$, as Γ is connected, the distance between x and y is a finite, n. However we know $\mathcal{M}_n \subset M$ doesn't miss y, therefore neither does M. So M misses no vertex. $\square$

This solves the case were we have a matching that misses only finitely many vertices.

Corollary 2.7.10. *If Γ is an infinite vertex transitive graph which is locally finite with a matching M missing only finitely many vertices, then Γ has a matching that misses no vertices.*

Proof. Let M be a matching that misses only finitely many vertices. Let $S \subset V(\Gamma)$ be the set of missed vertices. Pick any vertex $x \in S$. As S is finite let $m = \min_{y \in S} d(x, y)$.

As Γ is an infinite vertex transitive graph there exists automorphisms ϕ_n that maps x to a vertex y such that $d(x, y) \geq n+m$. Define a sequence of matching $M_n = \phi_n(M)$. Note by the triangle inequality M_n misses no vertex in $B(x, n)$. Therefore by Lemma 2.7.9 we obtain a matching that misses no vertex in Γ. $\square$

By Corollary 2.7.8 and 2.7.10 we deduce a proof of Theorem 2.7.1.

Chapter 3

2-Groups, Representations and Characters

The story about how 2-groups where originally formed is very long and spans over multiple decades. A good review of this story and a great introductory paper on the area is presented by Baez and Lauda [1], who made the area accessible to novices. Group theory has been a highly influential area of mathematics, however some may argue that a group is only the tip of the algebraic iceberg. Fundamentally group theory is the study of symmetries, as highlighted by Cayley. A 2-group hopes to capture the idea of symmetries between symmetries, allowing for the group laws to be weakened up to natural transformation. We give a whistle stop tour of this area in the first section, so the reader has all the relevant definition at hand for the later sections.

The concept of a 2-representation of a 2-group was introduced and studied by many authors, for this we mainly used Barrett and Mackaay [3]. In this work, 2-representations where taken into $2-\mathrm{Vect}_{\mathbb{K}}$, which was introduced by Kapranov and Voevodsky [26]. Ganter and Kapranov [16] studied 2-representations of finite groups, who introduced the 2-character. simultaneously to Barrett [4]. In the case of a finite group Osorno [46] developed explicit formulas for the 2-characters.

We review the Maclane strictification, and cover some work in progress of Hristova and the author. Maclane strictification says a generic 2-group is equivalent to a strict 2-group. The first major result of this chapter is a new correspondence between a skeletal 2-group and a crossed module.

Theorem 3.2.3 A skeletal 2-group given by (G, H, α) is equivalent to crossed module $\mathcal{K} = (A \overset{\partial}{\to} B)$ given by:

- group $B = G \times \mathrm{Mor}_{Set}(G, H)/H$ where $(X, [\theta^1]) \otimes (Y, [\theta^2]) = (XY, [\theta])$ with

$$\theta(z) = \alpha(X, Y, z) + {}^X\theta^2(z) + \theta^1(Yz),$$

- group $A = \mathrm{Mor}_{Set}(G, H)/H \times H$ where $([\theta_1], a) \otimes ([\theta_2], b) = ([\theta_1 + \theta_2], a + b)$,
- map $\partial : A \to B$ where $\partial([\Phi], h) = (1_G, [\Phi])$, and
- group action $B \hookrightarrow A$ given by $(X, [\theta]) : ([\Phi], h) \mapsto ([{}^X\Phi(X^{-1}-)], {}^Xh)$.

Note here that if G and H are finite our crossed module uses finite groups also. Whereas all the current known ways to do this all go via infinite free groups. We remind the reader of this technique after the proof of Theorem 3.2.3 and relate it to our work in Corollary 3.2.5.

The next section is comprised of the results of Rumynin and the author in [48]. However in this section we have choosen to prove the core theorems in a new way only using the knowledge of 2-groups, without needing module categories. To get the perspective of module categories we refer the reader to the paper [48]. In this section we look at representations of 2-groups into 2-vector space, in the sense of Kapranov and Voevodsky [26]. We expand on the work of Gunnells, Rose and Rumynin [21] to relate the space of 2-representations in this sense to a certain Burnside ring. To any representation Ganter and Kapranov [16] showed us, using 2-traces, how to associate a 2-character. This brings us to our first major result where we associate this 2-character to a mark homomorphism of the Burnside ring.

Theorem 3.3.5 (Rumynin Wendland [48]) Let $\mathcal{K} = (A \xrightarrow{\partial} B)$ be a crossed module, P be the subgroup of $\pi_1(\mathcal{K})$ generated by $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$. Let $\alpha := \mathfrak{X}(\mathbf{b}, \mathbf{a}, h)$ considered as a group homomorphism $2\text{-Rep}^1(\mathcal{K}_P) \to \mathbb{K}^\times$. If the order of $\pi_1(\mathcal{K})$ is finite and invertible in the field $\mathbb{K}$, then

$$\mathfrak{X}(\mathbf{b}, \mathbf{a}, h) = f_P^\alpha.$$

Orsorno [46] gives explicit formulas for this 2-character in terms of the cohomological data of the 2-representation. We take this lead to do the same for our mark homomorphisms. To this end we first review work done in the author's masters thesis, which gives an explicit version of the Shapiro Isomorphism [44]. We then work with a special class of crossed modules, of the form $\mathcal{K} = (1 \xrightarrow{\partial} G)$ with G a finite group. In this setting we derive a formula for the 2-character..

Theorem 3.3.8 (Rumynin Wendland [48]) Let $\mathbf{a}, \mathbf{b} \in B$ be commuting elements, Θ a degree one 2-representation of B, $\mu \in Z^2(B, \mathbb{K}^\times)$ a cocycle such that $[\mu] = \{\Theta\}$.

Then

$$\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta,B\rangle)=\mu(\mathbf{b},\mathbf{a}^{-1})\mu(\mathbf{a}^{-1},\mathbf{b})^{-1}.$$

Which we then use to rederive the formula originally given by Orsorno.

Theorem 3.3.11 ([45, Theorem 1]) Let Θ be a 2-representation of B that corresponds to a B-set X and a cohomology class $[\theta]$ for some cochain $\theta\in Z^2(B,(\mathbb{K}^\times)^X)$. Then

$$\mathfrak{X}_\Theta(\mathbf{b},\mathbf{a})=\sum_{x\in X,\ x=\mathbf{a}\cdot x=\mathbf{b}\cdot x}\frac{\theta^x(\mathbf{b},\mathbf{a}^{-1})}{\theta^x(\mathbf{a}^{-1},\mathbf{b})}=\sum_{x\in X,\ x=\mathbf{a}\cdot x=\mathbf{b}\cdot x}\frac{\theta^x(\mathbf{b},\mathbf{a}^{-1})\theta^x(\mathbf{a},\mathbf{b}\mathbf{a}^{-1})}{\theta^x(\mathbf{a},\mathbf{a}^{-1})\theta^x(1,1)}$$

for any commuting $\mathbf{a},\mathbf{b}\in B$.

The chapter is organised in the following manner. In the first section we review the definitions of a category, bicategory, group cohomology, crossed module and 2-representation. We show how a skeletal 2-group can be related to group cohomology data, as well as show how a strict 2-group is equivalent to one coming from crossed module.

The second section deals with the Maclane Strictification. This is a theorem that shows how all 2-groups are equivalent to a strict 2-group including skeletal ones. To this end we prove Theorem 3.2.3 and relate this to what is already known in the field in Corollary 3.2.5.

The third section mainly deals with the work of Rumynin and the author [48]. We first build the correspondence between the space of 2-representations and the Burnside ring. Then we show that a 2-character is a specific mark homomorphism of the Burnside ring in Theorem 3.3.5. We then review the work done in the author's masters thesis, and prove Theorem 3.3.6 which is an explicit formula for the Shapiro isomorphism. We use this to prove Theorem 3.3.8 and Theorem 3.3.11 which are explicit formulas for 2-characters in terms of the cohomological data which gives the 2-representation. In the last section we discuss future possible work in this field.

3.1 Preliminaries

Throughout this section we recall a lot of definitions, we assume no knowledge of the area for the reader. It is advised an informed reader skip sections which they are likely to be knowledgable about.

3.1.1 Categories

We first review some basic concepts from category theory.

Definition 3.1.1. A *category* $\mathcal{C}$ shall consist of the following data

- a collection of objects $\mathrm{Ob}(\mathcal{C})$,
- for each two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a collection of morphisms $\mathrm{Mor}(X, Y)$,
- a composition map $\circ : \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$ for every $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, and
- an identity morphism $1_X \in \mathrm{Mor}(X, X)$ for each $X \in \mathrm{Ob}(\mathcal{C})$,

such that composition is associative $c \circ (b \circ a) = (c \circ b) \circ a$ and identity morphisms are unital $1_Y \circ a = a = a \circ 1_X$ for $a \in \mathrm{Mor}(X, Y)$, $b \in \mathrm{Mor}(Y, Z)$ and $c \in \mathrm{Mor}(Z, W)$.

Example 3.1.2. There are many natural examples of categories, we detail a couple that will be used later in this section.

- The category Set with objects being sets, morphisms just maps of sets and composition being composition of set maps.
- The category Grp with objects being groups, morphisms as homomorphisms and composition being composition of homomorphism. This has a subcategory AbGrp of abelian groups.
- Given a group G we can define a category with $\mathrm{Ob}(G) = \{\star\}$ and $\mathrm{Mor}(\star, \star) = G$ with composition being the group operation.
- Given a field $\mathbb{K}$ we can define a category $\mathrm{Vect}_{\mathbb{K}}$ whose objects are finite dimensional $\mathbb{K}$ vector spaces and morphisms are linear maps.

When multiple categories are being considered we specify the category as a subscript. For example if we want ϕ to be a set map between two groups G and H we may let $\phi \in \mathrm{Mor}_{\mathrm{Set}}(G, H)$ whereas if we let $\phi \in \mathrm{Mor}_{\mathrm{Grp}}(G, H)$ it would be a homomorphism of groups.

Given a category $\mathcal{C}$, the opposite category $\mathcal{C}^{op}$ has $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$ where $\mathrm{Mor}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Mor}_{\mathcal{C}}(Y, X)$. For $f \in \mathrm{Mor}_{\mathcal{C}^{op}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}^{op}}(Y, Z)$ we define the composition $g \circ_{\mathcal{C}^{op}} f := f \circ_{\mathcal{C}} g \in \mathrm{Mor}_{\mathcal{C}}(Z, X) = \mathrm{Mor}_{\mathcal{C}^{op}}(X, Z)$.

We say $a \in \mathrm{Mor}(X, Y)$ is an *isomorphism* if there exists $a^{-1} \in \mathrm{Mor}(Y, X)$ such that $a \circ a^{-1} = 1_Y$ and $a^{-1} \circ a = 1_X$. We say $\mathcal{C}$ is a *skeletal category* if all isomorphisms are automorphisms.

Definition 3.1.3. A *functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ contains the following data

- a map $\mathcal{F} : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, and
- a family of maps $\mathcal{F} : \mathrm{Mor}(X, Y) \to \mathrm{Mor}(\mathcal{F}(X), \mathcal{F}(Y))$,

such that $\mathcal{F}$ preserves composition $\mathcal{F}(b \circ a) = \mathcal{F}(b) \circ \mathcal{F}(a)$ and identity $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$. A *contravariant functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}^{op}$ is just a functor however will sometimes be refered to by $\mathcal{F} : \mathcal{C} \to \mathcal{D}$.

There are many natural examples of functors, we will come on to define some of them. Define the *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ where all the maps are just the identity. Here we exhibit one example that will be useful motivation for later.

Example 3.1.4. Given a representation $\rho : G \to \mathrm{Gl}(V)$ of a group G into a $\mathbb{K}$ vector space V, one can interpret ρ as a functor $\tilde{\rho} : G \to \mathrm{Vect}_{\mathbb{K}}$. This is where $\tilde{\rho}(\star) = V$ and $\tilde{\rho}(g) = \rho(g) \in \mathrm{Gl}(V) = \mathrm{Mor}(V, V)$.

One can compose functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ where the maps of $\mathcal{F} \circ \mathcal{G}$ are the composition of the maps in $\mathcal{F}$ and $\mathcal{G}$.

Definition 3.1.5. A *natural transformation* $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ between two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is a collection of maps $\Phi_X \in \mathrm{Mor}(\mathcal{F}(X), \mathcal{G}(X))$, called its components, such that for each $a \in Hom(X, Y)$ we have that $\Phi_Y \circ \mathcal{F}(a) = \mathcal{G}(a) \circ \Phi_X$, equivalently the following diagram commutes.

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\mathcal{F}(a)"] \arrow[d, "\Phi_X"] & \mathcal{F}(Y) \arrow[d, "\Phi_Y"] \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(a)"] & \mathcal{G}(Y)
\end{tikzcd}$$

One can compose natural transformations $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ with $\Psi : \mathcal{G} \Rightarrow \mathcal{H}$ by simply composing their components for each object $X \in \mathcal{C}$. This will be called vertical composition. It is also possible to compose natural transformations $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ with $\Psi : \mathcal{H} \Rightarrow \mathcal{K}$, where $\mathcal{F}, \mathcal{G} : \mathcal{D} \to \mathcal{E}$ and $\mathcal{H}, \mathcal{K} : \mathcal{C} \to \mathcal{D}$, to get natural transformation $\Phi \otimes \Psi : \mathcal{F}\mathcal{H} \Rightarrow \mathcal{G}\mathcal{K}$. This is called the *Godement product* or horizontal composition and is given by the following composition which can be defined in two equivalent ways (Φ is a natural transformation and the following diagram is the coherence of Φ for the morphisms $\Psi(X) : \mathcal{H}(X) \to \mathcal{K}(X)$, therefore commutes).

$$\begin{tikzcd}
\mathcal{F}\mathcal{H}(X) \arrow[r, "\mathcal{F}(\Psi_X)"] \arrow[d, "\Phi_{\mathcal{H}(X)}"'] \arrow[dr, "(\Phi\otimes\Psi)_X"] & \mathcal{F}(\mathcal{K}(X)) \arrow[d, "\Phi_{\mathcal{K}(X)}"] \\
\mathcal{G}\mathcal{H}(X) \arrow[r, "\mathcal{G}(\Psi_X)"'] & \mathcal{G}\mathcal{K}(X)
\end{tikzcd}$$

A natural transformation is a *natural isomorphism* if each of its components are isomorphisms. It will be convenient in future to say when two categories convey the same information, for this we introduce the notion of equivalent categories.

Definition 3.1.6. We say a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* if there exists a functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$, and natural isomorphisms $\Phi : \mathcal{G} \circ \mathcal{F} \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\Psi : \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathcal{D}}$. We then say the categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent.*

Finding the functor $\mathcal{G}$ and natural transformations Φ and Ψ can be cumbersome so the following is useful.

Proposition 3.1.7. *[30, Theorem IV.4.1] A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if*

- *for every $Y \in Ob(\mathcal{D})$ there exists $X \in \mathcal{C}$ with isomorphisms $a \in Mor(\mathcal{F}(X), Y)$ (essentially surjective), and*
- *each $\mathcal{F} : Mor(X, Y) \to Mor(\mathcal{F}(X), \mathcal{F}(Y))$ is a bijection (full and faithfull).*

An equivalence of categories can help understanding larger categories by looking at simplier categories which they are equivalent to.

Example 3.1.8. Let $\mathcal{C}$ be a category. We define an equivalence $\sim$ on $\mathrm{Ob}(\mathcal{C})$ where $X \sim Y$ if there exists isomorphism $a \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. Note that if $X \sim Z$ and $Y \sim W$ using $a \in \mathrm{Mor}(X, Z)$ and $b \in \mathrm{Mor}(Y, W)$ respectively, then there exists isomorphism $f \in \mathrm{Mor}_{\mathrm{Set}}(\mathrm{Mor}_{\mathcal{C}}(Z, W), \mathrm{Mor}_{\mathcal{C}}(X, Y))$ where $f(c) = b \circ c \circ a$ with inverse $f^{-1}(c) = b^{-1} \circ c \circ a^{-1}$. We define the *skeleton* of $\mathcal{C}$ called $\mathcal{C}^{\mathrm{Skel}}$ as a sub-category of $\mathcal{C}$. For each class of equivalent objects choose a representative and set $\mathrm{Ob}(\mathcal{C}^{\mathrm{Skel}}) \subset \mathrm{Ob}(\mathcal{C})$ to be this collection. Let $\mathrm{Mor}_{\mathcal{C}^{\mathrm{Skel}}}(X, Y) := \mathrm{Mor}_{\mathcal{C}}(X, Y)$.

The equivalence of categories is then given by inclusion $\mathcal{F} : \mathcal{C}^{\mathrm{Skel}} \to \mathcal{C}$. Which is essentially surjective as all objects $Y \in \mathrm{Ob}(\mathcal{C})$ have an isomorphic object $X \in \mathcal{C}^{\mathrm{Skel}}$ and full and faithful by definition. Up to isomorphism, the skeleton category is unique and the choice of objects does not matter.

For $\mathrm{Vect}_{\mathbb{K}}$ we can make $\mathrm{Vect}_{\mathbb{K}}^{\mathrm{Skel}}$ have objects $\mathbb{K}^n \in \mathrm{Ob}(\mathrm{Vect}_{\mathbb{K}}^{\mathrm{Skel}})$ where $n \in \mathbb{N}$ and $\mathrm{Mor}_{\mathrm{Vect}_{\mathbb{K}}^{\mathrm{Skel}}}(\mathbb{K}^n, \mathbb{K}^m)$ are $n \times m$ matrices.

Later on it will be useful to apply more structure to our categories. For this we define the notion of a product category. For categories $\mathcal{C}_i$ with $i \in \{1, \ldots, n\}$ we can define the *product category* $\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$ with $\mathrm{Ob}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n) = \mathrm{Ob}(\mathcal{C}_1) \times \mathrm{Ob}(\mathcal{C}_2) \times \ldots \times \mathrm{Ob}(\mathcal{C}_n)$ and morphisms $\mathrm{Mor}((X_1, X_2, \ldots, X_n), (Y_1, Y_2, \ldots, Y_n)) =$

$\mathrm{Mor}_{\mathcal{C}_1}(X_1, Y_1) \times \mathrm{Mor}_{\mathcal{C}_2}(X_2, Y_2) \times \ldots \times \mathrm{Mor}_{\mathcal{C}_n}(X_n, Y_n)$ with composition defined component wise and identity morphism $1_{(X_1,X_2,\ldots,X_n)} = (1_{X_1}, 1_{X_2}, \ldots, 1_{X_n})$.

Here we use the notation $-$ to allow for an argument, for example one could say we have functor $X \times - : \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ for some $X \in \mathrm{Ob}(\mathcal{C})$. By this we mean the functor taking $(X \times -) : Y \mapsto (X, Y)$ for $Y \in \mathrm{Ob}(\mathcal{D})$ and $(X \times -) : a \mapsto (1_X, a)$ for $a \in Hom_{\mathcal{D}}(Y, Z)$ where $(1_X, a) \in \mathrm{Mor}_{\mathcal{C}\times\mathcal{D}}((X, Y), (X, Z))$. In Example 3.1.8 f and f^{-1} would be $b \circ - \circ a$ and $b^{-1} \circ - \circ a^{-1}$ respectively.

Definition 3.1.9. A *monoidal category* is a category $\mathcal{C}$ with the additional data

- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ where we will write $\otimes(a, b) =: a \otimes b$,
- a unit object $1_{\mathcal{C}} \in \mathcal{C}$,
- a natural isomorphisms between the two functors

$$(- \otimes -) \otimes -, - \otimes (- \otimes -) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

  $\mathrm{Ass} : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$ with components $\mathrm{Ass}(X, Y, Z) \in \mathrm{Mor}((X \otimes Y) \otimes Z, X \otimes (Y \otimes Z))$, therefore for maps $f \in \mathrm{Mor}(X, A)$, $g \in \mathrm{Mor}(Y, B)$ and $h \in \mathrm{Mor}(Z, C)$ we have

$$\mathrm{Ass}(A, B, C) \circ ((f \otimes g) \otimes h) = (f \otimes (g \otimes h)) \circ \mathrm{Ass}(X, Y, Z) \in \mathrm{Mor}((X \otimes Y) \otimes Z, A \otimes (B \otimes C)),$$

- a natural isomorphism between the two functors

$$1_{\mathcal{C}} \otimes -, - : \mathcal{C} \to \mathcal{C}$$

  $\lambda : 1_{\mathcal{C}} \otimes - \Rightarrow -$ with component $\lambda(X) \in \mathrm{Mor}(1_{\mathcal{C}} \otimes X, X)$ called the left unitor, and

- a natural isomorphism between the two functors

$$- \otimes 1_{\mathcal{C}}, - : \mathcal{C} \to \mathcal{C}$$

  $\rho : - \otimes 1_{\mathcal{C}} \Rightarrow -$ with component $\rho(X) \in \mathrm{Mor}(X \otimes 1_{\mathcal{C}}, X)$ called the right unitor,

such that the data above abide by the following axioms

- The triangle axiom giving

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "\mathrm{Ass}_{X,1,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

commutes.

- The pentagon axiom giving

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z) \otimes W \arrow[dl, "\mathrm{Ass}(X,Y,Z) \otimes 1_W"'] \arrow[dr, "\mathrm{Ass}(X \otimes Y, Z, W)"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "\mathrm{Ass}(X, Y \otimes Z, W)"] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "\mathrm{Ass}(X,Y,Z \otimes W)"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "1_X \otimes \mathrm{Ass}(Y,Z,W)"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

commutes.

Anything with a tensor product structure naturally becomes a monoidal category. For example $\mathrm{Vect}_{\mathbb{K}}$ is a monoidal category with respect to the tensor product. Here we have non-trivial associator Ass being the formal isomorphisms between $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$, similar for the unitors.

We call a monoidal category *strict* if Ass, λ and ρ are identities. For a skeletal monoidal category we say it is *special* if λ and ρ are identities.

One useful motivating example is how we look at groups as a monoidal category, as we explicate below.

Example 3.1.10. Let $(G, \cdot)$ be a group. Define a category $\mathcal{C}_G$ which has $\mathrm{Ob}(\mathcal{C}_G) := G$ where

$$\mathrm{Mor}_{\mathcal{C}_G}(g,h) = \begin{cases} \{1_g\} & \text{if } g = h \\ \emptyset & \text{otherwise} \end{cases}$$

with 1_g being the identity morphism which fully describes the composition map $\circ$. We can make $\mathcal{C}_G$ a monoidal category where $\otimes : \mathcal{C}_G \times \mathcal{C}_G \to \mathcal{C}_G$ is given by the group operation so

$$\otimes : \begin{array}{ccc} g \otimes h & \mapsto & g \cdot h \\ 1_g \otimes 1_h & & 1_{g \cdot h}. \end{array}$$

This has unity object 1_G and trivial maps for Ass, λ and ρ. Therefore it obeys the triangle and pentagon axioms trivially, moreover this is a strict skeletal category.

However there are other similar ways to represent groups as a monoidal category.

Example 3.1.11. Let $(G, \cdot)$ be a group with generating set $\mathcal{S} = \mathcal{S}^{-1}$ such that $\langle \mathcal{S} \rangle = G$. Let $W^{\mathcal{S}}$ be bracketed strings of words in $\mathcal{S}$. Formally let

$$W_n^{\mathcal{S}} = \begin{cases} \{\epsilon\} & \text{if } n = 0 \\ \mathcal{S} & \text{if } n = 1 \\ \{(w_1) * (w_2) | w_1 \in W_a^{\mathcal{S}} \ w_2 \in W_b^{\mathcal{S}} \ \text{ with } a, b > 0 \ \text{ and } a + b = n\} & \text{otherwise} \end{cases}$$

where notationally brackets around a single letter get dropped. Then let $W^{\mathcal{S}} = \cup_{n=0}^{\infty} W_n^{\mathcal{S}}$. Therefore if $a, b, c \in \mathcal{S}$ then $a * (b * c)$ is distinct from $(a * b) * c$ in $W^{\mathcal{S}}$. There is an evaluation map $\phi : W^{\mathcal{S}} \to G$ where you replace $*$ by the operation $\cdot$ and evaluate as an element of G as $\mathcal{S} \subset G$. Formally define it as

$$\phi(w) = \begin{cases} 1_G & \text{if } w = \epsilon \\ s & \text{if } w = s \in \mathcal{S} \\ \phi(w_1) \cdot \phi(w_2) & \text{if } w = (w_1) * (w_2) \end{cases} .$$

Now define category $\mathcal{C}_{(G,\mathcal{S})}$ where $\mathrm{Ob}(\mathcal{C}_{(G,\mathcal{S})}) = W^{\mathcal{S}}$ and

$$\mathrm{Mor}_{\mathcal{C}_{(G,\mathcal{S})}}(w_1, w_2) = \begin{cases} \{f_{w_1,w_2}\} & \text{if } \phi(w_1) = \phi(w_2) \\ \emptyset & \text{otherwise} \end{cases} .$$

Composition is forced, so if $\phi(w_1) = \phi(w_2) = \phi(w_3)$ then $\circ : \mathrm{Mor}(w_2, w_3) \times \mathrm{Mor}(w_1, w_2) \to \mathrm{Mor}(w_1, w_3)$ is given by $f_{w_2,w_3} \circ f_{w_1,w_2} = f_{w_1,w_3}$ making $f_{w,w}$ the unit morphism in $\mathrm{Mor}(w, w)$. Note here that every morphism is an isomorphism as if $\phi(w_1) = \phi(w_2)$ then there exists f_{w_1,w_2} and f_{w_2,w_1} such that $f_{w_i,w_{i\pm1}} \circ f_{w_{i\pm1},w_i} = f_{w_i,w_i}$.

We can make $\mathcal{C}_{(G,\mathcal{S})}$ a monoidal category by defining $\otimes : \mathcal{C}_{(G,\mathcal{S})} \times \mathcal{C}_{(G,\mathcal{S})} \to \mathcal{C}_{(G,\mathcal{S})}$ as

$$\otimes : \begin{array}{ccc} w_1 \otimes w_2 & \mapsto & (w_1) * (w_2) \\ f_{w_1,w_2} \otimes f_{w_3,w_4} & & f_{(w_1)*(w_3),(w_2)*(w_4)} \end{array} .$$

Take a word w^1 such that $\phi(w^1) = 1_G$ (Note that w^1 could be ϵ however need not be). We have non-trivial associator $\mathrm{Ass}(w_1, w_2, w_3) = f_{((w_1)*(w_2))*(w_3),(w_1)*((w_2)*(w_3))}$. Potentially non-trivial unitors $\lambda(w) = f_{(w^1)*(w),(w)}$ and $\rho(w) = f_{(w)*(w^1),(w)}$. A simple check confirms the triangle and pentagon axioms hold. This is neither strict nor is it skeletal.

We alluded to these two examples being similar, let us explain in what fashion now.

Definition 3.1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. Then a *monoidal functor* $\mathcal{F}$ consists of the following data:

- a functor of categories
$$\mathcal{F} : \mathcal{C} \to \mathcal{D},$$
- an isomorphism
$$\epsilon : 1_{\mathcal{D}} \to \mathcal{F}(1_{\mathcal{C}}), \quad \text{and}$$
- a natural isomorphism between the following functors
$$\mathcal{F}(-) \otimes_{\mathcal{D}} \mathcal{F}(-), \mathcal{F}(- \otimes_{\mathcal{C}} -) : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$$
called $\mu : \mathcal{F}(-)\otimes_{\mathcal{D}}\mathcal{F}(-) \Rightarrow \mathcal{F}(-\otimes_{\mathcal{C}}-)$ with components $\mu(X, Y) \in \mathrm{Mor}(\mathcal{F}(X)\otimes_{\mathcal{D}} \mathcal{F}(Y), \mathcal{F}(X \otimes_{\mathcal{C}} Y))$.

Such that the following axioms hold:

- Associativity, which says
$$\begin{tikzcd}
(\mathcal{F}(X) \otimes_{\mathcal{D}} \mathcal{F}(Y)) \otimes_{\mathcal{D}} \mathcal{F}(Z) \arrow[r, "\mathrm{Ass}_{\mathcal{D}}(\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z))"] \arrow[d, "\mu(X,Y)\otimes_{\mathcal{D}}1_{\mathcal{F}(Z)}"] & \mathcal{F}(X) \otimes_{\mathcal{D}} (\mathcal{F}(Y) \otimes_{\mathcal{D}} \mathcal{F}(Z)) \arrow[d, "1_{\mathcal{F}(X)}\otimes_{\mathcal{D}}\mu(Y,Z)"] \\
\mathcal{F}(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} \mathcal{F}(Z) \arrow[d, "\mu(X\otimes_{\mathcal{C}}Y,Z)"] & \mathcal{F}(X) \otimes_{\mathcal{D}} \mathcal{F}(Y \otimes_{\mathcal{C}} Z) \arrow[d, "\mu(X,Y\otimes_{\mathcal{C}}Z)"] \\
\mathcal{F}((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) \arrow[r, "\mathcal{F}(\mathrm{Ass}_{\mathcal{C}}(X,Y,Z))"] & \mathcal{F}(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{tikzcd}$$
commutes.
- Unitality, which says
$$\begin{tikzcd}
1_{\mathcal{D}} \otimes_{\mathcal{D}} \mathcal{F}(X) \arrow[r, "\epsilon\otimes_{\mathcal{D}}1_{\mathcal{F}(X)}"] \arrow[d, "\lambda_{\mathcal{D}}(\mathcal{F}(X))"] & \mathcal{F}(1_{\mathcal{C}}) \otimes_{\mathcal{D}} \mathcal{F}(X) \arrow[d, "\mu(1_{\mathcal{C}},X)"] \\
\mathcal{F}(X) & \mathcal{F}(1_{\mathcal{C}} \otimes_{\mathcal{C}} X) \arrow[l, "\mathcal{F}(\lambda_{\mathcal{C}}(X))"]
\end{tikzcd}$$
and
$$\begin{tikzcd}
\mathcal{F}(X) \otimes_{\mathcal{D}} 1_{\mathcal{D}} \arrow[r, "1_{\mathcal{F}(X)}\otimes_{\mathcal{D}}\epsilon"] \arrow[d, "\rho_{\mathcal{D}}(\mathcal{F}(X))"] & \mathcal{F}(X) \otimes_{\mathcal{D}} \mathcal{F}(1_{\mathcal{C}}) \arrow[d, "\mu(X,1_{\mathcal{C}})"] \\
\mathcal{F}(X) & \mathcal{F}(X \otimes_{\mathcal{C}} 1_{\mathcal{C}}) \arrow[l, "\mathcal{F}(\rho_{\mathcal{C}}(X))"]
\end{tikzcd}$$
commutes.

Definition 3.1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ monoidal functors then a *monoidal natural transformation* is a natural transformation $\Phi : \mathcal{C} \Rightarrow \mathcal{D}$ such that

$$\begin{tikzcd}
\mathcal{F}(X) \otimes_{\mathcal{D}} \mathcal{F}(Y) \arrow[r, "\Phi(X)\otimes_{\mathcal{D}}\Phi(Y)"] \arrow[d, "\mu_{\mathcal{F}}(X,Y)"] & \mathcal{G}(X) \otimes_{\mathcal{D}} \mathcal{G}(Y) \arrow[d, "\mu_{\mathcal{G}}(X,Y)"] \\
\mathcal{F}(X \otimes_{\mathcal{C}} Y) \arrow[r, "\Phi(X\otimes_{\mathcal{C}} Y)"] & \mathcal{G}(X \otimes_{\mathcal{C}} Y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
& 1_{\mathcal{D}} \arrow[dl, "\epsilon_{\mathcal{F}}"'] \arrow[dr, "\epsilon_{\mathcal{G}}"] & \\
\mathcal{F}(1_{\mathcal{C}}) \arrow[rr, "\Phi(1_{\mathcal{C}})"] & & \mathcal{G}(1_{\mathcal{C}})
\end{tikzcd}$$

commute.

Therefore we can extend the notion of equivalence of categories to equivalence of monoidal categories by just requiring that all functors and natural transformations are monoidal.

Example 3.1.14. Let $(G, \cdot)$ be a group with generating set $\mathcal{S}$. Then we want to show $\mathcal{C}_G$ and $\mathcal{C}_{(G,\mathcal{S})}$, as introduced in examples 3.1.10 and 3.1.11 respectively, are monoidally equivalent. Pick $\sigma \in \mathrm{Mor}_{\mathsf{Set}}(G, W^{\mathcal{S}})$ such that $\phi(\sigma(g)) = g$, which exists as $\mathcal{S}$ generates G. Define a functor $\mathcal{F} : \mathcal{C}_G \to \mathcal{C}_{(G,\mathcal{S})}$ given by

$$\mathcal{F} : \begin{array}{c} g \\ 1_g \end{array} \mapsto \begin{array}{c} \sigma(g) \\ f_{\sigma(g),\sigma(g)} \end{array}.$$

Note this is a functor of categories as $\mathcal{F}(1_g) = f_{\sigma(g),\sigma(g)}$ giving that it preserves identities and observing

$$\begin{aligned}
\mathcal{F}(1_g \circ_{\mathcal{C}_G} 1_g) &= \mathcal{F}(1_g) \\
&= f_{\sigma(g),\sigma(g)} \\
&= f_{\sigma(g),\sigma(g)} \circ_{\mathcal{C}_{(G,\mathcal{S})}} f_{\sigma(g),\sigma(g)} \qquad \text{as } f_{\sigma(g),\sigma(g)} \text{ is a unit morphism} \\
&= \mathcal{F}(1_g) \circ_{\mathcal{C}_{(G,\mathcal{S})}} \mathcal{F}(1_g)
\end{aligned}$$

we get that it preserves composition also. Let $\epsilon = f_{w^1,\sigma(1_G)}$ which exists as $\phi(w^1) = 1_G = \phi(\sigma(1_G))$. Define $\mu(g,h) = f_{\sigma(g)*\sigma(h),\sigma(g\cdot h)}$, which exists as

$$\begin{aligned}
\phi(\sigma(g \cdot h)) &= g \cdot h \\
&= \phi(\sigma(g)) \cdot \phi(\sigma(h)) \\
&= \phi(\sigma(g) * \sigma(h)).
\end{aligned}$$

This gives us that $\mathcal{F}$ is a monoidal functor. Note that $\mathcal{F} : \mathrm{Mor}_{\mathcal{C}_G}(g, g) \to \mathrm{Mor}_{\mathcal{C}_{(G,\mathcal{S})}}(\sigma(g), \sigma(g))$ is a bijection so $\mathcal{F}$ is full and faithful. Moreover for all $w \in W^{\mathcal{S}}$ $\phi(w) \in G$ therefore $f_{w,\sigma(\phi(w))}$ is an isomorphism and so $\mathcal{F}$ is full, giving that $\mathcal{F}$ is a monoidal equivalence of categories.

We will use monoidal categories as a simpler way to think of a higher categories, which will be described in the next subsection. However first let us name an interesting property monoidal categories have.

Definition 3.1.15. As $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, we have the *interchange law* which states

$$\begin{aligned} (b \circ a) \otimes (d \circ c) &= \otimes (b \circ a, d \circ c) \\ &= \otimes ((b, d) \circ (a, c)) \\ &= \otimes (b, d) \circ \otimes (a, c) \\ &= (b \otimes d) \circ (a \otimes c). \end{aligned}$$

This is a direct corollary of $\otimes$ preserving composition but has some interesting applications as we will see in the further sections.

3.1.2 2-Cateogries and Bicategories

Within this section we introduce 2-categories. Here the definitions get very complex in full generality, however we only need these definitions in specific cases which we explicate at the end of this subsection.

Definition 3.1.16. A *bicategory* $\mathcal{M}$ consists of the following data

- a collection of objects $\mathcal{M}_0$,
- for each $X, Y \in \mathcal{M}_0$ a category $\mathcal{M}_1(X, Y)$ where for $A, B \in \mathrm{Ob}(\mathcal{M}_1(X, Y))$ (which we abuse notation later and say $A, B \in \mathcal{M}_1(X, Y)$) we define $\mathcal{M}_2(A, B) := \mathrm{Mor}_{\mathcal{M}_1(X,Y)}(A, B)$,
- a unit object $1_X \in \mathcal{M}_1(X, X)$ for each $X \in \mathcal{M}_0$,
- a family of functors known as composition

$$\diamond = \Diamond_{X,Y,Z} : \mathcal{M}_1(X, Y) \times \mathcal{M}_1(Y, Z) \to \mathcal{M}_1(X, Z)$$

  where if the indices are clear from context we write $\Diamond_{X,Y,Z}(A, B) =: A \diamond B$,
- between functors

$$\Diamond_{X,Y,Y}(-, 1_Y), - : \mathcal{M}_1(X, Y) \to \mathcal{M}_1(X, Y)$$

we have a family of natural isomorphisms called the right unitor

$$RUn_{X,Y} : \diamondsuit_{X,Y,Y}(-, 1_Y) \Rightarrow -,$$

- between functors

$$\diamondsuit_{X,X,Y}(1_X, -), - : \mathcal{M}_1(X,Y) \to \mathcal{M}_1(X,Y)$$

we have a family of natural isomorphisms called the left unitor

$$LUn_{X,Y} : \diamondsuit_{X,X,Y}(1_X, -) \Rightarrow -, \quad \text{and}$$

- a family of natural isomorphisms between functors $\diamondsuit_{X,Z,W} \circ (\diamondsuit_{X,Y,Z} \times 1_{\mathcal{M}_1(Z,W)})$, and $\diamondsuit_{X,Y,W} \circ (1_{\mathcal{M}_1(X,Y)} \times \diamondsuit_{Y,Z,W})$ as maps $\mathcal{M}_1(X,Y) \times \mathcal{M}_1(Y,Z) \times \mathcal{M}_1(Z,W) \to \mathcal{M}_1(X,W)$ called the associator

$$\mathrm{Ass}_{X,Y,Z,W} : \diamondsuit_{X,Z,W} \circ (\diamondsuit_{X,Y,Z} \times 1_{\mathcal{M}_1(Z,W)}) \Rightarrow \diamondsuit_{X,Y,W} \circ (1_{\mathcal{M}_1(X,Y)} \times \diamondsuit_{Y,Z,W}).$$

Such that

- the pentagon axiom holds which says

$$\begin{tikzcd}
((\mathcal{M}_1(X,Y) \diamond \mathcal{M}_1(Y,Z)) \diamond \mathcal{M}_1(Z,W)) \diamond \mathcal{M}_1(W,V) \arrow[d, Rightarrow, "\mathrm{Ass}_{X,Y,Z,W} \diamond 1_{\mathcal{M}_1(W,V)}"] \arrow[dr, Rightarrow, "\mathrm{Ass}_{X,Z,W,V}"] & \\
(\mathcal{M}_1(X,Y) \diamond (\mathcal{M}_1(Y,Z) \diamond \mathcal{M}_1(Z,W))) \diamond \mathcal{M}_1(W,V) \arrow[d, Rightarrow, "\mathrm{Ass}_{X,Y,W,V}"] & (\mathcal{M}_1(X,Y) \diamond \mathcal{M}_1(Y,Z)) \diamond (\mathcal{M}_1(Z,W) \diamond \mathcal{M}_1(W,V)) \arrow[dd, Rightarrow, "\mathrm{Ass}_{X,Y,Z,V}"'] \\
\mathcal{M}_1(X,Y) \diamond ((\mathcal{M}_1(Y,Z) \diamond \mathcal{M}_1(Z,W)) \diamond \mathcal{M}_1(W,V)) \arrow[dr, Rightarrow, "1_{\mathcal{M}_1(X,Y)} \diamond \mathrm{Ass}_{Y,Z,W,V}"] & \\
& \mathcal{M}_1(X,Y) \diamond (\mathcal{M}_1(Y,Z) \diamond (\mathcal{M}_1(Z,W) \diamond \mathcal{M}_1(W,V)))
\end{tikzcd}$$

commutes, and

- the triangle axiom holds which says

$$\begin{tikzcd}
(\mathcal{M}_1(X,Y) \diamond 1_Y) \diamond \mathcal{M}_1(Y,Z) \arrow[rr, Rightarrow, "\mathrm{Ass}_{X,Y,Y,Z}"] \arrow[dr, Rightarrow, "RUn_{X,Y} \diamond 1_{\mathcal{M}_1(Y,Z)}"'] & & \mathcal{M}_1(X,Y) \diamond (1_Y \diamond \mathcal{M}_1(Y,Z)) \arrow[dl, Rightarrow, "1_{\mathcal{M}_1(X,Y)} \diamond LUn_{X,Y}"] \\
& \mathcal{M}_1(X,Y) \diamond \mathcal{M}_1(Y,Z) &
\end{tikzcd}$$

commutes.

A bicategory is called a *2-category* if Ass, RUn and LUn are all identities. Here $A \in \mathcal{M}_1(X,Y)$ is called a 1-isomorphism if there exists $A^{-1} \in \mathcal{M}_1(X,Y)$ such that $A \diamond A^{-1} = 1_X$ and $A^{-1} \diamond A = 1_Y$. There is also a weaker notion, known as 1-equivalence. We say $A \in \mathcal{M}_1(X,Y)$ is a 1-equivalence if there exists $A^{-1} \in \mathcal{M}_1(X,Y)$ and isomorphisms $\phi \in \mathcal{M}_2(A \diamond A^{-1}, 1_X)$ and $\psi \in \mathcal{M}_2(A^{-1} \diamond A, 1_Y)$.

Definition 3.1.17. A bicategory $\mathcal{K}$ is called a *2-group* if $\mathcal{K}_0$ is a one element set, all 1-morphisms are 1-equivalences and all 2-morphisms are 2-isomorphisms. It is a *strict* 2-group if each 1-morphism is a 1-isomorphism.

Here one should note that we have used group notation instead of function notation. This choice is on purpose due to the following proposition.

Proposition 3.1.18. *A bicategory with a single object can be described as a monoidal category.*

Proof. Let $\mathcal{M}$ be a bicategory with $\mathcal{M}_0 = \{*\}$. We will now define a monoidal category $\mathcal{M}_\otimes$ where

- the category $\mathcal{M}_\otimes := \mathcal{M}_1(*,*)$,
- the tensor product $\otimes := \diamondsuit_{*,*,*} : \mathcal{M}_1(*,*) \times \mathcal{M}_1(*,*) \to \mathcal{M}_1(*,*)$,
- the unit object is $1_{\mathcal{M}_\otimes} := 1_* \in \mathcal{M}_1(*,*)$,
- the associator Ass $:=$ $\mathrm{Ass}_{*,*,*,*}$ where the functors

$$(-\otimes-)\otimes- = \diamondsuit_{*,*,*}\circ(\diamondsuit_{*,*,*}\times 1_{\mathcal{M}_1(*,*)}) : \mathcal{M}_1(*,*)\times\mathcal{M}_1(*,*)\times\mathcal{M}_1(*,*) \to \mathcal{M}_1(*,*)$$

$$-\otimes(-\otimes-) = \diamondsuit_{*,*,*}\circ(1_{\mathcal{M}_1(*,*)}\times\diamondsuit_{*,*,*}) : \mathcal{M}_1(*,*)\times\mathcal{M}_1(*,*)\times\mathcal{M}_1(*,*) \to \mathcal{M}_1(*,*)$$

  so the associator Ass $:=$ $\mathrm{Ass}_{*,*,*,*} : (-\otimes-)\otimes- \Rightarrow -\otimes(-\otimes-)$ is a natural isomorphism,

- the left unitor $\lambda := LUn_{*,*} : 1_{\mathcal{M}_\otimes} \otimes - \Rightarrow -$ is a natural isomorphisms, and
- the right unitor $\rho := RUn_{*,*} : - \otimes 1_{\mathcal{M}_\otimes} \Rightarrow -$ is also a natural isomorphism.

Here the axioms are identical when explicating them for $Ass_{*,*,*,*}$, and $LUn_{*,*}$ and $RUn_{*,*}$. □

Therefore 2-groups can be expressed as monoidal categories, which is the approach we shall take in this thesis. We will also use a generalisation of $\mathrm{Vect}_{\mathbb{K}}$ described below.

Example 3.1.19. Let $\mathbb{K}$ be a field. We describe the bicategory $2-\mathrm{Vect}^{\mathbb{K}}$ which has

- objects $2-\mathrm{Vect}^{\mathbb{K}}_0 = \mathbb{N}$ where we assume $0 \in \mathbb{N}$,

- the category $2-\mathrm{Vect}_1^{\mathbb{K}}(m,n)$ is as follows
  - $\mathrm{Ob}(2-\mathrm{Vect}_1^{\mathbb{K}}(m,n))$ is the set of $n\times m$-matrices $(V_{i,j})_{i,j}$, where $V_{i,j}$ is a finite-dimensional $\mathbb{K}$-vector spaces in the i'th, j'th position, where if $m=0$ or $n=0$ this is a trivial category containing one object,
  - 2-morphisms $\mathrm{Mor}((V_{i,j})_{i,j},(W_{i,j})_{i,j}) = 2-\mathrm{Vect}_2^{\mathbb{K}}((V_{i,j})_{i,j},(W_{i,j})_{i,j})$ is the set of $n\times m$-matrices of linear maps $(\phi_{i,j}: V_{i,j}\to W_{i,j})_{i,j}$,
  - composition $\circ: 2-\mathrm{Vect}_2^{\mathbb{K}}((V_{i,j}),(W_{i,j}))\times 2-\mathrm{Vect}_2^{\mathbb{K}}((U_{i,j})_{i,j},(V_{i,j})_{i,j}) \to 2-\mathrm{Vect}_2^{\mathbb{K}}((U_{i,j})_{i,j},(W_{i,j})_{i,j})$ is of matricies being coordinatewise composition of linear maps, i.e. $(\phi_{i,j})_{i,j}\circ(\psi_{i,j})_{i,j} = (\phi_{i,j}\circ\psi_{i,j})_{i,j}$, and
  - for $(V_{i,j})_{i,j}\in 2-\mathrm{Vect}_1^{\mathbb{K}}(m,n)$ we have $1_{(V_{i,j})_{i,j}} = (1_{V_{i,j}})_{i,j} \in 2-\mathrm{Vect}_2^{\mathbb{K}}((V_{i,j})_{i,j},(V_{i,j})_{i,j})$ where $1_{V_{i,j}}$ is the identity on $V_{i,j}$.
- composition bifunctor $\diamondsuit_{m,n,p}: 2-\mathrm{Vect}_1^{\mathbb{K}}(m,n)\times 2-\mathrm{Vect}_1^{\mathbb{K}}(n,p)\to 2-\mathrm{Vect}_1^{\mathbb{K}}(m,p)$ defined analogously to matrix multiplication where sum is replaced by direct product and multiplication by tensor product, i.e.

$$(V_{i,j})_{i,j}\diamond(W_{i,j})_{i,j} := (\oplus_{k=1}^{n} V_{i,k}\otimes W_{k,j})_{i,j},$$

- the identity object $1_n := (\delta_{i,j}^n)_{i,j}$ where

$$\delta_{i,j}^n = \begin{cases}\mathbb{K} & \text{if } i=j\\ 0 & \text{otherwise}\end{cases},$$

- non-trivial associativity condition $Ass_{m,n,p,q}$ coming from the associativities of tensor product and direct product with

$$((U_{i,j})_{i,j}\diamond(V_{i,j})_{i,j})\diamond(W_{i,j})_{i,j} = (\oplus_{l=1}^{p}(\oplus_{k=1}^{n}U_{i,k}\otimes V_{k,l})\otimes W_{l,j})_{i,j}, \quad \text{and}$$

$$(U_{i,j})_{i,j}\diamond((V_{i,j})_{i,j}\diamond(W_{i,j})_{i,j}) = (\oplus_{k=1}^{n}U_{i,k}\otimes(\oplus_{l=1}^{p}V_{k,l}\otimes W_{l,j}))_{i,j},$$

  making $Ass_{m,n,p,q}$ a natural isomorphism given by reordering sums and tensor products, and
- non-trivial unitality conditions coming from the isomorphisms $\mathbb{K}\otimes V\cong V$ and $V\otimes\mathbb{K}\cong V$ where if $1_n = (\delta_{i,j}^n)_{i,j}$ we have

$$(V_{i,j})_{i,j}\diamond 1_n = (\oplus_{k=1}^{n}(V_{i,k}\otimes\delta_{k,j}^n))_{i,j} = (V_{i,j}\otimes\mathbb{K})_{i,j}, \quad \text{and similiary}$$

$$1_n\diamond(V_{i,j})_{i,j} = (\oplus_{k=1}^{n}(\delta_{i,k}^n\otimes V_{k,j}))_{i,j} = (\mathbb{K}\otimes V_{i,j})_{i,j}.$$

One might ask if there is motivation for this definition to be the extension of $\mathrm{Vect}_{\mathbb{K}}$. This comes from the area of module categories. However for simplicity of this thesis it will not be persued here. We refer an interested reader to Rumynin and Wendland [48, Theorem 1.2].

We build up definitions for bicategories so we can meaningfully talk about equivalences of bicategories and 2-representations.

Definition 3.1.20. Let $\mathcal{M}$ and $\mathcal{N}$ be two bicategories. A 2-functor $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ between bicategories consists of the following data:

- a function $\mathcal{F}^0 : \mathcal{M}_0 \to \mathcal{N}_0$,
- a family of functors $\mathcal{F}^1_{X,Y} : \mathcal{M}_1(X,Y) \to \mathcal{N}_1(\mathcal{F}^0(X), \mathcal{F}^0(Y))$,
- a family of 2-isomorphisms $\mathcal{F}^2_X : 1_{\mathcal{F}^0(X)} \Rightarrow F^1_{X,X}(1_X)$, and
- a family of compatibility conditions which are natural isomorphisms between the following functors

  $$\begin{matrix} \diamondsuit_{\mathcal{F}^0(X),\mathcal{F}^0(Y),\mathcal{F}^0(Z)} \circ (F^1_{X,Y} \circ F^1_{Y,Z}), \\ F^1_{X,Z} \circ \diamondsuit_{X,Y,Z} \end{matrix} : \mathcal{M}_1(X,Y) \times \mathcal{M}_1(Y,Z) \to \mathcal{N}_1(\mathcal{F}^0(X), \mathcal{F}^0(Z)),$$

  called

  $$F^2_{X,Y,Z} : \diamondsuit_{\mathcal{F}^0(X),\mathcal{F}^0(Y),\mathcal{F}^0(Z)} \circ (F^1_{X,Y} \circ F^1_{Y,Z}) \Rightarrow F^1_{X,Z} \circ \diamondsuit_{X,Y,Z}.$$

Set $\mathcal{M}_{X,Y} := \mathcal{M}_1(X,Y)$ and $\mathcal{N}_{X,Y} := \mathcal{N}_1(X,Y)$ then the following axioms hold.

- Associativity, for functors from $\mathcal{M}_{X,Y} \times \mathcal{M}_{Y,Z} \times \mathcal{M}_{Z,W} \to \mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(W)}$ we have that

$$\begin{tikzcd}
(\mathcal{N}_{\mathcal{F}^0(X)),\mathcal{F}^0(Y)} \diamond \mathcal{N}_{\mathcal{F}^0(Y),\mathcal{F}^0(Z)}) \diamond \mathcal{N}_{\mathcal{F}^0(Z),\mathcal{F}^0(W)} \arrow[dr, Rightarrow, "Ass_{\mathcal{F}^0(X),\mathcal{F}^0(Y),\mathcal{F}^0(Z),\mathcal{F}^0(W)}"] \arrow[dd, Rightarrow, "\mathcal{F}^2_{X,Y,Z} \diamond 1_{\mathcal{N}_{\mathcal{F}^0(Z),\mathcal{F}^0(W)}}"] & \\
& \mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)} \diamond (\mathcal{N}_{\mathcal{F}^0(Y),\mathcal{F}^0(Z)} \diamond \mathcal{N}_{\mathcal{F}^0(Z),\mathcal{F}^0(W)}) \arrow[dd, Rightarrow, "1_{\mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)}} \diamond \mathcal{F}^2_{Y,Z,W}"'] \\
\mathcal{F}^1_{X,Z}(\mathcal{M}_{X,Y} \diamond \mathcal{M}_{Y,Z}) \diamond \mathcal{N}_{\mathcal{F}^0(Z),\mathcal{F}^0(W)} \arrow[dd, Rightarrow, "\mathcal{F}^2_{X,Z,W}"] & \\
& \mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)} \diamond \mathcal{F}^1_{Y,W}(\mathcal{M}_{Y,Z} \diamond \mathcal{M}_{Z,W}) \arrow[dd, Rightarrow, "\mathcal{F}^2_{X,Y,W}"] \\
\mathcal{F}^1_{X,W}((\mathcal{M}_{X,Y} \diamond \mathcal{M}_{Y,Z}) \diamond \mathcal{M}_{Z,W}) \arrow[dr, Rightarrow, "\mathcal{F}^1_{X,W}(Ass_{X,Y,Z,W})"] & \\
& \mathcal{F}^1_{X,W}(\mathcal{M}_{X,Y} \diamond (\mathcal{M}_{Y,Z} \diamond \mathcal{M}_{Z,W}))
\end{tikzcd}$$

commutes.

- Unitality, for functors from $\mathcal{M}_{X,Y} \to \mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(W)}$ we have that

$$\begin{tikzcd}
1_{\mathcal{F}^0(X)} \diamond \mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)} \arrow[r, Rightarrow, "\mathcal{F}^2_X \diamond 1_{\mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)}}"] \arrow[d, Rightarrow, "LUn_{\mathcal{F}^0(X)),\mathcal{F}^0(Y)}"] & \mathcal{F}^1_{X,X}(1_X) \diamond \mathcal{N}_{\mathcal{F}^0(X)),\mathcal{F}^0(Y)} \arrow[d, Rightarrow, "\mathcal{F}^2_{X,X,Y}"] \\
\mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)} & \mathcal{F}^1_{X,Y}(1_X \diamond \mathcal{M}_{X,Y}) \arrow[l, Rightarrow, "\mathcal{F}^1_{X,Y}(LUn_{X,Y})"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)} \diamond 1_{\mathcal{F}^0(Y)} \arrow[r, Rightarrow, "1_{\mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)}} \diamond \mathcal{F}^2_Y"] \arrow[d, Rightarrow, "RUn_{\mathcal{F}^0(X)),\mathcal{F}^0(Y)}"] & \mathcal{N}_{\mathcal{F}^0(X)),\mathcal{F}^0(Y)} \diamond \mathcal{F}^1_{Y,Y}(1_Y) \arrow[d, Rightarrow, "\mathcal{F}^2_{X,Y,Y}"] \\
\mathcal{N}_{\mathcal{F}^0(X),\mathcal{F}^0(Y)} & \mathcal{F}^1_{X,Y}(\mathcal{M}_{X,Y} \diamond 1_Y) \arrow[l, Rightarrow, "\mathcal{F}^1_{X,Y}(RUn_{X,Y})"]
\end{tikzcd}$$

commute.

Definition 3.1.21. Let $\mathcal{M}$ and $\mathcal{N}$ be two bicategories. Let $\mathcal{F}, \mathcal{G} : \mathcal{M} \to \mathcal{N}$ be two 2-functors. Then a natural 2-transformation $\Phi : \mathcal{F} \Rightarrow \mathcal{G}$ contains the following data

- a family of 1-morphisms $\Phi^1_X \in \mathcal{D}_1(\mathcal{F}^0(X), \mathcal{G}^0(X))$, and
- a family of natural transformations between the functors

$$\mathcal{F}^1_{X,Y} \diamond \Phi^1_Y, \Phi^1_X \diamond \mathcal{G}^1_{X,Y} : \mathcal{M}_1(X,Y) \to \mathcal{N}_1(\mathcal{F}^0(X), \mathcal{G}^0(Y)),$$

  where on morphisms $\mathcal{F}^1_{X,Y} \diamond \Phi^1_Y : A \mapsto \mathcal{F}^1_{X,Y}(A) \diamond 1_{\Phi^1_Y}$ and $\Phi^1_X \diamond \mathcal{G}^1_{X,Y} : A \mapsto 1_{\Phi^1_X} \diamond \mathcal{G}^1_{X,Y}(A)$, called

$$\Phi^2_{x,y} : \mathcal{F}^1_{X,Y} \diamond \Phi^1_Y \Rightarrow \Phi^1_X \diamond \mathcal{G}^1_{X,Y}.$$

Where this data obeys the following

- a pentagon axiom which says

$$\begin{tikzcd}
 & (\mathcal{F}^1_{X,Y}(A) \diamond \mathcal{F}^1_{Y,Z}(B)) \diamond \Phi^1_Z \arrow[dl, "(1_{\mathcal{F}^1_{X,Y}(A)} \diamond \Phi^2_{Y,Z}) \circ Ass_{X,Y,Z,Z}(-)"'] \arrow[dr, "\mathcal{F}^2_{X,Y,Z}(A,B) \diamond 1_{\Phi^2_Z}"] & \\
\mathcal{F}^1_{X,Y}(A) \diamond (\Phi^1_Y \diamond \mathcal{G}^1_{Y,Z}(B)) \arrow[d, "(\Phi^2_{X,Y} \diamond 1_{\mathcal{G}^1_{Y,Z}(B)}) \circ Ass_{X,Y,Y,Z}(-)^{-1}"] & & \mathcal{F}^1_{X,Z}(A \circ B) \diamond \Phi^1_Z \arrow[d, "\Phi^2_{X,Z}(A \circ B)"] \\
(\Phi^1_X \diamond \mathcal{G}^1_{X,Y}(A)) \diamond \mathcal{G}^1_{Y,Z}(B) \arrow[rr, "(1_{\Phi^2_X} \diamond \mathcal{G}^2_{X,Y,Z}(A,B)) \circ Ass_{X,X,Y,Z}(-)"'] & & \Phi^1_X \diamond \mathcal{G}^1_{X,Z}(A \circ B)
\end{tikzcd}$$

commutes, where the arguments of the associtivity maps aren't given however can be read from the domain of the maps, and

- a unitary axiom which says

$$
\begin{tikzcd}
\mathcal{F}^1_{X,X}(1_X) \diamond \Phi^1_X \arrow[rr, "\Phi^2_{X,X}(1_X)"] \arrow[d, "\mathcal{F}^1_X \diamond 1_{\Phi^1_X}"] & & \Phi^1_X \diamond \mathcal{G}^1_{X,X}(1_X) \arrow[d, "1_{\Phi^1_X} \diamond \mathcal{G}^1_X"'] \\
1_{\mathcal{F}^0(X)} \diamond \Phi^1_X \arrow[dr, "LUn(\Phi^1_X)"] & & \Phi^1_X \diamond 1_{\mathcal{G}^0(X)} \arrow[dl, "RUn(\Phi^1_X)"'] \\
& \Phi^1_X &
\end{tikzcd}
$$

commutes.

A natural 2-transformation is a natural 2-isomorphism if all Φ^1_X are 1-equivalences and all $\Phi^2_{X,Y}$ are natural isomorphisms.

Two bicategories $\mathcal{M}$ and $\mathcal{N}$ are equivalent if there exists 2-functors $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ and $\mathcal{F}^{-1} : \mathcal{N} \to \mathcal{M}$ with natural 2-isomorphisms $\Phi : \mathcal{F} \circ \mathcal{F}^{-1} \Rightarrow 1_{\mathcal{N}}$ and $\Phi' : \mathcal{F}^{-1} \circ \mathcal{F} \Rightarrow 1_{\mathcal{M}}$.

Although these definitions contain a lot of data, in most applications we will not use much of it or it will be turned into data that is more human friendly.

3.1.3 Group cohomology

In this subsection we introduce group cohomology and show how we can use it to represent special skeletal 2-groups.

Let G be a group and H a $\mathbb{Z}G$-module. We write G in multiplicative notation with identity 1_G and $x, y, z, w \in G$ for generic elements. Whereas, we use additive notation for H with identity 0_H and $a, b, c \in H$. When manipulating formulas we capitalise elements that are fixed. We write the group action of x on a as ${}^x a$.

Definition 3.1.22. We define chain groups

$$C_n(G, H) = \mathrm{Mor}_{\mathsf{Set}}(G^n, H),$$

with group operation being pointwise addition of functions. We call elements of $C_n(G, H)$ n-cochains. Note we take G^0 to be a trivial group. We have (boundary) maps $d_n : C_{n-1}(G,H) \to C_n(G,H)$ defined by

$$d_n(\mu)(x_1,\dots,x_n) = {}^{x_1}\mu(x_2,\dots x_n) + \sum_{i=1}^{n-1}(-1)^i\mu(x_1,\dots,x_{i-1},x_ix_{i+1},x_{i+2},\dots,x_n) \\ + (-1)^n\mu(x_1,\dots,x_{n-1}).$$

For $C_n(G,H)$ define subgroups

$$Z^n(G,H) = ker(d_{n+1}) \quad \text{and} \quad B^n = \begin{cases} 0 & \text{if } n = 0 \\ im(d_n) & \text{if } n \in \mathbb{N}\backslash\{0\} \end{cases}$$

where we call elements n-cocyles and n-coboundaries respectively. We have that $d_{n+1} \circ d_n = 0$ therefore we can define

$$H^n(G,H) := Z^n(G,H)/B^n(G,H)$$

as the n'th cohomology group.

Cocycles are very useful to classify 2-groups and 2-representations as we will see throughout this thesis. We start with a motivating example.

Example 3.1.23. Let $\alpha \in Z^3(G,H)$ be a normalised 3-cocyle, i.e.,

$${}^x\alpha(y,z,w)+\alpha(x,yz,w)+\alpha(x,y,z) = \alpha(xy,z,w)+\alpha(x,y,zw) \text{ with } \alpha(x,1_G,y) = 0_H$$

for any $x,y,z,w \in G$. From this you can deduce $\alpha(1_G,z,w) = \alpha(x,y,1_G) = 0_H$ (take $x = y = 1_G$ or $z = 1_G$ respectively). Now define a 2-group $\mathcal{G}(G,H,\alpha) := \mathcal{G}$ (which we represent as a monoidal category) where

- we will have $Ob(\mathcal{G}) = G$,
- we set
$$\text{Mor}(x,y) = \begin{cases} H & \text{if } x = y \\ \emptyset & \text{otherwise} \end{cases}$$
where composition is the group operation of H. Therefore 1_x, the identity morphism on $x \in G$, is actually 0_H as an element of H, we will refer to this element as 1_x throughout. Regularly we will use elements of H to refer to morphisms notationally for $a \in H$ when considering this as a morphisms in $\text{Mor}(x,x)$ we will refer to this as $a_x \in \text{Mor}(x,x)$.
- The group operation of G defines $\otimes$ on objects ($x \otimes y = xy$). We have the following equality for morphisms
$$a_x \otimes b_y = (a + {}^xb)_{xy} \in \text{Mor}(xy,xy).$$

- We set $\lambda(x) = \rho(x) = 0_H$.

- However, we have a non-trivial associator given by Ass := α i.e. $\mathrm{Mor}(xyz, xyz) \ni \alpha(x,y,z) : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$ (notationally we will use $\alpha(x,y,z)$ to refer to the group element and the morphism as it is clear from context where it lies). We have that α is a natural transformation of the two functors

$$(- \otimes -) \otimes - : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G} \quad \text{and} \quad - \otimes (- \otimes -) : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}.$$

  As H is abelian, and for $a, b, c \in H$ we have that

$$\begin{aligned} \alpha(x,y,z) \circ ((a_x \otimes b_y) \otimes c_z) &= \alpha(x,y,z) \circ (a + {}^x b)_{xy} \otimes c_z \\ &= (\alpha(x,y,z) + a + {}^x b + {}^{xy} c)_{xyz} \\ &= (a + {}^x(b + {}^y c) + \alpha(x,y,z))_{xyz} \\ &= (a_x \otimes (b_y \otimes c_z)) \circ \alpha(x,y,z). \end{aligned}$$

  It is nice to note here that computationally $\otimes$ is associative on group elements of H, however we still have non-trivial associator.

- The triangle axiom

$$\begin{tikzcd} (X \otimes 1) \otimes Y \arrow[rr, "\alpha_{X,1,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\ & X \otimes Y & \end{tikzcd}$$

  is equivalent to checking $(1_x \otimes \lambda(y)) \circ \alpha(x, 1_G, y) = (\rho(x) \otimes 1_y)$. Though we can translate this into a calculation in H with $1_y = 0_H = 1_x$ and as α is normalised and from definition we have $\rho(x) = \lambda(y) = \alpha(x, 1_G, y) = 0_H$. Then applying our definition of $\otimes$ we get that

$$\begin{aligned} (1_x \otimes \lambda(y)) \circ \alpha(x, 1_G, y) &= ((0_H)_x \otimes (0_H)_y)) \circ (0_H)_{xy} \\ &= (0_H + {}^x 0_H + 0_H)_{xy} \\ &= (\rho(x) \otimes 1_y). \end{aligned}$$

- The pentagon axiom

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z) \otimes W \arrow[dl, "\alpha(X,Y,Z)\otimes 1_W"'] \arrow[dr, "\alpha(X\otimes Y,Z,W)"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "\alpha(X,Y\otimes Z,W)"] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "\alpha(X,Y,Z\otimes W)"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "1_X\otimes\alpha(Y,Z,W)"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

is equivalent to checking

$$\alpha(x,y,zw) \circ \alpha(xy,z,w) = 1_x \otimes \alpha(y,z,w) \circ \alpha(x,y \otimes z,w) \circ \alpha(x,y,z) \otimes 1_w$$

which if we observe

$$\begin{aligned}
\alpha(x,y,zw) \circ \alpha(xy,z,w) &= \alpha(x,y,zw) + \alpha(xy,z,w) \\
&= {}^{x}\alpha(y,z,w) + \alpha(x,yz,w) + \alpha(x,y,z) \\
&= ((0_H)_x \otimes \alpha(y,z,w)) \circ \alpha(x,yz,w) \circ (\alpha(x,y,z) \otimes (0_H)_w) \\
&= 1_x \otimes \alpha(y,z,w) \circ \alpha(x,y \otimes z,w) \circ \alpha(x,y,z) \otimes 1_w
\end{aligned}$$

we get the pentagon axiom holds.

This 2-group is skeletal as the only morphisms are automorphisms. Moreover it is special as it has trivial unitors. In fact this is exactly how we generate all special skeletal 2-groups.

Theorem 3.1.24. *[7, Theorem 1] Let $\mathcal{G}$ be a special skeletal 2-group. There exists a triple (G,H,α) with H being a $\mathbb{Z}G$-module and $\alpha \in Z^3(G,H)$ such that $\mathcal{G} \cong \mathcal{G}(G,H,\alpha)$ with the correct identifications.*

Proof. Define a magma $(G,\cdot)$ where $G = \mathrm{Ob}(\mathcal{G})$ and binary operation $\cdot := \otimes$ where $X, Y \in \mathrm{Ob}(\mathcal{G})$ we have $X \cdot Y := X \otimes Y$. Note that

- we have $1_{\mathcal{G}}$ where $\rho_X : X \otimes 1_{\mathcal{G}} \to X$ and $\lambda_X : 1_{\mathcal{G}} \otimes X \to X$ are the identity morphisms. However as we are in a skeletal category that means that $1_{\mathcal{G}} \otimes X = X = X \otimes 1_{\mathcal{G}}$. Giving that $1_{\mathcal{G}} = 1_G$ is a two sided identity of $(G,\cdot)$.

- as for 2-groups 1-morphisms are 1-equivalences we have that for all $X \in \mathrm{Ob}(\mathcal{G})$ we have a $X^{-1} \in \mathrm{Ob}(\mathcal{G})$ and isomorphisms $\Phi \in \mathrm{Mor}(X \otimes X^{-1}, 1_{\mathcal{G}})$ and $\Phi' \in \mathrm{Mor}(X^{-1} \otimes X, 1_{\mathcal{G}})$. However as $\mathcal{G}$ is skeletal this gives us that $X \otimes X^{-1} = 1_{\mathcal{G}} = X^{-1} \otimes X$, meaning that $(G,\cdot)$ has inverses.

- Lastly the associator $\mathrm{Ass} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is an isomorphism therefore as it is skeletal we have that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ making $(G, \cdot)$ associative.

Therefore $(G, \cdot)$ is a group.

Now define a magma $(H, \cdot)$ where $H := \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ and $\cdot := \circ$. Note that

- we have $0_H := 1_{1_{\mathcal{G}}} \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ where $0_H \circ A = A = A \circ 0_H$ for all $A \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ making 0_H a two sided inverse of $(H, \cdot)$.
- as $\circ$ is associative this makes $(H, \cdot)$ associative.
- as $\mathcal{G}$ is a 2-group all 2-morphisms are 2-isomorphisms meaning that for all $A \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ we have $A^{-1} \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ such that $A \circ A^{-1} = 0_H = A^{-1} \circ A$ giving $(H, \cdot)$ inverses.
- for all $A, B \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ we have that

$$\begin{aligned} A \circ B &= (0_H \otimes A) \circ (B \otimes 0_H) && \text{as we have trivial unitors} \\ &= (B \circ A) \otimes (0_H \circ 0_H) && \text{by the interchange law} \\ &= B \circ A && \text{as we have trivial unitors} \end{aligned}$$

  giving that $(H, \cdot)$ is commutative.

Therefore H is an abelian group. An interesting side fact is that $\circ$ and $\otimes$ agree when restricted to $\mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$,

$$\begin{aligned} B \circ A &= (1_{\mathcal{G}} \otimes B) \circ (A \otimes 1_{\mathcal{G}}) \\ &= (1_{\mathcal{G}} \circ A) \otimes (B \circ 1_{\mathcal{G}}) \\ &= A \otimes B \end{aligned}$$

given that $\circ$ is commutative on $\mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ means that $A \circ B = A \otimes B$.

It is also handy to observe at this point that as $\otimes$ is a functor it preserves inverses, therefore $1_X \otimes 1_Y = 1_{X \otimes Y}$. For any $X \in \mathrm{Ob}(\mathcal{G})$ we can get a composition preserving bijection between $- \otimes 1_X : \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}}) \to \mathrm{Mor}(X, X)$. This is a bijection as it has inverse $\mathrm{Ass}^{-1}_{1_{\mathcal{G}}, X, X^{-1}} \circ (- \otimes 1^{-1}_X) \circ \mathrm{Ass}_{1_{\mathcal{G}}, X, X^{-1}} : \mathrm{Mor}(X, X) \to \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$, as we observe

$$\begin{aligned}
\mathrm{Ass}^{-1}_{1_{\mathcal{G}},X,X^{-1}} \circ (- \otimes 1_{X^{-1}}) \circ \\
\mathrm{Ass}_{1_{\mathcal{G}},X,X^{-1}}(- \otimes 1_X(A)) &= \mathrm{Ass}^{-1}_{1_{\mathcal{G}},X,X^{-1}} \circ ((A \otimes 1_X) \otimes 1_{X^{-1}}) \circ \mathrm{Ass}_{1_{\mathcal{G}},X,X^{-1}} \\
&= \mathrm{Ass}^{-1}_{1_{\mathcal{G}},X,X^{-1}} \circ \mathrm{Ass}_{1_{\mathcal{G}},X,X^{-1}} \circ (A \otimes (1_X \otimes 1_{X^{-1}})) \\
&= A \otimes 0_H \\
&= A
\end{aligned}$$

because Ass is a natural transformation. It is composition preserving as

$$\begin{aligned}
(A \circ B) \otimes 1_X &= (A \circ B) \otimes (1_X \circ 1_X) \\
&= (A \otimes 1_X) \circ (B \otimes 1_X)
\end{aligned}$$

by the interchange law. Therefore we can use this to identify each $\mathrm{Mor}(X, X)$ with H and have composition defined by $\cdot$. This identification above makes it meaningful to talk about the associator as a map $\mathrm{Ass} : G^3 \to H$. Moreover by the unitor axiom

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "\mathrm{Ass}_{X,1,Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

and as $\rho_X = 0_H = \lambda_X$ (as $\mathcal{G}$ is special) therefore $(1_X \otimes 1_Y) = (1_X \otimes 1_Y) \circ \mathrm{Ass}(X, 1_G, Y)$ making $\mathrm{Ass}(X, 1_G, Y) = 0_H$ normalised. However from the pentagon axiom

$$\begin{tikzcd}
& ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "\mathrm{Ass}(X,Y,Z) \otimes 1_W"'] \arrow[dr, "\mathrm{Ass}(X \otimes Y, Z, W)"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "\mathrm{Ass}(X, Y \otimes Z, W)"] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "\mathrm{Ass}(X,Y,Z \otimes W)"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "1_X \otimes \mathrm{Ass}(Y,Z,W)"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

we have that

$$(1_X \otimes \mathrm{Ass}(Y, Z, W)) \circ \mathrm{Ass}(X, Y \otimes Z, W) \circ (\mathrm{Ass}(X, Y, Z) \otimes 1_W) = \mathrm{Ass}(X \otimes Y, Z, W) \circ \mathrm{Ass}(X, Y, Z \otimes W)$$

for any $X, Y, Z, W \in G$. From this you can deduce $\mathrm{Ass}(1_G, Z, W) = \mathrm{Ass}(X, Y, 1_G) = 0_H$ by taking $X = Y = 1_G$ or $Z = 1_G$ respectively.

Next we define the action of G on H by $X : A \mapsto 1_X \otimes A \otimes 1_{X^{-1}} := {}^X A \in$

$\mathrm{Mor}(X \otimes 1_{\mathcal{G}} \otimes X^{-1}, X \otimes 1_{\mathcal{G}} \otimes X^{-1}) = \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$. Note this is a group action as

$$\begin{aligned} {}^{X}({}^{Y}A) &= {}^{X}(1_Y \otimes A \otimes 1_{X^{-1}}) \\ &= 1_X \otimes 1_Y \otimes A \otimes 1_{Y^{-1}} \otimes 1_{X^{-1}} \\ &= 1_{XY} \otimes A \otimes 1_{(XY)^{-1}} \\ &= {}^{XY}A. \end{aligned}$$

For these checks we choose not to bracket anything, as the introduction and remove of any associator will be trivial as $\mathrm{Ass}(1_G, X, Y) = \mathrm{Ass}(X, 1_G, Y) = \mathrm{Ass}(X, Y, 1_G) = 0_H = 1_{XY}$. Furthermore note that for any $A \in \mathrm{Mor}(X, X)$ and $B \in \mathrm{Mor}(Y, Y)$ we have that

$$\begin{aligned} A \otimes B &= (A \otimes 1_X) \otimes (B \otimes 1_Y) \\ &= A \otimes ((1_X \otimes B) \otimes 1_{X^{-1}}) \otimes (1_X \otimes 1_Y) \\ &= (A \circ {}^{X}B) \otimes 1_{XY} \end{aligned}$$

as $\circ$ and $\otimes$ agree on $\mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$.

Therefore it remains to check that $\mathrm{Ass} \in Z^3(G, H)$ however this follows now directly from the pentagon axiom

$$(1_X \otimes \mathrm{Ass}(Y, Z, W)) \circ \mathrm{Ass}(X, Y \otimes Z, W) \circ (\mathrm{Ass}(X, Y, Z) \otimes 1_W) = \mathrm{Ass}(X \otimes Y, Z, W) \circ \mathrm{Ass}(X, Y, Z \otimes W)$$

from which when using the definition above we get

$${}^{X}\mathrm{Ass}(Y, Z, W) + \mathrm{Ass}(X, YZ, W) + \mathrm{Ass}(X, Y, Z) = \mathrm{Ass}(XY, Z, W) + \mathrm{Ass}(X, Y, ZW).$$

So set $\alpha := \mathrm{Ass} \in Z^3(G, H)$. $\square$

Remark. If we have two special skeletal categories $\mathcal{G} = (G, H, \alpha)$ and $\mathcal{H} = (G, H, \beta)$ where α and β differ by a 2-boundary, i.e., $\alpha = \beta + \delta(\tau)$ for $\tau \in C^2(G, H)$ so

$$\begin{aligned} \alpha(x, y, z) =& \beta(x, y, z) + d(\tau)(x, y, z) \\ =& \beta(x, y, z) + {}^{x_1}\tau(y, z) - \tau(xy, z) + \tau(x, yz) - \tau(x, y) \end{aligned}$$

Then $\mathcal{G}$ is equivalent to $\mathcal{H}$. To define this equivalence, let the functor $\mathcal{F}$ on the categories be given by the identity when we associate objects to elements of G and morphisms to H. Let the identity isomorphism $\epsilon : 1_G \to 1_G$ be the identity, 1_H. Then the natural isomorphism $\mu : \mathcal{F}(-) \otimes_{\mathcal{H}} \mathcal{F}(-) \Rightarrow \mathcal{F}(- \otimes_{\mathcal{G}} -)$ has components

$\tau(-,-)$. Then the unitary axiom holds trivially and the associativity axiom holds as

$$\alpha(x,y,z) + \tau(xy,z) + \tau(x,y) = \tau(x,yz) + {}^{x}\tau(y,z) + \beta(x,y,z).$$

3.1.4 Crossed modules

In this subsection we introduce crossed modules and show how we can use these to represent strict 2-groups.

Definition 3.1.25. A *crossed module* contains the following data

- groups A and B,
- an action of B on A written as $f \in B$ acts on $\gamma \in A$ by $f : \gamma \mapsto {}^{f}\gamma$, and
- a group homomorphism $\partial : A \to B$.

Such that the following axioms hold

(CM1) ${}^{\partial(\gamma)}\delta = \gamma\delta\gamma^{-1}$, for all $\gamma, \delta \in A$, and

(CM2) $\partial({}^{f}\gamma) = f\partial(\gamma)f^{-1}$, for all $\gamma \in A, f \in B$.

This shall be written as $\mathcal{K} = (A \xrightarrow{\partial} B)$.

Notationally we will use $\gamma, \delta, \epsilon \in A$ and $f, g, h, k \in B$ both with multiplicative notation with units 1_A and 1_B respectively. We define its fundamental groups as $\pi_2(\mathcal{K}) := \ker(\partial)$ and $\pi_1(\mathcal{K}) := \mathrm{coker}(\partial) = B/Im(\partial)$.

Example 3.1.26. Let G be any group then we have a crossed module given by $\mathrm{Inn} : G \to \mathrm{Aut}(G)$ where $\mathrm{Inn}(x) : y \mapsto xyx^{-1}$ maps elements of G to their corresponding inner automorphism. The action of $\mathrm{Aut}(G)$ on G is given by the automorphism. Here (CM1) holds from definition and (CM2) follows as

$$\begin{aligned}\phi\partial(x)\phi^{-1}(y) &= \phi\partial(x)(\phi^{-1}(y)) \\ &= \phi(x\phi^{-1}(y)x^{-1}) \\ &= \phi(x)y\phi(x)^{-1} \\ &= \mathrm{Inn}(\phi(x))(y).\end{aligned}$$

We get that $\pi_1(\mathcal{K}) = Out(G)$ the outer automorphisms and $\pi_2(\mathcal{K}) = Z(G)$ the centre of the group.

Example 3.1.27. For a more concrete example let $B = \langle x, y | x^4, y^2, yxy^{-1} = x^{-1}\rangle = D_8$ be the dihedral group of order 8 and $A = \langle x | x^4\rangle = C_4$ the cyclic group of order 4. Let $\partial : A \to B$ be the inclusion $x \mapsto x$ and let B act on A by conjugation, so

${}^{y}x = x^{-1}$ and ${}^{x}x = x$. Then both (CM1) and (CM2) hold by definition. We have $\pi_1(\mathcal{K}) = \langle y | y^2 \rangle = C_2$ being the cyclic group of order 2 and $\pi_2(\mathcal{K}) = 1$ being trivial. Note this works more generally for any normal subgroup $N \leq G$.

Definition 3.1.28. Given two crossed modules $\mathcal{K} = (A \xrightarrow{\partial} B)$ and $\mathcal{K}' = (A' \xrightarrow{\partial'} B')$ a homomorphisms $\phi : \mathcal{K} \to \mathcal{K}'$ of crossed modules consists of group homomorphisms $\phi_A : A \to A'$ and $\phi_B : B \to B'$, such that these commute with the crossed mappings $\partial' \circ \phi_A = \phi_B \circ \partial$ and the group action $\phi_A({}^{b}a) = {}^{\phi_B(b)}\phi_A(a)$.

Example 3.1.29. To any crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$, we have the crossed module $\overline{\mathcal{K}} = (1 \xrightarrow{\partial} \pi_1(\mathcal{K}))$ with trivial group action and image. There exists the quotient homomorphism

$$\phi : \mathcal{K} \to \overline{\mathcal{K}}, \quad \phi_B : b \mapsto b\partial(A), \text{ and } \phi_A : a \mapsto 1.$$

Then as ϕ_A is trivial and $\partial|_1$ is trivial we have that $\partial \circ \phi_A = \phi_B \circ \partial$ and $\phi_A({}^{b}a) = {}^{\phi_B(b)}\phi_A(a)$.

Definition 3.1.30. A map of crossed modules $\phi : \mathcal{K} \to \mathcal{K}'$ is an equivalence of crossed modules if

- ϕ_A induces an isomorphisms of kernels, i.e., $\phi_A|_{Ker(\partial)} : Ker(\partial) \to Ker(\partial')$ is a well defined isomorphism.
- ϕ_B induces an isomorphism of cokernels, i.e., $\widehat{\phi_B} : Im(\partial)\backslash B \to Im(\partial')\backslash B'$ defined by $Im(\partial)b \mapsto Im(\partial')\phi_B(b)$ (which is well defined as $\partial' \circ \phi_A = \phi_B \circ \partial$) is an isomorphism.

When two crossed modules are equivalent we get the following exact sequences.

$$\begin{tikzcd}
 & & A \arrow[r, "\partial"] \arrow[dd, "\phi_A"] & B \arrow[dd, "\phi_B"] \arrow[dr] & & \\
0 \arrow[r] & \pi_2(\mathcal{K}) = \pi_2(\mathcal{K}') \arrow[ur] \arrow[dr] & & & \pi_1(\mathcal{K}) = \pi_1(\mathcal{K}') \arrow[r] & 0 \\
 & & A' \arrow[r, "\partial'"] & B' \arrow[ur] & &
\end{tikzcd}$$

Example 3.1.31. Take any crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ and group H with automorphisms $\phi \in \mathrm{Aut}(H)$. Then you can define a crossed module $\mathcal{K}' = (A \times H \xrightarrow{\partial \times \phi} B \times H)$. We have that $\mathcal{K}$ is equivalent to $\mathcal{K}'$ via the projection maps, i.e., $\phi_A(a) = (a, 1)$ and $\phi_B(b) = (b, 1)$.

Remark. For each crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ we can associate an element of $H^3(\pi_1(\mathcal{K}), \pi_2(\mathcal{K}))$. In fact this classifies equivalent crossed modules first proven by

Holt and Mac Lane [24]. A sketch of the construction is included in Brown [6] and reviewed by Thomas [51].

Definition 3.1.32. Given a crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$, we define a sub-crossed module $\mathcal{K}' = (A' \xrightarrow{\partial} B')$ as subgroups $A' \leq A$, $B' \leq B$ such that $\partial(A') \leq B'$.

Crossed modules give rise to strict 2-groups, which it is convenient to keep in bicategory notation. This follows from the construction given by Forrester-Barker [13].

Example 3.1.33. Let $\mathcal{K} = (A \xrightarrow{\partial} B)$ be a crossed module. We now construct 2-group $\widetilde{\mathcal{K}}$ where

- the objects $\widetilde{\mathcal{K}}_0 = \{\star\}$ is a one point set,
- the objects of the category $\widetilde{\mathcal{K}}_1(\star, \star) = B$,
- for 1-morphisms $f, g \in B$ the 2-morphisms in the category $\widetilde{\mathcal{K}}_1(\star, \star)$ are

$$\widetilde{\mathcal{K}}_2(f,g) := \{f \xRightarrow{\gamma} g\} \ = \ \{ \begin{tikzcd} \star \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, "\Uparrow\gamma"] & \star \end{tikzcd} \ \mid \ \gamma \in A,\ g = \partial(\gamma) f\},$$

  with composition defined by the product in γ

$$[g \xRightarrow{\delta} h] \circ [f \xRightarrow{\gamma} g] = \begin{tikzcd} \star \arrow[r, bend left=70, "h"] \arrow[r, "g" description] \arrow[r, bend right=70, "f"'] \arrow[r, phantom, shift left=3, "\Uparrow\delta"] \arrow[r, phantom, shift right=3, "\Uparrow\gamma"] & \star \end{tikzcd} = \begin{tikzcd} \star \arrow[r, bend left=60, "h"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, "\Uparrow\delta\gamma"] & \star \end{tikzcd} \quad \text{and}$$

  making the identity morphisms of $\widetilde{\mathcal{K}}_2(f,f)$ be given by 1_A,
- the composition bifunctor defined as

$$[f \xRightarrow{\gamma} g] \diamond [h \xRightarrow{\delta} k] = \begin{tikzcd} \star \arrow[r, bend left=60, "g"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, "\Uparrow\gamma"] & \star \arrow[r, bend left=60, "k"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, "\Uparrow\delta"] & \star \end{tikzcd} = \begin{tikzcd} \star \arrow[r, bend left=60, "gk"] \arrow[r, bend right=60, "fh"'] \arrow[r, phantom, "\Uparrow\gamma^f\delta"] & \star \end{tikzcd},$$

- with identity $1_\star = 1_B$, and
- trivial unitors and associators as this is a strict 2-group.

Note that composition works as

$$\begin{aligned}\partial(\gamma^f\delta)fh &= \partial(\gamma)\partial(^f\delta)fh \\ &= \partial(\gamma)f\partial(\delta)f^{-1}fh \\ &= gk.\end{aligned}$$

Lemma 3.1.34. *An equivalence of the crossed modules induces an equivalence of there corresponding 2-groups.*

Proof. Let two crossed modules $\mathcal{K} = (A \xrightarrow{\partial} B)$ and $\mathcal{K}' = (A \xrightarrow{\partial'} B)$ where $\phi : \mathcal{K} \to \mathcal{K}'$ is an equivalence. Construct equivalence $\Phi : \tilde{\mathcal{K}} \to \tilde{\mathcal{K}'}$ by

- the only 0-object $\Phi(*) = *$.
- for $b \in \tilde{\mathcal{K}}_1(*, *)$ map $\Phi(b) = \phi_B(b) \in \tilde{\mathcal{K}'}_1(*, *)$.
- let $f, g \in B$ be 1-morphisms with 2-morphism $\gamma : f \Rightarrow g$ so $g = \partial(\gamma)f$. Map $\Phi(\gamma) := \phi_A(\gamma) : \phi_B(f) \Rightarrow \phi_B(g)$ where $\phi_B(g) = \phi_B(\partial(\gamma)f) = \partial'(\phi_A(\gamma))\phi_B(f)$.
- with trivial compatibility conditions, as both bicategories have trivial associators and unitors.

Note that Φ preserves composition as ϕ_A and ϕ_B are homomorphisms, so $\Phi(\delta \circ \gamma) = \phi_A(\delta\gamma) = \phi_A(\delta)\phi_A(\gamma) = \Phi(\delta) \circ \Phi(\gamma)$ similarly for $\Phi(f \circ g) = \Phi(f) \circ \Phi(g)$. Similarly for composition we have

$$\begin{aligned}\Phi(\gamma \diamond \delta) &= \phi_A(\gamma^f\delta) \\ &= \phi_A(\gamma)^{\phi_B(f)}\phi_A(\delta) \\ &= \Phi(\gamma) \diamond \Phi(\delta).\end{aligned}$$

Note this proves that a homomorphisms of crossed modules induces a homomorphisms of strict 2-groups.

If we consider Φ as a functor of monoidal categories, it suffices to show Φ is essentially surjective, full and faithfull. This is equivalent to showing

1. for every $g' \in \tilde{\mathcal{K}'}(*, *)$ there exists $g \in \tilde{\mathcal{K}}(*, *)$ with isomorphism $\gamma \in \tilde{\mathcal{K}'}_2(\Phi(g), g')$.
2. each $\Phi : \tilde{\mathcal{K}}(g, h) \to \tilde{\mathcal{K}'}(\Phi(g), \Phi(h))$ is a bijection.

Recall that as $\widehat{\phi_B} : Im(\partial)\backslash B \to Im(\partial')\backslash B'$ is an isomorphism. For $b' \in B'$ there exists $b \in B$ such that $Im(\partial')\phi_B(b) = Im(\partial')b'$ so there exists $a' \in A'$ such that $\phi_B(b) = \partial'(a')b'$. As all 2-morphisms are 2-isomorphisms we get an isomorphism

$a' \in \tilde{\mathcal{K}}'(\Phi(b), b')$ giving property (1).

Let $g, h \in B$ and $a' \in A'$ such that $\phi_B(g) = \partial'(a')\phi_B(f)$. So $gf^{-1}Im(\partial) \in Ker(\widehat{\phi_B})$, but as $\widehat{\phi_B}$ is an isomorphism we have that $gf^{-1} = \partial(a) \in Im(\partial)$. Therefore $g = \partial(a)f$ and $\phi_B(g) = \phi_B(\partial(a))\phi_B(f) = \partial'(\phi_A(a))\phi_B(f)$. So we have $\partial'(a'\phi_A(a)^{-1}) = 1_{B'}$ giving $a' = \phi_A(a)k$ with $k \in Ker(\partial')$. However as $\phi_A|_{Ker(\partial)} : Ker(\partial) \to Ker(\partial')$ is an isomorphism there exists a unique $a'' \in Ker(\partial)$ such that $k = \phi_A(a'')$ giving that $a' = \phi_A(aa'')$ with $g = \partial(a)h = \partial(a)\partial(a'')h = \partial(aa'')h$. Therefore $\Phi : \tilde{\mathcal{K}}(g, h) \to \tilde{\mathcal{K}}'(\Phi(g), \Phi(h))$ is surjective.

Let $g, h \in B$ and $a' \in A'$ such that $\phi_B(g) = \partial'(a')\phi_B(f)$, with $a_1, a_2 \in A$ such that $\phi_A(a_1) = \phi_A(a_2) = a'$ and $g = \partial(a_1)f = \partial(a_2)f$. Then $\partial(a_1a_2^{-1}) = 1_B$ so $a_1a_2^{-1} \in Ker(\partial)$. Observe that $\phi_A(a_1a_2^{-1}) = \phi_A(a_1)\phi_A(a_2)^{-1} = (a')(a')^{-1} = 1_{A'}$ however as $\phi_A|_{ker(\partial)}$ is an isomorphism this gives us $a_1 = a_2$. Therefore $\Phi : \tilde{\mathcal{K}}(g, h) \to \tilde{\mathcal{K}}'(\Phi(g), \Phi(h))$ is injective. Giving us that Φ is an equivalence. □

As with the special skeletal cases, the strict case must arise in such a fashion. In [7, Chapter 1] a method of going from a strict 2-group to a crossed module is described. We first remind the reader of this technique for later use. Here it is convenient to switch back to monoidal category notation.

Proposition 3.1.35. *([7]) Let $\mathcal{G}$ be a strict 2-group. There exists crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ such that $\mathcal{G} \cong \widetilde{\mathcal{K}}$ as 2-groups.*

Proof. Suppose we have a strict 2-group $\mathcal{G}$. Let $\circ : \mathrm{Mor}(Y, Z) \times \mathrm{Mor}(X, Y) \to \mathrm{Mor}(X, Z)$ be composition and $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be the tensor product. Let $1_X \in \mathrm{Mor}(X, X)$ be the identity with respect to $\circ$ so that $a \circ 1_X = a = 1_Y \circ a$ for all $a \in \mathrm{Mor}(X, Y)$. Let ${}^{-1} : \mathcal{G} \to \mathcal{G}$ be the inverse with respect to $\otimes$. We abuse notation here and let

$$X^{-1} \otimes X = 1_{\mathcal{G}} \in Ob(\mathcal{G})$$

for $X \in Ob(\mathcal{G})$ and

$$(a \otimes a^{-1}) = 1_{1_{\mathcal{G}}} =: \mathbf{1}_{\mathcal{G}} \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$$

for all $a \in \mathrm{Mor}(X, Y)$ with $a^{-1} \in \mathrm{Mor}(X^{-1}, Y^{-1})$. Note $\mathbf{1}_{\mathcal{G}} \otimes a = a \otimes \mathbf{1}_{\mathcal{G}} = a \in \mathrm{Mor}(X, Y)$. As $\mathcal{G}$ is strict $\otimes$ is associative and unital on objects and morphisms.

Given $a \in \mathrm{Mor}(Y, Z)$ and $b \in \mathrm{Mor}(X, Y)$ then we get

$$\begin{aligned} a \circ b =&(1_Y \otimes (1_Y^{-1} \otimes a)) \circ (b \otimes \mathbf{1}_{\mathcal{G}}) \\ =&(1_Y \circ b) \otimes ((1_Y^{-1} \otimes a) \circ \mathbf{1}_{\mathcal{G}}) && \text{by the exhange law} \\ =&b \otimes (1_Y^{-1} \otimes a) \end{aligned}$$

which similarly we get

$$\begin{aligned} a \circ b =&(a \otimes \mathbf{1}_{\mathcal{G}}) \circ (1_Y \otimes (1_Y^{-1} \otimes b)) \\ =&(a \circ 1_Y) \otimes (\mathbf{1}_{\mathcal{G}} \circ (1_Y^{-1} \otimes b)) && \text{by the exhange law} \\ =&a \otimes (1_Y^{-1} \otimes b). \end{aligned}$$

So for $a \in \mathrm{Mor}(Y, Z)$ and $b \in \mathrm{Mor}(X, Y)$ we have that

$$b \otimes (1_Y^{-1} \otimes a) = a \otimes (1_Y^{-1} \otimes b) = a \circ b. \tag{3.1}$$

Note the following two interesting facts:

- If $Y = 1_{\mathcal{G}}$ then we have $\mathbf{1}_{\mathcal{G}}^{-1} = \mathbf{1}_{\mathcal{G}} \in \mathrm{Mor}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ and

$$a \otimes b = b \otimes a \qquad \text{for } a \in \mathrm{Mor}(1_{\mathcal{G}}, Z) \text{ and } b \in \mathrm{Mor}(X, 1_{\mathcal{G}}). \tag{3.2}$$

- For any $a \in \mathrm{Mor}(X, Y)$ we have that $1_X \otimes a^{-1} \otimes 1_Y \in \mathrm{Mor}(Y, X)$ is a's inverse under composition. This is due to the following

$$\begin{aligned} (1_X \otimes a^{-1} \otimes 1_Y) \circ a =&(1_X \otimes a^{-1} \otimes 1_Y) \otimes (1_Y^{-1} \otimes a) \\ =&1_X \end{aligned}$$

  and similarly as $1_X \otimes a^{-1} \otimes 1_Y \in \mathrm{Mor}(Y, X)$ we have

$$\begin{aligned} a \circ (1_X \otimes a^{-1} \otimes 1_Y) =&a \otimes (1_X^{-1} \otimes (1_X \otimes a^{-1} \otimes 1_Y)) \\ =&1_Y. \end{aligned}$$

Let $B := Ob(\mathcal{G})$ and $A := \prod_{X \in Ob(\mathcal{G})} \mathrm{Mor}(1_{\mathcal{G}}, X)$ which both get a group structure from $\otimes$. Define homomorphisms $\partial : A \to B$ by $a \in \mathrm{Mor}(1_{\mathcal{G}}, X) \mapsto X =: \partial(a)$, and group action of B on A by $Y : a \mapsto 1_Y \otimes a \otimes 1_Y^{-1} =: {}^Y a$. From the definition for $a \in \mathrm{Mor}(1_{\mathcal{G}}, X)$ we get that

$$\partial({}^Y a) = \partial(1_Y \otimes a \otimes 1_Y^{-1}) = YXY^{-1} = Y\partial(a)Y^{-1}.$$

To deduce ${}^{\partial(a)}b = aba^{-1}$, take $a, b \in A$ with $a \in \mathrm{Mor}(1_{\mathcal{G}}, X)$. Then $1_X^{-1} \otimes a \in$

$\mathrm{Mor}(X^{-1}, 1_{\mathcal{G}})$ therefore from 3.2 we have $b \otimes (1_X^{-1} \otimes a) = (1_X^{-1} \otimes a) \otimes b$, rearranging this gives the first equality of

$$aba^{-1} = 1_X b 1_X^{-1} = {}^X b = {}^{\partial(a)} b$$

the second of which come from the definition of the action, B on A, and ∂ respectively. Therefore $\mathcal{K} = (A \xrightarrow{\partial} B)$ gives rise to a crossed module, which induces the strict 2-group $\widetilde{\mathcal{K}}$.

If we interpret $\widetilde{\mathcal{K}}$ as a monoidal category we construct a monoidal functor $\mathcal{F} : \widetilde{\mathcal{K}} \to \mathcal{G}$. Which is defined on objects by setting $\mathcal{F}(X) = X$ as $\mathrm{Ob}(\widetilde{\mathcal{K}}) = B = \mathrm{Ob}(\mathcal{G})$. To define it on morphisms note

$$\begin{aligned} \mathrm{Mor}_{\widetilde{\mathcal{K}}}(X, Y) &= \{a \in \mathrm{Mor}_{\mathcal{G}}(1_{\mathcal{G}}, X') |\ Y = \partial(a) X\} \\ &= \{a \in \mathrm{Mor}_{\mathcal{G}}(1_{\mathcal{G}}, X') |\ Y = X'X\} \\ &= \mathrm{Mor}_{\mathcal{G}}(1_{\mathcal{G}}, X') && \text{where } X'X = Y. \end{aligned}$$

To make it explicitly clear where the objects live we write $(a)_{X,Y} \in \mathrm{Mor}_{\widetilde{\mathcal{K}}}(X, Y)$ for $a \in \mathrm{Mor}_{\mathcal{G}}(1_{\mathcal{G}}, YX^{-1})$. So $\mathcal{F} : (a)_{X,Y} \mapsto a \otimes 1_X \in \mathrm{Mor}_{\mathcal{G}}(X, X'X) = \mathrm{Mor}_{\mathcal{G}}(X, Y)$.

So let us show this is a functor of categories note that $1_X \in \mathrm{Mor}_{\widetilde{\mathcal{K}}}(X, X)$ is given by $1_{1_{\mathcal{G}}} \in \mathrm{Mor}_{\mathcal{G}}(1_{\mathcal{G}}, 1_{\mathcal{G}})$ therefore

$$\begin{aligned} \mathcal{F}(1_X) &= \mathcal{F}((1_{1_{\mathcal{G}}})_{X,X}) \\ &= 1_{1_{\mathcal{G}}} \otimes 1_X \\ &= 1_X. \end{aligned}$$

Note that

$$\begin{aligned} \mathcal{F}((b)_{Y,Z} \circ_{\widetilde{\mathcal{K}}} (a)_{X,Y}) &= \mathcal{F}((b \cdot_A a)_{X,Z}) \\ &= \mathcal{F}((b \otimes_{\mathcal{G}} a)_{X,Z}) \\ &= b \otimes_{\mathcal{G}} a \otimes_{\mathcal{G}} 1_X \\ &= b \otimes_{\mathcal{G}} 1_Y \otimes_{\mathcal{G}} 1_Y^{-1} \otimes_{\mathcal{G}} a \otimes_{\mathcal{G}} 1_X \\ &= (b \otimes_{\mathcal{G}} 1_Y) \circ_{\mathcal{G}} (a \otimes_{\mathcal{G}} 1_X) && \text{by 3.1} \\ &= \mathcal{F}((b)_{Y,Z}) \circ_{\mathcal{G}} \mathcal{F}((a)_{X,Y}) \end{aligned}$$

where $b \otimes_{\mathcal{G}} a \in \mathrm{Mor}(1_{\mathcal{G}}, (ZY^{-1})(YX^{-1})) = \mathrm{Mor}(1_{\mathcal{G}}, ZX^{-1})$, therefore $\mathcal{F}$ is a functor of categories. Also note that

$$\begin{aligned}
\mathcal{F}((a)_{X,Y} \otimes_{\widetilde{\mathcal{K}}} (b)_{W,Z}) &= \mathcal{F}((a \cdot_A {}^X b)_{XW,YZ}) \\
&= a \otimes_{\mathcal{G}} 1_X \otimes_{\mathcal{G}} b \otimes_{\mathcal{G}} 1_X^{-1} \otimes_{\mathcal{G}} 1_{XW} \\
&= a \otimes_{\mathcal{G}} 1_X \otimes_{\mathcal{G}} b \otimes_{\mathcal{G}} 1_W \\
&= \mathcal{F}((a)_{X,Y}) \otimes_{\mathcal{G}} \mathcal{F}((b)_{W,Z})
\end{aligned}$$

where $a \cdot_A {}^X b \in \mathrm{Mor}(1_{\mathcal{G}}, (YX^{-1})X(WZ^{-1})X^{-1}) = \mathrm{Mor}(1_{\mathcal{G}}, (YW)(XZ)^{-1})$, giving that $\mathcal{F}$ preserves $\otimes$ exactly, therefore $\mu_{\mathcal{F}}$ can be trivial. Also note that $\mathcal{F}(1_{\mathcal{G}}) = 1_{\mathcal{G}}$ giving $\epsilon_{\mathcal{F}}$ is also trivial. Meaning that associativity and unitality hold trivially. We have that $\mathcal{F}$ is surjective on objects so essentially surjective. As the map $- \otimes 1_X : \mathcal{G} \to \mathcal{G}$ has inverse $- \otimes 1_X^{-1} : \mathcal{G} \to \mathcal{G}$ we have that $\mathcal{F}$ is full and faithful. Giving $\mathcal{F}$ is an equivalence of monoidal categories. $\square$

Someone who has read the original source might note that we made a slightly different choice for our group A. This is just to make the functor $\mathcal{F}$ have a more canonical form given our set up.

3.1.5 2-Representations

We remind the reader a group G can be turned into a category where $Ob(G) = \{*\}$ and $\mathrm{Mor}_G(*, *) = G$ with composition being defined by the group operation of G. Then a linear representation θ of G can be thought of as a functor $\theta : G \to \mathrm{Vect}_{\mathbb{K}}$. Similarly we define a 2-representation.

Definition 3.1.36. A *2-representation* of a 2-group $\mathcal{G}$ is a 2-functor $\Theta : \mathcal{G} \to 2 - \mathrm{Vect}^{\mathbb{K}}$.

A map of linear representations θ_1 and θ_2 of G is simply a natural transformation of these two functors. Similarly we define a map of 2-representations.

Definition 3.1.37. A *map of 2-representations* $\Theta_1, \Theta_2 : \mathcal{G} \to 2 - \mathrm{Vect}^{\mathbb{K}}$, is a 2-natural transformation $\Phi : \Theta_1 \Rightarrow \Theta_2$.

So for a given 2-group, $\mathcal{G}$, we can form the category of 2-representations, $2 - \mathrm{Rep}(\mathcal{G})$ with each object being a 2-representation and morphisms between two objects being maps of 2-representations.

Remark. Here we have made some choices with our generalisation. These are by no means the only legitimate choices. We have chosen to map into an analogue of finite dimensional vector spaces. However other authors have chosen to generalise projective representations [14, 17, 47], or map into infinite dimensional vector spaces

[2]. We have chosen to study 2-groups, which we will make some assumption about. However influential work was done by Mazorchuk-Miemietz [35, 36, 37, 38, 39, 40] on finitary 2-categories and their 2-representations. The role of the field $\mathbb{K}$ will not be discussed much, other than mild assumptions on its characteristic, however some authors [49, 54] study these over particular fields.

For a 2-group given by a crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ we summarise the information a 2-representation $R : \tilde{\mathcal{K}} \to 2 - \mathrm{Vect}^{\mathbb{K}}$ entails.

1. A number $n = \Theta^0(*)$. We call this number the *degree* of Θ.
2. A 1-morphism $\Theta^1(b) = \Theta^1_{*,*}(b) = (V_{i,j}) : n \to n$ for every $b \in B$. The dimensions of these vector spaces form a matrix $\dim(\Theta^1(b)) \in \mathbb{N}^{n\times n}$.
3. 2-Isomorphisms $\Theta^1(b,a) = \Theta^1_{*,*}(b \xRightarrow{a} \partial(a)b) = (\varphi_{i,j}) : \Theta^1(b) \Rightarrow \Theta^1(\partial(a)b)$ for all $a \in A$ and $b \in B$ that are subject to vertical multiplicativity rule $\Theta^1(\partial(a_1)b, a_2)\Theta^1(b, a_1) = \Theta^1(b, a_2a_1)$.
4. A 2-isomorphism $\Theta^2_* = (\theta_{i,j}) : 1_n \Rightarrow \Theta(1_B)$.
5. 2-Isomorphisms $\Theta^2(b_1, b_2) = \Theta^2_{*,*,*}(b_1, b_2) = (\psi_{i,j}) : \Theta^1(b_1) \diamond \Theta^1(b_2) \Rightarrow \Theta^1(b_1b_2)$ for every pair $b_1, b_2 \in B$.

Such that the following axioms hold.

1. The pentagon axiom: $\Theta^2(b_1b_2, b_3) \circ (\Theta^2(b_1, b_2) \diamond Id_{\Theta^2(b_3)}) = \Theta^2(b_1, b_2b_3) \circ (Id_{\Theta^2(b_1)} \diamond \Theta^2(b_2, b_3)) \circ \mathrm{Ass}(\Theta^1(b_1), \Theta^1(b_2), \Theta^1(b_3))$ as maps $(\Theta^1(b_1) \diamond \Theta^1(b_2)) \diamond \Theta^1(b_3) \Rightarrow \Theta^1(b_1b_2b_3)$.
2. The left triangle axiom: $\Theta^2(b, 1_B) = LUn(\Theta^1(b)) \circ (Id_{\Theta^1(b)} \diamond (\Theta^2_*)^{-1})$ as maps $\Theta^1(b) \diamond \Theta^1(1_B) \Rightarrow \Theta^1(b)$.
3. The right triangle axiom: $\Theta^2(1_B, b) = RUn(\Theta^1(b)) \circ ((\Theta^2_*)^{-1} \diamond Id_{\Theta^1(b)})$ as maps $\Theta^1(1_B) \diamond \Theta^1(b) \Rightarrow \Theta^1(b)$.
4. The naturality condition: $\Theta^2(\partial(a_1)b_1, \partial(a_2)b_2) \circ (\Theta^1(b_1, a_1) \diamond \Theta^1(b_2, a_2)) = \Theta^1(b_1b_2, a_1{}^{b_1}a_2) \circ \Theta^2(b_1, b_2)$ as maps $\Theta^1(b_1) \circ \Theta^1(b_2) \Rightarrow \Theta^1(\partial(a_1)b_1\partial(a_2)b_2)$.

We call the 2-representation Θ unital if Θ^2_* is an identity and strict if all $\Theta^2(b_1, b_2)$ and Θ^2_* are identities.

As $\Theta^1(b_1) \diamond \Theta^1(b_2)$ is 2-isomorphic to $\Theta^1(b_1b_2)$, we get a map $dim \circ \Theta^1$ to be a group homomorphism from B to S_n. Moreover, as we have 2-isomorphisms $\Theta^1(b,a) : \Theta^1(b) \Rightarrow \Theta^1(\delta(a)b)$ we get a permutation action of $\pi_1(\mathcal{K}) = B/\partial(A)$ on the finite set $\{1, 2, \ldots n\}$.

Definition 3.1.38. A homomorphism of 2-representations $\psi : \Theta \to \Theta'$ is a natural 2-transformation of 2-functors. An equivalence of 2-representations is a natural 2-isomorphism of 2-functors.

Definition 3.1.39. Let $2\text{-Rep}^n(\mathcal{G})$ be the class of equivalence classes of 2-representations of 2-group $\mathcal{G}$ of degree n.

For a 2-group $\tilde{\mathcal{K}}$ coming from a crossed module $\mathcal{K}$ we write $2\text{-Rep}^n(\mathcal{K})$ for $2\text{-Rep}^n(\tilde{\mathcal{K}})$. To understand these sets better, it is useful to be able to break down representations into simpler forms.

Definition 3.1.40. For two 2-representations $\Theta_1, \Theta_2 : \mathcal{G} \to 2 - \mathrm{Vect}^{\mathbb{K}}$ we define the direct product $\Theta_1 \boxplus \Theta_2 : \mathcal{G} \to 2 - \mathrm{Vect}^{\mathbb{K}}$ similarly to classical representations.

- If $\Theta_1(*) = [n]$ and $\Theta_2(*) = [m]$ then $\Theta_1 \boxplus \Theta_2(*) = [n+m]$.
- Where $\Theta_1 \boxplus \Theta_2(* \xrightarrow{b} *)$ is the block sum of $\Theta^1(* \xrightarrow{b} *)$ and $\Theta^2(* \xrightarrow{b} *)$.
- With $\Theta_1 \boxplus \Theta_2(b_1 \overset{a}{\Rightarrow} b_2)$ is the block sum of $\Theta^1(b_1 \overset{a}{\Rightarrow} b_2)$ and $\Theta^2(b_1 \overset{a}{\Rightarrow} b_2)$.

Then a 2-representation Θ is irreducible if it is not equivalent to a direct product of 2 other representations. For a 2-representation given by a crossed module $\mathcal{K}$ being irreducible is equivalent to the action of $\pi_1(\mathcal{K}) = B/\partial(A)$ on $\{1, 2, \dots, n\}$ being transitive. This is because of our choice of set up, you can see we have effectively assumed a strong version of the Artin–Wedderburn theorem where our representations must break down into 'matrix' rings.

Remark. In this setting we only require equivalence to a direct product, instead of an isomorphism. This is because of the rigidity of our set up, and can differ in other set ups.

Definition/Lemma 3.1.41. For two 2-representations $\Theta_1, \Theta_2 : \mathcal{G} \to 2 - \mathrm{Vect}^{\mathbb{K}}$ we define the 2-tensor product $\Theta_1 \boxtimes \Theta_2 : \mathcal{G} \to 2 - \mathrm{Vect}^{\mathbb{K}}$.

- If $\Theta_1(*) = [n]$ and $\Theta_2(*) = [m]$ then $\Theta_1 \boxtimes \Theta_2(*) = [nm]$. For convenience we set the coordinates to $\{(a, b) | 1 \leq a \leq n,\ 1 \leq b \leq m\}$.
- Where
$$\Theta_1 \boxtimes \Theta_2(* \xrightarrow{b} *)_{(i,j),(i',j')} = \Theta_1(* \xrightarrow{b} *)_{i,i'} \otimes \Theta_2(* \xrightarrow{b} *)_{j,j'}.$$
- With
$$\Theta_1 \boxtimes \Theta_2(b_1 \overset{a}{\Rightarrow} b_2)_{(i,j),(i',j')} = \Theta_1(b_1 \overset{a}{\Rightarrow} b_2)_{i,i'} \otimes \Theta_2(b_1 \overset{a}{\Rightarrow} b_2)_{j,j'}$$

This construction satisfies the universality properties you would expect.

For a 2-group $\mathcal{G}$ we define a sub 2-group to be a subcategory $\mathcal{H} \leq \mathcal{G}$ where the monoidal structure is closed, $h_1 \otimes h_2 \in \mathcal{H}$, and the monoidal identity is in $\mathcal{H}$, $1_{\mathcal{G}} \in \mathcal{H}$. Therefore for sub-2-groups we get 2-representations of $\mathcal{H}$ via restriction. So restriction is a functor $\mathrm{Res} : 2-\mathrm{Rep}(\mathcal{G}) \to 2-\mathrm{Rep}(\mathcal{H})$. However, we can also go in the other direction with induction. We will give the restricted form of induction for 2-groups arising from crossed modules as presented in [48], however a more general form can be found in a paper of Rumynin and Young [49]. Rumynin and Young show that this form of induction is a biadjoint to restriction under certain assumptions however conjecture it in generality [49, Proposition 3.5].

Definition 3.1.42. Let $\mathcal{K}' = (A' \xrightarrow{\partial} B')$ be a sub-crossed module of $\mathcal{K} = (A \xrightarrow{\partial} B)$ such that $A' = A$ and $m := |B' : B| < \infty$. Then for a 2-representation Θ of $\mathcal{K}'$ we define the induced 2-representation $\Theta \uparrow_{\mathcal{K}'}^{\mathcal{K}} =: \Phi$ where,

- If $\Theta^0(*) = n$ then $\Phi^0(*) = nm$. For convenience later pick a transversal $T = \{t_1, t_2, \ldots, t_m\}$ then define a bijection $\{1, 2, \ldots, nm\} \to \{(i,t) | 1 \leq i \leq n,\ t \in T\}$ and use (i,t) as the coordinates of the matrices.
- Then set
$$\Phi^1(b) := V_{(i,t),(i',t')} = \begin{cases} (\Theta^1(b'))_{i,i'} & \text{if } bt = t'b' \text{ with } b' \in B' \\ 1 & \text{otherwise} \end{cases}.$$
- Similarly define
$$\Phi^1(b,a) := \psi_{(i,t),(i',t')} = \begin{cases} (\Theta^1(b', {}^{(t')^{-1}}a))_{i,i'} & \text{if } bt = t'b' \text{ with } b' \in B' \\ 1 & \text{otherwise} \end{cases}.$$

  Note that for $a_1, a_2 \in A$ and $b \in B$ we have that if $bt = t'b'$ with $b' \in B'$ then
$$\partial(a_1)bt = \partial(a_1)t'b' = t'((t')^{-1}\partial(a_1)t')b' = t'\partial({}^{(t')^{-1}}a_1)b'.$$

  So the vertical multiplicative rule $\Phi^1(\delta(a_1)b, a_2)\Phi^1(b, a_1) = \Phi^1(b, a_2a_1)$ is resolved by the multiplicative rule for Θ mainly that $\Theta^1(\delta({}^{(t')^{-1}}a_1)b', {}^{(t')^{-1}}a_2)\Theta^1(b', {}^{(t')^{-1}}a_1)$ $= \Theta^1(b', {}^{(t')^{-1}}(a_2a_1))$.
- With 2-isomorphism
$$\Phi^2_* := \theta_{(i,t),(i',t')} = \begin{cases} (\Theta^2_*)_{i,i'} & \text{if } t = t' \\ id_1 & \text{otherwise} \end{cases}.$$

- With 2-isomorphisms

$$\Phi^2(b_1, b_2) := \psi_{(i,t),(i',t')} = \begin{cases} (\Theta^2(b'_2, b'_1))_{i,i'} & \text{if } b_2 b_1 t = b_2 \tilde{t} b'_1 = t' b'_2 b'_1 \\ id_1 & \text{otherwise} \end{cases}.$$

Which allows us to understand the irreducible representations of a 2-group arising from a crossed module as induced 1-dimensional representations of sub-crossed modules.

Theorem 3.1.43. *(Rumynin and Wendland, [48]) Let $\mathcal{K} = (A \xrightarrow{\partial} B)$ be a crossed module.*

1. *If Θ is a 2-representation of $\tilde{\mathcal{K}}$, then there exist irreducible 2-representations $\Theta_1, \ldots, \Theta_n$ such that $\Theta \cong \Theta_1 \oplus \ldots \oplus \Theta_n$.*
2. *The 2-representations $\Theta_1, \ldots, \Theta_n$ of $\tilde{\mathcal{K}}$ are unique up to permutation and equivalence.*
3. *If Φ is an irreducible 2-representation of $\tilde{\mathcal{K}}$, then there exists a subgroup of finite*
   of $\tilde{\mathcal{K}'}$ where $\mathcal{K}' = (A \xrightarrow{\partial} B')$ is a crossed submodule such that $\Phi \cong \Theta \uparrow_{\tilde{\mathcal{K}'}}^{\tilde{\mathcal{K}}}$.
4. *The pair (B', Θ) is unique up to conjugation by an element of B.*

This is proven using module categories, which we do not explore in this thesis. However, intuitively you can find the irreducible representations by dividing the set $\{1, \ldots, n\}$ into orbits under the action of $\pi_1(\mathcal{K}) = B/\partial(A)$. Then as these actions are transitive we know they come from coset actions. Note that Theorem 3.1.43 is a generalisation of the decomposition results in 1-representation theory without any restrictions on the field and only assuming $\pi_1(\mathcal{G})$ is finite. Equally it provides a strong version of the induction theory's in 1-representation theory where this applies to representations rather than characters.

3.2 Maclane Strictification

MacLane Strictification gives us a way to go from any 2-group to an equivalent strict 2-group. Here we pay particular attention to the method of proof (we follow [12, Theorem 1.8.5] for the proof), as we will explicate this later, on our skeletal 2-groups.

Theorem 3.2.1. *[30] Any monoidal category $\mathcal{C}$ is equivalent to a strict monoidal category.*

Proof. [12, Theorem 1.8.5] Given a monoidal category $\mathcal{C}$ we describe $\mathcal{RE}(\mathcal{C})$. Where the objects of $\mathcal{RE}(\mathcal{C})$ are functors F of the underlying category $\mathcal{C}$ (note: not monoidal functors) with functorial isomorphisms $\zeta(X,Y) : F(X) \otimes Y \xrightarrow{\sim} F(X \otimes Y)$ (this is the map $c_{X,Y}$ in the notes we are following). Such that the following diagram commutes.

$$\begin{tikzcd}
& (F(X) \otimes Y) \otimes Z \arrow[dl, "\zeta(X,Y)\otimes Id_Z"'] \arrow[dr, "\mathrm{Ass}(F(X),Y,Z)"] & \\
F(X \otimes Y) \otimes Z \arrow[d, "\zeta(XY,Z)"] & & F(X) \otimes (Y \otimes Z) \arrow[d, "\zeta(X,YZ)"] \\
F((X \otimes Y) \otimes Z) \arrow[rr, "F(\mathrm{Ass}(X,Y,Z))"] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Given two objects (F^1, ζ^1) and (F^2, ζ^2) we have morphisms between the two given by natural transformations $\Phi : F \Rightarrow F'$ such that the follow diagram holds.

$$\begin{tikzcd}
F^1(X) \otimes Y \arrow[r, "\zeta^1(X,Y)"] \arrow[d, "\Phi(X)\otimes Id_Y"] & F^1(X \otimes Y) \arrow[d, "\Phi(X\otimes Y)"] \\
F^2(X) \otimes Y \arrow[r, "\zeta^2(X,Y)"] & F^2(X \otimes Y)
\end{tikzcd}$$

Where composition of morphisms is just vertical composition of natural transformations. This becomes a monoidal category where the tensor product of two objects is given by $(F^1, \zeta^1) \otimes (F^2, \zeta^2) = (F^1F^2, \zeta)$ where ζ is given by the following composition

$$F^1F^2(X) \otimes Y \xrightarrow{\zeta^1(F^2(X),Y)} F^1(F^2(X) \otimes Y) \xrightarrow{F^1(\zeta^2(X,Y))} F^1F^2(X \otimes Y).$$

The tensor product of morphisms is horizontal composition of natural transformations or the Godement product. Which we remind the reader, for $\Phi^1 : F^1 \Rightarrow G^1$ and $\Phi^2 : F^2 \Rightarrow G^2$ is given by the following composition, which can be defined in two equivalent ways.

$$\begin{tikzcd}
F^1F^2(X) \arrow[r, "F^1(\Phi^2(X))"] \arrow[d, "\Phi^1(F^2(X))"'] \arrow[dr, "(\Phi^1\otimes\Phi^2)(X)"] & F^1(G^2(X)) \arrow[d, "\Phi^1(G^2(X))"] \\
G^1F^2(X) \arrow[r, "G^1(\Phi^2(X))"'] & G^1G^2(X)
\end{tikzcd}$$

Let

$$\begin{aligned}
&(F^1F^2, \tilde{\zeta}) := (F^1, \zeta^1) \otimes (F^2, \zeta^2), && (F^1F^2F^3, \tilde{\tilde{\zeta}}) := (F^1F^2, \tilde{\zeta}) \otimes (F^3, \zeta^3), \\
&(F^2F^3, \hat{\zeta}) := (F^2, \zeta^2) \otimes (F^3, \zeta^3), \quad \text{and} && (F^1F^2F^3, \hat{\hat{\zeta}}) := (F^1, c^1) \otimes (F^2F^3, \hat{\zeta})
\end{aligned}$$

then

$$\begin{aligned}
\hat{\hat{\zeta}}(X,Y) &= F^1(\hat{\zeta}(X,Y)) \circ \zeta^1(F^2F^3(X),Y) \\
&= F^1(F^2(\zeta^3(X,Y)) \circ \zeta^2(F^3(X),Y)) \circ \zeta^1(F^2F^3(X),Y) \\
&= F^1F^2(\zeta^3(X,Y)) \circ (F^1(\zeta^2(F^3(X),Y)) \circ \zeta^1(F^2F^3(X),Y)) \\
&= F^1F^2(\zeta^3(X,Y)) \circ \tilde{\zeta}(F^3(X),Y) \\
&= \tilde{\tilde{\zeta}}(X,Y)
\end{aligned}$$

giving

$$((F^1,\zeta^1)\otimes(F^2,\zeta^2))\otimes(F^3,\zeta^3) = (F^1,\zeta^1)\otimes((F^2,\zeta^2)\otimes(F^3,\zeta^3)).$$

To make $\mathcal{RE}(\mathcal{C})$ strict we set the associativity and identity morphisms to be trivial. This means that the pentagon and triangle axioms hold trivially.

Then map $\mathcal{F} : \mathcal{C} \to \mathcal{RE}(\mathcal{C})$ is defined by the following: on $X \in Ob(\mathcal{C})$ we have $\mathcal{F} : X \mapsto (X\otimes -, 1_X \otimes -, ass_{X,-,-})$ where ass is the associator, and on $f \in \mathrm{Mor}(X,Y)$ we map $\mathcal{F} : f \mapsto f\otimes -$ by which we mean the natural transformation $\Phi(Z) = f\otimes 1_Z \in \mathrm{Mor}(X\otimes Z, Y\otimes Z)$. This becomes a monoidal functor where $\mathcal{F}_{X,Y} : \mathcal{F}(X)\otimes\mathcal{F}(Y) \to \mathcal{F}(X\otimes Y)$ is given by the natural transformation $ass_{X,Y,-}$.

□

Next we use this proof to go from a skeletal 2-group to a strict 2-group.

Example 3.2.2. Suppose we have a special skeletal coherent 2-group $\mathcal{G}$ given by (G,H,α). Then take an object in $\mathcal{RE}(\mathcal{G})$ which is a functor $F : \mathcal{G} \to \mathcal{G}$ and a set of maps $\zeta(x,y) : F(x)\otimes y \xrightarrow{\sim} F(x\otimes y)$. Note that as $\mathcal{G}$ is skeletal the existence of such $\zeta(y,z)$ tells us that $F(y)\otimes z = F(y\otimes z)$ therefore $F_G := F(1_G) \in G$. This full dictates what F does to objects in $\mathcal{G}$ as $F(x) = F(1_G x) = F(1_G\otimes x) = F(1_G)\otimes x = F_G x$. We know F also permutes morphisms such that it preserves the identity and composition, so for each $x \in G$ we get an automorphism $F_x \in \mathrm{Aut}_{Grp}(H)$, where $a_x \in \mathrm{Mor}_{\mathcal{G}}(x,x)$ gets mapped to $(F_x(a))_{F_G x} \in \mathrm{Mor}_{\mathcal{G}}(F(x),F(x))$. Lastly $\zeta : G\times G \to H$ satisfies $F_{xyz}(\alpha(x,y,z)) + \zeta(xy,z) + \zeta(x,y) = \alpha(F_G x,y,z) + \zeta(x,yz)$ as the following diagram commutes.

$$\begin{tikzcd}[column sep=small]
 & ((F_G\otimes x)\otimes y)\otimes z \arrow[dl, "\zeta(x,y)\otimes 1_z"'] \arrow[dr, "\alpha(F_G x,y,z)"] & \\
(F_G\otimes(x\otimes y))\otimes z \arrow[d, "\zeta(xy,z)"] & & (F_G\otimes x)\otimes(y\otimes z) \arrow[d, "\zeta(x,yz)"] \\
F_G\otimes((x\otimes y)\otimes z) \arrow[rr, "F_{xyz}(\alpha(x,y,z))"] & & F_G\otimes(x\otimes(y\otimes z))
\end{tikzcd}$$

Suppose we have $\Phi \in \mathrm{Mor}_{\mathcal{RE}(\mathcal{G})}((F^1,\zeta^1),(F^2,\zeta^2))$, so $\Phi : F^1 \Rightarrow F^2$. We know Φ consists of morphisms $\Phi(x) : F^1(x) \xrightarrow{\sim} F^2(x)$, so we think of $\Phi \in \mathrm{Mor}_{Set}(G,H)$. For such a $\Phi(x)$ to exists we need that $F^1(1_G)x = F^2(1_G)x$ giving $F^1_G = F^2_G$. We also need that the following diagram commutes for each $a \in H$ and $x \in G$

$$\begin{tikzcd} F^1(x) \arrow[r, "F^1(a_x)"] \arrow[d, "\Phi(x)"] & F^1(x) \arrow[d, "\Phi(x)"] \\ F^2(x) \arrow[r, "F^2(a_x)"] & F^2(x) \end{tikzcd}$$

which gives us that $\Phi(x) + F^1_x(a) = F^2_x(a) + \Phi(x)$ however as we are in H which is an abelian group we have that $F^1_x = F^2_x$ therefore $F^1 = F^2 =: F$.

For Φ to be a morphism in $\mathcal{RE}(\mathcal{G})$ we need the following diagram to commute

$$\begin{tikzcd} (F_G x)y \arrow[r, "\zeta^1_{x,y}"] \arrow[d, "\Phi(x)\otimes 1_Y"] & F_G(xy) \arrow[d, "\Phi(xy)"] \\ (F_G x)y \arrow[r, "\zeta^2_{x,y}"] & F_G(xy) \end{tikzcd}$$

which gives us the condition $\Phi(xy) + \zeta^1(x,y) = \zeta^2(x,y) + \Phi(x)$.

Now consider $\mathcal{F} : \mathcal{G} \to \mathcal{RE}(\mathcal{G})$ the embedding of $\mathcal{G}$ into $\mathcal{RE}(\mathcal{G})$. Then let $X \in G$ and set $\mathcal{F}(X) =: (F^X, \zeta^X)$ to be an object in $\mathcal{RE}(\mathcal{G})$. From definition $F^X(y) = X \otimes y = Xy$ and $F^X(a_y) = 1_X \otimes a_y = ({}^X a)_{Xy}$ giving $F^X_G = X$ and $F^X_y(a) = {}^X a$. Then $\zeta^X(y,z) = \alpha(X,y,z)$ and

$${}^X\alpha(y,z,w) + \zeta^X(yz,w) + \zeta^X(y,z) = \alpha(Xy,z,w) + \zeta^X(y,wz)$$

holds as it is just the cocycle condition

$${}^X\alpha(y,z,w) + \alpha(X,yz,w) + \alpha(X,y,z) = \alpha(Xy,z,w) + \alpha(X,y,zw).$$

With $\mathcal{F}(A_X) =: \Phi^{A_X}$ we get $\Phi^{A_X}(y) = A_X \otimes 1_y = (A)_{Xy}$ for all $y \in G$.

This is an equivalence of categories, all objects are isomorphic to something in the image. So for a generic element (F,ζ) there exists an $X \in G$ such that $\theta \in \mathrm{Mor}_{\mathcal{RE}(\mathcal{C})}((F^X, \alpha(X,-,-)),(F,\zeta))$ giving $F^X = F$ and $\zeta(y,z) = \alpha(X,y,z) + \theta(yz) -$

$\theta(y)$. Note that for any $a \in H$ we have that $\tilde{\theta}(y) = \theta(y) + a$ gives the same ζ.

$$\begin{aligned} Ob(\mathcal{RE}(\mathcal{C})) &= \{(F^X, \zeta) \mid \exists\, \theta \in \mathrm{Mor}_{Set}(G, H) \text{ with } \zeta(y,z) = \alpha(X,y,z) + \theta(yz) - \theta(y)\} \\ &= G \times (\mathrm{Mor}_{Set}(G,H)/(\theta \sim \theta + a \mid a \in H)) \\ &= G \times \mathrm{Mor}_{Set}(G,H)/H. \end{aligned}$$

Note that any $\theta \in \mathrm{Mor}_{Set}(G, H)$ gives rise to a ζ obeying

$$^X\alpha(y,z,w) + \zeta(yz,w) + \zeta(y,z) = \alpha(Xy,z,w) + \zeta(y,zw)$$

as

$$\begin{aligned} \alpha(Xy,z,w) + \zeta(y,zw) &= \alpha(Xy,z,w) + \alpha(X,y,zw) - \theta(y) + \theta(yzw) \\ &= {}^X\alpha(y,z,w) + \alpha(X,yz,w) + \alpha(X,y,z) - \theta(y) + \theta(yzw) \\ &= {}^X\alpha(y,z,w) + \alpha(X,yz,w) - \theta(yz) + \theta(yzw) \\ &\qquad + \alpha(X,y,z) - \theta(y) + \theta(yz) \\ &= {}^X\alpha(y,z,w) + \zeta(yz,w) + \zeta(y,z). \end{aligned}$$

Let $(X, [\theta^1]) \otimes (Y, [\theta^2]) =: (XY, [\theta])$. Then let $\zeta^1, \zeta^2, \zeta : G \times G \to H$ be the functions θ^1, θ^2 and θ represent. Then composition above gives

$$\begin{aligned} \zeta(z,w) &= {}^X\zeta^2(z,w) + \zeta^1(Yz,w) \\ &= {}^X\alpha(Y,z,w) + \alpha(X,Yz,w) + {}^X\theta^2(zw) \\ &\quad + \theta^1(Yzw) - {}^X\theta^2(z) - \theta^1(Yz) \\ &= \alpha(XY,z,w) + \alpha(X,Y,zw) + {}^X\theta^2(zw) \qquad\qquad \text{by the cocycle condition} \\ &\quad + \theta^1(Yzw) - \alpha(X,Y,z) - {}^X\theta^2(z) - \theta^1(Yz) \end{aligned}$$

making

$$\theta(z) = \alpha(X,Y,z) + {}^X\theta^2(z) + \theta^1(Yz).$$

Let $\mathrm{Mor}((X,[\theta^1]),(X,[\theta^2])) \ni \Phi : G \to H$, with $\zeta^1, \zeta^2 : G \times G \to H$ being the functions θ^1 and θ^2 represent. Choose representatives $\theta^1(1_G) = \theta^2(1_G) = 0_H$. Then we know

$$\begin{aligned} \Phi(yz) - \Phi(y) &= \zeta^2(y,z) - \zeta^1(y,z) \\ &= \theta^2(yz) + \theta^1(y) - \theta^1(yz) - \theta^2(y) \end{aligned}$$

which when substituting $y = 1_G$ we get $\Phi(z) - \Phi(1_G) = \theta^2(z) - \theta^1(z)$ (which without the choice of representative could be summerised as $\Phi(z) = \Phi(1_G) + (\theta^2(z) - \theta^2(1_G)) -$

$(\theta^1(z) - \theta^1(1_G)))$ which up to choice of $\Phi(1_G) \in H$ uniquely determines Φ. This gives

$$\begin{aligned}\mathrm{Mor}((X,\theta^1),(X,\theta^2)) = \{\Phi_a : G \to H|\ \Phi_a(z) = a + (\theta^2(z) - \theta^2(1_G)) \\ - (\theta^1(z) - \theta^1(1_G)) \text{ for } z \in G\} \\ = H.\end{aligned}$$

Given $\Phi_a \in \mathrm{Mor}((X,[\theta^1]),(X,[\theta^2]))$ and $\Phi_b \in \mathrm{Mor}((g,[\theta^2]),(g,[\theta^3]))$ then

$$\begin{aligned}\Phi_a \circ \Phi_b(X) =& a + (\theta^2(X) - \theta^2(1_G)) - (\theta^1(X) - \theta^1(1_G)) \\ &+ b + (\theta^3(X) - \theta^3(1_G)) - (\theta^2(X) - \theta^2(1_G)) \\ =& (a+b) + (\theta^3(X) - \theta^3(1_G)) - (\theta^1(X) - \theta^1(1_G))\end{aligned}$$

meaning that $\Phi_a \circ \Phi_b = \Phi_{a+b} \in \mathrm{Mor}((g,[\theta^1]),(g,[\theta^3]))$.

Note that for $\Phi_a \in \mathrm{Mor}((X,[\theta^1]),(X,[\theta^2]))$ and $\Phi_b \in \mathrm{Mor}((Y,[\theta^*]),(Y,[\theta^{**}]))$ define $\Phi_a \otimes \Phi_b =: \Phi_c \in \mathrm{Mor}((XY,[\theta']),(XY,[\theta'']))$. For clarity of calculations pick all representatives so that 1_G is mapped to 0_H. So we have that

$$\begin{aligned}\Phi_c(z) &= c + \theta''(z) - \theta'(z) \\ &= c + \alpha(X,Y,z) + {}^X\theta^{**}(z) + \theta^2(Yz) \\ &\quad - \alpha(X,Y,z) - {}^X\theta^*(z) - \theta^1(Yz) \\ &= c + {}^X\theta^{**}(z) + \theta^2(Yz) - {}^X\theta^*(z) - \theta^1(Yz).\end{aligned}$$

By examining the Godement product we have that

$$\begin{aligned}\Phi_a \otimes \Phi_b(z) &= {}^X\Phi_b(z) + \Phi_a(Yz) \\ &= {}^Xb + {}^X\theta^{**}(z) - {}^X\theta^*(z) + a + \theta^2(Yz) - \theta^1(Yz) \\ &= (a + {}^Xb) + {}^X\theta^{**}(z) + \theta^2(Yz) - {}^X\theta^*(z) - \theta^1(Yz),\end{aligned}$$

making $c = a + {}^Xb$.

We have in summary the following data.

- The objects in $\mathcal{C}$ are the following

$$Ob(\mathcal{C}) = G \times \mathrm{Mor}_{Set}(G,H)/H.$$

- Tensor products on objects are defined by $(X,[\theta^1]) \otimes (Y,[\theta^2]) = (XY,[\theta])$ where

$$\theta(z) = \alpha(X,Y,z) + {}^X\theta^2(z) + \theta^1(Yz).$$

- The morphisms are

$$\mathrm{Mor}((X,[\theta^1]),(X,[\theta^2])) = \left\{ \Phi_a : G \to H \;\middle|\; \Phi_a(y) = \begin{array}{c} a + (\theta^2(y) - \theta^2(1_G)) \\ -(\theta^1(y) - \theta^1(1_G)) \end{array} \right\} = H.$$

- Given $\Phi_b \in \mathrm{Mor}((X,[\theta^1]),(X,[\theta^2]))$ and $\Phi_a \in \mathrm{Mor}((X,[\theta^2]),(X,[\theta^3]))$ then $\Phi_a \circ \Phi_b = \Phi_{a+b} \in \mathrm{Mor}((X,[\theta^1]),(X,[\theta^3]))$.

- Where if $\Phi_a \in \mathrm{Mor}((X,[\theta^1]),(X,[\theta^2]))$ and $\Phi_b \in \mathrm{Mor}((Y,[\theta']),(Y,[\theta'']))$ then $\Phi_a \otimes \Phi_b = \Phi_{a+{}^X b}$.

Note if we let

$$\begin{array}{ll} (XY,[\theta^*]) := (X,[\theta^1]) \otimes (Y,[\theta^2]), & (XYZ,[\theta^{**}]) := (XY,[\theta^*]) \otimes (Z,[\theta^3]), \\ (YZ,[\theta^\triangle]) := (Y,[\theta^2]) \otimes (Z,[\theta^3]), \quad \text{and} & (XYZ,[\theta^{\triangle\triangle}]) := (X,[\theta^1]) \otimes (YZ,[\theta^\triangle]) \end{array}$$

then

$$\begin{aligned} \theta^{**}(w) &= \alpha(XY,Z,w) + {}^{XY}\theta^3(w) + \theta^*(Zw) \\ &= \alpha(XY,Z,w) + {}^{XY}\theta^3(w) + \alpha(X,Y,Zw) + {}^X\theta^2(Zw) + \theta^1(YZw) \\ &= \alpha(X,Y,Z) + \alpha(X,YZ,w) + {}^X\alpha(Y,Z,w) + {}^{XY}\theta^3(w) + {}^X\theta^2(Zw) + \theta_1(YZw) \\ &= \alpha(X,Y,Z) + \alpha(X,YZ,w) + {}^X\theta^\triangle(w) + \theta_1(YZw) \\ &= \alpha(X,Y,Z) + \theta^{\triangle\triangle}(w) \end{aligned}$$

which gives $[\theta^{**}] = [\alpha(X,Y,Z) + \theta^{\triangle\triangle}] = [\theta^{\triangle\triangle}]$ making $Ob(\mathcal{C})$ associative under the $\otimes$ operation. Further more if we let $\theta^{const}(g) = 0_H$ be the constant function then set

$$(X,[\theta^1]) := (1_G,[\theta^{const}]) \otimes (X,[\theta]) \quad \text{and} \quad (X,[\theta^2]) := (X,[\theta]) \otimes (1_G,[\theta^{const}])$$

then

$$\begin{aligned} \theta^1(y) &= \alpha(1_G,X,y) + {}^{1_G}\theta(y) + \theta^{const}(Xy) \\ &= \theta(y) \\ \theta^2(y) &= \alpha(X,1_G,y) + {}^X\theta^{const}(y) + \theta(1_G y) \\ &= \theta(y) \end{aligned}$$

giving $[\theta^1] = [\theta^2] = [\theta]$ making $(1_G,[\theta^{const}])$ an identity element under $\otimes$. Lastly for any $(X,[\theta]) \in Ob(\mathcal{RE}(\mathcal{C}))$ set

$$-\bar{\theta}(z) = {}^{X^{-1}}(\alpha(X,X^{-1},z) + \theta(X^{-1}z))$$

then if

$$(1_G, [\theta^1]) := (X, [\theta]) \otimes (X^{-1}, [\bar{\theta}]) \text{ and } (1_G, [\theta^2]) := (X^{-1}, [\bar{\theta}]) \otimes (X, [\theta])$$

we have

$$\begin{aligned}
\theta^1(z) &= \alpha(X, X^{-1}, z) + {}^X\bar{\theta}(z) + \theta(X^{-1}z) \\
&= \alpha(X, X^{-1}, z) - (\alpha(X, X^{-1}, z) + \theta(X^{-1}z)) + \theta(X^{-1}z) \\
&= 0_H \\
\theta^2(z) &= \alpha(X^{-1}, X, z) + {}^{X^{-1}}\theta(z) + \bar{\theta}(Xz) \\
&= \alpha(X^{-1}, X, z) + {}^{X^{-1}}\theta(z) - {}^{X^{-1}}(\alpha(X, X^{-1}, Xz) + \theta(X^{-1}Xz)) \\
&= \alpha(X^{-1}, X, X^{-1}(Xz)) - {}^{X^{-1}}\alpha(X, X^{-1}, Xz) \\
&= \alpha(1_G, X^{-1}, Xz) - \alpha(X^{-1}, 1_G, Xz) - \alpha(X^{-1}, X, X^{-1}) \\
&= -\alpha(X^{-1}, X, X^{-1})
\end{aligned}$$

giving that $[\theta^2] = [\theta^{const} - \alpha(X^{-1}, X, X^{-1})] = [\theta^{const}] = [\theta^1]$ making $(X^{-1}, [\bar{\theta}]) = (X, [\theta])^{-1}$. Therefore $Ob(\mathcal{RE}(\mathcal{C}))$ is a group under the operation of $\otimes$.

Combining this with the proof of Proposition 3.1.35 we get the following.

Theorem 3.2.3. *A skeletal 2-group given by (G, H, α) is equivalent to crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ given by:*

- *group $B = G \times Mor_{Set}(G, H)/H$ where $(X, [\theta^1]) \otimes (Y, [\theta^2]) = (XY, [\theta])$ with*

$$\theta(z) = \alpha(X, Y, z) + {}^X\theta^2(z) + \theta^1(Yz),$$

- *group $A = Mor_{Set}(G, H)/H \times H$ where $([\theta_1], a) \otimes ([\theta_2], b) = ([\theta_1 + \theta_2], a + b)$,*
- *map $\partial : A \to B$ where $\partial([\Phi], h) = (1_G, [\Phi])$, and*
- *group action $B \hookrightarrow A$ given by $(X, [\theta]) : ([\Phi], h) \mapsto ([{}^X\Phi(X^{-1}-)], {}^Xh)$.*

Proof. Following on from the proof of Proposition 3.1.35 and Example 3.2.2, let $\mathcal{C}$ be the category constructed from (G, H, α). We have that $B := Ob(\mathcal{C}) = G \times \mathrm{Mor}_{Set}(G, H)/H$ where $(X, [\theta^1]) \otimes (Y, [\theta^2]) = (XY, [\theta])$ with

$$\theta(z) = \alpha(X, Y, z) + {}^X\theta^2(z) + \theta^1(Yz)$$

which we have shown forms a group above.

We have that

$$
\begin{aligned}
A := & \prod_{(x,[\theta])\in B} \mathrm{Mor}((1_G,[\theta^{const}]),(x,[\theta])) \\
= & \prod_{[\theta]\in \mathrm{Mor}_{Set}(G,H)/H} \mathrm{Mor}((1_G,[\theta^{const}]),(1_G,[\theta])) \\
= & \mathrm{Mor}_{Set}(G,H)/H \times H
\end{aligned}
$$

Let $(1_G,[\theta_1])\otimes(1_G,[\theta_2]) =: (1_G,[\theta])$, then

$$\theta(X) = \alpha(1_G,1_G,X) + {}^{1_G}\theta^2(X) + \theta^1(1_G X) = \theta^1(X) + \theta^2(X).$$

Let $\Phi_a \in \mathrm{Mor}((1_G,[\theta^{const}]),(1_G,[\theta_1]))$, and $\Phi_b \in \mathrm{Mor}((1_G,[\theta^{const}]),(1_G,[\theta_2]))$ then $\Phi_a\otimes\Phi_b = \Phi_{a+{}^{1_G}b} = \Phi_{a+b}$. Giving $([\theta_1],a)\otimes([\theta_2],b) = ([\theta_1+\theta_2],a+b)$.

With the map $\partial : A \to B$ being $\partial([\Phi],h) = (1_G,[\Phi])$.

The group action $* : B \hookrightarrow A$ is given by

$$
\begin{aligned}
(X,[\theta]) : ([\Phi],h) \mapsto & \; 1_{(X,[\theta])}\otimes([\Phi],h)\otimes 1^{-1}_{(X,[\theta])} \\
& = 1_{(X,[\theta])}\otimes([\Phi],h)\otimes 1_{(X^{-1},[\bar{\theta}])} \\
& := ([\Phi'],\Phi_{0_H}\otimes\Phi_h\otimes\Phi_{0_H})
\end{aligned}
$$

Then to calculate Φ' is equivalent to calculating $(X,[\theta])\otimes(1_G,[\Phi])\otimes(X^{-1},[\bar{\theta}]) =: (1_G,[\Phi'])$. So from the calculations of associativity of A we get that

$$
\begin{aligned}
\Phi'(y) &= \alpha(X1_G,X^{-1},y) + {}^{X1_G}\bar{\theta}(y) + \alpha(X,1_G,X^{-1}y) + {}^X\Phi(X^{-1}y) + \theta(1_G X^{-1}y) \\
&= \alpha(X,X^{-1},y) + {}^X\bar{\theta}(y) + {}^X\Phi(X^{-1}y) + \theta(X^{-1}y) \\
&= \alpha(X,X^{-1},y) - (\alpha(X,X^{-1},y) + \theta(X^{-1}y)) + {}^X\Phi(X^{-1}y) + \theta(X^{-1}y) \\
&= {}^X\Phi(X^{-1}y)
\end{aligned}
$$

so we have that $(X,[\theta]) : ([\Phi],h) \mapsto ([{}^X\Phi(X^{-1}-)],{}^Xh)$. Note this is a group action as

$$
\begin{aligned}
(X,[\theta^1])\cdot((Y,[\theta^2])\cdot([\Phi],h)) &= (X,[\theta^1])\cdot([{}^Y\Phi(Y^{-1}-)],{}^Yh) \\
&= ([{}^{XY}\Phi(Y^{-1}X^{-1}-)],{}^{XY}h) \\
&= (XY,[\theta])\cdot([\Phi],h).
\end{aligned}
$$

□

Note here we recover, $\pi_1(A, B, \partial, \star) = coker(\partial) = G$ and $\pi_2(A, B, \partial, \star) = ker(\partial) = H$ as we would expect.

MacLane then later Joyal and Street describe how to obtain a crossed module $\mathcal{K}_\mathcal{G}$ from the data (G, H, α) [25, 31]. However these methods leave the data as infinite free groups. We recall the procedure below following Joyal and Street [25].

Example 3.2.4. Let $F \coloneqq F_{G\setminus\{1_G\}}$ be the free group on the set $G \setminus \{1_G\}$ and $R \coloneqq F_{(G\setminus\{1_G\})^2}$ be the free group on the set $G \times G \setminus \{(1_G, x), (x, 1_G) \mid x \in G\}$. For any elements $x, y \in G \setminus \{1_G\}$ write $\{x\}$ and $\{x, y\}$ for their corresponding words in F and R respectively. Let $\sigma : G \to F$ and $\tau : G^2 \to R$ be the inclusion functions (of sets) satisfying

$$\sigma(x) = \begin{cases} 1_F & \text{if } x = 1_G \\ \{x\} & \text{otherwise} \end{cases} \quad \text{and} \quad \tau(x, y) = \begin{cases} 1_R & \text{if } x = 1_G \text{ or } y = 1_G \\ \{x, y\} & \text{otherwise} \end{cases}$$

for all $x, y \in G$. We use σ and τ when we want to refer to the basis elements of F and R respectively but we are unsure if the argument of the function is the identity. The universal property of free groups gives rise to a surjective group homomorphism $\varphi : F \to G$, having the property $\varphi(\{x\}) = x$, for all $x \in G \setminus \{1_G\}$ (so $\varphi \circ \sigma$ is the identity). Again by the universal property of free groups we obtain a homomorphism

$$\psi : R \to F, \text{ such that } \psi(\{x, y\}) = \{x\}\{y\}\sigma(xy)^{-1}, \text{ for all } x, y \in G \setminus \{1_G\}.$$

Thus, we have a short exact sequence:

$$1 \to R \xrightarrow{\psi} F \xrightarrow{\varphi} G \to 1. \tag{3.3}$$

Recall H is the $\mathbb{Z}G$-module which is part of the data of the special 2-group $\mathcal{G}$. Let $\pi : H \to H \times R$ be the projection onto the first coordinate, i.e., $\pi : a \mapsto (a, 1_R)$, and let $\partial : H \times R \to R$ be the map on $H \times R$ induced by ψ, i.e., $\partial : (a, \{x, y\}) \to \psi(\{x, y\}) = \{x\}\{y\}\sigma(xy)^{-1}$. Then the short exact sequence above induces an exact sequence of the form:

$$1 \to H \xrightarrow{\pi} H \times R \xrightarrow{\partial} F \xrightarrow{\varphi} G \to 1. \tag{3.4}$$

For every $X \in G$ we may define an endomorphism $\eta_X : H \times R \to H \times R$, given by

$$\eta_X((a, \{y, z\})) = ({}^X a + \alpha(X, y, z), \tau(X, y)\tau(Xy, z)\tau(X, yz)^{-1}).$$

Note the following

$$\begin{aligned}
\eta_X\eta_Y(a,\{z,w\}) =&\eta_X({}^Y a+\alpha(Y,z,w),\tau(Y,z)\tau(Yz,w)\tau(Y,zw)^{-1})\\
=&({}^{XY}\cdot a+{}^X\alpha(Y,z,w)+\alpha(X,Y,z)+\alpha(X,Yz,w)-\alpha(X,Y,zw),\\
&\tau(X,Y)\tau(XY,z)\tau(X,Yz)^{-1}\tau(X,Yz)\tau(XYz,w)\\
&\tau(X,Yzw)^{-1}\tau(X,Yzw)\tau(XY,zw)^{-1}\tau(X,Y)^{-1})\\
=&(XY\cdot a+\alpha(XY,z,w),\tau(X,Y)\tau(XY,z)\tau(XYz,w)\\
&\tau(XY,zw)^{-1}\tau(X,Y)^{-1})\\
=&(0_A,\tau(X,Y))\eta_{XY}(a,\{z,w\})(0_A,\tau(X,Y)^{-1})
\end{aligned}$$

since α is a 3-cocycle. So we have that

$$\eta_x\eta_y = c(0_A,\tau(x,y))\eta_{xy},\ \text{for all } x,y\in G, \tag{3.5}$$

where $c(k)$ denotes the conjugation by k in $H\times R$. Equation (3.5) implies that η_x is not only an endomorphism, but in fact an automorphism of $H\times R$ [10, 9.4]. Thus, we obtain a group homomorphism

$$\eta : F \to \mathrm{Aut}(H\times R),\ \eta : \{x\}\mapsto \eta_x.$$

In particular, we have an action $\star : F\times(H\times R)\to H\times R$. Write ${}^{\{x\}}k := \{x\}\star k = \eta_x(k)$, $k\in H\times R$. Note the following

$$\begin{aligned}
{}^{\partial((a,\{x,y\}))}(b,\{z,w\}) =&{}^{\{x\}\{y\}\sigma(xy)^{-1}}(b,\{z,w\})\\
=&\eta_x\eta_y\eta_{xy}^{-1}(b,\{z,w\})\\
=&c(\{x,y\})(b,\{z,w\}) &&\text{from (3.5)}\\
=&(a,\{x,y\})(a',\{z,w\})(a,\{x,y\})^{-1} &&\text{as } A \text{ is abelian}
\end{aligned}$$

and

$$\begin{aligned}
\partial({}^{\{x\}}(a,\{y,z\})) =&\partial(x\cdot a+\alpha(x,y,z),\tau(x,y)\tau(xy,z)\tau(x,yz)^{-1})\\
=&\{x\}\{y\}\sigma(xy)^{-1}\sigma(xy)\{z\}\sigma(xyz)^{-1}\sigma(xyz)\sigma(yz)^{-1}\{x\}^{-1}\\
=&\{x\}\{y\}\{z\}\sigma(yz)^{-1}\{x\}^{-1}\\
=&\{x\}\partial(\{y,z\})\{x\}^{-1}
\end{aligned}$$

Therefore this action satisfies conditions (CM1) and (CM2) as in Definition 3.1.25 and thus $\mathcal{K}_{\mathcal{G}} = (H\times R, F, \partial, \star)$ is a crossed module. This gives rise to a strict 2-group $\widehat{\mathcal{G}}$.

We say that $\widehat{\mathcal{G}}$ is the *strictification* of $\mathcal{G}$. In summary

- $\mathcal{K}_{\mathcal{H}} = (\partial : H \times F_{(G\backslash\{1\})\times(G\backslash\{1\})} \to F_{G\backslash\{1\}})$, where
- $\partial(a, \{x, y\}) = \{x\}\{y\}\sigma(xy)^{-1}$, with
- ${}^{\{x\}}(a, \{y, z\}) = ({}^{x}h + \alpha(x, y, z), \{x, y\}\tau(xy, z)\tau(x, yz)^{-1})$,

where $\sigma(\{x\}) = \{x\}$ for all $x \in G\backslash\{1\}$ and $\sigma(\{1\}) = 1$, similarly $\tau(\{x, y\}) = \{x, y\}$ for all $x, y \in G\backslash\{1\}$ otherwise $\tau(\{1, x\}) = \tau(\{x, 1\}) = \tau(\{1, 1\}) = 1$. Note that

$$\begin{aligned}
\pi_2(\mathcal{K}_{\mathcal{G}}) &= Ker(\partial) \\
&= \{(a, \{x, y\}) \in H \times F_{(G\backslash\{1\})\times(G\backslash\{1\})} \mid \{x\}\{y\}\sigma(xy)^{-1} = 1\} \\
&= \{(a, \{1, 1\}) \in H \times F_{(G\backslash\{1\})\times(G\backslash\{1\})}\} \\
&\cong H
\end{aligned}$$

and

$$\begin{aligned}
\pi_1(\mathcal{K}_{\mathcal{G}}) &= Coker(\partial) \\
&= F_{G\backslash\{1\}}/\langle\{x\}\{y\}\sigma(xy)^{-1}\rangle \\
&= \langle x \in G\backslash\{1\} | \{x\}\{y\} = \{xy\} \text{ for all } y \in G\rangle \\
&\cong G.
\end{aligned}$$

Corollary 3.2.5. *The two crossed modules introduced in Theorem 3.2.3 and Example 3.2.4 are equivalent.*

3.3 Representations of Strict 2-groups

This section will review the results of Rumynin and the author [48]. Establishing how 2-representations are given by elements in the Burnside ring. Lastly some formulae will be presented for the 2-character.

3.3.1 The Burnside Ring

Let us consider a crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ such that $\pi_1(\mathcal{K}) = B/\partial(A)$ is a finite group. Let $\mathcal{S}(\mathcal{K})$ be the category of subgroups of $\pi_1(\mathcal{K})$ [21]. Objects of $\mathcal{S}(\mathcal{K})$ are subgroups of $\pi_1(\mathcal{K})$. The morphisms $\mathcal{S}(P, Q)$ are conjugations $\gamma_{\mathbf{x}} : P \to Q$, $\gamma_{\mathbf{x}}(\mathbf{a}) = \mathbf{x}\mathbf{a}\mathbf{x}^{-1}$, $\mathbf{x} \in \pi_1(\mathcal{K})$ whenever $\mathbf{x}P\mathbf{x}^{-1} \subseteq Q$. If $\mathbf{y}^{-1}\mathbf{x} \neq 1_G$ is in the centraliser of P, then $\gamma_{\mathbf{x}}$ and $\gamma_{\mathbf{y}}$ are the same conjugations, but we treat them as different morphisms in $\mathcal{S}(P, Q)$. In this respect we are different from the setup, studied by Gunnells, Rose and Rumynin [21], where these are the same morphism. The setups are not drastically different, so we can still use their results exercising a certain care.

A subgroup $P \leq \pi_1(\mathcal{K})$ has an inverse image $\bar{P} := (B \to \pi_1(\mathcal{K}))^{-1}(P)$. It gives a restricted sub-crossed module $\mathcal{K}_P := (A \xrightarrow{\partial} \bar{P})$ with $\pi_1(\mathcal{K}_P) \cong P$.

Let Φ be the functor "2-representations of degree one". It is a contravariant functor from $\mathcal{S}(\mathcal{K})$ to the category of abelian groups. On objects, $\Phi(P) := \text{2-Rep}_1(\widetilde{\mathcal{K}_P})$. Let us look at a morphism $\gamma_{\mathbf{x}} : P \to Q$. By picking a lifting $\dot{\mathbf{x}} \in B$, i.e., an element with $\dot{\mathbf{x}}\partial(A) = \mathbf{x}$, we get a conjugation morphism of crossed modules

$$\gamma_{\dot{\mathbf{x}}} : \mathcal{K}_P \to \mathcal{K}_Q, \quad P \ni \mathbf{g} \mapsto \dot{\mathbf{x}}\mathbf{g}\dot{\mathbf{x}}^{-1}, \quad A \ni a \mapsto {}^{\dot{\mathbf{x}}}a$$

and the corresponding homomorphism of 2-groups

$$\gamma_{\dot{\mathbf{x}}} : \widetilde{\mathcal{K}_P} \to \widetilde{\mathcal{K}_Q}, \quad P \ni \mathbf{g} \mapsto \dot{\mathbf{x}}\mathbf{g}\dot{\mathbf{x}}^{-1}, \quad \begin{tikzcd} \star \arrow[r, bend left=50, "\mathbf{g}_2"] \arrow[r, bend right=50, "\mathbf{g}_1"'] \arrow[r, phantom, "a\Uparrow"] & \star \end{tikzcd} \mapsto \begin{tikzcd} \star \arrow[r, bend left=50, "\dot{\mathbf{x}}\mathbf{g}_2\dot{\mathbf{x}}^{-1}"] \arrow[r, bend right=50, "\dot{\mathbf{x}}\mathbf{g}_1\dot{\mathbf{x}}^{-1}"'] \arrow[r, phantom, "{}^{\dot{\mathbf{x}}}a\Uparrow"] & \star \end{tikzcd}.$$

An element $a \in A$ gives another lifting $\partial(a)\dot{\mathbf{x}} \in G$ of $\mathbf{x}$ and a new 2-morphism $\gamma_{\partial(a)\dot{\mathbf{x}}} : \widetilde{\mathcal{K}_P} \to \widetilde{\mathcal{K}_Q}$. In essence, they differ by an inner 2-isomorphism determined by a.

Lemma 3.3.1. *(Rumynin Wendland [48, Lemma 4.1]) Let Θ be a 2-representation of $\widetilde{\mathcal{K}_Q}$. Then the 2-representations $\Theta \circ \gamma_{\dot{\mathbf{x}}}$ and $\Theta \circ \gamma_{\partial(a)\dot{\mathbf{x}}}$ of $\widetilde{\mathcal{K}_P}$ are equivalent.*

Lemma 3.3.1 applies to 2-representations, hence, $\gamma_{\dot{\mathbf{x}}}$ and $\gamma_{\partial(a)\dot{\mathbf{x}}}$ determine the same pull-back homomorphism that allows us to define the functor Φ on morphisms:

$$\Phi(\gamma_{\mathbf{x}}) := [\gamma_{\dot{\mathbf{x}}}] = [\gamma_{\partial(a)\dot{\mathbf{x}}}] : \Phi(Q) = \text{2-Rep}_1(\mathcal{K}_Q) \to \text{2-Rep}_1(\mathcal{K}_P) = \Phi(P).$$

In general, if the conjugations $\gamma_{\mathbf{x}}$ and $\gamma_{\mathbf{y}}$ are the same, these pull-backs can be different: $\Theta \circ \gamma_{\dot{\mathbf{x}}}$ and $\Theta \circ \gamma_{\dot{\mathbf{y}}}$ are not necessarily equivalent because the actions of 2-objects of $\widetilde{\mathcal{K}}_Q$ could be different. This necessitates our version of the category $\mathcal{S}(\mathcal{K})$ [21].

Example 3.3.2. Take any abelian group A, and any non-trivial abelian subgroup $B \leq \text{Aut}(A)$, e.g., $A = C_n \times C_n = \langle a \rangle \times \langle b \rangle$ with $B = C_2 = \langle \phi \rangle$ such that $\phi(a) = b$ and $\phi(b) = a$. Then we have crossed module $\mathcal{K} = (A \xrightarrow{\partial} B)$ with $\partial(x) = 1$ for all $x \in A$. This satisfies (CM1) and (CM2), as A is abelian and ∂ is trivial. Then as $B = \pi_1(\mathcal{K})$ is abelian we have for any two $x, y \in B$ that xy^{-1} lies in the centraliser of any subgroup, however they can clearly have different actions.

Despite this slight difference, the functor Φ still leads to the generalised Burnside ring $\mathbb{B}_{\mathbb{A}}(\mathcal{K}) := \mathbb{B}^{\Phi}_{\mathbb{K}}(\pi_1(\mathcal{K}))$ with coefficients in a commutative ring $\mathbb{K}$ [21]. The $\mathbb{K}$-basis of $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ consists of pairs $\langle \Theta, P\rangle$ where P is a subgroup of $\pi_1(\mathcal{K})$, Θ is a degree one 2-representation of $\widetilde{\mathcal{K}_P}$. In each $\pi_1(\mathcal{K})$-conjugacy class of such pairs we choose one representative because

$$\langle \Theta, P\rangle = \langle \Phi(\gamma_{\mathbf{x}})(\Theta), \mathbf{x}^{-1}P\mathbf{x}\rangle$$

for all $\mathbf{x} \in \pi_1(\mathcal{K})$. We can also write a pair $\langle \Theta, P\rangle$ with an arbitrary 2-representation Θ of $\widetilde{\mathcal{K}_P}$ but they can be rewritten as linear combinations of pairs with degree one 2-representations by the formulas

$$\langle \Theta_1 \boxplus \Theta_2, P\rangle = \langle \Theta_1, P\rangle + \langle \Theta_2, P\rangle \text{ and } \langle \Theta \uparrow^{\widetilde{\mathcal{K}}_P}_{\widetilde{\mathcal{K}}_Q}, P\rangle = \langle \Theta, Q\rangle.$$

The multiplication in $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ is $\mathbb{K}$-bilinear, defined on the basis by the formula

$$\langle \Theta, P\rangle \cdot \langle \Omega, Q\rangle = \sum_{P\mathbf{x}Q \in P\backslash \pi_1(\mathcal{K})/Q} \left\langle \begin{array}{c} \Phi(\gamma_1 : P \cap \mathbf{x}Q\mathbf{x}^{-1} \to P)(\Theta)\boxtimes \\ \Phi(\gamma_{\mathbf{x}^{-1}} : P \cap \mathbf{x}Q\mathbf{x}^{-1} \to Q)(\Omega) \end{array}, P \cap \mathbf{x}Q\mathbf{x}^{-1} \right\rangle.$$

Intuitively this formula is exactly that of the tensor product of two representations as we will see in the proof of Proposition 3.3.4. The Burnside ring $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ is isomorphic to the space of 2-representations with coefficients in $\mathbb{K}$ for a 2-group coming from a crossed module $\mathcal{K}$ [48, Proposition 4.2]. Here multiplication takes the role of the tensor product of representations and irreducible representations map $[\Theta \uparrow^{\widetilde{\mathcal{K}}}_{\widetilde{\mathcal{K}}_P}] \mapsto \langle \Theta, P\rangle$.

One can understand the formula for multiplication by looking at the action of $\pi_1(\mathcal{K}) = B/\partial(A)$ on $[n] \times [m]$. Two irreducible representations $[\Theta \uparrow^{\widetilde{\mathcal{K}}}_{\widetilde{\mathcal{K}}_P}]$ and $[\Omega \uparrow^{\widetilde{\mathcal{K}}}_{\widetilde{\mathcal{K}}_Q}]$ give rise to two transitive actions of $\pi_1(\mathcal{K})$ on $[n]$ and $[m]$ which are equivalent to coset actions of $\pi_1(\mathcal{K})/P$ and $\pi_1(\mathcal{K})/Q$. When $\pi_1(\mathcal{K})$ acts diagonally on $\{(\mathbf{x}P, \mathbf{y}Q) | \mathbf{x}, \mathbf{y} \in \pi_1(\mathcal{K})\}$ we get orbits which can be represented by pairs $[(P, \mathbf{x}Q)]$ with $\mathbf{x} \in \pi_1(\mathcal{K})$. Where $[(P, \mathbf{x}Q)] = [(pP, \mathbf{x}qQ)] = [(P, p^{-1}\mathbf{x}qQ)]$ so $\mathbf{x}$ is uniquely determined up to the double coset $P\mathbf{x}Q$.

Let us introduce *a mark homomorphism* [21, Lemma 1.2]. We need to assume that the order $|\pi_1(\mathcal{K})|$ is invertible in $\mathbb{K}$. Let $\alpha : \Phi(P) \to \mathbb{K}^{\times}$ be a group homomorphism. The corresponding mark is an $\mathbb{K}$-algebra homomorphism $f^{\alpha}_P : \mathbb{B}_{\mathbb{A}}(\mathcal{K}) \to \mathbb{K}$ given by the formula

$$f^{\alpha}_P(\langle \Theta, Q\rangle) = \frac{1}{|Q|} \sum_{\mathbf{g} \in X} \alpha(\Phi(\gamma_{\mathbf{g}} : P \to Q)(\Theta))$$

where $X = \{\mathbf{g} \in \pi_1(\mathcal{K}) \mid \mathbf{g}P\mathbf{g}^{-1} \subseteq Q\}$. The marks work magnificently for the ring $\mathbb{B}_{\mathbb{A}}(\mathcal{K})$ if all the groups $\Phi(P)$ are finite.

Proposition 3.3.3. *(Gunnels, Rose, and Rumynin [21, Corollary 1.3]) Suppose all $\Phi(P)$ are finite. Let N be the least common multiple of all the orders of elements in various $\Phi(P)$. If $\mathbb{K}$ is a field, containing a primitive N-th root of unity, then the mark homomorphisms define an isomorphism of $\mathbb{K}$-algebras*

$$\mathbb{B}_{\mathbb{A}}(\mathcal{K}) \xrightarrow{\cong} \oplus\mathbb{K} = \mathbb{K}^k$$

where k is the number of B-orbits on the disjoint unions $\dot{\cup}_P \Phi(P)$.

If $\varphi : \mathcal{K}' \to \mathcal{K}$ is a homomorphism of crossed modules, the pull-back of 2-representations ($\theta \mapsto \theta^\varphi$ where $\theta^\varphi(-) = \theta(\varphi(-))$) gives a homomorphism of Burnside rings $\varphi^* : \mathbb{B}_{\mathbb{A}}(\mathcal{K}) \to \mathbb{B}_{\mathbb{A}}(\mathcal{K}')$. Consider the quotient homomorphism of crossed modules

$$\varphi : \mathcal{K} \to \bar{\mathcal{K}} \coloneqq (1 \to \pi_1(\mathcal{K})), \quad B \ni \mathbf{b} \mapsto \mathbf{b}\partial(A), \quad A \ni \mathbf{a} \mapsto 1.$$

The Burnside ring $\mathbb{B}_{\mathbb{A}}(\bar{\mathcal{K}})$ is precisely the generalised Burnside ring of $\pi_1(\mathcal{K})$ studied by Gunnells, Rose and Rumynin [21], because there are no non-trivial 2-objects in $\widetilde{\bar{\mathcal{K}}}$. All the groups $\Phi(P) = H^2(P, \mathbb{K}^\times)$ are finite. Proposition 3.3.3 tells us that if $\mathbb{K}$ is a field, containing a primitive N-th root of unity, then the corresponding pull-back algebra homomorphism $\varphi^* : \mathbb{B}_{\mathbb{A}}(\bar{\mathcal{K}}) = \mathbb{K}^k \to \mathbb{B}_{\mathbb{A}}(\mathcal{K})$ is injective. Its image can be thought of as the 2-representations "trivial" on H.

3.3.2 Ganter-Kapranov 2-character

Let us recall the notion of a 2-categorical trace [16]. Let $\mathcal{C}$ be a bicategory, and $x \in \mathcal{C}_0$ one of its 0-object. The *2-categorical trace* of a 1-morphism $u \in \mathcal{C}_1(x, x)$ is the set $\mathbb{T}\mathrm{r}_x(u) \coloneqq \mathcal{C}_2(\mathbf{i}_x, u)$. It is instructive to observe that in the bicategory of 2-vector spaces $2-\mathrm{Vect}^{\mathbb{K}}$ a 1-morphism $u = (U_{i,j})$ is an $n \times n$-matrix of vector spaces, while its trace is the vector space

$$\mathbb{T}\mathrm{r}_n(u) = \bigoplus_i \mathrm{Mor}_{\mathbb{K}}(\mathbb{K}, U_{i,i}) \oplus \bigoplus_{i \neq j} \mathrm{Mor}_{\mathbb{K}}(0, U_{i,j}) \cong \bigoplus_i U_{i,i} \, .$$

Let Θ be a 2-representation of degree n of the 2-group $\widetilde{\mathcal{K}}$, where $\mathcal{K}$ is a crossed module. To define the 2-character of Θ we consider two elements $\mathbf{a}, \mathbf{b} \in B$ such that their images in the fundamental group commute: $\bar{\mathbf{a}}\bar{\mathbf{b}} = \bar{\mathbf{b}}\bar{\mathbf{a}} \in \pi_1(\mathcal{K})$ and $h \in A$ such that $\partial(h)\mathbf{a}\mathbf{b}\mathbf{a}^{-1} = \mathbf{b}$. This data gives a linear operator $\mathbb{X}_\Theta(\mathbf{b}, \mathbf{a}, h) : \mathbb{T}\mathrm{r}_n(\Theta^1(\mathbf{b})) \to$

$\mathbb{T}\mathrm{r}_n(\Theta^1(\mathbf{b}))$

$$\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)\left(\begin{tikzcd} n \arrow[r, bend left, "\Theta(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] & n \end{tikzcd} \right) = \begin{tikzcd}[column sep=large] n \arrow[rrr, bend left=70, "\Theta(\mathbf{b})"] \arrow[rrr, bend right=70, "\mathbf{i}_n"'] \arrow[r, bend left, "\Theta(\mathbf{a})"] \arrow[r, bend right, "\Theta(\mathbf{a})"'] \arrow[r, phantom, "\Uparrow Id"] & n \arrow[r, bend left, "\Theta(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] \arrow[r, phantom, shift left=8ex, "\Uparrow {[\mathbf{a},\mathbf{b},\mathbf{a}^{-1},h]}"] \arrow[r, phantom, shift right=8ex, "\Uparrow {[\mathbf{a},1,\mathbf{a}^{-1},1]^{-1}}"] & n \arrow[r, bend left, "\Theta(\mathbf{a}^{-1})"] \arrow[r, bend right, "\Theta(\mathbf{a}^{-1})"'] \arrow[r, phantom, "\Uparrow Id"] & n \end{tikzcd}$$

where $\Theta(\mathbf{b}) = \Theta^1_{\star,\star}(\mathbf{b})$ and $[\mathbf{a},\mathbf{b},\mathbf{a}^{-1},h]$ is a composition of the natural morphism $\Theta(\mathbf{a}) \diamond \Theta(\mathbf{b}) \diamond \Theta(\mathbf{a}^{-1}) \to \Theta(\mathbf{aba}^{-1})$ and the action $h\cdot : \Theta(\mathbf{aba}^{-1}) \to \Theta(\mathbf{b})$. Let us write a matrix of vector spaces $\Theta^1_{\star,\star}(\mathbf{a}) = (U_{i,j}(\mathbf{a}))$. Its dimension is the permutation matrix of some permutation $\sigma_{\mathbf{a}}$, e.g., $U_{i,j}(\mathbf{a}) \neq 0$ if and only if $j = \sigma_{\mathbf{a}}(i)$. Now the natural map $\mathbf{i}_n \to \Theta(\mathbf{a})\diamond\Theta(\mathbf{a}^{-1})$ is given by a collection of elements $x_i(\mathbf{a}) \in U_{i,\sigma_{\mathbf{a}}(i)}(\mathbf{a})$, $y_i(\mathbf{a}) \in U_{\sigma_{\mathbf{a}}(i),i}(\mathbf{a}^{-1})$ in a way that

$$\mathbb{K} = (\mathbf{i}_n)_{i,i} \ni 1_i \mapsto x_i(\mathbf{a}) \otimes y_i(\mathbf{a}) \in U_{i,\sigma_{\mathbf{a}}(i)}(\mathbf{a}) \otimes U_{\sigma_{\mathbf{a}}(i),i}(\mathbf{a}^{-1}).$$

Now we can write the key map in an elementary way:

$$\begin{array}{c} \mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)(\sum_i b_i) = h\cdot(\sum_i x_i(\mathbf{a}) \otimes b_{\sigma_{\mathbf{a}}(i)} \otimes y_i(\mathbf{a})) \\ \text{where} \quad \sum_i b_i \in \mathbb{T}\mathrm{r}_n(\Theta^1_{\star,\star}(\mathbf{b})) = \oplus_i U_{i,i}(\mathbf{b}). \end{array} \tag{3.6}$$

The 2-character value $\mathfrak{X}_\Theta(\mathbf{a},\mathbf{b},h)$ is the trace of this linear map:

$$\mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h) \coloneqq \mathrm{Tr}(\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)).$$

Let $\mathbb{G}$ be the set of all triples $(\mathbf{a},\mathbf{b},h) \in B \times B \times A$ such that $\partial(h)\mathbf{ab} = \mathbf{ba}$. The group B acts on the set $\mathbb{G}$ by conjugation.

Proposition 3.3.4. *(Rumynin Wendland [48]) For any 2-representation Θ the function $\mathfrak{X}_\Theta : \mathbb{G} \to \mathbb{K}$ is constant on B-orbits. If Ψ is another 2-representation, then*

$$\mathfrak{X}_{\Psi\boxtimes\Theta}(\mathbf{b},\mathbf{a},h) = \mathfrak{X}_\Psi(\mathbf{b},\mathbf{a},h) \cdot \mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h).$$

Proof. Let us prove the first statement, i.e., that $\mathfrak{X}_\Theta({}^{\mathbf{g}}\mathbf{b}, {}^{\mathbf{g}}\mathbf{a}, {}^{\mathbf{g}}h) = \mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h)$ for all $\mathbf{g} \in B$. We have a natural "conjugation" linear map $\Gamma_{\mathbf{g}} : \mathbb{T}r_n(\Theta^1(\mathbf{b})) \to \mathbb{T}r_n(\Theta^1({}^{\mathbf{g}}\mathbf{b}))$

given by the formula

$$\Gamma_{\mathbf{g}}\left(\begin{tikzcd} n \arrow[r, bend left, "\Theta^1(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] & n \end{tikzcd} \right) = \begin{tikzcd} n \arrow[rrr, bend left=60, "\Theta^1(\mathbf{gbg}^{-1})"] \arrow[rrr, bend right=60, "\mathbf{i}_n"'] \arrow[r, bend left, "\Theta^1(\mathbf{g})"] \arrow[r, bend right, "\Theta^1(\mathbf{g})"'] \arrow[r, phantom, "\Uparrow Id"] & n \arrow[r, bend left, "\Theta^1(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] \arrow[r, phantom, shift left=6ex, "\Uparrow {[\mathbf{g},\mathbf{b},\mathbf{g}^{-1},1]}"] \arrow[r, phantom, shift right=6ex, "\Uparrow {[\mathbf{g},1,\mathbf{g}^{-1},1]^{-1}}"] & n \arrow[r, bend left, "\Theta^1(\mathbf{g}^{-1})"] \arrow[r, bend right, "\Theta^1(\mathbf{g}^{-1})"'] \arrow[r, phantom, "\Uparrow Id"] & n \end{tikzcd}$$

where $\Theta(\mathbf{b}) = \Theta^1_{\star,\star}(\mathbf{b})$ and $[\mathbf{g}, \mathbf{b}, \mathbf{g}^{-1}, 1]$ is a composition of the natural morphism $\Theta(\mathbf{g}) \diamond \Theta(\mathbf{b}) \diamond \Theta(\mathbf{g}^{-1}) \to \Theta(\mathbf{gbg}^{-1})$. So $\Gamma_{\mathbf{g}}$ is a change of basis matrix, therefore we get the following equality

$$\mathbb{X}_\Theta({}^{\mathbf{g}}\mathbf{b}, {}^{\mathbf{g}}\mathbf{a}, {}^{\mathbf{g}}h) = \Gamma_{\mathbf{g}} \mathbb{X}_\Theta(\mathbf{b}, \mathbf{a}, h) \Gamma_{\mathbf{g}}^{-1}.$$

To show this, we remind the reader of what these maps do in terms of 3.6,

$$\begin{aligned} \Gamma_{\mathbf{g}}(\sum_i b_i) &= \sum_i x_i(\mathbf{g}) \otimes b_{\sigma_{\mathbf{g}}(i)} \otimes y_i(\mathbf{g}), \\ \mathbb{X}_\Theta(\mathbf{b}, \mathbf{a}, h)(\sum_i b_i) &= h \cdot (\sum_i x_i(\mathbf{a}) \otimes b_{\sigma_{\mathbf{a}}(i)} \otimes y_i(\mathbf{a})), \text{ and} \\ \Gamma_{\mathbf{g}}^{-1}(\sum_i b_i) &= \sum_i x_i(\mathbf{g}^{-1}) \otimes b_{\sigma_{\mathbf{g}^{-1}}(i)} \otimes y_i(\mathbf{g}^{-1}). \end{aligned}$$

Before calculating the formulas above note that if $(x_i(\mathbf{a}))_{i,\sigma_{\mathbf{a}}(i)} \otimes (y_i(\mathbf{a}))_{\sigma_{\mathbf{a}}(i),i} \mapsto (1)_{i,i}$, and $(x_i(\mathbf{b}))_{i,\sigma_{\mathbf{b}}(i)} \otimes (y_i(\mathbf{b}))_{\sigma_{\mathbf{b}}(i),i} \mapsto (1)_{i,i}$, then

$$(x_i(\mathbf{a}))_{i,\sigma_{\mathbf{a}}(i)} \otimes (x_{\sigma_{\mathbf{a}}(i)}(\mathbf{b}))_{\sigma_{\mathbf{a}}(i),\sigma_{\mathbf{b}}\sigma_{\mathbf{a}}(i)} \otimes (y_{\sigma_{\mathbf{a}}(i)}(\mathbf{b}))_{\sigma_{\mathbf{b}}\sigma_{\mathbf{a}}(i),\sigma_{\mathbf{a}}(i)} \otimes (y_i(\mathbf{a}))_{\sigma_{\mathbf{a}}(i),i} \mapsto (1)_{i,i},$$

which gives that $x_i(\mathbf{a}) \otimes x_{\sigma_{\mathbf{a}}(i)}(\mathbf{b}) = x_i(\mathbf{ab})$ and $y_{\sigma_{\mathbf{a}}(i)}(\mathbf{b}) \otimes y_i(\mathbf{a}) = y_i(\mathbf{ab})$, so continuing with our calculation we have

$$
\begin{aligned}
\Gamma_{\mathbf{g}}\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)\Gamma_{\mathbf{g}}^{-1}(\sum_i b_i) &= \Gamma_{\mathbf{g}}\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)(\sum_i x_i(\mathbf{g}^{-1})\otimes b_{\sigma_{\mathbf{g}^{-1}}(i)}\otimes y_i(\mathbf{g}^{-1})) \\
&= \Gamma_{\mathbf{g}}(h\cdot(\sum_i x_i(a)\otimes x_{\sigma_{\mathbf{a}}(i)}(\mathbf{g}^{-1})\otimes b_{\sigma_{\mathbf{g}^{-1}}\sigma_{\mathbf{a}}(i)} \\
&\qquad\qquad \otimes y_{\sigma_{\mathbf{a}}(i)}(\mathbf{g}^{-1})\otimes y_i(\mathbf{a}))) \\
&= \Gamma_{\mathbf{g}}(h\cdot(\sum_i x_i(\mathbf{ag}^{-1})\otimes b_{\sigma_{\mathbf{ag}^{-1}}(i)}\otimes y_i(\mathbf{ag}^{-1})) \\
&= \sum_i x_i(\mathbf{g})\otimes h\cdot(x_{\sigma_{\mathbf{g}}(i)}(\mathbf{ag}^{-1})\otimes b_{\sigma_{\mathbf{ag}^{-1}}\sigma_{\mathbf{g}}(i)} \\
&\qquad\qquad \otimes y_{\sigma_{\mathbf{g}}(i)}(\mathbf{ag}^{-1}))\otimes y_i(\mathbf{g}) \\
&= {}^{\mathbf{g}}h\cdot(\sum_i x_i(\mathbf{g})\otimes(x_{\sigma_{\mathbf{g}}(i)}(\mathbf{ag}^{-1})\otimes b_{\sigma_{\mathbf{ag}^{-1}}\sigma_{\mathbf{g}}(i)} \\
&\qquad\qquad \otimes y_{\sigma_{\mathbf{g}}(i)}(\mathbf{ag}^{-1})\otimes y_i(\mathbf{g})) \\
&= {}^{\mathbf{g}}h\cdot(\sum_i x_i(\mathbf{gag}^{-1})\otimes b_{\sigma_{\mathbf{gag}^{-1}}(i)}\otimes y_i(\mathbf{gag}^{-1}) \\
&= \mathbb{X}_\Theta({}^{\mathbf{g}}\mathbf{b},{}^{\mathbf{g}}\mathbf{a},{}^{\mathbf{g}}h)(\sum_i b_i)
\end{aligned}
$$

hence, the linear maps $\mathbb{X}_\Theta({}^{\mathbf{g}}\mathbf{b},{}^{\mathbf{g}}\mathbf{a},{}^{\mathbf{g}}h)$ and $\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)$ have the same trace. Let us prove the second statement now. Writing matrices of vector spaces as $\Theta^1_{\star,\star}(\mathbf{b}) = (U_{i,j}(\mathbf{b}))$ and $\Psi^1_{\star,\star}(\mathbf{b}) = (V_{i,j}(\mathbf{b}))$, we observe that $\mathbb{T}\mathrm{r}_{nm}(\Theta\boxtimes\Psi)^1_{\star,\star}(\mathbf{b}) \cong \sum_{i,t} U_{i,i}(\mathbf{b})\otimes V_{t,t}(\mathbf{b}) \cong (\sum_i U_{i,i}(\mathbf{b}))\otimes(\sum_t V_{t,t}(\mathbf{b})) \cong \mathbb{T}\mathrm{r}_n\Theta^1_{\star,\star}(\mathbf{b})\otimes\mathbb{T}\mathrm{r}_m\Psi^1_{\star,\star}(\mathbf{b})$. Identifying the 2-traces under these maps, we can observe that $\mathbb{X}_{\Theta\boxtimes\Psi}(\mathbf{b},\mathbf{a},h) = \mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)\otimes\mathbb{X}_\Psi(\mathbf{b},\mathbf{a},h)$ that implies the second statement. $\square$

A character table of a finite group has rows and columns. Usually one thinks of columns as characters, yet it is often instructive to think of rows as characters. Applying this way of thinking to the 2-characters we can use Proposition 3.3.4 to conclude that a G-conjugacy class of triples $(\mathbf{a},\mathbf{b},h)$ with $\partial(h)\mathbf{ab} = \mathbf{ba}$ determines a ring homomorphism

$$\mathfrak{X}(\mathbf{b},\mathbf{a},h) : \mathbb{B}_{\mathbb{Z}}(\mathcal{K}) \to \mathbb{K}, \quad [\Theta] \mapsto \mathfrak{X}_\Theta(\mathbf{b},\mathbf{a},h).$$

It can be extended by $\mathbb{K}$-linearity to a $\mathbb{K}$-algebra homomorphism $\mathfrak{X}(\mathbf{b},\mathbf{a},h) : \mathbb{B}_{\mathbb{K}}(\mathcal{K}) \to \mathbb{K}$. Both versions of $\mathfrak{X}$ should be called a *Ganter-Kapranov 2-character*. In the finite case (i.e., under assumptions of Proposition 3.3.3) the Ganter-Kapranov 2-character must be one of the marks. Which one?

Theorem 3.3.5. *(Rumynin Wendland [48]) In the notations above, let P be the subgroup of $\pi_1(\mathcal{K})$ generated by $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$. Let $\alpha := \mathfrak{X}(\mathbf{b},\mathbf{a},h)$ considered as a group* *homomorphism* 2-Rep$^1(\mathcal{K}_P) \to \mathbb{K}^\times$*. If the order of* $\pi_1(\mathcal{K})$ *is finite and invertible in the field* $\mathbb{K}$*, then*

$$\mathfrak{X}(\mathbf{b}, \mathbf{a}, h) = f_P^\alpha.$$

Proof. It suffices to check this equality on irreducible 2-representations. So let us consider $\langle \Theta, Q\rangle \in \mathbb{B}_{\mathbb{A}}(\mathcal{K})$, which corresponds to 2-representations $\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}}$. Let $T = \{t_1, \ldots, t_m\}$ be a traversal of $Q \leq \pi_1(\mathcal{K})$. Note that

$$(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^1(\mathbf{b})_{t,t'} = \begin{cases} \Theta^1(\mathbf{b}') & \text{if } \mathbf{b}t = t'\mathbf{b}' \text{ with } \mathbf{b}' \in Q \\ 1 & \text{otherwise} \end{cases}$$

so for a component of $(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^1(\mathbf{b})$ to contribute to $\mathbb{T}\mathrm{r}_n(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^1(\mathbf{b})$ we need $\mathbf{b}t = t\mathbf{b}'$ with $\mathbf{b}' \in Q$, i.e., $t^{-1}\mathbf{b}t \in Q$. Moreover if we let $\sigma(t) = t'$, where $\mathbf{a}t = t'\mathbf{a}'$ with $\mathbf{a}' \in Q$, represent $\mathbf{a}$'s action on the set of cosets. Then we know that

$$\mathbb{X}_{\Theta\uparrow_{\mathcal{K}_Q}^{\mathcal{K}}}(\mathbf{b}, \mathbf{a}, h)(\sum_i b_i) = h \cdot (\sum_i x_i(\mathbf{a}) \otimes b_{\sigma(i)} \otimes y_i(\mathbf{a}))$$

therefore for the transversal t to contribute to

$$\mathfrak{X}_{\Theta\uparrow_{\mathcal{K}_Q}^{\mathcal{K}}}(\mathbf{b}, \mathbf{a}, h) := \mathrm{Tr}(\mathbb{X}_{\Theta\uparrow_{\mathcal{K}_Q}^{\mathcal{K}}}(\mathbf{b}, \mathbf{a}, h)).$$

We need that $\sigma(t) = t$, i.e. $t^{-1}\mathbf{a}t \in Q$.

So take $t \in T$ and $\mathbf{a}, \mathbf{b} \in \pi_1(\mathcal{K})$ such that ${}^{t^{-1}}\mathbf{a}, {}^{t^{-1}}\mathbf{b} \in Q$ and $h \in A$ such that $\partial(h)\mathbf{ab} = \mathbf{ba}$. Now consider t's components of $\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}}$,

$$\begin{aligned}
(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^1(\mathbf{b})_{t,t} &= \Theta^1({}^{t^{-1}}\mathbf{b}), \\
(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^1(\mathbf{a})_{t,t} &= \Theta^1({}^{t^{-1}}\mathbf{a}), \\
(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^2(x, y)_{t,t} &= \Theta^2({}^{t^{-1}}x, {}^{t^{-1}}y), \qquad \text{for } x, y \in \{\mathbf{a}, \mathbf{a}^{-1}, \mathbf{b}, \mathbf{b}^{-1}, 1\} \text{ and} \\
(\Theta \uparrow_{\mathcal{K}_Q}^{\mathcal{K}})^2(\mathbf{aba}^{-1}, h)_{t,t} &= \Theta^2({}^{t^{-1}}(\mathbf{aba}^{-1}), {}^{t^{-1}}h)_{t,t},
\end{aligned}$$

giving the composition of $\mathbb{X}_{\Theta\uparrow_{\mathcal{K}_Q}^{\mathcal{K}}}(\mathbf{b}, \mathbf{a}, h)$ in the t component is exactly that of $\mathbb{X}_\Theta({}^{t^{-1}}\mathbf{b}, {}^{t^{-1}}\mathbf{a}, {}^{t^{-1}}h)$. So we have that

$$\begin{aligned}
\mathfrak{X}_{\Theta\uparrow_{\mathcal{K}_Q}^{\mathcal{K}}}(\mathbf{b}, \mathbf{a}, h)(\langle \Theta, Q\rangle) &= \sum_{t \in T, {}^{t^{-1}}\mathbf{a}, {}^{t^{-1}}\mathbf{b} \in Q} \mathbb{X}_\Theta({}^{t^{-1}}\mathbf{b}, {}^{t^{-1}}\mathbf{a}, {}^{t^{-1}}h) \\
&= \frac{1}{|Q|} \sum_{g \in \pi_1(\mathcal{K}), g^{-1}Pg \subset Q} \mathbb{X}_\Theta({}^{g^{-1}}\mathbf{b}, {}^{g^{-1}}\mathbf{a}, {}^{g^{-1}}h) \quad \text{where } P = \langle \mathbf{a}, \mathbf{b}\rangle \\
&= \frac{1}{|Q|} \sum_{g \in \pi_1(\mathcal{K}), gPg^{-1} \subset Q} \mathbb{X}_\Theta({}^{g}\mathbf{b}, {}^{g}\mathbf{a}, {}^{g}h).
\end{aligned}$$

Next examine f_P^α with $\alpha := \mathfrak{X}(\mathbf{b}, \mathbf{a}, h) : \text{2-Rep}^1(\mathcal{K}_P) \to \mathbb{K}^\times$ being a group homomorphism,

$$\begin{aligned}
f_P^\alpha(\langle \Theta, Q \rangle) =& \frac{1}{|Q|} \sum_{g \in \pi_1(\mathcal{K}), gPg^{-1} \subset Q} \alpha(\Phi(\gamma_g : P \to Q)(\Theta)) \\
=& \frac{1}{|Q|} \sum_{g \in \pi_1(\mathcal{K}), gPg^{-1} \subset Q} \alpha(\theta^g) \qquad\qquad \text{where } \theta^g(x) = \theta({}^g x) \\
=& \frac{1}{|Q|} \sum_{g \in \pi_1(\mathcal{K}), gPg^{-1} \subset Q} \mathbb{X}_\Theta({}^g\mathbf{b}, {}^g\mathbf{a}, {}^g h),
\end{aligned}$$

giving us the required equality. □

3.3.3 Shapiro isomorphism

Let G be a group, $H \leq G$ its subgroup, M a $\mathbb{Z}H$-module. Shapiro's lemma [44] asserts isomorphisms in homology and cohomology:

$$H^*(G, \mathrm{Coind}_H^G(M)) \cong H^*(H, M), \quad H_*(G, \mathrm{Ind}_H^G(M)) \cong H_*(H, M).$$

The standard proof goes via a quasiisomorphism of the corresponding complexes. It does not give an explicit formula that we require for cohomology. Hence, we supply an explicit chain homotopy

$$\psi : C^n(H, M) \to C^n(G, \mathrm{Coind}_H^G(M)).$$

Choose a right transversal $T = \{\mathbf{t}_1, \mathbf{t}_2 \ldots\}_j$ to H in G such that $\mathbf{t}_1 = 1_G$. The coinduced module $\mathrm{Coind}_H^G(M)$ is the set of all H-equivariant functions $f : G \to M$. Such a function is uniquely determined by its values on T. The right transversal allows us to identify the coinduced module $\mathrm{Coind}_H^G(M)$ with the set of all functions $f : T \to M$. The cochains $C^n(H, M)$ are also functions $\mu : H^n \to M$. Given elements $\mathbf{g}_1, \ldots, \mathbf{g}_n \in G$ and $\mathbf{t} \in T$, there exist elements $\mathbf{h}_1, \ldots, \mathbf{h}_n \in H$ and $\mathbf{s}_0 = \mathbf{t}, \mathbf{s}_1, \ldots, \mathbf{s}_n \in T$ uniquely determined by the following equations:

$$\begin{gathered}
\mathbf{s}_0\mathbf{g}_1 \cdots \mathbf{g}_n = \mathbf{h}_1\mathbf{s}_1\mathbf{g}_2 \cdots \mathbf{g}_n = \ldots = \mathbf{h}_1 \cdots \mathbf{h}_k\mathbf{s}_k\mathbf{g}_{k+1} \cdots \mathbf{g}_n = \ldots \\
= \mathbf{h}_1 \cdots \mathbf{h}_{n-1}\mathbf{s}_{n-1}\mathbf{g}_n = \mathbf{h}_1 \cdots \mathbf{h}_n\mathbf{s}_n.
\end{gathered}$$

We use these elements to define ψ on a cochain $\mu \in C^n(H, M)$:

$$\psi(\mu)(\mathbf{g}_1, \ldots, \mathbf{g}_n)(\mathbf{t}) \coloneqq \mu(\mathbf{h}_1, \ldots, \mathbf{h}_n).$$

In the opposite direction we define a map for arbitrary elements $\mathbf{h}_1, \dots, \mathbf{h}_n \in H$:

$$\phi : C^n(G, \mathrm{Coind}_H^G(M)) \to C^n(H, M), \quad \phi(\theta)(\mathbf{h}_1, \dots, \mathbf{h}_n) = \theta(\mathbf{h}_1, \dots, \mathbf{h}_n)(1_G).$$

We are ready for the main result of this section.

Theorem 3.3.6. *(Rumynin Wendland [48]) Let $H \leq G$ be groups, M a $\mathbb{Z}H$-module. The above defined maps ϕ and ψ are isomorphisms of the cochain complexes $C^*(H, M)$ and $C^*(G, \mathrm{Coind}_H^G(M))$ in the homotopic category.*

Proof. Observe that for $\mathbf{g}_1, \dots, \mathbf{g}_n \in H$ and $\mathbf{t} = 1$ we get $\mathbf{h}_j = \mathbf{g}_j$. Hence,

$$\phi(\psi(\mu))(\mathbf{g}_1, \dots, \mathbf{g}_n) = \psi(\mu)(\mathbf{g}_1, \dots, \mathbf{g}_n)(1_G) = \mu(\mathbf{g}_1, \dots, \mathbf{g}_n)$$

proving that $\phi \circ \psi$ is equal to the identity. In the opposite direction, $\psi \circ \phi$ is only homotopic to the identity:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-1}(P, \mathrm{Coind}_H^G(M)) \arrow[r, "d^{n-1}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\phi\circ\psi"] \arrow[dl] & C^{n}(P, \mathrm{Coind}_H^G(M)) \arrow[r, "d^{n}"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\phi\circ\psi"] \arrow[dl, "\varpi^{n-1}"'] & C^{n+1}(P, \mathrm{Coind}_H^G(M)) \arrow[r] \arrow[d, shift right, "1"'] \arrow[d, shift left, "\phi\circ\psi"] \arrow[dl, "\varpi^{n}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & C^{n-1}(P, \mathrm{Coind}_H^G(M)) \arrow[r, "d^{n-1}"] & C^{n}(P, \mathrm{Coind}_H^G(M)) \arrow[r, "d^{n}"] & C^{n+1}(P, \mathrm{Coind}_H^G(M)) \arrow[r] & \cdots
\end{tikzcd}$$

where the homotopy ϖ is define by

$$\varpi^n(\theta)(\mathbf{g}_1, \dots, \mathbf{g}_n)(\mathbf{t}) = \sum_{j=0}^{n} (-1)^{j+1} \theta(\mathbf{h}_1, \dots, \mathbf{h}_j, \mathbf{s}_j, \mathbf{g}_{j+1}, \dots, \mathbf{g}_n)(1).$$

Let us verify that $\psi \circ \phi - 1 = \varpi^n \circ d^n + d^{n-1} \circ \varpi^{n-1}$. Let us first examine the left hand side of this equality:

$$\begin{aligned}
(\psi \circ \phi - 1)(\theta)(\mathbf{g}_1, \dots, \mathbf{g}_n)(\mathbf{t}) &= \psi \circ \phi(\theta)(\mathbf{g}_1, \dots, \mathbf{g}_n)(\mathbf{t}) - \theta(\mathbf{g}_1, \dots, \mathbf{g}_n)(\mathbf{t}) \\
&= \theta(\mathbf{h}_1, \dots, \mathbf{h}_n)(1) - \theta(\mathbf{g}_1, \dots, \mathbf{g}_n)(\mathbf{t}).
\end{aligned}$$

Now we scrutinise the first term of the right hand side. It is useful to pay attention which $\mathbf{s}_j$ appears in terms of the final expression because it tells you from which term of the second expression it originates. We label the lines to help observe the cancellations:

$$\varpi^n(d^n(\theta))(\mathbf{g}_1,\ldots,\mathbf{g}_n)(\mathbf{t}) = \sum_{j=0}^{n}(-1)^{j+1}d^n(\theta)(\mathbf{h}_1,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) =$$

$$-\,\theta(\mathbf{g}_1,\ldots,\mathbf{g}_n)(\mathbf{s}_0) \tag{3.7}$$

$$+\sum_{j=1}^{n}(-1)^{j+1}\theta(\mathbf{h}_2,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(\mathbf{h}_1)\,+ \tag{3.8}$$

$$+\sum_{j=2}^{n}\sum_{k=1}^{j-1}(-1)^{j+k+1}\theta\begin{pmatrix}\ldots\mathbf{h}_{k-1},\mathbf{h}_k\mathbf{h}_{k+1},\mathbf{h}_{k+2},\ldots\\ \ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_j\ldots\end{pmatrix}(1) \tag{3.9}$$

$$-\sum_{j=1}^{n}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_{j-1},\mathbf{h}_j\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) \tag{3.10}$$

$$+\sum_{j=0}^{n-1}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_{j-1},\mathbf{s}_j\mathbf{g}_{j+1},\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(1) \tag{3.11}$$

$$+\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}(-1)^{j+k}\theta\begin{pmatrix}\ldots\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots\\ \ldots,\mathbf{g}_{k-1},\mathbf{g}_k\mathbf{g}_{k+1},\mathbf{g}_{k+2}\cdots\end{pmatrix}(1) \tag{3.12}$$

$$+\sum_{j=0}^{n-1}(-1)^{j+n}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_{n-1})(1) \tag{3.13}$$

$$+\,\theta(\mathbf{h}_1,\ldots,\mathbf{h}_n)(1). \tag{3.14}$$

Lines (3.7) and (3.14) contribute to the left hand side. Lines (3.10) and (3.11) cancel because $\mathbf{s}_j\mathbf{g}_{j+1} = \mathbf{h}_{j+1}\mathbf{s}_{j+1}$. The remaining lines cancel with the second term (line labels correspond to their cancelling counterparts):

$$
\begin{aligned}
&d^{n-1}(\varpi^{n-1}(\theta))(\mathbf{g}_1,\ldots,\mathbf{g}_n)(\mathbf{t}) = \\
&\quad \varpi^{n-1}(\theta)(\mathbf{g}_2,\ldots,\mathbf{g}_n)(\mathbf{h}_1\mathbf{s}_1) + \sum_{k=1}^{n-1}(-1)^k\varpi^{n-1}(\theta)(\mathbf{g}_1,\ldots,\mathbf{g}_k\mathbf{g}_{k+1}\ldots)(\mathbf{t}) \\
&\qquad + (-1)^n\varpi^{n-1}(\theta)(\mathbf{g}_1,\ldots,\mathbf{g}_{n-1})(\mathbf{t}) \\
&\qquad = \sum_{j=1}^{n}(-1)^j\theta(\mathbf{h}_2,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_n)(\mathbf{h}_1) \quad (3.8) \\
&\qquad + \sum_{k=1}^{n-1}\sum_{j=0}^{k-1}(-1)^{k+j+1}\theta\begin{pmatrix} \ldots\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots \\ \ldots,\mathbf{g}_{k-1},\mathbf{g}_k\mathbf{g}_{k+1},\mathbf{g}_{k+2}\ldots \end{pmatrix}(1) \\
&\qquad\qquad (3.12) \\
&\qquad + \sum_{k=1}^{n-1}\sum_{j=k+1}^{n-1}(-1)^{k+j}\theta\begin{pmatrix} \ldots\mathbf{h}_{k-1},\mathbf{h}_k\mathbf{h}_{k+1},\mathbf{h}_{k+2},\ldots \\ \ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots \\ \ldots,\mathbf{g}_n \end{pmatrix}(1) \\
&\qquad\qquad (3.9) \\
&\qquad + \sum_{j=0}^{n-1}(-1)^{n+j+1}\theta(\mathbf{h}_1,\ldots,\mathbf{h}_j,\mathbf{s}_j,\mathbf{g}_{j+1},\ldots,\mathbf{g}_{n-1})(1). \\
&\qquad\qquad (3.13)
\end{aligned}
$$

□

3.3.4 Osorno Formula

In this section we investigate the special case of trivial A. Thus $B = \pi_1(\mathcal{K})$ is a finite group. We will write B for $\mathcal{K}$ where appropriate, e.g., 2-Rep(B) = 2-Rep$(\mathcal{K})$ etc. The degree one 2-representations of B are in bijection with elements of the Schur multiplier over $\mathbb{K}$:

Proposition 3.3.7. *(Elgueta [11, 5.3]) The group of degree one 2-representations* $(\text{2-Rep}_1(B), \boxtimes)$ *is isomorphic to* $H^2(B, \mathbb{K}^\times)$ *where the multiplicative group* $\mathbb{K}^\times$ *is a trivial* $\mathbb{Z}G$*-module.*

Proof. Let $\Theta \in \text{2-Rep}_1(B)$, this tells us $\Theta^0(*) = 1$, and $\Theta^1(b) : (\mathbb{K}) \to (\mathbb{K})$ for all $b \in B$. Moreover as $A = 1$, $\Theta^1(b, a)$ is predetermined as $\Theta^1(b, 1)\Theta^1(b, 1) = \Theta^1(b, 1)$ giving $\Theta^1(b, 1) = 1_{\mathbb{K}} \in \mathbb{K}$. From the triangle axiom we have that $\Theta^2(b, 1_B)^{-1} = \Theta^2_* = \Theta^2(1_B, b)^{-1}$ for all $b \in B$. So Θ is uniquely determined by its maps $\Theta^2(b_1, b_2) : (\mathbb{K}) \circ (\mathbb{K}) = (\mathbb{K}) \Rightarrow (\mathbb{K})$, which can be thought of as a map $\Theta^2 = \mu_\Theta : B \times B \to \mathbb{K}^\times$.

As Θ abides by the pentagon axiom we have that

$$\Theta^2(b_1b_2,b_3)\circ(\Theta^2(b_1,b_2)\diamond Id_{\Theta^2(b_3)}) = \begin{array}{c}\Theta^2(b_1,b_2b_3)\circ(Id_{\Theta^2(b_1)}\diamond\Theta^2(b_2,b_3))\\ \circ\alpha(\Theta^1(b_1),\Theta^1(b_2),\Theta^1(b_3))\end{array}$$

giving that

$$\mu_\Theta(b_2,b_3)\mu_\Theta(b_1b_2,b_3)^{-1}\mu_\Theta(b_1,b_2b_3)\mu_\Theta(b_1,b_2)^{-1} = 1_{\mathbb{K}}.$$

So we get a well defined map $\phi : 2\text{-Rep}_1(G) \to H^2(G,\mathbb{K}^\times)$ by $\Theta \mapsto [\mu_\Theta]$. The definition of tensor product of representations gives us $\mu_{\Theta\boxtimes\Phi}(b_1,b_2) = \mu_\Theta(b_1,b_2)\mu_\Phi(b_1,b_2)$. This is surjective as for any class $[\mu]\in H^2(G,\mathbb{K}^\times)$, we have from above μ gives rise to a 2-representation.

Suppose we have $\Theta,\Phi\in 2\text{-Rep}_1(G)$ such that there exists a map $C^1(B,\mathbb{K}^\times)\ni\nu : B\to\mathbb{K}^\times$ such that

$$\mu_\Phi(b_1,b_2)\nu(b_2)\nu(b_1b_2)^{-1}\nu(b_1) = \mu_\Theta(b_1,b_2).$$

Then ν defines a natural transformation between $\Theta,\Phi : \tilde{B}\to\text{Vect}_{\mathbb{K}}$ where the components have values $\nu(b)$. The condition $\mu_\Phi(b_1,b_2)\nu(b_2)\nu(b_1) = \nu(b_1b_2)\mu_\Theta(b_1,b_2)$ gives us the compatibility condition

$$\begin{tikzcd}
\Theta^1(b_1)\diamond\Theta^1(b_2) \arrow[r, Rightarrow, "\nu_{b_1}\diamond\nu_{b_2}"] \arrow[d, Rightarrow, "\Theta^2(b_1,b_2)"] & \Phi^1(b_1)\diamond\Phi^1(b_2) \arrow[d, Rightarrow, "\Phi^2(b_1,b_2)"] \\
\Theta^1(b_1b_2) \arrow[r, Rightarrow, "\nu_{b_1b_2}"] & \Phi^1(b_1b_2)
\end{tikzcd}$$

so Θ and Φ are equivalent as 2-representations making ϕ injective, thus bijective. □

Since A is trivial we drop h from the notation for the Ganter-Kapranov 2-character: $\mathfrak{X}(\mathbf{b},\mathbf{a}) := \mathfrak{X}(\mathbf{b},\mathbf{a},1)$. Let us compute its value on a degree one 2-representation:

Theorem 3.3.8. *(Rumynin Wendland [48]) Let $\mathbf{a},\mathbf{b}\in B$ be commuting elements, Θ a degree one 2-representation of B, $\mu\in Z^2(B,\mathbb{K}^\times)$ a cocycle such that $[\mu]=\{\Theta\}$. Then*

$$\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta,B\rangle) = \mu(\mathbf{b},\mathbf{a}^{-1})\mu(\mathbf{a}^{-1},\mathbf{b})^{-1}.$$

Proof. We remind the reader of the definition of the 2-character.

$$\mathbb{X}_\Theta(\mathbf{b},\mathbf{a},h)\left(\begin{tikzcd} n \arrow[r, bend left, "\Theta(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] & n \end{tikzcd} \right) = \begin{tikzcd} n \arrow[r, bend left, "\Theta(\mathbf{a})"] \arrow[r, bend right, "\Theta(\mathbf{a})"'] \arrow[r, phantom, "\Uparrow Id"] \arrow[rrr, bend left=60, "\Theta(\mathbf{b})"] \arrow[rrr, bend right=60, "\mathbf{i}_n"'] \arrow[rrr, phantom, bend left=35, "\Uparrow {[\mathbf{a},\mathbf{b},\mathbf{a}^{-1},h]}"] \arrow[rrr, phantom, bend right=35, "\Uparrow {[\mathbf{a},1,\mathbf{a}^{-1},1]^{-1}}"] & n \arrow[r, bend left, "\Theta(\mathbf{b})"] \arrow[r, bend right, "\mathbf{i}_n"'] \arrow[r, phantom, "\Uparrow v"] & n \arrow[r, bend left, "\Theta(\mathbf{a}^{-1})"] \arrow[r, bend right, "\Theta(\mathbf{a}^{-1})"'] \arrow[r, phantom, "\Uparrow Id"] & n \end{tikzcd}$$

where $\Theta(\mathbf{b}) = \Theta^1_{\star,\star}(\mathbf{b})$ and $[\mathbf{a},\mathbf{b},\mathbf{a}^{-1},h]$ is a composition of the natural morphism $\Theta(\mathbf{a}) \diamond \Theta(\mathbf{b}) \diamond \Theta(\mathbf{a}^{-1}) \to \Theta(\mathbf{aba}^{-1})$ and the action $h\,\cdot : \Theta(\mathbf{aba}^{-1}) \to \Theta(\mathbf{b})$. In our case $h = 1 \in A$ is always trivial so as in the notation of Proposition 3.3.7, we have that the map $[\mathbf{a},\mathbf{b},\mathbf{a}^{-1},h] = \mu(\mathbf{a},\mathbf{b})\mu(\mathbf{ab},\mathbf{a}^{-1}) = \mu(\mathbf{a},\mathbf{ba}^{-1})\mu(\mathbf{b},\mathbf{a}^{-1})$. For the final computation note the following two interesting applications of the cocycle condition $\mu(\mathbf{b}_2,\mathbf{b}_3)\mu(\mathbf{b}_1\mathbf{b}_2,\mathbf{b}_3)^{-1}\mu(\mathbf{b}_1,\mathbf{b}_2\mathbf{b}_3)\mu(\mathbf{b}_1,\mathbf{b}_2)^{-1} = 1$

$$1 = \mu(1,\mathbf{b})\mu(\mathbf{a},\mathbf{b})^{-1}\mu(\mathbf{a},\mathbf{b})\mu(\mathbf{a},1)^{-1} = \mu(1,\mathbf{b})\mu(\mathbf{a},1)^{-1} \qquad \text{with } \mathbf{b}_1 = \mathbf{a}, \mathbf{b}_2 = 1, \mathbf{b}_3 = \mathbf{b} \tag{3.15}$$

$$1 = \mu(a^{-1},\mathbf{b})\mu(1,\mathbf{b})^{-1}\mu(\mathbf{a},\mathbf{a}^{-1}\mathbf{b})\mu(\mathbf{a},\mathbf{a}^{-1})^{-1} \qquad \text{with } \mathbf{b}_1 = \mathbf{a}, \mathbf{b}_2 = \mathbf{a}^{-1}, \mathbf{b}_3 = \mathbf{b}. \tag{3.16}$$

So we have that

$$\begin{aligned}
\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta, B\rangle) &= [\mathbf{a},\mathbf{b},\mathbf{a}^{-1},1] \circ [\mathbf{a},1,\mathbf{a}^{-1},1] \\
&= (\mu(\mathbf{a},\mathbf{ba}^{-1})\mu(\mathbf{b},\mathbf{a}^{-1}))(\mu(\mathbf{a},\mathbf{a}^{-1})^{-1}\mu(\mathbf{a},1)^{-1}) \\
&= \mu(\mathbf{b},\mathbf{a}^{-1})(\mu(1,\mathbf{b})^{-1}\mu(\mathbf{a},\mathbf{a}^{-1}\mathbf{b})\mu(\mathbf{a},\mathbf{a}^{-1})^{-1}) && \text{as } \mathbf{ab} = \mathbf{ba} \text{ and (3.15)} \\
&= \mu(\mathbf{b},\mathbf{a}^{-1})\mu(\mathbf{a}^{-1},\mathbf{b})^{-1} && \text{by (3.16).}
\end{aligned}$$

□

Occasionally in the literature different choices are made in the definition of $\mathfrak{X}$, then the formula for $\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta, G\rangle)$ in Theorem 3.3.8 changes to its reciprocal. Choices leading to the reciprocal are using right representations instead of left ones or using $\mathbf{a}^{-1}\mathbf{ba}$ in the definition of Ganter-Kapranov 2-character. We are ready to derive a formula for an irreducible 2-representation:

Corollary 3.3.9. *(Rumynin Wendland [48]) Let Θ be a degree one 2-representation of a subgroup $P \leq B$, $\mu \in Z^2(P,\mathbb{K}^\times)$ a cocycle such that $\{\Theta\} = [\mu]$. Let T be a right* *transversal to P in B. If ${}^{\mathbf{t}}\mathbf{a} := \mathbf{tat}^{-1}$ then*

$$\begin{aligned}\mathfrak{X}(\mathbf{b},\mathbf{a})(\langle\Theta,P\rangle) &= \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu(({}^{\mathbf{t}}\mathbf{a})^{-1},\ {}^{\mathbf{t}}\mathbf{b})} \\ &= \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{b})({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu(1,1)}.\end{aligned}$$

Proof. If $\mathbf{g}$ and $\mathbf{h}$ commute, then similarly to (3.15) and (3.16) we get the following identity

$$\mu(\mathbf{g},\mathbf{hg}^{-1}) = \mu(\mathbf{g},\mathbf{g}^{-1}\mathbf{h}) = \mu(\mathbf{g}^{-1},\mathbf{h})^{-1}\mu(\mathbf{g},\mathbf{g}^{-1})\mu(1,\mathbf{h}) = \mu(\mathbf{g}^{-1},\mathbf{h})^{-1}\mu(\mathbf{g},\mathbf{g}^{-1})\mu(1,1).$$

Using Theorem 3.3.8, Theorem 3.3.5 and the definition of the mark homomorphism we compute the character:

$$\begin{aligned}\mathfrak{X}(\mathbf{b},\mathbf{a})(\Theta,P) &= \frac{1}{|P|}\sum_{\mathbf{g}\in G,\ {}^{\mathbf{g}}\mathbf{b},{}^{\mathbf{g}}\mathbf{a}\in P} \mu({}^{\mathbf{g}}\mathbf{b},({}^{\mathbf{g}}\mathbf{a})^{-1})\mu(({}^{\mathbf{g}}\mathbf{a})^{-1},\ {}^{\mathbf{g}}\mathbf{b})^{-1} \\ &= \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{b},{}^{\mathbf{t}}\mathbf{a}\in P} \mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu(({}^{\mathbf{t}}\mathbf{a})^{-1},\ {}^{\mathbf{t}}\mathbf{b})^{-1} \\ &= \sum_{\mathbf{t}\in T,\ {}^{\mathbf{t}}\mathbf{a},{}^{\mathbf{t}}\mathbf{b}\in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{b})({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu({}^{\mathbf{t}}\mathbf{a},({}^{\mathbf{t}}\mathbf{a})^{-1})\mu(1,1)}.\end{aligned}$$

□

Corollary 3.3.9 allows us to compute the value of the Ganter-Kapranov 2-character on any 2-representation in terms of its decorated B-set [21], i.e. a finite B-set X, decorated with a cocycle $\mu_x \in Z^2(B_x,\mathbb{K}^\times)$ at every point $x \in X$. An alternative data describing a representation is a cocycle on a permutation module [45, Proposition 1]. To describe we need a notation $(\mathbb{K}^\times)^X$ for the permutation $\mathbb{Z}B$-module of all the functions $f : X \to \mathbb{K}^\times$. Such a function f is given by a collection of its values $(f(x)) = (\alpha_x)_{x\in X}$, i.e., non-zero field elements $\alpha_x \in \mathbb{K}^\times$. The action is left: $\mathbf{b}\cdot(\alpha_x) = (\alpha_{\mathbf{b}\cdot x})$. On the level functions it is given by $[\mathbf{b}\cdot f](x) = f(\mathbf{b}^{-1}\cdot x)$.

Proposition 3.3.10. *([45, Prop. 1] and [11, 5.4]) There is a one-to-one correspondence between equivalence classes of 2-representations of B over $\mathbb{K}$ and pairs (X, $[\theta]$) where X is a finite B-set and $[\theta] \in H^2(B,(\mathbb{K}^\times)^X)$.*

Proof. Theorem 3.1.43 associates to a 2-representation Θ a unique (up to conjugacy and an isomorphism) a collection (P_i,Φ_i) of pairs a subgroup P_i and a degree one 2-representation Φ_i of P_i so that

$$\Theta \cong \boxplus_i \Phi_i \uparrow_{P_i}^{B}.$$

Proposition 3.3.7 gives cohomology classes $\{\Phi_i\} \in H^2(P_i, \mathbb{K}^\times)$. The permutation module $(\mathbb{K}^\times)^{B/P_i}$ is naturally isomorphic to the coinduced module $\mathrm{Coind}_{P_i}^B(\mathbb{K}^\times)$, thus, we can use Shapiro isomorphism (see Theorem 3.3.6) to get unique cohomology classes $\psi(\{\Phi_i\}) \in H^2(B, (\mathbb{K}^\times)^{B/P_i})$. We have associated the set and the cohomology class

$$X \coloneqq \coprod_i B/P_i, \quad [\theta] \coloneqq \bigoplus_i \psi(\{\Phi_i\}) \in \bigoplus_i H^2(B, (\mathbb{K}^\times)^{B/P_i}) \cong H^2(B, (\mathbb{K}^\times)^X)$$

to Θ. All these steps are reversible. $\square$

Given a finite B-set X, $x \in X$ and a cochain $\theta \in C^2(B, (\mathbb{K}^\times)^X)$, we write $\theta^x \in C^2(B, \mathbb{K}^\times)$ for the component cochains. We have $\theta(\mathbf{g}, \mathbf{h})(x) = \theta^x(\mathbf{g}, \mathbf{h})$ on the level of functions $X \to \mathbb{K}^\times$. We are ready to give our proof of Osorno Formula:

Theorem 3.3.11. *([45, Theorem 1]) Let Θ be a 2-representation of B that corresponds to a B-set X and a cohomology class $[\theta]$ for some cochain $\theta \in Z^2(B, (\mathbb{K}^\times)^X)$. Then*

$$\mathfrak{X}_\Theta(\mathbf{b}, \mathbf{a}) = \sum_{x \in X,\ x = \mathbf{a}\cdot x = \mathbf{b}\cdot x} \frac{\theta^x(\mathbf{b}, \mathbf{a}^{-1})}{\theta^x(\mathbf{a}^{-1}, \mathbf{b})} = \sum_{x \in X,\ x = \mathbf{a}\cdot x = \mathbf{b}\cdot x} \frac{\theta^x(\mathbf{b}, \mathbf{a}^{-1})\theta^x(\mathbf{a}, \mathbf{b}\mathbf{a}^{-1})}{\theta^x(\mathbf{a}, \mathbf{a}^{-1})\theta^x(1,1)}$$

for any commuting $\mathbf{a}, \mathbf{b} \in B$.

Proof. The component θ^x is not a cocycle, in general. Yet for the terms in the formula it works as a cocycle: the restriction $\theta^x \mid_{<\mathbf{a},\mathbf{b}>}$ is a cocycle on $< \mathbf{a}, \mathbf{b} >$ since $x = \mathbf{a} \cdot x = \mathbf{b} \cdot x$. Thus, the second and the third expressions are equal.

Since $\mathfrak{X}_{\Theta \boxplus \Psi}(\mathbf{b}, \mathbf{a}) = \mathfrak{X}_\Theta(\mathbf{b}, \mathbf{a}) + \mathfrak{X}_\Psi(\mathbf{b}, \mathbf{a})$ and the second expression is additive on B-orbits. It suffices to prove the theorem under an assumption that Θ is irreducible. Without loss of generality $\Theta = \Psi \uparrow_P^B$ for a degree one 2-representation of some subgroup P and $X = B/P$. Let $\mu \in Z^2(P, \mathbb{K}^\times)$ a cocycle such that $\{\Psi\} = [\mu]$. A right transversal T (with $\mathbf{t}_0 = 1$) to P in B is in natural bijection with X via $\mathbf{t} \mapsto \mathbf{t}^{-1}P$. We use T and μ to decorate X with cocycles:

$$\mu_{\mathbf{t}^{-1}P} \in Z^2(\mathbf{t}^{-1}P\mathbf{t}, \mathbb{K}^\times), \quad \mu_{\mathbf{t}^{-1}P}(\mathbf{g}, \mathbf{h}) \coloneqq \mu({}^{\mathbf{t}}\mathbf{g}, {}^{\mathbf{t}}\mathbf{h}).$$

By Corollary 3.3.9,

$$\mathfrak{X}_\Theta(\mathbf{b}, \mathbf{a}) = \sum_{\mathbf{t} \in T,\ {}^{\mathbf{t}}\mathbf{a}, {}^{\mathbf{t}}\mathbf{b} \in P} \frac{\mu({}^{\mathbf{t}}\mathbf{b}, ({}^{\mathbf{t}}\mathbf{a})^{-1})}{\mu(({}^{\mathbf{t}}\mathbf{a})^{-1}, {}^{\mathbf{t}}\mathbf{b})} = \sum_{\mathbf{t} \in T,\ \mathbf{a}\mathbf{t}^{-1}P = \mathbf{b}\mathbf{t}^{-1}P = \mathbf{t}^{-1}P} \frac{\mu_{\mathbf{t}^{-1}P}(\mathbf{b}, \mathbf{a}^{-1})}{\mu_{\mathbf{t}^{-1}P}(\mathbf{a}^{-1}, \mathbf{b})} \ .$$

The cohomology classes of the cocycles $\mu_{\mathbf{t}^{-1}P}$ and θ are related via Shapiro isomorphisms with different subgroups: $[\mu_{\mathbf{t}^{-1}P}] = \phi_{\mathbf{t}^{-1}P\mathbf{t}}([\theta])$. Each term of the last sum depends only on cohomology class $[\mu_{\mathbf{t}^{-1}P}]$. Hence, we may assume that $\mu_{\mathbf{t}^{-1}P} = \phi_{\mathbf{t}^{-1}P\mathbf{t}}(\theta)$ without loss of generality. The condition $\mathbf{a}, \mathbf{b} \in \mathbf{t}^{-1}P\mathbf{t}$ ensures that

$$\mu_{\mathbf{t}^{-1}P}(\mathbf{a}, \mathbf{b}) = \phi_{\mathbf{t}^{-1}P\mathbf{t}}(\theta)(\mathbf{a}, \mathbf{b}) = \theta(\mathbf{a}, \mathbf{b})(\mathbf{t}^{-1}) = \theta^{\mathbf{t}^{-1}P}(\mathbf{a}, \mathbf{b})$$

facilitating the last in the proof:

$$\mathfrak{X}_{\Theta}(\mathbf{b}, \mathbf{a}) = \sum_{\mathbf{t} \in T,\ \mathbf{a}\mathbf{t}^{-1}P = \mathbf{b}\mathbf{t}^{-1}P = \mathbf{t}^{-1}P} \frac{\theta^{\mathbf{t}^{-1}P}(\mathbf{b}, \mathbf{a}^{-1})}{\theta^{\mathbf{t}^{-1}P}(\mathbf{a}^{-1}, \mathbf{b})} = \sum_{x \in X,\ \mathbf{a} \cdot x = \mathbf{b} \cdot x = x} \frac{\theta^{x}(\mathbf{b}, \mathbf{a}^{-1})}{\theta^{x}(\mathbf{a}^{-1}, \mathbf{b})} \ .$$

$\square$

3.4 Conclusion

The author anticipates two main areas of further study here. First we would like to use Theorem 3.2.3 to translate the work of Ganter and Usher [17] from the language of skeletal 2-groups into that of crossed modules. This work is currently being pursued by Hristova and the author. Secondly the author would like to see a nice method to summarise the data for a 2-representation of a crossed module. Then one could use this data to write an explicit formula for the 2-character, as done by Osorno [46]. The trouble lies in correctly defining the relationship between the cohomological data of $\pi_1(\mathcal{K})$ and the action of H.

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