The micro-world of cographs*

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Abstract

Cographs constitute a small point in the atlas of graph classes. However, by zooming in on this point, we discover a complex world, where many parameters jump from finiteness to infinity. In the present paper, we identify several milestones in the world of cographs and create a hierarchy of graph parameters grounded on these milestones.

Keywords: cographs; graph parameters; well-quasi-ordering

1 Introduction

Large things are seen from a distance, but to examine small things, one needs to look up-close. Cographs constitute a small class. In particular, it has zero entropy [5], i.e.

$$\lim_{n\to\infty} \frac{\log_2 X_n}{\binom{n}{2}} = 0,$$

where X_n is the number of n-vertex labelled graphs in this class. Also, cographs are simple structurally. In particular, the clique-width of any cograph is at most 2, implying polynomial-time solutions for a variety of NP-hard problems, when restricted to cographs. In other words, in the continuum of hereditary classes the cographs constitute a tiny point. In the present paper, we analyse this point with a "magnifying glass", trying to spot the details. With a closer look at this class we discover a complex world and observe that many important parameters can be arbitrarily large within cographs. This is the case, for instance, for chromatic number, co-chromatic number, matching number, tree-width, linear clique-width and many others. Interestingly, these parameters jump to infinity on specific subclasses of cographs. The existence of such "critical points" is due to the fact that the class of cographs is well-quasi-ordered under the induced subgraph relation [19]. This implies, as we show in the paper, that for every parameter κ which is unbounded in the class of cographs, there exists a finite collection $M(\kappa)$ of inclusion-wise minimal hereditary

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subclasses of cographs, where κ can be arbitrarily large. This observation suggests a simple way of comparing two parameters: a parameter κ_1 is stronger than a parameter κ_2 if for every class $X \in M(\kappa_1)$ there exists a class $Y \in M(\kappa_2)$ such that $Y \subseteq X$. In other words, κ_1 is stronger than κ_2 if the family of classes where κ_1 is bounded contains the family of classes where κ_2 is bounded.

For some parameters, identifying minimal classes is an easy task. For instance, since cographs are perfect, the chromatic number is bounded if and only if the clique number is bounded and hence the class of complete graphs is the only minimal hereditary subclass of cographs where the chromatic number is unbounded. However, in general, identifying minimal classes is far from being trivial, as the example of linear clique-width shows. The authors of [14] develop a sophisticated approach to show that there exist precisely two minimal hereditary subclasses of cographs where linear clique-width is unbounded: the class of (P_4, C_4)-free graphs, also known as the quasi-threshold [47] or trivially perfect [32] graphs, and the class of their complements.

In the present paper, we characterize a variety of other graphs parameters in terms of minimal hereditary subclasses of cographs where these parameters are unbounded, which is the content of Section 3. In Section 2 we introduce basic terminology and notation used throughout the paper. In particular, in Section 2.1 we describe a collection of subclasses of cographs that play a critical role in our study, and Section 2.2 is devoted to well-quasi-ordering and related notions. Finally, Section 4 concludes the paper with a number of open problems.

2 Preliminaries

All graphs in this paper are simple, i.e., finite, undirected, without loops and without multiple edges. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Two sets $A, B \subseteq V(G)$ are said to be *complete* to each other if every possible edge between them appears in G, and *anticomplete* to each other if they are no vertex of A is adjacent to a vertex on B.

As usual, P_n, C_n, K_n denote a chordless path, a chordless cycle and a complete graph with n vertices, respectively. Also, $K_{n,m}$ is a complete bipartite graphs with parts of size n and m.

A *clique* in a graph is a subset of pairwise adjacent vertices and an *independent set* is a subset of pairwise non-adjacent vertices. The Ramsey number $R(p, q)$ is the smallest natural number such that any graph with $R(a, b)$ vertices contains a clique of size a or an independent set of size b.

The complement of a graph G is denoted by $\overline{G}$. Given two graphs G and H, we denote by

- $G \cup H$ the disjoint union of G and H. The disjoint union of p copies of G will be denoted by pG.

- $G \times H$ the join of G and H, i.e., the graph obtained from $G \cup H$ by adding all possible edges between G and H. In other words, $G \times H = \overline{\overline{G} \cup \overline{H}}$.

We say that a graph G is H-free if G does not contain a copy of H as an induced subgraph.

A class of graphs is *hereditary* if it is closed under taking induced subgraphs. It is well-known (and not difficult to see) that a class is hereditary if and only if it can be characterized in terms of minimal forbidden induced subgraphs.

For a parameter κ, a class X is said to be κ-bounded if there exists a constant C such that for any $G \in X, \kappa(G) \leq C$, and κ-unbounded otherwise.

2.1 Cographs

A graph G is a *cograph* if every induced subgraph of G with at least two vertices is either disconnected or the complement of a disconnected graph. Alternatively, G is a cograph if it can be obtained from one-vertex graphs by recursively applying the operations of disjoint union and join. It is clear from the definition that cographs constitute a hereditary class, and that it is self-complementary in the sense that the complement of a cograph is again a cograph.

Cographs have been introduced independently by many researchers, but perhaps the first comprehensive study of this class was reported in [16]. That paper presents various characterisations of the class of cographs, one of which states that it is precisely the class of P_4-free graphs.

Since the discovery of cographs, this class has attracted the attention of thousands of researchers both within mathematics and beyond (see, e.g., [34]). Cographs are closely related to some other mathematical structures, such as separable permutations [1] or read-once Boolean functions [33], and they inspired the introduction of several related notions and classes of graphs, such as bi-cographs [30] or graphs with few P_4s [10].

Cographs constitute a subclass of several important graph classes, such as permutation graphs and perfect graphs. On the other hand, they also contain a number of important classes as subclasses, such as threshold graphs [39] and quasi-threshold graphs [47]. We will use special notation for these and some other subclasses of cographs as follows:

- $\mathcal{Q}$ the class of *quasi-threshold graphs*, i.e., (P_4, C_4)-free graphs,
- $\mathcal{T}$ the class of *threshold graphs*. This is the class of $(P_4, C_4, 2K_2)$-free graphs, i.e., the intersection of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$.
- $\mathcal{U}$ the class of P_3-free graphs, i.e., graphs every connected component of which is a clique.
- $\mathcal{K}$ the class of complete graphs.
- $\mathcal{F}$ the class of star forests, i.e., graphs every connected component of which is a star. This is the class of (P_4, C_4, K_3)-free graphs, i.e., the class of bipartite graphs in $\mathcal{Q}$.
- $\mathcal{M}$ the class of graphs of vertex degree at most 1. This is the class of (P_3, K_3)-free graphs, i.e., the class of bipartite graphs in $\mathcal{U}$.

$\mathcal{B}$ the class of complete bipartite graphs (an edgeless graph is counted as complete bipartite with one part being empty). This is the class of $(\overline{P}_3, K_3)$-free graphs, i.e., the class of bipartite graphs in $\overline{\mathcal{U}}$.

$\mathcal{S}$ the class of *stars*, i.e., graphs of the form $K_{1,n}$ and their induced subgraphs.

Since the complement of a cograph is again a cograph, with every subclass $\mathcal{X}$ of cographs we associate the subclass $\overline{\mathcal{X}}$ of complements of graphs in $\mathcal{X}$.

As we mentioned earlier, cographs enjoy many nice properties. For the purpose of the present paper, the most important one is well-quasi-orderability, which we define in the next section.

2.2 Well-quasi-orderings and beyond

A binary relation $\leq$ on a set W is a *quasi-order* (also known as *preorder*) if it is reflexive and transitive. Two elements $x, y \in W$ are said to be *comparable* with respect to $\leq$ if either $x \leq y$ or $y \leq x$. Otherwise, x and y are *incomparable*. A set of pairwise comparable elements is called a *chain* and a set of pairwise incomparable elements an *antichain*. If $x \leq y$ and $y \not\leq x$, we write $x < y$. A chain $x_1 > x_2 > \ldots$ is called *strictly decreasing*. A quasi-order $(W, \leq)$ is a *well-quasi-ordering* ("wqo", for short) if it contains neither infinite strictly decreasing chains nor infinite antichains.

The celebrated result of Robertson and Seymour [44] states that the set of all simple graphs is well-quasi-ordered with respect to the minor relation. However, the induced subgraph relation is not a wqo, as the cycles create an infinite antichain with respect to this relation. On the other hand, with some restrictions, it may become a wqo, which is the case, for instance, for cographs [19].

A dive in the literature reveals that, in fact, cographs enjoy the stronger property of *better-quasi-ordering* ("bqo", for short) under the induced subgraph relation. The full definition of bqo is technical, and outside of the scope of this paper (see, e.g., [6] for a short introduction). The fact that cographs are bqo can be derived as follows.

A map $f : (X, \leq) \to (Y, \preceq)$ is called a *quasi-embedding* if, for all $a, b \in X$, $f(a) \preceq f(b) \implies a \leq b$. It is immediate from the definitions that, if there exists a quasi-embedding $X \to Y$, and Y is wqo, then X is wqo. This remains true when replacing "wqo" with "bqo" (see, e.g., [6], Lemma 5.3). In [19], Damaschke proves that cographs are wqo by producing a quasi-embedding to the set of finite trees labelled using 4 labels, ordered by tree embedding. The fact that finite labelled trees are wqo is the statement of Kruskal's famous tree theorem. Nash-Williams proved in [40] that infinite (in addition to finite) trees are bqo, and this was later strengthened by Laver (in [36], Theorem 2.2) to labelled infinite trees, provided the set of labels is bqo (which is the case for any finite set).

The additional strength of bqo (as opposed to just wqo) can appear subtle at first, but it is in fact very effective, and allows us to derive concrete results about cographs. Let $(X, \leq)$ be a quasi-order. A subset $L \subseteq X$ is a *lower closed set* if for all $a \in L$ and $b \in X$, $b \leq a$ implies $b \in L$. We denote by $\mathcal{L}(X)$ the set of lower closed sets of X. The strength of bqo can be summarised with the following proposition (see, e.g., [6]).

Proposition 1. *Suppose $(X, \leq)$ is bqo. Then $(X, \leq)$ is wqo, and $(\mathcal{L}(X), \subseteq)$ is bqo.*

As an immediate consequence, we draw the following conclusion.

Corollary 1. *The set of hereditary subclasses of cographs is wqo by inclusion.*

This implies, in particular, that for every parameter κ which is unbounded in the class of cographs, there is a finite collection $M(\kappa)$ of inclusion-wise minimal hereditary subclasses of cographs where this parameter is unbounded. Moreover, every subclass of cographs in which κ is unbounded contains one of the minimal classes.

Now let κ_1 and κ_2 be two graph parameters. We will say that κ_1 is *stronger* than κ_2 if the family of κ_1-bounded hereditary classes contains the family of κ_2-bounded hereditary classes. We can naturally adapt this definition when restricting ourselves to a class $\mathcal{X}$ of graphs by saying κ_1 is stronger than κ_2 in $\mathcal{X}$ if the family of κ_1-bounded hereditary subclasses of $\mathcal{X}$ contains the family of κ_2-bounded hereditary subclasses of $\mathcal{X}$. For the remainder of the paper, when talking about the strength of parameters, we will mean "strength in the class of cographs" unless otherwise specified.

By analogy with graph classes characterised by minimal forbidden induced subgraphs, we can compare two parameters from their sets $M(\kappa)$ as follows.

Lemma 1. *Parameter κ_1 is stronger than κ_2 if and only if for every minimal hereditary class $\mathcal{F}_1 \in M(\kappa_1)$, there is a minimal hereditary class $\mathcal{F}_2 \in M(\kappa_2)$ such that $\mathcal{F}_2 \subseteq \mathcal{F}_1$.*

Proof. To prove the "if" direction, assume that for every minimal hereditary class $\mathcal{F}_1$ where κ_1 is unbounded, there is a minimal hereditary class $\mathcal{F}_2$ where κ_2 is unbounded such that $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Now let $\mathcal{X}$ be a κ_2-bounded hereditary class. Since any minimal class where κ_1 is unbounded contains a class where κ_2 is unbounded, it follows $\mathcal{X}$ cannot contain any minimal class where κ_1 is unbounded, and so by Corollary 1, $\mathcal{X}$ is κ_1-bounded, showing κ_1 is stronger than κ_2.

Conversely, suppose κ_1 is stronger than κ_2, and let $\mathcal{F}_1$ be a minimal hereditary class where κ_1 is unbounded. Since κ_1 is stronger, κ_2 is also unbounded in $\mathcal{F}_1$, and by Corollary 1, $\mathcal{F}_1$ contains a minimal class $\mathcal{F}_2$ where κ_2 is unbounded, as required. □

One consequence of this result is that the strength relation defined on the set of parameters is a quasi-order in the class of cographs. Moreover, from better-quasi-orderability of cographs we derive that this relation is a well-quasi-order.

Corollary 2. *The set of graph parameters is wqo by their strength in the class of cographs.*

Proof. Note that the set of classes where a parameter is bounded is downwards closed under inclusion. The claim then follows immediately from Proposition 1. □

3 Graph parameters

We start by reporting some known results or results that readily follows from known results. In particular, directly from Ramsey's Theorem we derive the following conclusion.

Proposition 2. *The class $\mathcal{K}$ of complete graphs and the class of $\mathcal{S}$ of stars are the only two minimal hereditary classes of graphs of unbounded maximum vertex degree.*

To report more results, we denote by

$\alpha(G)$ the *independence number* of G, i.e., the size of a maximum independent set in G,

$\omega(G)$ the *clique number* of G, i.e., the size of a maximum clique in G,

$\chi(G)$ the *chromatic number* of G, i.e., the minimum number of subsets in a partition of $V(G)$ such that each subset is an independent set,

$y(G)$ the *clique partition* (also known as *clique cover*) *number*, i.e., the minimum number of subsets in a partition of $V(G)$ such that each subset is a clique.

Clearly, the class $\mathcal{K}$ of complete graphs is the only minimal hereditary class of unbounded clique number, i.e., by forbidding a complete graph we obtain a class of bounded clique number. Also, it is not difficult to see that $\mathcal{K}$ is a minimal hereditary class of unbounded chromatic number. However, it is not the only minimal hereditary class of unbounded chromatic number, i.e., forbidding a complete graph does not guarantee a bound on the chromatic number. Moreover, as shown by Erdős [22] chromatic number is unbounded even in the class of $(C_3, C_4, \ldots, C_k)$-free graphs for any value of k, which means that in the universe of hereditary classes chromatic number cannot be characterized by means of minimal classes where this parameter is unbounded. On the other hand, when we restrict ourselves to cographs such a characterization is possible, which is due to the fact that cographs are perfect, and hence $\omega(G) = \chi(G)$ for any cograph G. As a result, we obtain the following conclusion.

Proposition 3. *The class $\mathcal{K}$ of complete graphs is the only minimal hereditary subclass of cographs of unbounded clique number and chromatic number.*

The *degeneracy* of a graph G is the smallest value of k such that every induced subgraph of G has a vertex of degree at most k. It is not difficult to see that the class $\mathcal{K}$ of complete graphs and the class of $\mathcal{B}$ of complete bipartite graphs are minimal hereditary classes of unbounded degeneracy. However, these are not the only minimal classes, because forbidding a complete graph and a complete bipartite graph does not guarantee a bound on the degeneracy. To explain this, we observe that the degeneracy of G is bounded from below by $\chi(G) - 1$ and from above by the tree-width of G. Therefore, degeneracy and tree-width are unbounded in the class of $(C_3, C_4, \ldots, C_k)$-free graphs for any value of k, and for $k \geq 4$ the set of forbidden induced subgraphs include both a complete graph C_3 and a complete bipartite graph C_4. This discussion shows that, similarly to chromatic number, in the universe of all hereditary classes neither degeneracy nor tree-width admit a characterization in terms of minimal classes where these parameters are unbounded. On the other hand, again similarly to chromatic number, such a characterization is possible when restricting to cographs, and it is presented in the next claim.

Proposition 4. *The class $\mathcal{K}$ of complete graphs and the class of $\mathcal{B}$ of complete bipartite graphs are the only two minimal hereditary subclasses of cographs of unbounded degeneracy and tree-width.*

Proof. To prove the claim, it suffices to show that for any s and p, the tree-width of $(P_4, K_s, K_{p,p})$-free graphs is bounded by a constant. For this, we refer the reader to the following result from [8]: for every t, p, s, there exists a $z = z(t, p, s)$ such that every graph with a (not necessarily induced) path of length at least z contains either an induced P_t or an induced $K_{p,p}$ or a clique of size s. From this result it follows that $(P_4, K_s, K_{p,p})$-free graphs do not contain (not necessarily induced) paths of length $z(4, p, s)$. It is well known (see, e.g., [25]) that graphs of bounded path number (the length of a longest path) have bounded tree-width. □

The *matching number* of a graph G is the size of a maximum matching in G. The following result was proved in [18].

Lemma 2. *For any natural numbers s, t and p, there is a number $N(s, t, p)$ such that every graph with a matching of size at least $N(s, t, p)$ contains either a clique K_s or an induced bi-clique $K_{t,t}$ or an induced matching pK_2.*

A natural corollary from this result is the following characterization of the matching number in terms of minimal hereditary classes where this parameter is unbounded.

Theorem 1. *$\mathcal{M}$, $\mathcal{B}$ and $\mathcal{K}$ are the only three minimal hereditary classes of graphs of unbounded matching number.*

The *vertex cover number* of a graph G is the size of a minimum vertex cover in G. It is well known that the vertex cover number is never smaller than the matching number and never larger than twice the matching number. Therefore, the characterization of matching number given in Theorem 1 applies to the vertex cover number as well.

Theorem 2. *$\mathcal{M}$, $\mathcal{B}$ and $\mathcal{K}$ are the only three minimal hereditary classes of graphs of unbounded vertex cover number number.*

The *neighbourhood diversity* of a graph was introduced in [35] and can be defined as follows.

Definition 1. Let us say that two vertices x and y are *similar* if there is no vertex z distinguishing them (i.e., if there is no vertex z adjacent to exactly one of x and y). Vertex similarity is an equivalence relation. We denote by $nd(G)$ the number of similarity classes in G and call it the *neighbourhood diversity* of G.

Neighbourhood diversity was characterized in [38] by means of nine minimal hereditary classes of graphs where this parameter is unbounded. Six of these minimal classes contain a P_4. Therefore, when restricted to cographs, neighbourhood diversity can be characterized by three minimal classes as follows.

Theorem 3. *$\mathcal{M}$, $\overline{\mathcal{M}}$, and $\mathcal{T}$ are the only three minimal hereditary subclasses of cographs of unbounded neighbourhood diversity.*

3.1 Co-chromatic number

The *co-chromatic number* of G, denoted $z(G)$, is the minimum number of subsets in a partition of $V(G)$ such that each subset is either a clique or an independent set [23]. It is not difficult to see that the co-chromatic number can be arbitrarily large in the class of P_3-free graphs, where each graph is a disjoint union of cliques. Therefore, it is also unbounded in the complements of P_3-free graphs, also known as complete multipartite graphs. In what follows, we show that these are the only two minimal subclasses of cographs of unbounded co-chromatic number.

Lemma 3. *Let n, m, t be positive integers with $t \geq 2$. If G is a $(nK_t, \overline{mK_t})$-free cograph, then $z(G) \leq 2^{m+n-1}(t-1)$.*

Proof. Call a partition of $V(G)$ *good* if it contains at least $t-1$ cliques and $t-1$ independent sets (empty sets in the partition may count as either). We prove by induction on $m+n$ that G admits a good partition into $2^{m+n-1}(t-1)$ sets, each of which is a clique or an independent set.

If $m+n=2$ ($n=m=1$), then G is K_t-free. Hence $\chi(G)=\omega(G)\leq t-1$; we add empty sets to the partition until we reach $2(t-1)$ sets in total. This makes the partition good, and we have proved the basis for the induction. In general, put $G' := G$. We are in one of the following three cases:

(a) $G' = G_1 \cup G_2$, and both G_1 and G_2 are K_t-free, OR $G' = G_1 \times G_2$, and both G_1 and G_2 are $\overline{K_t}$-free.

(b) $G' = G_1 \cup G_2$, and both G_1 and G_2 contain a K_t, OR $G' = G_1 \times G_2$, and both G_1 and G_2 contain a $\overline{K_t}$.

(c) $G' = G_1 \cup G_2$, G_1 contains a K_t and G_2 is K_t-free, OR $G' = G_1 \times G_2$, G_1 contains a $\overline{K_t}$ and G_2 is $\overline{K_t}$-free.

As long as we are in case (c), iteratively put $G' := G_1$. We end up with a graph G' in either case (a) or (b). Note first that any good partition of G' extends to a good partition of G without increasing the number of sets. Indeed, at each step, G_2 was either K_t-free and anticomplete to the rest of the graph or $\overline{K_t}$-free and complete to the rest of the graph. The disjoint union of all K_t-free G_2s is again K_t-free and hence can be partitioned into at most $t-1$ independent sets, and we take the union of each of these sets with one of the independent sets in the good partition of G' injectively. Similarly, the join of the $\overline{K_t}$-free G_2s can be partitioned into at most $t-1$ cliques, each of which we join to one of the cliques in the good partition of G' injectively.

Now, if G' is in case (a), then G' is K_t-free or $\overline{K_t}$-free and we act like in the base case to obtain a good partition of G' (and therefore of G) in $2(t-1)$ sets. If G' is in case (c), then G_1 and G_2 are both either $(n-1)K_t$-free or $\overline{(m-1)K_t}$-free. In either case, the inductive hypothesis applies, and we have a good partition of G' of size at most

$$2^{m+n-2}(t-1) + 2^{m+n-2}(t-1) = 2^{m+n-1}(t-1).$$

Like before, this extends to a partition of G, concluding the proof. □

Lemma 3 naturally leads to the following conclusion.

Theorem 4. *The class $\mathcal{U}$ of P_3-free graphs and the class $\overline{\mathcal{U}}$ of $\overline{P}_3$-free graphs are the only two minimal hereditary subclasses of cographs of unbounded co-chromatic number.*

3.2 Lettericity

The notion of letter graphs was introduced in [41]. Recently, an intriguing connection between letter graphs and geometric grid classes of permutations [2] has been identified in [4]. We define the notion of letter graphs and the related parameter, known as lettericity, as follows.

Let A be a finite alphabet, $D \subseteq A^2$ and $w = w_1w_2 \ldots w_n$ a word over A (repetitions allowed). The letter graph $G(D, w)$ associated to w has $\{1, 2, \ldots, n\}$ as its vertex set, and two vertices $i < j$ are adjacent if and only if the ordered pair (w_i, w_j) belongs to D. A graph G is said to be a *letter graph* if there exist an alphabet A, a subset $D \subseteq A^2$ and a word $w = w_1w_2 \ldots w_n$ over A such that G is isomorphic to $G(D, w)$.

The role of D is to decode (transform) a word into a graph and therefore we refer to D as a decoder. Every graph G is trivially a letter graph over the alphabet $A = V(G)$ with the decoder $D = \{(v, w), (w, v) : \{v, w\} \in E(G)\}$. The *lettericity* of G, denoted $\ell(G)$, is the minimum k such that G is representable as a letter graph over an alphabet of k letters.

To give a less trivial example, consider the alphabet $A = \{a, b\}$ and the decoder $D = \{(a, a), (a, b)\}$. Then the word *ababababab* describes the graph represented in Figure 1. This graph can be constructed from a single vertex by means of two operations: adding a dominating vertex (corresponds to adding letter a as a prefix) or adding an isolated vertex (corresponds to adding letter b as a prefix). The class of all graphs that can be constructed by means of these two operations coincides with the class of threshold graphs defined in Section 2 as $(2K_2, C_4, P_4)$-free graphs [39]. The above discussion shows that a graph is threshold if and only if it is a letter graph over the alphabet $A = \{a, b\}$ with the decoder $D = \{(a, a), (a, b)\}$.

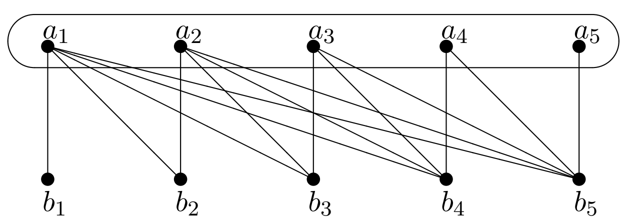

Figure 1: The letter graph of the word *ababababab* (the oval represents a clique). We use indices to indicate in which order the a-letters and the b-letters appear in the word.

Lemma 4. $\ell(nK_2) = n$.

Proof. First, it is not difficult to see that $\ell(nK_2) \leq n$, since n letters suffice (one letter per edge). Assume $\ell(nK_2) < n$, then there must exist a letter a representing at least 3 vertices of the graph. Clearly, $(a, a) \notin S$, since otherwise a triangle arises. Then the neighbour of the middle a is different from a, say b. If this neighbour appears before the middle a, it must also be adjacent to the last a. If it appears after the middle a, it must also be adjacent to the first a. In both case, b has at least two neighbours. Therefore, $\ell(nK_2) \geq n$. □

The above theorem shows that the lettericity is unbounded in the class $\mathcal{M}$ of graphs of vertex degree at most 1. Therefore, it is also unbounded in the class $\overline{\mathcal{M}}$, since $\ell(G) = \ell(\overline{G})$.

Theorem 5. *$\mathcal{M}$ and $\overline{\mathcal{M}}$ are the only two minimal hereditary subclasses of cographs of unbounded lettericity.*

Proof. To prove the theorem, we will show that for any natural numbers $p, t \geq 2$, the lettericity of a $(P_4, pK_2, \overline{tK_2})$-free graph G is at most 2^{p+t-3}. This will be shown by induction on $p + t$. Moreover, we will show that G can be represented with a decoder D containing a source letter, i.e., a letter a such that $(a, b) \in D$ for any letter b, and a sink letter, i.e., a letter b such that $(b, a) \notin D$ for any letter a.

If $p = t = 2$, then G is a threshold graph and its lettericity is at most 2, because any threshold graph can be represented over the decoder $D = \{(a, a), (a, b)\}$. In this decoder, a is a source letter and b is a sink letter.

Assume that every $(P_4, pK_2, \overline{tK_2})$-free graph with $p + t \leq k$ can be represented as a letter graph over an alphabet of at most 2^{p+t-3} letters with a decoder containing a source vertex a and a sink vertex b. Consider now a $(P_4, pK_2, \overline{tK_2})$-free graph G with $p+t = k+1$.

The presence of source and sink letters in the decoder allows us to assume that G has neither dominating nor isolated vertices. Indeed, if v is dominating, then a word for G can be constructed from a word for $G - v$ by adding a source letter as a prefix, and if v is isolated, then a word for G can be constructed from a word for $G - v$ by adding a sink letter as a prefix. Therefore, in the rest of the proof we assume that G has neither isolated nor dominating vertices.

Case 1: G is disconnected. Denote by G_1 a connected component of G and by G_2 the rest of the graph. Observe that each of G_1 and G_2 contains a K_2, since otherwise G has an isolated vertex. Therefore, each of G_1 and G_2 is $(p-1)K_2$-free and hence we can apply induction to each of G_1 and G_2. In other words, G_1 can be represented by a word ω_1 over an alphabet A_1 of size at most 2^{p+t-4} with a decoder containing a source vertex a_1 and a sink vertex b_1, and G_1 can be represented by a word ω_2 over an alphabet A_2 of size at most 2^{p+t-4} with a decoder containing a source vertex a_2 and a sink vertex b_2 (we assume that A_1 and A_2 are disjoint). Then the word $\omega = \omega_1\omega_2$ represents G over the alphabet $A_1 \cup A_2$ of size at most 2^{p+t-3} with the decoder $D = D_1 \cup D_2$. In this decoder, vertex b_2 is a sink vertex. To guarantee the presence of a source vertex, we add to D the pair (a_2, c) for every vertex $c \in A_1$. This extension transforms a_2 into a source vertex and does not change the graph represented by the word ω, since every letter from A_1 appears in ω before any appearance of a_2.

Case 2: G is connected. In this case, $\overline{G}$ is disconnected and $(P_4, tK_2, \overline{pK_2})$-free. A similar argument as above gives a representation for $\overline{G}$ with at most 2^{p+t-3} letters, and complementing the corresponding decoder produces one for G (note that when doing that, sink letters become source letters and vice-versa).

□

3.3 Boxicity

The notion of boxicity was introduced in [43] and has become the subject of research in a vast literature (see e.g. [24, 45]). The *boxicity* $\mathrm{box}(G)$ of a graph G is the minimum dimension in which G can be represented as an intersection graph of hyper-rectangles. Equivalently, it is the smallest number of interval graphs on the same set of vertices whose intersection is G. The next lemma was shown in [43]; we give here a proof for the sake of completeness.

Lemma 5. $\mathrm{box}(\overline{nK_2}) = n$.

Proof. To see that $\mathrm{box}(\overline{nK_2}) \leq n$, note that K_{2n} without an edge is an interval graph, and $\overline{nK_2}$ is the intersection of n such graphs. Conversely, note that two different matched non-edges in $\overline{nK_2}$ cannot belong to the same interval graph (since the corresponding four vertices would induce a C_4, which is not an interval graph). Hence we need at least n interval graphs to obtain $\overline{nK_2}$ as an intersection. □

Lemma 6. *Let G_1 and G_2 be two graphs. Then*

$$\mathrm{box}(G_1 \cup G_2) \leq \max(\mathrm{box}(G_1), \mathrm{box}(G_2)) \text{ and } \mathrm{box}(G_1 \times G_2) \leq \mathrm{box}(G_1) + \mathrm{box}(G_2).$$

Moreover, if G_2 is a clique, then $\mathrm{box}(G_1 \times G_2) = \mathrm{box}(G_1)$.

Proof. Suppose $G_1 = \bigcap_{i=1}^{s} A_i$ and $G_2 = \bigcap_{i=1}^{t} B_i$ where the A_i and B_i are interval graphs, and assume without loss of generality that $s \geq t$. Put $C_i = A_i \cup B_i$ for $1 \leq i \leq t$ and $C_i = A_i \cup K_{|G_2|}$ for $t < i \leq s$. Put $D_i = A_i \times K_{|G_2|}$ for $1 \leq i \leq s$ and $D_i = K_{|G_1|} \times B_{i-s}$ for $s < i \leq s + t$.

The C_i and D_i are interval graphs, and with the obvious labellings of C_i and D_i, we have $G_1 \cup G_2 = \bigcap_{i=1}^{s} C_i$ and $G_1 \times G_2 = \bigcap_{i=1}^{s+t} D_i$.

For the final claim, if $G_2 = K_{|G_2|}$ is a clique, then $G_1 \times G_2 = \bigcap_{i=1}^{s} (A_i \times K_{|G_2|})$, and each of those is an interval graph. □

Theorem 6. $\overline{\mathcal{M}}$ *is the only minimal hereditary subclass of cographs of unbounded boxicity.*

Proof. Let $n \geq 2$. We prove by induction on n that $(P_4, \overline{nK_2})$-free graphs have boxicity at most 2^{n-2}. The result is true for $n = 2$, since (P_4, C_4)-free graphs are known to be interval graphs (see, e.g., [13]).

For the induction step, suppose the result is true for some $n \geq 2$, and let G be a cograph that is $\overline{(n+1)K_2}$-free. By Lemma 6, we may assume that G is connected, and in particular that $G = G_1 \times G_2$ where neither of the cographs G_1 or G_2 is a clique. But then G_1 and G_2 each have a $\overline{K_2}$, and so they are both $\overline{nK_2}$-free. The induction hypothesis applies, and another application of Lemma 6 gives us that $\text{box}(G) \leq \text{box}(G_1)+\text{box}(G_2) \leq 2^{n-2}+2^{n-2} = 2^{n-1}$ as required. □

3.4 H-index

The H-index $h(G)$ of a graph G is the largest $k \geq 0$ such that G has k vertices of degree at least k. This parameter is important in the study of dynamic algorithms [21]. Clearly, H-index is unbounded for cographs, since it is unbounded for complete graphs. To characterize this parameter in terms of minimal subclasses of cographs with unbounded H-index, we start with a helpful lemma.

Lemma 7. *Let $G_1, \ldots, G_t$ be graphs. Then*

$$h(\bigcup_{i=1}^{t} G_i) \leq \sum_{i=1}^{t} h(G_i), \text{ and } h(G_1 \times G_2) \leq \min(h(G_1) + |V(G_2)|, h(G_2) + |V(G_1)|).$$

Proof. For the first bound, note that for any j, $1 + \sum_i h(G_i) > h(G_j)$. In particular, by definition of the H-index, each G_j has at most $h(G_j)$ vertices of degree $1 + \sum_i h(G_i)$ or more, and so $\bigcup_j G_j$ has at most $\sum_j h(G_j)$ vertices of degree at least $1 + \sum_i h(G_i)$, from which the claim follows.

For the other bound, note that $G_1 \times G_2$ has at most $|V(G_2)|$ vertices of degree at least $h(G_1) + |V(G_2)| + 1$ coming from G_2, and at most $h(G_1)$ coming from G_1, since[1] $\deg_{G_1 \times G_2}(v) = \deg_{G_1}(v) + |V(G_2)|$ for any $v \in G_1$, and G_1 does not have more than $h(G_1)$ vertices of degree $h(G_1) + 1$. By definition of the H-index, we obtain that $h(G_1 \times G_2) \leq h(G_1) + |V(G_2)|$, and the claim follows by symmetry. □

Theorem 7. *$\mathcal{K}$, $\mathcal{B}$ and the class $\mathcal{F}$ of star forests are the only minimal hereditary subclasses of cographs of unbounded H-index.*

Proof. One can check that those are, indeed, minimal hereditary classes of unbounded H-index. To see they are the only ones, let $p, q, r, s \geq 1$. We will show by induction on $p + r$ that if G avoids K_p, $K_{q,q}$ and $rK_{1,s}$, then the H-index of G is bounded by a constant $H(p, q, r, s)$. For the base case, note that if $p = 1$, this is trivial, and if $r = 1$, then G is $(K_p, K_{1,s})$-free and therefore the maximum vertex degree in G is bounded by $R(p, s)$. This in turn implies that $h(G) \leq R(p, s)$. We may thus assume $p, r \geq 2$.

If $G = G_1 \times G_2$ is a join of non-empty graphs, then not both G_1 and G_2 have more than $R(p, q)$ vertices. Indeed, if both do, then either one of them contains a clique of size p, which is forbidden, or they both have independent sets of size q, which again cannot happen since $K_{q,q}$ is forbidden. Without loss of generality, we may assume that $|V(G_2)| \leq R(p, q)$. In this case, by Lemma 7, $h(G) \leq h(G_1) + R(p, q)$. Since $|V(G_2)| \geq 1$, G_1 is K_{p-1}-free, so by the induction hypothesis, $h(G_1)$ is bounded by $H(p - 1, q, r, s)$.

[1] When a vertex v appears in more than one graph, we write $\deg_G(v)$ for the degree of v in graph G.

If $G = \bigcup_{i=1}^{t} G_i$ is a union of connected graphs, we may write $G = G_1 \cup \ldots G_l \cup G'$, where $G_1, \ldots, G_l$ each have a $K_{1,s}$, and G' is $K_{1,s}$-free (we may have $l = 0$). Since K_p and $K_{1,s}$ are forbidden for G', the maximum vertex degree, and hence the H-index of G', is bounded by $R(p,s)$. Moreover, if $l \geq 2$ and so two of the components of G do have a $K_{1,s}$, then we may write G as the union of two graphs that are $(r-1)K_{1,s}$-free, and by Lemma 7, $h(G) \leq 2H(p,q,r-1,s)$. Finally, if only one component has a $K_{1,s}$, then that component is a join of non-empty graphs and we obtain, again by Lemma 7 and from the previous paragraph, $h(G) \leq H(p-1,q,r,s) + R(p,q) + R(p,s)$.

Combining the above, we obtain

$$H(p,q,r,s) \leq \max(H(p-1,q,r,s) + R(p,q) + R(p,s), 2H(p,q,r-1,s)).$$

□

3.5 Achromatic number

A *complete k-colouring* is a partition of G into k independent sets (the "colour classes") such that any two independent sets in the partition have at least one edge between them. The *achromatic number* $\psi(G)$ of a graph G is the maximum number k such that G admits a complete k-colouring. Computing this parameter is a difficult task even for cographs and interval graphs [12].

Note that the class $\mathcal{K}$ of complete graphs and the class $\mathcal{M}$ of matchings have unbounded achromatic number. Indeed, this is clear for complete graphs, and we note that $\binom{n}{2}K_2$ admits a complete n-colouring where each edge of the matching joins two of the colour classes. We claim that among cographs, those are the only minimal classes of unbounded achromatic number. To show this, we start with a short lemma.

Lemma 8. *Let r, $s \in \mathbb{N}$. The class of (K_r, sK_2, P_4)-free graphs has bounded neighbourhood diversity.*

Proof. From Theorem 3, the only minimal subclasses of cographs where neighbourhood diversity is unbounded are $\mathcal{M}$, $\overline{\mathcal{M}}$ and $\mathcal{T}$. K_r belongs to both $\overline{\mathcal{M}}$ and $\mathcal{T}$, while sK_2 belongs to $\mathcal{M}$. □

We are now ready to prove the main result of this section.

Theorem 8. *$\mathcal{K}$ and $\mathcal{M}$ are the only minimal hereditary subclasses of cographs of unbounded achromatic number.*

Proof. It suffices to show that for any r, $s \in \mathbb{N}$, the class of (K_r, sK_2, P_4)-free graphs has bounded achromatic number. Let G be a graph in this class. By Lemma 8, the class has bounded neighbourhood diversity. In other words, there is a constant k (independent of G) such that the vertex set of G can be partitioned into k similarity classes, each similarity class being a clique or an independent set. Moreover, since the size of cliques is bounded by r, we may further assume that each of these similarity classes is an independent set. Let G'

be the quotient of G by this partition, i.e., the graph whose vertices are the independents sets, with two vertices being adjacent if and only if the corresponding sets are complete to each other.

Now consider a t-colouring of G, and interpret the colours as vertices of the complete graph K_t. From each edge e of G', we obtain a complete bipartite subgraph of K_t as follows: if the edge e in G' joins independent sets A_1 and A_2, then the two sets are complete to each other, so the sets of colours $I_1, I_2 \subseteq V(K_t)$ appearing in A_1 and A_2 respectively are disjoint. The complete bipartite graph B^e corresponding to e has I_1 and I_2 as its parts. With this set-up, the t-colouring is complete if any only if the edges of the graphs ${B^e}_{e\in E(G')}$ cover the edges of K_t. From [26], we need at least $\lceil \log_2(t) \rceil$ complete bipartite graphs to cover K_t. It follows that $t \leq 2^{|E(G')|} \leq 2^{\binom{k}{2}}$, as required.

□

3.6 Contiguity

The notion of contiguity was introduced in [31] and was motivated by the need of compact representations of graphs in computer memory. One approach to achieving this goal is finding a linear order of the vertices in which the neighbourhood of each vertex forms an interval. Not every graph admits such an ordering, in which case one can relax this requirement by looking for an ordering in which the neighbourhood of each vertex can be split into at mots k intervals. The minimum value of k which allows a graph G to be represented in this way is the *contiguity* of G, denoted $\mathrm{cont}(G)$.

In [17], it was shown that contiguity of n-vertex cographs is $\Theta(\log n)$, implying that this parameter is unbounded in the class of cographs. In what follows, we identify two minimal hereditary subclasses of cographs of unbounded contiguity.

Lemma 9. *Contiguity is unbounded in the class $\mathcal{Q}$ of (P_4, C_4)-free graphs and in the class of their complements.*

Proof. Let G be a graph and v a vertex of G. In a linear order of $V(G)$, the number of intervals representing the neighbourhood of v differs from the number of intervals representing the non-neighbourhood of G by at most 1. Therefore, the contiguity is bounded in a class X of graphs if and only if it is bounded in the class of complements of graphs in X. Thus, it suffices to prove the lemma only for (P_4, C_4)-free graphs, also known as quasi-threshold graphs.

Every quasi-threshold graph can be recursively constructed from one-vertex graphs by applying one of the following two operations: disjoint union of two quasi-threshold graphs G and H, denoted $G \cup H$, and addition of a dominating vertex v to a quasi-threshold graph G, denoted $v \times G$.

Let G be a quasi-threshold graph of contiguity k. In particular, for any linear order L of $V(G)$, there exists a vertex u whose neighbourhood consists of at least k intervals in L. To prove the lemma, we will show that the contiguity of the graph $H = v \times (G \cup G \cup G)$ is strictly greater than k.

Let L be an arbitrary linear order of $V(H)$, and consider the order L^v that we obtain by restricting L to $H - v$, as well as the orders L_1, L_2 and L_3 that we obtain by further restricting L^v to the vertices of each of the three copies of G. Find vertices $u_1, u_2, u_3 \in V(H)$ belonging to each of the copies of G such that in its respective copy, the neighbourhood of u_i consists of at least k intervals in L_i. Since L_i is a restriction of L^v, the neighbourhood of u_i in $H - v$ still consists of at least k intervals in L^v (the number of intervals cannot increase when removing vertices).

Now, the neighbourhood of u_i in H consists of those at least k intervals in L^v, together with v. Note that v can only be adjacent to (or inside) at most one of these intervals. Moreover, since the u_i have disjoint neighbourhoods in $H - v$, v cannot be adjacent to intervals coming from all three neighbourhoods. In other words, there is an $i \in \{1, 2, 3\}$ such that u_i has a neighbourhood consisting of at least $k + 1$ intervals in L (one of which consists only of v). Since L was arbitrary, this shows the contiguity of H is at least $k + 1$, as required.

□

Lemma 10. *For any pair of graphs $H \in Free(P_4, C_4)$ and $K \in Free(P_4, 2K_2)$, there is a constant $c(H, K)$ such that the contiguity of (P_4, H, K)-free graphs is at most $c(H, K)$.*

Proof. We prove the lemma by induction on $|V(H)|+|V(K)|$. For the basis of the induction we observe that if one of H and K consists of two vertices, then the statement is obvious.

Now assume that both H and K contain more than two vertices and let G be a (P_4, H, K)-free graph. Below we analyse various cases depending on the structure of H and K. Our analysis is based on the following observations (the first one can be derived by restricting orders like in the previous lemma, and the second immediately follows by a double complementation argument):

(a) if G is disconnected and $G_1, \dots, G_p$ are the components of G, then $\mathrm{cont}(G) = \max_i \mathrm{cont}(G_i)$;

(b) if G is connected and $G_1, \dots, G_p$ are the co-components (components of the complement) of G, then $\mathrm{cont}(G) \leq \max_i \mathrm{cont}(G_i) + 2$.

Assume first that H contains a dominating vertex v and let $H' = H - v$. By the inductive assumption, there is a constant $c(H', K)$ bounding the contiguity of (P_4, H', K)-free graphs. If G is connected, then each co-component of G is H'-free and hence by (b), $\mathrm{cont}(G) \leq c(H', K) + 2$. If G is disconnected, then as in the previous sentence, the contiguity of each component of G is at most $c(H', K) + 2$ and hence by (a), the contiguity of G is at most $c(H', K) + 2$.

If K contains an isolated vertex, then the arguments are similar. Therefore, in the rest of the proof we assume that H is disconnected and K is the complement of a disconnected graph. We represent H as $H' \cup H''$, where H' is a component of H and H'' is the rest of the graph. Similarly, we represent $K = K' \times K''$, where K' is a co-component of K and K'' is the rest of the graph.

Assume without loss of generality that G is disconnected. If each of the components of $G'_0 := G$ is H'-free, then by the inductive assumption the contiguity of each component, and hence of G'_0, is at most $c(H', K)$. Suppose now that one of the components of G'_0 contains H' as an induced subgraph. Denote that component by G'_1, and the rest of the graph by G_1. Note that each of the components of G_1 is H''-free, and hence, by (a), G_1 has contiguity at most $c(H'', K)$. Applying similar arguments to G'_1, we see that either all of its co-components are K'-free, or it can be expressed as the join of two graphs G'_2 and G_2 such that G'_2 is disconnected and contains K' as an induced subgraph, and G_2 has contiguity bounded by a constant depending on H and one of K', K''.

Continue in this way for as long as possible. We produce two sequences G_i and G'_i such that $G'_i = G'_{i+1} \star G_{i+1}$, where $\star$ stands for $\cup$ when i is even and $\times$ when i is odd, G'_i is connected and contains H' when i is odd/disconnected and contains K' when i is even, and all G_i have contiguity uniformly bounded by some constant depending only on H and K. Since $|G'_i|$ strictly decreases as i increases, there exists a k such that every component or co-component of G'_k (according to whether k is even or odd respectively) is H', respectively K'-free. Put $G_{k+1} := G'_k$.

Assuming without loss of generality that k is even, we have, by construction, that $G = G_1 \cup (G_2 \times (G_3 \cup \ldots (G_k \times G_{k+1})))$, and each G_i has contiguity bounded by, e.g., $c'(H, K) := \max(c(H, K'), c(H, K''), c(H', K), c(H'', K)) + 2$.

Let $L_i, 1 \leq i \leq k+1$, be a linear order on the vertices of G_i that witnesses a contiguity of at most $c'(H, K)$, and consider the linear order on $V(G)$ given by the concatenation $L := L_1 L_3 \ldots L_{k+1} L_k \ldots L_4 L_2$. We claim that this order witnesses a contiguity of at most $c'(H, K) + 2$ for G. Indeed, the neighbourhood in G of any vertex $v \in G_i$ consists of its neighbours in G_i, together with some of the G_j, as follows:

- If i is even, the neighbourhood outside of G_i of v consists of $\bigcup_{j>i} V(G_j) \cup \bigcup_{\substack{j<i \\ j \text{ even}}} V(G_j)$.
- If i is odd, the neighbourhood outside of G_i of v consists of $\bigcup_{\substack{j<i \\ j \text{ even}}} V(G_j)$.

Note that each of the indexed unions above corresponds to an interval in L. Thus the neighbourhood of v consists of at most $c(H, K) := c'(H, K) + 2$ intervals in L, as required. □

Combining the two lemmas above we obtain the main result of this section as follows.

Theorem 9. *The class $\mathcal{Q}$ of quasi-threshold graphs and the class of their complements are the only two minimal hereditary subclasses of cographs of unbounded contiguity.*

4 Concluding remarks and open problems

Let us bring together the different pieces of our analysis and draw a hierarchy of the parameters studied in this paper. Each parameter κ is presented in Figure 2 together with its collection $M(\kappa)$ of minimal hereditary subclasses of cographs where κ is unbounded, and the parameters are compared by their strength.

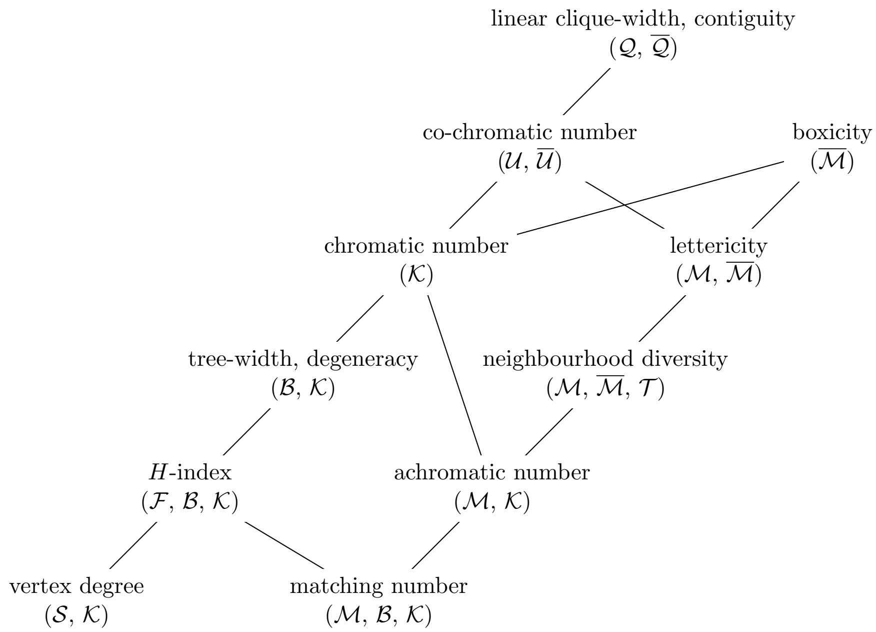

Figure 2: A Hasse diagram of graph parameters within the universe of cographs. For each parameter, the minimal hereditary subclasses of cographs where the parameter is unbounded are listed in parentheses.

There are many other interesting parameters that are unbounded in the class of cographs, such as Dilworth number [29], distinguishing number [9], shrub-depth [28], rank [15], metric dimension [46], etc. However, surprisingly, there are not so many “interesting” subclasses of cographs that appear in the characterisation of those parameters. For instance, Dilworth number, distinguishing number and shrub-depth can be characterised without extending the (already small) set of classes studied in this paper. What makes those classes special?

It is not difficult to show that any class $\mathcal{X}$ appearing in the set $M(\kappa)$ for some parameter κ is *atomic*, in the sense that it cannot be written as the union of two proper subclasses. This property is equivalent to the *joint embedding property*, whereby if $\mathcal{X}$ contains G and H, then it must contain a graph containing both G and H as induced subgraphs (Fraïssé [27] studied these notions, albeit in a more general setting). Conversely, for any atomic class $\mathcal{X}$, one can cook up a parameter $\kappa_{\mathcal{X}}$ with $M(\kappa_{\mathcal{X}}) = \{\mathcal{X}\}$. However, even when restricting our search to atomic classes, only a select few seem to occur when studying “natural” parameters. Understanding this phenomenon is a challenging research problem.

One more challenging research direction deals with algorithmic problems. As we mentioned earlier, computing the achromatic number is an NP-complete problem for cographs.

The same is true for the related problem of computing harmonious colouring [7]. Two more problems that remain NP-complete for cographs are k-path partition [7] and induced subgraph isomorphism [11]. Moreover, each of these problems has been shown to be NP-complete in the class of quasi-threshold graphs. Is that the minimal class where the problems are NP-complete?

For the problem of computing the achromatic number, the answer to the above question is 'no'. Indeed, in the proof of the NP-completeness of this problem given in [12], 3-PARTITION reduces to an instance of ACHROMATIC NUMBER on a cograph consisting of several connected components, each of which is a star, except for one component consisting of two cliques sharing a vertex. Clearly, this is a quasi-threshold graph, but this graph avoids many other quasi-threshold graphs as induced subgraphs, for instance $3K_3$. Therefore, the problem remains NP-complete for $3K_3$-free quasi-threshold graph. Is this class minimal? The answer again is 'no', as the reader can easily find more quasi-threshold graphs that are not contained in the described graph. On the other hand, due to well-quasi-orderability of cographs, there must exist a minimal class where the problem is NP-complete, and the number of such classes must be finite. Identifying minimal classes for this and other problems that are NP-complete for cographs is an attractive and ambitious topic for future research.

Finally, another series of questions stems from our observation in Section 2.2 that cographs are bqo. There is a rich and beautiful theory behind this notion, originally introduced by Nash-Williams [40]. However, it seems that bqo properties under the induced subgraph relation have not yet been studied in depth. In particular, as far as the authors are aware, many fundamental questions in this area remain unanswered, the most immediate being: is every wqo class of graphs in fact bqo?

Note that this is not the case for quasi-orders in general. For instance, the so-called *Rado structure* [42] is a wqo, but its power set is not wqo under inclusion. In fact, this structure is in a certain sense universal with this property [37], so a first step towards answering the question would be to determine whether there exists a Rado structure of graphs under induced subgraphs. We also note that bqo of graphs under the *minor* relation is an open problem (see, e.g., [20]).

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