Optimal Voting Order under Sequential Voting Schemes

by

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Declarations

Parts of this thesis have been previously published by the author in the following:

- Alpern, S., Chen, B., Lee, V. Pan, C. (2021), 'Optimizing voting order on sequentialjuries: A sealed card model',*Available at SSRN*.

There is no research that is performed in collaboration during the development of this thesis but does not form part of the thesis.

Chapter 1

Introduction

Voting theory is a branch of social choice theory, which studies the process and procedures of the collective decision. Social choice theory aims to capture the features of transformation from individual judgements to collective judgements using applied mathematics. These studies are not limited to analysing cases such as legislation committees, expert panels and boards, but also summarising general approaches and models using mathematical modelling and proofs. Social choice theory impacts many subjects, such as political science, economics, applied mathematics, sociology, and even computer science. In addition to helping to comprehend the process of collective decision-making, these results and models can be applied to designing organisational structures and mechanisms of social welfare. This study focuses on one particular aspect of collective decision making: the optimal voting order under generalised sequential voting schemes.

The origin of this research goes as far back as the Condorcet Jury Theorem (CJT) (Condorcet 1785). In the CJT model, a jury needs to decide between two alternative states of nature, saying A "guilty" or B "innocent" in a legal trial. The theorem states that if the jurors are competent (better than random but worse than perfect), the probability of a correct by majority rule is higher than that of each individual juror, and this probability will converge to one when the number of jurors approaches infinity. The results of the CJT are cited in the works of political science to show the advantages of the democratic system. However, the idealistic assumptions of the CJT, especially requiring homogeneous competence and independence among jurors, should not be neglected. Although there is a large body of literature on the extensions through relaxing the assumptions about CJT (see Section 1.1.2), CJT under sequential voting schemes still lacks attention beyond works of Dekel & Piccione (2000), Sørensen & Ottaviani (2001), and Alpern & Chen (2017a,b). When jurors' abilities are homogeneous in classic CJT, the voting order makes no difference to the verdict. Under the sequential voting scheme with heterogeneous abilities, the voting order is of significance as the voters need to know the ascribed votes of two states (number of votes for states A and B) and the votes' composition (how individual jurors voted). Thus, the voting sequences concerning jurors' abilities significantly impact the probability of the correct verdict in sequential voting schemes.

1.1 Literature review

The problems discussed in this thesis lie in the general social choice theory context studied by many political scientists and public choice theorists. In the first subsection, the common voting rules will be introduced as the fundamental analysis tool for Chapters 3–6. Although preference voting (with choices more than two) is not the primary concern of this study, works relating to voting rules in that they are significant to form the foundation of collective decision theory. In the second subsection, CJT as a vital theme of this research will be discussed in detail, especially in terms of the relaxation of assumptions.

1.1.1 Voting rules

Before formally introducing the CJT, the three most common voting rules are presented to provide preparatory knowledge of this topic.

Unanimity rule

The unanimity scheme is that a collective decision is reached, only when all committee members agree with the motion or no one rejects it. There are two advantages to this rule. The first advantage is that this scheme is Pareto optimal. According to Wicksell (1958), every member of the committee decides on whether the issue is beneficial to them or not. Therefore, this scheme can obtain Pareto efficiency, since no individual's interests are compromised. This efficiency guarantees that all committee members will be at an advantage without putting any one member at a disadvantage. The unanimity scheme in political science corresponds to a completely competitive market in economics, where every participant agrees to the same market price. The second advantage of this scheme is that the free will of every member is guaranteed. Buchanan (1986) believes that the unanimity scheme is the only one that does not undermine fairness. However, Reisman (1989) points out that the unanimity scheme, while a good voting rule, may be expensive. The primary reason is that this rule requires a long time to reach an agreement. The second disadvantage is that this may lead to elays or even blackmail, as every member has the right to veto or reject the issue. Sah & Stiglitz (1985) point out another drawback of unanimity rule: it can be harder to reach agreements in larger groups, as subgroups may not agree with each other. In conclusion, the application of this scheme is limited to collective decision making in small groups.

Simple majority rule

The simple majority rule, as the most common voting rule, states that if more than half of the committee agrees with the issue, a final decision can be made. In America, according to Strøm et al. (1990), most states' legislations and practices are passed using this method. This voting rule also has two important advantages. Buchanan & Tullock (1962) state that the primary merit is that the simple majority voting rule will reduce both the external and decision costs of collective decision making. The second advantage of this rule is its high efficiency. Unless the number of the committee is even, the committee will conclude whether they are satisfied with the motion or not as a whole unless abstaining is allowed. However, this simple majority voting rule suffers from several shortcomings. The first is that the scheme neglects the strength of preference. Dahl (1956) thinks that this scheme is only fair in terms of the intensity of its unfairness. The second shortcoming is discussed in the famous Arrow's impossibility theorem. When the options are more than two, there is no ranked voting system can transfer ranked preferences of individuals into a clear ordered social choice satisfying the some democratic requirements for its procedure, like non-dictatorship and Pareto efficiency. The third disadvantage is the Borda effect. Under some circumstances, using the simple majority rule may lead to results that are unsatisfactory for most members. The fourth drawback is that this rule may inspire strategic voting behaviour. Rather than following their preferences, voters may change their votes to favour their groups. The simple majority rule may be not ideal for the voting with preference, it still can be used for studying jury voting theories with two alternatives.

Weighted average majority scheme

This is a variation of the simple majority scheme that attempts to overcome the disadvantages of the simple majority voting rule. The European Council by Tsebelis & Garrett (2000) uses the qualified majority voting rule to make decisions (except for motions regarding fiscal policy, free flow of personnel and right of employers, which require unanimity). In EU15, 62 out of 87 can pass a motion. The common practice is that voters are divided into groups based on their importance. Then votes are reallocated by groups. This division of voters helps solve the strength of preference in the simple majority voting rule. However, this scheme fails to resolve other shortcomings of simple majority voting. Furthermore, when the alternatives are more than two, aggregating different preferences of individuals may be impossible, which means it may fail to reach equilibrium and cause a cycle among all alternatives. Sen (1966) introduced value restriction to systematically solve this problem.

1.1.2 Condorcet Jury Theorem

In this subsection, a summary of the underlying assumptions of the CJT will be given, and the recent works related to these assumptions will be discussed. There are two key assumptions regarding the CJT: the competence and independence assumptions. First, the Condorcet ability is defined as the probability of correct decision $p \in [0, 1]$. The competence assumption requires every juror to have a fixed ability larger than a half ($p > 1/2$). Secondly, the independence assumption states that individual decisions made by Condorcet jurors are statistically independent. Today, there are many versions of the Condorcet Jury Theorem. Nevertheless, the main results of the CJT can be summarised into three main parts.

- The probability that an institution of decision-makers would collectively make the right choice is higher than the probability that any single member of the group would make such a choice.
- This superiority of the collective decision over the individual decision monotonically increases with the size of the institution.
- When the size of this institution approaches infinity, this probability will approach certainty (Limit $\lim_{n\to\infty} p_n = 1$, where n is the number of jury members).

The following will discuss these assumptions and recent works related to them. Table 1.1 provides a summary of the underlying assumptions regarding CJT. The research interest of this study is sequential voting with heterogeneous abilities, which relaxes both individual competence and independence assumptions of the classic CJT. These relaxations have been highlighted in bold in Table 1.1. Two subsections discuss the respective violations of these two assumptions.

Heterogeneous abilities

The assumption of homogeneous abilities in the CJT is too strong to find real-life cases. However, this assumption is the key to making all three CJT statements valid. A

Nature of Problem	**Features of Jury**	**Voting behaviour**
binary alternatives	**homogeneous abilities**	**independent voting**
symmetric alternatives	identical goals	honest voting

Table 1.1: Assumptions of the Condorcet Jury Theorem

numerical example is easy to find if we violate this assumption. For example, considering a jury of size three, their Condorcet abilities are (0.99,0.60,0.60). Supposing that the decision rule is the simple majority (more than half of the jury votes for the same alternative), the probability of the correct verdict is 0.835. The decision by majority rule is worse than that of the most senior juror. Therefore, it already violates the superiority of group decision by CJT. Furthermore, we can reduce the accuracy even further by increasing both the size of the jury and the variance within the abilities set. For instance, based on the above jury, say we add two more jurors with lower abilities (0.5,0.5). Now, the Condorcet abilities set is (0.99,0.60,0.60,0.5,0.5), and the probability of the correct verdict is now 0.793 (less than 0.835). Thus, it violates the relation between the size of the institution and the correctness of the collective decision.

For the reasons mentioned above, the existing literature attempts to prove the robustness of the validity of limit $\lim_{n\to\infty} p_n = 1$ when faced with a heterogeneous abilities jury. Boland et al. (1989) proposed the concept of an indirect majority rule to replace the simple majority rule to maintain the CJT limit. Paroush (1998) proved that for the jurors with different abilities, the necessary and sufficient conditions for the CJT is the mean of jurors is larger than a half. However, he still emphasised the importance of independence on survival of CJT. Similar results also have been found in the work of Fey (2003). He proved the last part of CJT to hold in a jury with adequately large size "as long as the average competence of the voters is greater than the fraction of votes needed for passage" (p. 28). He used the supermajority rule, also known as the qualified majority rule (details see Chapter 2). The validity of the CJT when the supermajority rule rather than the simple majority rule is used as the basis for the jury's decision has also been discussed in Nitzan & Paroush (1984), Ben-Yashar & Paroush (2000) and Kanazawa (1998). In summary, the last part of the Condorcet Jury Theorem holds for juries with heterogeneous abilities using supermajority rules, although independence remains essential. This study concerns violating homogeneities of abilities and independence. Based on the above studies, the last part of CJT fails to survive in this study. However, the interesting relationship between the majority verdict's correctness and the size of the jury (a variation of the second part of CJT) will be discussed under different sequential voting schemes.

Correlated jurors

One of the controversial arguments that limit the application of the CJT is the difficulty of being independent. Actually, on most occasions, discussion between jurors has already happened before the decision-making process. This interaction among the jurors may have different effects on the verdict. Scholars like Kahneman et al. (2011) believe that conversations and discussions among the decision-making institution members can harm the correctness of the final decision. This statement is supported by Surowiecki (2005), celebrated author of 'wisdom of crowds'. In his opinion, social psychology factors such as herding will damage the independence of individual judgement. These factors will lead to a decrease in the credibility of the group's final decision. However, others argue that the process of learning through information aggregation may help the jurors make better decisions than independent voting. The recent works on correlation among jurors can be divided into two categories. The first is the *opinion leader* model. Boland et al. (1989) introduced a model with one leader and corresponding followers. They advocate that the public opinion leader maybe not always beneficial for the group decision making. Other notable works in this category include extensions with weighted voting rules by Berg (1994) and the condition on the superiority opinion leader by Estlund (1994). The second category uses *exchangeable* random variables. Ladha (1992) and Berg (1993) found that, given the restricted correlation of conditional probability, the CJT still holds. The underlying features of both models are homogeneous abilities and correlation requiring same covariances. Although both results are clear, the condition on covariance is so strong that it may be contrary to the original intention of introducing correlation. More recently, Peleg & Zamir (2012) focused on the effect of correlated jurors and the jury size on the verdict's correctness. Pivato (2017) conducted a comprehensive survey on correlated jurors under dichotomous decision settings. He found that decreasing average covariance is the key to maintain the last part of CJT under more complex settings (polychotomous decision problems).

Sequential voting provides another way to consider correlated jurors. Here, sequential voting is defined by Alpern & Chen (2017*b*) a voting scheme under which labelled jurors take turns to cast their votes with the knowledge of the votes of a given subset of previous jurors. Take simultaneous voting as an example. The given subset is empty for every juror. Nevertheless, for the roll-call voting scheme, the given subset consists of all the previous jurors' votes. Six different sequential voting schemes exist for a jury of three (see Chapter 2). Most existing literature follows the research direction of Condorcet and focuses on one type of sequential voting scheme, namely simultaneous voting. Among the other five voting schemes, only roll-call voting has raised some attention, especially in the works of Dekel & Piccione (2000) and Ottaviani & Sørensen (2001).

Dekel & Piccione (2000) studied sequential voting under symmetric binary settings. They found weak and strong equilibriums for some sequential and simultaneous voting schemes. The underlying finding is that, although sequential voting is superior to simultaneous voting for decision-making as jurors under roll-call voting hold more information, this kind of voting (roll-call voting) is still no better than simultaneous voting in terms of information collection during group decision-making. The fundamental reason for this phenomenon is herding. Therefore, it is important to avoid or minimise the effect of herding. Contrary to the results of this thesis, they believe that the voting order does not matter as jurors will cast their votes based on their own beliefs. They also pointed out two future research directions. The first is that more possible states could be added into the model. More choices can widen the potential applications of this theory (for example, the primary elections). The second potential research direction is the introduction of other methods to the model signalling process, such as the common value. In large group decision-making, one voter may be close to others. The introduction of the common value implies that some jurors are better informed concerning the preferences of those voters. This topic has been studied intensively by Fain et al. (2017). This thesis follows the latter direction by introducing a continuous signalling model with heterogeneous abilities, which reached the opposite conclusion that voting order is of great significance.

Although the jurors in sequential voting are more informative than Condorcet simultaneous voting, it still suffers from the possibility of information cascades and herding. When the jurors with higher ability vote early, these effects will be more significant. Under these circumstances, less able jurors are blind to their own private information to choose the alternative which has already a consensus by previous jurors. Bikhchandani et al. (1992) studied the possibilities of information cascades and its consequence (the possibilities of wrong collective decision). They believed that the analysis of information cascades can be a tool to explain many well-known phenomena like herding. Ottaviani & Sørensen (2001) found the similar result that the improvement of abilities on some specialists may result in more serious herding behaviour and deteriorate the quality of aggregate information collected by the team leader. This is a direct contradiction to the traditional rule of thumb, seniority rule. They focused on solving the problem of herding in public decision making. They believed that individual experts may hold their private information under a certain sequence of speaking in order to maintain their reputation. Thus, they aim to find the optimal speaking sequence to mitigate the effects of herding and enhance information aggregation for the whole group. They concluded that optimising the sequence of speaking could enhance information aggregation. However, exceptions exist. Thus, they do not find a consistent optimal speaking sequence that can be applied to all circumstances.

The works of Acemoglu et al. (2011), Acemoglu & Ozdaglar (2011) Gale & Kariv (2003), Lobel & Sadler (2015) and Lobel & Sadler (2016) provide another method to deal with information updating processes. They use different Bayesian social learning models to describe the mental world for individuals and inference process of these individuals. The main research question for them is finding the optimal way that social observation integrates with the process of Bayesian inference for rational individuals. Different from most models in this field, Song (2016) introduced the cost of observation making as no longer exogenous. He showed that the sufficient condition for individual making the right decision through expanding observation by Acemoglu et al. (2011) no longer holds. This introduction of monetary transfer brings the suggestion of changing the models used in this thesis such that the vote of previous juror(s) is no longer cost-less information. They need to pay a cost to know this information and are allowed to strategically select the information sets. This model will be for my future research.

Back to the jury voting problem without preferences, the recent papers of Alpern & Chen (2017*a*,*b*) introduced new signals models to bring more possibilities of private information. Alpern & Chen (2017*a*) studied the multiple signals setting model using the roll-call scheme. The voters are heterogeneous in terms of ability, and the distribution of the signals is no longer binary, unlike in works related to the CJT. They argued that the experiences, education levels, and other attributes of voters that could enhance their competence would affect the probability of finding the true state. Thus, jurors' abilities should, realistically, be different. Unlike Ottaviani & Sørensen (2001), Alpern and Chen found the consistent optimal voting order in roll-call voting. They suggested that, for a jury of three, the median-ability juror should vote first and the sequence of voting for the other two jurors is immaterial. Another finding of Alpern and Chen's work is that voting using the seniority rule that the voters vote in decreasing order of ability is superior to the anti-seniority rule in roll-call voting. These two findings inspire this study to explore sequential majority voting with knowledge of the previous voter, which is similar to roll-call voting but with a more complicated information flow. Similar patterns have been found in Chapter 5. Alpern & Chen (2017*b*) conducted a study on a casting-vote scheme: $n-1$ jurors vote simultaneously and, if there is a tie, the last juror casts the pivotal vote. The problem is who should be the last voter according to their abilities. Alpern and Chen proved algebraically that the juror with median ability should cast the deciding vote. This finding inspires the study on who should take an initial public vote under the inverse structure of the cast voting (see Chapter 6).

Diversity and degree of independence are essential to make the idealised CJT applicable to the real world. This study intends to identify a set of conditions that uniquely typify class of solutions to the generalised sequential voting schemes with heterogeneous abilities and different private information.

The remaining of the thesis is structured as follows. Chapter 2 will formally formulate the model and provide the basic concepts, notations, and fundamental theories. In Chapter 3, we consider roll-call voting with a concrete ability setting called the sealed card problem. In Chapter 4, we study the optimal voting order under sequential voting with an independent voter. In Chapter 5, we consider the order of voting, which can maximise reliability under sequential voting with knowledge of the previous juror. Chapter 6 conducts a study on the optimal voting order for sequential voting with an initial public vote. In Chapter 7, we answer the question of which voting scheme we should choose to maximise the probability of a correct collective decision among all six sequential voting schemes. Chapter 8 provides a summary of contributions, potential future directions and concluding remarks.

Chapter 2

Preliminaries

In this chapter, we will first define a general framework of sequential voting with a jury of three. We will then define the basic concepts and notation used in the thesis under both discrete and continuous signal models.

2.1 Sequential voting schemes

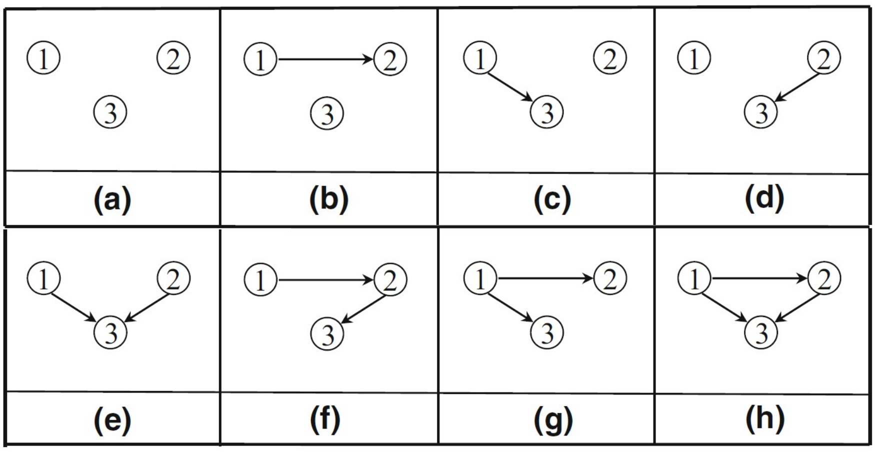


Figure 2.1: Sequential voting schemes for a jury of three (Alpern & Chen 2017*b*)

We will start with the following assumption. It is impossible for voter i to know the vote of voter j and for voter j to know simultaneously the vote of voter i. This time-ordered information transfer indicates that no two voters cannot know each other's votes at the same time. In general, the cycle of voters is not allowed, each of whom knows the vote of his previous voter. An acyclic graph (a directed graph) with no cycles fits this situation. A sequential voting scheme can be modelled as a particular directed acyclic graph with a finite number of nodes. The nodes represent the voters. The arc from node i to node j implies that voter j knows the vote of voter i when he casts his votes.

The foundation characteristic of a finite directed acyclic graph is that all the nodes can be numbered such that any arc from a smaller numbered node to a larger numbered node is admissive (Bang-Jensen & Gutin 2008). The number shows the chronologically arranged order of the events. By this characteristic, we give meaning to the arc which is the relation between each voter i of $\{0, 1, \ldots, n\}$ as knowledge set $K(i) \subseteq \{1, \ldots, i-1\}$ under sequential voting schemes. The contents of the information set vary depending on the specific sequential voting scheme. Knowledge in classic roll-call voting can be described as follows: when voter i casts his vote, he does so knowing votes of all voters j in the knowledge set $K(i)$.

There are several approaches to enumerating a finite directed acyclic graph. This thesis considers the time-consistent numbering of nodes. Take a jury of three as an example. As mentioned in the previous paragraph, a sequential voting scheme depends on the specific descriptions of $K(1)$, $K(2)$ and $K(3)$. There are $2^{(i-1)}$ subsets for the set $\{1, \ldots, i-1\}$. Thus, we have $2^0 \times 2^1 \times 2^{(3-1)} = 8$ possible sequential voting schemes. Figure 2.1 provides the details of these eight sequential voting schemes.

As shown in Figure 2.1, there are six different non-isomorphic voting schemes. The graphs (b) (c) and (d) are isomorphic. The voting scheme (a) is known as simultaneous voting, which has been extensively studied after the Condorcet research, such as the consequences of strategic voting in the CJT model studied by Austen-Smith & Banks (1996) and extension on correlated voters by Ladha (1992). The voting scheme (h) is called the roll-call voting scheme, and scheme (e) is called the casting-vote scheme. Voting schemes (h) and (e) have been thoroughly studied in the works of Alpern & Chen (2017*a*,*b*).

As well as the three schemes mentioned above, another three voting schemes also exist. The isomorphic schemes (b), (c) and (d) are sequential voting with an independent voter (type I). Scheme (f) is sequential voting with knowledge of previous voter (type II), and scheme (g) is sequential voting with an initial public vote (type III). This thesis mainly concerns these three voting schemes as well as one particular discrete variation of the roll-call voting called, the sealed card problem. This thesis examines the phenomenon of jurors voting for the alternative they believe is the more likely (here referred to as the honest voting strategy, see 2.2.1) and aims to find an optimal voting order related to jurors' abilities hat maximises the probability of a correct verdict.

2.2 Basic concepts and notations

There are two models in this work, a continuous model proposed in Alpern & Chen (2017*a*) and a discrete model known as the sealed card problem. Although the two models have different measures of ability and signal distribution, they share the same foundation. To summarise the models here, suppose a jury consisting of three members sequentially votes for the true Nature N of the states between two equiprobable alternatives, state A and state B (colour $R(ed)$ or colour $B(lack)$ for the sealed card problem). The purpose of the jury is to make a collective decision between these two alternatives.

A priori probability θ is defined as the conditional probability of state A depending on the vote(s) of the previous juror(s) (see equations (2.10) and (2.11) for a simple example). When $\theta = 1/2$, this indicates equal probabilities between A and B. Each juror apart from the first (or independent) juror uses the Bayes' formula to update this probability via the votes of previous jurors. If $\theta = 1/2$, the juror will flip a fair coin to decide which state he should vote for. Aside from this trivial case, the juror will vote for the state with higher conditional probability ($\theta > 1/2$ for state A while $\theta < 1/2$ for state B). *A priori* probability θ_0 for the first juror is equal to the unconditional probability of state A, $\Pr[A] = \theta_0$. If we assume that two states are equally likely, we have $\theta_0 = 1/2$ and this is called *neutral binary settings*.

The ability of juror is either a probability of making the correct decision or a proxy of this probability (see Section 2.2.2). The private information is a distribution of signals (samples). The juror with higher ability will have samples with higher quality. Both of these depend on a specific decision problem. The only stochastic part of θ is the private information (his signal s). The signal s indicating when the juror cannot make a decision ($\theta = 1/2$) is called *threshold* τ (see Section 2.2.4). By backward calculation of the signal with $\theta = 1/2$, we can solve for the threshold for each juror. Section 2.2.3 will elaborate on definitions of the signal distributions in different voting problems.

The probability of vote for each juror is determined by this threshold. After we know how each juror decides his vote (a set of thresholds), we can calculate the probability of the correct verdict (reliability) through scenario analysis. Reliability is the standard of comparison for different voting schemes and orders.

In the following, we will discuss voting behaviour followed by the above definitions and procedures in details under two different models.

2.2.1 Voting behaviour: honest voting and strategic voting

Lull (1283), cited in McLean & Urken (1995), pointed out that voters' voting behaviours may not be consistent with his preferences and noted that people would vote strategically under specific voting rules and procedures. In his work "art of elections" in 1299, he stated a belief that every voter should swear that they are telling the truth before voting. Inspired by Lull (1299), Cusanus (1434) cited in McLean & Urken (1995) stated that jurors should vote under the guidance of conscience and morality, and that the outcome of this would be most efficient. He believed that voters should vote independently rather than sequentially as in Lull's voting method. He proposed a method of comparing every pair of alternatives. This method resembles the famous Borda count. Borda (1784) proposed the Borda count to overcome the disadvantage of simple majority voting. However, his voting scheme could easily be manipulated by voting strategically to change the outcome. He argued that voting should only be carried out by honest people. Dodgson (1876) found that strategic behaviour is common in practice by conducting systematic research on elections and committees. He argued that the decision-making process involves making strategies under a specific voting rule. Arrow et al. (1951) remarked that people would find profitable strategies in the voting process rather than reveal their preference in the most social choice mechanisms. Satterthwaite (1975) and Gibbard (1973) proposed the Gibbard-Satterthwaite Theorem, which proved the necessity of strategic voting in theory. Nevertheless, scholars in the field of mechanism design found a way to escape from the impossibility conclusion of the Gibbard-Satterthwaite Theorem by limiting the classes of preferences. Vickrey (1960) believed that the independence condition is the key to solve the strategy problem. He introduced the class of quasi-linear preferences, whose utility functions are linearly dependent on money. By restricting the preferences, he successfully designed the Vickrey-Groves-Clarke (VCG) class of strategy-proof mechanisms. The above works show that, in preference voting, strategic voting is common and fails to act as a truth-telling device when the voting scheme is sequential. For this reason, this thesis only admits honest voting behaviour to avoid the problems caused by strategic voting.

Here, the definition of honest voting is a strategy profile in which every member of the jury votes for the state that they believe to be most likely given the conditional probability of the state based on their private information and the votes of the previous jurors by Alpern & Chen (2017*b*). The mathematical definition of this voting behaviour will be discussed in the following Section 2.2.2. This idea is similar to Dekel & Piccione (2000) in a different sequential model.

2.2.2 Juror abilities

Continuous model

In the classic (Condorcet) model, the measurement of each juror's ability is the probability of ascertaining the true state, denoted as p. As a binary signal (either A or B) is provided as each juror's private information, p is the probability of receiving the right signal corresponding to the actual state of Nature. This probability can be observed through the historical records of correct decisions made by individual jurors. One can consider this as the empirical probability in the probability theory. The range of this probability p is $[1/2, 1]$. When $p = 1/2$, the juror knows nothing about the true state and guesses it at random. When $p = 1$, he receives concrete information, and he knows the true state.

Alpern & Chen (2017*b*) proposed the model here with more sophisticated private information than the binary signals in the classic Condorcet model. The key components of the continuous model here are ability a and signal s. Different from the Condorcet binary signals (for state A or for state B), each juror in this model receives a signal s in the interval $[-1, 1]$. The private information of each juror is no longer the discrete A, B but a continuous signal $s \in [-1, 1]$. A higher positive signal indicates that state A is more likely for each juror, while a lower negative signal indicates that state B is more likely. When the signal is equal to 0, the juror is neutral about the two states. Section 2.2.3 will provide a more detailed description for this signal distribution.

Unlike the traditional Condorcet's model with probability p and binary signals, the model here uses the parameter ability a which acts as a *proxy* for p but with continuous signal s. The ability a and signal s are positively related. A juror with higher ability a will have a higher probability of obtaining a stronger signal indicating the correct state. Like the Condorcet juror who has the probability of correct decision p, the juror in this model has the ability a, ranging from 0 to 1. The set of ability profiles is $\mathcal{A} = [0, 1]^n$. The real number a is linearly correlated to the conditional probability p_τ of the state A, given a positive signal s. p_τ is an analogue or a sequential version of p. If we define p as a *posteriori* probability indicating the state of Nature A, then $p = p_\tau$. The mathematical expression of relation between p_τ and ability a is given by the equation (2.1) (See last part of Section 2.2.3 for derivation). For each juror, the conditional probability of the state A (p_τ), given that a signal s by his ability a and a threshold τ, $\tau \in [-1, +1]$, is:

$$p_\tau = \Pr[A|s \geq \tau] = \frac{1}{2} + \frac{1+\tau}{4}a. \tag{2.1}$$

To better understand this relation, let us start the analysis with a simpler version of the equation (2.1) where $\tau = 0$,

$$p_0 = \frac{1}{2} + \frac{1}{4}a. \tag{2.2}$$

A juror with ability $a = 0$, in both the simpler setting in equation (2.2) and the more general setting in equation (2.1), $p_0 = p_\tau = 1/2$ is the same value as *a priori* of the state A under neutral binary settings. Therefore, this type of juror is without any ability, and their private information is an empty set (and therefore useless). On the contrary, if a juror has the full ability $a = 1$, they will have get a higher probability of ascertaining the true state. In equation (2.2), the probability that the voter's opinion is correct is 3/4 when a reaches the maximum value. For the general setting case in equation (2.1), the situation is somewhat complex as we are also considering the thresholds τ and signal s. When the juror receives a very high signal above the threshold τ (close to 1), their conditional probability is almost certain ($p_\tau = 1$). On the other hand, if the juror receives a very weak signal above the threshold, say $\tau = -1$, he gets almost nothing from his private information and expect the settings of the two states ($p_{-1} = p_{min} = 1/2$). Introduction of the continuous signal model brings sampling process into the classic Condorcet model.

Discrete model (the sealed card problem)

Now we introduce a discrete model called the sealed card problem. The model's parameters are the size n of the jury, which is an odd number to allow the existence of a majority verdict, and the size $D = 2m$ of the deck of cards, where m are red (R), and m are black (B). Also, the ability $a_i \in \{1, \ldots, D-1\}$ of each juror i is known. The model is dynamic: the first step is a stochastic choice of a card removed hidden (*sealed*) from the deck. The colour of the sealed card, R or B, is the state of Nature ($N \in \{R, B\}$) that the jurors will figure out accurately. This binary setting is similar to the continuous model. Then each juror who votes in position $i \in \{1, \ldots, n\}$ is allowed to see a_i cards drawn randomly from the remaining deck of $D-1$ cards, which we will call the *card pool* from now on. Denoted by $\Lambda = \{1, 2, \ldots, D-1\}$, the set of juror *abilities*, where $D-1$ is the size of the card pool. A juror with ability a suggests that when it is his round of voting, he draws a cards from the card pool. An *ability set* $A \subseteq \Lambda$ is a set of n distinct abilities, considered as an *unordered jury*. In other words, no two jurors have exactly the same ability. Thus the set $\mathcal{A}_n$ of all unordered juries of size n is as follows:

$$\mathcal{A}_n = \{A \subseteq \Lambda : |A| = n\},$$

where we use $|A|$ to denote the cardinality of set A. We list the abilities of an unordered jury in increasing order for convenience, such as $A = \{4, 5, 6\}$. Let Π_n denote the set of all $n!$ rank orderings (permutations) of $\{1, \ldots, n\}$. A *jury* J of size n is an *ordered* set of n distinct abilities. Let $\mathcal{J}_n$ denotes the set of all juries of size n. There is a natural way to apply an ordering $r \in \Pi_n$ to an ability set $A \in \mathcal{A}_n$ to obtain a jury J. For example, if $n = 3$, $r = (2, 1, 3)$ and $A = \{5, 7, 9\}$, then we can write $r(A) = (7, 5, 9)$.In other words, the elements of set A are written in the rank order of r. Thus we can think of each rank order r as a mapping of ability sets into juries, $r : \mathcal{A}_n \to \mathcal{J}_n$. In this work, we only consider the sealed card problem under the roll-call voting scheme using simple majority rule.

2.2.3 Signal distributions

Continuous model

The underlying assumption is that there are two states of Nature: A and B. Each juror has private information about the state of Nature. In this model, this information is a continuous signal s in the interval $[-1, +1]$. When juror receives positive signals, he believes that state A is more likely. Likewise, negative signals are indications of state B. The neutral signal is simply $s = 0$. The intensity of the signal depends on the absolute value of the signal.

The ability of each juror lies in the interval $[0, 1]$. When the juror's ability is close to 1, he generally ascertains the state of Nature better. In classic the Condorcet model, the ability p is defined as a *posteriori* probability with binary signal of state A indicating the state of Nature A. In this model, we define the signal s in a way such that the conditional probability of the state A is a linear function of juror's ability a. We denote $f_A(a, s)$ ($f_B(a, s)$) as the probability density function (PDF) of the juror when the state of Nature is A (B), given that his ability is a and his private information (signal) is $s \in [-1, 1]$. We make the simplest assumption about this PDF: it is linear function with the slope a linear function of the ability a. Equations (2.3) and (2.4) provide the forms of the density functions on signal $s \in [-1, 1]$:

When Nature is state A,

$$f_A(a, s) = \frac{1 + as}{2}, \ s \in [-1, 1]; \tag{2.3}$$

When Nature is state B,

$$f_B(a, s) = \frac{1 - as}{2}, \ s \in [-1, 1]. \tag{2.4}$$

$f_A(.)$ and $f_B(.)$ satisfy the requirements for the PDF. For any $a \in [0, 1]$, the integral on the whole interval $[-1, +1]$ is 1. For the trivial case, when the ability of the juror is equal to 0, both $f_A(0, s) = f_B(0, s) = 1/2$ (indicating that the juror is no more effective than tossing a fair coin to guess the state of Nature among two states) on the interval $[-1, +1]$. The corresponding cumulative distribution functions of the signal s when the Nature lies in state A or state B are given by the equation (2.5):
When Nature is state A

$$F_A\left(a, s\right) = \frac{(s+1)(as-a+2)}{4}, \; s \in [-1, 1]. \tag{2.5}$$

Similarly, when Nature is state B;

$$F_B\left(a, s\right) = \frac{(s+1)(a-as+2)}{4}, \; s \in [-1, 1]. \tag{2.6}$$

After knowing his private signal, the juror is interested in understanding the probability that he is in state A or state B. That is, based on his private signal and the previous voting, the likelihood that the true state of Nature is A or B. The conditional probability of state A can be computed using *a posteriori* probability and signal based on Bayes' Law:

$$\Pr[A|s] = \frac{\theta f_A\left(a, s\right)}{\theta f_A\left(a, s\right) + (1-\theta) f_B\left(a, s\right)} = \frac{\theta + as\theta}{2as\theta - as + 1}.$$

Under the neutral conditional $\theta_0 = 1/2$, the $\Pr[A|s]$ for the first juror can be further simplified:

$$\Pr[A|s] = \frac{\theta_0 + as\theta_0}{2as\theta_0 - as + 1} = \frac{as+1}{2}.$$

The trivial case is $a = 0$, we will obtain $\Pr[A|s] = 1/2$. The conditional probability of state A is the same as the neutral setting indicating for no information updating during the whole process. We can easily convert our definition of the ability to the classic definition by Condorcet using the integral:

$$p_{Condorcet} = \int_0^1 f_A\left(a, s\right) ds = 1 - F_A\left(a, 0\right) = \frac{1}{2} + \frac{1}{4}a.$$

This is the same as the equation (2.2). It is easy to check that our definition of ability will lead the minimal probability of being correct for each juror is a half when $a = 0$.

The conditional probability formula (2.1) can be derived from $p_{Condorcet}$ as well:

$$\begin{aligned}\Pr[A|s \geq \tau] =& \frac{\theta_0(1 - F_A(a,\tau))}{\theta_0(1 - F_A(a,\tau)) + (1-\theta_0)(1 - F_B(a,\tau))} \\ =& \frac{1}{2} + \frac{1+\tau}{4}a.\end{aligned}$$

Discrete model

For the sealed card problem, given ability a, the probability of randomly sampling X red cards out of a total m red cards in the card pool is as follows:

$$\Pr[X|m,a] = \frac{\binom{m}{X}\binom{m-1}{a-X}}{\binom{2m-1}{a}}. \tag{2.7}$$

In other words, the random variable X follows a hypergeometric distribution. On the basis of the number of red and black cards in his sample and votes of the previous juror, juror i in this round adopts Bayes' Law to determine the more likely state of Nature. If R and B are equally probable, the juror randomizes his vote. He then votes this way (R or B) and returns his sample to the card pool. After the last juror cast his vote, the collective decision (the verdict of the colour) is determined by majority voting, denoted by V, in the odd voting order. We intend to make a comparison among all the voting orders of the jurors by given abilities with the intention of maximizing the possibility (which we called the *reliability*) that the verdict $V \in \{R, B\}$ is correct. For example, if $n = 3$ and the abilities are different, say 1, 2, 3, there are $3! = 6$ ability orders to be taken into account.

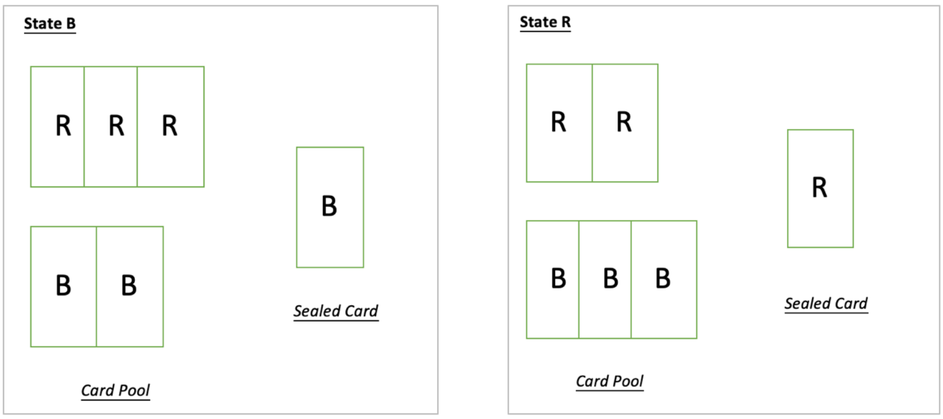


Figure 2.2: The two states of Nature, $D = 6$

Consider, for instance, a deck of $D = 6$ cards, with the two states of Nature illustrated in Figure 2.2. If juror $i = 1$ with ability $a_1 = 1$ samples a red card, he possesses the knowledge that he is more likely in State B and consequently votes B. Suppose juror $i = 2$ with ability $a_2 = 4$ obtains a sample with two red and two black cards. He would still think that these two states of Nature are still equally likely based solely on his private information, and would vote randomly, providing that he votes first. Nevertheless, if he possesses the knowledge that the first juror voted B with the ability of $a_1 = 1$, this information raises the conditional probability that Nature is B to above one half, and he would also vote B. This kind of imitative behaviour is a typical heading. Providing that there are three black cards in the second juror's sample, he would vote R.

2.2.4 Thresholds

In this section, we will discuss the threshold for each juror. Firstly, the definition of the threshold will be provided. Then, the threshold's determination process will be presented, given the assumption that voting behaviour is honest. A vote is said to be *honest* if every juror votes for the most likely alternative, given the information offered by his signal, previous voting, and the *a priori* probability of the two states of Nature. Two models here assume that each juror votes honestly. Following the classic definition of game theory, we define the juror's strategy as the threshold τ. This threshold depends on his private information signal s or the number of red card in the sample in the sealed card problem) and all the available information (the updated probabilities of the two states and the votes of the previous voters). This threshold plays the role of the decision rule for individual jurors. When the signal s (or the number of red card) is higher than or equal to τ, he votes for state A (or R). The formal definition of τ will be introduced in Section 2.2.4 and determined by equation (2.9). The collection of all the jurors involved in this voting game is called a strategy profile.

The specific formulas for jurors under different sequential voting schemes are distinct. Details of the thresholds under different voting schemes will also be elaborated on in the corresponding chapters. Here, we will consider the simplest voting structure under the continuous signal model. Then an analysis of threshold in the discrete model will be provided.

Decision rule and thresholds in the continuous model

We define the threshold as the minimal signal s required for the juror to vote for state A based on the given votes of the previous juror(s) (θ). Under honest voting, we can easily calculate the honest threshold for each juror using θ. For the first juror with ability a given the *a priori* probability θ_0, he just needs solve the equality (2.8) for signal s,

$$\Pr[A|\theta_0] = \frac{\theta_0 f_A\,(a,s)}{\theta_0 f_A\,(a,s) + (1-\theta_0) f_B\,(a,s)} = \frac{\theta_0 + as\theta_0}{2as\theta_0 - as + 1} = \frac{1}{2}. \tag{2.8}$$

By solving s, we have $\tau = (1-2\theta_0)/a$. There are two trivial cases for the thresholds as the range for the signal is $[-1,+1]$. If we have $\tau = (1-2\theta_0)/a$ is larger than the unit, the juror will always vote for state B. Similarly, when $\tau = (1-2\theta_0)/a$ is less than negative unit, the juror will always vote for state B. These two extreme cases of τ are the mathematical expression of herding behaviours, under which circumstances the juror is completely blind about his private information and follows the votes of previous jurors. Although we may not have the value for ability $a = 0$, we can always take the right-hand limit of zero for it. Now we have the explicit formula for the first juror's honest threshold given his ability a and the *a priori* probability θ_0.

$$\tau_a(\theta_0) = \begin{cases} 0, & \text{if } \theta_0 = 1/2; \\ -1, & \text{if } \ a < 2\theta_0 - 1 \text{ and } \ \theta_0 > 1/2; \\ +1, & \text{if } \ a < 2\theta_0 - 1 \text{ and } \ \theta_0 > 1/2; \\ (1-2\theta_0)/a, & \text{otherwise.} \end{cases} \tag{2.9}$$

The exact determination of τ for the juror n given the voting behaviour of the previous juror(s) $(1, 2, 3, ..., n-1)$ is different for different sequential voting schemes. However, we can always use equation (2.9) by substituting θ_0 with θ.

In summary, τ is determined by θ, and θ contains all the information gained by observing the behaviour of the previous juror. The following will provide a simple example of how to calculate τ for the second juror under the simplest voting scheme in sequential voting schemes.

Threshold under different voting schemes in the continuous model

As shown in Section 2.1, this thesis mainly concerns schemes (b), (c) and (d) or sequential voting with an independent voter (type I), scheme (f) or sequential voting with knowledge of the previous voter (type II), and scheme (g) or sequential voting with an initial public vote (type III). Although different voting schemes have different information aggregation processes, the sequential voting part includes at least two layers. We call this duo structure. Therefore, we will start with an analysis of this duo structure.

Duo structure

Figure 2.3 is a visual representation of this basic structure of sequential voting. One

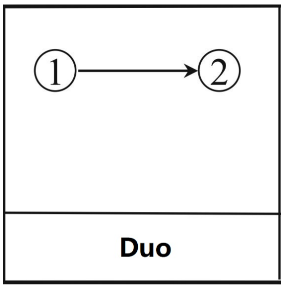


Figure 2.3: Duo structure in the sequential voting

juror votes first and passes his vote to the second juror (indicated by the arrow). In this duo sequential structure, the juror in the first layer acts as an independent voter. His only information that is the states of Nature are equally likely, just as with any independent juror. From the equation (2.9), it is clear that the threshold for the first juror is $x = 0$ given that his a *priori* probability $\theta_0 = 1/2$. The *a priori* probability for the second juror can be calculated using the Bayes' Law,

$$\theta(A) = \frac{1 - F_A(0)}{(1 - F_A(0)) + (1 - G_A(0))} = \frac{2 + a}{4}. \tag{2.10}$$

When $\theta > 1/2$, the state A is more likely. From equation (2.10), any positive value of a will result in favouring for state A. Similarly, if the first juror votes for state B, then a *priori* probability for the second juror is

$$\theta(B) = \frac{2 - a}{4}. \tag{2.11}$$

When $\theta < 1/2$, the state B is more likely. From equation (2.11), any positive value of a will result in favouring for state B. Based on equations (2.9) (2.10), and (2.11), the honest threshold y_A for the juror in the second layer is:

$$y_A = y_A(a, b) = \begin{cases} -1, & \text{if } b \leqslant a/2 \text{ ;} \\ -a/(2b), & \text{otherwise.} \end{cases} \tag{2.12}$$

By symmetry, we have

$$y_B = y_B(a, b) = -y_A(a, b). \tag{2.13}$$

From the honest threshold equation (2.12) for the second juror, it may be noted that if the ability of the second layer juror is small enough that $b \leqslant a/2$, then he will copy the vote of the previous juror, neglecting all his private information (signal s) and other public information. Therefore, we call this condition *herding* for the second juror. This condition composes the following voting ordering set (a, b):

$$h_2 = \{(a, b) \in [0, 1]^2 : b \leqslant a/2\} \tag{2.14}$$

When this herding occurs (satisfying h_2) with a jury of three ($n = 3$), the verdict is determined. There is no need for the independent juror in type I, the third juror in type II or the other juror in the second layer in type III to cast his vote. For a jury of three, only sequential voting with knowledge of the previous juror (type II) has third layer. The threshold for the third juror under type II will be explicated in Chapter 5.

Voting threshold in the discrete model

The above discussion centres around the continuous model. Now, let us come back to the sealed card problem. When it is the juror's round to vote before he knows that specific contents of his sample, there is a probability θ, conditional on votes of the previous juror(s) that the sealed card is red. We call this probability the *public bias.* We call it public as it is conditioned only on public information (voting) and not private information (cards in the juror's sample). The first juror is estimated at $\theta = 1/2$, or $\theta_1 = 1/2$ to indicate it is the first juror's public bias. It is $1/2$ because, as the primary deck has an equal number of red and black cards, the randomly chosen sealed card has a one-half probability of being red. After juror i votes R, the bias θ_{i+1} moves up (from θ_i) for the next juror $i + 1$; if he votes B, it moves down. The change from θ_i to θ_{i+1} does not depend directly on the cards drawn by juror i, as this is not known by juror $i + 1$. The precise method where θ changes after casting a vote is determined by Bayes' Law. We note that the public bias θ_{i+1} could be calculated by anyone (not necessarily a juror) who knows the abilities of the first i jurors and how they voted.

Next, suppose it is the turn of a juror with ability a to vote given that the public bias is θ. By the number X of red cards in his sample of a cards, the probability of $N = R$ (sealed card red) is either $> 1/2$, $< 1/2$ or in rare cases $= 1/2$. The corresponding vote will be to vote B, R, or randomize equiprobably. Before seeing his sample of cards he can calculate a threshold τ such that if $X > \tau$ the probability of B is $> 1/2$; if $X < \tau$ the probability of B is $< 1/2$; if $X = \tau$ the probability of B is $1/2$. His corresponding votes will be B, R, or to randomize. Note that $\tau = 3.1$ and $\tau = 3.2$ are equal thresholds because they generate the same votes. For convenience, we can take $\tau = 3.5$ in such cases for uniqueness. So generally $\tau = (k + 1)/2$ for a non-negative integer k. However, if, say, $X = 4$ results in equiprobable R and B, and thus randomized voting, we say that $\tau = 4$. If $\tau = a + 1/2$, with a denoting his ability, the juror will certainly vote R because he cannot have X more than a. In such a case ($\tau > a$) we call this situation herding – the juror can cast his vote without even looking at his sample. Similarly if $\tau < 0$, the juror casts his vote B with certainty. Herding might be transitory in our model: one juror might herd while the next juror (with a higher ability) might not. Given that we admit jurors without ability $a = 0$, they would always herd. In general, herding is less likely for jurors with higher ability.

2.2.5 Reliabilities

Continuous model

For every jury $J \in \mathcal{J}$, we define its *reliability* $Q(J) \in [0, 1]$ as the probability that the jury gets the verdict correct, say verdict A when $N = A$. Thus $Q : \mathcal{J} \to [0, 1]$ is a map from juries to probabilities. In other words, the reliability Q is the probability of a correct majority verdict under a certain voting scheme with a certain voting order. We will use the simultaneous voting scheme as a starter. In the following illustrative examples, there are three jurors with ability a,b and c in voting order (a, b, c). We denote the reliability $Q_{simultaneous}(a, b, c)$ for the simultaneous voting scheme and $Q(a, b, c)$ for the roll-call voting scheme with order (a, b, c). Given that the Nature is in state A, three voting orders $(A, A, \text{ any})$, (A, B, A), and (B, A, A) will lead to a correct verdict. Note that if the first two jurors vote for the same state, the verdict has already been reached by majority rule, no matter how the last juror votes. Similarly, the three voting sequences (B, B, any), (B, A, B) and (A, B, B) lead to the correct verdict if the state of Nature is B. The formula for $q_A(a, b, c)$ is the summation of probabilities of the voting patterns $(A, A, \text{ any})$, (A, B, A) and (B, A, A) when the Nature is A. Here, the thresholds for all three jurors are just 0 when the state of Nature is equally likely. The following part shows the calculation of the reliability $Q(a, b, c)$. Let $q_A(a, b, c)$ be the probability of the verdict being A when the Nature is A. Similarly, we define $q_B(a, b, c)$. Then, given an arbitrary *a priori* probability θ (the probability of state A before voting), we have:

$$Q(a, b, c) = \theta q_A(a, b, c) + (1 - \theta) q_B(a, b, c).$$

Because of the neutral settings $\theta = 1/2$, then,

$$Q(a, b, c) = \frac{1}{2}(q_A(a, b, c) + q_B(a, b, c)).$$

By symmetry,

$$Q(a, b, c) = q_A(a, b, c) = q_B(a, b, c).$$

Now we have all the ingredients to calculate the reliability of the simultaneous voting scheme.

$$\begin{aligned} q_A(a,b,c)_{Simultaneous} &= (1-F_A(a,0))(1-F_A(b,0)) + (1-F_A(a,0))F_A(b,0) \\ &\quad (1-F_A(c,0)) + F_A(a,0)(1-F_A(b,0))(1-F_A(c,0)) \\ &= a/8 + b/8 + c/8 - (abc)/32 + 1/2. \end{aligned}$$

In the following part, we will evaluate the reliability for a jury of size $n = 3$ under the roll-call voting scheme. Under roll-call voting, the first juror with ability a votes first, followed by the juror with ability b, then the juror with ability c. The first juror's threshold is simply 0, as he is under the neutral setting. From the second juror, the threshold depends on the previous juror's vote. Here, we denote the y and z as the second and third juror thresholds respectively. The subscript indicates the vote for the previous juror. For examples, y_A means that the first juror votes for state A and z_{AB} means that the first juror votes for state A and the second juror votes for the state B. Given all the thresholds, we can derive the formula for the $q_A(a,b,c)$,

$$\begin{aligned} q_A(a,b,c)_{roll-call} =& (1-F_A(a,0))(1-F_A(b,y_A)) + (1-F_A(a,0))F_A(b,y_A) \\ & (1-F_A(c,z_{AB})) + F_A(a,0)(1-F_A(b,y_B))(1-F_A(c,z_{BA})). \end{aligned}$$

Due to the different sequential voting structures, the formula for reliability Q varies from scheme to scheme. $q_A(a,b,c)_{Simultaneous}$ and $q_A(a,b,c)_{roll-call}$ are just two illustrative examples. Details of the reliabilities under each voting scheme will be presented in the corresponding chapters.

Discrete model

Similarly, we will define the reliability of the sealed card problem here. Then we will introduce other measures of voting performance, followed by the formula for reliability. For every jury $J \in \mathcal{J}_n$, we define its *reliability* $Q(J) \in [0,1]$ as the chance that the jury gets the verdict correctly, that is, verdict $V = R$ if Nature is R (the same as the continuous model). Thus, $Q : \mathcal{J}_n \to [0,1]$ is a mapping from juries to probabilities. The problem we intend to solve is which voting order can obtain highest reliability given an ability set. For every rank ordering $r^* \in \Pi_n$, we define its *optimality fraction* $\phi(r^*)$ as the fraction of ability sets $A \in \mathcal{A}_n$ for which it provides the highest reliability, that is, $Q(r^*(A)) \geq Q(r(A))$ for all $r \in \Pi_n$. Note that this function ϕ depends on the deck size. More accurately,

$$\phi(r^*) = \frac{|\{A \in \mathcal{A}_n : Q(r^*(A)) \geq Q(r(A)) \text{ for all } r \in \Pi_n\}|}{|\mathcal{A}_n|}. \tag{2.15}$$

For $r \in \Pi_n$, we define its *average reliability* $\bar{Q}(r)$ as a measure of how reliable this voting order is for a random set of abilities. More accurately, $\bar{Q}(r)$ is the average value of $Q(r(A))$ for ability sets $A \in \mathcal{A}_n$. It is natural to assume a uniform probability distribution on $\mathcal{A}_n$. Thus, $\bar{Q} : \Pi_n \to [0,1]$ is a mapping from rank orderings to probabilities. In the continuous model, we also use the discretized ability sets to calculate the average reliability. Of course, later research can work on other distributions based on the data in some specific areas.

In the following part, we derive formulas for the reliability of a jury of size $n = 3$. Similar analysis has been conducted in the continuous model. Here, we consider state A in the continuous model as corresponding to state R in the sealed card problem. For every jury $J \in \mathcal{J}_n$ of public bias of θ, the reliability $Q(J)$ is calculated as follows:

$$\begin{aligned} Q(a,b,c) = \theta(&\Pr[RR] + \Pr[RBR] + \Pr[BRR]) \\ &+ (1-\theta)(\Pr[BB] + \Pr[BRB] + \Pr[RBB]). \end{aligned}$$

We use the hypergeometric probability represented in equation (2.7) and its cumulative distribution function (CDF) to compute the above probabilities. When τ is a half-integer, we introduce the floor function $floor$ taking the largest integer less than or equal to τ. Let integer-valued random variables Y and Z be defined as follows for all τ:

$$\begin{aligned} \Pr[Y \leq \tau | a] &= \sum_{i=0}^{floor(\tau)} \frac{\binom{m}{i}\binom{D-m}{a-i}}{\binom{D-1}{a}}, \\ \Pr[Z \leq \tau | a] &= \sum_{i=0}^{floor(\tau)} \frac{\binom{m-1}{i}\binom{D-m+1}{a-i}}{\binom{D-1}{a}}. \end{aligned}$$

Suppose the hidden card is black (i.e., $N = B$) and hence there are m red cards and $m-1$ black cards in the card pool. Then for a jury $J = (a,b,c)$, we can calculate the following probabilities, where $\tau(\sigma)$ denotes the threshold of the juror who has the prior voting sequence σ:

$$\begin{aligned}
\Pr[RR] &= (1 - \Pr[Y \leq \tau|a])(1 - \Pr[Y \leq \tau_r|b]), \\
\Pr[RBR] &= (1 - \Pr[Y \leq \tau|a]) \Pr[Y \leq \tau(R)|b](1 - \Pr[Y \leq \tau(RB)|c]), \\
\Pr[BRR] &= \Pr[Y \leq \tau|a](1 - \Pr[Y \leq \tau(B)|b])(1 - \Pr[Y \leq \tau(BR)|c]), \\
\Pr[BB] &= \Pr[Z \leq \tau|a] \Pr[Z \leq \tau(B)|b], \\
\Pr[BRB] &= \Pr[Z \leq \tau|a](1 - \Pr[Z \leq \tau(B)|b]) \Pr[Z \leq \tau(BR)|c], \\
\Pr[RBB] &= (1 - \Pr[Z \leq \tau|a]) \Pr[Z \leq \tau(B)|b] \Pr[Z \leq \tau(RB)|c].
\end{aligned}$$

There are some trivial cases. For instance, if all the jurors have ability $a = 0$, then the reliability $Q = 0.5$.

Chapter 3

The Sealed Card Problem

The sealed card problem is described as follows. A (sealed) card is randomly selected from a deck with evenly distributed cards (m red and m black). Each juror can draw a certain amount of cards equivalent to his ability and cast his vote for the color of the sealed card, on basis of his sample and the votes of prior juror(s). We have found that the Alpern-Chen order is with the highest reliability in most cases. This voting order generally is also better than secret ballot (simultaneous voting). When simultaneous voting is better, the abilities of the jurors have a tendency to be alike. An analogue of Alpern-Chen order also has high performance for larger juries in terms of reliability. Example 3.1 is an illustration to give a taste of this model and the significance of voting order under the roll-call voting scheme.

Example 3.1. Suppose there is a deck with $D = 4$ cards, two red and two black. There are three jurors, and juror $i = 1, 2, 3$ can draw i cards. Note that juror 3 who can sample all the remaining cards will possess the knowledge of the color of the sealed card. Therefore, if he votes first or second, the later jurors will copy him to generate a perfect majority verdict with the accuracy of 1. There are only two orderings for us to consider where juror 3 votes last. It will be easiest to calculate the probability from the counterexamples when the verdict is not correct. Suppose $N = R$, we calculate the possibility w with prior voting BB: Firstly, we consider the voting order $(1, 2, 3)$. Juror 1 votes B if he samples a red card, which has probability $1/3$. The only case where juror 2 will not copy is if he samples two black cards with probability $1/3$. Therefore, the possibility w is $(1/3)(2/3) = 2/9$ and the reliability is $7/9$.

Now, consider the voting order $(2, 1, 3)$. If the Juror 1 draws one red and one black, he will randomize. In other cases, he votes for R. Therefore, he votes B with probability $(2/3)(1/2) = 1/3$. If juror 2 selects a red, he votes B as well. Given that he samples a black, he votes R. The probability he samples a red is $1/3$; so the probability of BB is $(1/3)(1/3) = 1/9$ and consequently the reliability for the jury $(2, 1, 3)$ is $1 - 1/9 = 8/9$.

Example 3.1 has demonstrated the significance of voting order. In particular, it should be noted that the Alpern-Chen order $(2, 3, 1)$ is among the optimal orders.

3.1 Optimal voting orders for $n = 3$

The results on optimal voting order with the jury of size three will be presented as follows. Firstly, we start in Section 3.1.1 with the research on the smaller deck. In this section, every juror draws an even number of cards from the deck (even-only ability). In Section 3.1.2, we will show our findings on the optimality fractions ϕ. Section 3.1.3 studies optimal voting orderings regarding the average reliability $\bar{Q}$. Besides Section 3.1.1, the rest of this chapter conduct an analysis on the normal 52-card deck with full ability sets (both even and odd).

3.1.1 On reliability for small decks

In this subsection, we will present that when the deck size D satisfies $D \leq 16$ and all jurors are with even abilities, the Alpern-Chen order is with the highest reliability among all six orders. It should be noticed that given that the first juror has even ability, this setting will admit ties (the state of Nature is still evenly likely even after drawing sampling), where the juror randomizes his vote. The consequence of random voting has a tendency to smooth out some aspects of the reliability function and provide more consistent optimal reliability patterns. These findings are shown below.

Observation 3.2. *For three-member jury of even-only abilities and deck size $D \leq 16$, the Alpern-Chen ordering gives the strictly highest reliability.*

Many existing literatures discuss the superiority between seniority and anti-seniority order. For small decks, we are able to provide a definitive answer to this relation.

Observation 3.3. *For three-member jury of even-only abilities with $a < b < c$ and deck size $D \leq 16$, the seniority ordering (c, b, a) has the higher reliability than the anti-seniority ordering (a, b, c).*

Observation 3.4. *Suppose D satisfies $D \leq 16$, and we have a three-member jury of even abilities x, a, b, with $a < b$. Then the ordering (x, b, a) has a higher reliability than (x, a, b). That is, it is always better for the last two jurors to vote in decreasing order of ability.*

Note that Observation 3.4 can be applied to the situation where the first voter is already appointed by external factors (for instance most often in the board of a listed company, the chairman (not necessarily with highest ability) voice his opinion in the first place). Next, the member with the lower ability between the remaining two is supposed to cast his vote lastly.

3.1.2 On optimality fraction

In this subsection, the optimality fraction for the six orders for the jury of size three is presented. Recall that this is the fraction of the latent ability sets for which the order provides the top reliability.

Optimality fraction						
Deck size / Voting Orders	$D = 6$	8	...	26	...	52
(a, b, c)	0/10=0.0	0.029	...	0.006	...	0.007
(a, c, b)	8/10=0.8	0.620	...	0.298	...	0.200
(b, a, c)	0/10=0.0	0.000	...	0.000	...	0.000
(b,c,a)	**9/10=0.9**	**0.800**	...	**0.652**	...	**0.630**
(c, a, b)	6/10=0.6	0.429	...	0.167	...	0.148
(c, b, a)	6/10=0.6	0.486	...	0.305	...	0.253
Sum	2.9	2.371	...	1.427	...	1.238

Table 3.1: Optimality fraction for deck size up to 52, $(a < b < c)$

Table 3.1 summarizes results on the performance of six voting orders respect to their optimality fraction ϕ for decks of sizes range from 6 to 52 (for complete findings the reader is referred to Table B.2 in the Appendix). To provide taste of our results, observe Table 3.2, which matches the case of deck size $D = 6$. Note that for $D = 6$, there are $\binom{6-1}{3} = 10$ ability sets:

$$\begin{aligned}\mathcal{A}_3 = \{&\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\\ &\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\}.\end{aligned}$$

Among these ten ability sets, the Alpern-Chen order (b, c, a) dominates on nine of the ten ability sets. Table 3.2 illustrates how the records for the column $D = 6$ in Table 3.1 have been determined. For each order (column), one compares the rows (ability sets) where that order is optimal. The bottom row provides the fraction of the ten ability sets where the order is optimal. This is by our definition its optimality fraction. It should be noticed that this fraction admits the value, which is higher than one because we allow ties in reliability.

Ability sets \ Orderings	(a,b,c)	(a,c,b)	(b,a,c)	(b,c,a)	(c,a,b)	(c,b,a)
{1,2,3}	0	✓	0	0	0	0
{1,2,4}	0	✓	0	✓	0	0
{1,2,5}	0	✓	0	✓	✓	✓
{1,3,4}	0	0	0	✓	0	0
{1,3,5}	0	✓	0	✓	✓	✓
{1,4,5}	0	✓	0	✓	✓	✓
{2,3,4}	0	0	0	✓	0	0
{2,3,5}	0	✓	0	✓	✓	✓
{2,4,5}	0	✓	0	✓	✓	✓
{3,4,5}	0	✓	0	✓	✓	✓
Counts	0	8	0	9	6	6
Fraction	0	0.8	0	0.9	0.6	0.6

Table 3.2: Optimality fraction for deck size $D = 6$, $(a < b < c)$

Moreover, Figure 3.1 make a comparison for the optimality fractions with all deck sizes, in which the vertical axis is the optimality fraction, and the horizontal axis is the deck size. We have the following result:

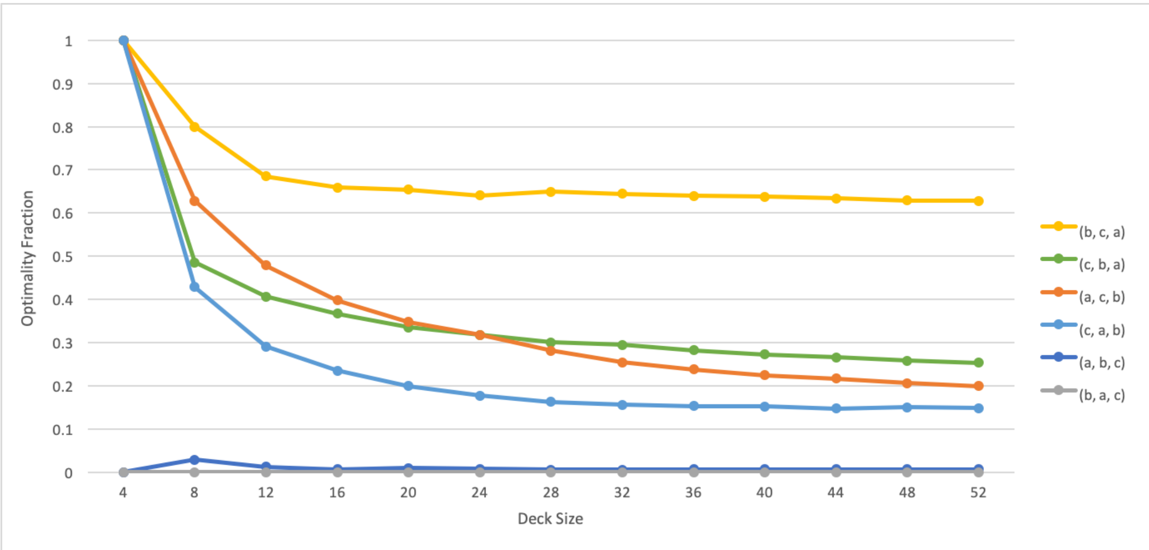


Figure 3.1: Optimality fractions of the six voting orderings

Observation 3.5. *Given any deck of size $D \leq 52$, the Alpern-Chen ordering (b, c, a) has the highest optimality fraction.*

It should be noticed that order (b, a, c) is never optimal. Above Observations are statements on the cases with a finite number. The exhaustive analysis is used as the "proofs" for these cases.

3.1.3 On average reliability

Given a deck of size D with jury size three, without any knowledge of the abilities for all jurors, we intend to figure out which of the six voting orders is with the highest average reliability $\bar{Q}$. That is defined as the average value of reliabilities for the same voting order divided by all possible ability sets under any deck size less than or equivalent to 52.

For every voting order r, the average reliability is calculated within the range from the smallest deck size $D = 4$ until the largest $D = 52$, via exhaustive search approach over all possible ability sets. Figure 3.2 shows the results for these deck sizes.

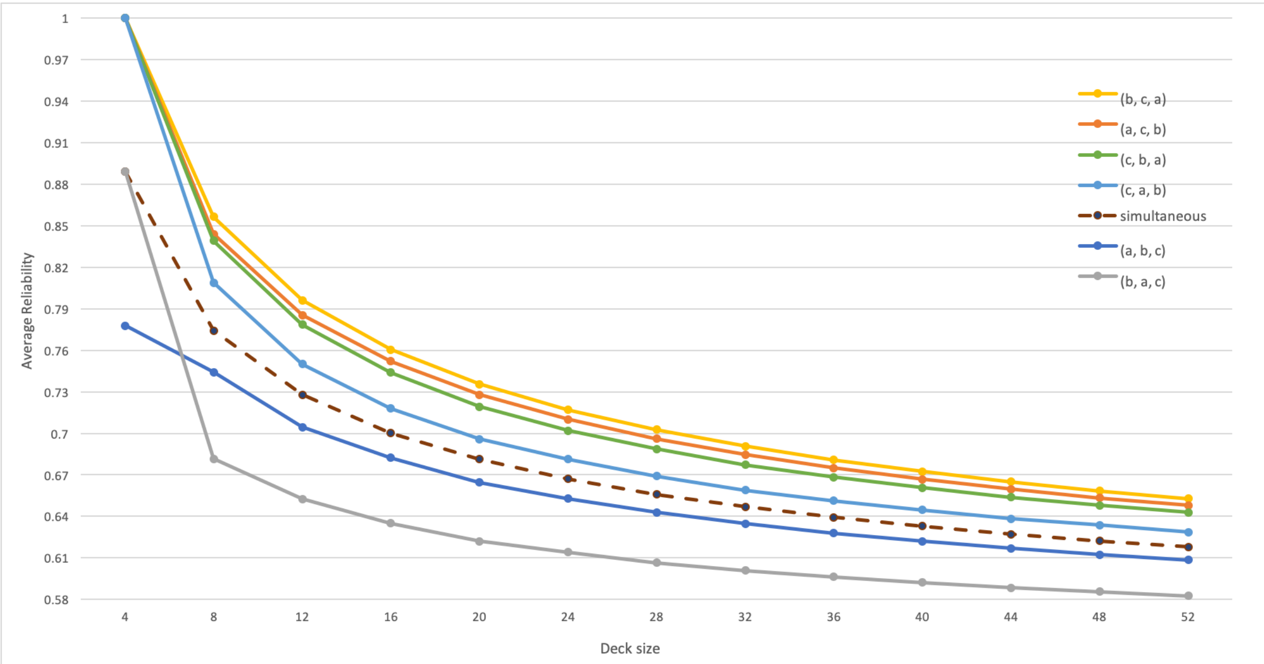


Figure 3.2: Average reliability of the six voting orderings

In Figure 3.2, the legend on the right side demonstrates that the six voting orders listed from the highest average reliability to the lowest. Table 3.3 provides several cases under different deck sizes . The full data is provided in Table B.3 in Appendix.

Average reliability								
Voting order \ Deck size	$D = 4$	6	8	...	26	...	50	52
(a, b, c)	$7/9 = 0.78$	0.768	0.744	...	0.648	...	0.610	0.608
(a, c, b)	1	0.901	0.844	...	0.703	...	0.651	0.648
(b, a, c)	$8/9= 0.89$	0.703	0.682	...	0.609	...	0.584	0.582
(b, c, a)	1	0.910	0.856	...	0.710	...	0.655	0.653
(c, a, b)	1	0.867	0.809	...	0.673	...	0.631	0.628
(c, b, a)	1	0.890	0.839	...	0.695	...	0.645	0.643

Table 3.3: Average reliabilities for deck size up to 52

Given that deck size $D = 4$, there exists merely one ability set $\{1, 2, 3\}$. Therefore, the average reliability $\bar{Q}$ and reliability Q has the same value for this particular ability set. The reliabilities for this ability set has already calculated in Example 3.1. Observation 3.6 summarises the pattern from Table 3.3 and Figure 3.2.

Observation 3.6. *Given any deck of size $D \leq 52$, the Alpern-Chen ordering (b, c, a) has an average reliability larger than any other voting order.*

3.2 Sequential vs. simultaneous voting scheme

Up to now, the analysis is around roll-call voting scheme from a *internal* view, comparing the reliabilities of six sequential voting orders for a fixed set of abilities. In this section, the analysis on sequential voting is from a *external* view, by measuring the performance (reliability) with the more traditional *simultaneous voting scheme*. This voting approach has been extensively studied by Condorcet (1785), under which every juror cast a vote on the basis of merely on his own private information (his sample drawn from the remaining deck, here), without knowing any information of other jurors. Due to this feature, this voting scheme is also knowns as independent voting. We make a comparison of the reliability of roll-call voting via the optimal (Alpern-Chen) order and that of simultaneous voting. The intuitive reasoning may conclude that sequential voting is *always* better because jurors are better informed in this case. However, as demonstrated in this section, indeed roll-call voting has *generally* better performance, but not *always*. The obstacle with roll-call voting, particularly for a large jury, is that a high ability juror who is not a perfect predictor (sometimes makes the wrong judgements) may improperly have an effect on subsequent jurors through herding. Then, their information is not helpful for the collective decision at all.

To demontrate this reasoning in a simple setting but with a larger jury, suppose that there is a deck of size D, one juror of ability 3 and all the rest ($n-1$ of them) of ability 1. Suppose the juror of ability 3 votes first. He is with some probability $p = p_3$ that at least two of the cards from his sample are the opposite colour to the sealed card, and that he consequently casts his vote accurately. Essentially, this probability is the same as the juror's probability of being correct p. Then, all subsequent juror, who obtains a signal that is right with a probability $p_1 < p_3$, will follow voting behaviour of the first juror and consequently the collective decision by majority voting, for any jury size n, will be right with probability $p = p_3$). This is when the first juror makes the right decision. Known from the Condorcet Jury Theorem under the setting of Owen et al. (1989), all jurors are with p which has the value at least p_1), then for sufficiently large jury size n, the probability of the correct verdict(reliability) approaches to 1, will be larger than that of reliability p_3 of the roll-call voting. For sufficiently large juries, the Condorcet Jury Theorem guarantees that independent voting is better than roll-call voting.

Go back to the typical small jury which this study mainly concerns. To make a comparison of these two voting schemes with jury size three, we start the analysis via recalling Example 3.1, under which suppose a deck of two red and two black cards, and three jurors with abilities 1, 2, and 3 takes turns to vote. In the case, the juror of ability 3 (called juror 3) casts his vote firstly or secondly. Then his perfect vote will be followed by subsequent two jurors to generate a perfect collective decision by majority voting (reliability is 1). The other voting orders $(1, 2, 3)$ and $(2, 1, 3)$ were presented in Example 3.1 with reliabilities 7/9 and 8/9, respectively. Note that roll-call voting in the optimal Alpern-Chen ordering $(2, 3, 1)$ is with reliability 1. Then, we analyze the reliability of simultaneous (independent) voting. By symmetry, we could make an assumption that the sealed card is R. We wonder the probability that the verdict is not correct, B. Because the juror with ability 3 is a perfect predictor whose sample is two blacks and a red and results in the right vote of R. Therefore, the wrong verdict B will be generated given that the other two juror votes wrongly. The juror with ability 1 will cast his vote B if and only if he has one red card in his sample with probability $1/3$. The juror with ability 2 will his vote B if he has one red and one black card in his sample and then randomizes to vote B. This case is with probability $(2/3)\ (1/2) = 1/3$. Therefore, the reliability of simultaneous voting under this setting is $1 - (1/3)^2 = 8/9$. This value is inferior to the perfect reliability 1 of roll-call voting in the optimal (Alpern-Chen) ordering. Note that simultaneous voting in this example is superior to roll-call voting with the ability order $(2, 1, 3)$.

Here is an illustration when the relative reliability of these two voting schemes can be another story: Given that three jurors have the common ability 1, simultaneous voting is superior to roll-call voting. For clarity of the proof, we make the assumption that all jurors vote in the precisely same order under both voting schemes. However, in the roll-call voting scheme, every juror possesses the knowledge of all prior jurors' votes. We denote *bl* and *re* for the colour in the jurors' sample, while B and R for votes he casts. For instance, we would say 'the second juror sees *re* in his sample but he casts his votes R as he allows the voting behaviour of the first juror'. The trivial ability set is used in this case with all jurors having the same ability 1. There is merely one card in the sample of every juror. Here, we consider the extreme case where the abilities of jurors are homogeneous. It should be noticed that the proof of the following findings is without any calculating but qualitative analysis.

Theorem 3.7. *For any deck size* $D \geq 4$ *and ability set* $\{1, 1, 1\}$, *simultaneous voting has a higher reliability than sequential voting. Furthermore, for any total private information (draws of card for all players), simultaneous voting is at least as likely as sequential voting to have the correct verdict* (N) ; *for some total private information vectors, it is more likely.*

Proof. We prove the second statement, which indicates the first statement. We start with dividing the private information vector into three circumstances. In the first two circumstances, two voting schemes will results in the same collective decision. Under the rest circumstance, simultaneous voting will always provide the collective decision that has higher probability be true. On the other hand, the roll-call voting wiil generate the collective decision with lower probability to be true.

Case 1: draws re,re,_ (or bl,bl,_). By symmetry, we make the assumption that there is merely an *re.* in the first jurors' sample. When they vote simultaneously, the first two jurors will cast their votes BB (with verdict $V = B$). The same is still true, given that the second juror possesses the knowledge of the first juror's vote. Therefore, under these circumstances, two voting schemes will provide the same collective decision.

Case 2: re,bl,re (or bl,re,bl). By symmetry, we make the assumption that the cards in samples of the three jurors are re, bl, re. When they vote simultaneously, voting will be BRB, and the collective decision is $V = B$. When they take turns to vote, and the first juror cast his vote B. Then the second juror with the knowledge that one black and one red card in their samples will randomize and equally likely cast his vote R or B. Under both circumstances, the third juror will cast his vote B. Therefore, the voting order and process are different under different circumstances. The collective decision is the same.

Case 3: re,bl,bl (or bl,re,re). Again we assume re, bl, bl by symmetry. It should be noticed that the sealed card may be $N = R$. This more likely state of Nature R is the verdict when jurors vote simultaneously with B, R, R. Next we consider roll-call voting. The first juror casts his vote B and the second juror randomizes between R and B. He casts his vote B, which provides collective decision $V = B$. with some positive probability. Therefore, we will obtain the collective B, with positive probability where the collective decision is more likely to be wrong. Therefore, with these samples, the jury's collective decision with roll-call voting is more likely to be wrong than the jury with simultaneous voting.

This case analysis is the second part of the proof because these three cases cover full possibilities. The reliability of every voting scheme can be calculated through the Law of Total Probability. The comparison asserted above for conditional reliability shows the claim for reliability in the first sentence.

We will also show the proof with the algebraic calculation: When $p = (D/2)/(D-1)$, the number of cards of the other colour of the sealed card divided by the size of the remaining deck, the reliability of sequential voting is calculated through $\left(p + 3p^2 - 2p^3\right)/2$ and the reliability for the simultaneous voting is $p^3 + 3p^2(1-p)$. The difference between these two voting scheme:

$$\left(p + 3p^2 - 2p^3\right)/2 - \left(p^3 + 3p^2(1-p)\right) = (1/2)\ p\ (2p-1)\ (p-1),$$

which has a positive value when $1/2 < p < 1$, for any value of D. □

Now we move to the more general case of the perfectly homogeneous juries (k, k, k), the reader might think that simultaneous voting is always better than roll-call voting in terms of reliability. Here is a counterexample.

Observation 3.8. *Suppose we have a deck of $D = 4$ cards (two red, two black) and three jurors of common ability 2. Then sequential voting has a higher reliability ($21/27$) than simultaneous voting ($20/27$).*

Proof. We start the analysis with roll-call voting. If the first two jurors vote the identical colour, call them Y, then the collective decision by majority voting is already generated. Suppose first two jurors vote for a different colour, say W, Y. The first vote W may result from a tie sample. However, given that the second juror also has a tie in his sample, he follows the first juror's vote and votes W. Therefore, the vote of Y is right with certainty, providing that two cards with identical colour indicate the sealed card must be the other colour (Y). This reasoning follows that the third juror will cast his vote Y. as well. Hence, the collective decision is dependent on the votes of the first two jurors. In this case, their positions are equivalent. Thus, the only case when the roll-call voting provides the wrong verdict is when the samples of the first two jurors both are distinct, the first juror randomizes with the wrong result and the second juror follows. The probability of distinct samples for first two jurors is 2/3, hence the probability of a wrong collective decision is $(2/3)^2\,(1/2) = 2/9$, hence the reliability of roll-call voting is $1 - 2/9 = 7/9 = 21/27$.

In simultaneous (independent) voting, the only situation a juror voting inaccurately is identical if he has a distinct colour of cards in his sample and has the incorrect result for his randomizing. This case is with probability $q = (2/3)\,(1/2) = 1/3$, so $p = 2/3$. Simultaneous voting with $p = 2/3$ is with reliability $p^3 + 3p^2q = 20/27$, the probability that two or three jurors receive the accurate signal (the other colour rather than the that of the sealed card). □

According to the Condorcet Jury Theorem where simultaneous juries' reliability approach to one when the size of jury approaches infinity, herding may result in a limiting upper bound of reliability of roll-call voting which is less than one. For juries of size three, we have demonstrated that the ability set $\{1, 2, 3\}$ of Example 3.1 is with roll-call reliability higher than simultaneous. For perfectly homogeneous juries $\{k, k, k\}$, the superiority of reliabilities is inconclusive:simultaneous voting better for $k = 1$ but roll-call voting better for $k = 2$.

In the rest of this subsection, we will demonstrate that roll-call voting is generally with higher reliability. We will also show that the ability sets in which simultaneous is superior are comparatively homogeneous. These findings are discovered through numerical methods.

3.2.1 Which voting scheme is generally more reliable?

Up to now, we have shown that for specific deck sizes and ability sets either roll-call voting or simultaneous voting has better performance in terms of reliability. We do not have a consistent conclusion on which voting scheme can always generate a more reliable result. Nevertheless, in this Subsection, we will show that the Alpern-Chen roll-call voting is always superior to simultaneous voting with respects to both the average reliability and the optimality fraction.

Average reliability

Recall that the average reliability is defined as follows: for fixed deck size D, the average value of its reliability over all admitting ability sets with different abilities. The average reliability measures the expected performance of the reliability for a jury chosen randomly. It is noticed that in Example 3.1, $D = 4$ and the only distinct-ability set is $\{1, 2, 3\}$. Thus, average reliability is identical to the reliability for this ability set. There we have already demonstrated that roll-call voting with Alpern-Chen order (or any order in which the juror of ability 3 votes first or second) has reliability 1, and simultaneous voting has reliability 8/9. For values of D from 6 to 52, the results on the average reliabilities are shown in two of the curves (yellow line for sequential Alpern-Chen and brown dashed-line for simultaneous) in Figure 3.2. We have found the following pattern.

Observation 3.9. *For deck sizes D, $4 \leq D \leq 52$, the average reliability of sequential voting in the Alpern-Chen order is higher than that of simultaneous voting. The difference exceeds* 0.035*.*

Optimality fraction

In comparison between roll-call and simultaneous voting, for fixed deck size D, we wonder which can provide higher reliability in terms of the fraction of ability sets. In particular, let $\hat{\phi}(D)$ denote the fraction of ability sets for which simultaneous voting is with higher or equivalent reliability. Recall from the analysis above of Example 3.1 for simultaneous voting in the first part of this subsection that for $D = 4$ there is only one ability set $\{1, 2, 3\}$ and roll-call voting is with the higher reliability, Therefore $\hat{\phi}(4) = 0$. Through simple calculation, we observe that $\hat{\phi}(D) = 0$ for all $D < 12$. The computational results for $D = 12, 14, \ldots, 52$ are presented in Table B.4 in the Appendix. To put it in a nutshell, we have the following observation:

Observation 3.10. *The Alpern-Chen sequential voting scheme has a higher optimality fraction than simultaneous voting scheme. More specifically, the fraction of ability sets with deck size D, $\hat{\phi}(D)$ satisfies the following:*

1. For D satisfying $4 \leq D \leq 10$, $\hat{\phi}(D) = 0$.

2. For D satisfying $12 \leq D \leq 52$, $\hat{\phi}(D) \leq 0.0268$.

3.2.2 Which ability sets do better with each scheme?

In Section 3.2.1, we try to solve the problem *how many* ability sets simultaneous voting is with a higher reliability than the Alpern-Chen roll-call voting. Now we try to address the problem *for which* ability sets simultaneous voting is superior. We are not concerned with providing a specific *list* of these sets, but rather a feature of them. Our results show that a jury's feature that most decides which voting scheme has better performance is the heterogeneity level of the abilities. Here, we use the standard deviation of the ability set or the *spread* $\delta = y - w$ (range), the difference between its highest ability y and its lowest ability w to measure this level. Higher heterogeneity generally indicates that roll-call voting is highly possible to be better. We start with considering merely *middle centered* ability sets, those ability sets containing the middle ability $m = D/2$. These ability sets can be presented in a two-dimensional vector, as in Figure 3.3. Next, we try to address more general setting, but still, use the spread as the measurement of heterogeneity. It should be noticed that in this subsection we regress to the restriction on the distinct abilities, removing common-ability examples of $\{1, 1, 1\}$ and $\{2, 2, 2\}$ considered at the beginning of the subsection for simplicity.

Spread analysis for centered ability sets

One approach to present where these ability sets are located is to generate a two dimensional (w, y) vectors for *centered ability sets*. These ability sets are of the form $\{w, m, y\}$, $m = D/2$, with $w < m < y$, within which the middle ability is $m = D/2$. These integer pairs (w, y) is used to characterize the ability set $\{w, m, y\}$. The ability set possesses the property that simultaneous voting is superior to sequential voting in the Alpern-Chen order (m, y, w) is labelled (w, y) with a small dot. These ability sets are *dotted*. Figure 3.3 presents these ability sets for the standard deck size $D = 52$, $m = 26$. As shown in Figure 3.3 all the dotted points are located in the triangle under the line $d = y - w = D/4 + 1 = 14$, which encompasses the one eighth most homogeneous points (ability sets) in term of the *spread* measurement $y - w$, (range). In Figure 3.3, a line with constant standard deviation passes the point of ability set $\{15, 26, 29\}$($\sigma = 7.37$) as the parting with all the dotted points lying under. The most homogeneous point $(25, 27)$, (ability set $\{25, 26, 27\} = \{m - 1, m, m + 1\}$,) is labelled as dotted point. Nevertheless, this point is not labelled as the dotted point in Example 3.1 with $D = 4$ with unique ability set $\{1, 2, 3\}$, under which the roll-call voting is better. However it is dotted for sufficiently large deck size D. Also note that when $D = 52$, most of the dotted points have both coordinates odd - this changes to all such points for smaller deck sizes. Recall that jurors with odd abilities cannot draw an equal number of cards of each color – ties are impossible. The lack of ties seems to favor simultaneous voting. The generalizations of observations regarding Figure 3.3 ($D = 52$) that hold for other values of D are listed in the following Observation.

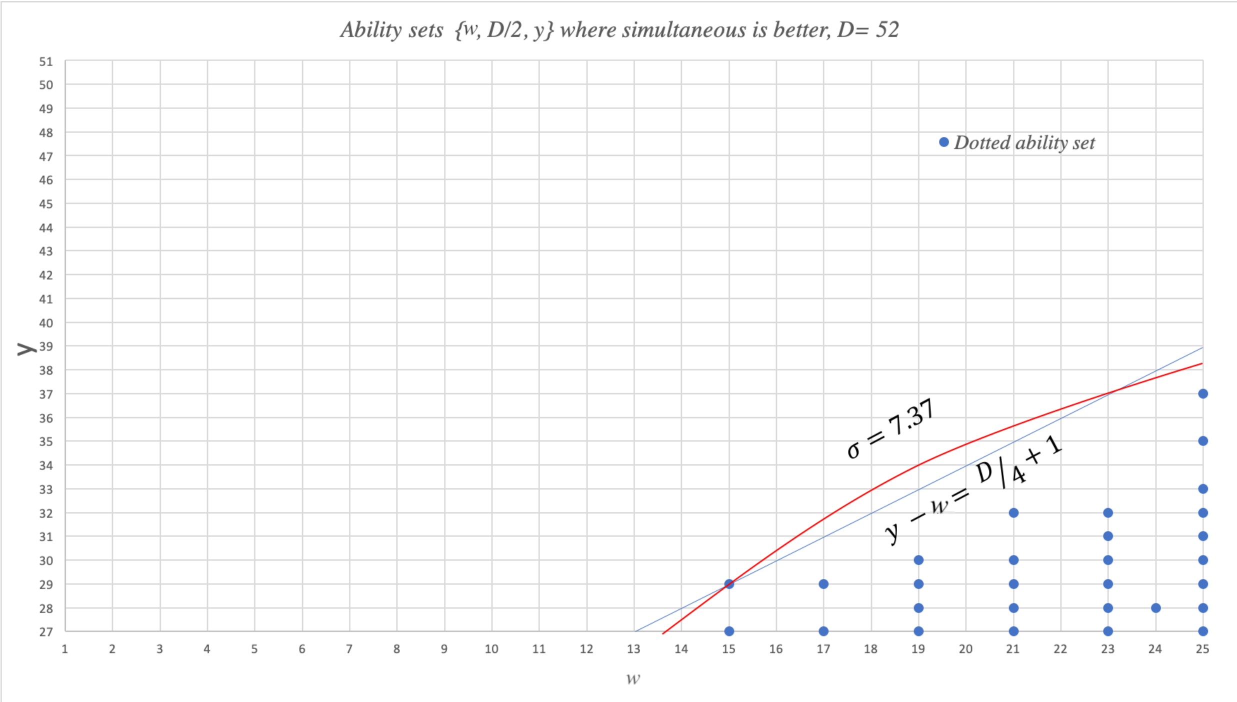


Figure 3.3: Scatter plot of dotted ability sets $\{w, m, y\}$ for $D = 52$.

Observation 3.11. *For $D \equiv 2m \leq 52$, consider ability sets of the form $A = \{w, m, y\}$ with $1 \leq w < m < y < D$. We say that A is **dotted** (as in Figure 3.3) if the reliability of simultaneous voting with abilities A is greater than or equal to the reliability of sequential voting in the Alpern-Chen order. Then*

1. *If $D \leq 10$ or D is not divisible by 4, then there are no dotted points, that is, the Alpern-Chen sequential voting is always more reliable.*
2. *Suppose D satisfies $12 \leq D \leq 52$ and is divisible by 4. Then all dotted ability sets A are in the triangular set $y \leq w + D/4 + 1$ (with the lowest $1/8$ of spread $y - w$). In particular, the most homogeneous ability set $\{m-1, m, m+1\}$ is dotted. Furthermore if $D \leq 32$ then all dotted points (ability sets) have both w and y odd.*

The full data regarding this Observation for $D < 52$ are shown in Figure B.1 in the Appendix.

Spread analysis for general ability sets

Here, we analyse the normal deck size $D = 52$ and arbitrary ability sets $\{a, b, c\}$ with $a < b < c$. The spread is defined as $\delta = c - a$. Figure 3.4 shows all the values of δ starting from 2 to 20, the number of all possible ability sets labelled "#(δ)" in the figure (blue) and the number of these where roll-call voting is better than simultaneous voting. The orange line indicated the difference between the total number of the ability and the dotted ability sets labelled "#Seq(δ)" in the figure. Though the upper bound for δ is $51 - 1 = 50$, when $\delta \geq 16$, dotted ability sets do not exist. Thus, we limit the upper bound of δ to 20. For instance, given that $\delta = 2$, there are 50 ability sets (of the form $\{w, w + 1, w + 2\}$). Among these ability sets, 23 of them are dotted with better performance of simultaneous voting.

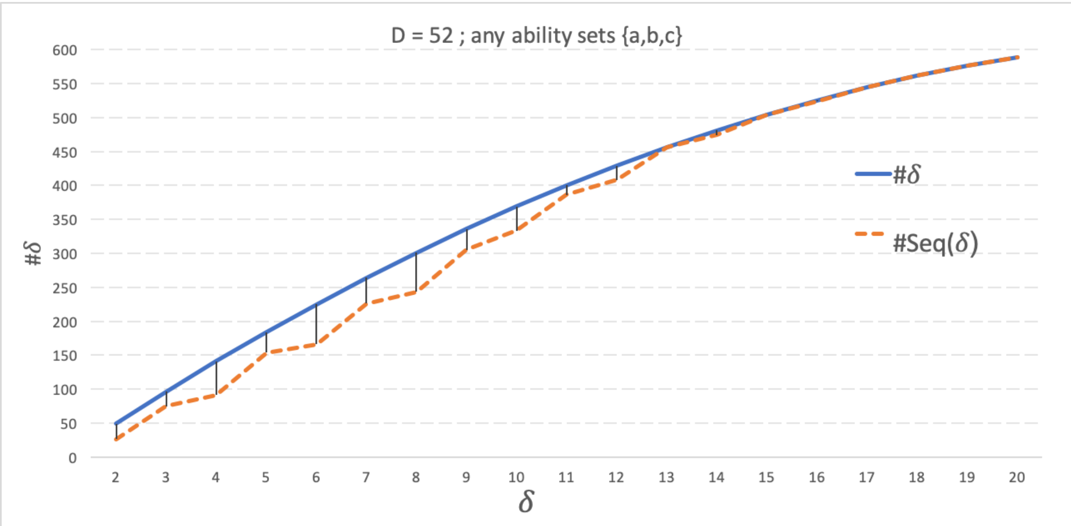


Figure 3.4: Relationship between δ, $\#\delta$, and $\#seq(\delta)$ for fixed $D = 52$.

In summary, when the jury is with high degree of heterogeneity of abilities, the additional reliability of roll-call voting over Condorcet simultaneous voting is more significant.

3.2.3 Reliability and heterogeneity

The work of Kanazawa (1998) has demonstrated that the reliability of Condorcet simultaneous voting, for juries of fixed average of p, is increasing in the standard deviation σ of the jury. This part shows that this conclusion still holds, even for the roll-call voting. Additionally, the reliability difference between roll-call and simultaneous voting also increases with heterogeneity (σ).

In Figure 3.5 , we fix the deck size at $D = 52$ and juries with ability sets with a fixed mean of 26. For every ability set $A = \{a, b, c\}$ we draw two points: $(\sigma_A, Q(b, c, a))$ and $(\sigma_A, Q_{sim}(A))$, both on the same vertical line with horizontal coordinate σ_A. The horizontal coordinate σ ranges from 1 (the only ability set $\{25, 26, 27\}$) to 25 (the only set $\{1, 26, 51\}$). By Observation 3.11, the second part, the orange point is below $\sigma = 1(\{25, 26, 27\}$ is dotted) the blue point. It should be noticed that for any ability set comprising juror with the highest ability c (51), the reliability of the roll-call voting is 1(as he votes secondly). There exits 13 such ability sets, of the form $(a, b) = (w, 27 - w)$, for $w = 1, \ldots, 13$. These are the 13 points at level (reliability) 1.

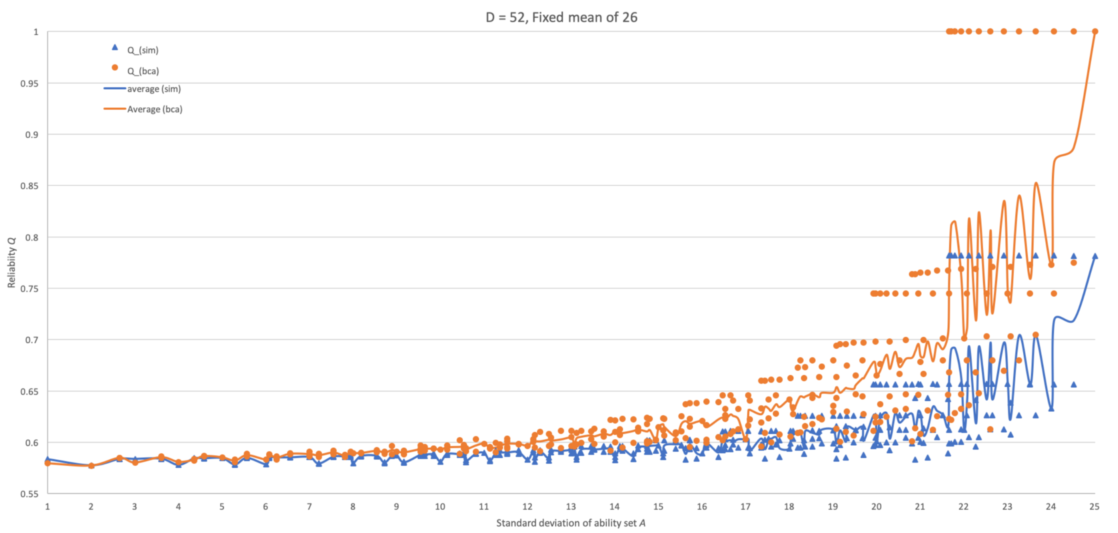


Figure 3.5: Plots of $Q_{sim}(A)$ and $Q(b, c, a)$ against σ of ability sets $\{a, b, c\}$

.

Our observations regarding reliability increasing with heterogeneity are on the basis of the pattern of the orange dots (sequential reliability) and the blue triangles (simultaneous reliability) in Figure 3.5. We also plot the line with respect to the averages of the reliabilities for every standard deviation for both sequential and simultaneous reliability. When the standard deviation is greater than 22, there are 13 ability sets comprising the ability juror of ability 51 who can see all the cards in the remaining deck. For these ability sets, the reliability under roll-call scheme is 1 as the juror of ability 51 votes second and the last juror will always follow him. For the same ability sets, the reliability is around 0.78 if they vote simultaneously. This is the reason why the lines are volatile between the standard deviation of 22 and 25. When the average of the ability sets is fixed to 26, both lines increase with larger standard deviation. This result provides the support to the finding of Grofman & Feld (1988) that the diversity of voices is an approach to the right collective decision with democratic majority rules. Figure 3.5 also shows that when the standard deviation is very small, the blue line is higher than the orange line. For example, when the standard deviation is unit, ability set 25,26,27, the blue dot is above the orange dot. When the standard deviation increases, the superiority of roll-call voting (orange line) over Condorcet simultaneous voting (blue line) increases. This shows that higher level heterogeneity is required for more significant additional reliability of roll-call voting over simultaneous voting. The reason for this phenomenon is herding. The jurors with higher ability (higher quality of samples) in sequential voting have a greater influence on the reliabilities through the herding than those in the simultaneous voting. This influence become more significant with the increase of the most able juror's ability, especially the jurors with perfect information.

3.3 Average Reliability for $n = 5$

The jury size three observations make us think whether these findings can be extended to the larger jury. This section concerns the performance of roll-call voting under different voting order with jury size five. The exhaustive numerical approach is adopted here to obtain the average reliability $\bar{Q}(r)$ for all orderings $r \in \Pi_5$.

Voting orderings	$D = 12$	$D = 14$	$\dots$	$D = 30$
(d, a, e, c, b)	0.822	0.801	$\dots$	0.718
(d, b, e, c, a)	0.817	0.798	$\dots$	0.717
(c, a, e, d, b)	0.815	0.795	$\dots$	0.715
(c, b, e, d, a)	0.802	0.785	$\dots$	0.713
(d, c, e, b, a)	0.801	0.783	$\dots$	0.710

Table 3.4: Average reliabilities of the highest five orderings

Among all $\Pi_5 = 120$ orders, Table 3.4 provides details of the reliabilities for the top five and bottom three orders, ranging from $D = 12$ to 30. These rankings are consistent, starting from $D = 12$. This observation leads to the following finding:

Observation 3.12. *Given a deck size between 12 and 30, and a five-member jury of abilities $\{a, b, c, d, e\}$ with $0 < a < b < c < d < e < D$, the voting orders with the five highest reliabilites all have the most able juror as the third (middle) voter.*

Ordering \ voter	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}
r^1 (Best)	d	a	e	c	b
r^2	d	b	e	c	a
r^3	c	a	e	d	b
r^4	c	b	e	d	a
r^5	d	c	e	b	a
v_i = average	$v_1 = \frac{3d+2c}{5}$	$v_2 = \frac{2a+2b+c}{5}$	$v_3 = e$	$v_4 = \frac{b+2c+2d}{5}$	$v_5 = \frac{3a+2b}{5}$

Table 3.5: Average ability of i^{th} juror to vote, among top five ordering

Table 3.5 shows the top five voting orders in terms of reliability. These identical top five order holds until deck $D = 52$. It also shows the average ability v_i of the i^{th} juror to vote, for different voting orders. We are intended to make a comparison our findings on juries of size five with the finding in our earlier work for juries of size three. In order to do this, we first divide the five jurors in voting order into three subgroups: Early, Middle and Late (E, M, L). Providing we had the jury with the size of multiples of three, this division will be easier. To transform five to three, we must consider let the second juror partly belong to E and partly belong to M. A similar division applies to the fourth juror. In order to guarantee the equal fractional number of jurors in every group, we let the second juror fractionally $2/3$ belong to E and $1/3$ belong M, and correspondingly for the fourth juror. This process let $5/3$ jurors into every the three subgroups, with total weight $3(5/3) = 5$ for the three groups, as required. Subsequently, the fairly fractional weight for every group is $5/3$. We can divide by $5/3$ to obtain the average abilities E, M, L for the three subgroups on the top five ranking order:

$$
\begin{aligned}
E &= \frac{3}{5}\left(v_1 + \frac{2}{3}v_2\right), \\
M &= \frac{3}{5}\left(\frac{1}{3}v_2 + v_3 + \frac{1}{3}v_4\right), and \\
L &= \frac{3}{5}\left(\frac{2}{3}v_4 + v_5\right).
\end{aligned}
$$

Comparing the average abilities among three subgroups, we calculate

$$
\begin{aligned}
25(E - L) &= 5(d - a) + 4(c - b) > 0, \text{ and} \\
25(M - E) &= 15e - (7d + 5c + b + 2a) \geq 15(e - d) > 0.
\end{aligned}
$$

Thus, we have shown the following

Observation 3.13. *Given any deck size* $12 \leq D \leq 30$*, for any five-member jury with abilities* $0 < a < b < c < d < e < D$*, we have*

$$L < E < M,$$

which implies that the highest five voting orderings in terms of average ability jointly form the weighted group order of (E, M, L) *— precisely the Alpern-Chen ordering.*

3.4 Conclusions

The celebrated paper of Alpern & Chen (2017*a*) shows that when jurors with distinct abilities vote sequentially, the probability of correct collective decision between two states of Nature is influenced by the voting sequence. In that paper, the ability is defined as a factor regarding the juror's private information quality. In this chapter, we introduce a new measurement of a juror's ability, which is the number of cards he can draw from a predetermined deck of cards. The introduction of this model brings a discrete voting game called the sealed card problem.

Under discussion in this model, we have shown that providing that the two states are equally probable with a jury of size three, then the optimal voting order is predominantly median ability first, then the highest ability, then the lowest ability, known as the Alpern-Chen voting order. When the deck of cards is of small size ($D \leq 16$), the Alpern-Chen order dominates all other six voting orders in terms of the majority verdict's reliability for any jury with even-only abilities. For any medium-size decks ($D \leq 52$), the Alpern-Chen Ordering is optimal for more juries than any other voting orders. Our findings also show that for each deck sizes, the average reliability of Alpern-Chen ordering, taken over all possible juries, is superior to any other order.

After determining the optimal ordering, we compare reliability with other voting schemes fixing this voting order. We make a comparison of the reliability of the roll-call voting with the traditional simultaneous voting scheme. A spread of the set of juror abilities with larger spread representing more heterogeneous juries is introduced. Our findings indicate that given that sufficiently heterogeneous juries, roll-call voting is superior to simultaneous voting. In contrast, when juries are sufficiently homogeneous, simultaneous voting provides more reliable performance. Similarly, our results show that a jury's reliability of fixed average ability increase with its heterogeneity.

For a jury of size five, we have demonstrated that the highest five voting orders feature the ablest juror votes in the middle (third). Our results also show that by assigning the abilities in the following fashion: median voters to vote earlier, high ability voters to vote in the middle, and low ability to vote last, the Alpern-Chen order can still be at least an exceptional heuristic approach to generating high reliability.

Chapter 4

Sequential voting with an independent voter

Under sequential voting with an independent juror (SVI), the voting is divided into two parts, the roll-call voting part and an independent juror. In the roll-call voting part, one juror votes first, followed by another juror knowing the vote of the first juror. This duo voting structure is the same voting scheme that was discussed in Section 2.3. The independent juror makes his own decision knowing nothing about the votes of others. Although the timing is vital in the sequential voting schemes, the independent juror has a time-invariability property. It makes no difference whether the independent juror votes first, second, or third as shown in Figure 4.1. The voting rule here is the simple majority rule introduced in Section 1.1.1. When more than half of the jurors vote for the same alternative, the verdict is generated. Under this voting scheme, the information flow occurs within the roll-call voting part.

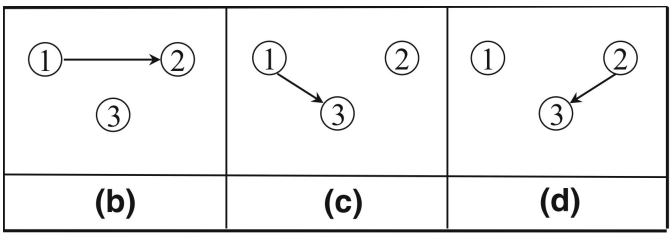


Figure 4.1: Sequential voting with an independent juror

Under SVI, we are mainly concerned with the following question:

Who should be the independent juror, and what voting order of the sequential part according to the abilities of jurors such that the probability of correct verdict under majority rule is maximised?

Order of jurors and reliability under SVI

In this part, we will provide the notation of voting order under SVI along with the reliability formula under this voting scheme. Let $(\{v_1, (v_2, v_3)\})$ be a voting order under SVI. Then, the first position belongs to the independent juror v_1. The second position v_2 and the third position v_3 are the first and second layers in the roll-call part. For example $r = (\{c, (a, b)\})$, the independent juror is c and the other jurors follows the order a then b.

The following shows the calculation of the reliability $Q(\{c, (a, b)\})$ under this voting scheme. Let $q_A(\{c, (a, b)\})$ be the probability of the verdict being A when the Nature is A. Similarly, we define $q_B(\{c, (a, b)\})$. Then, given an arbitrary *a priori* probability θ_0, we have:

$$Q(\{c, (a, b)\}) = \theta_0 q_A(\{c, (a, b)\}) + (1 - \theta_0) q_B(\{c, (a, b)\}).$$

When $\theta_0 = 1/2$,

$$Q(\{c, (a, b)\}) = \frac{1}{2}[q_A(\{c, (a, b)\}) + q_B(\{c, (a, b)\})].$$

By symmetry,

$$Q(\{c, (a, b)\}) = q_A(\{c, (a, b)\})) = q_B(\{c, (a, b)\}).$$

Thus, the calculation of $Q(\{c, (a, b)\})$ just requires $q_A(\{c, (a, b)\})$ as they have the same value. The formula for $q_A(\{c, (a, b)\})$ is the summation of probabilities of voting order $(A, A,$ any$)$, (A, B, A) and (B, A, A) when the Nature is A:

$$\begin{aligned} q_A(\{c, (a, b)\}) &= \Pr[A, A, any] + \Pr[A, B, A] + \Pr[B, A, A] \\ &= (1 - F_A(c, 0))(1 - F_A(a, 0)) + (1 - F_A(c, 0))F_A(a, 0)(1 - F_A(b, y_B))) \\ &+ F_A(c, 0)(1 - F_A(a, 0))(1 - F_A(b, y_A)). \end{aligned}$$

When we substitute y_A and y_B formulas to our linear continuous signal model using equations (2.12) and (2.13), we need to pay attention to whether or not the pair (a, b) belongs to h_2 (i.e. $b \leqslant a/2$ in equation (2.14)). Then,

$$Q(\{c,(a,b)\}) = \begin{cases} c/4 + 1/2, & \text{if } (a,b) \in h_2 \ (b \leqslant a/2); \\ Q_1(\{c,(a,b)\}), & (a,b) \notin h_2, \end{cases} \tag{4.1}$$

Where

$$Q_1(\{c,(a,b)\}) = \frac{64a + 16a^2 + 16ab + 16ac + 8bc + bc^3 - 4a^2bc - 12c^2}{128a}.$$

4.1 Optimal voting order under SVI

In this section, we will discuss the optimal voting order under SVI, along with proofs. The condition of the optimal ordering consists of two parts. The first is that the juror with the highest ability takes the independent juror position. For the roll-call voting part, the juror with the least ability casts his vote first, followed by the juror with middle ability. We have the following proposition:

Proposition 4.1. *Given that jurors' abilities are* $0 \leqslant a < b < c \leqslant 1$*, the highest ability juror should be the independent voter. For the roll-call voting part, the voting should be in anti-seniority order (increasing order) unless herding exists.*

Proposition 4.1 can be applied to an organizational context. Consider that there are two senior employees with different levels of experience and one junior employee. The optimal decision rule for them is that the junior employee reports his decision to the less experienced senior employee. If they are in agreement, a collective decision is made. If not, they need to consult the more experienced employee for an independent opinion. This simple application is consistent with business practice. The unique feature of this proposition is that it requires only a specific order of jury (such as the most able and middle juror), instead of a specific numerical value of the ability. The only necessary condition is the prevention of herding. We can always use equations (2.1) or (2.2) to calculate the condition on the empirical probability of making a right decision.

Proof of Proposition 4.1
Case 1 without herding

The idea of the proof is simple: first, we will show that for the two-layer voting structure, voting should be in anti-seniority order. Second, we will show that the juror of the highest ability c should be the independent voter. The equations for $\Delta_{4.1}$, $\Delta_{4.2}$ and $\Delta_{4.3}$ define the differences between anti-seniority order and seniority order when we fix the independent juror:

$$\Delta_{4.1} = Q(\{a,(b,c)\}) - Q(\{a,(c,b)\}),$$
$$\Delta_{4.2} = Q(\{b,(a,c)\}) - Q(\{b,(c,a)\}),$$
$$\Delta_{4.3} = Q(\{c,(a,b)\}) - Q(\{c,(b,a)\}).$$

Substitute with the reliability formula for each order,

$$\Delta_{4.1} = \frac{b(c-b)(2c-b)}{32c^2},$$
$$\Delta_{4.2} = \frac{a(c-a)(2c-a)}{32c^2},$$
$$\Delta_{4.3} = \frac{a(b-a)(2b-a)}{32b^2}.$$

We have $0 \leqslant a < b < c \leqslant 1$. Therefore, $\Delta_{4.1}$ $\Delta_{4.2}$ and $\Delta_{4.3}$ are positive:

$$Q(\{a,(b,c)\}) > Q(\{a,(c,b)\}),$$
$$Q(\{b,(a,c)\}) > Q(\{b,(c,a)\}),$$
$$Q(\{c,(a,b)\}) > Q(\{c,(b,a)\}).$$

Now we have proven the second part of the statement. For the roll-call part, the juror with the higher ability between two jurors should always vote last. Then we merely need to compare $Q(\{a,(b,c)\})$, $Q(\{b,(a,c)\})$ and $Q(\{c,(a,b)\})$. In this step, we try to solve the problem of who should be the independent juror. To complete this task, we define the differences of $\Delta_{4.4}$ and $\Delta_{4.5}$,

$$\Delta_{4.4} = Q(\{b,(a,c)\}) - Q(\{a,(b,c)\}),$$
$$\Delta_{4.5} = Q(\{c,(a,b)\}) - Q(\{b,(a,c)\}).$$

Substitute with the reliability formula into $\Delta_{4.4}$ and $\Delta_{4.5}$, we have:

$$\Delta_{4.4} = \frac{b(b-a)}{8c},$$
$$\Delta_{4.5} = \frac{b(c-b)(2c+b)}{32c^2}.$$

By the initial setting of the ability ($0 \leqslant a < b < c \leqslant 1$), $\Delta_{4.4}$ and $\Delta_{4.5}$ are positive. Therefore, we have the following inequalities:

$$Q(\{c,(a,b)\}) > Q(\{b,(a,c)\}) > Q(\{a,(b,c)\}).$$

Now we have partially solved the second part of the problem regarding who should take an independent judge's position: the one with the highest ability.

Case 2 with herding

The herding condition $(v_2, v_3) \in h_2$ implies $v_3 < v_2/2$. This means that for the roll-call part, $v_3 < v_2$. In other words, the herding condition indicates that for the roll-call part, the two jurors follow the seniority rule. Thus, we have proved the first part of the proposition. Now we need to address the problem of who should be the independent juror. On basis of equation (4.1), we have:

$$Q(\{a,(c,b)\}) = a/4 + 1/2,$$

$$Q(\{b,(c,a)\}) = b/4 + 1/2,$$

$$Q(\{c,(b,a)\}) = c/4 + 1/2.$$

Because $a < b < c$, we have the following inequalities:

$$Q(\{c,(b,a)\}) > Q(\{b,(c,a)\}) > Q(\{a,(c,b)\}).$$

Therefore, the independent juror should be the juror with the highest ability. Based on the analysis of case 2, the second part of Proposition 4.1 has been proven.

4.2 Comparison with the simultaneous voting scheme

The CJT is based on simultaneous voting. Therefore, it is necessary to compare simultaneous voting with the SVI voting scheme (sequential voting with an independent voter, type I). Based on results of the previous section, we need to compare the optimal voting order under SVI voting (the highest ability juror is the independent voter, and the other two jurors vote in anti-seniority order) with the simultaneous voting scheme.

Figure 4.2 shows the result. The blue dots mean that that the optimal voting order under the SVI voting scheme is better than the simultaneous voting scheme. The green dots mean that simultaneous voting is superior to the optimal voting order under the SVI voting scheme. From the mixed Figure 4.2, we know that when the difference in abilities is large enough, the optimal voting order under SVI voting scheme is better.

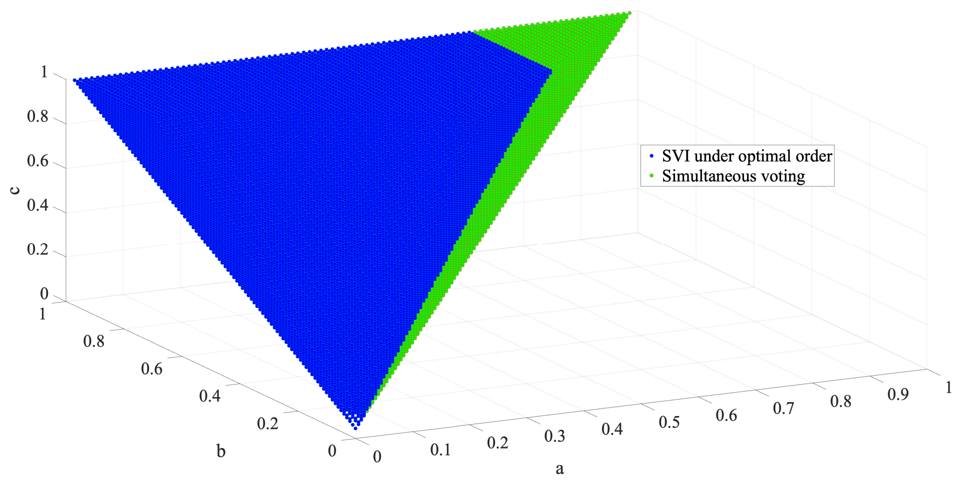


Figure 4.2: SVI (type I) voting scheme and the simultaneous voting scheme

Now, we focus on the boundary of the simultaneous part. From numerical experiments illustrated in Figure 4.3, the maximum difference between the highest ability juror and lowest ability juror is approximately 0.3.

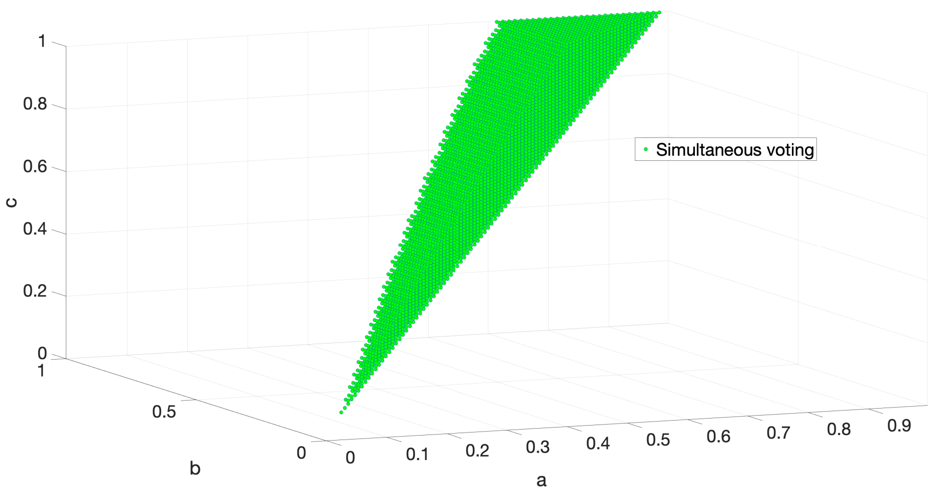


Figure 4.3: SVI (type I) voting scheme and simultaneous voting scheme

We can solve this using the boundary condition when the reliability of optimal voting under SVI is equal to that of simultaneous voting. First, the optimal voting order under

SVI voting is achieved when the highest ability juror is the independent voter and the other two jurors vote in anti-seniority order. The corresponding reliability is:

$$Q(\{c,(a,b)\}) = \frac{64b + 16ab + 8ac + 16bc + a^3c - 12a^2 + 16b^2 - 4ab^2c}{128b}. \tag{4.2}$$

The reliability for the simultaneous voting scheme (see equation (2.2.5)) is,

$$Q(abc) = a/8 + b/8 + c/8 - (abc)/32 + 1/2. \tag{4.3}$$

The difference between equations (4.2) and (4.3) is,

$$\frac{a(a^2c - 12a + 8c)}{128b}.$$

When the difference is zero, we can obtain the boundary equation:

$$a^2c - 12a + 8c = 0. \tag{4.4}$$

We notice that equation (4.4) is not related to b, and it is a quadratic equation in terms of a. By rearranging equation (4.4), we can obtain the explicit expression for c:

$$c = \frac{12a}{a^2 + 8}.$$

The difference between c and a is given by

$$c - a = \frac{12a}{a^2 + 8} - a.$$

This difference should be positive under the initial setting $c > a$. The solution is $0 < a < 6 - 2\sqrt{7}$. Now we want to ascertain the maximum range of the ability, which is $c - a$. We just need to know the monotonicity of $c - a = \frac{12a}{a^2+8} - a$ on the interval $0 < a < 6 - 2\sqrt{7}$. The derivative of this difference is:

$$\frac{32 - a^4 - 28a^2}{(a^2 + 8)^2},$$

which is obviously positive when $0 < a < 6 - 2\sqrt{7}$.

Therefore, the maximum difference between highest ability juror c and lowest ability juror a is achieved when a approaches $6 - 2\sqrt{7}$. When $a = 6 - 2\sqrt{7}$, then $c - a = 2\sqrt{7} - 5 \approx 0.3$. On the other hand, when a approaches 0, the difference $c - a$ approaches 0.

Voting Scheme	Counts(#)	Proportion(%)
SVI	90697	86.28%
Simultaneous	14607	13.87%

Table 4.1: Comparison between SVI (b,c,d) and simultaneous voting (a)

In conclusion, in most cases, sequential voting is better than simultaneous voting. From Table 4.1, it can be seen that sequential voting with an independent voter using optimal voting order is superior to simultaneous voting in terms of reliability in 86.28% of the total cases. This advantage will decline with an increase in the ability of highest-ability juror. The condition for superiority of simultaneous voting is $c < \frac{12a}{a^2+8}$.

4.3 Conclusions

Under the sequential voting with an independent voter and for a fixed set of jury abilities, the jury's reliability can be improved by rearranging the voting sequence. This arrangement requires two parts. The first part regards who takes the position of the outsider (independent juror). The second regards that which order the remaining two jurors should follow. For a jury of three, the optimal order is always as follows: highest ability takes the independent position, and the other two jurors follow an anti-seniority order unless herding exists. For sufficiently heterogeneous juries, which can be described as a linear fractional-cubic function, sequential voting with an independent is more reliable than simultaneous voting. However, SVI's superiority will decrease when the ability of the highest juror increases statistically.

Chapter 5

Sequential voting with knowledge of previous voter

In sequential voting with knowledge of the previous voter (SVKP in Figure 5.1), the jurors take turns to vote. Each juror can only see the vote of the adjacent juror. That is, all but the first juror have restricted information of previous jurors. This voting scheme looks similar to the roll-call voting graph (h) in Figure 5.2. It may be noticed that there is an additional arc in the graph (h). As we have defined in Section 2.1, the main characteristic of a sequential voting scheme is the knowledge sets of the jury. In Figure 5.1, the knowledge sets of the first and remaining two jurors are $K(0)$ and $K(1)$ respectively. In Figure 5.2, the third juror knows both the votes of the first and the second juror, In Figure 5.1, the third juror only knows the vote of the second juror and does not know that of the first juror. To be more precise, the knowledge set is $K(2)$ for the last juror (all the votes of previous jurors), $K(1)$ for the second juror (only the votes of the first juror), and $K(0)$ for the first juror (nothing about the votes of other jurors). In summary, the third juror in the roll-call voting scheme knows the previous two jurors' votes and abilities and the private information signal s. Unlike the third juror in the roll-call voting scheme, the third juror under SVKP does not know the first juror's vote but still has all the other information. Therefore, the underlying difference is that the last juror under SVKP does know how the first juror has voted. This uncertainty faced by the third juror will make the voting process more complicated.

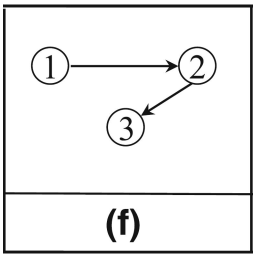


Figure 5.1: Voting Scheme (f): SVKP

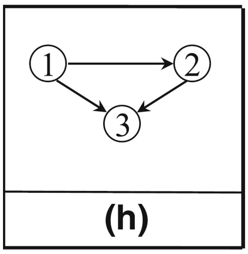


Figure 5.2: Voting scheme (h): Roll-call

Under SVKP, our main question is:

What is the optimal voting order in terms of jurors' abilities such that the probability of correct verdict is maximised?

Thresholds for the third juror under sequential voting with knowledge of the previous voter

As discussed in Section 2.2.4, the thresholds of first two jurors are the same as the duo voting structure. Now let us calculate the voting threshold for the third juror. To do this, we use Bayes' Rule to calculate the probability of state A given that the second voter vote for the state A. Here, U indicates that we do not know the vote of the first juror. UA represents the vote information for the third voter when he knows the second juror has voted for state A but knows nothing about first juror's vote. The numerator is the probability that the first juror votes A or B when the second juror votes for state A and the state of nature is A. The denominator is the total probability when the second juror votes for state A. Equation 5.1 shows the details of the calculation:

$$\begin{aligned}
\theta(UA) &= \frac{\theta(A)(1 - F_A(b, y_A)) + \theta(B)(1 - F_A(b, y_B))}{T(AA) + T(BA)}, \\
T(AA) &= \theta(A)(1 - F_A(b, y_A) + (1 - \theta(A))(1 - F_B(b, y_A))), \\
T(BA) &= \theta(B)(1 - F_A(b, y_B) + (1 - \theta(B))(1 - F_B(b, y_B))).
\end{aligned} \tag{5.1}$$

where $\theta(A) = \frac{2+a}{4}$ and $\theta(B) = \frac{2-a}{4}$ which are the *posterior* probabilities of the state A after the first juror votes for state A (resp. B); y_A and y_B are provided in the Section 2. Thus, based on equations (5.1) and (2.9), the honest threshold z_{XA} for the third juror in the equation:

$$Z_{UA} = Z_{UA}(a,b,c) = \begin{cases} -1, & \text{if } c \leqslant \rho(a,b) \ , \\ -\rho(a,b)/c, & \text{otherwise,} \end{cases} \tag{5.2}$$

where

$$\rho(a,b) = \frac{(a^2+4b^2)}{8b}. \tag{5.3}$$

Example

The following example considers a jury of three under sequential voting with knowledge of the previous voter. Suppose that the first juror has ability 2/3 and the second juror has ability 3/4. Given that the second juror votes A, the conditional probability of A is calculated using the following two methods, the algebraic and numerical methods. The first method use the equations to calculate the probability directly.

The numerical method by Monte Carlo simulation (Hammersley & Handscomb 1964) is based on the following procedure. Generate state of nature (equiprobable A or B) , then signal s_1 and vote of the first juror (U, this can either be $U = A$ or $U = B$). Finally, signal s_2 and vote of the second juror (either be $Y = A$ or $Y = B$). If the second juror votes A ($Y = A$), we calculate the fraction of the simulations had $N = A$. So far, we have obtained one trial of simulation. The approximation of $\theta(UA)$ is calculated by averaging a large number of trials. $\theta(AA)$ and $\theta(BA)$ are also calculated to ensure that every step of the simulation can be seen from the programming process.

Table 5.1: Simulation results for one trial

	#	$\#N = A$	Ratio(simulation)	Ratio(theory)	difference
$\theta(AA)$	386719	291644	0.75414965	0.75324675	0.0009029
$\theta(BA)$	113568	71237	0.62753924	0.62711864	0.0004206
$\theta(XA)$	5000287	291644	0.72534565	0.72453703	0.0008086

Table 5, the difference between the estimated and theoretical values for $\theta(AA)$ and $\theta(BA)$ is comparatively small. We note that this is just the data from one trial.

Ten thousand trials were conducted to improve accuracy. This process will reduce random error using Monte Carlo simulation theory, and the numerical value will converge to the actual value. Figure 5.3 shows that the average value of simulations (blue) approaches the theoretical value (red) when the number of trials increases. This example provides an intuitive way to show the calculation of the thresholds through simulation.

Now we know the threshold formula for the third juror under SVKP. We note that the third juror vote is of significance, or as we call it, pivotal, when the second juror

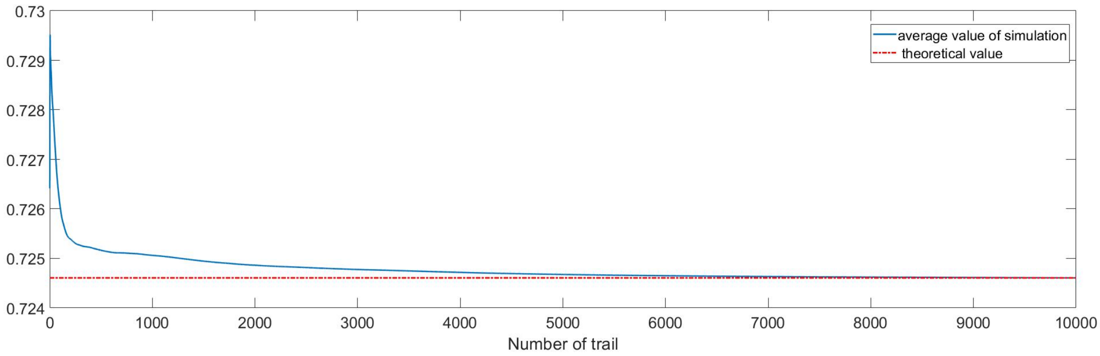


Figure 5.3: Average of the simulation

does not copy the vote of the first juror. When $(a, b) \notin h_2$, we have the equation (5.4) for the third juror under SVKP:

$$Z_{UB} = -Z_{UA} = Z_{UB}(a,b,c) = \begin{cases} 1, & \text{if } c \leqslant \rho(a,b) \ , \\ \rho(a,b)/c, & \text{otherwise,} \end{cases} \tag{5.4}$$

where

$$\rho(a,b) = \frac{(a^2+4b^2)}{8b}.$$

which is positive as $0 \leq a < b \leq 1$. Similar to Section 2, we have the *herding* condition for the third juror:

$$h_3 = \{(a,b,c) \in [0,1]^3 : c \leqslant \frac{(a^2+4b^2)}{8b}\}. \tag{5.5}$$

which is useful when h_2 does not happen. The verdict is then determined by the latter two jurors and not related to the first juror at all.

Juror ordering and reliability

Similar to Chapter 4, we will provide the notation here for the orderings under SVKP. We have two duo voting structures (see Section 2.2.4) facing different θ. The structure of SVKP is such that one of the jurors votes first, followed by the remaining two. The latter two jurors vote in the same fashion. For example, in $r = (a,\ b,\ c)$, the first juror is a followed by juror b and then juror c.

The following part shows the calculation of the reliability $Q(a,b,c)$ under SVKP.

From Section 2, for $\theta_0 = 1/2$, we know that

$$\begin{aligned}
Q(a,b,c) &= \frac{1}{2}(q_A(a,b,c) + q_B(a,b,c)) \\
&= q_A(a,b,c) = q_B(a,b,c) \\
&= \Pr[A,A,any] + \Pr[A,B,A] + \Pr[B,A,A] \\
&= (1 - F_A(a,0))(1 - F_A(b,y_A)) + (1 - F_A(a,0))F_A(b,y_A) \\
&(1 - F_A(c,z_{UB})) + F_A(a,0)(1 - F_A(b,y_B))(1 - F_A(c,z_{UA})).
\end{aligned}$$

This formula is similar to the reliability formula under the roll-call voting scheme in Chapter 2. The main difference is the $z_{UA}(z_{UB})$ for the last juror, as discussed in the previous section. When we substitute $y_A(y_B)$ and $z_{uA}(z_{UB})$ formulas to our linear continuous signal model, we need to be careful. Herding can exist either in the second or the third juror. In other words, we are required to check whether the pair (b,c) belongs to h_2 and whether the triple of (a,b,c) belongs to h_3. Then we have the reliability equation under SVKP with the order (a,b,c):

$$Q(a,b,c) = \begin{cases} a/4 + 1/2, & \text{if } (a,b) \in h_2 \ (a \leqslant b/2), \\ \dfrac{a^2 + 4b^2 + 8b}{16b}, & \text{if } (a,b,c) \in h_3 (c \leqslant \rho(a,b)) \ , \\ Q_2(a,b,c), & \text{otherwise,} \end{cases} \tag{5.6}$$

where

$$Q_2(a,b,c) = \frac{f_1(a,b,c)}{8192b^3c},$$

and

$$\begin{aligned}
f_1(a,b,c) =&8a^5 + 16a^4b + 16a^3b^4 + 64a^3b^2c^2 + 128a^2b^3 + 256a^2b^2c + 64ab^6 \\
&+ 1024ab^3c^5 + 256b^4 + 1024b^4c + 1024b^3c^2 + 4096b^3c - a^7 - 4a^5b^2 \\
&- 256ab^4c^2 - 384ab^4 - 512ab^2c^2 - 64a^3b^2.
\end{aligned}$$

5.1 Solutions to SVKP

In this section, we will discuss solutions to SVKP. Lemma 5.1 states the optimal order for the latter juror under the condition that no herding happens (i.e $(a,b) \notin h_2$ and $(a,b,c) \notin h_3$). This is the best solution for the later duo voting structure. Proposition 5.2 provides the optimal solution to SVKP. Similar results have been found in Alpern & Chen (2017a) under the roll-call voting scheme. Proposition 5.3 tries to address the problem of whether the seniority order supported by the Dekel & Piccione (2000) (decreasing order of ability) is better than anti-seniority order supported by Ottaviani & Sørensen (2001) (increasing order of ability). We will first state all three results followed by proofs.

Lemma 5.1. *Given that the jurors' abilities satisfy* $0 \leqslant a < b < c \leqslant 1$*, if there is no herding* ($(a,b) \notin h_2$ *and* $(a,b,c) \notin h_3$)*, no matter who votes first, the latter two jurors should follow seniority order (decreasing order).*

Proposition 5.2. *Given that the jurors' abilities satisfy* $0 \leqslant a < b < c \leqslant 1$*, the Alpern-Chen order (median-high-low) is optimal.* $Q(b,c,a)$ *has the highest reliability among the six different orderings.*

Proposition 5.3. *Given that the jurors' abilities satisfy* $0 \leqslant a < b < c \leqslant 1$*, seniority ordering* (c,b,a) *outperforms the anti-seniority ordering* (a,b,c)*.*

5.1.1 Seniority order for latter duo

In this part, we will prove Lemma 5.1's optimal voting order for the latter two jurors: seniority order. The following proof assumes that, for jurors, their abilities satisfy $0 \leqslant a < b < c \leqslant 1$.

The idea of the proof is simple. There are three types of orderings: highest ability first, middle ability first, and lowest ability first. For each type of voting order, we will show that seniority order is always better for the latter two jurors than anti-seniority order. Now, we define $\Delta_{5.1}$, $\Delta_{5.2}$ and $\Delta_{5.3}$:

$$\Delta_{5.1} = Q(a,c,b) - Q(a,b,c),$$

$$\Delta_{5.2} = Q(b,c,a) - Q(b,a,c),$$

$$\Delta_{5.3} = Q(c,b,a) - Q(c,a,b).$$

After substituting with the reliability formula for each order, we can express these differences. The following contents are divided into three parts to show signs of $\Delta_{5.1}$, $\Delta_{5.2}$ and $\Delta_{5.3}$.

Lowest ability juror vote first

The explicit expression for the $\Delta_{5.1}$ is,

$$\Delta_{5.1} = \frac{f_2(a,b,c)}{8192 b^3 c^3},$$

where

$$\begin{aligned}f_2(a,b,c) =& a^7c^2 + 8a^5b^2 + 16a^4b^2c + 48a^3b^4c^2 + 128a^2b^3c^2 + 64ab^2c^6 + 128ab^2c^4\\ &+ 256b^2c^5 - a^7b^2 - 8a^5c^2 - 16a^4bc^2 - 128a^2b^2c^3 - 48a^3b^2c^4 - 64ab^6c^2\\ &- 128ab^4c^2 - 256b^5c^2.\end{aligned}$$

By factorisation, we have:

$$f_2(a,b,c) = (c-b)f_3(a,b,c).$$

As $b < c$, we just need to show $f_3(a,b,c) > 0$, where

$$\begin{aligned}f_3(a,b,c) =& a^7b + a^7c + 64ab^5c^2 + 64ab^4c^3 + 64ab^3c^4 + 128ab^3c^2 + 64ab^2c^5 + 128ab^2c^3\\ &+ 256b^4c^2 + 256b^3c^3 + 256b^2c^4 - 8a^5b - 8a^5c - 16a^4bc - 48a^3b^3c^2\\ &- 48a^3b^2c^3 - 128a^2b^2c^2.\end{aligned}$$

Using $a < b$ to reduce minuend and $b < c$ to enlarge subtrahend, we have

$$\begin{aligned}f_3(a,b,c) >& a^8 + a^7c + 64a^6c^2 + 64a^5c^3 + 64a^4c^4\\ &+ (368a^4c^2 + 384a^3c^3 + 128a^2c^4 - 32a^3c^5 - 16a^5c)\\ &> 368a^4c^2 + 384a^3c^3 + 128a^2c^4 - 32a^3c^5 - 16a^5c.\end{aligned}$$

Denote

$$\begin{aligned}f_4(a,b,c) =& 368a^4c^2 + 384a^3c^3 + 128a^2c^4 - 32a^3c^5 - 16a^5c\\ =& 16a^2c(a^2(c-a+22c) + 2ac^2(12-c^2) + 8c^3).\end{aligned}$$

As $0 < a < c < 1$, we have $c - a > 0$ and $12 - c^2 > 0$. Therefore, $f_4(a,b,c) > 0$. Then,

$$Q(a,c,b) > Q(a,b,c) \tag{5.7}$$

Median ability juror vote first

The explicit expression for the $\Delta_{5.2}$ is,

$$\Delta_{5.2} = \frac{f_5(a,b,c)}{8192a^3c^3},$$

where,

$$\begin{aligned}
f_5(a,b,c) =& 48a^4b^3c^2 + 128a^3b^2c^2 + 8a^2b^5 + 16a^2b^4c + 64a^2bc^6 + 128a^2bc^4 + 256a^2c^5 \\
&+ b^7c^2 - 64a^6bc^2 - 256a^5c^2 - 128a^4bc^2 - a^2b^7 - 48a^2b^3c^4 \\
&- 128a^2b^2c^3 - 16ab^4c^2 - 8b^5c^2.
\end{aligned}$$

By factorization, we have:

$$f_5(a,b,c) = (c-a)f_6(a,b,c).$$

As $a < c$, we just need to show $f_6(a,b,c) > 0$, where

$$\begin{aligned}
f_6(a,b,c) =& 64a^5bc^2 + 64a^4bc^3 + 256a^4c^2 + 64a^3bc^4 + 128a^3bc^2 + 256a^3c^3 + 64a^2bc^5 \\
&+ 128a^2bc^3 + 256a^2c^4 + ab^7 + b^7c - 48a^3b^3c^2 - 48a^2b^3c^3 - 128a^2b^2c^2 \\
&- 8ab^5 - 16ab^4c - 8b^5c
\end{aligned}$$

The no herding condition implies that $a > c/2$. Also, we have $0 < a < b < c < 1$. We know $b > c/2$. Use $b > c/2$ to reduce minuend and $b < c$ to enlarge subtrahend. Then we have

$$\begin{aligned}
f_6(a,b,c) >& 32a^5c^3 + 32a^4c^4 + 256a^4c^2 + 192a^2c^4 + 320a^3c^3 - 16a^3c^5 - 16a^2c^65 \\
&- 24ac^{-}8c^6 + ((ac^7)/128 + c^8/128) \\
&> 32a^5c^3 + 32a^4c^4 + 256a^4c^2 + 192a^2c^4 + 320a^3c^3 - 16a^3c^5 - 16a^2c^6 \\
&- 24ac^5 - 8c^6.
\end{aligned}$$

Denote

$$f_7(a,b,c) = 32a^5c^3 + 32a^4c^4 + 256a^4c^2 + 192a^2c^4 + 320a^3c^3 - 16a^3c^5 - 16a^2c^6 - 24ac^5 - 8c^6.$$

Use $a > c/2$ to reduce minuend and $a < c$ to enlarge subtrahend of $f_7(a,b,c)$. Then, we have,

$$f_7(a,b,c) > c^6(72 - 29c^2) > 0.$$

Since $f_7(c)$ are positive. Then,

$$Q(b,c,a) > Q(b,a,c). \tag{5.8}$$

Highest ability juror vote first

The explicit expression for the $\Delta_{5.3}$ is,

$$\Delta_{5.3} = \frac{f_8(a,b,c)}{8192a^3b^3},$$

where

$$\begin{aligned} f_8(a,b,c) =& 48a^4b^2c^3 + 128a^3b^2c^2 + 64a^2b^6c + 256a^2b^5 + 128a^2b^4c + 16a^2bc^4 + 8a^2c^5 \\ &+ b^2c^7 - 64a^6b^2c - 256a^5b^2 - 128a^4b^2c - 48a^2b^4c^3 - 128a^2b^3c^2 - a^2c^7 \\ &- 16ab^2c^4 - 8b^2c^5. \end{aligned}$$

By factorization, we have

$$f_8(a,b,c) = (b-a)f_9(a,b,c).$$

As $0 < a < b < c < 1$, we just need to show $f_9(a,b,c) > 0$ where,

$$\begin{aligned} f_9(a,b,c) =& 64a^5b^2c + 64a^4b^3c + 256a^4b^2 + 64a^3b^4c + 256a^3b^3 + 128a^3b^2c + 64a^2b^5c \\ &+ 256a^2b^4 + 128a^2b^3c + ac^7 + bc^7 - 48a^3b^2c^3 - 48a^2b^3c^3 - 128a^2b^2c^2 \\ &- 16abc^4 - 8ac^5 - 8bc^5. \end{aligned}$$

If the abilities of the jurors satisfy the condition $0 < a \leqslant b \leqslant c \leqslant 1$, the minimum of $f_9(a,b,c)$ is obtained when $a = b = 1/2, c = 1$, namely $f_9(a,b,c)_{min} = f_9(1/2,1/2,1) = 0$. However, the abilities of the jurors actually satisfy $0 < c/2 < a < b < c < 1$. Therefore, $f_9(a,b,c) > 0$, namely, we have

$$Q(c,b,a) > Q(c,a,b). \tag{5.9}$$

Summary

From inequalities (5.7), (5.8) and (5.9), we know that given that the jurors' abilities satisfy $0 \leqslant a < b < c \leqslant 1$ and there is no herding ($(a,b) \notin h_2$ and $(a,b,c) \notin h_3$), no matter who votes first, the latter two jurors should follow seniority order.

5.1.2 Optimal voting order under SVKP

From Lemma 5.1, we know that the latter two jurors should follow seniority order. Thus, we just need to compare $Q(a,c,b)$, $Q(b,c,a)$, and $Q(c,b,a)$. Proposition 5.2 states that the Alpern-Chen order is optimal. Thus, the following proof will show that $Q(b,c,a)$ is higher than the other two. Firstly, we define the $\Delta_{5.4}$ and $\Delta_{5.5}$ as:

$$\Delta_{5.4} = Q(b,c,a) - Q(a,c,b),$$

$$\Delta_{5.5} = Q(b,c,a) - Q(c,b,a).$$

We start with the calculation of the $\Delta_{5.4}$:

$$\Delta_{5.4} = \frac{g_1(a,b,c)}{8192abc^3},$$

where,

$$\begin{aligned} g_1(a,b,c) =& a^8 + 4a^6c^2 + 64a^4c^2 + 64a^2b^4c^2 + 384a^2c^4 + 256ab^3c^2 + 8b^6 + 16b^5c \\ &+ 16b^4c^4 + 128b^3c^3 + 64b^2c^6 + 256bc^5 - 8a^6 - 16a^5c - 64a^4b^2c^2 \\ &- 16a^4c^4 - 256a^3bc^2 - 28a^3c^3 - 64a^2c^6 - 256ac^5 - b^8 - 4b^6c^2 \\ &- 64b^4c^2 - 384b^2c^4. \end{aligned}$$

By factorization, we have:

$$g_1(a,b,c) = (b-a)g_2(a,b,c).$$

As $a < b$, we just need to show $g_2(a,b,c) > 0$,

$$\begin{aligned} g_2(a,b,c) =& 8a^5 + 8a^4b + 16a^4c + 60a^3b^2c^2 + 8a^3b^2 + 16a^3bc + 16a^3c^4 + 60a^2b^3c^2 \\ &+ 8a^2b^3 + 16a^2b^2c + 16a^2bc^4 + 192a^2bc^2 + 128a^2c^3 + 8ab^4 + 16ab^3c \\ &+ 16ab^2c^4 + 192ab^2c^2 + 128abc^3 + 64ac^6 + 8b^5 + 16b^4c + 16b^3c^4 \\ &+ 128b^2c^3 + 64bc^6 + 256c^5 - a^7 - a^6b - a^5b^2 - 4a^5c^2 - a^4b^3 \\ &- 4a^4bc^2 - a^3b^4 - 64a^3c^2 - a^2b^5 - ab^6 - 4ab^4c^2 - 384ac^4 \\ &- b^7 - 4b^5c^2 - 64b^3c^2 - 384bc^4. \end{aligned}$$

Clearly, $g_2(0,0,0) = 0$. Thus, by continuity, the infimum of $g_2(a,b,c)$ on the region must be 0. This numerical result includes the two additional conditions implied by the no herding conditions $a > c/2$ and $b > c/2$. All herding cases are discussed in Section 5.1.4. This indicates the inequality (5.10):

$$Q(b,c,a) \geqslant Q(a,c,b). \tag{5.10}$$

The explicit expression of $\Delta_{5.5}$ is:

$$\Delta_{5.5} = \frac{g_3(a,b,c)}{8192ab^3c^3},$$

where,

$$\begin{aligned}g_3(a,b,c) =& 64a^2b^6c^2 + 512a^2b^2c^4 + 256ab^5c^2 + 8b^8 + 16b^7c + 48b^4c^6 + 128b^3c^5 + 4b^2c^8 \\ &+ 64b^2c^6 + c^{10} - 512a^2b^4c^2 - 64a^2b^2c^6 - 256ab^2c^5 - b^10 - 4b^8c^2 - 48b^6c^4 \\ &- 64b^6c^2 - 128b^5c^3 - 16bc^7 - 8c^8.\end{aligned}$$

By factorization, we have:

$$g_3(a,b,c) = (c-b)g_4(a,b,c).$$

As $b < c$, we just need to show $g_4(a,b,c) > 0$

$$\begin{aligned}g_4(a,b,c) =& 512a^2b^3c^2 + 512a^2b^2c^3 + b^9 + b^8c + 5b^7c^2 + 5b^6c^3 + 53b^5c^4 + 40b^5c^2 \\ &+ 53b^4c^5 + 168b^4c^3 + 5b^3c^6 + 168b^3c^4 + 5b^2c^7 + 40b^2c^5 + bc^8 + c^9 \\ &- 64a^2b^5c^2 - 64a^2b^4c^3 - 64a^2b^3c^4 - 64a^2b^2c^5 - 256ab^4c^2 \\ &- 256ab^3c^3 - 256ab^2c^4 - 8b^7 - 24b^6c - 24bc^6 - 8c^7.\end{aligned}$$

Clearly, $g_4(0,0,0) = 0$. Thus, by continuity, the infimum of $g_4(a,b,c)$ on the region must be 0. This indicates the inequity (5.11):

$$Q(b,c,a) \geqslant Q(c,b,a). \tag{5.11}$$

From inequalities (5.11), (5.10) and Proposition 5.2, we know that if the jurors' abilities satisfy $0 \leqslant a < b < c \leqslant 1$, the Alpern-Chen order is the optimal order with the highest reliability.

5.1.3 Seniority order for the triple

In this section, we will show that seniority voting order (decreasing in terms of ability) has higher reliability of the verdict than anti-seniority order (increasing order of abilities), given that the ability of a juror are distinct $0 \leqslant a < b < c \leqslant 1$. Firstly, we define the $\Delta_{5.6}$ as follows:

$$\begin{aligned}\Delta_{5.6} &= Q(c,b,a) - Q(a,b,c)\\ &= \frac{g_5(a,b,c)}{8192ab^3c},\end{aligned}$$

where,

$$\begin{aligned}g_5(a,b,c) =& a^8 + 4a^6b^2 + 64a^4b^2 + 384a^2b^4 + 64a^2b^2c^4 + 256ab^2c^3 + 64b^6c^2\\ &+ 256b^5c + 16b^4c^4 + 128b^3c^3 + 16bc^5 + 8c^6 - 8a^6 - 16a^5b - 16a^4b^4\\ &- 64a^4b^2c^2 - 128a^3b^3 - 256a^3b^2c - 64a^2b^6 - 256ab^5 - 384b^4c^2\\ &- 4b^2c^6 - 64b^2c^4 - c^8.\end{aligned}$$

By factorisation, we have

$$g_5(a,b,c) = (c-a)g_6(a,b,c).$$

As $a < c$, we just need to show $g_6(a,b,c) > 0$, where,

$$\begin{aligned}g_6(a,b,c) =& 8a^5 + 16a^4b + 8a^4c + 16a^3b^4 + 60a^3b^2c^2 + 16a^3bc + 8a^3c^2\\ &+ 16a^2b^4c + 128a^2b^3 + 60a^2b^2c^3 + 192a^2b^2c + 16a^2bc^2\\ &+ 8a^2c^3 + 64ab^6 + 16ab^4c^2 + 128ab^3c + 192ab^2c^2 + 16abc^3\\ &+ 8ac^4 + 64b^6c + 256b^5 + 16b^4c^3 + 128b^3c^2 - 384b^4c - 4b^2c^5\\ &- 64b^2c^3 + 16bc^4 - c^7 + 8c^5 - a^7 - a^6c - 4a^5b^2 - a^5c^2\\ &- 4a^4b^2c - a^4c^3 - 64a^3b^2 - a^3c^4 - a^2c^5 - 384ab^4 - 4ab^2c^4 - ac^6.\end{aligned}$$

From $g_6(a,b,c)$, we collect the terms of order seven,

$$\begin{aligned}g_7(a,b,c) =& 16a^3b^4 + 60a^3b^2c^2 + 16a^2b^4c + 60a^2b^2c^3 + 64ab^6\\ &+ 16ab^4c^2 + 64b^6c + 16b^4c^3 - a^7 - a^6c - 4a^5b^2 - a^5c^2\\ &- 4a^4b^2c - a^4c^3 - a^3c^4 - a^2c^5 - 4ab^2c^4 - ac^6\\ &- 4b^2c^5 - c^7.\end{aligned}$$

The no herding condition indicates $b/2 < a$ and $c/2 < b$. Using $b/2 < a$ to reduce minuend and $a < b$ to enlarge subtrahend, we have,

$$g_7(a,b,c) > 1/2(58b^7 + 126b^6c + 29b^5c^2 + 60b^4c^3 \\ - 10b^3c^4 - 10b^2c^5 - 2bc^6 - 2c^7).$$

Denote

$$g_8(a,b,c) = 58b^7 + 126b^6c + 29b^5c^2 + 60b^4c^3 - 10b^3c^4 - 10b^2c^5 - 2bc^6 - 2c^7.$$

Rearrange $g_8(a,b,c)$

$$\begin{aligned} g_8(a,b,c) =& 58b^7 + 126b^6c + 29b^5c^2 + 60b^4c^3 - 10b^3c^4 - 10b^2c^5 - 2bc^6 - 2c^7 \\ =& 10(8b^6c - b^3c^4) + (56b^6c - 7b^2c^5/2) + (26b^4c^3 - 13b^2c^5/2) \\ &+ 2(8b^4c^3 - c^3) + (58b^7 + 29b^5c^2 + 18b^4c^3 - 2c^7) \\ =& 10b^3c(8b^3 - c^3) + 7b^2c(16b^4 - c^4)/2 + 13b^2c^3(4b^2 - c^2)/2 \\ &+ 2bc^3(8b^3 - c^3) + (58b^7 + 29b^5c^2 + 18b^4c^3 - 2c^7). \end{aligned}$$

The no herding condition indicates $b/2 < a$ and $c/2 < b$. Thus,

$$\begin{aligned} g_8(a,b,c) =& 10b^3c(8b^3 - c^3) + 7b^2c(16b^4 - c^4)/2 + 13b^2c^3(4b^2 - c^2)/2 \\ &+ 2bc^3(8b^3 - c^3) + (58b^7 + 29b^5c^2 + 18b^4c^3 - 2c^7) \\ &> 0 + 0 + 0 + (58(c/2)^7 + 29(c/2)^5c^2 + 18(c/2)^4c^3 - 2c^7) \\ &= 31/64c \\ &> 0. \end{aligned}$$

Now we collect the terms of order five (the rest of the $g_6(a,b,c)$ after we collect the terms of order seven).

$$\begin{aligned} g_9(a,b,c) =& 8a^5 + 16a^4b + 8a^4c + 16a^3bc + 8a^3c^2 + 128a^2b^3 \\ &+ 192a^2b^2c + 16a^2bc^2 + 8a^2c^3 + 128ab^3c + 192ab^2c^2 \\ &+ 16abc^3 + 8ac^4 + 256b^5 + 128b^3c^2 + 16bc^4 + 8c^5 - 64a^3b^2 \\ &- 384ab^4 - 384b^4c - 64b^2c^3 \end{aligned}$$

For minimisation of $g_9(a,b,c)$, subject of $0 \leqslant a \leqslant b \leqslant c \leqslant 1$ and $c - 2a \geqslant 0$, the

Kuhn-Tucker conditions for the potential minimal points are as follows:

$$\begin{aligned}
&40a^4+64a^3b+256ab^3+32a^3c+48a^2bc+384ab^2c\\
&+128b^3c+24a^2c^2+32abc^2+192b^2c^2+16ac^3+16bc^3\\
&+8c^4-192a^2b^2-384b^4-\lambda_1+\lambda_2-2\lambda_5=0,\\
&16a^4-128a^3b+384a^2b^2-1536ab^3+1280b^4+16a^3c\\
&+384a^2bc+384ab^2c-1536b^3c+16a^2c^2+384abc^2+384b^2c^2\\
&+16ac^3-128bc^3+16c^4-\lambda_2+\lambda_3+\lambda_5-2\lambda_6=0,\\
&8a^4+16a^3b+192a^2b^2+128ab^3+16a^3c+32a^2bc\\
&+384ab^2c+256b^3c+24a^2c^2+48abc^2+32ac^3+64bc^3\\
&+40c^4-384b^4-192b^2c^2-\lambda_3+\lambda_4+\lambda_6=0,\\
&\lambda_1a=0,\lambda_2(b-a)=0,\lambda_3(c-b)=0,\\
&\lambda_4(1-c)=0,\lambda_5(2a-b)=0,\lambda_6(2b-c)=0,\\
&\lambda_1,...,\lambda_6\geqslant 0;\ 0\leqslant a\leqslant b\leqslant c\leqslant 1\ ,b-2a\geqslant 0\ and\ c-2b\geqslant 0.
\end{aligned} \tag{5.12}$$

The condition (5.12) has a unique solution of (0,0,0,0,0,0,0,0,0) and $g_5(0,0,0)=0$. Hence we have $g_9(a,b,c)\geqslant 0$. Combining with $g_7(a,b,c)>0$, we have $g_6(a,b,c)>0$. That is, $\Delta_{5.6}>0$. The relative comparison between seniority order and anti-senority order brings us to a clear conclusion that seniority order is superior.

5.1.4 Herding cases

This section will discuss all the herding cases ($(a,b)\in h_2$ and $(a,b,c)\in h_3$) under SVKP.

Second juror herding

If herding exists for the second juror, the reliability is the same for the following two voters in anti-seniority order and seniority order. In this case, the second layer voters' threshold is either -1 or 1. The condition for this herding is that the second juror's ability is less than half that of the first juror. Therefore, the first two jurors must be in seniority order. In this case, we need to compare $Q(b,a,c)=b/4+1/2$ and $Q(c,a,b)=c/4+1/2$. Because $0\leqslant a<b<c\leqslant 1$, we have,

$$Q(c,a,b)>Q(b,a,c).$$

If herding exists for the second juror, the majority voting is meaningless as the first juror makes the decision. Therefore, we should avoid this kind of herding situation. This requires that the ability of the second juror is larger than half that of the first juror. In practice, we can use a different method to measure these abilities, such as working length or position. In conclusion, the more able juror should vote first if herding exists for the second juror.

Third juror herding

In this case, we consider cases where the second juror does not copy the previous juror's vote (also discussed in the previous section) but the third juror does. Thus, the first two jurors follow anti-seniority order. We only consider the cases $Q(a, c, b)$ and $Q(b, c, a)$:

$$\begin{aligned} Q(a, c, b) &= \frac{a^2 + 4c^2 + 8c}{16c}, \\ Q(b, c, a) &= \frac{b^2 + 4c^2 + 8c}{16c}. \end{aligned}$$

$Q(b, c, a)$ is clearly larger than $Q(a, b, c)$, as $0 \leqslant a < b < c \leqslant 1$. Thus, the Alpern-Chen order is optimal in this case.

Summary

Herding cases follow simple reasoning. The experience from two cases is quite straight forward as well. When the herding happens, whether for the second or the third juror, we should let the juror with the lowest ability follow the behaviour of the juror with the highest ability, i.e. (c, a, b) or (b, c, a). In other words, when herding exists, the best practice is letting the least able juror imitate the ablest juror's behaviour.

5.2 Comparison with the simultaneous voting scheme

The most notable feature of simultaneous voting lies in its unique threshold for each juror, which is independent of the other jurors' votes and voting behaviour. In SVKP, the juror knows the previous juror's vote (additional information). It is interesting to look at the effect of this change on the performance of the jury. Based on Proposition 5.3, we will first compare the optimal order under SVKP. Then, we will compare the average reliability and optimal fraction under six different voting orders.

In this section, we will compare the performance of the optimal voting order under SVKP and simultaneous voting in term of reliability. As shown in Figure 5.4, we calculate the reliability of all the points in Alpern-Chen order under SVKP and simultaneous voting in the tetrahedron with vertex (0, 0, 0), (1, 1, 1), (0, 0, 1), (0, 1, 1) which contains all the ability sets (a, b, c) satisfy the condition $0 < a < b < c < 1$ with step 0.01 starting from (0.01, 0.02, 0.03) to (0.97, 0.98, 0.99).

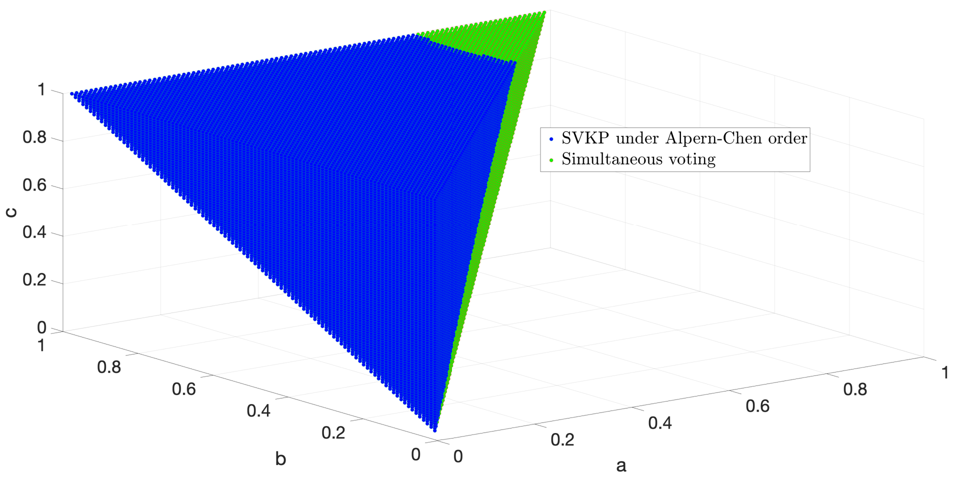


Figure 5.4: SVKP and the simultaneous voting scheme

The blue dots mean that the optimal voting order under SVKP is better than the simultaneous voting scheme. The green dots mean that the simultaneous voting is superior to the optimal voting order under SKVP. From the mixed Figure 5.4, we know that when the difference in abilities is large enough, the optimal voting order (Alpern-Chen order) under SVKP voting scheme is superior. Figure 5.5 provides the boundary of the simultaneous part. Similar to the SVI voting scheme, the maximum difference is around 0.3.

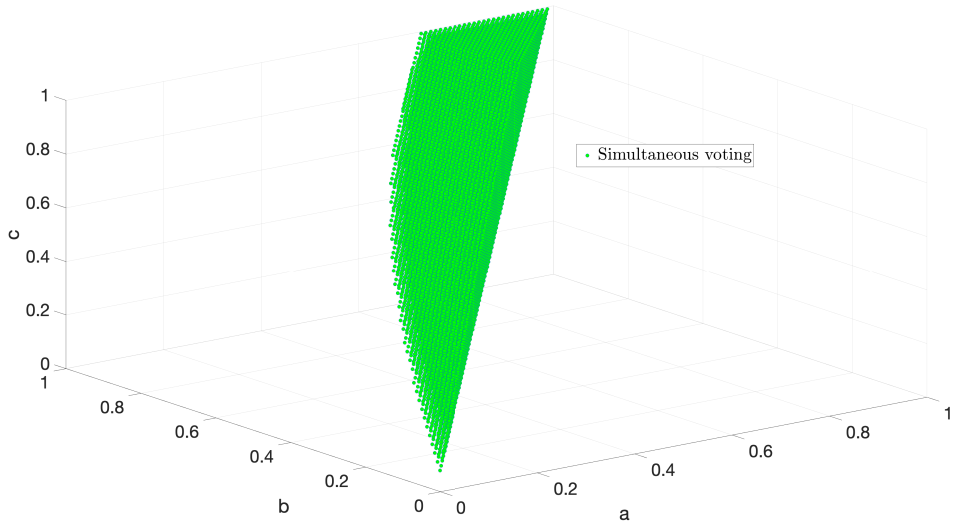


Figure 5.5: Comparison between optimal order under SVKP and the simultaneous voting scheme (simultaneous voting part)

Voting Scheme	Counts(#)	Proportion(%)
SVKP	133936	87.29%
Simultaneous	19489	12.70%

Table 5.2: Comparison between SVKP (f) and simultaneous voting (a)

Table 5.2 shows that in almost 90% of cases SVKP is better than simultaneous voting, which is an overwhelming advantage. The optimal voting order under SVKP is achieved when the voting order is done according to Alpern-Chen ordering. Alpern-Chen ordering dictates that the middle-ability juror votes first, followed by the highest ability juror, and the lowest juror votes last. The corresponding reliability is:

$$\begin{aligned} Q(b,c,a) =&(64a^2b^3c^2 - 256a^2bc^4 - 512a^2bc^2 + 1024a^2c^3 + 256ab^2c^2 + 1024abc^3 \\ &+ 1024ac^4 + 4096ac^3 - b^7 - 4b^5c^2 + 8b^5 + 16b^4c + 16b^3c^4 - 64b^3c^2 \\ &+ 128b^2c^3 + 64bc^6 - 384bc^4 + 256c^5)/(8192ac^3). \end{aligned} \tag{5.13}$$

The reliability of the simultaneous voting scheme has already been calculated in the previous chapter. The difference between equations (5.13) and (4.3) is,

$$
\begin{aligned}
Q(b,c,a) - Q(abc) =& (64a^2b^3c^2 - 512a^2bc^2 + 256ab^2c^2 - b^7 - 4b^5c^2 \\
&+ 8b^5 + 16b^4c + 16b^3c^4 - 64b^3c^2 + 128b^2c^3 \\
&+ 64bc^6 - 384bc^4 + 256c^5)/(8192ac^3).
\end{aligned} \tag{5.14}
$$

When the difference is zero, we can obtain the boundary equation:

$$
\begin{aligned}
&64a^2b^3c^2 - 512a^2bc^2 + 256ab^2c^2 - b^7 - 4b^5c^2 + 8b^5 + 16b^4c \\
&+ 16b^3c^4 - 64b^3c^2 + 128b^2c^3 + 64bc^6 - 384bc^4 + 256c^5 = 0.
\end{aligned} \tag{5.15}
$$

By solving the quadratic equation (5.15) in terms of a, we can obtain the explicit expression for a:

$$
a = \frac{256b^2c^2 + \sqrt{65536b^4c^4 - 4\left(64b^3c^2 - 512bc^2\right)g_{10}}}{128\left(8bc^2 - b^3c^2\right)}, \tag{5.16}
$$

where,

$$
\begin{aligned}
g_{10} =& -b^7 - 4b^5c^2 + 8b^5 + 16b^4c + 16b^3c^4 - 64b^3c^2 + 128b^2c^3 \\
&+ 64bc^6 - 384bc^4 + 256c^5.
\end{aligned}
$$

The other solution is negative. This quadratic equation explains the shape of the boundary.

Table 5.3: Average reliabilities for SVKP and simultaneous voting

Voting scheme and ordering	$\bar{Q}$	example	rank
(a,b,c) SVKP	0.688865240652835	(1, 2, 3)	1
(a,c,b) SVKP	0.688865240652774	(1, 3, 2)	2
(b,a,c) SVKP	0.688865240652321	(2, 1, 3)	5
(b,c,a) SVKP	**0.688865240652561**	**(2, 3, 1)**	**3**
(c,a,b) SVKP	0.688865240652319	(3, 1, 2)	6
(c,b,a) SVKP	0.688865240652555	(3, 2, 1)	4
Simultaneous	0.685350386718749	(1, 2, 3)	7

Another measure of performance is the average reliability. We want to know which order is better than the simultaneous voting scheme. As shown in Table 5.3, all six voting orders under SVKP are better than simultaneous voting.

To sum up, SVKP under optimal voting order is better than simultaneous voting in terms of the optimal fraction with spread 0.75. In terms of average reliability, all six orders under SVKP are superior to simultaneous voting.

5.3 Larger juries

The main results of Proposition 5.2 (Alpern-Chen order as optimal, with the highest ability juror vote first followed by the median ability juror and the lowest ability juror vote last) and Proposition 5.3 (seniority order outperforms anti-seniority order) have been proved algebraically only for juries of three. The natural question to ask is whether these patterns will still hold for a larger jury. Due to explosion in the combinations of orderings and the computation of thresholds, the algebraic method seems out of reach for larger juries. Thus, numerical methods through simulations are adopted to study the patterns of a larger jury.

Jury size of five

We started with a jury of five using a numerical method to determine the highest reliability under honest voting. Take the jury $\{0.1, 0.2, 0.3, 0.4, 0.5\}$ as an example. For simplicity, we denote them by $\{1, 2, 3, 4, 5\}$ for the rest of this section, which can be considered as an arbitrary ranking of the abilities among members of the jury as shown in Tables 5.4 and 5.5. There are $5! = 120$ voting orders for such a jury. One million random juries (a, b, c, d, e) were generated to conduct a trial, where a, b, c, d and e were chosen independently and uniformly in $[0, 1]$. One million trials were conducted to estimate the fraction of the orders that are optimal. We count the number of a particular voting ordering (among 120) being optimal divided by the total number of simulation ability sets. Only 10 of 120 orders are optimal in a significant fraction of cases, which are presented in Table 5.4. The first two orders occupied around 80% of the total cases. The similarity between the first two orders is that the three most able jurors vote in increasing order of ability (anti-seniority order). The latter two judges, meanwhile vote in seniority order. In other words, the least able among the five jurors vote in decreasing order of abililty.. This is called Ascending-Descending Order (ADO), which can be seen as a general form of Alpern-Chen order for a large jury, as proposed by Alpern & Chen (2020). ADO states that for a jury with an odd number of members $(2i + 1)$, the ablest $i + 1$ jurors vote in ascending order of ability while the least able i jurors vote in descending order. This ordering means that the juror with median ability votes first, the juror with the highest ability votes in the third (for a jury of five), and the juror with the lowest ability votes last. ADO is an extension of median-high-low (Alpern-Chen) ordering. Table 5.4 indicates that this ADO ordering, along with anti-seniority order (AO) for the first three, is a useful heuristic for a jury size of five.

Voting Scheme	Counts(#)	Proportion(%)
(c, d, e, b, a) (3 4 5 2 1)	43532	43.532%
(c, d, e, a, b) (3 4 5 1 2)	37702	37.702%
(b, d, e, c, a) (2 4 5 3 1)	11083	11.083%
(d, a, e, b, c) (4 1 5 2 3)	1912	1.912%
(c, a, e, b, d) (3 1 5 2 4)	1361	1.361%
(e, b, d, a, c) (5 2 4 1 3)	1267	1.267%
(c, e, d, a, b) (3 5 4 1 2)	1244	1.244%
(d, b, e, a, c) (4 2 5 1 3)	1219	1.219%
(b, c, e, d, a) (2 3 5 4 1)	603	0.603%
(d, c, e, a, b) (4 3 5 1 2)	73	0.073%

Table 5.4: Optimal fraction for jury of five under SVKP

Another critical measure of performance for the voting order of a random jury is *average reliability*. This is the average value of Q for all ability sets in the simulation. Table 5.5 shows that, among 120 orders, there are precisely ten with average reliabilities above 80%. These average reliabilities are calculated by taking the average of one thousand simulations. Each simulation contains one million trials. From Table 5.5, it can be seen that the top two orders keep the median juror in the first position. Although Tables 5.5 and 5.4 show different results, it is better to let the juror with median ability vote first in a jury of size five.

Table 5.6 looks at seniority ordering (SO), anti-seniority (AO), and Ascending-Descending Order (ADO). ADO is no longer optimal among the three orders also seen in Table 5.5 concerning average reliability. However, SO still dominates the AO in terms of $\bar{Q}$. Furthermore, among one billion cases, there are none where SO is inferior to AO among one billion cases. The superiority of SO over AO holds for a jury of five.

Table 5.5: Average reliabilities for jury of size five

Voting Scheme	$\bar{Q}$	Ranking	example
(c, e, a, b, d)	0.852980176	1	3 5 1 2 4
(c, a, e, b, d)	0.833206006	2	3 1 5 2 4
(a, c, b, e, d)	0.827385278	3	1 3 2 5 4
(b, d, c, e, a)	0.825768673	4	2 4 3 5 1
(c, e, b, d, a)	0.824029849	5	3 5 2 4 1
(d, b, e, c, a)	0.822842006	6	4 2 5 3 1
(a, b, e, d, c)	0.820527564	7	1 2 5 4 3
(b, e, a, d, c)	0.810914093	8	2 5 1 4 3
(b, c, e, a, d)	0.810440233	9	2 3 5 1 4
(d, a, b, c, e)	0.800732075	10	4 1 2 3 5

Table 5.6: Average reliabilities for AO, SO, and ADO

Voting Scheme	$\bar{Q}$	example
SO(e, d, c, b, a)	0.771197681	5 4 3 2 1
ADO(c, d, e, a, b)	0.708619934	3 4 5 1 2
AO(a, b, c, d, e)	0.696238213	1 2 3 4 5

5.4 Conclusions

The main result of this chapter is an extension of the Alpern-Chen ordering under the roll-call voting scheme. The Alpern-Chen theorem states that when jurors' abilities are heterogeneous, the median ability juror should vote first in the roll-call voting scheme. In this scheme, the third juror knows the votes of both of the previous jurors, while in SVKP, he only knows the second juror's vote. This degradation of information disclosure does not affect the Alpern-Chen theorem, indicating that the theorem is robust. Furthermore, under SVKP, the seniority order is superior to anti-seniority order for juries with different abilities. Seniority order also has the advantage in the duo voting structure. No matter who votes first, the latter two jurors should vote in decreasing order of ability. FIn herding cases, we should let the lowest ability juror vote, followed by the juror with the highest ability, to achieve higher reliability. For a jury of five, ADO has the highest optimal fraction, and the order (c, e, a, b, d) has the highest average reliability.

Chapter 6

Sequential voting with an initial public vote

In sequential voting with an initial public vote (SVP), as shown in Figure 6.1, one juror votes first and then the rest of the jury simultaneously announce their votes. The latter two jurors share the same initial information, which is the vote of the first juror.

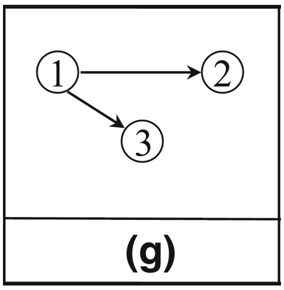


Figure 6.1: Sequential voting with an initial public vote (SVP)

In SVP, duo voting structures happen at the same time. One juror votes first, followed by two other jurors. The latter two jurors vote independently. Let $(v_1,\ \{v_2, v_3\})$ be a voting order under SVP. For example $r = (a,\ \{b,\ c\})$, the first juror having ability a while the latter two jurors have abilities b and c, respectively. Here, $r = (a,\ \{b,\ c\})$ is equivalent to $r = (a,\ \{c,\ b\})$ as the second layer jurors have the same positions. Once we understand this variation on the duo structure, we can calculate the reliability $Q(a, \{b, c\})$. Based on the analysis in Section 2.2.5, we know that $Q(a, \{b, c\}) = q_A(a, \{b, c\})$.

The formula for $q_A(a, \{b, c\})$ is the summation of probabilities of voting patterns $(A, \{A, B\}), (A, \{B, A\}), (A, \{A, A\})$ and $(B, \{A, A\})$ when the Nature is state A. There are potential cases: either the first juror votes for state A or the first juror votes for state B. When the first juror votes for state A, at least one of the latter two jurors should also vote state for A $((A,\{A,B\}),(A,\{B,A\})$, and $(A,\{A,A\})$). For simpler calculation, in this case, we just need to exclude the case when the latter two jurors both vote for state B $((A,\{B,B\}))$. When the first juror votes for state B, the latter two jurors must vote for state A $((B,\{A,A\}))$. Denote v_1 as the vote of the first juror. The simplified formula is as follows:

$$\begin{aligned}q_A(a,\{b,c\}) &= (1-\Pr[v_1=A](1-\Pr[ABB|v_1=A]))+\Pr[v_1=B](1-\Pr[BAA|v_1=B])\\ &= (1-F_A(a,0))(1-F_A(b,y_A)F_A(c,y_A))\\ &+ F_A(a,0)(1-F_A(b,y_B))(1-F_A(c,y_B)).\end{aligned}$$

When we substitute the y_A (y_B) formula to our linear continuous signal model, we have two cases depending on whether the pairs (a,b) or (a,c) belong to h_2. Then,

$$Q(a,\{b,c\}) = \begin{cases} a/4+1/2, & \text{if } (a,b) \text{ or } (a,c)\in h_2 \ (b\leqslant a/2 \text{ or } c\leqslant a/2)\ ; \\ Q_3(a,\{b,c\}), & \text{otherwise,}\end{cases} \tag{6.1}$$

where,

$$Q_3(a,\{b,c\}) = \frac{f_1(a,b,c)}{512bc},$$

and,

$$\begin{aligned}f_1(a,b,c) =& 4a^3b^2+4a^3c^2+16a^2b+16a^2c+64b^2c+64bc^2+256bc+64abc\\ &-a^5-16ab^2c^2-32ab^2-32ac^2.\end{aligned}$$

6.1 Optimal voting order under SVP

In this section, we will discuss the optimal voting under SVP. There are three different propositions, but all emphasise the importance of seniority. In other words, all three propositions say that the juror with the highest ability should take the initial public vote. Proposition 6.1 is a strong statement that includes the optimal voting order with ranking and the conditions for it. Proposition 6.2 states that if we relax the restrictions for the juror with the lowest ability, we can still have the same optimal voting order (highest-ability juror goes first) but lose the certainty of ranking for the other two voting orders. Proposition 6.3 states that if any herding happens $(a,b)\in h_2$ or $(a,c)\in h_2$, the optimal voting order is still that the ablest juror casts the initial public vote. The following contents will be presented in this way: Firstly, three prepositions will be provided along with the proofs for strong form and weak form. Then we will discuss herding cases.

Proposition 6.1. *(Strong form) Given that the jurors' abilities satisfy* $0.8165 \leqslant a < b < c \leqslant 1$*, the reliability is highest when c votes first and lowest when a votes first,* $Q(c,\{a,b\}) > Q(b,\{a,c\}) > Q(a,\{b,c\})$*.*

Proposition 6.2. *(Weak form) Given that the jurors' abilities satisfy* $0.521 \leqslant a < b < c \leqslant 1$*, the reliability is highest when c vote first,* $Q(c,\{a,b\}) > Q(b,\{a,c\})$ *and* $Q(c,\{a,b\}) > Q(a,\{b,c\})$*.*

Proposition 6.3. *Given that the jurors' abilities satisfy* $0 < a < b < c < 1$ *and there is herding behaviour among the jurors, the reliability is highest when c votes first and lowest when b votes first,* $Q(c,\{a,b\}) > Q(b,\{a,c\})$*.*

Proof of Proposition 6.1

The following proof assumes that for jurors, their abilities have $0.8165 \leqslant a < b < c \leqslant 1$. The idea of this proof is simple: We have three orders: the highest ability juror voting first, the middle ability juror voting first and the lowest ability juror voting first. We will show that the highest ability first superior to the middle ability juror first. Then, we will show that the middle ability juror taking the initial public vote is better than the lowest ability. By combining these two claims, we can inference that the highest ability juror first is better than the other two orders.

Firstly, we define $\Delta_{6.1}$, $\Delta_{6.2}$ and $\Delta_{6.3}$:

$$\begin{aligned}\Delta_{6.1} &= Q(c,\{a,b\}) - Q(b,\{a,c\}),\\ \Delta_{6.2} &= Q(c,\{a,b\}) - Q(a,\{b,c\}),\\ \Delta_{6.3} &= Q(b,\{a,c\}) - Q(a,\{a,b\}).\end{aligned}$$

Substitute with the reliability formula for each order. We will have the explicit equation for $\Delta_{6.1}$, $\Delta_{6.2}$ and $\Delta_{6.3}$:

$$\Delta_{6.1} = \frac{f_2(a,b,c)}{512abc},$$

where,

$$\begin{aligned}f_2(a,b,c) =& 32a^2b^2 + 4a^2c^4 + 16ac^3 + b^6 + 4b^2c^4 + 16bc^3\\ &- 4a^2b^4 - 32a^2c^2 - 16ab^3 - 4b^4c^2 - 16b^3c - c^6,\end{aligned}$$

$$\Delta_{6.2} = \frac{f_3(a,b,c)}{512abc},$$

$$
\begin{aligned}
f_3(a,b,c) =& a^6 + 32a^2b^2 + 4a^2c^4 + 16ac^3 + 4b^2c^4 + 16bc^3 \\
& - 4a^4b^2 - 4a^4c^2 - 16a^3b - 16a^3c - 32b^2c^2 - c^6,
\end{aligned}
$$

and,

$$
\Delta_{6.3} = \frac{f_4(a,b,c)}{512abc},
$$

where,

$$
\begin{aligned}
f_4(a,b,c) =& a^6 + 4a^2b^4 + 32a^2c^2 + 16ab^3 + 4b^4c^2 + 16b^3c \\
& - 4a^4b^2 - 4a^4c^2 - 16a^3b - 16a^3c - b^6 - 32b^2c^2.
\end{aligned}
$$

It should be noticed that $\Delta_{6.2} = \Delta_{6.1} + \Delta_{6.3}$. Therefore, we just need to prove that both signs of $\Delta_{6.1}$ and $\Delta_{6.3}$ are positive. We start with the numerator of $\Delta_{6.1}$:

$$
\begin{aligned}
f_2(a,b,c) =& ((4a^2b^3 + 4a^2b^2c + 4a^2bc^2 + 4a^2c^3 + 3b^3c^2 + 3b^2c^3 - b^5 - b^4c \\
& - bc^4 - c^5) + (16ab^2 + 16ac^2 + 16abc + 16b^2c + 16bc^2 - 32a^2b \\
& - 32a^2c))(c-b),
\end{aligned}
$$

For $f_2(a,b,c)$, we collect the terms of order five:

$$
\begin{aligned}
& 4a^2b^3 + 4a^2b^2c + 4a^2bc^2 + 4a^2c^3 + 3b^3c^2 + 3b^2c^3 - b^5 - b^4c - bc^4 - c^5 \\
=& (4a^2b^3 - b^5) + (4a^2b^2c - b^4c) + (4a^2bc^2 - bc^4) + (3b^2c^3 - c^5) + 4a^2c^3 \\
& + 3b^3c^2 \\
=& b^3(2a-b)(2a+b) + b^2c(2a-b)(2a+b) + bc^2(2a-c)(2a+c) \\
& + c^3(\sqrt{3}b - c)(\sqrt{3}b + c) + 4a^2c^3 + 3b^3c^2 \\
>& b^3(2 \times 0.8165 - 1)(2a+b) + b^2c(2 \times 0.8165 - 1)(2a+b) + bc^2(2 \times 0.8165 \\
& - 1)(2a+c) + c^3(\sqrt{3} \times 0.8165 - c)(\sqrt{3}b + c) + 4a^2c^3 + 3b^3c^2 \\
>& 0.
\end{aligned}
$$

We have proven that the sum of all terms of order five for $f_2(a,b,c)$ is positive. Now we just need to prove the sign of the sum of all terms of order three for $f_2(a, b, c)$:

$$\begin{aligned}
&16ab^2 + 16ac^2 + 16abc + 16b^2c + 16bc^2 - 32a^2b - 32a^2c \\
=&16(ab^2 + ac^2 + abc + b^2c + bc^2 - 2a^2b - 2a^2c) \\
>&16b(ab^2 + ab^2 + ab^2 + b^2b + bb^2 - 2a^2b - 2a^2) \\
=&16(a(3b^2 - 2a) + 2b(b - a)(b + a)) \\
>&16a(3b^2 - 2a) \\
>&16ab^2(3 \times 0.8165^2 - 2) \\
>&0
\end{aligned}$$

Then, we have:

$$Q(c, \{a, b\}) > Q(b, \{a, c\}), \tag{6.2}$$

Next, we just need to know the relation between $Q(b, \{a, c\})$ and $Q(a, \{a, b\})$ to know the ranking among $Q(c, \{a, b\})$, $Q(b, \{a, c\})$ and $Q(a, \{a, b\})$.

We only consider the sign of the $\Delta_{6.3}$. Then, we just need to focus on the numerators as we have $a, b, c > 0$.

$$\begin{aligned}
f_4(a, b, c) =&(b - a)(3a^3b^2 + 4a^3c^2 + 3a^2b^3 + 4a^2bc^2 + 16a^2b + 16a^2c + 4ab^2c^2 \\
&+ 16ab^2 + 16abc + 4b^3c^2 + 16b^2c - a^5 - a^4b - ab^4 - 32ac^2 - b^5 \\
&- 32bc^2)
\end{aligned}$$

Denote

$$\begin{aligned}
d_1 =&3a^3b^2 + 4a^3c^2 + 3a^2b^3 + 4a^2bc^2 + 16a^2b + 16a^2c + 4ab^2c^2 + 16ab^2 + 16abc \\
&+ 4b^3c^2 + 16b^2c - a^5 - a^4b - ab^4 - 32ac^2 - b^5 - 32bc^2.
\end{aligned}$$

Take the partial derivative with respect to a, we have:

$$\begin{aligned}
\partial_a(d_1(a, b, c)) =&(9a^2b^2 - 5a^4 - 4a^3b) + (4b^2c^2 - b^4) \\
&+12a^2c^2 + 6ab^3 + 8abc^2 + (32ab + 32ac + 16b^2 + 16bc - 32c^2) \\
=&(5a^2(b + a)(b - a) + 4a^2b(b - a)) + b^2(2c + b)(c + c - b)) \\
&+12a^2c^2 + 6ab^3 + 8abc^2 + 16(2ab + 2ac + b^2 + bc - 2c^2) \\
>&16(6 \times 0.8165^2 - 2) \\
>&0
\end{aligned}$$

Therefore, $d_1(a, b, c)$ is monotonically increasing in a. Similarly, we can take the partial derivative with respect to b. Then, we have:

$$\begin{aligned}\partial_b(d_1(a,b,c)) =&(6a^3b-a^4)+(12b^2c^2-4ab^3-5b^4)+9a^2b^2+4a^2c^2\\&+8abc^2+(16a^2+32ab+16ac+32bc-32c^2)\\=&a^3(5b+b-a)+4b^2(c^2-ab)+5b^2(c+b)(c-b)+3b^2c^2+9a^2b^2\\&+4a^2c^2+8abc^2+16(a^2+2ab+ac+2bc-2c^2)\\>&16(6\times 0.8165^2-2)\\>&0.\end{aligned}$$

Therefore, $d_1(a,b,c)$ is monotonically increasing in b as well. Because $d_1(a,b,c)$ is monotonically increasing in b and $a<b$, we have:

$$d_1(a,b,c)>d_1(a,a,c),$$

where,

$$d_1(a,a,c)=32a^3+2a^5+48a^2c-64ac^2+16a^3c^2.$$

Take the partial derivative of $d_1(a,a,c)$ with respect to c, we have:

$$\begin{aligned}\partial_c(d_1(a,a,c)) =&16a(2a^2c+3a-8c)\\<&16a(2a^2+3a-8a)\\=&16a^2(a-5/2)\\<&0.\end{aligned}$$

Therefore, $d_1(a,a,c)$ is monotonically decreasing in c. Additionally, $d_1(a,b,c)$ is monotonically increasing in a and b. And, we have $0.8165\leqslant a<b<c\leqslant 1$. Thus, we know

$$d_1(a,b,c)>d_1(a,a,c)\geqslant d_1(a,a,1)\geqslant d_1(0.8165,0.8165,1)=0.0046407.$$

Thus, the sign of $\Delta_{6.3}$ is positive. In other words, we have

$$Q(b,\{a,c\})>Q(a,\{b,c\}) \tag{6.3}$$

Given that $\Delta_{6.2}=\Delta_{6.1}+\Delta_{6.3}$, we know that the sign of the $\Delta_{6.2}$ is positive as well. In other wordss, we have:

$$Q(c,\{a,b\})>Q(a,\{b,c\}). \tag{6.4}$$

Inequalities (6.2) and (6.4) indicates that the reliability is highest when the highest ability juror votes first.
From inequalities (6.2) and (6.3), we have:

$$Q(c, \{a, b\}) > Q(b, \{a, c\}) > Q(a, \{b, c\}).$$

Proof of Proposition 6.2

The following proof assumes that for jurors, their abilities have $0.521 \leqslant a < b < c \leqslant 1$. The idea of the proof of Proposition 6.2 is the same as the first part of the proof of Proposition 6.1. We have three orders: highest ability first, middle ability first, and lowest ability first. We will show that the highest ability first is better than the other two orders. Here, Kuhn-Tucker conditions are applied to find the potential minimal points. Recall, the definition of $\Delta_{6.1}$ and $\Delta_{6.1}$:

$$\begin{aligned}\Delta_{6.1} &= Q(c,\{a,b\}) - Q(b,\{a,c\}),\\ \Delta_{6.2} &= Q(c,\{a,b\}) - Q(a,\{b,c\}).\end{aligned}$$

Substitute with the reliability formula for each order. We have the explicit expression for $\Delta_{6.1}$ and $\Delta_{6.1}$:

$$\Delta_{6.1} = \frac{f_2(a,b,c)}{512abc},$$

where,

$$\begin{aligned}f_2(a,b,c) =& 32a^2b^2 + 4a^2c^4 + 16ac^3 + b^6 + 4b^2c^4 + 16bc^3\\ &- 4a^2b^4 - 32a^2c^2 - 16ab^3 - 4b^4c^2 - 16b^3c - c^6,\end{aligned}$$

and,

$$\Delta_{6.2} = \frac{f_3(a,b,c)}{512abc},$$

where,

$$\begin{aligned}f_3(a,b,c) =& a^6 + 32a^2b^2 + 4a^2c^4 + 16ac^3 + 4b^2c^4 + 16bc^3\\ &- 4a^4b^2 - 4a^4c^2 - 16a^3b - 16a^3c - 32b^2c^2 - c^6.\end{aligned}$$

We only consider the signs of the $\Delta_{6.1}$ and $\Delta_{6.2}$. Then, we just need to focus on the numerators as we have $a,b,c>0$. By factorisation, we have:

$$\begin{aligned}f_2(a,b,c) =& (4a^2b^3 + 4a^2b^2c + 4a^2bc^2 + 4a^2c^3 + 3b^3c^2 + 3b^2c^3 - b^5 - b^4c - bc^4\\ &- c^5 + 16a^2b + 16ac^2 + 16abc + 16b^2c + 16bc^2 - 32a^2b - 32a^2c)(c-b).\end{aligned}$$

Because $b < c$, we only need to consider the other factor of $f_2(a,b,c)$. Denote

$$\begin{aligned}f_5(a,b,c) =& 4a^2b^3 + 4a^2b^2c + 4a^2bc^2 + 4a^2c^3 + 3b^3c^2 + 3b^2c^3 - b^5 - b^4c - bc^4\\ &- c^5 + 16a^2b + 16ac^2 + 16abc + 16b^2c + 16bc^2 - 32a^2b - 32a^2c.\end{aligned}$$

Similarly, by factorisation, we have:

$$
\begin{aligned}
f_3(a,b,c) =&(4a^3b^2+3a^3c^2+4a^2b^2c+3a^2c^3+4ab^2c^2+4b^2c^3-a^5-a^4c-ac^4\\
&-c^5+16a^2b+16a^2c+16abc+16ac^2+16bc^2-32ab^2-32b^2c)(c-a).
\end{aligned}
$$

Because $a<c$, we only need to consider the other factor of $f_3(a,b,c)$. Denote

$$
\begin{aligned}
f_6(a,b,c) =&4a^3b^2+3a^3c^2+4a^2b^2c+3a^2c^3+4ab^2c^2+4b^2c^3-a^5-a^4c-ac^4\\
&-c^5+16a^2b+16a^2c+16abc+16ac^2+16bc^2-32ab^2-32b^2c.
\end{aligned}
$$

For minimisation of $f_5(a,b,c)$, subject of $0.521 \leqslant a \leqslant b \leqslant c \leqslant 1$ and $c-2a \geqslant 0$, the Kuhn-Tucker conditions for the potential minimal points are as follows:

$$
\begin{aligned}
&8ab^3+8ab^2c+8abc^2+8ac^3+16b^2+16bc+16c^2\\
&-64ab-64ac-\lambda_1+\lambda_2-2\lambda_5=0,\\
&32ab+12a^2b^2+16ac+32bc+8a^2bc-4b^3c+16c^2+4a^2c^2+9b^2c^2\\
&+6bc^3-32a^2-5b^4-c^4-\lambda_2+\lambda_3=0,\\
&16ab+16b^2+4a^2b^2+32ac+32bc+8a^2bc+6b^3c+12a^2c^2+9b^2c^2\\
&-32a^2-b^4-4bc^3-5c^4-\lambda_3+\lambda_4+\lambda_5=0,\\
&\lambda_1(a-0.521)=0, \lambda_2(b-a)=0, \lambda_3(c-b)=0,\\
&\lambda_4(1-c)=0, \lambda_5(2a-c)=0,\\
&\lambda_1,...,\lambda_5 \geqslant 0;\ 0.521 \leqslant a \leqslant b \leqslant c \leqslant 1 \ \mathit{and}\ c-2a \geqslant 0.
\end{aligned} \tag{6.5}
$$

The condition (6.5) has a unique solution of (0.521,1,1,19.6604,39.0179,19.509,0,0) and $f_5(0.521,0.521,0.521) = 2.95371$. Hence we have $f_5(a,b,c) > 0$. That is $\Delta_{6.1} > 0$, given $0.521 \leqslant a \leqslant b \leqslant c \leqslant 1$. Similarly, for minimisation of $f_6(a,b,c)$, subject of $0.521 \leqslant a \leqslant b \leqslant c \leqslant 1$ and $c-2a \geqslant 0$, the Kuhn-Tucker conditions for the potential minimal points are as follows:

$$
\begin{aligned}
&32ab + 12a^2b^2 + 32ac + 16bc + 8ab^2c + 16c^2 + 9a^2c^2 \\
&+ 4b^2c^2 + 6ac^3 - 5a^4 - 32b^2 - 4a^3c - c^4 - \lambda_1 + \lambda_2 - 2\lambda_5 = 0, \\
&16a^2 + 8a^3b + 16ac + 8a^2bc + 16c^2 + 8abc^2 + 8bc^3 \\
&- 64ab - 64bc - \lambda_2 + \lambda_3 = 0, \\
&16a^2 + 16ab + 4a^2b^2 + 32ac + 6a^3c + 32bc + 8ab^2c + 9a^2c^2 \\
&+ 12b^2c^2 - a^4 - 32b^2 - 4ac^3 - 5c^4 - \lambda_3 + \lambda_4 + \lambda_5 = 0, \\
&\lambda_1(a - 0.521) = 0, \lambda_2(b - a) = 0, \lambda_3(c - b) = 0, \\
&\lambda_4(1 - c) = 0, \lambda_5(2a - c) = 0, \\
&\lambda_1, ..., \lambda_5 \geqslant 0;\ 0.521 \leqslant a \leqslant b \leqslant c \leqslant 1\ ;\ c - 2a \geqslant 0.
\end{aligned} \tag{6.6}
$$

The condition (6.6) has three solutions of $(0.521, 1, 1, 19.6604, 0.151437, 19.509, 0, 0)$, $(0.521, 0.526302, 0.526302, 19.7553, 0, 19.696, 0, 0)$, and $(0.521, 1, 1, 48.4042, 0, 53.194, 10.4554, 0)$. Among these three candidates, $f_6(0.521, 1, 1) = 0.0270767$ is the minimal. Hence we have $f_6(a, b, c) > 0$. That is, $\Delta_{6.2}$ is positive given $0.521 \leqslant a \leqslant b \leqslant c \leqslant 1$. Therefore, both $\Delta_{6.1}$ and $\Delta_{6.2}$ are positive. Then,

$$
Q(c, \{a, b\}) > Q(b, \{a, c\}) \tag{6.7}
$$

$$
Q(c, \{a, b\}) > Q(a, \{b, c\}) \tag{6.8}
$$

In other words, the reliability is highest when the highest ability juror votes first.

6.1.1 Proof of Proposition 6.3 (the herding case)

The following proof assumes that, for jurors in a jury of three, their abilities satisfy $0 \leqslant a < b < c \leqslant 1$. Under this circumstance, the threshold for the later two voters is either -1 or 1. The condition of herding is that the ability of the latter juror is less than or equal to half the previous juror. This condition indicates that herding only happens when juror b or c votes first. Based on the equation (6.1), the explicit expressions of the reliability for all possible voting orders:

$$
\begin{aligned}
Q(b, \{a, c\}) &= b/4 + 1/2, \\
Q(c, \{a, b\}) &= c/4 + 1/2.
\end{aligned}
$$

Given that the jurors' abilities satisfy $0 < b < c \leqslant 1$ and there is herding behaviour among the jurors, the reliability is highest when c votes first and lowest when b votes first, $Q(c, \{a, b\}) > Q(b, \{a, c\})$.

6.1.2 Exceptions

Readers may wonder about the performance of the jury when the conditions in Proposition 6.1 and Proposition 6.2 no longer hold. This section, along with appendix A, will discuss these cases. Among these cases, the optimal voting order under SVP can be obtained when either the juror with the lowest ability votes first rather than just the juror with the highest ability votes first. In other words, the optimal voting order is no longer unique. Figure 6.2 shows the optimal voting order with the step of 0.01. The red dots mean that the optimal voting order is obtained when the lowest ability juror (juror a) votes first. The blue dots mean that the optimal voting order is obtained when the highest ability juror (juror c) votes first.

From the decomposed Figure 6.2, we know that when the lowest juror is able enough, the optimal voting order is obtained when the highest ability juror votes first. Furthermore, when abilities of juror b and juror c are close and that of the juror a is small, the optimal voting order is achieved when the lowest ability juror votes first.

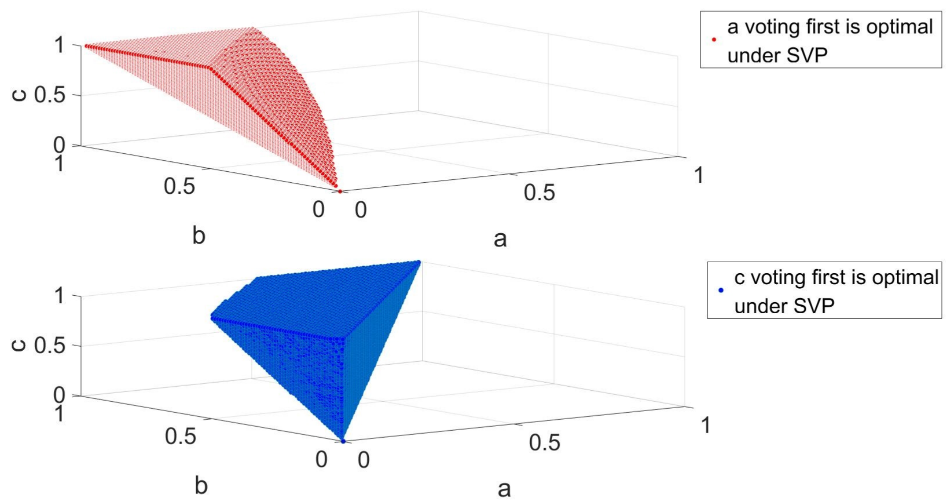


Figure 6.2: SVP optimal voting order decomposed

6.2 Comparison with the simultaneous voting scheme

The underlying assumption of the Condorcet Jury Theorem requires is that jurors are independent. SVP is a typical voting scheme that violates this assumption as we consider one of the simultaneous voting jurors voting first. We now investigate whether this violation is beneficial or not. Here, we compare the performance of sequential voting with simultaneous voting in terms of reliability. Based on the results from the previous section, we need to compare the optimal voting order under SVP voting scheme (either the lowest ability juror voting first or the highest ability juror voting first) with the simultaneous voting scheme. Figure 6.3 shows the result.

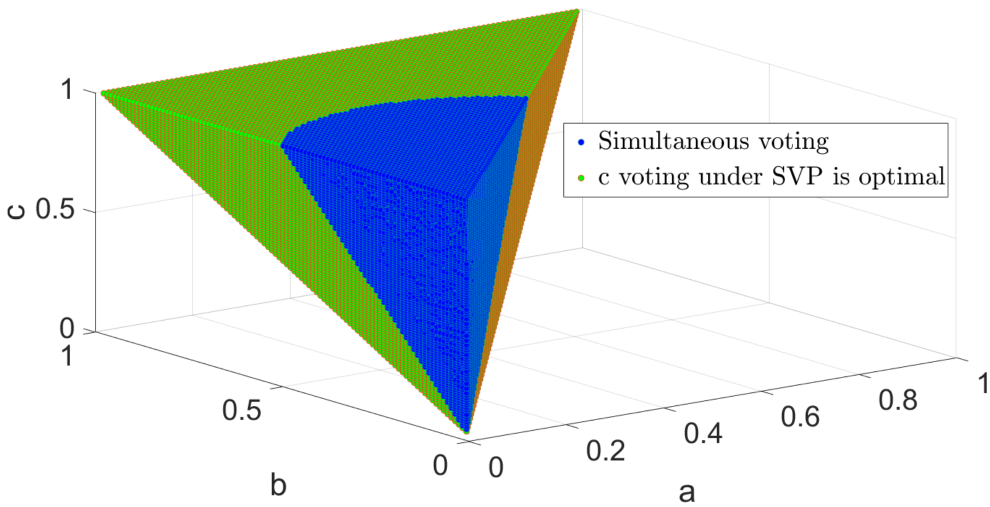


Figure 6.3: Comparision between SVP and the simultaneous voting scheme

The blue dots mean that the optimal voting order (seniority order) under SVP is superior to the simultaneous voting scheme. It should be noted that there are no red dots, which we use to represent the optimal order being when the lowest ability juror votes first. When the optimal voting order has the lowest ability juror voting first, the simultaneous voting scheme is better than all others. The green dots mean that simultaneous voting is superior to the optimal voting order under SVP. Figure 6.3 indicates that when the difference in abilities is large enough, the optimal voting order (seniority order) under SVP is superior. Figure 6.4 provides the boundary of the sequential part.

We care about the reliability when the juror with the highest ability votes first. The

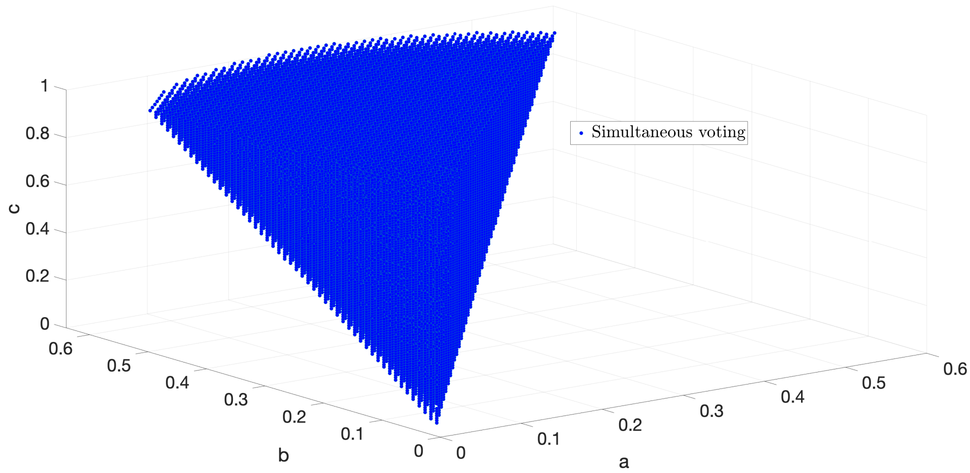


Figure 6.4: SVP (Type III) voting scheme and simultaneous voting scheme

corresponding reliability is:

$$\begin{aligned} Q(c,\{a,b\}) =& (-16a^2b^2c + 64a^2b + 4a^2c^3 - 32a^2c + 64ab^2 + 64abc + 256ab \\ &+ 16ac^2 + 4b^2c^3 - 32b^2c + 16bc^2 - c^5)/(512ab). \end{aligned} \tag{6.9}$$

The reliability for the simultaneous voting scheme has already been calculated in Chapters 4 and 5. The difference between equations (6.9) and (4.3) is:

$$Q(c,\{a,b\}) - Q(abc) = \frac{c\left(4a^2c^2 - 32a^2 + 16ac + 4b^2c^2 - 32b^2 + 16bc - c^4\right)}{512ab}. \tag{6.10}$$

When the difference is zero, we can achieve the boundary equation:

$$4a^2c^2 - 32a^2 + 16ac + 4b^2c^2 - 32b^2 + 16bc - c^4 = 0.$$

When $c = 1$ we get the equation:

$$16a - 28a^2 + 16b - 28b^2 = 1.$$

By rearrangement, we have

$$(a - 2/7)^2 + (b - 2/7)^2 = (5/14)^2. \tag{6.11}$$

The equation (6.11) is a centre-radius form of the circle equation. For different values of c, we can obtain a series of circle equations. These equations explain the shape of the boundary shown in Figure 6.4.

Voting Scheme	Counts(#)	Proportion(%)
SVP	54341	33.86%
Simultaneous	106126	66.13%

Table 6.1: Comparison between SVP (g) and simultaneous voting (a)

To sum up, we note that we have reached the opposite conclusion to a similar comparison in Chapters 4 and 5. As shown in Table 6.1, under most scenarios, simultaneous voting is better than SVP. For juries with adequately homogeneous abilities, which can be described as the areas outsides a series of circles, SVP has higher reliability than simultaneous voting. This advantage increases with the growth of the ability of the ablest juror.

6.3 Larger jury

The main result of Proposition 6.2 has been proved algebraically only for juries of three. The natural question to ask is whether this pattern will still hold for a larger jury. Due to the explosion in terms of combinations of orders and the computation of thresholds the algebraic method seems out of reach for larger juries. Thus, numerical methods are adopted to study the patterns of a larger jury.

6.3.1 Jury of five

We started with a jury of five using a numerical method to determine the highest reliability under honest voting. We divide the ability interval [0,1] into 20 sub-intervals to form the pool of the abilities, namely 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85,0.9 and 0.95. Then five jury members are selected from this ability pool: 11628 combinations in total. Table 6.2 shows that the highest reliability is achieved either when the highest ability juror votes first or, the lowest ability juror votes first.

In the majority of instances, the highest reliability is still achieved when the highest ability juror votes first. However, this advantage is no longer overwhelming.

Table 6.2: Frequency for jury of five under SVP

	Frequency
a,{ b,c,d,e }	3297
e,{ a,b,c,d }	8331

6.3.2 Large jury on SVP with uniformly distributed abilities

This section provides another numerical method for further studying a larger jury. Given that we take a jury of size n with uniformly distributed abilities, we can determine numerically which juror should be given the initial public vote to maximize reliability under honest voting. We divide the ability interval $[0, 1]$ into n subintervals of length $1/n$ and give one juror i the ability of the midpoint of the i th interval, so that $a_i = (2i - 1)/(2n)$ for the ith juror in the jury of size n. As an example, when $n = 5$, the abilities of the five jurors are 0.1, 0.3, 0.5, 0.7 and 0.9.

For each jury of size n, let $\widehat{Q}_n$ denote the reliability of the simultaneous voting scheme and $Q_n[i]$ denote the reliability under the sequential voting scheme (g) with the initial public vote given to the ith juror, the one with the ability a_i. We then define the increment reliability of SVP and the simultaneous voting scheme:

$$\Delta(n, a_i) = Q_n[i] - \widehat{Q}_n.$$

By comparing this with the reliability of simultaneous voting, it is easier to know which juror should vote first and the difference between simultaneous voting and the optimal voting order under SVP. For fixed n, the reliability of giving the initial public vote to juror i is maximised when $\Delta(n, a_i)$ is maximized over a_i. Figure 6.5 plots for $n = 3, 5, 7$ the incremental reliability $\Delta(n, a_i)$ when the initial public vote on the jury of size n is given to the juror of ability $a_i, i = 1, 2, 3 \ldots n$. For each n, the plotted points are connected by straight lines.

The curve for $n = 3$ has three plot points at abilities $1/6, 1/2$ and $5/6$. As we have learned from Proposition 6.2, the juror with the highest ability 5/6 will achieve the highest reliability, as shown in the curve for $n = 3$. For a jury of size $n = 5$, the abilities of the jurors are $0.1, 0.3, 0.5, 0.7, 0.9$. The incremental reliabilities increases with the increase of the ability of the initial public voter. The incremental reliability is maximised when the highest ability juror (ability 0.9) has the initial public vote. Similarly, for $n = 7$, this pattern continues. The incremental reliabilities monotonically increase along with the increase of the ability of the initial public voter. The reliability

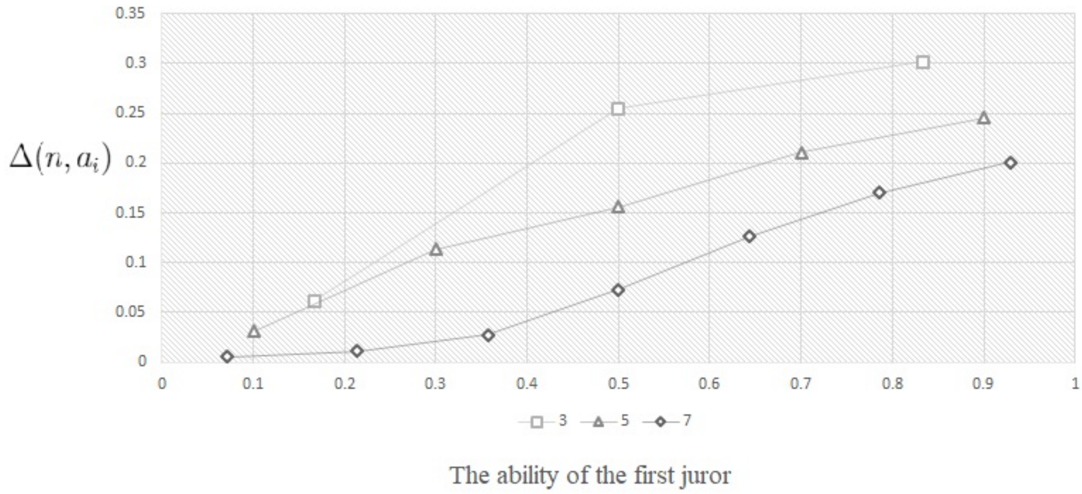


Figure 6.5: Incremental reliability as a function of initial public voter ability ($n = 3, 5, 7$)

is maximised when the juror of the highest ability (ability 13/14) is given the initial public vote.

However, for $n \geqslant 9$, the patterns do not continue, as shown in Figure 6.6, for juries of size $n = 9, 11, 13, 15, 17, 19, 21, 23$, where incremental (or absolute) reliability first decreases with in the ability of the initial public voter and then increases. To distinguish between the curves for different values of n, note that, at the right legends, the curves are $n = 9, 11, 13, 15, 17, 19, 21, 23$, counting from the top.

The lowest initial public vote's reliability is achieved when the third-ranking voter (starting from the lowest) casts the initial public vote. For example, for $n = 9$, the lowest reliability is achieved when the third voter casts the initial public vote. When the juror with comparatively low ability votes first, his vote (public information) actually degrades the quality of collective decision (worse than simultaneous voting). For $n \leqslant 23$, reliability is maximised when the juror with the highest ability is given the initial public vote. However, this advantage gradually narrows as jury size increases. For $n = 23$, the difference between the voter with the lowest ability taking the initial public vote and the voter with the highest ability taking the initial public vote almost disappears. The effect of setting an example is not always positive as even the juror with highest ability still has a chance of getting wrong choice. Furthermore, when n= 23, sequential voting's optimal reliability under initial public voting no longer dominates the simultaneous voting scheme. The whole increment reliability curve is below zero.

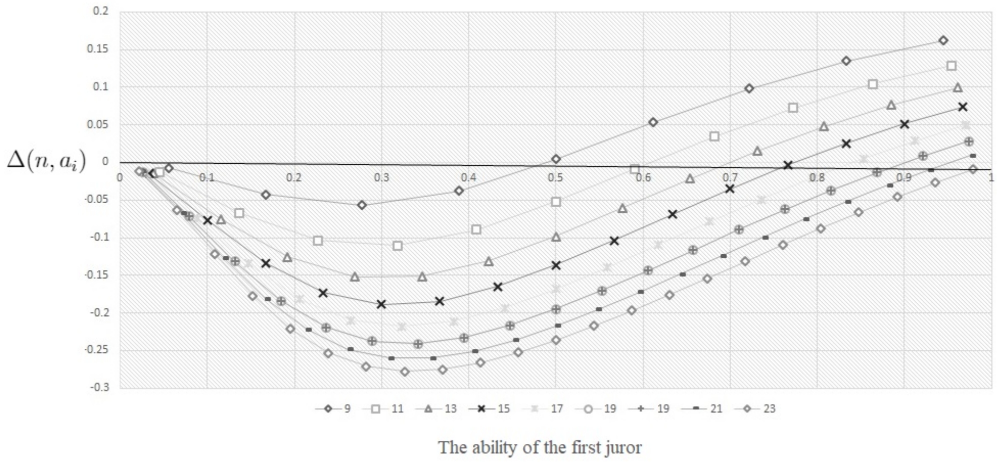


Figure 6.6: Incremental reliability as a function of initial public voter ability ($n = 9, 11, 13, 15, 17, 19, 21, 23$)

This observation indicates that for the sufficiently large jury $n \geq 23$, the performance of the Condorcet jury ($p_{min} > 1/2$) is better than SVP. Consider the following scenario: given n appropaching infinity, the reliability of the Condorcet jury will approach to one $\lim_{n\to\infty} p_n = 1$. We pull out one juror and let him vote first, which is the same voting scheme described here. Suppose this first juror votes state A, and a juror gets a signal to vote for state B, then randomises. Now suppose that the state of Nature is A. By the definition of the Condorcet juror p, he will still vote for state B with probability $1 - p$ (incorrect vote). Under this circumstance, all the incorrect jurors who vote for state B without looking the vote of the first juror and half the correct jurors who will vote for state A with his private signal (randomisation as mentioned) vote B. As a result, B is the majority verdict. Then, the reliability of the jury decreases from near one (CJT) to p.

6.4 Conclusions

The main results on a jury of three under SVP can be categorised into two parts:

- When the minimal ability is lager than 0.521, the probability of a correct verdict is maximised when the agent of highest ability has the initial public vote (including in herding cases).
- The performance of SVP is worse than simultaneous voting in most cases.

These two main results can be extended beyond a jury of three. Due explosion in terms of combinations of orders and the computation of thresholds, we adopt the simulation method rather than the algebraic method. From the numerical results, when the jury size is a medium ($n \leq 23$), we find through simulation that the highest reliability is achieved when the juror with highest ability votes first. Furthermore, the preferable candidate for the initial public juror is either the juror with the lowest ability or the juror with the highest ability. In other words, we tell the public that the first vote either brings insignificant or very significant information, with nothing in between. For the general public, an example is not necessary as it is highly likely that this limited information may mislead the public.

Chapter 7

Comparison

Up to now, we have analysed three sequential voting schemes internally (SVI, SVKP, SVP) by comparing reliabilities of different sequential voting orders of a fixed set of abilities. This section analyses the sequential voting scheme from an external point of view by comparing all the generalised sequential voting schemes for a jury of three. From Section 2.1, we know that there are six distinct voting schemes for a jury of three, namely the simultaneous voting scheme (Condorcet), roll-call voting, casting voting, SVI (Type I, sequential voting with an independent voter), SVKP (Type II, sequential voting with knowledge of the previous voter) and SVP (Type III, sequential voting with an initial public vote).

Now we define the information index of a scheme as sum of known votes for all jurors. In other words, the total number of arcs in Figure 2.1. We start with the simplest case where simultaneous voting originated from Condorcet has an information index of zero. Under simultaneous voting, every juror's decision is based on his private information without other jurors' information. Roll-call voting has an information index of 3, the highest information index among all voting schemes. Under roll-call voting, the information index is 0 for the first juror who votes without any prior vote from others; 1 for the second juror (the vote of the first juror) and 2 for the third juror (the votes of the first juror and second juror). The SVP has an information index of 1, under which the independent juror and the first juror in the duo voting structure have 0 while the second juror in the duo voting structure has 1. SVKP has an information index of 2 (0 for the first juror, 1 for the second juror and 1 for the third juror). SVP has an information index of 2, under which the initial public juror has 0 while the latter two jurors voting simultaneously have 1.

In this section, the findings will be presented on two main categories of questions for juries of three. Section 7.1 addresses the question of which sequential voting scheme has the highest reliability given that they vote in the *optimal order* under each scheme given the *fixed ability set* of the jury. Section 7.2 discusses the question of which sequential voting scheme has the highest reliability given that they vote in the *optimal order* given the random ability set of the jury.

7.1 Fixed jury

A large body of literature on CJT focuses on simultaneous voting due to the voting scheme's independence property. This thesis investigates whether sequential voting is superior to simultaneous voting or secret ballot in terms of information aggregation. However, the degree of information exposure must be carefully designed to avoid herding. The existence of a juror with high ability making the wrong decision may jeopardise the whole decision chain through herding, even if a less able juror has the correct private information. In this situation, the truth is hidden by the strong *prior* probability. There are two methods of controlling the information flow: using the appropriate sequential voting method or the order of voting. These two methods have different application contexts. Here we focus on the fixed jury. We ask the simple question: When the jury's abilities are known, which sequential voting scheme has the highest reliability if we use the optimal order under each voting scheme?

7.1.1 Comparison among generalised sequential voting schemes

Table 7.1 shows the optimal fractions of six different voting schemes for a jury of three under a typical trial. As can be seen in Table 7.1, the sequential voting schemes family (SVI, SVKP SVP, roll-call voting and casting voting) is superior to the simultaneous voting scheme. Furthermore, among five different sequential voting schemes the one with the highest information index, roll-call voting, is the most reliable. The information index may explain why roll-call voting occupies more than four-fifths. The remaining fifth is taken by the casting voting, which is the most reliable of the information index 2 family (SVP and casting voting). Table 7.1 uses the exhaustive method for the distinctive jury with step 0.01. We have further narrowed the step length to check the optimal fraction. When the minimal step is 0.001, the optimal fraction stays the same.

Table 7.1 presents an overall ranking of the optimal fraction among six different sequential voting schemes. Among five sequential voting family members, the casting voting scheme and SVP are close to Condorcet (simultaneous voting). SVP pulls one member of the Condorcet jury out and lets him vote first publicly, more like a sequential

Table 7.1: Frequency for fixed jury size of three under all six voting schemes with optimal order

	Frequency	Percentage
SVI (Type I)	0	0
SVKP (Type II)	0	0
SVP (Type III)	0	0
Simultaneous	0	0
Roll-call voting	132348	81.85%
Casting voting	29352	18.15%

voting family. On the other hand, the casting voting scheme is closer to simultaneous voting as it pulls one member of the Condorcet jury out and let him vote last. The last juror can see all the available information, the votes of all the previous jurors, and their abilities. The comparison between simultaneous voting and SVP has been discussed in Chapter 6, which indicates that simultaneous voting or secret ballot is generally 66.13% better than SVP. However, casting voting with the median juror voting lastly is generally superior to the simultaneous voting scheme. We prove this through the difference between the reliabilities under each voting scheme and its corresponding optimal order. The reliability of casting voting with the median juror voting lastly is:

$$Q(\{a,c\},b) = \frac{1}{32}\left(-\frac{4(a-b)^2}{c(ab-4)} + c(4-ab) + 4(a+b+4)\right). \tag{7.1}$$

The reliability for the simultaneous voting scheme has already been calculated in the previous Chapter 2. The difference between equations (7.1) and (4.3) is:

$$Q(\{a,c\},b) - Q(abc) = \frac{(a-b)^2}{8c(4-ab)}. \tag{7.2}$$

Given that $0 < a, b < 1$, the difference is clearly positive. The casting voting with its optimal is objectively superior to simultaneous voting. This phenomenon has two consequences. The first consequence is that simply changing the order of the jurors will have different effects on the probability of a correct collective decision (the jury's reliability). If we can control the information flow properly, we can improve the performance of the jury. The second consequence is that in contrast to intuition, the juror with middle ability should make the final decision when there is a tie rather than the most senior juror with highest ability (see Alpern & Chen (2017*b*)).

Let us go return to which ability sets have the better performance with the roll-call voting scheme and which perform better under the casting voting scheme. Figure 7.1 shows a more intuitive way representing the ability sets for roll-call voting and casting voting being optimal.

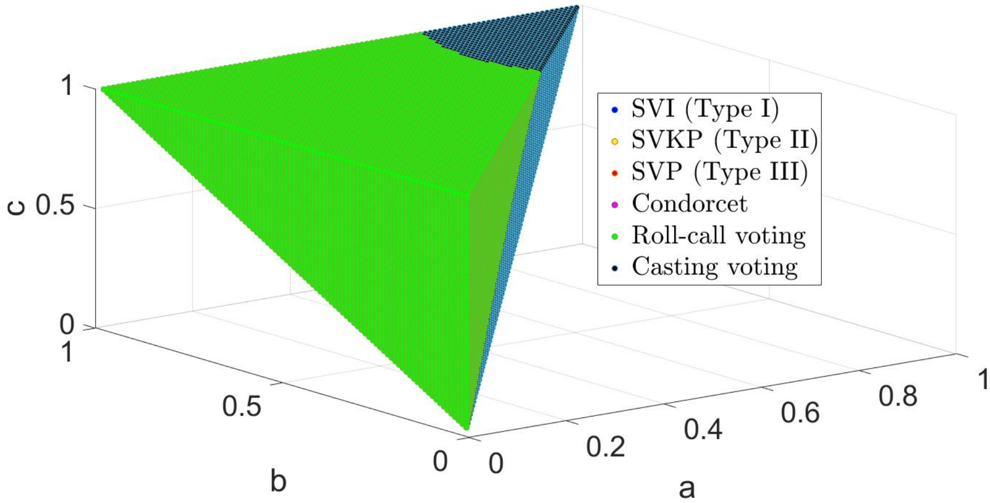


Figure 7.1: Comparison among six different sequential voting schemes

In Figure 7.1, the vertical axis represents the ability of the juror c (the highest ability) and the horizontal axis represents the abilities a (the lowest) and b (the median) respectively. The green dots mean that roll-call voting (with the Alpern-Chen ordering) is optimal among all six different voting schemes, occupying 81.85%. The blue dots mean that the casting voting scheme with the median voter casting the casting vote is optimal among all six voting schemes, occupying 18.15%. These blue dots are concentrated in the upright corner of the inverted tetrahedron directed from the origin point (0,0,0) to the vertex (1,1,1). This implies that when the abilities of jurors are close or relatively high, the preferred the voting scheme is casting voting scheme. More precisely, we can use the roll-call with the Alpern-Chen ordering to find out the boundary condition. The reliability of the roll-call with median-high-low ordering is:

$$\begin{aligned} Q(a,b,c) =& \Big(\frac{4(a-2b)^3}{c\,(a^2+2ab-8)} + 4(a+2b)^2 + 64b \\ & + (a-2b)(-8+a^2+2ab)c\Big)/(128b). \end{aligned} \tag{7.3}$$

The difference between equations (7.1) and (7.3) is:

$$
\begin{aligned}
Q(a,b,c) - Q(\{a,c\},b) =& a(\frac{20a^3b - 8a^2\left(3b^2+2\right) - 32ab + 64b^2}{c(ab-4)\left(a^2+2ab-8\right)} \\
& 4a + \left(a^2-8\right)c)/(128b).
\end{aligned}
$$

Thus, the boundary condition is:

$$
\frac{4\left(5a^3b - 2a^2\left(3b^2+2\right) - 8ab + 16b^2\right)}{c(ab-4)\left(a^2+2ab-8\right)} = (8-a^2)c - 4a,
$$

which is the quadratic equation for c.

7.1.2 SVI, SVKP, and SVP

Although SVI, SVKP, and SVP are never optimal among the six sequential voting schemes, we still need to know the relative performance for these three voting schemes as they are the focus of this study. Based on previous sections' results, we need to compare the optimal voting order under three different sequential voting schemes. That is, comparisions among SVI (Type I, the highest ability juror is the independent voter and the other two jurors vote in anti-seniority order), SVKP (Type II, the highest ability juror votes first, followed by the median ability juror, and the lowest ability juror vote last), and SVP (Type III, the highest ability juror votes first). Figure 7.2 shows the result. It should be noted that the additional condition for the comparison in this part is $0.521 < a < b < c < 1$ (able jury) to make sure the optimal voting order for the SVP (Type III) voting scheme is exclusive (see Proposition 6.2).

In Figure 7.2, the red dots mean that the optimal voting scheme is SVI. There are no blue dots, which means that the SVP voting scheme is excluded from the graph. This means that SVP with the optimal voting order (the highest ability juror voting first) is never optimal compared with the optimal orders under SVI and SVKP.

The yellow dots mean that the optimal sequential voting scheme under SVKP is superior to the optimal voting order under SVI. From the mixed Figure 7.2, it is not easy to see the pattern for the boundary condition.

However, from the decomposed Figure 7.3, we see that the boundary condition is a regular shape. The boundary condition can be achieved by solving the equation (7.4) under the condition that $0.521 < a < b < c < 1$:

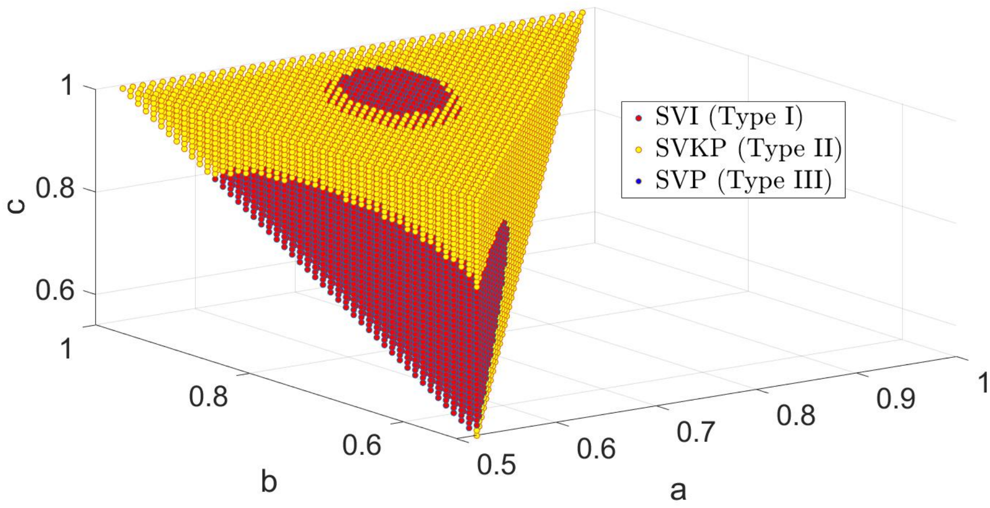


Figure 7.2: The optimal voting scheme among SVI, SVKP and SVP

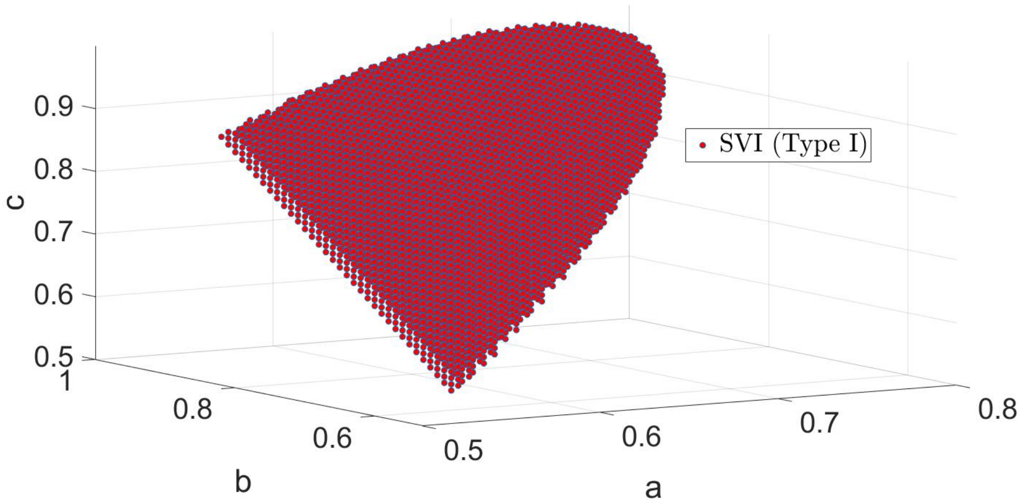


Figure 7.3: Decomposed graph for SVI voting scheme

$$\begin{aligned} 16b^3c(16a + c^3 - 4c) + 128b^2c^2(2a + c) + 64bc^3(-8a + c^3 - 6c) \\ -b^7 - 4b^5(c^2 - 2) + 16b^4c + 256c^5 = 0. \end{aligned} \tag{7.4}$$

The yellow dots envelop the red dots. This shape means that, except in extreme cases where the juror with the highest ability dominates other jurors, in most cases we should use SVKP with Alpern-Chen ordering. However, when there is an experienced expert in the jury, we should use SVI. Furthermore, the experienced expert with relatively high ability (close to 1) should be the independent voter. The other two jurors adopt duo roll-call voting in anti-seniority order.

Now we relax our assumption for the abilities of the jury. The condition for the juror with minimal ability is merely larger than zero $0 < a < b < c < 1$. Table 7.2 shows the results.

Table 7.2: Frequency for general jury of three under SVI, SVKP and SVP

	Counts(#)	Proportion(%)	Information index
SVI (Type I)	32308	19.98%	1 (0,0,1)
SVKP (Type II)	116425	72.00%	2 (0,1,1)
SVP (Type III)	12964	8.020%	2 (0,1,1)

Table 7.2 indicates that SVKP with information index 2 (sequential voting with knowledge of the previous voter) still prevails.

7.1.3 The simultaneous voting scheme

The CJT is based on simultaneous voting. Comparing simultaneous voting with type I, type II, and type III voting schemes will provide insights into how information exposure affects reliability.

Based on the result from the previous section, we just need to compare the optimal voting orders under type I and type II voting schemes with the simultaneous voting scheme.

Table 7.3: Frequency for able jury size three under SVI, SVKP, SVP and simultaneous voting schemes

	Counts(#)	Proportion(%)	Information index
SVI (Type I)	4595	24.94%	1 (0,0,1)
SVKP (Type II)	3678	20.01%	2 (0,1,1)
SVP (Type III)	0	0	2 (0,1,1)
Condorcet(Simultaneous)	10151	55.10%	0 (0,0,0)

As shown in Figure 7.4, the purple dots mean that the simultaneous voting scheme is the optimal among the three. The red dots mean that the optimal voting scheme is

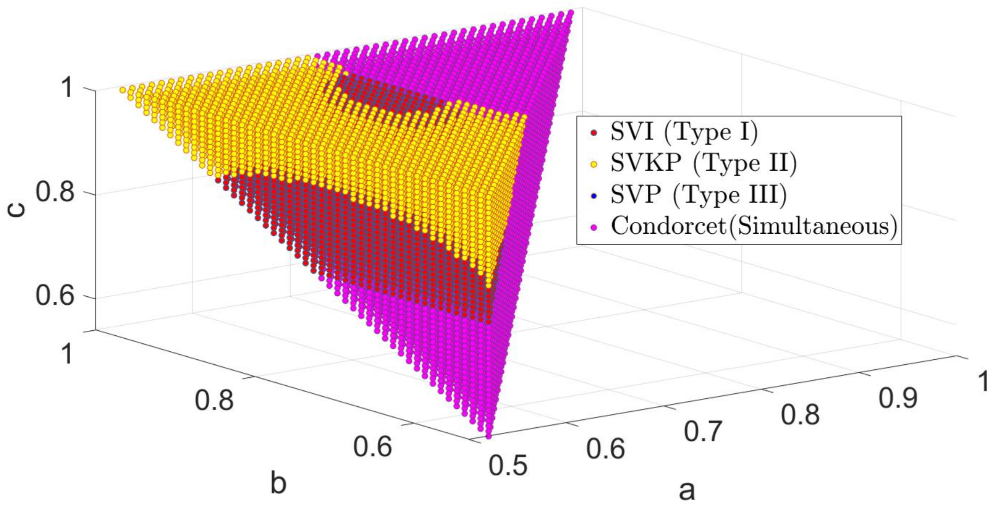


Figure 7.4: The optimal voting scheme among SVI, SVKP, SVP and simultaneous voting schemes

SVI. The yellow dots mean that optimal sequential voting scheme under SVKP. From the mixed Figure 7.4 , it is hard to see the pattern for the boundary condition. Nevertheless, if we separate the simultaneous voting scheme from the mixed graph, the boundary condition is regularly shaped. By studying Figure 7.5, we see that when the abilities are large and close to 1 or the abilities are close, we should choose the simultaneous voting scheme. The most substantial difference for the abilities is approximately 0.3.

Now we relax our assumption for the abilities of the jury. The condition now is $0 < a < b < c < 1$. Table 7.4 indicates that SVKP still prevails, even after we include simultaneous voting (Condorcet Jury voting).

Table 7.4: Frequency for general jury of three under SVI, SVKP, SVP and simultaneous voting schemes

	Counts(#)	Proportion(%)	Information index
SVI (Type I)	21205	13.11%	1 (0,0,1)
SVKP (Type II)	112958	69.86%	2 (0,1,1)
SVP (Type III)	12964	8.020%	2 (0,1,1)
Condorcet(Simultaneous)	14573	9.010%	0 (0,0,0)

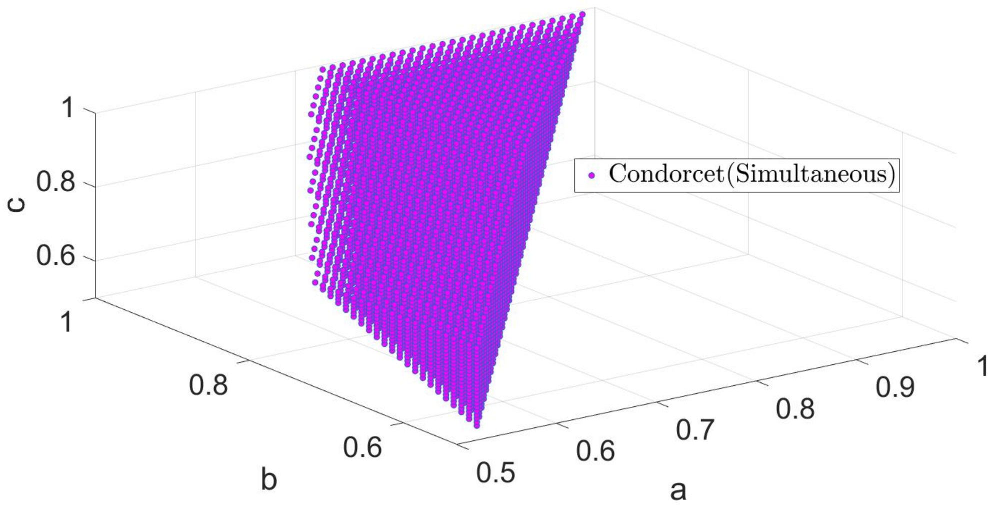


Figure 7.5: Decomposed graph: the optimal voting scheme among SVI, SVKP, SVP and simultaneous voting schemes for simultaneous voting scheme

7.2 Random jury

In the previous section, we discussed the problem for a fixed jury under where the jury's abilities are known. Now, we consider another question: when the abilities of the jury are unknown (randomly generated), among six different sequential voting schemes, which scheme has the highest reliability given that they vote in the *optimal order* of each voting scheme? The answer to this question try to address the problem of which sequential voting scheme is more robust in terms of voting orders for the random jury.

In this section, we compare the performance of the six different sequential voting schemes by simulation. Each sequential voting schemes adopts the optimal voting order. That is, SVI (the highest ability juror is the independent voter, and the other two jurors vote in anti-seniority order), SVKP (the highest ability juror votes first, followed by the median ability juror, and the lowest ability juror vote last), SVKP (either the highest ability juror or the lowest ability juror votes first) and roll-call voting (Alpern-Chen ordering).

We generated one million random ability sets of juries of three for each sequential voting scheme under optimal ordering and counted the frequency of the particular ordering being optimal. This number divided by one million is an approximation of the optimal fraction of the specific ordering. The optimal fraction is the fraction of potential ability sets for which the ordering provides the highest reliability. Table 7.5 shows the average data for one million trials. By Hammersley & Handscomb (1964), this process will generate a relatively accurate estimation for the optimal fraction for each sequential voting scheme.

Table 7.5: Optimal fraction for optimal order

	Counts(#)	Proportion(%)	Information index
SVI (Type I)	0	0	1 (0,0,1)
SVKP (Type II)	615583.25	61.55833%	2 (0,1,1)
SVP (Type III)	0	0	2 (0,1,1)
Condorcet(Simultaneous)	0	0	0 (0,0,0)
Roll-call voting	813921.5	81.39215%	3 (0,1,2)
Casting voting	186078.5	18.60785%	2 (0,0,2)

From Table 7.5, we can see that the sum of the optimal fraction of roll-call voting and casting voting is 1. However, we still can see that SVKP occupies around 60% of the total cases. When SVKP is optimal, it has the same reliability as roll-call voting.

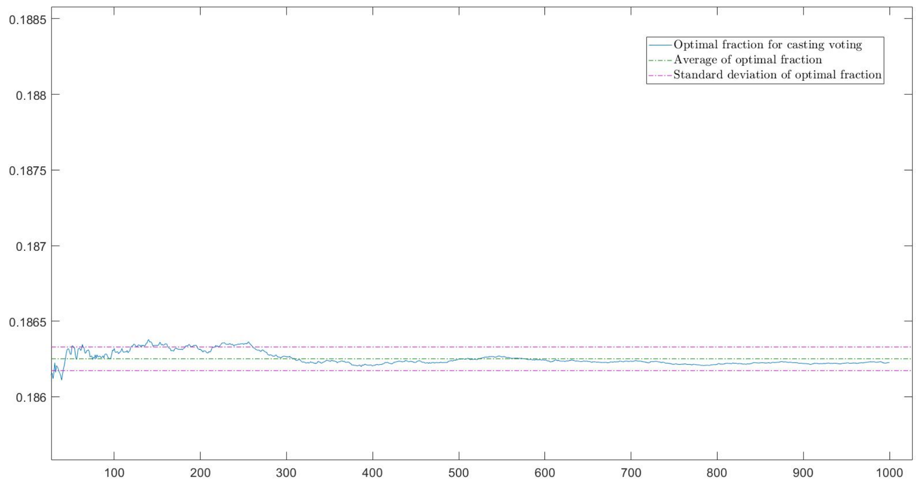


Figure 7.6: Average optimal fraction of the simulation for casting voting

Figures 7.6 and 7.7 show that one million trials were repeated one thousand times to improve accuracy. As shown in Figures 7.6 and 7.7, the optimal fractions for both sequential voting schemes tend to be stable with the increase of trials. This convergence indicates that the estimation of the optimal fraction is comparatively reliable.

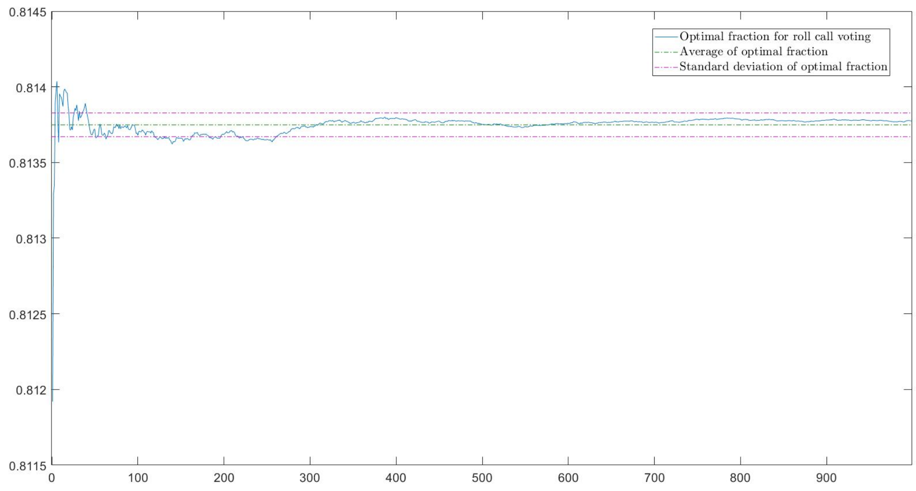


Figure 7.7: Average optimal fraction of the simulation for roll-call voting

In summary, Table 7.5 and Figures 7.6 and 7.7 further confirm the results in the last section for the fixed jury.

Table 7.6: Average reliabilities with ranking

	$\bar{Q}$	Ranking	Information Index
Roll-call voting	0.712486	1	3 (0,1,2)
SVKP (Type II)	0.711652	2	2 (0,1,1)
Casting voting	0.706455	3	2 (0,0,2)
SVP (Type III)	0.691805	4	2 (0,1,1)
SVI (Type I)	0.690398	5	1 (0,0,1)
Condorcet (Simultaneous)	0.683508	6	0 (0,0,0)

Another important measure of performance for the six different sequential voting schemes is *average reliability*. This is the average value of Q for all ability sets in the simulation. Table 7.6 shows that among the six sequential voting schemes, the roll-call voting scheme with Alpern-Chen order and information index 3 has the highest average reliability. Furthermore, simultaneous voting with information index 0 has the lowest average reliability. The ranking of the average reliability is positively correlated to the information index. For the same information index group (Casting voting, SVP and SVP), it is better to inform more people (SVKP) instead just let more people get more information (SVP and casting voting). Another consequence is that among all the voting scheme sharing the same information index, lowest-ability juror should vote last. Finally, by comparing SVI and simultaneous voting, we know that by adding a duo structure into the simultaneous voting, the average performance of jury is improved. Sequential voting enhances the probability of correct verdict by better information aggregation in terms of the average reliability.

7.3 Herding in generalised sequential voting schemes

In this section, we try to see the extent to which the superiority of sequential voting can be explained as a consequence of herding. Under the simultaneous voting scheme, herding does not exist as no one knows any voters' information, either votes or private signals. The introduction of the sequential brings more information, as described in the previous section. However, the cost of more information is the risk of herding. If the prior voter's ability is overwhelmingly higher compared to one juror, he may ignore his private information and copy the behaviour of the previous juror (h_2 and h_3 conditions). This herding behaviour may explain why sequential is better than simultaneous voting. The following will discuss herding under different voting schemes. Now, we define a relative measurement of the reliability of different voting schemes. For each jury under the particular sequential voting scheme, let $\dot{Q}$ denote the reliability of simultaneous voting scheme and $Q[.]$ denote the reliability under the particular sequential voting scheme. For example, $Q[SVI]$ is the reliability of adopting sequential voting with an independent juror. Then the sequential voting performance is measured by the difference between the reliabilities of the sequential and simultaneous voting schemes. Take sequential voting with an independent juror as an example:

$$\Delta(improvement) = Q[SVI] - \dot{Q}.$$

Besides this relative measure, we also use the average reliability as the interval measurement with sequential voting scheme for groups with and without herding. It should be noticed that the analysis of this section is based on the exhaustive method with step 0.01.

Herding under sequential voting with an independent voter

Under sequential voting with an independent voter, the herding occurs in the roll-call part. As we know from Chapter 4, in the roll-call part, we should use anti-seniority order. Under the optimal voting order, herding does not exist. Therefore, the superiority of SVI over simultaneous voting results from a better information aggregation of the SVI rather than herding.

Herding under sequential voting with the knowledge of the previous voter
In SVKP, the herding can either happen in the first or the second duo structure. The herding conditions are $(v_1, v_2) \in h_2(v_2 < v_1/2)$ and $(v_1, v_2, v_3) \in h_3$ $(v_3 \leqslant \frac{(v_1^2+4v_2^2)}{8v_2^2})$. There are three orderings $r = (b, a, c)$, $r = (c, a, b)$ and $r = (c, b, a)$ where herding of the first duo is possible while three orderings $r = (a, c, b)$, $r = (b, c, a)$ and $r = (c, b, a)$ where herding of the second duo is possible. As was revealed in Chapter 5, the optimal order is Alpern-Chen ordering. Thus, the following analysis is based on the $r = (b, c, a)$ (the optimal voting order). We use the sample with the average ability 1.5 for the whole jury as this group is the largest sample across different average ability groups.

Group	Average reliability	$\Delta(improvement)$
Herding	0.724617427	0.039245121
No herding	0.681063145	-0.002698256

Table 7.7: Comparison between herding and no herding groups under SVKP with average abilities of jury 1.5

Table 7.7 shows that the average reliability of the herding group is higher than the group without herding, which indicates that herding has a positive effect on the accuracy of the collective decision. The average ability measures the performance internally as we compare the absolute value of the reliability. For the relative performance, we use the $\Delta(improvement)$ because simultaneous voting eliminates all the effects of herding. The herding group is better than the group without herding with a positive sign on average. Now we take a close look at the herding group. There are only 3 cases out of 1110 cases with negative signs of the $\Delta(improvement)$.

Jury	(b, c, a)	$\Delta(improvement)$
(0.35, 0.56, 0.59)	0.682228595	-0.000519249
(0.35, 0.57, 0.58)	0.682531229	-0.000972928
(0.36, 0.54, 0.60)	0.680980833	-0.000586966

Table 7.8: Cases: Δ with negative sign

From the above three cases, along with Table 7.10, we know that herding plays an important role in the superiority of sequential voting over simultaneous voting, especially for the positive sign of the Δ. However, the existence of counter examples indicates that herding cannot fully explain this superiority.

Group	$\#\ \Delta(improvement)[+]$	$\#\Delta(improvement)[-]$
Herding	1107	3
No herding	31	115

Table 7.9: Comparison between herding and no herding group under SVKP for signs of Δ with mean 1.5

Restricting to the cases where the mean ability is 1.5 may neglect some extreme cases involving herding like jury (0.01, 0.98, 0.99). If we look at all 1000000 samples from the numerical simulation without any restrictions on mean, we still find that herding cannot explain 28% of the superiority of Alpern-Chen ordering under SVKP over simultaneous voting.

Group	$\#\ \Delta(improvement)[+]$	$\#\Delta(improvement)[-]$
Herding	381407 (72 %)	19404 (4 %)
No herding	142021 (28 %)	457123(96 %)

Table 7.10: Comparison between herding and no herding group under SVKP for signs of Δ

Herding under sequential voting with an initial public vote

In SVP a jury size of three, herding only exists when the first juror has the highest or the middle ability. As the optimal voting order under SVP is the juror with the highest ability voting first, we focus on this voting order. The herding condition $(v_1, v_2) \in h_2$ or $(v_1, v_3) \in h_2$ implies $v_2 < v_1/2$ or $v_3 < v_1/2$. For a fair comparison, the average abilities of a jury are the same. The largest sample has an average value of 1.5 for the whole jury. The Table 7.11 shows the comparison between the herding and no herding groups using two different measurements.

Group	Average reliability	$\Delta(improvement)$
Herding	0.706291866	0.020833376
No herding	0.662749759	-0.021093011

Table 7.11: Comparison between herding and no herding groups under SVP with average abilities of jury 1.5

For the interval comparison, the herding group's average reliability is better than that of the group without herding. This finding implies that the herding does improve the performance of the jury. For the relative comparison, the average value of the Δ of herding group is higher than that of the group without herding. The sign of the herding group is positive, while the sign of the group without herding is negative. The reason for this phenomenon may be that herding improves the performance of the jury. However, if we look at the herding case by case, we find that 234 cases among a total of 1045 have positive signs.

Interestingly, if we check all the data with a positive $\Delta > 0$, we find that only 118 out of 85702 (0.2%) do not have herding phenomena. This ratio provides further evidence that herding may help explain why SVP is better than simultaneous voting in some cases. Furthermore, for these 118 exceptions, the latter two jurors' abilities are similar, and they are all close to the herding condition.

Chapter 8

Summary and concluding remarks

In this chapter, we will summarize overall work on generalized sequential voting and access the contributions, based on research questions of the sealed card problem, SVI, SVKP and SVP tackled in the previous chapters. In the end, we will have a discussion on several future directions and some concluding remarks.

8.1 Conclusions

Although there is a large body of literature on CJT, the sequential voting has not received enough attention, where studies on the traditional simultaneous voting are still mainstream. This thesis has addressed several fundamental problems regarding sequential voting, with the intention of finding a set of optimal voting orders under generalised sequential voting schemes with heterogeneous abilities under continuous signal model and eventually providing comprehensive solutions applicable to three agents or even more agents. The specific problems have been tackled are about the optimal voting order for a jury of three or even larger under the sealed card model, SVI, SVKP and SVP, relative performance between simultaneous and sequential voting and numerical comparisons of the six sequential voting schemes under the optimal voting orders.

The sealed card problem considers roll-call voting with a concrete ability setting. A (sealed) card is randomly drawn from a deck with an equal number of red and black cards. According to his ability, ever juror samples a particular number of cards and then votes for the colour of the sealed card based on their samples and the votes of the previous jurors. The results of our study show that Alpern-Chen (median-high-low) ordering dominates all other six voting orders. Compared to simultaneous voting, Alpern-Chen ordering is superior in most cases, except when jurors' abilities are similar.

An analogue of this voting order can be extended to a larger jury.

Under sequential voting with an independent voter, two of the jurors take turns to vote, and another juror votes independently. For a jury with heterogeneous abilities on the interval $[0, 1]$, maximum reliability is achieved when the juror with the highest ability takes the independent vote and the other two jurors adopt duo roll-call voting with anti-seniority order (increasing) unless herding. Compared with traditional simultaneous voting, sequential voting with an independent voter adopting the above order is more reliable, taking up 86.28% of the total cases. Simultaneous voting is better only when the least able juror lies in the interval $[0, 6 - 2\sqrt{7}]$ regarding their ability.

In the sequential voting with knowledge of the previous juror, the later juror only knows the adjacent juror's vote, which is limited information, and they must guess the vote(s) before the adjacent juror (if one exists). The probability of a correct verdict (reliability) is maximised when the jury adopts Alpern-Chen ordering (median, high, low). Compared to Condorcet voting (simultaneous voting), sequential voting with knowledge of previous voter using Alpern-Chen ordering is more reliable, occupying 87.29% of the total cases. We also find that the seniority ordering (SO) dominates anti-seniority ordering (AO) under this sequential voting scheme. The superiority of SO and another analogue of Alpern-Chen ordering called Ascending-Descending Order (ADO) still holds true for a larger jury.

In terms of sequential voting with an initial public vote, for an able jury of three (where the minimal ability is larger than 0.521), the probability of a correct verdict is maximised when the agent of highest ability has the initial public vote. For medium-sized juries ($n \leq 23$), we find through simulation that the highest-ability juror should vote first. Compared with simultaneous voting, sequential voting with an initial public vote and highest ability taking the first vote is inferior, occupying 66.13% of the total cases for able jury of three. For medium-sized jury ($n \leq 23$), the numerical simulation shows that ablest juror still should vote first.

As for the question of which voting scheme we should choose to maximise the probability of correct collective decision among all six sequential voting schemes, for the fixed juror, the sequential voting schemes family (SVI(type I), SVKP(type II), SVP (type III), roll-call voting and casting voting), 92.47%, is generally better than simultaneous voting, 7.53%, in terms of frequency. The herding may be one of reasons why the sequential voting schemes family is more reliable than the simultaneous voting.

This work provides comprehensive solutions to three agents sequential voting schemes under continuous signal distribution rather than binary private information in a large body of Condorcet Jury Theorem literature. The main contribution is summarised in the Table 8.1.

Sequential voting schemes (SV)	**Optimal voting order**
Sequential voting with an independent voter (SVI, type I)	**Proposition 4.1** Given that abilities of jurors satisfy $0 < a < b < c < 1$, the highest ability juror should be the independent voter. For the roll-call voting part, the voting order should be **anti-seniority order(AO)** (increasing order) unless herding exists.
Sequential voting with knowledge of the previous voter (SVKP, type II)	**Proposition 5.2** Given that abilities of jurors satisfy $0 < a < b < c < 1$, the **Alpern-Chen order (median high-low)** is the optimal order. $Q(b, c, a)$ has the highest reliability among six different orderings.
Sequential voting with initial public vote (SVP, type III)	**Proposition 6.2** Given that abilities of jurors satisfy $0.521 < a < b < c < 1$, the reliability is highest when the juror of highest ability votes first (weak form).

Table 8.1: Optimal voting order for SVI, SVKP and SVP (type I, II, III)

8.2 Future work

This thesis discusses research topics explicitly on generalised sequential voting in juries of three. Potential research directions under this topic are listed and discussed as follows:

Random jury with random order We have discussed the performance of each sequential voting schemes using optimal order internally in the last section. However, we can only know the optimal order by knowing the full picture of the ability sets. However, what if we do not know the ability sets? Consider the following case. The superior court has launched a new voting rule for the district court. The efficiency of the new policy depends on reliability. However, it is unrealistic to ask every district court to meet the same standard regarding the jury's abilities. In most cases, we do not know the ability set of the jury. Under this circumstance, we need a sequential voting scheme that can still guarantee relatively high performance when the jury's abilities are unknown. Consider the following research question as a more specific example: Which voting scheme is best with a random jury (x, y, z), where x, y, z are chosen independently and uniformly in $[0, 1]$? Here, x, y, z are the first, second and third juror's abilities to vote.

Transfer payoff No one can obtain information without a cost. For example, the first juror may charge the second juror for the information he provides considering the reputation measured by ability (public information). It is natural to consider a model that quantifies the cost for different jurors according to their abilities. Inspired by Song (2016), the introduction of monetary transfer will also change the voting behaviour. The potential research direction may change from the relation between the reliabilities and voting orders to designing incentives.

Weighted average So far, we have only considered the cases in which each juror has the same weight. It is reasonable to assume that all the jurors in the jury share the same weight if all are equipped with the same probability of making the correct decision. However, a more common case is that the experts will have different abilities to discern the state of Nature as, their education, experience, and other attributes may vary. Rae (1969), Straffin (1977) and Fishburn & Gehrlein (1977) show that the simple majority rule is not the optimal decision rule in most cases. It may be inappropriate to use the same weight for each juror. Therefore, a more general case is that for each juror, the weight is different. There are many situations where the weights of members of the decision-making body are different. In political science, the number of representatives in parliament is determined by the size of the district. This distribution of representation means that different districts will have different influence over the outcome of a bill. For example, the council of the European Union and the United States Electoral College. In sequential voting, weighted average voting rule has drawn little attention (outside the works of Berend & Kontorovich (2014)). Further research could investigate the optimal voting order in generalised sequential voting schemes. Specifically, future researchers may consider investigating the following questions: If the weights satisfy $w_1 \leq w_2 \leq \cdots \leq w_n$ and the jurors' abilities satisfy $a_1 \leq a_2 \leq \cdots \leq a_n$, does there exist an optimal voting order that is independent of the ability distribution of the jurors? If it does exist, what is it? Is the optimal order's reliability superior to that of the simultaneous voting considering the weights and quota? The answers to these questions could be extremely illuminating.

Probability model for receiving the signals So far, the arc represents the determinant information transfer. However, the information transfer may be random. For instance, if the nodes represent the sensor, the transformation between sensors is not always successful. For this reason, uncertainty regarding the information transfer will be introduced into the model. Take the SVI voting scheme as an example. As shown in

Figure 8.1, the solid arrow represents that the second juror knows the vote of the first juror for sure, while the arrow with a line of dashes means that the third juror knows the vote of the first juror with a certain probability.

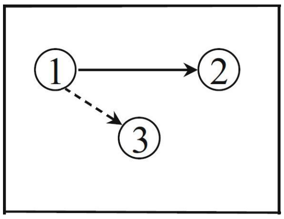


Figure 8.1: Uncertainty Model

Strategic voting behaviour In honest voting, the juror votes for the state that they consider most likely based on his private information and observations from other jurors. The motivation for this kind of voting behaviour is that the juror merely wants his verdict is correct rather than the jury's verdict as a group. This assumption indicates that he is the only pivot voter in the voting process. Nevertheless, recent works by Austen-Smith & Banks (1996) and McLennan (1998) argue that this traditional assumption is unreasonable. Another highly related line of notable works is those of Feddersen & Pesendorfer (Feddersen & Pesendorfer 1996, 1997, 1998, 1999). These scholars believe that consolidating the private signals and common goals may motivate the juror to vote strategically. Inspired by the work of Alpern & Chen (2017*a*), a combination of private signals and a common goal maximising the probability of correct verdict accounts for strategic voting behaviour. Take the following case of a jury of three under the roll-call voting scheme as an example. Consider a jury where two junior jurors have exactly same low ability and one senior juror has high ability. The two junior jurors vote first, and the senior juror votes last. One junior juror votes for certain state first say, Guilty. In the second round, the other junior juror will vote for the same state, as his ability is the same as the junior juror before and he possesses additional information, i.e. that previous voters voted Guilty. The conclusion of the juror's assessment from all existing information is Guilty. However, if he votes for the Guilty, the jury's verdict is already determined by simple majority rule, and the vote of a senior juror does not matter at all. Under this circumstance, our juror may decide to vote Innocent and leave the pivotal vote to the senior juror. By adopting this strategy, the possibility of correct jury verdict increases. Further research could consider the central controller to customise the thresholds based on the jury's ability to address strategic voting behaviour and to improve the reliability of a particular jury further.

Remark The primary assumption in this study is a linear continuous signal distribution to model the sampling process. As signal models develop, we may find that other models better fit the jury's attributes. This study is the first step toward the signal distribution-free model. Alternatively, we may find that similar results hold for a family of signal distribution models.

On the other hand, from the game theory perspective, this study uses Bayesian probability to capture the learning process in the voting procedure. Furthermore, the performances of six sequential voting schemes determined in this study are under specific numerical settings. Due to the lack of data, we are not sure whether jurors in real use this strategy or not. For instance, in behavioural science, people use a heuristic method rather than mathematical calculation under certain circumstances. As a continuation of this study, we will use the behaviour labs to test these sequential voting schemes' performances with real data as case studies. By analysing real data, we will identify the heuristic methods jurors use to update their information. We can then modify the solution under each voting scheme concerning the voting behaviour from data and evaluate the performance of the sequential voting scheme and ordering policy further to enhance the probability of the correct collective decision.

Appendix A

Slice graphs of exceptions under SVP

In this part, we will present the details of the exception cases by using the slice graphs. We transfer three dimensions to two dimensions via this method. The slice graphs provide the details for different lower bound of the abilities. When $a = 0.1$ and the optimal voting order is lowest ability juror voting first, the maximal difference between b and c is 0.4. When $a = 0.2$ and the optimal voting order is lowest ability juror voting first, the maximal difference between b and c is 0.3. When $a = 0.3$ and the optimal voting order is lowest ability juror voting first, the maximal difference between b and c is 0.2. When $a = 0.4$ and the optimal voting order is lowest ability juror voting first, the maximal difference between b and c is 0.1. When $a = 0.5$ and the optimal voting order is lowest ability juror voting first, the maximal difference between b and c is almost 0. To sum up, the lowest ability juror first being optimal requires the other two jurors with similar abilities. The tolerance for the difference decreases when the lower bound of the lowest ability juror increases. The relation between them is linearly negative correlated.

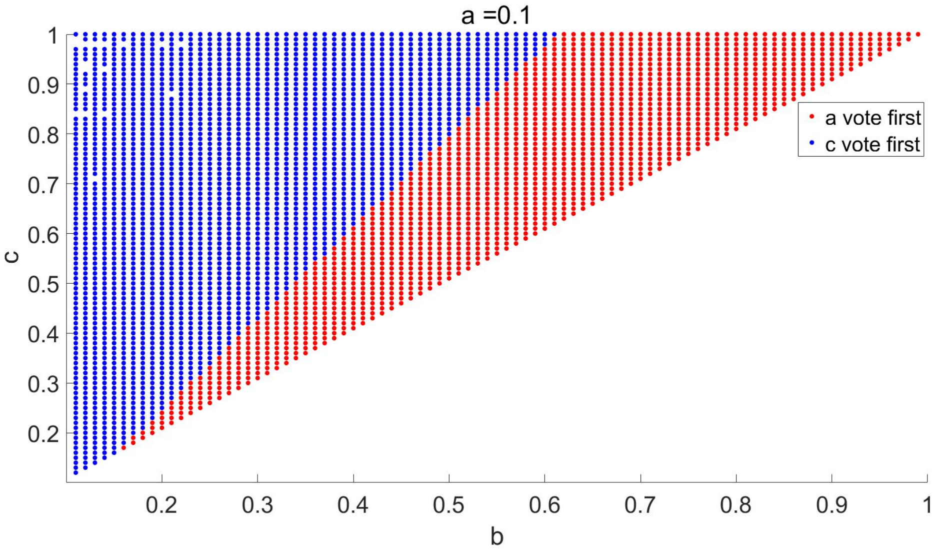


Figure A.1: SVP (Type III) Optimal voting order $a = 0.1$

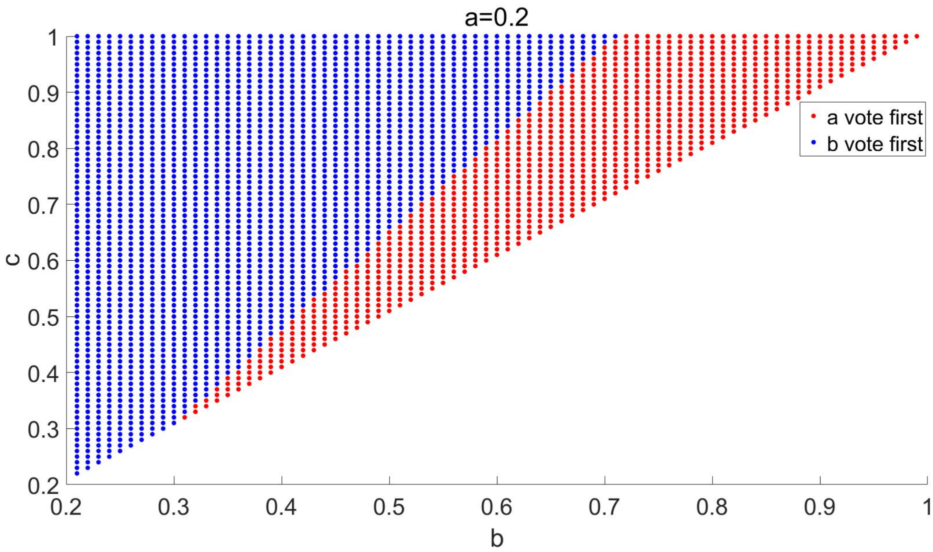


Figure A.2: SVP (Type III) Optimal voting order $a = 0.2$

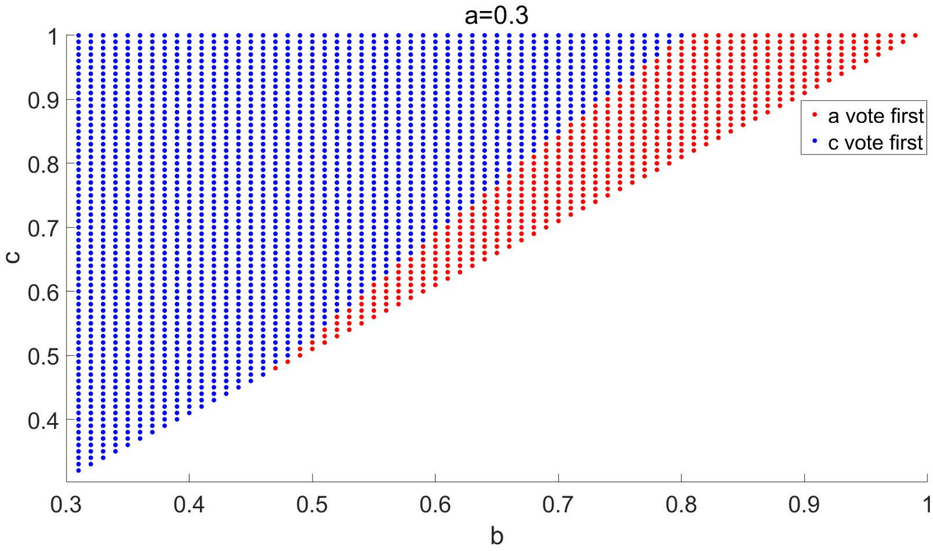


Figure A.3: SVP (Type III) Optimal voting order $a = 0.3$

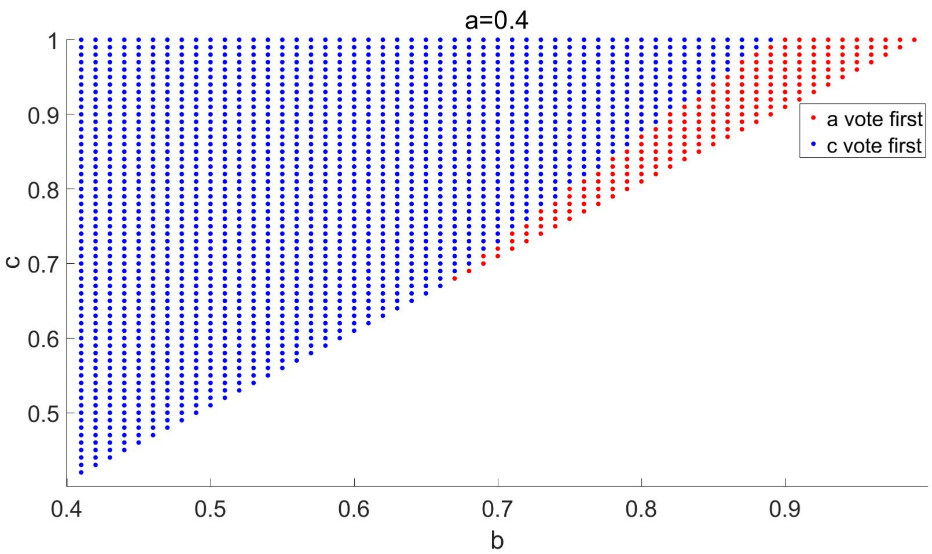


Figure A.4: SVP (Type III) Optimal voting order $a = 0.4$

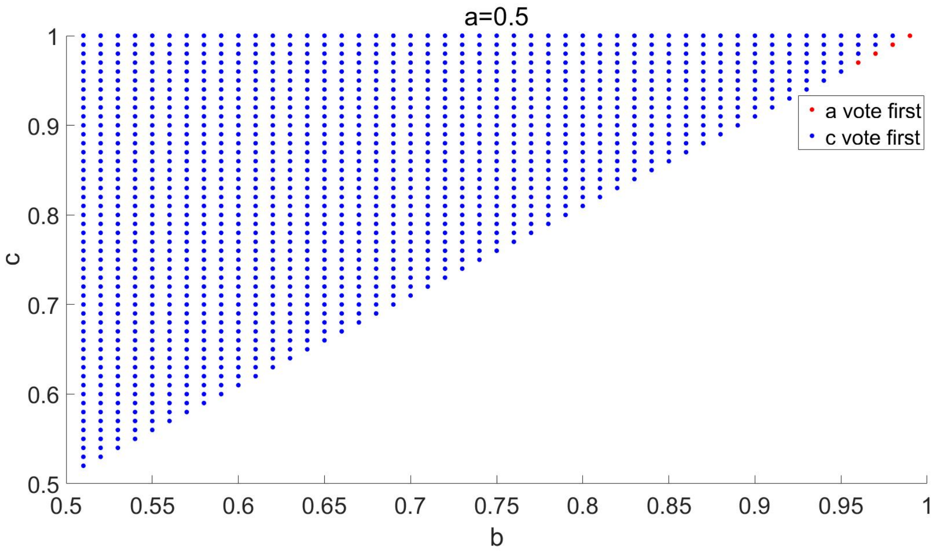


Figure A.5: SVP (Type III) Optimal voting order $a = 0.5$

Appendix B

Data for sealed card problem

Table B.1: Reliability Q for $D \leq 16$ and even ability

		Voting Ordering									
D	Ability sets	(b, c, a)[1]	$>$	(b, a, c)	$\mid$	(a, c, b)	$>$	(a, b, c)	$<$	(c, b, a)	$>$	(c, a, b)
8	$\{2, 4, 6\}$	**0.7878**		0.6834		0.7551		0.6933		0.7469		0.7143
	$\{2, 4, 6\}$	**0.6882**		0.6352		0.6705		0.6403		0.6718		0.6429
	$\{2, 4, 8\}$	**0.7751**		0.6526		0.7531		0.6628		0.7275		0.7222
10	$\{2, 6, 8\}$	**0.8016**		0.6429		0.7531		0.6987		0.7538		0.7222
	$\{4, 6, 8\}$	**0.8016**		0.7109		0.7751		0.7197		0.7606		0.7285
	$\{2, 4, 6\}$	**0.6383**		0.6075		0.6296		0.6107		0.6328		0.6082
	$\{2, 4, 8\}$	**0.6860**		0.6172		0.6723		0.6228		0.6575		0.6515
	$\{2, 4, 10\}$	**0.7686**		0.6344		0.7521		0.6453		0.7294		0.7273
	$\{2, 6, 8\}$	**0.7007**		0.6082		0.6741		0.6469		0.6801		0.6515
	$\{2, 6, 10\}$	**0.7863**		0.6082		0.7521		0.6714		0.7361		0.7273
12	$\{2, 8, 10\}$	**0.8099**		0.6515		0.7521		0.7022		0.7578		0.7273
	$\{4, 6, 8\}$	**0.7015**		0.6592		0.6883		0.6637		0.6867		0.6586
	$\{4, 6, 10\}$	**0.7863**		0.6794		0.7686		0.6869		0.7363		0.7308
	$\{4, 8, 10\}$	**0.8099**		0.6620		0.7686		0.7177		0.7628		0.7318
	$\{6, 8, 10\}$	**0.8099**		0.7257		0.7863		0.7342		0.7691		0.7379
	$\{2, 4, 6\}$	**0.6125**		0.5893		0.6032		0.5916		0.6085		0.5874

[1] The boldface under (b, c, a) corresponds to Proposition **??**, the one " $<$ " covers Proposition **??**, and the three " $>$ "s indicate Proposition **??**.

	$\{2,4,8\}$	**0.6422**	0.5959	0.6330	0.5997	0.6224	0.6166
	$\{2,4,10\}$	**0.6851**	0.6053	0.6743	0.6115	0.6608	0.6573
	$\{2,4,12\}$	**0.7647**	0.6223	0.7515	0.6338	0.7308	0.7308
	$\{2,6,8\}$	**0.6522**	0.5874	0.6350	0.6177	0.6413	0.6166
	$\{2,6,10\}$	**0.6959**	0.5874	0.6743	0.6305	0.6674	0.6573
	$\{2,6,12\}$	**0.7778**	0.5874	0.7515	0.6549	0.7342	0.7308
	$\{2,8,10\}$	**0.7087**	0.6166	0.6767	0.6512	0.6853	0.6573
	$\{2,8,12\}$	**0.7935**	0.6166	0.7515	0.6768	0.7421	0.7308
	$\{2,10,12\}$	**0.8155**	0.6573	0.7515	0.7047	0.7605	0.7308
14	$\{4,6,8\}$	**0.6514**	0.6286	0.6433	0.6315	0.6472	0.6234
	$\{4,6,10\}$	**0.6956**	0.6398	0.6856	0.6439	0.6677	0.6621
	$\{4,6,12\}$	**0.7778**	0.6602	0.7647	0.6673	0.7351	0.7308
	$\{4,8,10\}$	**0.7089**	0.6257	0.6879	0.6643	0.6905	0.6630
	$\{4,8,12\}$	**0.7935**	0.6284	0.7647	0.6889	0.7422	0.7308
	$\{4,10,12\}$	**0.8155**	0.6667	0.7647	0.7169	0.7641	0.7308
	$\{6,8,10\}$	**0.7103**	0.6732	0.6991	0.6775	0.6965	0.6695
	$\{6,8,12\}$	**0.7935**	0.6943	0.7778	0.7009	0.7424	0.7371
	$\{6,10,12\}$	**0.8155**	0.6750	0.7778	0.7290	0.7688	0.7388
	$\{8,10,12\}$	**0.8155**	0.7352	0.7935	0.7435	0.7748	0.7444
	$\{2,4,6\}$	**0.5949**	0.5764	0.5875	0.5782	0.5919	0.5734
	$\{2,4,8\}$	**0.6157**	0.5814	0.6089	0.5842	0.6007	0.5952
	$\{2,4,10\}$	**0.6435**	0.5877	0.6353	0.5920	0.6262	0.6224
	$\{2,4,12\}$	**0.6847**	0.5968	0.6759	0.6037	0.6630	0.6615
	$\{2,4,14\}$	**0.7621**	0.6137	0.7511	0.6256	0.7333	0.7333
	$\{2,6,8\}$	**0.6214**	0.5734	0.6075	0.5985	0.6168	0.5952
	$\{2,6,10\}$	**0.6511**	0.5734	0.6366	0.6069	0.6323	0.6224
	$\{2,6,12\}$	**0.6932**	0.5734	0.6760	0.6197	0.6673	0.6615
	(2,6,14)	**0.7725**	0.5734	0.7511	0.6439	0.7338	0.7333
	$\{2,8,10\}$	**0.6597**	0.5952	0.6388	0.6223	0.6468	0.6224
	$\{2,8,12\}$	**0.7025**	0.5952	0.6759	0.6357	0.6745	0.6615
	$\{2,8,14\}$	**0.7841**	0.5952	0.7511	0.6612	0.7377	0.7333
	$\{2,10,12\}$	**0.7143**	0.6224	0.6786	0.6542	0.6888	0.6615
	$\{2,10,14\}$	**0.7986**	0.6224	0.7511	0.6807	0.7464	0.7333
	$\{2,12,14\}$	**0.8195**	0.6615	0.7511	0.7065	0.7623	0.7333
	$\{4,6,8\}$	**0.6249**	0.6081	0.6189	0.6101	0.6219	0.6017
	$\{4,6,10\}$	**0.6518**	0.6157	0.6453	0.6184	0.6326	0.6273

	$\{4, 6, 12\}$	**0.6932**	0.6267	0.6848	0.6308	0.6681	0.6643
16	$\{4, 6, 14\}$	**0.7725**	0.6472	0.7621	0.6542	0.7338	0.7333
	$\{4, 8, 10\}$	**0.6564**	0.6034	0.6434	0.6335	0.6517	0.6281
	$\{4, 8, 12\}$	**0.7025**	0.6049	0.6853	0.6464	0.6747	0.6651
	$\{4, 8, 14\}$	**0.7841**	0.6078	0.7621	0.6711	0.7385	0.7333
	$\{4, 10, 12\}$	**0.7139**	0.6305	0.6879	0.6650	0.6930	0.6660
	$\{4, 10, 14\}$	**0.7986**	0.6332	0.7621	0.6905	0.7464	0.7333
	$\{4, 12, 14\}$	**0.8195**	0.6698	0.7621	0.7165	0.7651	0.7333
	$\{6, 8, 10\}$	**0.6607**	0.6415	0.6543	0.6442	0.6569	0.6343
	$\{6, 8, 12\}$	**0.7024**	0.6532	0.6942	0.6567	0.6751	0.6701
	$\{6, 8, 14\}$	**0.7841**	0.6748	0.7725	0.6805	0.7394	0.7339
	$\{6, 10, 12\}$	**0.7148**	0.6381	0.6967	0.6753	0.6977	0.6715
	$\{6, 10, 14\}$	**0.7986**	0.6425	0.7725	0.6998	0.7466	0.7340
	$\{6, 12, 14\}$	**0.8195**	0.6776	0.7725	0.7261	0.7687	0.7341
	$\{8, 10, 12\}$	**0.7166**	0.6825	0.7065	0.6867	0.7034	0.6773
	$\{8, 10, 14\}$	**0.7986**	0.7041	0.7841	0.7102	0.7469	0.7419
	$\{8, 12, 14\}$	**0.8195**	0.6843	0.7841	0.7367	0.7731	0.7439
	$\{10, 12, 14\}$	**0.8195**	0.7418	0.7986	0.7500	0.7789	0.7492

Table B.2: Optimality Fraction ϕ for $D \leq 52$

D \ r	(a,b,c)	(a,c,b)	(b,a,c)	(b,c,a)	(c,a,b)	(c,b,a)	*Sum*
4	0.000	1.000	0.000	1.000	1.000	1.000	4.000
6	0.000	0.800	0.000	0.900	0.600	0.600	2.900
8	0.029	0.629	0.000	0.800	0.429	0.486	2.371
10	0.012	0.524	0.000	0.750	0.345	0.440	2.071
12	0.012	0.479	0.000	0.685	0.291	0.406	1.873
14	0.007	0.444	0.000	0.675	0.248	0.385	1.759
16	0.007	0.398	0.000	0.659	0.235	0.367	1.666
18	0.010	0.372	0.000	0.651	0.212	0.351	1.597
20	0.009	0.348	0.000	0.654	0.199	0.335	1.546
22	0.008	0.328	0.000	0.650	0.182	0.329	1.498
24	0.007	0.318	0.000	0.640	0.177	0.318	1.461
26	0.006	0.298	0.000	0.652	0.167	0.305	1.427
28	0.005	0.281	0.000	0.650	0.162	0.300	1.399
30	0.005	0.264	0.000	0.647	0.162	0.297	1.375
32	0.006	0.255	0.000	0.644	0.156	0.295	1.356
34	0.006	0.247	0.000	0.643	0.151	0.291	1.338
36	0.006	0.238	0.000	0.640	0.153	0.282	1.319
38	0.006	0.230	0.000	0.641	0.150	0.278	1.305
40	0.007	0.225	0.000	0.638	0.152	0.272	1.293
42	0.007	0.220	0.000	0.637	0.148	0.268	1.280
44	0.006	0.217	0.000	0.634	0.147	0.266	1.270
46	0.006	0.211	0.000	0.633	0.148	0.261	1.260
48	0.006	0.206	0.000	0.629	0.151	0.258	1.250
50	0.007	0.204	0.000	0.629	0.147	0.256	1.242
52	0.007	0.200	0.000	0.630	0.148	0.253	1.238

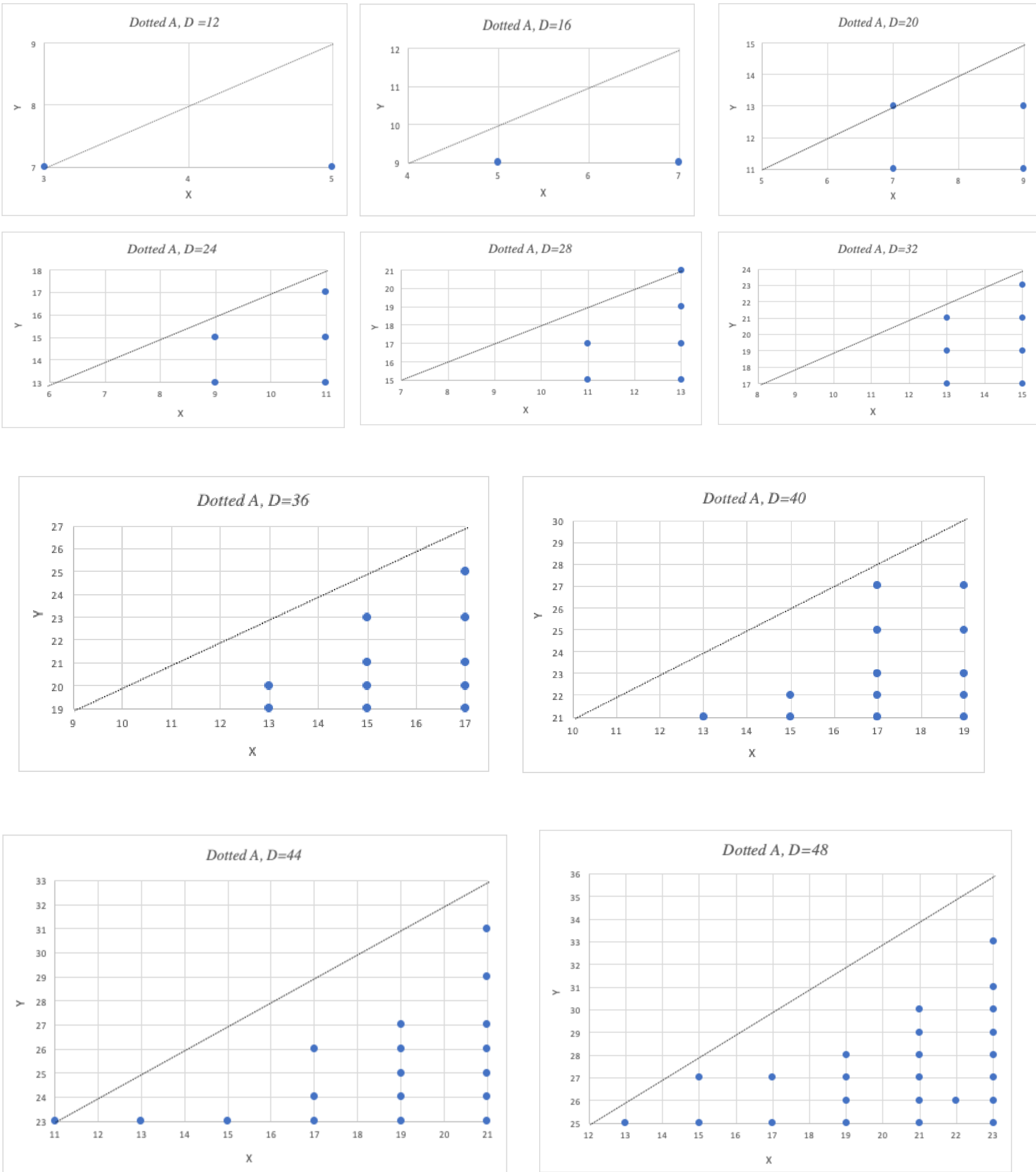
Table B.3: Average reliability $\bar{Q}$ for $D \leq 52$

D \ r	(a,b,c)	(a,c,b)	(b,a,c)	(b,c,a)	(c,a,b)	(c,b,a)	*simultaneous*
4	0.7778	1	0.8889	1	1	1	0.888999
6	0.7678	0.9006	0.7025	0.9096	0.86695	0.8896	0.8146
8	0.744128	0.843855	0.681486	0.85628	0.8088	0.839024	0.774127
10	0.725061	0.812857	0.659651	0.822672	0.772841	0.803265	0.747348
12	0.704583	0.785297	0.652501	0.796018	0.750056	0.77844	0.727762
14	0.692853	0.768276	0.640532	0.776651	0.730099	0.758285	0.712546
16	0.682228	0.751981	0.635073	0.760459	0.717897	0.743916	0.700239
18	0.672185	0.738815	0.627091	0.746923	0.705303	0.730055	0.689993
20	0.664594	0.727777	0.621887	0.735475	0.695876	0.719259	0.681275
22	0.658326	0.718764	0.616938	0.725789	0.687751	0.709833	0.67373
24	0.652772	0.710041	0.614211	0.716997	0.681314	0.70195	0.667109
26	0.647514	0.702963	0.60912	0.709481	0.673368	0.69456	0.661234
28	0.642993	0.696167	0.60641	0.702621	0.669046	0.688619	0.655971
30	0.638499	0.689947	0.603337	0.696407	0.663914	0.682538	0.651218
32	0.634848	0.684518	0.600855	0.690692	0.658975	0.677337	0.646897
34	0.631287	0.679688	0.598015	0.685594	0.654575	0.672615	0.642943
36	0.62775	0.674934	0.596187	0.680797	0.651382	0.668366	0.639308
38	0.624959	0.670851	0.59369	0.676422	0.647283	0.664191	0.635949
40	0.622059	0.666824	0.592177	0.672309	0.644644	0.660627	0.632833
42	0.619261	0.663077	0.590193	0.66847	0.641029	0.657066	0.62993
44	0.616911	0.659695	0.588524	0.664908	0.638197	0.653857	0.627218
46	0.614529	0.656505	0.586695	0.661634	0.63587	0.650837	0.624676
48	0.612361	0.653408	0.58549	0.65846	0.633715	0.648017	0.622286
50	0.610391	0.650569	0.583772	0.655492	0.630596	0.645235	0.620035
52	0.608398	0.647872	0.582477	0.652719	0.62865	0.64278	0.617907

Table B.4: fraction of ability sets in which $Q_{sim} > Q(b, c, a)$ given D

D	$\hat{\phi}(D)$
12	0.0242
14	0.0210
16	0.0220
18	0.0235
20	0.0268
22	0.0248
24	0.0226
26	0.0204
28	0.0191
30	0.0175
32	0.0165
34	0.0163
36	0.0163
38	0.0156
40	0.0165
42	0.0173
44	0.0168
46	0.0178
48	0.0179
50	0.0181
52	0.0187

Figure B.1: Graphs of Dotted A for $12 \leq D < 52$.



Bibliography

Acemoglu, D., Dahleh, M. A., Lobel, I. & Ozdaglar, A. (2011), 'Bayesian learning in social networks', *The Review of Economic Studies* **78**(4), 1201–1236.

Acemoglu, D. & Ozdaglar, A. (2011), 'Opinion dynamics and learning in social networks', *Dynamic Games and Applications* **1**(1), 3–49.

Alpern, S. & Chen, B. (2017*a*), 'The importance of voting order for jury decisions by sequential majority voting', *European Journal of Operational Research* **258**(3), 1072–1081.

Alpern, S. & Chen, B. (2017*b*), 'Who should cast the casting vote? Using sequential voting to amalgamate information', *Theory and Decision* **83**(2), 259–282.

Alpern, S. & Chen, B. (2020), 'Optimizing voting order on sequential juries: A median voter theorem', *arXiv preprint arXiv:2006.14045* .

Arrow, K. J. et al. (1951), An extension of the basic theorems of classical welfare economics, *in* 'Proceedings of the second Berkeley symposium on mathematical statistics and probability', The Regents of the University of California.

Austen-Smith, D. & Banks, J. S. (1996), 'Information aggregation, rationality, and the Condorcet Jury Theorem', *American Political Science Review* **90**(1), 34–45.

Bang-Jensen, J. & Gutin, G. Z. (2008), *Digraphs: theory, algorithms and applications*, Springer Science & Business Media.

Ben-Yashar, R. & Paroush, J. (2000), 'A nonasymptotic Condorcet Jury Theorem', *Social Choice and Welfare* **17**(2), 189–199.

Berend, D. & Kontorovich, A. (2014), Consistency of weighted majority votes, *in* 'Proceedings of the 27th International Conference on Neural Information Processing Systems-Volume 2', pp. 3446–3454.

Berg, S. (1993), 'Condorcet's Jury Theorem, dependency among jurors', *Social Choice and Welfare* **10**(1), 87–95.

Berg, S. (1994), 'Evaluation of some weighted majority decision rules under dependent voting', *Mathematical Social Sciences* **28**(2), 71–83.

Bikhchandani, S., Hirshleifer, D. & Welch, I. (1992), 'A theory of fads, fashion, custom, and cultural change as informational cascades', *Journal of Political Economy* **100**(5), 992–1026.

Boland, P. J., Proschan, F. & Tong, Y. L. (1989), 'Modelling dependence in simple and indirect majority systems', *Journal of Applied Probability* **26**(1), 81–88.

Borda, J. d. (1784), 'Mémoire sur les élections au scrutin', *Histoire de l'Academie Royale des Sciences* .

Buchanan, J. M. (1986), *Liberty, market and state: Political economy in the 1980s*, Wheatsheaf Books.

Buchanan, J. M. & Tullock, G. (1962), *The calculus of consent*, Vol. 3, University of Michigan Press.

Condorcet, M. D. (1785), 'Essay on the application of analysis to the probability of majority decisions', *Imprimerie Royale* .

Cusanus, N. (1434), On Catholic Harmony, *in* I. McLean & A. Urken, eds, 'Classics of Social Choice', University of Michigan Press.

Dahl, R. A. (1956), *A preface to democratic theory*, Vol. 115, University of Chicago Press.

Dekel, E. & Piccione, M. (2000), 'Sequential voting procedures in symmetric binary elections', *Journal of political Economy* **108**(1), 34–55.

Dodgson, C. (1876), A method of taking votes on more than two issues, *in* D. Black, ed., 'The theory of committees and elections', Kluwer Academic Publishers.

Estlund, D. M. (1994), 'Opinion leaders, independence, and Condorcet's jury theorem', *Theory and Decision* **36**(2), 131–162.

Fain, B., Goel, A., Munagala, K. & Sakshuwong, S. (2017), Sequential deliberation for social choice, *in* 'International Conference on Web and Internet Economics', Springer, pp. 177–190.

Feddersen, T. J. & Pesendorfer, W. (1996), 'The swing voter's curse', *The American Economic Review* pp. 408–424.

Feddersen, T. & Pesendorfer, W. (1997), 'Voting behavior and information aggregation in elections with private information', *Econometrica: Journal of the Econometric Society* pp. 1029–1058.

Feddersen, T. & Pesendorfer, W. (1998), 'Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting', *American Political Science Review* **92**(1), 23–35.

Feddersen, T. & Pesendorfer, W. (1999), 'Elections, information aggregation, and strategic voting', *Proceedings of the National Academy of Sciences* **96**(19), 10572–10574.

Fey, M. (2003), 'A note on the Condorcet jury theorem with supermajority voting rules', *Social Choice and Welfare* **20**(1), 27–32.

Fishburn, P. C. & Gehrlein, W. V. (1977), 'An analysis of voting procedures with non-ranked voting', *Behavioral Science* **22**(3), 178–185.

Gale, D. & Kariv, S. (2003), 'Bayesian learning in social networks', *Games and Economic Behavior* **45**(2), 329–346.

Gibbard, A. (1973), 'Manipulation of voting schemes: A general result', *Econometrica* **41**(4), 587–601.

Grofman, B. & Feld, S. L. (1988), 'Rousseau's general will: a Condorcetian perspective', *American Political Science Review* **82**(2), 567–576.

Hammersley, J. M. & Handscomb (1964), *Monte Carlo Methods*, Chapman and Hall.

Kahneman, D., Lovallo, D. & Sibony, O. (2011), 'Before you make that big decision', *Harvard Business Review* **89**(6), 50–60.

Kanazawa, S. (1998), 'A brief note on a further refinement of the Condorcet Jury Theorem for heterogeneous groups', *Mathematical Social Sciences* **35**(1), 69–73.

Ladha, K. K. (1992), 'The Condorcet Jury Theorem, free speech, and correlated votes', *American Journal of Political Science* **36**(3), 617–634.

Lobel, I. & Sadler, E. (2015), 'Information diffusion in networks through social learning', *Theoretical Economics* **10**(3), 807–851.

Lobel, I. & Sadler, E. (2016), 'Preferences, homophily, and social learning', *Operations Research* **64**(3), 564–584.

Lull, R. (1283), Blanquera, *in* I. McLean & A. Urken, eds, 'Classics of Social Choice', University of Michigan Press.

Lull, R. (1299), The Art of Elections, *in* I. McLean & A. Urken, eds, 'Classics of Social Choice', University of Michigan Press.

McLean, I. & Urken, A. (1995), *Classics of social choice*, University of Michigan Press.

McLennan, A. (1998), 'Consequences of the Condorcet Jury Theorem for beneficial information aggregation by rational agents', *American Political Science Review* **92**(2), 413–418.

Nitzan, S. & Paroush, J. (1984), 'The significance of independent decisions in uncertain dichotomous choice situations', *Theory and Decision* **17**(1), 47–60.

Ottaviani, M. & Sørensen, P. (2001), 'Information aggregation in debate: who should speak first?', *Journal of Public Economics* **81**(3), 393–421.

Owen, G., Grofman, B. & Feld, S. L. (1989), 'Proving a distribution-free generalization of the Condorcet Jury Theorem', *Mathematical Social Sciences* **17**(1), 1–16.

Paroush, J. (1998), 'Stay away from fair coins: A condorcet jury theorem', *Social Choice and Welfare* **15**(1), 15–20.

Peleg, B. & Zamir, S. (2012), 'Extending the Condorcet Jury Theorem to a general dependent jury', *Social Choice and Welfare* **39**(1), 91–125.

Pivato, M. (2017), 'Epistemic democracy with correlated voters', *Journal of Mathematical Economics* **72**, 51–69.

Rae, D. W. (1969), 'Decision-rules and individual values in constitutional choice', *American Political Science Review* **63**(1), 40–56.

Reisman, D. (1989), *The political economy of James Buchanan*, Springer.

Sah, R. K. & Stiglitz, J. E. (1985), 'Human fallibility and economic organization', *The American Economic Review* **75**(2), 292–297.

Satterthwaite, M. A. (1975), 'Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions.', *Journal of Economic Theory* **10**(2), 187–217.

Sen, A. K. (1966), 'A possibility theorem on majority decisions', *Econometrica: Journal of the Econometric Society* pp. 491–499.

Song, Y. (2016), 'Social learning with endogenous observation', *Journal of Economic Theory* **166**, 324–333.

Sørensen, P. N. & Ottaviani, M. (2001), 'Information aggregation in debate: who should speak first?', *Journal of Public Economics* **81**(3), 393–421.

Straffin, P. D. (1977), 'Majority rule and general decision rules', *Theory and Decision* **8**(4), 351–360.

Strøm, K., Strøm, K. et al. (1990), *Minority government and majority rule*, Cambridge University Press.

Surowiecki (2005), *The wisdom of crowds*, New York: Anchor Books.

Tsebelis, G. & Garrett, G. (2000), 'Legislative politics in the european union', *European Union Politics* **1**(1), 9–36.

Vickrey, W. (1960), 'Utility, strategy, and social decision rules', *The Quarterly Journal of Economics* **74**(4), 507–535.

Wicksell, K. (1958), *A new principle of just taxation*, Springer.