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DECOMPOSITION OF INTEGRAL METRIC CURRENTS

PAOLO BONICATTO, GIACOMO DEL NIN, AND ENRICO PASQUALETTO

ABSTRACT. In the setting of complete metric spaces, we prove that integral currents can be decomposed as a sum of indecomposable components. In the special case of one-dimensional integral currents, we also show that the indecomposable ones are exactly those associated with injective Lipschitz curves or injective Lipschitz loops, therefore extending Federer's characterisation to metric spaces. Moreover, some applications of our main results will be discussed.

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1. INTRODUCTION

Currents have been widely used in geometric measure theory and calculus of variations as a weak setting to formulate a variety of geometric problems, especially within the theory of *normal* and *integral* currents developed by Federer and Fleming [10]. While the initial theory was set in Euclidean spaces, Ambrosio and Kirchheim [5] extended it to *metric currents*, defined on complete metric spaces. Since then there has been a significant effort to understand which results, previously proven for Euclidean currents, could be extended to the metric setting. Among these we mention: Smirnov's result on the decomposition of Euclidean normal 1-currents into solenoidal charges [16], later extended by Paolini and Stepanov to the metric setting [13, 14]; isoperimetric inequalities, first proved by Federer and Fleming [10] and then by Almgren in a sharp form [2], extended by Wenger to metric spaces admitting a cone-type inequality [17, 18].

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Two useful results in the Euclidean case concern the structure of integral currents, which is particularly nice:

- (i) Every integral k -current T admits a decomposition into countably many *indecomposable* components T_n , with

$$T = \sum_n T_n, \quad \mathbf{N}(T) = \sum_n \mathbf{N}(T_n),$$

where $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$ is the normal mass;

- (ii) In the particular case of 1-currents, the indecomposable ones are precisely given by (currents associated with) Lipschitz curves which are either injective or injective loops, and as a consequence every integral 1-current is the countable sum of such curves.

These results are stated in Federer's book [9, 4.2.25], and a hint of a proof is provided. A detailed proof of these results was however missing for a long time, and it appears that only recently the proof of the structure of integral 1-currents in the Euclidean case has been written down (see [12, Proposition 1.3.16] and [8, Theorem 2.5]), even though the characterisation of the indecomposable ones as injective Lipschitz curves or loops seems to be missing. Up to our knowledge, the proof of the decomposition in indecomposable components was instead never explicitly written out, even in the Euclidean setting.

The aim of the present work is to rigorously prove both results in the full generality of the Ambrosio–Kirchheim setting: we will work in a complete metric space (X, d) , with the assumption that the cardinality of every set is an Ulam number. The possibility of proving both results in a complete metric space without any further geometric assumption on the space stems from the fact that an integral current is already given, and we are just looking for a suitable decomposition *within* the current itself. More precisely, the strategy is as follows: we isometrically embed X into a Banach space; here we have at our disposal the isoperimetric inequality, which is the main tool allowing us to prove the decomposition; then we just observe that the decomposition lives entirely within the current itself, and thus we can isometrically transport it back to the original metric space. We thus bypass the use of a polyhedral deformation theorem in metric spaces (whose statement would not even be clear), instead relying ultimately on the Euclidean one.

Concerning (i) (see Theorem 3.2) we present two proofs in Sections 3 and 4. For the first proof we take inspiration from [3], where the case of d -currents in \mathbb{R}^d has been essentially proven within the theory of indecomposable finite perimeter sets. For the second proof we stay closer to the original suggestion by Federer, and prove that a greedy algorithm is successful: we inductively remove from the current an indecomposable component with maximal mass, and show that the process ends in countably many steps. Regarding this strategy we are thankful to Giada Franz, whose notes [11] were an inspiration to implement a similar strategy for currents.

Concerning (ii) (see Theorem 5.3), an ε -approximation of integral 1-currents with Lipschitz curves was already proven in length spaces by Wenger in the boundaryless case [19] and Ambrosio–Wenger in the general case [6]. We take these works as a starting point and we first show that one can remove the ε by means of an Arzelà–Ascoli compactness argument; then, with the aid of the Decomposition Theorem 3.2, we show how it is possible to obtain injectivity, and this yields the desired description of indecomposable 1-currents.

In the final section we present some applications of the decomposition theorems. The first one, Proposition 6.1, is an extension to metric spaces of a lemma by Alberti–Bianchini–Crippa [1, Lemma 2.14] regarding boundaryless currents with support in a simple curve. The second one, Corollary 6.5, is an alternative proof of the characterisation of planar simple sets (defined in [3]) as those bounded by Jordan curves. The proof of this result relies on Proposition 6.4, namely

the observation that a finite perimeter set of finite measure is simple if and only if the current associated with its boundary is indecomposable.

Structure of the paper. The paper is organised as follows: we use Section 2 to collect useful, previously known results on metric currents. Such section contains also a detailed explanation of our notation and should be kept as a reference, while reading the paper. Section 3 contains the first proof of the decomposition result for integral k -currents (Theorem 3.2), based on a simple variational argument, while Section 4 is devoted to the presentation of an alternative, independent proof of the same result. Finally, in Section 5 we present the characterisation of indecomposable 1-currents (Theorem 5.3) and we conclude the paper with Section 6, which contains some applications of our main results.

Addendum. After we completed the manuscript, we learned that some of the results had been proven in [15, Appendix A]: there, the author proves that 1-currents can be written as a sum of (not necessarily simple) curves. Also, he briefly sketches the proof of the decomposition into indecomposable components of k -currents following Federer's hint as well.

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2. REMINDER ON METRIC CURRENTS

Aim of this section is to briefly recall the theory of metric currents, which was introduced by L. Ambrosio and B. Kirchheim in [5]. We only discuss those definitions and results that are sufficient for our purposes. In accordance with [5], in this paper we will always assume that the cardinality of any set is an Ulam number, which is consistent with the standard ZFC set theory. This guarantees that any finite Borel measure on a complete metric space has a separable support and is concentrated on a σ -compact set.

2.1. Main classes of metric currents. Let (X, d) be a complete metric space. Given any $k \in \mathbb{N}$, we denote by $\mathcal{D}^k(X)$ the family of all **metric k -dimensional differential forms** on X , namely

$$\mathcal{D}^k(X) := \text{LIP}_b(X) \times \text{LIP}(X)^k,$$

where $\text{LIP}(X)$ stands for the space of all Lipschitz, real-valued functions on X , while $\text{LIP}_b(X)$ is the set of bounded functions in $\text{LIP}(X)$. We denote by $\text{Lip}(f)$ the Lipschitz constant of $f \in \text{LIP}(X)$.

Definition 2.1 (Metric current [5]). *Let (X, d) be a complete metric space. Let $k \in \mathbb{N}$ be given. Then a **metric k -current** in X is a multilinear functional $T: \mathcal{D}^k(X) \rightarrow \mathbb{R}$ such that:*

- (i) *If $f \in \text{LIP}_b(X)$ and $\pi_i^n, \pi_i \in \text{LIP}(X)$ satisfy $\sup_{n \in \mathbb{N}} \text{Lip}(\pi_i^n) < +\infty$ and $\lim_n \pi_i^n(x) = \pi_i(x)$ for every $i = 1, \dots, k$ and $x \in X$, then it holds $T(f, \pi_1, \dots, \pi_k) = \lim_n T(f, \pi_1^n, \dots, \pi_k^n)$.*
- (ii) *Given any $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X)$, it holds $T(f, \pi_1, \dots, \pi_k) = 0$ whenever there exists an index $i = 1, \dots, k$ such that the function π_i is constant on some neighbourhood of $\{f \neq 0\}$.*

(iii) *There exists a finite Borel measure μ on X such that*

$$|T(f, \pi_1, \dots, \pi_k)| \leq \prod_{i=1}^k \text{Lip}(\pi_i) \int |f| d\mu, \quad \text{for every } (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X). \quad (2.1)$$

The minimal measure μ satisfying (2.1) is called the **mass measure** of T and denoted by $\|T\|$. The **total mass** of T is given by $M(T) := \|T\|(X)$. The space of all metric k -currents in X is denoted by $\mathcal{M}_k(X)$. It holds that $(\mathcal{M}_k(X), M)$ is a Banach space. The **support** $\text{spt}(T)$ of a given current $T \in \mathcal{M}_k(X)$ is defined as the support of its mass measure $\|T\|$.

The **boundary** of $T \in \mathcal{M}_k(X)$ is the functional $\partial T: \mathcal{D}^{k-1}(X) \rightarrow \mathbb{R}$ defined by

$$\partial T(f, \pi_1, \dots, \pi_{k-1}) := T(1, f, \pi_1, \dots, \pi_{k-1}), \quad \text{for every } (f, \pi_1, \dots, \pi_{k-1}) \in \mathcal{D}^{k-1}(X).$$

We say that T is **normal** provided ∂T is a metric $(k-1)$ -current in X . We denote by $\mathcal{N}_k(X)$ the space of all normal k -currents in X . Observe that one has $\partial(\partial T) = 0$ for every $T \in \mathcal{N}_k(X)$. It holds that $\mathcal{N}_k(X)$ is a Banach space if endowed with the **normal mass** N , which is given by

$$N(T) := M(T) + M(\partial T), \quad \text{for every } T \in \mathcal{N}_k(X).$$

Moreover, it follows from (2.1) that any current $T \in \mathcal{M}_k(X)$ can be uniquely extended to a multilinear real-valued functional (still denoted by T) on the $(k+1)$ -tuples $L^1(X, \|T\|) \times \text{LIP}(X)^k$ in such a way that the inequality in (2.1) is still valid. Therefore, given any metric k -current $T \in \mathcal{M}_k(X)$ and any Borel set $E \subseteq X$, we can define the **restriction** $T \llcorner E \in \mathcal{M}_k(X)$ as

$$T \llcorner E(f, \pi_1, \dots, \pi_k) := T(\mathbb{1}_E f, \pi_1, \dots, \pi_k), \quad \text{for every } (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X).$$

Another important operation is the **pushforward**: given (X, d_X) , (Y, d_Y) complete metric spaces and $\varphi \in \text{LIP}(X; Y)$, we associate to any $T \in \mathcal{M}_k(X)$ the current $\varphi\#T \in \mathcal{M}_k(Y)$ that is given by

$$\varphi\#T(f, \pi_1, \dots, \pi_k) := T(f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_k \circ \varphi), \quad \text{for every } (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(Y).$$

Note that pushforward and boundary commute, namely $\partial(\varphi\#T) = \varphi\#(\partial T)$ for all $T \in \mathcal{M}_k(X)$. The mass measure also behaves well under pushforward: $\|\varphi\#T\| \leq \text{Lip}(\varphi)^k \varphi\#\|T\|$ for all $T \in \mathcal{M}_k(X)$ and any Lipschitz map $\varphi \in \text{LIP}(X; Y)$, where $\varphi\#\|T\|$ denotes the usual pushforward of measures.

We say that a current $T \in \mathcal{M}_k(X)$ is **rectifiable** provided $\|T\|$ is concentrated on a countably \mathcal{H}^k -rectifiable set and vanishes on \mathcal{H}^k -negligible Borel sets, where \mathcal{H}^k stands for the k -dimensional Hausdorff measure on (X, d) . We denote by $\mathcal{R}_k(X)$ the space of all rectifiable k -currents in X . In addition, we say that a rectifiable current $T \in \mathcal{R}_k(X)$ is **integer-rectifiable** provided the following property holds: given a Lipschitz map $\varphi \in \text{LIP}(X; \mathbb{R}^k)$ and an open set $\Omega \subseteq X$, there exists a density function $\theta \in L^1(\mathbb{R}^k, \mathbb{Z})$ such that

$$\varphi\#(T \llcorner \Omega)(f, \pi_1, \dots, \pi_k) = \int \theta f \det \left(\frac{\partial \pi_i}{\partial x_j} \right)_{i,j} d\mathcal{L}^k, \quad \text{for every } (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(\mathbb{R}^k).$$

The existence of the partial derivatives $\frac{\partial \pi_i}{\partial x_j}$ is granted by the classical Rademacher's theorem. Finally, we say that a metric k -current is **integral** provided it is both integer-rectifiable and normal. We denote by $\mathcal{I}_k(X)$ the space of all integral k -currents in X , which is a \mathbb{Z} -module.

We recall the following definition, which is given in [9, Definition 4.2.25] for Euclidean currents.

Definition 2.2 (Indecomposable current). *Let (X, d) be a complete metric space. Then a given integral current $T \in \mathcal{I}_k(X)$ is said to be **decomposable** provided there exists a couple of non-zero integral currents $R, S \in \mathcal{I}_k(X)$ such that $T = R + S$ and $N(T) = N(R) + N(S)$. An integral current which is not decomposable is called **indecomposable**.*

2.2. Convergence of metric currents. There are two important notions of convergence for integral metric currents, as we are going to recall in this section.

Let (X, d) be a complete metric space. Let $T \in \mathcal{M}_k(X)$ and $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_k(X)$ be given. Then we say that T_n **weakly converges** to T as $n \rightarrow \infty$ (briefly, $T_n \rightharpoonup T$ as $n \rightarrow \infty$) provided it holds

$$T(f, \pi_1, \dots, \pi_k) = \lim_{n \rightarrow \infty} T_n(f, \pi_1, \dots, \pi_k), \quad \text{for every } (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X).$$

It turns out that, given any open set $\Omega \subseteq X$, the functional $\mathcal{M}_k(X) \ni T \mapsto \|T\|(\Omega) \in [0, \infty)$ is lower semicontinuous when the domain is equipped with the topology of the weak convergence. In particular, it holds that $\mathcal{M}_k(X) \ni T \mapsto \mathbf{M}(T)$ is lower semicontinuous. Moreover, the boundary operator is continuous with respect to the weak convergence of normal currents, namely it holds

$$T \in \mathcal{N}_k(X), \quad (T_n)_{n \in \mathbb{N}} \subseteq \mathcal{N}_k(X), \quad T_n \rightharpoonup T \text{ as } n \rightarrow \infty \implies \partial T_n \rightharpoonup \partial T \text{ as } n \rightarrow \infty. \quad (2.2)$$

Given a sequence $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_k(X)$, we will denote by $\sum_{n \in \mathbb{N}} T_n \in \mathcal{M}_k(X)$ the weak limit of the partial sums $\sum_{n=1}^N T_n$ as $N \rightarrow \infty$, whenever such limit exists. No ambiguity should occur, as in this paper we will just consider weakly converging series. Observe that if $T = \sum_{n \in \mathbb{N}} T_n$ exists, then for any $\Omega \subseteq X$ open we have $\|T\|(\Omega) \leq \varliminf_N \|\sum_{n=1}^N T_n\|(\Omega) \leq \lim_N \sum_{n=1}^N \|T_n\|(\Omega) = \sum_{n \in \mathbb{N}} \|T_n\|(\Omega)$. We thus conclude that

$$\|\sum_{n \in \mathbb{N}} T_n\| \leq \sum_{n \in \mathbb{N}} \|T_n\|, \quad \text{whenever } \sum_{n \in \mathbb{N}} \mathbf{M}(T_n) < \infty \text{ (and thus } \sum_{n \in \mathbb{N}} T_n \text{ exists),} \quad (2.3)$$

thanks to the outer regularity of the measures $\|T\|$ and $\sum_{n \in \mathbb{N}} \|T_n\|$. Given that the boundary operator ∂ is linear, it also holds that

$$T \in \mathcal{N}_k(X), \quad (T_n)_{n \in \mathbb{N}} \subseteq \mathcal{N}_k(X), \quad T = \sum_{n \in \mathbb{N}} T_n \implies \partial T = \sum_{n \in \mathbb{N}} \partial T_n. \quad (2.4)$$

Remark 2.3. In general, the element $\sum_{n \in \mathbb{N}} T_n$ depends on the ordering of the currents T_n in the sequence. However, this is not the case under the additional assumption that $C := \sum_{n \in \mathbb{N}} \mathbf{M}(T_n)$ is finite, since it grants that

$$\sum_{n \in \mathbb{N}} |T_n(f, \pi_1, \dots, \pi_k)| \stackrel{(2.1)}{\leq} C \sup_X |f| \prod_{i=1}^k \text{Lip}(\pi_i), \quad \text{for every } (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X).$$

In light of this observation, we will occasionally consider series of the form $\sum_{n \in I} T_n$, where $\{T_n\}_{n \in I}$ is a family of currents that is indexed over a countable set I whose ordering is not specified. \blacksquare

Proposition 2.4. *Let (X, d) be a complete metric space. Let $T \in \mathcal{N}_k(X)$ and $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{N}_k(X)$ be such that $T = \sum_{n \in \mathbb{N}} T_n$. Then it holds that $\mathbf{N}(T) = \sum_{n \in \mathbb{N}} \mathbf{N}(T_n)$ if and only if*

$$\|T\| = \sum_{n \in \mathbb{N}} \|T_n\|, \quad \|\partial T\| = \sum_{n \in \mathbb{N}} \|\partial T_n\|. \quad (2.5)$$

Proof. Sufficiency is obvious. To prove necessity, assume $\mathbf{N}(T) = \sum_{n \in \mathbb{N}} \mathbf{N}(T_n)$. We know from (2.3) and (2.4) that $\|T\| \leq \sum_{n \in \mathbb{N}} \|T_n\|$ and $\|\partial T\| \leq \sum_{n \in \mathbb{N}} \|\partial T_n\|$. For any $E \subseteq X$ Borel, one has

$$\begin{aligned} \mathbf{N}(T) &= \|T\|(X) + \|\partial T\|(X) = \|T\|(E) + \|T\|(X \setminus E) + \|\partial T\|(E) + \|\partial T\|(X \setminus E) \\ &\leq \sum_{n \in \mathbb{N}} \|T_n\|(E) + \|T_n\|(X \setminus E) + \|\partial T_n\|(E) + \|\partial T_n\|(X \setminus E) \\ &= \sum_{n \in \mathbb{N}} \|T_n\|(X) + \|\partial T_n\|(X) = \sum_{n \in \mathbb{N}} \mathbf{N}(T_n) = \mathbf{N}(T), \end{aligned}$$

which forces the identities $\|T\|(E) = \sum_{n \in \mathbb{N}} \|T_n\|(E)$ and $\|\partial T\|(E) = \sum_{n \in \mathbb{N}} \|\partial T_n\|(E)$. Thanks to the arbitrariness of E , we deduce that (2.5) is verified, yielding the sought conclusion. \square

On integral currents, another (more geometric) notion of convergence is given by the **flat norm**:

$$F(T) := \inf \left\{ M(R) + M(S) \mid R \in \mathcal{S}_k(X), S \in \mathcal{S}_{k+1}(X), T = R + \partial S \right\}, \quad \text{for every } T \in \mathcal{S}_k(X).$$

Observe that $F(T) \leq M(T) \leq N(T)$ holds for every $T \in \mathcal{S}_k(X)$.

In the classical theory of currents in Euclidean spaces, a fundamental result states that the weak convergence of currents – when restricted to integral currents having uniformly bounded normal mass – is metrised by the flat norm. This theorem has been generalised by Wenger [18] to the framework of *quasi-convex*, complete metric spaces admitting *local cone-type inequalities* (a class of spaces which contains, for instance, all Banach spaces):

Theorem 2.5 (Weak convergence and flat norm [18]). *Let $(X, \|\cdot\|)$ be a Banach space. Consider a sequence $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{S}_k(X)$ satisfying $\sup_n N(T_n) < \infty$. Then for any $T \in \mathcal{S}_k(X)$ it holds that*

$$T_n \rightarrow T \text{ as } n \rightarrow \infty \iff \lim_{n \rightarrow \infty} F(T_n - T) = 0.$$

2.3. Currentification of Lipschitz curves. Let (X, d) be a complete metric space. Then each Lipschitz curve $\gamma: [a, b] \rightarrow X$ can be naturally associated with a metric 1-current $[\![\gamma]\!]$ as follows:

$$[\![\gamma]\!](f, \pi) := \int_a^b f(\gamma(t))(\pi \circ \gamma)'(t) dt, \quad \text{for every } (f, \pi) \in \mathcal{D}^1(X).$$

We call $[\![\gamma]\!]$ the **currentification** of γ . It holds that $[\![\gamma]\!] \in \mathcal{S}_1(X)$ and $\|[\![\gamma]\!]\| \leq \gamma_{\#}(|\dot{\gamma}| \mathcal{L}^1|_{[a,b]})$, where $|\dot{\gamma}|$ is the metric speed of γ . If γ is injective, then $\|[\![\gamma]\!]\| = \mathcal{H}^1|_{\gamma([a,b])}$ and $M([\![\gamma]\!])$ coincides with the **length** of γ , namely

$$M([\![\gamma]\!]) = \mathcal{H}^1(\gamma([a, b])) = \ell(\gamma) := \int_a^b |\dot{\gamma}|(t) dt.$$

Moreover, the boundary of $[\![\gamma]\!]$ is given by

$$\partial[\![\gamma]\!] = \delta_{\gamma(b)} - \delta_{\gamma(a)},$$

where δ_x stands for the Dirac measure at $x \in X$. In particular, it holds that $M(\partial[\![\gamma]\!]) = 0$ if and only if γ is a **loop**, meaning that $\gamma(a) = \gamma(b)$, otherwise $M(\partial[\![\gamma]\!]) = 2$. By an **injective loop** we will mean a loop $\gamma: [a, b] \rightarrow X$ such that the restriction $\gamma|_{[a,b]}$ is injective.

As observed for instance in [14], it holds that the mapping

$$\text{LIP}([0, 1]; X) \ni \gamma \mapsto [\![\gamma]\!] \in \mathcal{M}_1(X) \quad \text{is continuous,} \quad (2.6)$$

where $\text{LIP}([0, 1]; X)$ and $\mathcal{M}_1(X)$ are equipped with the topology of the uniform convergence and the topology of the weak convergence of currents, respectively.

In the following we will need the version of Arzelà–Ascoli’s theorem stated below, that can be found in [13]. We include for completeness a direct proof.

Lemma 2.6 (Arzelà–Ascoli revisited [13, Proposition 2.1]). *Let (X, d) be a complete metric space. Consider a sequence $(\gamma_n)_n \subseteq \text{LIP}([0, 1]; X)$ of Lipschitz curves satisfying $L := \sup_n \text{Lip}(\gamma_n) < \infty$. Suppose that for every $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that*

$$\mathcal{L}^1(\{t \in [0, 1] : \gamma_n(t) \notin K\}) \leq \varepsilon, \quad \text{for every } n \in \mathbb{N}.$$

Then there exists a subsequence $(\gamma_{n_i})_i$ uniformly converging to some limit curve $\gamma \in \text{LIP}([0, 1]; X)$.

Proof. Given any integer $m \geq 1$, let us consider the family $\mathcal{I}^m := (I_j^m)_{j=1}^m$ of subintervals of $[0, 1]$ given by

$$I_j^m = \left[\frac{j-1}{m}, \frac{j}{m} \right], \quad \text{for every } j = 1, \dots, m.$$

We choose $\varepsilon = \frac{1}{2m}$ and consider a compact set K given by the assumptions of the Lemma. Since every subinterval $I \in \mathcal{I}^m$ has measure $\frac{1}{m}$, for every such I and for every $n \in \mathbb{N}$ there exists a point $x_n^I \in I$ such that $\gamma_n(x_n^I) \in K$. By compactness of both K and I , for every $I \in \mathcal{I}^m$ we can ensure that, up to a not relabelled subsequence in n ,

$$\begin{aligned} x_n^I &\rightarrow x_\infty^I, & \text{for some } x_\infty^I \in I, \\ \gamma_n(x_n^I) &\rightarrow y^I, & \text{for some } y^I \in K \end{aligned} \tag{2.7}$$

as $n \rightarrow \infty$. We now claim that $\gamma_n(x_\infty^I) \rightarrow y^I$ as $n \rightarrow \infty$. Indeed, by triangle inequality and the uniform Lipschitz property of γ_n we have

$$\mathbf{d}(\gamma_n(x_\infty^I), y^I) \leq \mathbf{d}(\gamma_n(x_\infty^I), \gamma_n(x_n^I)) + \mathbf{d}(\gamma_n(x_n^I), y^I) \leq L|x_\infty^I - x_n^I| + \mathbf{d}(\gamma_n(x_n^I), y^I),$$

which goes to zero by (2.7). We have thus ensured that a subsequence of γ_n converges on all points x_∞^I , $I \in \mathcal{I}^m$. Repeating the same procedure for every $m \in \mathbb{N}$, and finally extracting a diagonal subsequence, we find $(\gamma_{n_i})_{i \in \mathbb{N}}$ that converges on a dense subset of $[0, 1]$ as $i \rightarrow \infty$. By the uniform Lipschitz property the convergence is uniform and holds on the entirety of $[0, 1]$, thus concluding the proof. \square

2.4. Isoperimetric inequalities of Euclidean type. As proved by S. Wenger in [17], all Banach spaces admit an isoperimetric inequality of Euclidean type, as described in the following statement.

Theorem 2.7 (Isoperimetric inequality of Euclidean type [17]). *Given any $k \in \mathbb{N}$, there exists a constant $\tilde{D}_k > 0$ such that the following property is verified. If $(X, \|\cdot\|)$ is a Banach space and a current $T \in \mathcal{I}_k(X)$ satisfies $\partial T = 0$, then there exists $S \in \mathcal{I}_{k+1}(X)$ such that $\partial S = T$ and*

$$\mathbf{M}(S) \leq \tilde{D}_k \mathbf{M}(T)^{k+1/k}. \tag{2.8}$$

In the remaining part of this paper, we will need the following consequence of Theorem 2.7.

Corollary 2.8. *Given any $k \in \mathbb{N}$, there exists a constant $D_k > 0$ such that the following property is verified. If $(X, \|\cdot\|)$ is a Banach space, then it holds that*

$$\mathbf{F}(T) \leq D_k \mathbf{N}(T)^{k+1/k}, \quad \text{for every } T \in \mathcal{I}_k(X). \tag{2.9}$$

Proof. In the case where $\mathbf{N}(T) \geq 1$, we have that $\mathbf{F}(T) \leq \mathbf{N}(T) \leq \mathbf{N}(T)^{k+1/k}$. Hence, let us suppose that $\mathbf{N}(T) < 1$. If $k = 1$, then necessarily $\partial T = 0$, whence it follows from Theorem 2.7 that there exists $S \in \mathcal{I}_2(X)$ such that $\partial S = T$ and $\mathbf{M}(S) \leq \tilde{D}_1 \mathbf{M}(T)^2$. In particular, we obtain that

$$\mathbf{F}(T) \leq \mathbf{M}(S) \leq \tilde{D}_1 \mathbf{M}(T)^2 \leq \tilde{D}_1 \mathbf{N}(T)^2.$$

Let us now pass to the case where $k \geq 2$. Since $\partial(\partial T) = 0$, we know from Theorem 2.7 that there exists $R \in \mathcal{I}_k(X)$ such that $\partial R = \partial T$ and $\mathbf{M}(R) \leq \tilde{D}_{k-1} \mathbf{M}(\partial T)^{k/k-1}$. Since $\partial(T-R) = 0$, by using again Theorem 2.7 we obtain $S \in \mathcal{I}_{k+1}(X)$ such that $\partial S = T - R$ and $\mathbf{M}(S) \leq \tilde{D}_k \mathbf{M}(T - R)^{k+1/k}$.

Observe that $F(T) \leq M(R) + M(S)$, as it follows from $T = R + \partial S$. Therefore, it holds that

$$\begin{aligned}
F(T) &\leq M(R) + M(S) \leq \tilde{D}_{k-1} M(\partial T)^{k/k-1} + \tilde{D}_k M(T-R)^{k+1/k} \\
&\leq \tilde{D}_{k-1} M(\partial T)^{k/k-1} + \tilde{D}_k (M(T) + M(R))^{k+1/k} \\
&\leq \tilde{D}_{k-1} M(\partial T)^{k/k-1} + 2^{1/k} \tilde{D}_k M(T)^{k+1/k} + 2^{1/k} \tilde{D}_k M(R)^{k+1/k} \\
&\leq \tilde{D}_{k-1} M(\partial T)^{k/k-1} + 2^{1/k} \tilde{D}_k M(T)^{k+1/k} + 2^{1/k} \tilde{D}_k \tilde{D}_{k-1}^{k+1/k} M(\partial T)^{k+1/k-1} \\
&\leq \tilde{D}_{k-1} N(T)^{k/k-1} + 2^{1/k} \tilde{D}_k N(T)^{k+1/k} + 2^{1/k} \tilde{D}_k \tilde{D}_{k-1}^{k+1/k} N(T)^{k+1/k-1} \\
&\leq (\tilde{D}_{k-1} + 2^{1/k} \tilde{D}_k + 2^{1/k} \tilde{D}_k \tilde{D}_{k-1}^{k+1/k}) N(T)^{k+1/k},
\end{aligned}$$

where in the last inequality we used the fact that $N(T) < 1$ and $\frac{k+1}{k} < \frac{k}{k-1} < \frac{k+1}{k-1}$. All in all, we proved that the statement holds with $D_k := 1 + \tilde{D}_{k-1} + 2^{1/k} \tilde{D}_k + 2^{1/k} \tilde{D}_k \tilde{D}_{k-1}^{k+1/k}$, where $\tilde{D}_0 := 0$. \square

2.5. Behaviour of currents under isometric embeddings. Let (X, d_X) , (Y, d_Y) be complete metric spaces. Let $\iota: X \hookrightarrow Y$ be an isometry. In particular, the map ι is Lipschitz, thus to any $T \in \mathcal{M}_k(X)$ we can associate its pushforward $\iota_{\#}T \in \mathcal{M}_k(Y)$. We aim to show that currents in Y supported in $\iota(X)$ and currents in X can be canonically identified via the pushforward map $\iota_{\#}$.

Given any $T \in \mathcal{M}_k(X)$, it holds that

$$\|\iota_{\#}T\| = \iota_{\#}\|T\|, \quad (2.10)$$

as observed for instance in the line below [5, Eq. (2.4)]. In particular, we have $M(\iota_{\#}T) = M(T)$. Moreover, since $\iota(X)$ is closed in Y , we deduce that $\text{spt}(\iota_{\#}T) \subseteq \iota(X)$. Given that $\partial(\iota_{\#}T) = \iota_{\#}(\partial T)$, we also have that $\iota_{\#}T \in \mathcal{N}_k(Y)$ whenever $T \in \mathcal{N}_k(X)$, and in this case it holds $N(\iota_{\#}T) = N(T)$.

Lemma 2.9. *Let (X, d_X) , (Y, d_Y) be complete metric spaces and $\iota: X \hookrightarrow Y$ an isometry. Fix any*

$$(\mathcal{F}_X, \mathcal{F}_Y) \in \left\{ (\mathcal{M}_k(X), \mathcal{M}_k(Y)), (\mathcal{N}_k(X), \mathcal{N}_k(Y)), (\mathcal{R}_k(X), \mathcal{R}_k(Y)), (\mathcal{I}_k(X), \mathcal{I}_k(Y)) \right\}.$$

Then the pushforward map $\iota_{\#}$ is a bijection between \mathcal{F}_X and $\{T' \in \mathcal{F}_Y : \text{spt}(T') \subseteq \iota(X)\}$.

Proof. Let us first check that $\iota_{\#}$ is a bijection from $\mathcal{M}_k(X)$ to $\{T' \in \mathcal{M}_k(Y) : \text{spt}(T') \subseteq \iota(X)\}$. Let $T' \in \mathcal{M}_k(Y)$ be a given current satisfying $\text{spt}(T') \subseteq \iota(X)$. Given any $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X)$, by virtue of McShane's Extension Theorem we can find a $(k+1)$ -tuple $(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k) \in \mathcal{D}^k(Y)$ such that $\tilde{f}|_{\iota(X)} = f \circ \iota^{-1}$ and $\tilde{\pi}_i|_{\iota(X)} = \pi_i \circ \iota^{-1}$ for every $i = 1, \dots, k$. Then, let us define

$$T(f, \pi_1, \dots, \pi_k) := T'(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k). \quad (2.11)$$

The resulting operator $T: \mathcal{D}^k(X) \rightarrow \mathbb{R}$ is well-defined, because the expression appearing in the right-hand side of (2.11) does not depend on the specific choice of $(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k)$. Indeed, given another $(k+1)$ -tuple $(\tilde{f}', \tilde{\pi}'_1, \dots, \tilde{\pi}'_k) \in \mathcal{D}^k(Y)$ with the same properties, we may estimate

$$\begin{aligned}
&|T'(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k) - T'(\tilde{f}', \tilde{\pi}'_1, \dots, \tilde{\pi}'_k)| \\
&\leq |T'(\tilde{f} - \tilde{f}', \tilde{\pi}_1, \dots, \tilde{\pi}_k)| + \sum_{i=1}^k |T'(\mathbb{1}_{\iota(X)} \tilde{f}', \tilde{\pi}'_1, \dots, \tilde{\pi}'_{i-1}, \tilde{\pi}_i - \tilde{\pi}'_i, \tilde{\pi}_{i+1}, \dots, \tilde{\pi}_k)| \\
&= |T'(\tilde{f} - \tilde{f}', \tilde{\pi}_1, \dots, \tilde{\pi}_k)| \leq \prod_{i=1}^k \text{Lip}(\tilde{\pi}_i) \int |\tilde{f} - \tilde{f}'| d\|T'\| = \prod_{i=1}^k \text{Lip}(\tilde{\pi}_i) \int_{\iota(X)} |\tilde{f} - \tilde{f}'| d\|T'\| = 0,
\end{aligned}$$

where we employed the locality property of metric currents; cf. [5, Theorem 3.5(iii)]. Therefore, the definition (2.11) of T is well-posed. Moreover, one can readily check that T is a current on X and that $\iota_{\#}T = T'$. Finally, the injectivity of $\iota_{\#}$ can be achieved by means of similar arguments.

In order to complete the proof it remains to check that, given $T \in \mathcal{M}_k(X)$, it holds that $T \in \mathcal{F}_X$ if and only if $\iota_{\#}T \in \mathcal{F}_Y$, where $(\mathcal{F}_X, \mathcal{F}_Y) \in \{(\mathcal{N}_k(X), \mathcal{N}_k(Y)), (\mathcal{R}_k(X), \mathcal{R}_k(Y)), (\mathcal{S}_k(X), \mathcal{S}_k(Y))\}$. The case of normal currents follows from the observation, already made in the paragraph preceding this lemma, that $\partial(\iota_{\#}T) = \iota_{\#}(\partial T)$. The case of rectifiable currents is a consequence of (2.10), as well as of the fact that countably \mathcal{H}^k -rectifiable sets and \mathcal{H}^k -negligible sets are preserved under isometries. Concerning integral currents, let us prove that if $\iota_{\#}T$ is integral, then T is integral; the converse implication is much easier to check, by means of similar arguments. Suppose $\iota_{\#}T$ is integral. The previous cases ensure that T is normal and rectifiable, thus it is only left to show that T is integer-rectifiable. Fix any $\varphi \in \text{LIP}(X; \mathbb{R}^k)$ and $\Omega \subseteq X$ open. By using (componentwise) McShane's Extension Theorem, we can find $\psi \in \text{LIP}(Y; \mathbb{R}^k)$ such that $\psi|_{\iota(X)} = \varphi \circ \iota^{-1}$. Since $\iota(\Omega)$ is a relatively open subset of $\iota(X)$, we can find an open set $\Omega' \subseteq Y$ such that $\iota(\Omega) = \Omega' \cap \iota(X)$. Now fix a metric differential form $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(\mathbb{R}^k)$. Notice that $\mathbb{1}_{\Omega'}|_{\iota(X)} = \mathbb{1}_{\Omega} \circ \iota^{-1}$, $f \circ \psi|_{\iota(X)} = f \circ \varphi \circ \iota^{-1}$, and $\pi_i \circ \psi|_{\iota(X)} = \pi_i \circ \varphi \circ \iota^{-1}$ for every $i = 1, \dots, k$. Hence, (2.11) yields

$$\begin{aligned} \varphi_{\#}(T \llcorner \Omega)(f, \pi_1, \dots, \pi_k) &= T(\mathbb{1}_{\Omega} f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_k \circ \varphi) = \iota_{\#}T(\mathbb{1}_{\Omega'} f \circ \psi, \pi_1 \circ \psi, \dots, \pi_k \circ \psi) \\ &= \psi_{\#}((\iota_{\#}T) \llcorner \Omega')(f, \pi_1, \dots, \pi_k). \end{aligned}$$

This means that $\varphi_{\#}(T \llcorner \Omega) = \psi_{\#}((\iota_{\#}T) \llcorner \Omega')$, whence it follows that T is integer-rectifiable. \square

Corollary 2.10. *Let (X, d_X) , (Y, d_Y) be complete metric spaces and let $\iota: X \hookrightarrow Y$ be an isometry. Fix $T \in \mathcal{S}_k(X)$ and suppose there exists a sequence $(T'_i)_{i \in \mathbb{N}} \subseteq \mathcal{S}_k(Y)$ such that $\iota_{\#}T = \sum_{i \in \mathbb{N}} T'_i$ and $\mathbf{N}(\iota_{\#}T) = \sum_{i \in \mathbb{N}} \mathbf{N}(T'_i)$. Then there exists a sequence $(T_i)_{i \in \mathbb{N}} \subseteq \mathcal{S}_k(X)$ such that*

$$T = \sum_{i \in \mathbb{N}} T_i, \quad \mathbf{N}(T) = \sum_{i \in \mathbb{N}} \mathbf{N}(T_i). \quad (2.12)$$

Proof. Given any $i \in \mathbb{N}$, denote by $T_i \in \mathcal{S}_k(X)$ the unique current such that $\iota_{\#}T_i = T'_i$, whose existence stems from Lemma 2.9. By arguing as we did in the proof of Lemma 2.9, we can easily show that $T = \sum_{i \in \mathbb{N}} T_i$. Indeed, given any $(f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X)$ and $(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k) \in \mathcal{D}^k(Y)$ such that $\tilde{f}|_{\iota(X)} = f \circ \iota^{-1}$ and $\tilde{\pi}_i|_{\iota(X)} = \pi_i \circ \iota^{-1}$ for every $i = 1, \dots, k$, it holds that

$$T(f, \pi_1, \dots, \pi_k) = \iota_{\#}T(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k) = \sum_{i \in \mathbb{N}} T'_i(\tilde{f}, \tilde{\pi}_1, \dots, \tilde{\pi}_k) = \sum_{i \in \mathbb{N}} T_i(f, \pi_1, \dots, \pi_k).$$

Moreover, we have that $\mathbf{N}(T) = \mathbf{N}(\iota_{\#}T) = \sum_{i \in \mathbb{N}} \mathbf{N}(T'_i) = \sum_{i \in \mathbb{N}} \mathbf{N}(T_i)$, whence (2.12) follows. \square

Remark 2.11. Under the same assumptions of Lemma 2.9, for any $T \in \mathcal{S}_k(X)$ it holds that

$$T \text{ is indecomposable} \iff \iota_{\#}T \text{ is indecomposable.}$$

The validity of this claim is an immediate consequence of Lemma 2.9. \blacksquare

3. DECOMPOSITION OF INTEGRAL CURRENTS

In this section we obtain our main Decomposition Theorem 3.2, which states that every integral k -current T in an arbitrary complete metric space can be written as an (at most countable) weak sum $\sum_i T_i$ of *indecomposable* integral k -currents T_i , having the property that $\mathbf{N}(T) = \sum_i \mathbf{N}(T_i)$. An alternative proof of this fact will be presented in Section 4. Before passing to Theorem 3.2, we briefly recall a classical result concerning the embeddability of metric spaces into Banach spaces.

Remark 3.1. Any given metric space (X, d) can be isometrically embedded into a Banach space. For instance, this can be achieved via the **Kuratowski embedding**, that we are going to recall.

Denote by $C_b(X)$ the space of all bounded, continuous, real-valued functions defined on X , which is a Banach space if endowed with the usual pointwise operations and the supremum norm

$$\|f\|_{C_b(X)} := \sup_X |f|, \quad \text{for every } f \in C_b(X).$$

Given any point $\bar{x} \in X$, we define the map $\iota: X \hookrightarrow C_b(X)$ as

$$\iota(x) := d(x, \cdot) - d(\bar{x}, \cdot), \quad \text{for every } x \in X.$$

Then ι is an isometry. Note that ι is highly non-canonical, as it depends on the chosen \bar{x} . \blacksquare

Theorem 3.2 (Decomposition of integral metric currents). *Let (X, d) be a complete metric space. Let $T \in \mathcal{S}_k(X)$ be given. Then there exists an at most countable family $\{T_i\}_{i \in I} \subseteq \mathcal{S}_k(X)$ of indecomposable integral k -currents such that*

$$T = \sum_{i \in I} T_i, \quad \mathbf{N}(T) = \sum_{i \in I} \mathbf{N}(T_i).$$

Proof. By taking Remark 3.1, Corollary 2.10, and Remark 2.11 into account, we can assume without loss of generality that X is a Banach space. Fix any constant $\alpha \in (1, \frac{k+1}{k})$ and let us denote $\theta := (k+1)/(\alpha k) - 1 > 0$. We also define the family \mathcal{P} as

$$\mathcal{P} := \left\{ \{T_i\}_{i \in \mathbb{N}} \subseteq \mathcal{S}_k(X) \mid T = \sum_{i \in \mathbb{N}} T_i, \mathbf{N}(T) = \sum_{i \in \mathbb{N}} \mathbf{N}(T_i) \right\}.$$

Notice that \mathcal{P} is non-empty, as it contains $\{T, 0, 0, \dots\}$. To achieve the claim, we aim to show that

$$M := \sup \left\{ \sum_{i \in \mathbb{N}} F(T_i)^{1/\alpha} \mid \{T_i\}_{i \in \mathbb{N}} \in \mathcal{P} \right\} \quad (3.1)$$

is finite and attained.

STEP 1: Compactness argument and lower semicontinuity. Choose any $(\{T_i^n\}_{i \in \mathbb{N}})_{n \in \mathbb{N}} \subseteq \mathcal{P}$ such that $\lim_n \sum_{i \in \mathbb{N}} F(T_i^n)^{1/\alpha} = M$. Without loss of generality, we can suppose that for any $n \in \mathbb{N}$ the sequence $(\mathbf{N}(T_i^n))_{i \in \mathbb{N}}$ is non-increasing. Observe that the very definition of \mathcal{P} gives

$$\mathbf{N}(T_i^n) \leq \sum_{j \in \mathbb{N}} \mathbf{N}(T_j^n) \leq \mathbf{N}(T), \quad \text{for every } i, n \in \mathbb{N}. \quad (3.2)$$

The measures $\|T\|$ and $\|\partial T\|$ are tight (recall that the cardinality of X is an Ulam number by assumption). Given that $\|T_i^n\| \leq \|T\|$ and $\|\partial T_i^n\| \leq \|\partial T\|$ for every $i, n \in \mathbb{N}$ by Proposition 2.4, we deduce that for any $i \in \mathbb{N}$ the sequences $(\|T_i^n\|)_n$ and $(\|\partial T_i^n\|)_n$ are tight, thus the compactness properties of $\mathcal{A}_k(X)$ and the closure of $\mathcal{S}_k(X)$ (cf. [5, Theorem 5.2] and [5, Theorem 8.5]) grant the existence of a sequence $(T_i)_{i \in \mathbb{N}} \subseteq \mathcal{S}_k(X)$ such that (up to a not relabelled subsequence in n) it holds that $T_i^n \rightarrow T_i$ as $n \rightarrow \infty$ for every $i \in \mathbb{N}$. By using (2.2), we deduce that $\partial T_i^n \rightarrow \partial T_i$ as $n \rightarrow \infty$, thus the lower semicontinuity of \mathbf{M} yields

$$\mathbf{N}(T_i) = \mathbf{M}(T_i) + \mathbf{M}(\partial T_i) \leq \liminf_{n \rightarrow \infty} \mathbf{M}(T_i^n) + \liminf_{n \rightarrow \infty} \mathbf{M}(\partial T_i^n) \leq \liminf_{n \rightarrow \infty} \mathbf{N}(T_i^n), \quad \text{for every } i \in \mathbb{N}.$$

In particular, by using Fatou's lemma we obtain that

$$\sum_{i \in \mathbb{N}} \mathbf{N}(T_i) \leq \sum_{i \in \mathbb{N}} \liminf_{n \rightarrow \infty} \mathbf{N}(T_i^n) \leq \liminf_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \mathbf{N}(T_i^n) \stackrel{(3.2)}{\leq} \mathbf{N}(T). \quad (3.3)$$

Moreover, by applying Theorem 2.5 we deduce that $\lim_n F(T_i^n) = F(T_i)$ for every $i \in \mathbb{N}$.

STEP 2: Uniform estimates of tails. We claim that

$$\lim_{j \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \sum_{i \geq j} F(T_i^n) = \lim_{j \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \sum_{i \geq j} F(T_i^n)^{1/\alpha} = 0. \quad (3.4)$$

Given that the sequence $i \mapsto \mathbf{N}(T_i^n)$ is non-increasing for any $n \in \mathbb{N}$, one has that

$$j \mathbf{N}(T_j^n) \leq \sum_{i \leq j} \mathbf{N}(T_i^n) \stackrel{(3.2)}{\leq} \mathbf{N}(T), \quad \text{for every } j, n \in \mathbb{N}. \quad (3.5)$$

Thanks to (3.5), we can choose $j_0 \in \mathbb{N}$ such that $D_k \mathbf{N}(T_j^n)^{k+1/k} \leq 1$ for every $n \in \mathbb{N}$ and $j \geq j_0$, with D_k as in Corollary 2.8. Then for any $n \in \mathbb{N}$ and $j \geq j_0$ it holds $\mathbf{F}(T_j^n) \leq 1$ by (2.9), whence

$$\begin{aligned} \sum_{i \geq j} \mathbf{F}(T_i^n) &\leq \sum_{i \geq j} \mathbf{F}(T_i^n)^{1/\alpha} \stackrel{(2.9)}{\leq} D_k^{1/\alpha} \sum_{i \geq j} \mathbf{N}(T_i^n)^{\theta+1} \stackrel{(3.5)}{\leq} \frac{D_k^{1/\alpha} \mathbf{N}(T)^\theta}{j^\theta} \sum_{i \geq j} \mathbf{N}(T_i^n) \\ &\stackrel{(3.2)}{\leq} \frac{D_k^{1/\alpha} \mathbf{N}(T)^{\theta+1}}{j^\theta} =: \lambda_j. \end{aligned}$$

Consequently, we have that $\overline{\lim}_n \sum_{i \geq j} \mathbf{F}(T_i^n) \leq \overline{\lim}_n \sum_{i \geq j} \mathbf{F}(T_i^n)^{1/\alpha} \leq \lambda_j$. Since $\lim_j \lambda_j = 0$, we conclude that (3.4) is verified. As a consequence, it holds that $\sum_{i \geq j} \mathbf{F}(T_i) \leq \lambda_j$ for all $j \geq j_0$ and

$$\sum_{i \in \mathbb{N}} \mathbf{F}(T_i)^{1/\alpha} = \lim_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \mathbf{F}(T_i^n)^{1/\alpha} = M < \infty. \quad (3.6)$$

STEP 3: $\{T_i\}_{i \in I}$ belongs to \mathcal{P} . We aim to prove that $T = \sum_{i \in \mathbb{N}} T_i$. Fix any $j \geq j_0$ and $n \in \mathbb{N}$. Notice that

$$\mathbf{F}\left(T - \sum_{i \in \mathbb{N}} T_i\right) = \mathbf{F}\left(\sum_{i \in \mathbb{N}} T_i^n - T_i\right) \leq \sum_{i < j} \mathbf{F}(T_i^n - T_i) + \sum_{i \geq j} \mathbf{F}(T_i^n) + \sum_{i \geq j} \mathbf{F}(T_i)$$

for every $n \in \mathbb{N}$, thus accordingly

$$\mathbf{F}\left(T - \sum_{i \in \mathbb{N}} T_i\right) \leq \lambda_j + \overline{\lim}_{n \rightarrow \infty} \sum_{i < j} \mathbf{F}(T_i^n - T_i) + \overline{\lim}_{n \rightarrow \infty} \sum_{i \geq j} \mathbf{F}(T_i^n) \leq 2\lambda_j + \sum_{i < j} \overline{\lim}_{n \rightarrow \infty} \mathbf{F}(T_i^n - T_i) = 2\lambda_j.$$

By letting $j \rightarrow \infty$ we conclude that $\mathbf{F}(T - \sum_{i \in \mathbb{N}} T_i) = 0$, which means that $T = \sum_{i \in \mathbb{N}} T_i$. Observe that $\mathbf{N}(T) = \sum_{i \in \mathbb{N}} \mathbf{N}(T_i)$, as one inequality is granted by (2.3), while the converse one by (3.3). Finally, we have that $\sum_{i \in \mathbb{N}} \mathbf{F}(T_i)^{1/\alpha} = M$ as a consequence of (3.6).

STEP 4: Indecomposability of T_i . We argue by contradiction: suppose that the current T_j is decomposable for some index $j \in \mathbb{N}$. Then we can find two non-zero currents $R, S \in \mathcal{I}_k(X)$ such that $T_j = R + S$ and $\mathbf{N}(T_j) = \mathbf{N}(R) + \mathbf{N}(S)$, which guarantees that $\{T_i\}_{i \in \mathbb{N} \setminus \{j\}} \cup \{R, S\} \in \mathcal{P}$. On the other hand, since \mathbf{F} is subadditive and $(0, \infty) \ni t \mapsto t^{1/\alpha}$ is strictly concave, we conclude that

$$\mathbf{F}(R)^{1/\alpha} + \mathbf{F}(S)^{1/\alpha} + \sum_{i \in \mathbb{N} \setminus \{j\}} \mathbf{F}(T_i)^{1/\alpha} > \mathbf{F}(R+S)^{1/\alpha} + \sum_{i \in \mathbb{N} \setminus \{j\}} \mathbf{F}(T_i)^{1/\alpha} = \sum_{i \in \mathbb{N}} \mathbf{F}(T_i)^{1/\alpha} = M,$$

which contradicts the maximality of M . Therefore, the statement is eventually achieved. \square

Remark 3.3. Some comments are in order:

- (i) The decomposition given by Theorem 3.2 is in general not unique, as can be seen considering the currents associated with two perpendicular long segments intersecting at their midpoints. Moreover, there is no maximality property for the components similar to [3, Theorem 1].
- (ii) Contrarily to the case of finite perimeter sets considered in [3], corresponding essentially to d -dimensional currents in \mathbb{R}^d (for which the mass norm and the flat norm coincide), it is not possible to replace the flat norm in (3.1) with the mass norm. As an example, we can consider a countable sum of loops of lengths ℓ_i , where $\sum_i \ell_i < \infty$ but $\sum_i \ell_i^\beta = \infty$ for every $\beta < 1$. \blacksquare

4. AN ALTERNATIVE PROOF OF THE DECOMPOSITION

We present a second proof of Theorem 3.2, which is based on the approach suggested by Federer in [9, 4.2.25]: if T is decomposable then we keep splitting it in two pieces, until we can not split them anymore. To make sure that such a process ends we actually ensure that at every step we remove an indecomposable component with a significant mass, which is the content of the following lemma.

Lemma 4.1 (Existence of a big indecomposable component). *Let (X, d) be a complete metric space. Let $T \in \mathcal{S}_k(X)$ be given. Then there exists $T_1 \in \mathcal{S}_k(X)$ such that $\mathbf{N}(T) = \mathbf{N}(T_1) + \mathbf{N}(T - T_1)$, T_1 is indecomposable, and $\mathbf{N}(T_1) \geq D_k^{-k} \left(\frac{\mathbf{F}(T)}{\mathbf{N}(T)} \right)^k$.*

Proof. As in the Proof of Theorem 3.2, we assume without loss of generality that X is a Banach space.

STEP 1: Lower bound on the biggest component. Let $m := \inf \{ \mathbf{N}(T_1) : (T_n)_n \in \mathcal{G}(T) \}$, where $\mathcal{G}(T)$ denotes the family of good partitions of T , that is those finite or countable sequences $(T_n)_{n=1}^N$ (possibly with $N = \infty$) such that

$$T = \sum_n T_n, \quad \mathbf{N}(T) = \sum_n \mathbf{N}(T_n),$$

and such that $\mathbf{N}(T_n)$ is non-increasing in n . We claim that $m \geq D_k^{-k} \left(\frac{\mathbf{F}(T)}{\mathbf{N}(T)} \right)^k$. Indeed by Corollary 2.8, for every good partition $(T_n)_n$ we have

$$\mathbf{F}(T) \leq \sum_n \mathbf{F}(T_n) \leq \sum_n D_k \mathbf{N}(T_n)^{\frac{k+1}{k}} \leq D_k \mathbf{N}(T_1)^{\frac{1}{k}} \sum_n \mathbf{N}(T_n) = D_k \mathbf{N}(T_1)^{\frac{1}{k}} \mathbf{N}(T). \quad (4.1)$$

STEP 2: Grouping together small components. Let $\mathcal{F}(T)$ be the family of those good partitions $(T_n)_{n=1}^N$ such that $\mathbf{N}(T_n) \geq \frac{m}{2}$ for every $n = 1, \dots, N$, except for at most one index. Then $\mathcal{F}(T)$ is nonempty since (T) belongs to $\mathcal{F}(T)$, and

$$(N-1) \frac{m}{2} \leq \sum_{n=1}^N \mathbf{N}(T_n) = \mathbf{N}(T). \quad (4.2)$$

Therefore the number of components of a good partition in $\mathcal{F}(T)$ is equibounded once we fix T . Moreover, from every good partition in $\mathcal{G}(T)$ we can obtain a good partition in $\mathcal{F}(T)$ just grouping different currents together if their norm is below $\frac{m}{2}$, and we can make it so that the biggest norm among all the elements remains the same. Indeed, given $(T_n)_n \in \mathcal{G}(T)$, let $a_n := \mathbf{N}(T_n)$, so that $(a_n)_n$ is non-increasing. By assumption $\sum_n a_n < \infty$. From elementary considerations we can find indices $1 \leq n_1 < \dots < n_p \leq \infty$ such that $a_n \geq m$ for every $1 \leq n \leq n_1$ and

$$\begin{aligned} \frac{m}{2} &\leq \sum_{n=n_j+1}^{n_{j+1}} a_n \leq m, \quad \text{for every } j = 1, \dots, p-1, \\ \sum_{n > n_p} a_n &\leq \frac{m}{2}. \end{aligned}$$

Accordingly, we define

$$\begin{aligned} \tilde{T}_j &:= \sum_{n=n_j+1}^{n_{j+1}} T_n, \quad \text{for every } j = 1, \dots, p-1, \\ \tilde{T}_p &:= \sum_{n > n_p} T_n. \end{aligned}$$

Then $(T_n)_{n=1}^{n_1} \cup (\tilde{T}_j)_{j=1}^p$ defines (up to reordering) a good partition in $\mathcal{F}(T)$. In particular,

$$\inf \{ \mathbf{N}(T_1) : (T_n) \in \mathcal{F}(T) \} \quad (4.3)$$

also coincides with m .

STEP 3: Existence of a minimum for (4.3). Let us consider a minimising sequence for (4.3), indexed by $j \in \mathbb{N}$, of good partitions in $\mathcal{F}(T)$, denoted $(T_n^j)_{n=1}^N$. We can assume that, without loss of generality, the cardinality of each partition is the same number N . This is possible, up to extracting a subsequence, thanks to (4.2). By compactness of integral currents (cf. [5, Theorem 5.2] and [5, Theorem 8.5]; notice that every element of the partition is a subcurrent of T , and thus the equitightness of the mass measures is satisfied) we obtain a subsequence (not relabelled) such that $T_n^j \rightarrow T_n^\infty$ as $j \rightarrow \infty$. We infer that $T = \sum_{n=1}^N T_n^\infty$, hence

$$\mathbf{N}(T) \leq \sum_{n=1}^N \mathbf{N}(T_n^\infty) \leq \sum_{n=1}^N \liminf_{j \rightarrow \infty} \mathbf{N}(T_n^j) \leq \liminf_{j \rightarrow \infty} \sum_{n=1}^N \mathbf{N}(T_n^j) = \mathbf{N}(T).$$

Therefore all inequalities are equalities, and in particular $(T_n^\infty)_{n=1}^N$ is a good partition of T . By lower semicontinuity the infimum in (4.3) is thus achieved.

STEP 4: Finding a big indecomposable component. We claim that at least one among the currents T_n^∞ with normal mass m is indecomposable. If not, we would obtain a good partition having a strictly lower energy than the infimum in (4.3), which is impossible. This concludes the proof. \square

Alternative proof of Theorem 3.2. We set $S_0 := T$, and inductively define T_n and S_n , $n \geq 0$, as follows: we take T_n as an indecomposable component of S_n with maximal norm, and then we set $S_{n+1} = S_n - T_n$. In particular by Lemma 4.1

$$\mathbf{N}(T_n) \geq D_k^{-k} \left(\frac{\mathbf{F}(S_n)}{\mathbf{N}(S_n)} \right)^k \geq D_k^{-k} \left(\frac{\mathbf{F}(S_n)}{\mathbf{N}(T)} \right)^k. \quad (4.4)$$

Either S_n is indecomposable for some $n \in \mathbb{N}$, and we stop, in which case we immediately obtain the desired decomposition; or we keep going for every $n \in \mathbb{N}$. In the latter case, we observe that $\mathbf{F}(S_n) \rightarrow 0$ as $n \rightarrow \infty$. This follows directly from (4.4), because $\sum_n \mathbf{N}(T_n) \leq \mathbf{N}(T) < \infty$, and therefore $\mathbf{N}(T_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus $T = \sum_{n=0}^{\infty} T_n$ and for every $p \in \mathbb{N}$

$$\mathbf{N}(T) \geq \mathbf{N}(S_p) + \sum_{n=0}^p \mathbf{N}(T_n) \geq \sum_{n=0}^p \mathbf{N}(T_n).$$

Passing to the limit as $p \rightarrow \infty$ we obtain additivity of the normal mass, and thus $T = \sum_n T_n$ is the desired decomposition. \square

Remark 4.2. As in the first proof of Theorem 3.2, the components obtained above are in general not uniquely determined, even up to reordering. \blacksquare

5. CHARACTERISATION OF INDECOMPOSABLE INTEGRAL 1-CURRENTS

In this section, we focus our attention on integral metric 1-currents. With the aid of the Decomposition Theorem 3.2, we can provide a full characterisation of the indecomposable integral 1-currents in an arbitrary complete metric space. More specifically, we show that they are exactly those currents induced by a Lipschitz curve which is either injective or an injective loop.

Lemma 5.1 (Indecomposable and non-cancelling implies injective). *Let (X, d) be a complete metric space and $\gamma : [0, 1] \rightarrow X$ a Lipschitz curve. Suppose that $[\gamma]$ is indecomposable and that $\mathbf{M}([\gamma]) = \ell(\gamma)$. Then γ is either injective or an injective loop.*

Proof. Without loss of generality, we assume γ is a constant-speed Lipschitz curve. Let us consider separately two cases.

- (i) Suppose $\gamma(s) = \gamma(t)$ for some $s, t \in (0, 1)$, with $s < t$. Consider the currents $R := \llbracket \gamma|_{[0,s]} \rrbracket + \llbracket \gamma|_{[t,1]} \rrbracket$ and $S := \llbracket \gamma|_{[s,t]} \rrbracket$. Notice that R, S are both non-trivial, since $0 < s < t < 1$ and γ is parametrized by constant-speed. Since $\llbracket \gamma \rrbracket = R + S$ and $\mathbf{N}(\llbracket \gamma \rrbracket) = \mathbf{N}(R) + \mathbf{N}(S)$, we deduce that $\llbracket \gamma \rrbracket$ is decomposable.
- (ii) Suppose there exists $t \in (0, 1)$ such that $\gamma(t) \in \{\gamma(0), \gamma(1)\}$. Similarly to the previous case, consider the non-null currents $R := \llbracket \gamma|_{[0,t]} \rrbracket$ and $S := \llbracket \gamma|_{[t,1]} \rrbracket$. Since $\llbracket \gamma \rrbracket = R + S$ and $\mathbf{N}(\llbracket \gamma \rrbracket) = \mathbf{N}(R) + \mathbf{N}(S)$, we deduce that $\llbracket \gamma \rrbracket$ is decomposable. \square

Before passing to the main decomposition result of this section (Theorem 5.3), let us recall its suboptimal version obtained by Ambrosio–Wenger [6] (see also [17] for the boundaryless case).

Lemma 5.2 (Almost optimal representation of integral 1-currents [6]). *Let (X, d) be a complete, length metric space. Fix any $T \in \mathcal{S}_1(X)$ and $\varepsilon > 0$. Then there exist finitely many 1-Lipschitz curves $\gamma_i: [0, a_i] \rightarrow X$, $i = 1, \dots, n$, such that $\partial T = \sum_{i=1}^n \partial \llbracket \gamma_i \rrbracket$, $\mathbf{M}(\partial T) = \sum_{i=1}^n \mathbf{M}(\partial \llbracket \gamma_i \rrbracket)$, and*

$$\mathbf{M}(T - \sum_{i=1}^n \llbracket \gamma_i \rrbracket) \leq \varepsilon \mathbf{M}(T), \quad \sum_{i=1}^n a_i \leq (1 + \varepsilon) \mathbf{M}(T).$$

By combining Theorem 3.2 and Lemma 5.2 with a compactness argument (based upon the Arzelà–Ascoli-type result stated in Lemma 2.6), we can obtain the following optimal representation theorem for integral metric 1-currents.

Theorem 5.3 (Optimal representation of integral 1-currents). *Let (X, d) be a complete metric space. Fix any $T \in \mathcal{S}_1(X)$. Then there exists a sequence $(\gamma_i)_i$ of injective Lipschitz curves or injective Lipschitz loops in X such that*

$$T = \sum_{i \in \mathbb{N}} \llbracket \gamma_i \rrbracket, \quad \mathbf{N}(T) = \sum_{i \in \mathbb{N}} \mathbf{N}(\llbracket \gamma_i \rrbracket).$$

Proof. By taking Remark 3.1 and Corollary 2.10 into account, we can assume without loss of generality that X is a Banach space. We subdivide the proof into several steps:

STEP 1: Monotone rearrangement and bounds on the length of curves. Fix any $(\varepsilon_j)_{j \in \mathbb{N}} \subseteq (0, 1)$ such that $\varepsilon_j \searrow 0$. For any $j \in \mathbb{N}$, by using Lemma 5.2 we can find indices $m_j, n_j \in \mathbb{N}$ with $m_j \leq n_j$ and constant-speed Lipschitz curves $\gamma_i^j: [0, 1] \rightarrow X$, $i = 1, \dots, n_j$, such that $\gamma_i^j(0) \neq \gamma_i^j(1)$ for every $i = 1, \dots, m_j$, $\gamma_i^j(0) = \gamma_i^j(1)$ for every $i = m_j + 1, \dots, n_j$, $\partial T = \sum_{i=1}^{m_j} \partial \llbracket \gamma_i^j \rrbracket$, $\mathbf{M}(\partial T) = \sum_{i=1}^{m_j} \mathbf{M}(\partial \llbracket \gamma_i^j \rrbracket)$, and

$$\mathbf{M}(T - \sum_{i=1}^{n_j} \llbracket \gamma_i^j \rrbracket) \leq \varepsilon_j, \quad \sum_{i=1}^{n_j} \ell(\gamma_i^j) \leq \mathbf{M}(T) + \varepsilon_j. \quad (5.1)$$

Since $\mathbf{M}(\partial \llbracket \gamma_i^j \rrbracket) = 2$ for every $i = 1, \dots, m_j$, we deduce that $m_j \leq \mathbf{M}(\partial T)/2$, thus (up to taking a not relabelled subsequence in j) we may assume that $m := m_1 = m_j$ for every $j \in \mathbb{N}$. Moreover, up to relabelling the curves $(\gamma_i^j)_{i=m+1}^{n_j}$, we may assume that $(\ell(\gamma_i^j))_{i=m+1}^{n_j}$ is non-increasing. Fix any point $\bar{x} \in \text{spt}(T)$ and set $\gamma_i^j(t) := \bar{x}$ for every $j \in \mathbb{N}$, $i > n_j$, and $t \in [0, 1]$. Given that (5.1) yields $\ell(\gamma_i^j) \leq \mathbf{M}(T) + 1$ for every $i, j \in \mathbb{N}$, up to taking a not relabelled subsequence in j we may assume that there exists $(\lambda_i)_{i \in \mathbb{N}} \subseteq [0, \mathbf{M}(T) + 1]$ such that $\lim_j \ell(\gamma_i^j) = \lambda_i$ for every $i \in \mathbb{N}$. Observe that $\lambda_i > 0$ for all $i \leq m$ and $(\lambda_i)_{i > m}$ is non-increasing, thus there exists a unique $i_0 \in \mathbb{N} \cup \{\infty\}$ with $i_0 > m$ such that $\lambda_i > 0$ for all $i < i_0$ and $\lambda_i = 0$ for all $i \geq i_0$.

STEP 2: Compactness of curves with non-infinitesimal length. Given $i < i_0$, we aim to apply Lemma 2.6 to the sequence $(\gamma_i^j)_j$. For any $\varepsilon > 0$, there exist $j_0 \in \mathbb{N}$ and a compact set $K \subseteq X$ containing \bar{x} such that $\varepsilon_j \leq \varepsilon$, $\ell(\gamma_i^j) \geq \lambda_i/2$ for all $j \geq j_0$, and $\|T\|(X \setminus K) \leq \varepsilon$. Using the first

equation in (5.1) and restricting to the set K , we obtain

$$\mathbf{M}(T \llcorner K - \sum_{i'=1}^{\infty} \llbracket \gamma_{i'}^j \rrbracket \llcorner K) \leq \varepsilon_j.$$

From the choice of the set K we also have that $\mathbf{M}(T \llcorner K) \geq \mathbf{M}(T) - \varepsilon$. From this, and by triangle inequality, we obtain

$$\mathbf{M}(T) - \varepsilon - \varepsilon_j \leq \mathbf{M}(T \llcorner K) - \varepsilon_j \leq \mathbf{M}(\sum_{i'=1}^{\infty} \llbracket \gamma_{i'}^j \rrbracket \llcorner K) \leq \sum_{i'=1}^{\infty} \mathbf{M}(\llbracket \gamma_{i'}^j \rrbracket \llcorner K). \quad (5.2)$$

Let us define, for every curve $\gamma_{i'}^j$, the set of bad points $B_{i'}^j := \{t \in [0, 1] : \gamma_{i'}^j(t) \notin K\}$. Then $\llbracket \gamma_{i'}^j \rrbracket \llcorner K = (\gamma_{i'}^j)_{\#} \llbracket (B_{i'}^j)^c, e_1, 1 \rrbracket$ and as a consequence

$$\mathbf{M}(\llbracket \gamma_{i'}^j \rrbracket \llcorner K) \leq \text{Lip}(\gamma_{i'}^j)(1 - \mathcal{L}^1(B_{i'}^j)) = \ell(\gamma_{i'}^j)(1 - \mathcal{L}^1(B_{i'}^j)).$$

Putting this together with (5.2) and the second inequality in (5.1), we obtain

$$\mathbf{M}(T) - 2\varepsilon \leq \sum_{i'=1}^{\infty} \mathbf{M}(\llbracket \gamma_{i'}^j \rrbracket \llcorner K) \leq \sum_{i'=1}^{\infty} \ell(\gamma_{i'}^j)(1 - \mathcal{L}^1(B_{i'}^j)) \leq \mathbf{M}(T) + \varepsilon - \sum_{i'=1}^{\infty} \ell(\gamma_{i'}^j) \mathcal{L}^1(B_{i'}^j)$$

for every $j \geq j_0$, which implies in particular that for every $i < i_0$ and every $j \geq j_0$

$$\frac{\lambda_i}{2} \mathcal{L}^1(B_i^j) \leq \ell(\gamma_i^j) \mathcal{L}^1(B_i^j) \leq \sum_{i'=1}^{\infty} \ell(\gamma_{i'}^j) \mathcal{L}^1(B_{i'}^j) \leq 3\varepsilon,$$

whence accordingly

$$\mathcal{L}^1(\{t \in [0, 1] : \gamma_i^j(t) \notin K\}) \leq \frac{6\varepsilon}{\lambda_i}, \quad \text{for every } j \geq j_0.$$

Also, we have that γ_i^j is $\ell(\gamma_i^j)$ -Lipschitz, thus in particular $(\mathbf{M}(T) + 1)$ -Lipschitz, for every $j \in \mathbb{N}$. Therefore, an application of Lemma 2.6 yields the existence of a family $(\gamma_i)_{i < i_0}$ of Lipschitz curves $\gamma_i : [0, 1] \rightarrow X$ with the property that, up to taking a further subsequence in j , it holds $\gamma_i^j \rightrightarrows \gamma_i$ uniformly as $j \rightarrow \infty$ for every $i < i_0$. Thanks to (2.6), we infer that

$$\llbracket \gamma_i^j \rrbracket \rightharpoonup \llbracket \gamma_i \rrbracket \quad \text{as } j \rightarrow \infty, \quad \text{for every } i \in \mathbb{N} \text{ such that } i < i_0. \quad (5.3)$$

STEP 3: Convergence of the non-infinitesimal part. In the case $i_0 \in \mathbb{N}$, it is an immediate consequence of (5.3) that the finite sums $\sum_{i < i_0} \llbracket \gamma_i^j \rrbracket$ converge (weakly in the sense of currents) to $\sum_{i < i_0} \llbracket \gamma_i \rrbracket$. We now aim to prove that the same conclusion remains valid when $i_0 = +\infty$: in this case, note first that

$$(i - m) \cdot \ell(\gamma_i^j) \leq \sum_{i'=m+1}^i \ell(\gamma_{i'}^j) \leq \mathbf{M}(T) + 1$$

for every $i, j \in \mathbb{N}$ with $m < i$. Furthermore, since the length functional ℓ is lower semicontinuous with respect to the uniform convergence, we see that $(i - m) \cdot \ell(\gamma_i) \leq (i - m) \underline{\lim}_j \ell(\gamma_i^j) \leq \mathbf{M}(T) + 1$ for every $i \in \mathbb{N}$ with $m < i$. Notice also that the flat norm F is continuous along weakly converging sequences in $\mathcal{S}_1(X)$ having bounded \mathbf{N} -norm (by Theorem 2.5). Consequently, we may estimate

$$\begin{aligned} F(\sum_{i < i'} \llbracket \gamma_{i'}^j \rrbracket - \llbracket \gamma_{i'} \rrbracket) &\leq \sum_{i < i'} F(\llbracket \gamma_{i'}^j \rrbracket - \llbracket \gamma_{i'} \rrbracket) \leq \sum_{i < i'} F(\llbracket \gamma_{i'}^j \rrbracket) + \sum_{i < i'} F(\llbracket \gamma_{i'} \rrbracket) \\ &\stackrel{(2.9)}{\leq} D_1 \sum_{i < i'} \mathbf{N}(\llbracket \gamma_{i'}^j \rrbracket)^2 + D_1 \sum_{i < i'} \mathbf{N}(\llbracket \gamma_{i'} \rrbracket)^2 \\ &\leq D_1 \sum_{i < i'} \ell(\gamma_{i'}^j)^2 + D_1 \sum_{i < i'} \ell(\gamma_{i'})^2 \\ &\leq D_1 \frac{\mathbf{M}(T) + 1}{i - m} \sum_{i < i'} (\ell(\gamma_{i'}^j) + \ell(\gamma_{i'})) \leq 2D_1 \frac{(\mathbf{M}(T) + 1)^2}{i - m}, \end{aligned}$$

for every $i, j \in \mathbb{N}$ with $m < i$. This guarantees that, given any $\varepsilon > 0$, there exists $i \in \mathbb{N}$ sufficiently large such that $m < i$ and $F(\sum_{i < i'} \llbracket \gamma_{i'}^j \rrbracket - \llbracket \gamma_{i'} \rrbracket) \leq \varepsilon$ for every $j \in \mathbb{N}$. As a consequence of (5.3)

and Theorem 2.5, there exists $j_0 \in \mathbb{N}$ such that $F(\llbracket \gamma_{i'}^j \rrbracket - \llbracket \gamma_{i'} \rrbracket) \leq \varepsilon/i$ for every $j \geq j_0$ and $i' \leq i$. Hence, we have $F(\sum_{i' < i_0} \llbracket \gamma_{i'}^j \rrbracket - \llbracket \gamma_{i'} \rrbracket) \leq 2\varepsilon$ for every $j \geq j_0$, thus using again Theorem 2.5 we get

$$\sum_i \llbracket \gamma_i^j \rrbracket \rightarrow \sum_i \llbracket \gamma_i \rrbracket, \quad \text{as } j \rightarrow \infty. \quad (5.4)$$

STEP 4: The infinitesimal part converges to zero. Since $\lambda_i = 0$ for all $i \geq i_0$ and $(\ell(\gamma_i^j))_{i \geq i_0}$ is non-increasing for all $j \in \mathbb{N}$, for any $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that $\ell(\gamma_i^j) \leq \varepsilon$ for every $i \geq i_0$ and $j \geq j_0$. Hence, Corollary 2.8 gives

$$\begin{aligned} F(\sum_{i \geq i_0} \llbracket \gamma_i^j \rrbracket) &\leq \sum_{i \geq i_0} F(\llbracket \gamma_i^j \rrbracket) \leq D_1 \sum_{i \geq i_0} \mathbf{N}(\llbracket \gamma_i^j \rrbracket)^2 = D_1 \sum_{i \geq i_0} \ell(\gamma_i^j)^2 \leq \varepsilon D_1 \sum_{i \geq i_0} \ell(\gamma_i^j) \\ &\leq \varepsilon D_1 (\mathbf{M}(T) + 1), \end{aligned}$$

for every $j \geq j_0$. This implies that $\lim_j F(\sum_{i \geq i_0} \llbracket \gamma_i^j \rrbracket) = 0$, thus accordingly one has $\sum_{i \geq i_0} \llbracket \gamma_i^j \rrbracket \rightarrow 0$ as $j \rightarrow \infty$ by Theorem 2.5.

STEP 5: Conclusion (decomposition). By recalling (5.4), we obtain that $\sum_{i=1}^{n_j} \llbracket \gamma_i^j \rrbracket = \sum_{i \in \mathbb{N}} \llbracket \gamma_i^j \rrbracket \rightarrow \sum_{i < i_0} \llbracket \gamma_i \rrbracket$ as $j \rightarrow \infty$. The first identity in (5.1) yields $\mathbf{M}(T - \sum_{i < i_0} \llbracket \gamma_i \rrbracket) \leq \lim_j \mathbf{M}(T - \sum_{i=1}^{n_j} \llbracket \gamma_i^j \rrbracket) = 0$, so that $T = \sum_{i < i_0} \llbracket \gamma_i \rrbracket$ and in particular $\mathbf{M}(T) \leq \sum_{i < i_0} \mathbf{M}(\llbracket \gamma_i \rrbracket)$. On the other hand, by exploiting the inequality $\mathbf{M}(\llbracket \gamma_i \rrbracket) \leq \ell(\gamma_i)$ for every $i \in \mathbb{N}$, the second identity in (5.1), the uniform convergence $\gamma_i^j \rightrightarrows \gamma_i$, and Fatou's lemma, we get

$$\sum_{i < i_0} \mathbf{M}(\llbracket \gamma_i \rrbracket) \leq \sum_{i < i_0} \ell(\gamma_i) \leq \sum_{i < i_0} \lim_{j \rightarrow \infty} \ell(\gamma_i^j) \leq \lim_{j \rightarrow \infty} \sum_{i < i_0} \ell(\gamma_i^j) \leq \mathbf{M}(T).$$

All in all, we have shown that

$$\mathbf{M}(T) = \sum_{i < i_0} \mathbf{M}(\llbracket \gamma_i \rrbracket) = \sum_{i < i_0} \ell(\gamma_i). \quad (5.5)$$

Finally, up to passing to a further subsequence in j , we can additionally assume that $x_i := \gamma_i^1(0) = \gamma_i^j(0)$ and $y_i := \gamma_i^1(1) = \gamma_i^j(1)$ for every $j \in \mathbb{N}$ and $i = 1, \dots, m$, which implies that $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$ for every $i = 1, \dots, m$. In particular, one has $\mathbf{M}(T) = \sum_{i=1}^m \mathbf{M}(\partial \llbracket \gamma_i \rrbracket) = \sum_{i < i_0} \mathbf{M}(\partial \llbracket \gamma_i \rrbracket)$ and thus $\mathbf{N}(T) = \sum_{i < i_0} \mathbf{N}(\llbracket \gamma_i \rrbracket)$.

STEP 6: Conclusion (injectivity). It remains to show that the Lipschitz curves $(\gamma_i)_i$ can be taken to be injective or injective loops. Observe that it is not restrictive at this point to assume that $\llbracket \gamma_i \rrbracket$ is indecomposable for every $i \in I$: indeed, applying Theorem 3.2 to each current $\llbracket \gamma_i \rrbracket$ (given by STEPS 1-5 above) we find a family of indecomposable currents $(S_i^j)_{j \in J} \subseteq \mathcal{A}_1(X)$ such that

$$\llbracket \gamma_i \rrbracket = \sum_{j \in J} S_i^j, \quad \mathbf{N}(\llbracket \gamma_i \rrbracket) = \sum_{j \in J} \mathbf{N}(S_i^j).$$

Applying now STEPS 1-5 above to the family $(S_i^j)_{j \in J}$ and exploiting their indecomposability, we conclude that we can assume $\llbracket \gamma_i \rrbracket$ indecomposable for every i . On the other hand, from (5.5) it follows that for every $i \in \mathbb{N}$ the curve γ_i satisfies the non-cancelling property $\mathbf{M}(\llbracket \gamma_i \rrbracket) = \ell(\gamma_i)$. A straightforward application of Lemma 5.1 yields the desired conclusion. \square

Lemma 5.4 (Injective implies indecomposable). *Let (X, d) be a complete metric space and let $\gamma : [0, 1] \rightarrow X$ be a Lipschitz curve, either injective or an injective loop. Then $\mathbf{M}(\llbracket \gamma \rrbracket) = \ell(\gamma)$ and $\llbracket \gamma \rrbracket$ is indecomposable.*

Proof. The non-cancelling property $\mathbf{M}(\llbracket \gamma \rrbracket) = \ell(\gamma)$ follows immediately from the injectivity of γ on $(0, 1)$, so it suffices to prove indecomposability.

- (i) Suppose first that γ is an injective Lipschitz curve: as a consequence of Theorem 5.3, to prove that $[\gamma]$ is indecomposable amounts to showing that if $[\gamma] = \sum_{i \in I} [\gamma_i]$ and $\mathbf{N}([\gamma]) = \sum_{i \in I} \mathbf{N}([\gamma_i])$ with γ_i simple, Lipschitz curves, then I must be a singleton. To this aim, notice that $\mathbf{M}(\partial[\gamma]) = 2$, thus there exists a unique $i \in I$ such that the curve γ_i is not a loop. We have $\partial[\gamma] = \partial[\gamma_i]$, so that $\gamma_i(0) = \gamma(0)$ and $\gamma_i(1) = \gamma(1)$. Since $\|[\gamma_i]\| \leq \|[\gamma]\|$ by Proposition 2.4, we deduce that $\gamma_i([0, 1]) \subseteq \gamma([0, 1])$: were this false, we would find by continuity an open interval $U \subseteq [0, 1] \setminus (\gamma_i^{-1} \circ \gamma)([0, 1])$. In turn, this would imply $0 < \|[\gamma_i]\|(\gamma_i(U)) \leq \|[\gamma]\|(\gamma(U)) = 0$, which is a contradiction: hence $\gamma_i([0, 1]) \subseteq \gamma([0, 1])$. We now claim that also the reverse inclusion holds true, i.e. $\gamma_i([0, 1]) \supseteq \gamma([0, 1])$. To prove this, we first note that γ is a continuous map from a compact to a Hausdorff space, hence closed, therefore a homeomorphism between $[0, 1]$ and $\gamma([0, 1])$. This implies that the map $\gamma^{-1} \circ \gamma_i: [0, 1] \rightarrow [0, 1]$ is continuous; since $\gamma^{-1}(\gamma_i(0)) = 0$ and $\gamma^{-1}(\gamma_i(1)) = 1$, we conclude that $\gamma^{-1} \circ \gamma_i$ is surjective on $[0, 1]$ and this yields the sought inclusion. We have thus shown that γ and γ_i are injective Lipschitz curves, with the same initial and final points and $\gamma_i([0, 1]) = \gamma([0, 1])$. We can conclude that $\ell(\gamma) = \ell(\gamma_i)$ and thus, by the non-cancelling property, $\mathbf{M}([\gamma]) = \mathbf{M}([\gamma_i])$: this implies that $I = \{i\}$, as desired, and this concludes the proof in this case.
- (ii) If γ is an injective loop, we argue in a similar way: again, as a consequence of Theorem 5.3, to prove that $[\gamma]$ is indecomposable amounts to showing that if $[\gamma] = \sum_{i \in I} [\gamma_i]$ and $\mathbf{N}([\gamma]) = \sum_{i \in I} \mathbf{N}([\gamma_i])$ with γ_i simple, Lipschitz curves, then I must be a singleton. Notice that $\mathbf{M}(\partial[\gamma]) = 0$, thus for every $i \in I$ the curve γ_i is a loop. Fix any $i \in I$: since $\|[\gamma_i]\| \leq \|[\gamma]\|$ by Proposition 2.4, we deduce that $\gamma_i([0, 1]) \subseteq \gamma([0, 1])$. As above, a short topological argument yields also the reverse inclusion, i.e. $\gamma_i([0, 1]) \supseteq \gamma([0, 1])$. Being a loop, γ can be identified with a continuous map $\gamma: \mathbb{S}^1 \rightarrow X$: as above, γ is then a closed map, hence a homeomorphism between \mathbb{S}^1 and $\gamma([0, 1])$. This implies that the map $\gamma^{-1} \circ \gamma_i: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is continuous and injective, thus (again by standard facts in topology) it must be surjective as well. We have thus shown that γ and γ_i are injective loops, with $\gamma_i([0, 1]) = \gamma([0, 1])$. We can thus deduce that $\ell(\gamma) = \ell(\gamma_i)$ and the conclusion follows as in the previous case, as a consequence of the non-cancelling property. \square

Remark 5.5. The following equivalence is satisfied: if (X, d) is a complete metric space and $\gamma: [0, 1] \rightarrow X$ is a Lipschitz curve, then

$$\gamma \text{ is injective or an injective loop} \iff \mathbf{M}([\gamma]) = \ell(\gamma) \text{ and } [\gamma] \text{ is indecomposable.}$$

This can be achieved by combining Lemma 5.1 with Lemma 5.4. \blacksquare

As an immediate consequence of Theorem 5.3, Lemma 5.1, and Lemma 5.4, we deduce the following characterisation of indecomposable 1-currents:

Corollary 5.6 (Characterisation of indecomposable integral 1-currents). *Let (X, d) be a complete metric space. Then the indecomposable integral 1-currents $T \in \mathcal{I}_1(X)$ are exactly those of the form $T = [\gamma]$, where the Lipschitz curve $\gamma: [0, 1] \rightarrow X$ is either injective or an injective loop.*

In particular, in the Euclidean setting, we obtain the characterisation of indecomposable 1-currents hinted in Federer's book, injectivity included.

The careful reader might have noticed that the result we build upon to prove Theorem 5.3 and Corollary 5.6, namely [6, Lemma 4.4], already makes use of Federer's claim. This opens up the possibility of a circular reasoning. However this is not the case, and we present two ways out. First we observe that Ambrosio and Wenger do not make use of the full result, namely

they do not need the injectivity of the curves. Therefore one can first prove the decomposition in Euclidean spaces without the injectivity claim (as done in [8] and [12]) and use that result to prove Ambrosio–Wenger, and then our result. As a second way out, we can also prove the result without injectivity, prove the decomposition given by Theorem 3.2 (which is unrelated to Federer’s injectivity claim) and then put together the two things similarly to what we did in STEP 6 of the proof of Theorem 5.3 to obtain injectivity.

6. APPLICATIONS

In this final section, we present some consequences of the decomposition and of the characterisation of indecomposable 1-currents.

6.1. Integral currents with support contained in a given curve. We commence with the following generalisation of [1, Lemma 2.14].

Proposition 6.1. *Let (X, d) be a complete metric space and let $\gamma: [0, a] \rightarrow X$ be a Lipschitz curve, parametrised by arc-length. Suppose that γ is either injective or an injective loop. Consider a 1-dimensional integral $T \in \mathcal{I}_1(X) \setminus \{0\}$, with $\text{spt}(T) \subseteq \gamma([0, a])$.*

- (i) *If $\partial T = 0$, then γ is an injective Lipschitz loop and $T = k\llbracket\gamma\rrbracket$, for some $k \in \mathbb{Z}$.*
- (ii) *If γ is injective and $\partial T = \partial\llbracket\gamma\rrbracket = \delta_{\gamma(a)} - \delta_{\gamma(0)}$, then $T = \llbracket\gamma\rrbracket$.*

Proof. Let us split the proof in some steps.

STEP 1: Decomposition. Consider the current $T \neq 0$ and assume $\partial T = 0$. Applying Theorem 5.3, we can write

$$T = \sum_{i \in I} \llbracket\sigma_i\rrbracket, \quad \text{with } \mathbf{M}(T) = \sum_{i \in I} \ell(\sigma_i) = \sum_{i \in I} \mathbf{M}(\llbracket\sigma_i\rrbracket) \quad (6.1)$$

for at most countably many injective Lipschitz loops $\sigma_i: \mathbb{S}^1 \rightarrow X$, $i \in I$, which we assume to be parametrised with constant speed.

STEP 2: γ is a loop. Fix any $i \in I$ such that σ_i is non-trivial: since $\text{spt}(T) \subseteq C = \gamma([0, a])$ we have $\sigma_i(\mathbb{S}^1) \subseteq \gamma([0, a])$ (because $\|\llbracket\sigma_i\rrbracket\| \leq \|\llbracket\gamma\rrbracket\|$ as measures by Proposition 2.4) and this forces γ to be a loop as well: if not, we would find a homeomorphism between $\mathbb{S}^1 \simeq \sigma_i(\mathbb{S}^1)$ (recall σ_i is injective, hence a homeomorphism onto its image) and a closed subinterval of $[0, a]$, a contradiction.

STEP 3: Homeomorphism. Since γ is injective, it is a homeomorphism between $[0, a]^* \simeq \mathbb{S}^1$ and C . The map $\gamma^{-1} \circ \sigma_i: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is thus a continuous, injective map and hence it is also surjective. We infer that $C = \sigma_i(\mathbb{S}^1)$.

STEP 4: Isometries. Let us now equip $C = \gamma([0, a]) = \sigma_i(\mathbb{S}^1)$ with the arc-length distance d_{al} (notice this depends only on the support C , not on the parametrisation). The map γ is an isometric homeomorphism between $(\mathbb{S}^1, ad_\theta/2\pi)$ and (C, d_{al}) , where d_θ denotes the distance on \mathbb{S}^1 . Similarly, since σ_i is injective, it is also an isometric homeomorphism between $(\mathbb{S}^1, ad_\theta/2\pi)$ and (C, d_{al}) . The map $\gamma^{-1} \circ \sigma_i: (\mathbb{S}^1, ad_\theta/2\pi) \rightarrow (\mathbb{S}^1, ad_\theta/2\pi)$ is thus an isometry and hence $\llbracket\gamma\rrbracket = \llbracket\gamma \circ (\gamma^{-1} \circ \sigma_i)\rrbracket = \llbracket\sigma_i\rrbracket$.

STEP 5: Conclusion in the boundaryless case. From (6.1), we deduce that

$$\mathbf{M}(T) = \sum_{i \in I} \mathbf{M}(\llbracket\sigma_i\rrbracket) = \#I \cdot \mathbf{M}(\llbracket\gamma\rrbracket),$$

hence the set of indices I is finite and this concludes the proof in the case $\partial T = 0$.

STEP 6: The case with boundary. The proof of Point (ii) can be done exploiting a similar strategy. First, one decomposes T as

$$T = \sum_{i=1}^k \llbracket \beta_i \rrbracket + \sum_{i \in I} \llbracket \sigma_i \rrbracket, \quad \text{with } \mathbf{M}(T) = \sum_{i=1}^k \mathbf{M}(\llbracket \beta_i \rrbracket) + \sum_{i \in I} \mathbf{M}(\llbracket \sigma_i \rrbracket), \quad (6.2)$$

where $\beta_i: [0, 1] \rightarrow X$ are finitely many injective Lipschitz curves, parametrised with constant speed, while σ_i are at most countably many injective Lipschitz loops. It is readily seen that $\text{spt}(T) \subseteq \gamma([0, a])$ implies that the loops σ_i are all trivial (proceed as in STEP 2: the map $\gamma^{-1} \circ \sigma_i$ would be a homeomorphism between S^1 and a closed subinterval of $[0, a]$); as for the curves β_i , from $\partial T = \partial \llbracket \gamma \rrbracket$ we infer that there exists one and only one $i_0 \in \{1, \dots, k\}$ such that β_{i_0} is non-trivial. Such β_{i_0} is thus an injective, constant-speed Lipschitz curve whose image is contained in $\gamma([0, a])$ and $\beta_{i_0}(1) = \gamma(a)$ and $\beta_{i_0}(0) = \gamma(0)$, hence (arguing similarly as above) we conclude $\beta_{i_0}(t) = \gamma(at)$ for all $t \in [0, 1]$, in particular $\llbracket \beta_{i_0} \rrbracket = \llbracket \gamma \rrbracket$ and the proof is completed also in this case. \square

6.2. On the structure of simple sets in \mathbb{R}^2 . In this paragraph, we show how it is possible to derive from the decomposition some consequences on the structure of particular sets of finite perimeter in \mathbb{R}^2 . The material presented here should be compared with [3], to which we refer the reader for the notation and for a more detailed analysis.

For any measurable set $A \subseteq \mathbb{R}^d$ we denote by T_A the canonical current associated with A (with unit density and orientation induced by \mathbb{R}^d); observe that if $\mathcal{L}^d(A) < +\infty$ then $T_A \in \mathcal{R}_d(\mathbb{R}^d)$ and if A , in addition, has also finite perimeter, then $T_A \in \mathcal{S}_d(\mathbb{R}^d)$. In this case, the current ∂T_A is the natural current associated with the reduced boundary $\partial^* A$ with orientation compatible with Stokes' Theorem (we refer the reader to [4] for a comprehensive treatment of the theory of sets of finite perimeter). A set A of finite perimeter is said to be **decomposable** if there exist two measurable sets B, C with $A = B \cup C$, with $\mathcal{L}^d(B \cap C) = 0$, $\mathcal{L}^d(B) > 0$, $\mathcal{L}^d(C) > 0$, and $\text{Per}(A) = \text{Per}(B) + \text{Per}(C)$. A set A of finite perimeter is said to be **indecomposable** if it is not decomposable. It is easily seen that A is indecomposable if and only if T_A is indecomposable. Further, we recall the following notion:

Definition 6.2 ([3, Definition 3] and [7, Proposition 2.17]). *A set $A \subseteq \mathbb{R}^d$ of finite perimeter is said to be **simple** if*

- (i) *either $A = \mathbb{R}^d$;*
- (ii) *or $\mathcal{L}^d(A) < +\infty$ and both A, A^c are indecomposable.*

We look for a criterion for simple sets in terms of the decomposability of its associated current. Before stating our result, we present the following observation.

Remark 6.3 (Indecomposability of a current and of its boundary). If $T \in \mathcal{S}_d(\mathbb{R}^d)$, the indecomposability of ∂T implies the indecomposability of T . Indeed, if $T = U + V$, with $\mathbf{N}(T) = \mathbf{N}(U) + \mathbf{N}(V)$, then we would have $\partial T = \partial U + \partial V$, with $\mathbf{N}(\partial T) = \mathbf{N}(\partial U) + \mathbf{N}(\partial V)$ and this necessarily implies $\partial U = 0$ or $\partial V = 0$. By the Constancy Lemma and finiteness of the mass, either $U = 0$ or $V = 0$, whence the indecomposability of T follows.

Observe that, except for the particular case mentioned above, there is no other implication between the indecomposability of T and the indecomposability of ∂T . In one direction, it is enough to consider the 1-integral current associated with two loops, which is decomposable but whose boundary is zero (hence indecomposable). Conversely, the 2-current associated with an annulus in \mathbb{R}^2 is indecomposable but its boundary is not. \blacksquare

We are finally ready to state and prove the following criterion:

Proposition 6.4. *Let $E \subseteq \mathbb{R}^d$ be a set of finite perimeter and of finite Lebesgue measure. The set E is simple if, and only if, the current ∂T_E is indecomposable.*

Proof. Assume ∂T_E is indecomposable. By Remark 6.3, the current T_E is indecomposable, hence E is indecomposable. It remains to show that E^c is indecomposable. Let us consider two sets A, B such that $E^c = A \cup B$ with $\text{Per}(E^c) = \text{Per}(A) + \text{Per}(B)$ and $\mathcal{L}^d(A \cap B) = 0$. In view of [3, Remark 1], we can assume that $\mathcal{L}^d(A) < \infty$ and $\mathcal{L}^d(B) = +\infty$. It can be readily checked that $B^c = E \cup A$ with $\mathcal{L}^d(E \cap A) = 0$, so we can consider the currents T_A, T_{B^c} which satisfy $T_{B^c} = T_E + T_A$, whence $\partial T_E = \partial T_{B^c} - \partial T_A$. Since $\text{Per}(E) = \text{Per}(E^c) = \text{Per}(A) + \text{Per}(B) = \text{Per}(A) + \text{Per}(B^c)$, by indecomposability of ∂T_E we deduce either $\partial T_{B^c} = 0$ or $\partial T_A = 0$ and both cases force $T_A = 0$, hence $\mathcal{L}^d(A) = 0$ and the proof is complete.

Conversely, let us suppose that E is simple and let $\partial T_E = U + V$, for some $U, V \in \mathcal{I}_{d-1}(\mathbb{R}^d)$ with $\partial U, \partial V = 0$ and with

$$\mathbf{M}(\partial T_E) = \mathbf{M}(U) + \mathbf{M}(V). \quad (6.3)$$

By the Isoperimetric Inequality (see Theorem 2.7) and the Constancy Lemma, one can uniquely determine $X, Y \in \mathcal{I}_d(\mathbb{R}^d)$ of finite mass such that $\partial X = U$ and $\partial Y = V$. By standard facts, $X = \llbracket \mathbb{R}^d, e_1 \wedge \dots \wedge e_d, \vartheta \rrbracket$ and $Y = \llbracket \mathbb{R}^d, e_1 \wedge \dots \wedge e_d, \psi \rrbracket$ for some functions $\vartheta, \psi \in \text{BV}(\mathbb{R}^d; \mathbb{Z})$. The goal is to show that $D\vartheta = 0$ or $D\psi = 0$. In view of (6.3), we can write

$$\mathcal{H}^{d-1}|_{\partial^* E} = |D\vartheta| + |D\psi| \quad (6.4)$$

as measures on \mathbb{R}^d ; testing (6.4) on the set $E^{(1)}$ (the set of density points of E) we deduce $|D\vartheta|(E^{(1)}) + |D\psi|(E^{(1)}) = 0$ and hence, in view of [3, Remark 2] and of the indecomposability of E , we infer ϑ and ψ are constant in E . Arguing similarly on $(E^c)^{(1)}$, we deduce that ϑ and ψ vanish in E^c (recall that $\mathcal{L}^d(E) < \infty$ and $\vartheta, \psi \in L^1(\mathbb{R}^d)$). The conclusion is thus $\vartheta = \alpha \mathbb{1}_E$ and $\psi = \beta \mathbb{1}_{E^c}$, for some $\alpha, \beta \in \mathbb{Z}$. By (6.3) we must have $|\alpha| + |\beta| = 1$, which forces either $\alpha = 0$ or $\beta = 0$, which readily implies either $D\vartheta = 0$ or $D\psi = 0$, and the proof is thus completed. \square

As a consequence, we deduce the following result about the structure of simple sets in \mathbb{R}^2 :

Corollary 6.5 ([3, Theorem 7]). *Let $E \subseteq \mathbb{R}^2$ be a set of finite perimeter and finite measure. Then E is simple if, and only if, its reduced boundary is equal (up to an \mathcal{H}^1 -negligible subset) to the image of an injective Lipschitz loop.*

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(P. Bonicatto) MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, ZEEMAN BUILDING, CV4 7HP COVENTRY, UK

Email address: Paolo.Bonicatto@warwick.ac.uk

(G. Del Nin) MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, ZEEMAN BUILDING, CV4 7HP COVENTRY, UK

Email address: Giacomo.Del-Nin@warwick.ac.uk

(E. Pasqualetto) SCUOLA NORMALE SUPERIORE, PIAZZA DEI CAVALIERI 7, 56126 PISA, ITALY

Email address: enrico.pasqualetto@sns.it