FINITE-DIMENSIONAL NEGATIVELY INVARIANT SUBSETS OF BANACH SPACES

Alexandre N. Carvalho [1,a], Arthur C. Cunha [2,a], José A. Langa [3,b], James C. Robinson[c]

[a] Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil.

[b] Departamento de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Apdo. de Correos 1160, 41080-Sevilla, Spain.

[c] Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK.

Abstract. We give a simple proof of a result due to Mañé (Springer LNM 898, 230–242, 1981) that a compact subset $\mathcal{A}$ of a Banach space that is negatively invariant for a map S is finite-dimensional if $\mathrm{D}S(x) = C(x) + L(x)$, where C is compact and L is a contraction (and both are linear). In particular, we show that if S is compact and differentiable then $\mathcal{A}$ is finite-dimensional. We also prove some results (following Málek et al. (Acta Appl. Math. **37**, 83–97, 1994) and Zelik (Rend. Accad. Naz. Sci. XL MMMMA **118**, 1–25, 2000)) that give bounds on the (box-counting) dimension of such sets assuming a 'smoothing property': in its simplest form this requires S to be Lipschitz from X into another Banach space Z that is compactly embedded in X. The resulting bounds depend on the Kolmogorov ε-entropy of the embedding of Z into X. We give applications to an abstract semilinear parabolic equation and the two-dimensional Navier–Stokes equations on a periodic domain.

1. Introduction

We want to estimate the box-counting dimension of attractors associated to dynamical systems on Banach spaces. In the continuous-time case, the standard abstract setting is the following. We say that a family of continuous maps $\{S(t)\}_{t\geq 0}$ from a Banach space X into itself is a *semigroup* if

(i) $S(0) = \mathrm{Id}_X$,

(ii) $S(t+s) = S(t)\,S(s)$, for all $t, s \geq 0$,

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E-mails. andcarva@icmc.usp.br(Carvalho); cunha.arthurc@gmail.com(Cunha); langa@us.es(Langa); j.c.robinson@warwick.ac.uk(Robinson).

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(iii) the map $[0,\infty)\times X \ni (t,x)\mapsto S(t)x\in X$ is continuous.

We say that a subset $\mathcal{A}\subset X$ is *invariant* under the action of the semigroup $S(\cdot)$ if $S(t)\mathcal{A}=\mathcal{A}$ for all $t\geq 0$, and we say that $\mathcal{A}$ *attracts* a subset D of X under the action of the semigroup if $\mathrm{dist}(S(t)D,\mathcal{A})\to 0$ as $t\to\infty$, where

$$\mathrm{dist}(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_X.$$

A subset $\mathcal{A}$ of X is said to be the *global attractor* for $S(\cdot)$ if it is compact, invariant and attracts all bounded subsets B of X under the action of $S(\cdot)$.

The global attractor for the semigroup $S(\cdot)$ is the same as the global attractor for the discrete semigroup $\{S^n : n=0,1,2,\cdots\}$, where we can take $S=S(t_0)$ for any $t_0>0$; so throughout this paper we consider only the discrete case. In fact our results are valid for compact sets $\mathcal{A}\subset X$ such that $\mathcal{A}\subseteq S(\mathcal{A})$, i.e. that are negatively invariant under S.

The earliest result on finite dimensionality of attractors for dynamical systems is due to Mallet-Paret in 1976 (see [16]), who considered separable Hilbert spaces. Mañé generalised this result to Banach spaces in 1980 (see [17]); his proof is taken up and somewhat improved by Carvalho et al. in [3]. All these three papers treat a map S whose derivative is everywhere equal to the sum of a compact map and a contraction, and the proofs all rely on using the compactness assumption to find a finite-dimensional subspace U such that the image under DS of the unit ball in U provides a good approximation of the image of the unit ball in X. The resulting dimension estimate involves the dimension of U, which means that, in practice, it is hard to use the results to give explicit bounds on the dimension of $\mathcal{A}$.

For explicit bounds the standard technique relies on setting the problem in a Hilbert space, and then one can obtain estimates using the theory of Lyapunov exponents, as developed by Constantin & Foias [8] (see also Chepyzhov & Vishik [5] or Carvalho et al. [2]).

Our starting point for this paper was the following question: if $\mathcal{A}$ is the attractor of a compact map, is it a finite-dimensional set? A relatively simple example shows that the answer to the question is generically *no*. Indeed, consider the map $S\colon \ell^2\to\ell^2$ given by

$$(S\boldsymbol{x})_j=\begin{cases} j^{-1}\frac{x_j}{|x_j|} & ,|x_j|>j^{-1}\\ x_j & ,|x_j|\leq j^{-1}.\end{cases}$$

This map is compact, but its attractor is the set

$$\{\boldsymbol{x}\in\ell^2 :\ |x_j|\leq 1/j\}$$

which is an infinite-dimensional subset of ℓ^2. However, it turns out that the answer to this question is *yes* if as well as being compact S is differentiable: in this case it follows that $\mathrm{D}S$ is compact, and the finite-dimensionality can then be obtained from Mañé's result [17]. In fact this holds whenever $\mathrm{D}S$ is the sum of a compact map and a contraction (in an appropriately uniform way over the attractor); here (see Theorem 3.1) we give a proof of this central fact that is much simpler than Mañé's argument.

However, our approach yields no explicit bound on the attractor dimension. In order to remedy this, we use ideas due to Málek et al. [15], and assume a quantitative smoothing estimate for the compact part of DS: we suppose that for $x \in \mathcal{A}$ we have $\mathrm{D}S(x) = C(x) + L(x)$, where $C(x)$ and $L(x)$ are both linear, that $\|L(x)\|_{\mathcal{L}(X)} < 1/4$, and that $C(x)$ satisfies

$$\|C(x)u\|_Z \le K\|u\|_X \qquad u \in X,$$

for some space Z that is compactly embedded in X. This enables us to give an explicit bound on the dimension of $\mathcal{A}$, which (see Zelik [23]) involves the Kolmogorov ε-entropy of the compact embedding of Z into X. Some of the arguments here are also inspired by those in the paper by Carvalho & Sonner [4], which uses such a quantitative smoothing property and a similar splitting to bound the dimension of pullback exponential attractors.

In Section 2 we recall the definition of the box-counting dimension, and give two simple lemmas that enable us to bound the box-counting dimension based on iterated coverings of $\mathcal{A}$. We then prove two results based on DS in Section 3. In the next section we relate properties of S to properties of DS, showing in particular in Corollary 4.2 that a differentiable compact map has a finite-dimensional attractor. In Section 5 we use the smoothing property from Málek et al. [15] and Zelik [23] to work directly with assumptions on the map S itself (see also Cholewa, Czaja, & Mola [7]).

We then discuss the Kolomogorov ε-entropy which enters the dimension bounds, giving a simple argument to bound this in the case of L^2-based Sobolev spaces. In the final two sections we apply the theory to two classical examples: abstract semilinear parabolic equations, and the two-dimensional Navier–Stokes equations on periodic domains.

Similar results to those in Section 5 of this paper can be found in the review article by Miranville & Zelik [18]; however, our aim here is to link the original approach of Mañé to more recent advances, and to tell something of a coherent 'story' about dimension estimates in this setting.

2. Bounding the box-counting dimension of negatively invariant sets via simple covering lemmas

For a precompact subset $\mathcal{A}$ of a normed space X (or more generally a metric space (X, d)), let $N_X[\mathcal{A}, \varepsilon]$ denote the minimum number of open ε-balls in X that are necessary to cover $\mathcal{A}$ and let $N_X^\bullet[\mathcal{A}, \varepsilon]$ denote the minimum number of open ε-balls in X centred at points of $\mathcal{A}$ that are necessary to cover $\mathcal{A}$. We will omit the X subscript when it is clear from the context.

Note that

$$N[\mathcal{A},\varepsilon] \le N^\bullet[\mathcal{A},\varepsilon] \le N[\mathcal{A},\varepsilon/2], \tag{2.1}$$

since given any cover of $\mathcal{A}$ by balls of radius $\varepsilon/2$, if

$$\mathcal{A} \cap B(x,\varepsilon/2) \ne \varnothing$$

then there exists $a \in \mathcal{A}$ such that $x \in B(a,\varepsilon/2)$ and so $B(x,\varepsilon/2) \subset B(a,\varepsilon)$.

The *(upper) box-counting dimension* of $\mathcal{A}$ in X, denoted by $\dim_B(\mathcal{A};X)$, is defined as

$$\dim_B(\mathcal{A};X) := \limsup_{\varepsilon\to 0^+} \frac{\log N_X[\mathcal{A},\varepsilon]}{-\log\varepsilon}. \tag{2.2}$$

Note that, because of (2.1), it is possible to replace $N_X[\mathcal{A},\varepsilon]$ in (2.2) by $N^\bullet_X[\mathcal{A},\varepsilon]$ and still obtain the same quantity, but we prefer $N_X[A,\varepsilon]$ here as it is less restrictive.

Essentially, this definition extracts the exponent in

$$N_X[\mathcal{A},\varepsilon] \sim \varepsilon^{-\dim_B(\mathcal{A};X)}.$$

A detailed treatment of the box-counting dimension of compact sets can be found in Falconer [12] and Robinson [20].

The lim sup in (2.2) can also be taken over a geometrically decreasing sequence: the proof of the following lemma can be found in Carvalho et al. [2], Lemma 4.1.

Lemma 2.1. *Given a compact subset $\mathcal{A}$ of X, $r>0$, and any $\eta \in (0,1)$*

$$\dim_B(\mathcal{A}) = \limsup_{k\to\infty} \frac{\log N[\mathcal{A},\eta^k r]}{-k\,\log\eta}.$$

We now prove two simple lemmas that allow us to obtain bounds on the box-counting dimension of $\mathcal{A}$ by applying S or $\mathrm{D}S$ to given coverings of $\mathcal{A}$. We write $B_X(x,r)$ for the open ball in X centred at x of radius r, dropping the X subscript when clear from the context.

Lemma 2.2. *Let $\mathcal{A}$ be a compact subset of a Banach space X that is negatively invariant for $S\colon X\to X$, i.e. $\mathcal{A} \subseteq S(\mathcal{A})$. If there exist $M \ge 1$, $0<\beta<1/2$, and $r_0>0$ such that for all $x\in\mathcal{A}$ and all $0<r\le r_0$*

$$N[S(B(x,r)),\beta r] \le M, \tag{2.3}$$

then

$$\dim_B(\mathcal{A}) \le \frac{\log M}{-\log 2\beta}. \tag{2.4}$$

If (2.3) holds for all x in some δ-neighbourhood of $\mathcal{A}$ then it is enough to take $0<\beta<1$ and the bound in (2.4) becomes $\log M/(-\log\beta)$.

Proof. If we only have (2.3) for balls centred at points of $\mathcal{A}$ then we need to make sure all the balls in our iterated covers are also centred at points of $\mathcal{A}$, so we have to work with $N^\bullet$.

Cover $\mathcal{A}$ with $N = N^\bullet[\mathcal{A}, r_0]$ balls of radius r_0, $\{B(x_i, r_0)\}_{i=1}^N$, with centres $x_i \in \mathcal{A}$. Apply S to every element of this cover. Since $\mathcal{A} \subseteq S(\mathcal{A})$, this provides a new cover of $\mathcal{A}$, $\{S(B(x_i, r_0))\}_{i=1}^N$. Using (2.3) each of these images can be covered by M balls of radius βr_0, with centres $y_{ij} \in X$; by enlarging these to balls of twice the radius we can take new centres x_{ij} to be in $\mathcal{A}$ once again, and so we obtain

$$N^\bullet[\mathcal{A}, 2\beta r_0] \leq M N^\bullet[\mathcal{A}, r_0].$$

Since the centres of the balls in this new cover lie in $\mathcal{A}$ and $2\beta < 1$ we can apply the same argument n times to obtain

$$N^\bullet[\mathcal{A}, (2\beta)^n r_0] \leq M^n N^\bullet[\mathcal{A}, r_0],$$

and Lemma 2.1 yields

$$\dim_B(\mathcal{A}) \leq \frac{\log M}{-\log 2\beta}.$$

If (2.3) holds in a δ-neighbourhood of $\mathcal{A}$ then take $r_0 < \delta$ and cover $\mathcal{A}$ with $N[\mathcal{A}, r_0]$ balls $\{B(x_i, r_0)\}_{i=1}^N$. Applying S as before we obtain a new cover of $\mathcal{A}$ that is contained in MN balls of radius βr_0; the centres of the balls that we need to retain in this cover lie within $\beta r_0 < r_0 < \delta$ of $\mathcal{A}$, and so we can repeat the argument as above, only this time obtaining a cover with balls of radius $\beta^n r_0$, which yields the improved bound given in the statement. $\square$

One way to obtain the bound in (2.3) is to have a similar bound on covers for the unit ball under the derivative $\mathrm{D}S(x)$ on the attractor. We need some uniformity in what it means for S to be differentiable 'on $\mathcal{A}$'. As shown in the Appendix, such uniform differentiability follows whenever S is continuously differentiable on an open neighbourhood of $\mathcal{A}$.

Definition 2.3. *We say that $S\colon X \to X$ is* uniformly differentiable for $x \in \mathcal{A}$ *if for every $x \in \mathcal{A}$ there exists a bounded linear map $\mathrm{D}S(x)\colon X \to X$ such that for every $\eta > 0$ there exists a positive constant $r_0(\eta) > 0$ such that*

$$\|S(x+h) - S(x) - \mathrm{D}S(x)h\| < \eta\|h\|, \qquad \text{for every } x \in \mathcal{A},\ h \in X \text{ with } \|h\| < r_0(\eta).$$

Using this definition we can now prove a result based on assumptions on $\mathrm{D}S$.

Lemma 2.4. *Let $\mathcal{A}$ be a compact subset of a Banach space X that is negatively invariant for a map $S\colon X \to X$ that is uniformly differentiable for $x \in \mathcal{A}$. If there exist $\alpha \in \big(0, 1/2\big)$ and $M \geq 1$ such that*

$$N[\mathrm{D}S(x)(B(0,1)), \alpha] \leq M, \qquad x \in \mathcal{A}, \tag{2.5}$$

then

$$\dim_B(\mathcal{A}) \leq \frac{\log M}{-\log 2\alpha}. \tag{2.6}$$

As in Lemma 2.2, if (2.5) *holds for all x in a δ-neighbourhood of $\mathcal{A}$ then we can take $\alpha \in (0,1)$ and then* $\dim_B(\mathcal{A}) \leq \log M/(-\log \alpha)$.

Proof. The uniform differentiability assumption implies for $\eta \in \big(0, 1/2 - \alpha\big)$ that

$$S(B(x,r)) \subseteq S(x) + \mathrm{D}S(x)B(0,r) + B(0,\eta r), \qquad x \in \mathcal{A},\ r \leq r_0(\eta);$$

combining this with the covering assumption in (2.5) it follows that

$$N[S(B(x,r)), (\alpha+\eta)r] \leq M,$$

for all $x \in \mathcal{A}$ and for all $0 < r \leq r_0(\eta)$. Lemma 2.2 now guarantees that

$$\dim_B(\mathcal{A}) \leq \frac{\log M}{-\log 2(\alpha+\eta)},$$

and since this holds for all $0 < \eta < 1/2 - \alpha$ the estimate in (2.6) follows.

The argument when (2.5) holds in a neighbourhood of $\mathcal{A}$ is essentially the same and uses the second part of Lemma 2.2. □

3. Finite-dimensional attractors from assumptions on DS

For our first results we make assumptions on the derivative of S; these results parallel the original ones in Mallet-Paret [16] and Mañé [17].

3.1. **The dimension is finite.** Our first theorem revisits the classical result of Mañé [17] (see also Carvalho et al. [3]) for a map whose derivative is the sum of a compact map and a contraction, but the proof is considerably simpler. However, our assumptions do not yield an explicit bound on the dimension.

We say that a map $L\colon X \to X$ is a λ-contraction if

$$\|Lx - Ly\| \leq \lambda\|x - y\| \qquad x, y \in X.$$

If L is linear then it is a λ-contraction if $\|L\|_{\mathcal{L}(X)} \leq \lambda$. [The proof of a result very similar to the following (and the subsequent Corollary 3.2 that allows for $\lambda \in (0,1)$) can be found in the paper by Dung & Nicolaenko [10]; however, they follow Mañé's argument quite closely, while our argument is significantly more direct.]

Theorem 3.1. *Let X be a Banach space and $\mathcal{A}$ a compact subset of X that is negatively invariant for a map $S\colon X \to X$ that is uniformly differentiable for $x \in \mathcal{A}$. Suppose that for each $x \in \mathcal{A}$ we can write*

$$\mathrm{D}S(x) = C_x + L_x,$$

where

- C_x *and* L_x *are both linear;*
- $C_x\colon X \to X$ *is compact for each* $x \in \mathcal{A}$*;*
- C_x *is continuous in* x *(on* $\mathcal{A}$*); and*
- *there exists* $0 < \lambda < 1/2$ *such that* $\|L_x\|_{\mathcal{L}(X)} \le \lambda$ *for every* $x \in \mathcal{A}$*.*

Then $\dim_B(\mathcal{A}) < \infty$*.*

Proof. Fix $\theta > 0$ such that $0 < \theta + \lambda < 1/2$. Since C_x is compact, for each $x \in \mathcal{A}$ there exists $M(x,\theta) > 0$ such that

$$N[C_x B(0,1), \theta/2] \le M(x,\theta).$$

Since C_x is continuous on the compact set $\mathcal{A}$ it is uniformly continuous, so there exists $\delta = \delta(\theta) > 0$ such that

$$\|x - x'\|_X < \delta,\ x, x' \in \mathcal{A} \qquad \Rightarrow \qquad \|C_x - C_{x'}\|_{\mathcal{L}(X)} < \theta/2.$$

It follows that if $\|x' - x\|_X < \delta$ with $x, x' \in \mathcal{A}$ then

$$\begin{aligned} C_{x'}B(0,1) &\subseteq C_x B(0,1) + [C_{x'} - C_x]B(0,1) \\ &\subseteq C_x B(0,1) + B(0,\theta/2), \end{aligned}$$

and so

$$N[C_{x'}B(0,1), \theta] \le M(x,\theta)$$

for every $x' \in B(x,\delta) \cap \mathcal{A}$.

Since

$$\mathcal{A} = \bigcup_{x \in \mathcal{A}} B(x,\delta) \cap \mathcal{A}$$

and $\mathcal{A}$ is compact we can find $x_1, \ldots, x_k \in \mathcal{A}$ such that

$$\mathcal{A} = \bigcup_{i=1}^{k} B(x_i,\delta) \cap \mathcal{A}.$$

It now follows that by taking

$$M^*(\theta) := \max_{i=1,\ldots,k} M(x_i,\theta)$$

we have

$$\sup_{x \in \mathcal{A}} N[C_x B(0,1), \theta] \le M^*(\theta),$$

with M^* independent of x.

Now

$$\begin{aligned} DS(x)B(0,1) &= [C_x + L_x]B(0,1) \\ &\subseteq C_x B(0,1) + L_x B(0,1) \\ &\subseteq \bigcup_{i=1}^{M^*(\theta)} B(y_i,\theta) + B(0,\lambda) \\ &\subseteq \bigcup_{i=1}^{M^*(\theta)} B(y_i,\theta+\lambda), \end{aligned}$$

for some $\tilde{y}_i \in X$, and then

$$\sup_{x\in\mathcal{A}} N[DS(x)B(0,1),\theta+\lambda] \le M^*(\theta).$$

From Lemma 2.4 we conclude that

$$\dim_B(\mathcal{A}) \le \frac{\log M^*(\theta)}{-\log 2(\theta+\lambda)} < \infty. \qquad \square$$

If we include a continuity condition on L_x then we can relax the requirement that $\lambda \in (0,1/2)$ to $\lambda \in (0,1)$.

Corollary 3.2. *Suppose that the hypotheses of Theorem 3.1 hold. Assume further that $\mathcal{A}$ is invariant ($S(\mathcal{A}) = \mathcal{A}$), and that L_x also continuous in x (on $\mathcal{A}$) with $\|L_x\|_{\mathcal{L}(X)} \le \lambda$ for every $x \in \mathcal{A}$, where $\lambda \in (0,1)$. Then $\dim_B(\mathcal{A}) < \infty$.*

Proof. If $\lambda \in [1/2,1)$ then choose k such that $\lambda^k < 1/2$ and apply Theorem 3.1 to the map $\tilde{S} := S^k$. Since

$$(C_2+L_2)\circ(C_1+L_1) = \underbrace{[C_2\circ(C_1+L_1)+L_2\circ C_1]}_{\text{compact}} + L_2\circ L_1$$

and

$$\mathrm{D}\tilde{S}(x) = \mathrm{D}S(S^{k-1}x)\circ \mathrm{D}S(S^{k-2}x)\circ\cdots\circ \mathrm{D}S(x)$$

it follows that $\mathrm{D}\tilde{S}(x) = \tilde{C}_x + \tilde{L}_x$ where $\tilde{C}_x$ and $\tilde{L}_x$ satisfy the conditions of the previous theorem, guaranteeing that $\dim_B(\mathcal{A}) < \infty$ as claimed. $\square$

We now discuss, briefly, how this method relates to that of Mañé [17]. Certain particular examples, such as the semilinear parabolic equation we treated here in Section 7.1 (see also Carvalho et al. [3]), generate a semigroup S that is continuously differentiable and for which it is possible to obtain a sequence of finite rank projections $\{P_n\}_{n\in\mathbb{N}}$ such that for points $x \in \mathcal{A}$ we have

$$\mathrm{D}S(x) = P_n \mathrm{D}S(x) + (I-P_n)\mathrm{D}S(x) = C_x + L_x,$$

where $C_x = P_n \mathrm{D}S(x)$ is compact and for sufficiently large n the operator $L_x = (I - P_n)\mathrm{D}S(x)$ is a contraction on X with contraction constant less than $1/2$. Therefore, in this case S satisfies hypotheses in Theorem 3.1 and its attractor $\mathcal{A}$ has finite box-counting dimension. We note that for each $x \in \mathcal{A}$ we have $C_x B(0,1) \subseteq U_x$, for some linear subspace $U_x \subseteq X$ with finite algebraic dimension $\dim(U_x) < \infty$; since $\mathcal{A}$ is compact, given $\varepsilon > 0$ there exists a linear subspace $U_\varepsilon \subseteq X$ with finite algebraic dimension such that

$$\mathrm{dist}\big(C_x B(0,1), U_\varepsilon\big) < \varepsilon, \qquad x \in \mathcal{A}.$$

Indeed, just as in the proof of Theorem 3.1, given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|x - x'\|_X < \delta,\ x, x' \in \mathcal{A} \qquad \Rightarrow \qquad \|C_x - C_{x'}\|_{\mathcal{L}(X)} < \varepsilon/2.$$

Now

$$\begin{aligned} C_{x'} B(0,1) &\subseteq C_x B(0,1) + [C_{x'} - C_x] B(0,1) \\ &\subseteq U_x + B(0, \varepsilon/2), \end{aligned}$$

and so

$$\mathrm{dist}\big(C_{x'} B(0,1), U_x\big) < \varepsilon, \qquad x' \in B(x,\delta) \cap \mathcal{A}.$$

Since $\mathcal{A} = \bigcup_{i=1}^k B(x_i, \delta) \cap \mathcal{A}$, if we set $U_\varepsilon := \Big[\bigcup_{i=1}^k U_{x_i}\Big]$, then we have

$$\mathrm{dist}\big(C_x B(0,1), U_\varepsilon\big) < \varepsilon, \qquad x \in \mathcal{A}. \tag{3.1}$$

Expression (3.1) essentially portrays the fundamental property needed to follow Mañé's approach: it provides a finite-dimensional subspace U that is a good approximation of the image under C_x of the unit ball in X. For more general situations (in which C_x is not necessarily a finite rank operator) (3.1) is restated and achieved using the compactness of operators C_x (see Carvalho et al. [3], Lemma 2.4).

3.2. **Bounds on the dimension: the smoothing method.** Theorem 3.1 in the previous section ensures that the box-counting dimension of $\mathcal{A}$ is finite, but it does not provide any explicit estimate on this dimension.

In order to give a bound on the box-counting dimension we now consider an auxiliary Banach space Z that is compactly embedded in X, and impose a Lipschitz continuity property between these spaces for the derivative $\mathrm{D}S$ of S, which here we refer to as the *smoothing property.* The bound we will obtain will involve the quantities

$$\mathcal{N}_\varepsilon := N_X[B_Z(0,1), \varepsilon],$$

i.e. the minimum number of ε-balls in X that are needed to cover the unit ball $B_Z(0,1)$ in the subspace Z. These are related to the Kolmogorov entropy numbers for the compact embedding of Z into X; we discuss this further in Section 6. This method is based on the techniques developed by Málek et al. [15] (see also Zelik [23]), making use of the choice of coverings from Efendiev et al. [11].

Theorem 3.3. *Let Z and X be two Banach spaces such that Z is compactly embedded in X, and let $\mathcal{A} \subset Z$ be a compact subset of X that is negatively invariant for a map $S\colon X \to X$ that is uniformly differentiable for $x \in \mathcal{A}$. Suppose in addition that for all $x \in \mathcal{A}$ we have $\mathrm{D}S(x) = C_x + L_x$, with $C_x \in \mathcal{L}(X, Z)$ and $L_x \in \mathcal{L}(X)$ such that there exist $K > 0$ and $0 < \lambda < 1/2$ satisfying*

$$\|C_x(u)\|_Z \leq K\|u\|_X, \qquad u \in X, \tag{3.2}$$

and

$$\|L_x\|_{\mathcal{L}(X)} < \lambda. \tag{3.3}$$

Then

$$\dim_B(\mathcal{A}; X) \leq \frac{\log \mathcal{N}_{\nu/K}}{-\log 2(\nu + \lambda)}, \qquad \textit{for } \nu \in \left(0, \frac{1}{2} - \lambda\right).$$

Proof. For any $x \in \mathcal{A}$, by (3.2) and (3.3) we have

$$\begin{aligned} DS(x)B_X(0,1) &\subseteq C_x B_X(0,1) + L_x B_X(0,1) \\ &\subseteq B_Z(0,K) + B_X(0,\lambda). \end{aligned}$$

But given $0 < \nu < 1/2 - \lambda$, if

$$B_Z(0,1) \subseteq \bigcup_{i=1}^{\mathcal{N}_{\nu/K}} B_X\big(x_i, \nu/K\big),$$

where $x_i \in X$, then

$$\begin{aligned} DS(x)B_X(0,1) &\subseteq \left[\bigcup_{i=1}^{\mathcal{N}_{\nu/K}} B_X\big(Kx_i, \nu\big)\right] + B_X(0,\lambda) \\ &\subseteq \bigcup_{i=1}^{\mathcal{N}_{\nu/K}} B_X\big(y_i, \nu + \lambda\big), \end{aligned}$$

with $y_i \in X$. So

$$N_X[DS(x)B_X(0,1), \nu + \lambda] \leq \mathcal{N}_{\nu/K}$$

and from Lemma 2.4 we conclude that

$$\dim_B(\mathcal{A}; X) \leq \frac{\log \mathcal{N}_{\nu/K}}{-\log 2(\nu + \lambda)}, \qquad \nu \in \big(0, 1/2 - \lambda\big). \qquad \square$$

In the next section we prove some results on the relationship between maps and their derivatives, which will allow us to deduce results on the dimension of $\mathcal{A}$ by imposing conditions on S rather than $\mathrm{D}S$.

4. Maps and their derivatives

In this section we relate properties of the original map S to properties of its derivative, and vice versa.

First we show that if S is compact and differentiable then $\mathrm{D}S$ is also compact.

Lemma 4.1. *Let X be a Banach space and suppose that $S\colon X \to X$ is a compact map. If S is Fréchet differentiable at $x \in X$ with derivative $\mathrm{D}S(x)$, then $\mathrm{D}S(x)\colon X \to X$ is compact.*

Proof. Suppose that the operator $\mathrm{D}S(x)$ is not compact. Then there exist $\varepsilon_0 > 0$ and a sequence $(y_n) \subset X$ such that $\|y_n\| \le 1$ and

$$\|\mathrm{D}S(x)y_n - \mathrm{D}S(x)y_m\| \ge \varepsilon_0, \qquad n \ne m.$$

By the definition of the derivative there exists $\delta > 0$ such that for all $\|h\| \le \delta$ we have

$$\|S(x+h) - S(x) - \mathrm{D}S(x)h\| \le \frac{\varepsilon_0}{4}\|h\|.$$

Choosing $\tau > 0$ such that $\|\tau y_n\| < \delta$ for all $n \in \mathbb{N}$ we obtain

$$\begin{aligned}\|S(x+\tau y_n) - S(x+\tau y_m)\| &= \|S(x+\tau y_n) - S(x) - \mathrm{D}S(x)(\tau y_n)\\ &\qquad + \mathrm{D}S(x)(\tau y_n) - \mathrm{D}S(x)(\tau y_m)\\ &\qquad + S(x) - S(x+\tau y_m) + \mathrm{D}S(x)(\tau y_m)\|\\ &\ge \tau\|\mathrm{D}S(x)y_n - \mathrm{D}S(x)y_m\|\\ &\qquad - \|S(x+\tau y_n) - S(x) - \mathrm{D}S(x)(\tau y_n)\|\\ &\qquad - \|S(x+\tau y_m) - S(x) - \mathrm{D}S(x)(\tau y_m)\|\\ &\ge \tau\varepsilon_0 - 2\frac{\tau}{4}\varepsilon_0 = \frac{\tau}{2}\varepsilon_0.\end{aligned}$$

Since $x + \tau y_n$ is a bounded sequence, this shows that $S(x+\tau y_n)$ can have no Cauchy subsequence, and so S is not compact, a contradiction. □

This result enables us to answer the question posed in the introduction in the affirmative if S is compact and differentiable.

Corollary 4.2. *If $\mathcal{A}$ is a compact subset of X that is negatively invariant under a map $S : X \to X$, and S is compact and uniformly differentiable for $x \in \mathcal{A}$ with $\mathrm{D}S$ continuous on $\mathcal{A}$, then $\dim_B(\mathcal{A}) < \infty$.*

In fact we can prove a 'compact map plus contraction' result in this setting too, although this is perhaps a little less natural, since we require a splitting $S = C + L$ in which both C and L are differentiable. First we show that if $L\colon X \to X$ is a λ-contraction (i.e. $\|Lx - Ly\| \le \lambda\|x - y\|$ for every $x, y \in X$) then so is $\mathrm{D}L$.

Lemma 4.3. *Suppose that $L\colon X \to X$ is a λ-contraction. If L is differentiable at $x \in X$ then $\mathrm{D}L(x) \in \mathcal{L}(X)$ is also a λ-contraction.*

Proof. Since

$$L(x+h) = L(x) + \mathrm{D}L(x)h + o(\|h\|)$$

then

$$\|\mathrm{D}L(x)h\| \leq \lambda\|h\| + o(\|h\|).$$

Given $\varepsilon > 0$ there is $\delta > 0$ such that for $\|h\| < \delta$ we have

$$\|\mathrm{D}L(x)h\| \leq (\lambda + \varepsilon)\|h\|.$$

Now given $h \in X$ let $h_1 := \frac{\delta}{2\|h\|}h$ and then $\|h_1\| < \delta$. Since $\mathrm{D}L(x)$ is linear we conclude

$$\|\mathrm{D}L(x)h\| \leq (\lambda + \varepsilon)\|h\|, \qquad \text{for all } \varepsilon > 0,$$

and so $\mathrm{D}L(x)$ is a λ-contraction. □

The following is now an immediate corollary of Theorem 3.1.

Corollary 4.4. *Suppose that $\mathcal{A}$ is a compact subset of X that is negatively invariant under a map $S\colon X \to X$, with $S = C + L$, where C is compact and L is a λ-contraction with $0 < \lambda < 1/2$. If C and L are uniformly differentiable for $x \in \mathcal{A}$, and $\mathrm{D}C$ is continuous on $\mathcal{A}$ then $\dim_B(\mathcal{A}) < \infty$.*

Note that with the additional assumption that $\mathrm{D}L$ is also continuous on $\mathcal{A}$ then we could increase the range of λ to $0 < \lambda < 1$ as in Corollary 3.2.

We can also transfer a smoothing property for C to one for $\mathrm{D}C$; but rather than argue this way in the next section we instead prove directly that a smoothing property for C can be used to bound the dimension of $\mathcal{A}$. Indeed, this was the original way that this property was used by Málek et al. [15] and Zelik [23].

5. Results using assumptions on S

We now prove a result in which we assume that S can be written in the form $S = C + L$, where C and L satisfy (5.1) and (5.2) below.

This method is a powerful tool in constructing exponential attractors in various settings (besides the autonomous case in Málek et al. [15] and Zelik [23], see Carvalho & Sonner [4] for the non-autonomous case and Caraballo & Sonner [1] for the random case) and then determining the dimension of attractors. As in Theorem 3.3 the estimates for the box-counting dimension are once again given in terms of the numbers $\mathcal{N}_\varepsilon$ related to the compact embedding of Z into X. The next theorem (which can also be found as Lemma 2.1 in [7]) uses a similar assumption to that in Carvalho & Sonner [4] (which generalises [15] and [23]; see also [18]) by allowing the additional contraction term L.

Theorem 5.1. *Let Z and X be two Banach spaces with Z compactly embedded in X, and suppose that $\mathcal{A} \subset Z$ is a compact set that is negatively invariant under a map $S\colon X \to X$. Suppose in addition that $S = C + L$, where $C\colon X \to Z$ and $L\colon X \to X$ are continuous maps and there exist some $K > 0$ and $\lambda \in (0, 1/2)$ such that for all $x, y \in \mathcal{A}$*

$$\|C(x) - C(y)\|_Z \le K\|x - y\|_X \tag{5.1}$$

and

$$\|L(x) - L(y)\|_X \le \lambda\|x - y\|_X. \tag{5.2}$$

Then

$$\dim_B(\mathcal{A}; X) \le \frac{\log \mathcal{N}_{\nu/K}}{-\log 2(\nu + \lambda)}, \qquad \textit{for any choice of } \nu \in \left(0\,, \frac{1}{2} - \lambda\right).$$

Proof. Given $0 < \nu < 1/2 - \lambda$, let $x_0 \in \mathcal{A}$ and $R > 0$ be such that

$$\mathcal{A} = B_X(x_0, R) \cap \mathcal{A}. \tag{5.3}$$

By the smoothing property for C in (5.1) and the definition of $\mathcal{N}_*$ we have

$$\begin{aligned} C(B_X(x_0, R) \cap \mathcal{A}) &\subseteq B_Z(C(x_0), RK) \\ &\subseteq \bigcup_{j=1}^{\mathcal{N}_{\nu/K}} B_X(y_j, \nu R), \end{aligned} \tag{5.4}$$

for some $y_j \in X$. Now, by the contraction property (5.2) for L we obtain

$$L(B_X(x_0, R) \cap \mathcal{A}) \subseteq B_X(L(x_0), \lambda R)$$

and then applying S in (5.3) we have

$$\begin{aligned} \mathcal{A} &= S(\mathcal{A}) \cap \mathcal{A} \\ &= \big[\, C(B_X(x_0, R) \cap \mathcal{A}) + L(B_X(x_0, R) \cap \mathcal{A}) \,\big] \cap \mathcal{A} \\ &= \left[\left(\bigcup_{j=1}^{\mathcal{N}_{\nu/K}} B_X(y_j, \nu R)\right) + B_X(L(x_0), \lambda R)\right] \cap \mathcal{A} \\ &= \bigcup_{j=1}^{\mathcal{N}_{\nu/K}} B_X\big(y_j + L(x_0)\,,\ (\nu + \lambda)R\big) \cap \mathcal{A} \\ &= \bigcup_{j=1}^{\mathcal{N}_{\nu/K}} B_X(z_j\,,\ 2(\nu + \lambda)R\,) \cap \mathcal{A}, \end{aligned}$$

for some $z_j \in \mathcal{A}$, i.e.,

$$\mathcal{A} = \bigcup_{j=1}^{\mathcal{N}_{\nu/K}} B_X(z_j\ ,\ 2(\nu+\lambda)R\ ) \cap \mathcal{A}.$$

Analogously, for each $n \in \mathbb{N}$, there exists a subset $V_n \subset \mathcal{A}$ with $\#V_n \le \mathcal{N}_{\nu/K}^n$ such that

$$\mathcal{A} = \bigcup_{z \in V_n} B_X(z\ ,\ [\ 2(\nu+\lambda)\ ]^n R) \cap \mathcal{A},$$

and the result follows. □

Note that in the above result we have estimated the box-counting dimension of the set $\mathcal{A}$ in the space X. However, when the contraction term L is absent (i.e. when $\lambda = 0$) then we have

$$\dim_B(\mathcal{A}; Z) = \dim_B(\mathcal{A}; X). \tag{5.5}$$

Indeed, if

$$\|S(x) - S(y)\|_Z \le K\|x-y\|_X, \qquad x, y \in \mathcal{A},$$

then the map $S\colon \mathcal{A} \to Z$ is Lipschitz and then

$$\dim_B(\mathcal{A}; Z) \le \dim_B(S(\mathcal{A}); Z) \le \dim_B(\mathcal{A}; X) \le \dim_B(\mathcal{A}; Z),$$

where the last inequality holds because Z is continuously embedded in X.

Remark 5.2. *There is another smoothing property that we could use in the above theorem and obtain the same bounds for the dimension. We assume that $\mathcal{A}$ is a compact subset of X that is negatively invariant under the map $S\colon X \to X$, but now we suppose that $S = C + L$ and for all $x, y \in \mathcal{A}$*

$$\|C(x) - C(y)\|_X \le K\|x-y\|_Y,$$

where X is compactly embedded in Y; once again we take L to be a λ-contraction on X for some $\lambda \in (0, 1/2)$. Then once again we obtain

$$\dim_B(\mathcal{A}; X) \le \frac{\log \mathcal{N}_{\nu/K}}{-\log 2(\nu+\lambda)}, \qquad \textit{for all } \nu \in (0, 1/2-\lambda)\,.$$

The proof of the result in this case is almost identical and uses the same argumentation.

6. Kolmogorov entropy numbers

In this section we recall the notion of the Kolmogorov entropy numbers for the compact embedding between two Banach spaces Z and X and discuss how it relates to the estimates for the box-counting dimension presented above. See Kolmogorov & Tikhomirov [14] for a detailed treatment of this subject.

The quantities $\mathcal{N}_\varepsilon = N_X[B_Z(0,1),\varepsilon]$ are related with the entropy numbers of the compact embedding between the spaces Z and X: the *Kolmogorov ε-entropy* of that embedding, denoted by $\mathcal{H}_\varepsilon(Z,X)$, is

$$\mathcal{H}_\varepsilon(Z,X) := \log_2 \mathcal{N}_\varepsilon.$$

For many function spaces it is possible to determine estimates for these numbers. In particular, for Sobolev spaces we generically have a polynomial growth in ε (as $\varepsilon \to 0^+$): if $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, then the embedding

$$W^{l_1,p_1}(\Omega) \hookrightarrow W^{l_2,p_2}(\Omega)$$

is compact if $l_1, l_2 \in \mathbb{R}$, $p_1, p_2 \in (1,\infty)$ with $l_1 > l_2$ and $l_1 - \frac{d}{p_1} > l_2 - \frac{d}{p_2}$. It is shown in Triebel [22] (Section 4.10.3, Remark 3) that in this case there exist constants $c_1, c_2 > 0$ such that

$$c_1 \varepsilon^{-\frac{d}{l_1-l_2}} \le \mathcal{H}_\varepsilon(W^{l_1,p_1}(\Omega)\ ,\ W^{l_2,p_2}(\Omega)) \le c_2 \varepsilon^{-\frac{d}{l_1-l_2}}, \quad \text{for all } \varepsilon > 0. \tag{6.1}$$

We therefore suppose that Z and X are spaces such that

$$c_1 \varepsilon^{-\gamma} \le \mathcal{H}_\varepsilon(Z,X) \le c_2 \varepsilon^{-\gamma}, \quad \text{for all } \varepsilon > 0 \tag{6.2}$$

for some $c_1, c_2 > 0$ and $\gamma > 0$, so that

$$\log \mathcal{N}_\varepsilon \le (\log 2^{c_2})\, \varepsilon^{-\gamma}. \tag{6.3}$$

Using the estimate in (6.3) for $\mathcal{N}_\varepsilon$ and the dimension estimate from Theorem 3.3 or Theorem 5.1, it follows that for $\lambda \in (0,1/2)$, for each choice of $\nu \in (0, \frac{1}{2} - \lambda)$ we have

$$\dim_B(\mathcal{A}; X) \le \frac{\log \mathcal{N}_{\nu/K}}{-\log 2(\nu+\lambda)} \le \frac{c(\nu/K)^{-\gamma}}{-\log 2(\nu+\lambda)}.$$

If we restrict to $\lambda \in (0,1/4)$ then, setting $\nu = \lambda$, we obtain, for example

$$\dim_{\mathrm{B}}(\mathcal{A}; X) \le c(\lambda) K^\gamma, \tag{6.4}$$

where $c(\lambda) = c\lambda^{-\gamma}/(-\log 4\lambda)$.

It seems useful to sketch an elementary proof of the estimate in (6.2) in the case that X is a separable Hilbert space and Z is the fractional power space $X^\alpha := D(A^\alpha)$ of a linear operator A (e.g. $X = L^2(\Omega)$, with Ω a smooth bounded domain in $\mathbb{R}^d$ and $A = -\Delta$ the Dirichlet Laplacian). Let A have eigenvalues $(\lambda_j)_{j=1}^\infty$ with $\lambda_{j+1} \ge \lambda_j$ and $\lambda_j \to \infty$ as $j \to \infty$, with orthonormal eigenfunctions $(e_j)_{j=1}^\infty$ that form a basis for X. We will assume that

$$\lambda_j \sim j^\theta, \tag{6.5}$$

i.e. there exist constants c_1 and c_2 such that $c_1 j^\theta \le \lambda_j \le c_2 j^\theta$. If $A = -\Delta$ on a bounded domain in $\mathbb{R}^d$, then $\theta = 2/d$ (see Davies [9]).

We want to cover the unit ball in X^α,

$$B_\alpha := \left\{ x = \sum_{j=1}^\infty x_j e_j : \sum_{j=1}^\infty \lambda_j^{2\alpha} |x_j|^2 \le 1 \right\},$$

with 2ε-balls in X.

Let n be the smallest integer such that $\lambda_{n+1}^{-\alpha} \le \varepsilon$. Then for every $x \in B_\alpha$ we have

$$\left\| \sum_{j=n+1}^\infty x_j e_j \right\|^2 = \sum_{j=n+1}^\infty |x_j|^2 \le \lambda_{n+1}^{-2\alpha} \left[\sum_{j=n+1}^\infty \lambda_j^{2\alpha} |x_j|^2 \right] \le \varepsilon^2;$$

so if we can cover the finite-dimensional set

$$E := \left\{ x = \sum_{j=1}^n x_j e_j : \sum_{j=1}^n \lambda_j^{2\alpha} |x_j|^2 \le 1 \right\}$$

with ε-balls in X we can cover B_α with the same number of 2ε-balls.

Lemma III.2.1 in Chepyzhov & Vishik [5] shows that we can cover the ellipse E, whose semiaxes are $\{\lambda_j^{-\alpha}\}_{j=1}^n$, using no more than

$$4^n \frac{\prod_{j=1}^n \lambda_j^{-\alpha}}{\varepsilon^n} \le 4^n \frac{\prod_{j=1}^n \lambda_j^{-\alpha}}{\lambda_{n+1}^{-\alpha n}} = 4^n \frac{\lambda_{n+1}^{\alpha n}}{\prod_{j=1}^n \lambda_j^\alpha}$$

ε-balls. The assumption that $\lambda_j \sim j^\theta$ yields, using the lower bound $n! \ge \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}$, that

$$\begin{aligned} \mathcal{N}_{2\varepsilon} &\le 4^n \frac{\lambda_{n+1}^{\alpha n}}{\prod_{j=1}^n \lambda_j^\alpha} \le (c_2/c_1)^{\alpha n} 4^n \frac{(n+1)^{n\theta\alpha}}{(n!)^{\theta\alpha}} \le (c_2/c_1)^{\alpha n} 4^n \left(\frac{n+1}{n} \right)^{n\theta\alpha} \frac{\mathrm{e}^{n\theta\alpha}}{(2\pi n)^{\theta\alpha/2}} \\ &\le (c_2/c_1)^{\alpha n} 4^n 2^{n\theta\alpha} \frac{\mathrm{e}^{n\theta\alpha}}{(2\pi n)^{\theta\alpha/2}} =: \frac{\beta^n}{(2\pi n)^{\theta\alpha/2}} \le \beta^n, \end{aligned}$$

where $\beta = 4(c_2/c_1)^\alpha 2^{\theta\alpha} \mathrm{e}^{\theta\alpha}$.

The ε-entropy is therefore bounded by

$$\mathcal{H}_{2\varepsilon}(X^\alpha, X) = \log \mathcal{N}_{2\varepsilon} \le n \log \beta;$$

since n is the smallest integer such that $\lambda_{n+1}^{-\alpha} \le \varepsilon$ and $\lambda_n \ge c_1 n^\theta$ it follows that $n \le c_1^{-1/\theta} \varepsilon^{-1/\alpha\theta}$, and therefore

$$\mathcal{H}_{2\varepsilon}(X^\alpha, X) \le C\varepsilon^{-1/\alpha\theta}.$$

In the case of Laplacian on a bounded domain in $\mathbb{R}^d$ we have $\theta = 2/d$ and then

$$\mathcal{H}_\varepsilon(X^\alpha, X) \le C\varepsilon^{-d/2\alpha}.$$

It follows easily using essentially the same argument that if $0 < \alpha < \beta$ we have

$$\mathcal{H}_\varepsilon(X^\beta, X^\alpha) \le C\varepsilon^{-d/2(\beta-\alpha)},$$

which in particular agrees with the upper bound in (6.1) in the case $p_1 = p_2 = 2$, $l_1 = 2\beta$, $l_2 = 2\alpha$.

7. Applications.

7.1. **Application 1: An abstract semilinear parabolic problem.** For the general abstract model we treat in this section we can apply either Theorem 3.1 or Theorem 3.3 to determine that the box-counting dimension of the associated attractor $\mathcal{A}$ is finite.

Let X be a Banach space, $A\colon D(A) \subset X \to X$ be a sectorial operator with $\operatorname{Re}\sigma(A) > a > 0$ and such that A has compact resolvent. By X^γ, with $\gamma \geq 0$, we represent the associated fractional power spaces of X. Now, for a fixed $\alpha \in (0,1)$, suppose $F\colon X^\alpha \to X$ is continuously differentiable, Lipschitz continuous in bounded subsets of X^α (with Lipschitz constant $L_{\alpha,B}$, for B a bounded subset of X^α). For $\beta \in (\alpha, 1)$, note that $F\colon X^\beta \to X$ satisfies the same hypothesis as before with Lipschitz constant replaced by $L_{\beta,D}$, for D any bounded subset in X^β. Suppose that the semigroup $\{S(t) : X^\alpha \to X^\alpha \ : \ t \geq 0\}$ associated to the abstract parabolic problem

$$\begin{cases} u_t + Au = F(u), \ \ t > 0 \\ u(0) = u_0 \in X^\alpha \end{cases}$$

has a global attractor $\mathcal{A} \subset X^\beta$. By hypothesis we know that

$$\|\mathrm{e}^{-At}\|_{\mathcal{L}(X^\alpha,\, X^\theta)} \leq c_{\theta-\alpha}\, t^{-(\theta-\alpha)}\, e^{-at}, \quad t > 0 \ , \ 0 \leq \alpha \leq \theta \leq \beta,$$

for some positive constants $c_\rho > 0$, $\rho \geq 0$ (for details see Henry [13], Theorem 1.4.3).

For $u \in \mathcal{A}$ we have

$$S(t)u = \mathrm{e}^{-At}u + \int_0^t \mathrm{e}^{-A(t-s)} F(S(s)u)\, ds \tag{7.1}$$

and differentiating this expression with respect to u, denoting it by $S_u(t)$, we obtain

$$S_u(t) = \mathrm{e}^{-At} + \int_0^t \mathrm{e}^{-A(t-s)} DF(S(s)u)\, S_u(s)\, ds.$$

So for any $t > 0$ and for any $v \in X^\alpha$ we have

$$\|S_u(t)\, v\|_{X^\beta} \leq \frac{c_{\beta-\alpha}}{t^{\beta-\alpha}}\, \|v\|_{X^\alpha} + \int_0^t \frac{c_\beta\, N}{(t-s)^\beta}\, \|S_u(s)\, v\|_{X^\beta}\, ds$$

where $N := \sup_{u \in \mathcal{A}} \{\|DF(u)\|_{\mathcal{L}(X^\beta, X)}\}$.

By a Volterra inequality (see Cholewa & Dlotko [6], formulas (1.2.21) and (1.2.30) in Lemma 1.2.9) we obtain for t_0 satisfying

$$\frac{c_\beta\, N\, t_0^{1-\beta}}{2^{\alpha-2\beta}} \left(\frac{1}{1-\beta+\alpha} + \frac{1}{1-\beta} \right) = 1 \tag{7.2}$$

that

$$\|S_u(t_0)\,v\|_{X^\beta} \leq 2c_{\beta-\alpha}\,t_0^{\alpha-\beta}\|v\|_{X^\alpha}, \quad \text{for all } v \in X^\alpha,$$

and this is precisely the smoothing property corresponding to the compact embedding (since the operator A has compact resolvent) of X^β into X^α with

$$K := 2c_{\beta-\alpha}\,t_0^{\alpha-\beta}.$$

Note that K is uniform with respect to $\mathcal{A}$. Then, applying Theorem 3.3 to $S := S(t_0)$ and by (5.5) it follows that

$$\dim_B(\mathcal{A}; X^\beta) = \dim_B(\mathcal{A}; X^\alpha) \leq \frac{\log \mathcal{N}_{\nu/K}}{-\log \nu}, \quad \text{for all } \nu \in \left(0, \frac{1}{2}\right).$$

Moreover, following steps in Corollary 3.4 in Carvalho et al. [3] we say that operator A is an admissible sectorial operator if there are a sequence of finite rank projections $\{P_n\}_{n\in\mathbb{N}}$, a sequence of positive real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_{n+1} \geq \lambda_n$ and $\lambda_n \to \infty$ as $n \to \infty$, and $M > 0$ such that

$$\|\mathrm{e}^{-At}(I - P_n)\|_{\mathcal{L}(X^\alpha, X^\theta)} \leq M t^{-(\theta-\alpha)}\mathrm{e}^{-\lambda_n t}, \qquad t > 0,\ 0 \leq \alpha \leq \theta \leq \beta.$$

If $Q_n := I - P_n$, we have $S_u(1) = P_n S_u(1) + Q_n S_u(1)$, and there exists $n_0 \in \mathbb{N}$ such that $\|Q_{n_0} S_u(1)\|_{\mathcal{L}(X^\alpha)} < 1/4$. Since $P_n S_u(1)$ is a compact operator we can apply Theorem 3.1 and then guarantee that $\mathcal{A}$ has finite box-counting dimension.

7.2. **Application 2: 2D Navier–Stokes equations.** In this final section we show how the smoothing property can be used to bound the dimension of the attractor for the two-dimensional Navier–Stokes equations on a periodic domain Q. We write the equation as

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mu A u + B(u,u) = f,$$

where $A = -P\Delta$ is the Stokes operator (P is the orthogonal projection onto divergence-free fields), $B(u,u) = P[(u\cdot\nabla)u]$ and $\mu > 0$; we use $|\cdot|$ for the L^2 norm. For more details on this standard setting see Temam [21], for example. These equations generate a semigroup $\{S(t)\}_{t\geq 0}$ on the space H of divergence-free functions in $L^2(Q)$; again, see Temam [21] or Robinson [19] for details. We use the norm $|\nabla u|$ on the space $\dot{H}_1(Q)$ consisting of functions with zero average over Q (this is equivalent to the standard H^1 norm due to the Poincaré inequality).

Estimates for this equation are usually given in terms of the quantity

$$G := |f|/\mu^2\lambda_1.$$

Although the dimension we will obtain for the attractor here is $\sim G^4$, which is worse than the best known estimate $\sim G^{2/3}(1 + \log G)^{1/3}$, this polynomial estimate requires only the relatively simple bounds from this paper rather than the full Hilbert-space theory in Temam [21].

First we recall the following estimates for solutions on the attractor (see Robinson [19], for example). We have

$$|u|^2 \le \mu^2 G^2 \qquad \text{and} \qquad |\nabla u|^2 \le \mu^2 \lambda_1 G^2, \qquad u \in \mathcal{A}.$$

Theorem 7.1. *If $S(t)$ denotes the time-t map of the semigroup generated by the 2D Navier–Stokes equations on H then there exists a time $t_0 > 0$ such that $S := S(t_0)$ satisfies*

$$\|Su_0 - Sv_0\|_{H^1} \le K\|u_0 - v_0\|_{L^2},$$

where $K = c\mu^{1/2}\lambda_1^{1/2}G^2$. Consequently

$$\dim_B(\mathcal{A}; H) \le c\mu\lambda_1 G^4.$$

Proof. The equation for the difference $w = u - v$ of two solutions is

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \mu A w + B(u, w) + B(w, v) = 0.$$

Take the inner product with w to give (using the fact that $(B(u, w), w) = 0$ and the Ladyzhenskaya inequality $\|w\|_{L^4} \le c|w|^{1/2}|\nabla w|^{1/2}$)

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 + \mu|\nabla w|^2 &\le -(B(w, v), w) \\
&\le \int |w||\nabla v||w| \le \|w\|_{L^4}^2 |\nabla v| \\
&\le c|w||\nabla w||\nabla v| \\
&\le \frac{c}{2\mu}|\nabla v|^2|w|^2 + \frac{\mu}{2}|\nabla w|^2;
\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 + \mu|\nabla w|^2 \le \frac{c}{\mu}|\nabla v|^2|w|^2 \le c\mu\lambda_1 G^2|w|^2.$$

Drop the second term on the left-hand side, and integrate from $t = 0$ to s to obtain

$$|w(s)|^2 \le \exp(c\mu\lambda_1 G^2 s)|w(0)|^2.$$

Now use this to integrate again from $t = 0$ to t^*, where $t^* = 1/\mu\lambda_1 G^2$:

$$\begin{aligned}
|w(t^*)|^2 + \mu\int_0^{t^*} |\nabla w(s)|^2 \,\mathrm{d}s &\le \left[c\mu\lambda_1 G^2 \int_0^{t^*} \mathrm{e}^{c\mu\lambda_1 G^2 s}\,\mathrm{d}s\right]|w(0)|^2 \\
&\le c|w(0)|^2,
\end{aligned}$$

with $c > 0$ independent of G. From this we take the estimate

$$\mu\int_0^{t^*} |\nabla w(s)|^2 \,\mathrm{d}s \le c|w(0)|^2.$$

Now take the inner product of the equation for differences with Aw to give (using Agmon's inequality $\|w\|_{L^\infty} \leq c|w|^{1/2}|Aw|^{1/2}$)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla w|^2 + \mu|Aw|^2 &\leq \int |u||\nabla w||Aw| + |w||\nabla v||Aw| \\ &\leq c|u|^{1/2}|\nabla u|^{1/2}|\nabla w|^{1/2}|Aw|^{3/2} + c|w|^{1/2}|\nabla v||Aw|^{3/2} \\ &\leq \frac{\mu}{2}|Aw|^2 + \frac{c}{\mu^3}|u|^2|\nabla u|^2|\nabla w|^2 + \frac{c}{\mu^3}|w|^2|\nabla v|^4.\end{aligned}$$

Since $|w|^2 \leq \lambda_1^{-1}|\nabla w|^2$ this gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla w|^2 + |Aw|^2 &\leq \frac{c}{\mu^3}\left[|u|^2|\nabla u|^2 + \lambda_1^{-1}|\nabla v|^4\right]|\nabla w|^2 \\ &\leq \frac{c}{\mu^3}[\mu^4\lambda_1 G^4]|\nabla w|^2 \\ &= [c\mu\lambda_1 G^4]|\nabla w|^2.\end{aligned}$$

Now integrate from $t = s$ to $t = t^*$ with $0 \leq s \leq t^*$ to give

$$|\nabla w(t^*)|^2 \leq |\nabla w(s)|^2 + [c\mu\lambda_1 G^4]\int_s^{t^*}|\nabla w(\tau)|^2\,\mathrm{d}\tau,$$

and then integrate again with respect to s from $s = 0$ to $s = t^*$ to give

$$\begin{aligned}t^*|\nabla w(t^*)|^2 &\leq \int_0^{t^*}|\nabla w(s)|^2\,\mathrm{d}s + [c\mu\lambda_1 G^4]\int_0^{t^*}\int_s^{t^*}|\nabla w(\tau)|^2\,\mathrm{d}\tau\,\mathrm{d}s \\ &\leq \left[1 + c\mu\lambda_1 G^4 t^*\right]\int_0^{t^*}|\nabla w(s)|^2\,\mathrm{d}s \\ &\leq c\left[1 + c\mu\lambda_1 G^4 t^*\right]|w(0)|^2.\end{aligned}$$

So

$$|\nabla w(t^*)|^2 \leq \left[\frac{c}{t^*} + c\mu\lambda_1 G^4\right]|w(0)|^2$$

and since $t^* = (\mu\lambda_1 G^2)^{-1}$ this is

$$|\nabla w(t^*)|^2 \leq \left[c\mu\lambda_1 G^2 + c\mu\lambda_1 G^4\right]|w(0)|^2,$$

which for G large this gives

$$|\nabla w(t^*)|^2 \leq \left[c\mu\lambda_1 G^4\right]|w(0)|^2.$$

This is the L^2–H^1 smoothing estimate that we need, with $K = c\mu^{1/2}\lambda_1^{1/2}G^2$.

Since the smoothing estimate is from L^2 into H^1, this means from (6.1) ($d = 2$, $l_1 = 1$, $l_2 = 0$) that $\gamma = 2$. Therefore the dimension bound in (6.4) is of the order of K^2, i.e. of the order of G^4. □

8. Appendix

In this section we shall guarantee that continuously differentiable maps are in fact uniformly differentiable maps and vice versa.

Proposition 8.1. *A map $S : X \to X$ is continuously differentiable on a neighbourhood $\mathcal{U}$ of a compact set $\mathcal{A}$ if and only if it is uniformly differentiable on a neighbourhood $\mathcal{V}$ of $\mathcal{A}$.*

Proof. Without loss of generality (since $\mathcal{A}$ is compact) let $\mathcal{U} = B_X(\mathcal{A}, \delta) := \bigcup_{x\in\mathcal{A}} B_X(x, \delta)$, for some $\delta > 0$. First note that given $\eta > 0$ there exists $r_0 = r_0(\eta) > 0$ and some neighbourhood $\mathcal{V}$ of $\mathcal{A}$ such that

$$\|DS(w) - DS(z)\| < \eta, \qquad \forall\ 0 < r \leqslant r_0, w \in \mathcal{V}, z \in \mathcal{A}, \|w - z\| < r. \tag{8.1}$$

Indeed, let $x \in \mathcal{A}$. Since $DS(\cdot)$ is continuous then given $\eta > 0$ there is $\delta_x = \delta_x(\eta) > 0$ such that

$$\|DS(w) - DS(x)\| < \frac{\eta}{2}, \qquad \forall w \in \mathcal{U},\ \|w - x\| < \delta_x. \tag{8.2}$$

Since $\mathcal{A}$ is a compact subset we have $\mathcal{A} \subseteq \bigcup_{i=1}^m B_X(x_i, \delta_{x_i}/2)$, with $x_i \in \mathcal{A}$ for each i. Let $r_0 := \min\{\delta, \delta_{x_1}/2, \cdots, \delta_{x_m}/2\}$ and define

$$\mathcal{V} := B_X(\mathcal{A}, r_0).$$

Clearly $\mathcal{V} \subseteq \mathcal{U}$. Now take $0 < r \leqslant r_0$ and any $w \in \mathcal{V}$ and $z \in \mathcal{A}$ with $\|w - z\| < r$. Hence $z \in B_X(x_i, \delta_{x_i}/2)$ for some $i = 1, \cdots, m$ and then $\|w-x_i\| \leqslant \|w-z\|+\|z-x_i\| < r+\delta_{x_i}/2 \leqslant \delta_{x_i}/2 + \delta_{x_i}/2 = \delta_{x_i}$, i.e., $w \in B_X(x_i, \delta_{x_i})$. Finally, from (8.2)

$$\|DS(w) - DS(z)\| \leqslant \|DS(w) - DS(x_i)\| + \|DS(x_i) - DS(z)\| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

proving (8.1).

Let $0 < r \leqslant r_0$, $x \in \mathcal{A}$ and $z \in \mathcal{V}$ with $\|x - z\| < r$. Note that for each $t \in [0, 1]$ the line $x_t := tz + (1 - t)x \in \mathcal{V}$ ($x_t \in B_X(x, r) \subseteq \mathcal{V}$). So from the fundamental theorem of calculus and (8.1) we have

$$\begin{aligned}
\|S(z) - S(x) - DS(x)(z - x)\| &= \left\| \int_0^1 DS(x_t)(z - x)dt - DS(x)(z - x) \right\| \\
&= \left\| \int_0^1 \big[DS(x_t)(z - x) - DS(x)(z - x)\big] dt \right\| \\
&\leqslant \int_0^1 \|DS(x_t) - DS(x)\|\|z - x\|dt \\
&< \int_0^1 \eta\|z - x\|dt \\
&= \eta\|z - x\|,
\end{aligned}$$

i.e.,

$$\|S(z)-S(x)-DS(x)(z-x)\| < \eta\|z-x\|, \qquad \text{for all } x \in \mathcal{A},\ z \in \mathcal{V},\ \|z-x\| < r,\ 0 < r \leqslant r_0.$$

Note that this is equivalent to the expression in Definition 2.3.

Now suppose that S is uniformly differentiable on $\mathcal{V}$. Then given $\eta > 0$ there is $r_0 = r_0(\eta) > 0$ such that

$$\|S(x+h) - S(x) - DS(x)h\| \leq \eta\|h\|, \qquad x \in \mathcal{V}, h \in X, \|h\| \leq r_0. \tag{8.3}$$

Given $x, y \in \mathcal{V}$ note that

$$\begin{aligned}\|DS(x) - DS(y)\|_{\mathcal{L}(X)} &= \sup_{\|w\|=1} \|DS(x)w - DS(y)w\| \\ &= \frac{2}{r_0} \sup_{\|w\|=1} \Big\|DS(x)(wr_0/2) - DS(y)(wr_0/2)\Big\|\end{aligned}$$

and so

$$\begin{aligned}&\Big\|DS(x)(wr_0/2) - DS(y)(wr_0/2)\Big\| \leq \\ &\qquad \leq 2\eta\|wr_0/2\| + \|S(x+wr_0/2) - S(x) + S(y) - S(y+wr_0/2)\| \\ &\qquad = r_0\eta + \|S(x+wr_0/2) - S(x) + S(x+z) - S(x+z+wr_0/2)\|,\end{aligned}$$

where $z = y - x$. Suppose that $\|z\| < r_0/2$. Then $\|z + wr_0/2\| < r_0$ and then using the differentiability (8.3) three times (in directions $wr_0/2$, z and $z + wr_0/2$) we obtain

$$\begin{aligned}\|S(x+wr_0/2) - S(x) + S(x+z) - S(x+z+wr_0/2)\| &\leq 2\eta\big(\|z\| + \|wr_0/2\|\big) \\ &< 2r_0\eta.\end{aligned}$$

Finally for any $x, y \in \mathcal{V}$ with $\|x - y\| < r_0/2$ we have

$$\begin{aligned}\|DS(x) - DS(y)\|_{\mathcal{L}(X)} &< \frac{2}{r_0}\big(r_0\eta + 2r_0\eta\big) \\ &= 6\eta,\end{aligned}$$

and then $DS(x)$ is uniformly continuous (with respect to x) on $\mathcal{V}$. □

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