Cartan connections for stochastic developments on sub-Riemannian manifolds

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Abstract

Analogous to the characterisation of Brownian motion on a Riemannian manifold as the development of Brownian motion on a Euclidean space, we construct sub-Riemannian diffusions on equinilpotentisable sub-Riemannian manifolds by developing a canonical stochastic process arising as the lift of Brownian motion to an associated model space. The notion of stochastic development we introduce for equinilpotentisable sub-Riemannian manifolds uses Cartan connections, which take the place of the Levi–Civita connection in Riemannian geometry. We first derive a general expression for the generator of the stochastic process which is the stochastic development with respect to a Cartan connection of the lift of Brownian motion to the model space. We further provide a necessary and sufficient condition for the existence of a Cartan connection which develops the canonical stochastic process to the sub-Riemannian diffusion associated with the sub-Laplacian defined with respect to the Popp volume. We illustrate the construction of a suitable Cartan connection for free sub-Riemannian structures with two generators and we discuss an example where the condition is not satisfied.

1 Introduction

Brownian motion, also called Wiener process, is a mathematical description of the animated and irregular motion of particles which are suspended, say, in a fluid. This process plays an important role in various areas of mathematics and is used, among other things, to describe more complicated stochastic processes, to model unknown forces in control theory, to give a rigorous path integral formulation of quantum mechanics, and it prominently features in mathematical finance.

Brownian motion $(b_t)_{t\geq 0}$ on a smooth Riemannian manifold M with the Laplace–Beltrami operator Δ_M is the unique continuous-time stochastic process on M whose infinitesimal motion is described by $\frac{1}{2}\Delta_M$, that is, for the heat semigroup $(P_t)_{t\geq 0}$ associated with $(b_t)_{t\geq 0}$ and for any function $f \in C_c^\infty(M)$, we have

$$\frac{1}{2}\Delta_M f(q) = \lim_{t\downarrow 0} \frac{P_t f(q) - f(q)}{t} .$$

We call $\frac{1}{2}\Delta_M$ the infinitesimal generator of Brownian motion on M. One of the many interesting features of Brownian motion is that it can be used to give a solution to the Dirichlet problem

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associated with Δ_M on a domain in M. In fact, it is even possible to uniquely characterise Brownian motion via the heat equation. Brownian motion also arises as the limit of a sequence of random walks on the manifold M.

Yet another alternative construction of Brownian motion uses the notion of anti-development of a curve in a Riemannian manifold. To a differentiable curve in the Riemannian manifold M of dimension n, we can associate a curve in the model space $\mathbb{R}^n$ via the Levi–Civita connection. By extending this correspondence to stochastic processes whose sample paths are almost surely continuous but nowhere differentiable it can be shown that a stochastic process on the manifold M of dimension n is a Brownian motion on M if and only if its anti-development is a standard Brownian motion on $\mathbb{R}^n$. In particular, if we take a standard Brownian motion on $\mathbb{R}^n$ we obtain a process in the orthonormal frame bundle $\mathcal{O}(M)$ which projects nicely to M to give a Brownian motion on the Riemannian manifold M.

For further details on and properties of Brownian motions on smooth Riemannian manifolds, see, for instance, Émery [13], Grigor'yan [16], Hsu [21], and Jørgensen [22], as well as [18] for a more exhaustive overview of the various characterisations of Brownian motion.

With Brownian motions on smooth Riemannian manifolds being well understood, we turn our attention to the sub-Riemannian setting. A sub-Riemannian manifold is a triple $(M, \mathcal{D}, g)$ consisting of a smooth manifold M together with a bracket generating distribution $\mathcal{D} \subset TM$ and a metric g on $\mathcal{D}$. As the natural generalisation of the Laplace–Beltrami operator in Riemannian geometry, the sub-Riemannian Laplacians, also called sub-Laplacians, on a sub-Riemannian manifold are defined as the divergence of the horizontal gradient. The divergence div_ν depends on a choice of a positive smooth measure ν on the manifold M, and the horizontal gradient $\operatorname{grad}_H f$ of a function $f \in C^\infty(M)$ is a smooth section of $\mathcal{D}$ such that, for any section $X \in \Gamma(\mathcal{D})$,

$$g(\operatorname{grad}_H f, X) = \mathrm{d}f(X) \ .$$

The sub-Laplacian Δ_ν with respect to the measure ν acting on smooth functions f on M is thus given by

$$\Delta_\nu f = \operatorname{div}_\nu(\operatorname{grad}_H f) \ . \tag{1.1}$$

Note that the horizontal gradient $\operatorname{grad}_H f$ is the unique smooth section of $\mathcal{D}$ satisfying (1.1) and that it only depends on the sub-Riemannian structure. However, unlike Riemannian geometry the measure ν is not canonically defined and several natural choices are possible. This includes the Popp measure that we define later.

For a local orthonormal frame $(X_1, \ldots, X_{k_1})$ of $\mathcal{D}$ with respect to g, we can locally write

$$\Delta_\nu = \sum_{i=1}^{k_1} X_i^2 + \sum_{i=1}^{k_1} \operatorname{div}_\nu(X_i) X_i \ . \tag{1.2}$$

Similar to Brownian motion on a Riemannian manifold, we can consider the continuous-time stochastic process on M whose infinitesimal generator is $\frac{1}{2}\Delta_\nu$. These processes, which we call sub-Riemannian diffusions, are significantly less well understood than Brownian motion, and many of their properties remain to be understood.

Following Métivier [23] and Ben Arous [7,8], the small-time asymptotics of sub-Riemannian diffusion processes have been studied, amongst others, by Bailleul, Mesnager, Norris [3], by Barilari, Boscain, Neel [4], by de Verdière, Hillairet, Trélat [11] as well as in [19] and [20]. Already at the level of small-time asymptotics, it is seen that sub-Riemannian diffusions show qualitatively different behaviours compared to Brownian motions. We further remark that since sub-Laplacians are in divergence form, the first-order small-time heat kernel asymptotics of the associated stochastic processes only depend on the underlying sub-Riemannian structure, and not on the measure ν.

For certain sub-Laplacians, the question of approximating the associated sub-Riemannian diffusions by random walks has been addressed by Boscain, Neel, Rizzi [10] and by Gordina, Laetsch [15].

In comparison to the various characterisations of Brownian motion on a Riemannian manifold, the one characterisation which appears to be missing for sub-Riemannian diffusions is as the development of a suitable model stochastic process. The first major obstacle to such a construction is that the notion of Levi–Civita connection does not carry over to the sub-Riemannian setting. Instead, we employ the notion of Cartan connections which is well-adapted to the graded structures appearing in the study of sub-Riemannian manifolds. The Cartan geometry approach to sub-Riemannian geometry works particularly well for equinilpotentisable sub-Riemannian manifolds. Moreover, for equiregular, and therefore also for equinilpotentisable, sub-Riemannian manifolds, there exists a smooth volume canonically associated with the sub-Riemannian structure, which is the so-called Popp volume. The objective of this article is to initiate the characterisation of sub-Riemannian diffusions via stochastic development by providing such a construction on a wide range of equinilpotentisable sub-Riemannian manifolds for the sub-Riemannian diffusion associated with the sub-Laplacian defined with respect to the Popp volume.

We stress that the work presented lies more on the differential geometry side and is mainly concerned with constructing suitable Cartan connections. The central ingredient needed from stochastic analysis is the correspondence between Stratonovich stochastic differential equations and infinitesimal generators, which is provided at the suitable point.

Cartan geometry makes the idea of a model tangent space rigorous. For example, the tangent space of a Riemannian manifold M of dimension n is the Euclidean space $\mathbb{R}^n$ and we can view a Cartan connection as a way of rolling this Euclidean space on the Riemannian manifold, see Wise [32]. It gives rise to a natural notion of the development of a curve $\gamma\colon [0,t] \to \mathbb{R}^n$ to M by simply saying that the contact point while rolling traces out γ, and the notion of stochastic development is defined in a similar way.

In the sub-Riemannian setting, the model tangent spaces are Lie algebras of nilpotent Lie groups known as the Carnot groups. The distribution $\mathcal{D}$ of a sub-Riemannian manifold $(M, \mathcal{D}, g)$ generates a filtration which is defined iteratively at $q \in M$, for $i \in \mathbb{N}$, by $\mathcal{D}_q^{-1} = \mathcal{D}_q$ and

$$\mathcal{D}_q^{-(i+1)} = \mathcal{D}_q^{-i} + \left[\mathcal{D}, \mathcal{D}^{-i}\right]_q \ .$$

The minimal $m \in \mathbb{N}$ such that $\mathcal{D}_q^{-m} = T_qM$ for every $q \in M$ is the step of the sub-Riemannian manifold, and the tuple of numbers $(k_1, k_2, \ldots, k_m) = (\dim \mathcal{D}_q^{-1}, \dim \mathcal{D}_q^{-2}, \ldots, \dim \mathcal{D}_q^{-m})$ is called the growth vector at $q \in M$. Using the filtration, we further define the associated grading of the tangent space T_qM at $q \in M$ by

$$\operatorname{gr}(T_qM) = \mathcal{D}_q^{-1} \oplus \mathcal{D}_q^{-2}/\mathcal{D}_q^{-1} \oplus \cdots \oplus \mathcal{D}_q^{-m}/\mathcal{D}_q^{-(m-1)} \ . \tag{1.3}$$

Each $\operatorname{gr}(T_qM)$ has the natural structure of a nilpotent Lie algebra as well as a natural horizontal metric defined on $\mathcal{D}_q^{-1}$. If $X \in \Gamma(\mathcal{D}^{-i})$, $Y \in \Gamma(\mathcal{D}^{-j})$, for $i, j \in \{1, \ldots, m\}$, are sections of the corresponding bundles, then the Lie algebra structure is defined by

$$\left[X + \Gamma\left(\mathcal{D}^{-i+1}\right), Y + \Gamma\left(\mathcal{D}^{-j+1}\right)\right] = [X, Y] + \Gamma\left(\mathcal{D}^{-i-j+1}\right) \ .$$

We notice that the metric g on $\mathcal{D}$ induces a metric on all of $\operatorname{gr}(T_qM)$ for $q \in M$ as follows. On the tensor product $\otimes_{i=1}^{l} \mathcal{D}_q^{-1}$ for $l \in \{2, \ldots, m\}$, we define a map

$$\pi_l\colon \bigotimes_{i=1}^{l} \mathcal{D}_q^{-1} \to \mathcal{D}_q^{-l}/\mathcal{D}_q^{-l+1}$$

given, for vector fields $X_1, \ldots, X_l$ on M extending vectors $v_1, \ldots, v_l \in T_q M$, by

$$\pi_l(v_1 \otimes \cdots \otimes v_l) = [X_1, [X_2, \ldots, [X_{l-1}, X_l] \ldots]](q) \mod \mathcal{D}_q^{-l+1} .$$

Using the metric induced by g on $\otimes_{i=1}^{l} \mathcal{D}^{-1}$, we identify $\mathcal{D}_q^{-l}/\mathcal{D}_q^{-l+1}$ with $(\ker \pi_l)^{\perp}$ for the restricted metric. This further gives rise to a metric on the whole of $\operatorname{gr}(T_q M)$, which plays an important role in Section 4. The space $\Lambda^n(T_q M)$ is then naturally isomorphic to $\Lambda^n \operatorname{gr}(T_q M)$, and the constructed metric allows us to choose a canonical element in $\Lambda^n \operatorname{gr}(T_q M)$. The image of this element under the canonical isomorphism is the Popp volume at $q \in M$. For further details, see Montgomery [24, Chapter 10].

A sub-Riemannian manifold is called equinilpotentisable if $\operatorname{gr}(T_q M)$ does not depend on the point $q \in M$ as a metric Lie algebra, that is, all the associated gradings are metrically isomorphic. In this case, we write $\mathfrak{n}_{-k} = \mathcal{D}_q^{-k}/\mathcal{D}_q^{-k+1}$ for $k \in \{1, \ldots, m\}$ as well as $\mathfrak{n} = \operatorname{gr}(T_q M)$ to emphasise this independence. Moreover, $k_1 = \dim \mathcal{D}_q^{-1}$ is then also independent of $q \in M$. The Carnot group which serves as model tangent space to an equinilpotentisable sub-Riemannian manifold is the simply connected Lie group whose Lie algebra is $\mathfrak{n}$. In the Cartan terminology, this Carnot group is called the nilpotent model. It represents the flat space and the curvature invariants measure how much a given sub-Riemannian manifold differs from its nilpotent model.

It is important to note that Carnot groups can possess several non-equivalent left-invariant sub-Riemannian metrics. We say that left-invariant sub-Riemannian metrics g_1 and g_2 on a Carnot group $(G, \mathcal{D})$ are equivalent if there exists a graded Lie group automorphism that maps g_1 to g_1. One of the simplest examples of a Carnot group with non-equivalent sub-Riemannian metrics is the 5-dimensional Heisenberg group H_5. Its Lie algebra $\mathfrak{h}_5$ has a natural filtration $\mathbb{R}^4 \oplus z\mathbb{R} = \mathcal{D}^{-1} \oplus \mathcal{D}^{-2}$, where z is an element of the centre of $\mathfrak{h}_5$ and where, for $v_1, v_2 \in \mathcal{D}$, the Lie brackets are given, in terms of a symplectic form ω on $\mathcal{D}$, by

$$[v_1, v_2] = \omega(v_1, v_2) z .$$

All automorphisms that preserve $\mathcal{D} = \mathbb{R}^4$ have to preserve ω up to multiplication by a scalar. Therefore, the group of $\mathcal{D}$-preserving automorphisms is equivalent to the conformal-symplectic group $CSp(\mathbb{R}^4)$. As all symplectic forms on $\mathbb{R}^4$ are equivalent, classification of metrics on $\mathbb{R}^4$ with respect to $CSp(\mathbb{R}^4)$ is equivalent to the classification of pairs $(g, [\omega])$ where g is a metric on $\mathbb{R}^4$ and $[\omega]$ is a conformal class of symplectic forms. By normalising g to an arbitrary fixed metric, our classification problem reduces to the classification of $[\omega]$ with respect to the orthogonal group $O(\mathbb{R}^4)$. Furthermore, instead of ω we consider the skew-symmetric operator $\omega \circ g^{-1}$. Using the action of $O(\mathbb{R}^4)$, we can normalise every skew-symmetric operator to the form

$$\begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \end{pmatrix},$$

where $\lambda_1 \geq \lambda_2 > 0$. We see that conformal classes $[\omega]$ are in one-to-one correspondence with the ratio $\lambda_1 : \lambda_2$. This means that the family of non-equivalent metrics on H_5 is one-dimensional.

A Cartan connection allows us to develop curves from the corresponding Carnot group to an equinilpotentisable sub-Riemannian manifold, and in the same spirit to define a notion of stochastic development, exactly like it was done in the Riemannian case.

There is a canonical sub-Riemannian diffusion on a Carnot group, arising as the lift of a Brownian motion on $\mathbb{R}^{k_1}$ and taking the place of the standard Brownian motion on $\mathbb{R}^n$, which we develop to give a sub-Riemannian diffusion on the corresponding sub-Riemannian manifold. The generator of the resulting stochastic process always has the same principal symbol which is uniquely defined by the metric g, while the first order term depends on the choice of the Cartan connection. As said previously, we are particularly interested in constructing Cartan connections which give rise to sub-Riemannian diffusions associated to sub-Laplacians defined with respect to the Popp volume.

Barilari and Rizzi [5] give a local formula for the Popp volume $\mathcal{P}$ and for the sub-Laplacian $\Delta_{\mathcal{P}}$ with respect to the Popp volume in terms of an adapted frame, that is, a frame $(X_1, \ldots, X_n)$ such that $X_1, \ldots, X_{k_i}$ span $\mathcal{D}^{-i}$ for $i \in \{1, \ldots, m\}$ and $X_1, \ldots, X_{k_1}$ are orthonormal in $\mathcal{D}$. On an equinilpotentisable sub-Riemannian manifold, the local formula for $\Delta_{\mathcal{P}}$ only involves the structure constants $c_{ij}^k \in C^\infty(M)$ of the adapted frame, which satisfy, for $i, j \in \{1, \ldots, n\}$,

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \ ,$$

and it is given by

$$\Delta_{\mathcal{P}} = \sum_{i=1}^{k_1} \left(X_i^2 - \sum_{l=1}^{n} c_{il}^l X_i \right) \ . \tag{1.4}$$

Restricting our attention to equinilpotentisable sub-Riemannian manifolds limits the classes of structures that we can consider. For instance, even contact structures in higher dimensions are not equinilpotentisable in general, and our setting excludes singular structures like the Martinet manifold. Moreover, even for a equinilpotentisable sub-Riemannian manifold, there does not always exist a Cartan connection which allows us to characterise the sub-Riemannian diffusion associated with $\Delta_{\mathcal{P}}$ in terms of a stochastic development. However, in Theorem 1.2, we provide a necessary and sufficient condition, which is proven in Section 4, for when we can obtain such Cartan connections. All the Cartan connections we construct are characterised by a torsion-free-like condition, which requires a certain part of the curvature two-form to vanish.

We remark that Baudoin, Feng, Gordina [6] consider the stochastic parallel transport with respect to the Bott connection on foliated manifolds, and that Angst, Bailleul, Tardif [2] use the Cartan geometry approach to define kinetic Brownian motion on Riemannian manifolds.

A Cartan connection in the sub-Riemannian setting is a $\mathfrak{g}$-valued one-form ω, where $\mathfrak{g}$ is a semi-direct product of the nilpotent Lie algebra $\mathfrak{n}$ and the Lie algebra $\mathfrak{h}$ of the Lie group H of infinitesimal symmetries of $\mathfrak{n}$ which is isomorphic to a subgroup of $SO(k_1)$. It should be noted that H can be the trivial group consisting only of the identity element, as it happens for distributions with the growth vector $(2, 3, 4, \ldots, \dim M - 1, \dim M)$, see Section 4. The connection form can be separated into its $\mathfrak{n}$-valued part $\omega_{\mathfrak{n}}$ and its $\mathfrak{h}$-valued part $\omega_{\mathfrak{h}}$. We choose a basis $\{A_\alpha \colon 1 \leq \alpha \leq \dim H\}$ of $\mathfrak{h}$ and consider the corresponding components ω^α of $\omega_{\mathfrak{h}}$. If we further choose a local coframe $(\theta^1, \ldots, \theta^n)$ of T^*M dual to the adapted frame $(X_1, \ldots, X_n)$, we can locally write

$$\omega^\alpha = \sum_{i=1}^{n} \Gamma_i^\alpha \theta^i \ ,$$

for some smooth functions Γ_i^α which are the Christoffel symbols of the connection $\omega_{\mathfrak{h}}$. Often we aggregate the first k_1 components into a single vector-valued function

$$\Gamma^\alpha = \begin{pmatrix} \Gamma_1^\alpha \\ \vdots \\ \Gamma_{k_1}^\alpha \end{pmatrix} \ ,$$

and similarly, we write

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{k_1} \end{pmatrix}$$

to shorten the notations and to simplify the formulae.

The main formula underlying our constructions of Cartan connections is provided in the following theorem.

Theorem 1.1. *For an equinilpotentisable sub-Riemannian manifold $(M, \mathcal{D}, g)$ with a symmetry group H and an adapted Cartan connection ω, the generator $\frac{1}{2}\Delta$ of the stochastic process on M obtained as the stochastic development of the canonical sub-Riemannian diffusion on the nilpotent model N is given, in an adapted frame $(X_1, \ldots, X_n)$ and in terms of the Christoffel symbols Γ^α, by*

$$\Delta = \sum_{i=1}^{k_1} X_i^2 + \sum_{\alpha=1}^{\dim H} (\Gamma^\alpha)^T A_\alpha X \; . \tag{1.5}$$

In particular, if the symmetry group H is trivial then the generator is just the sum of squares operator

$$\Delta = \sum_{i=1}^{k_1} X_i^2 \; .$$

Some remarks are needed at this point. Comparing (1.2) and (1.5), we call

$$\sum_{\alpha=1}^{\dim H} (\Gamma^\alpha)^T A_\alpha X$$

the local divergence term of the operator Δ. While Δ and the stochastic development of the canonical sub-Riemannian diffusion on the nilpotent group N are coordinate invariant objects, the formula (1.5) and the local divergence are not coordinate invariant. However, these expressions and Theorem 1.1 are still very convenient tools for proving various results and for studying Δ as well as the stochastic development.

Moreover, it is important to emphasise that not every orthonormal frame in $\mathcal{D}$ can be extended to an adapted frame, see the Engel example in Section 4. For this reason, the condition that the frame $(X_1, \ldots, X_n)$ is adapted is crucial. Similarly, the adapted frame bundle is not the same as the orthonormal frame bundle. The former is a reduction of the latter to a smaller group of symmetries $H \subset SO(k_1)$ which agrees with the graded structure. When we talk about local trivialisations in the following we mean local trivialisations of the adapted frame bundle.

The key result of this article is stated in the next theorem.

Theorem 1.2. *Suppose $(M, \mathcal{D}, g)$ is an equinilpotentisable sub-Riemannian manifold with a symmetry group H. Then there exists a Cartan connection ω such that the stochastic process arising as the stochastic development of the canonical sub-Riemannian diffusion on the nilpotent model N has generator $\frac{1}{2}\Delta_{\mathcal{P}}$ if and only if every one-dimensional sub-representation of H on $\mathfrak{n}_{-1}$ corresponds to a divergence-free vector field in $\mathcal{D}$.*

The proof of Theorem 1.2 is constructive and it gives an explicit description for a possible choice of a Cartan connection. The result particularly implies that there exist suitable Cartan connections for many interesting examples including all equinilpotentisable contact structures and structures with the growth vector $(2, 3, 5)$. The latter arise when rolling distributions of surfaces, and it can be applied to modelling rolling of spherical robots on an unknown non-flat ground such as soil. It should also be mentioned that 3D contact structures are always equinilpotentisable.

The article has the following structure. In Section 2, we provide an overview of Cartan geometry, with a focus on Cartan geometry on a sub-Riemannian manifold in Section 2.1, and as an illustration we show how to understand the Levi–Civita connection on a Riemannian manifold as a Cartan connection. In Section 3, we discuss how the development with respect to a Cartan connection of a curve in the model space can be characterised by a system of ordinary differential equations, and we use this to introduce a notion of stochastic development for equinilpotentisable sub-Riemannian manifolds. We further prove Theorem 1.1 and we show that the Cartan geometry approach recovers the characterisation of Brownian motion on a Riemannian manifold via stochastic development. In Section 4, we establish Theorem 1.2, and we illustrate the construction of a suitable Cartan connection for manifolds modelled by free nilpotent structures with two generators, which include 3D contact structures. We further provide an example of a manifold where a stochastic development of the canonical sub-Riemannian diffusion on the model space never gives rise to the stochastic process with generator $\frac{1}{2}\Delta_{\mathcal{P}}$.

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2 Overview of Cartan geometry

We provide a general overview of Cartan geometry with a focus towards Cartan geometry on a sub-Riemannian manifold. In this section, we use the Einstein summation convention.

Before giving the general definitions, we start by interpreting Riemannian geometry as a Cartan geometry. Let us consider a Riemannian manifold (M, g) of dimension n together with a connection ∇ on the tangent bundle TM. In a local orthonormal frame $(X_1, \ldots, X_n)$ of TM, the connection ∇ is uniquely characterised by the Christoffel symbols Γ_{ij}^k, for $i, j, k \in \{1, \ldots, n\}$, which are given by, for $i, j \in \{1, \ldots, n\}$,

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k \ .$$

If $(\theta^1, \ldots, \theta^n)$ is a local coframe dual to $(X_1, \ldots, X_n)$, we can equivalently define a connection via one-forms θ_i^j, for $i, j \in \{1, \ldots, n\}$, which satisfy

$$\nabla_X X_i = \theta_i^j(X) X_j \ .$$

Comparing this with the previous definition of a connection in terms of Christoffel symbols, we find that, for all $i, j \in \{1, \ldots, n\}$,

$$\theta_i^j = \Gamma_{ki}^j \theta^k \ .$$

The Levi–Civita connection is the unique torsion-free connection on TM which is metric. For this connection, the metric compatibility condition implies that, for all $i, j, k \in \{1, \ldots, n\}$, we have

$$\Gamma_{ij}^k + \Gamma_{ik}^j = 0 \ ,$$

which in turn gives the antisymmetry condition that, for all $i, j \in \{1, \ldots, n\}$,

$$\theta_i^j + \theta_j^i = 0 \ .$$

To describe the torsion-free property of the Levi–Civita connection in terms of the dual frame $(\theta^1, \ldots, \theta^n)$, we construct the Lie-algebra-valued one-form $\theta_{\mathfrak{h}} \in \Omega(M, \mathfrak{so}(n))$ and the one-form $\theta_{\mathfrak{n}} \in \Omega(M, \mathbb{R}^n)$ defined by, for $i, j \in \{1, \ldots, n\}$,

$$(\theta_{\mathfrak{h}})_j^i = \theta_j^i \quad \text{and} \quad (\theta_{\mathfrak{n}})^i = \theta^i \ .$$

We call $\theta_{\mathfrak{h}}$ the Levi–Civita gauge and $\theta_{\mathfrak{n}}$ the soldering gauge. Given a vector space V, we define an exterior product between $\mathrm{End}(V)$-valued one-forms and V-valued one-forms by requiring that, for $i, j \in \{1, \ldots, n\}$ and all $A_i \in \mathrm{End}(V)$ as well as all $v_j \in V$, we have

$$\left(A_i \otimes \theta^i\right) \wedge \left(v_j \otimes \theta^j\right) = A_i(v_j) \theta^i \wedge \theta^j \ .$$

The Levi–Civita gauge $\theta_{\mathfrak{h}}$ and the soldering gauge $\theta_{\mathfrak{n}}$ characterise the torsion-freeness of a metric connection as follows, see e.g. Sharpe [30].

Proposition 2.1. *A metric connection ∇ is torsion-free if and only if for any orthonormal coframe $(\theta^1, \ldots, \theta^n)$ the structure equations*

$$\mathrm{d}\theta_{\mathfrak{n}} + \theta_{\mathfrak{h}} \wedge \theta_{\mathfrak{n}} = 0 \tag{2.1}$$

are satisfied, that is, for all $i, j \in \{1, \ldots, n\}$, we have

$$\mathrm{d}\theta^i + \theta^i_j \wedge \theta^j = 0 \; .$$

Since the structure equations have to hold for any frame, let us consider what happens in a different frame. Applying a rotation $h \in C^1(M, SO(n))$ to the frame with respect to which $\theta_{\mathfrak{h}}$ and $\theta_{\mathfrak{n}}$ are defined yields a frame in which $\theta_{\mathfrak{n}}^{\text{new}}$ and $\theta_{\mathfrak{h}}^{\text{new}}$ are given by

$$\begin{aligned} \theta_{\mathfrak{n}}^{\text{new}} &= h^{-1}\theta_{\mathfrak{n}} \; , \\ \theta_{\mathfrak{h}}^{\text{new}} &= h^{-1}\,\mathrm{d}h + h^{-1}\theta_{\mathfrak{h}} h \; . \end{aligned} \tag{2.2}$$

These expressions show that $\theta_{\mathfrak{h}}$ is a pull-back of a principle $SO(n)$-connection via a trivialising section of the orthonormal frame bundle $\mathcal{O}(M)$, that is, if $s\colon M \to \mathcal{O}(M)$ is the local section of $\mathcal{O}(M)$ defined by the local orthonormal frame $(X_1, \ldots, X_n)$ then there exists a principle $SO(n)$-connection with one-form $\omega_{\mathfrak{h}}$ such that

$$\theta_{\mathfrak{h}} = s^*\omega_{\mathfrak{h}} \; .$$

When changing the local section from s to sh, we obtain $\theta_{\mathfrak{h}}^{\text{new}}$ given by (2.2). Similarly, the soldering gauge $\theta_{\mathfrak{n}}$ is a pull-back of the canonical soldering form $\omega_{\mathfrak{n}}$, that is,

$$\theta_{\mathfrak{n}} = s^*\omega_{\mathfrak{n}} \; .$$

The canonical soldering form $\omega_{\mathfrak{n}}$ can be defined on $\mathcal{O}(M)$ in a totally invariant manner. Let $\pi\colon \mathcal{O}(M) \to M$ be the projection mapping. Then, for all $v \in T\mathcal{O}(M)$ and all $f \in \mathcal{O}(M)$, we set

$$(\omega_{\mathfrak{n}})_f\,(v) = f^{-1}\,\mathrm{d}\pi(v) \; , \tag{2.3}$$

where $f \in \mathcal{O}(M)$ is considered as a map $f\colon \mathbb{R}^n \to T_{\pi(f)}M$.

We combine $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{n}}$ into a single matrix-valued one-form ω on the frame bundle $\mathcal{O}(M)$ given by

$$\omega = \begin{pmatrix} \omega_{\mathfrak{h}} & \omega_{\mathfrak{n}} \\ 0 & 0 \end{pmatrix} \; .$$

This one-form is an example of a Cartan connection. It takes values in the Lie algebra $\mathfrak{se}(n)$ corresponding to the special Euclidean Lie group $SE(n)$. The curvature two-form Ω associated with a Lie-algebra-valued one-form ω is given by

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \; , \tag{2.4}$$

for $[\cdot, \cdot]$ the commutator of two Lie-algebra-valued one-forms, which is defined by

$$[\omega_1, \omega_2](X, Y) = [\omega_1(X), \omega_2(Y)] + [\omega_2(X), \omega_1(Y)] \; .$$

In particular, we see that

$$[\omega_1, \omega_2] = [\omega_2, \omega_1] \; , \tag{2.5}$$

and in a local trivialisation $(\theta^1, \ldots, \theta^n)$, we have

$$\left[A_i \otimes \theta^i, B_j \otimes \theta^j\right] = [A_i, B_j] \otimes \theta^i \wedge \theta^j \ . \tag{2.6}$$

The condition (2.1) is then equivalent to the vanishing of the $\mathbb{R}^n$-valued part of the curvature two-form Ω, which is exactly given by the torsion of a metric connection.

Applying this whole language to the Euclidean space $\mathbb{R}^n$, we find that the Cartan connection constructed above is simply the Maurer–Cartan one-form $\omega_{SE(n)}$ of the special Euclidean group, that is, for $g \in SE(n)$, we have

$$\left(\omega_{SE(n)}\right)_g = (L_{g^{-1}})_* \ , \tag{2.7}$$

which has zero curvature. Note that the Cartan geometry contains information both about the model space $\mathbb{R}^n$ and about its symmetry group $SE(n)$.

Keeping the Riemannian geometry example in mind, we give a general overview of Cartan geometries which are generalisations of Klein geometries. Every homogeneous space can be identified with a quotient G/H of a Lie group G by a subgroup H. The Maurer–Cartan form ω_G on G can be thought of as a connection on the symmetry group G of the homogeneous space G/H with values in the Lie algebra $\mathfrak{g}$ of G. It is defined exactly as in (2.7) with G instead of $SE(n)$. In Cartan geometry, we replace G/H by a H-principle bundle P over a manifold M and the Maurer–Cartan form ω_G by a $\mathfrak{g}$-valued one-form on P. We adopt the convention to denote the Lie algebra associated with a Lie group by the corresponding Gothic letter.

Definition 2.2. *Given a smooth manifold M, a Lie group G and a subgroup $H \subset G$, a Cartan geometry (P, ω) on M modelled on $(\mathfrak{g}, \mathfrak{h})$ consists of the following data.*

1. *A right principle H-bundle $\pi\colon P \to M$.*
2. *A $\mathfrak{g}$-valued one-form ω on P, called a Cartan connection, which satisfies*
   (a) *$\omega_p\colon T_pP \to \mathfrak{g}$ is an isomorphism for all $p \in P$,*
   (b) *$R_h^*\omega = \mathrm{Ad}_{h^{-1}}\,\omega$ for all $h \in H$, and*
   (c) *$\omega(X^*) = X$ for all $X^* \in \Gamma(TP)$ and $X \in \mathfrak{h}$ which are related by, for all $f \in C^\infty(P)$ and all $p \in P$,*
   $$\left(X^* f\right)(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(p \exp(tX)\right).$$

The homogeneous space G/H is called the model space for the corresponding Cartan geometry.

In the case of a Riemannian manifold M of dimension n, a Cartan geometry is modelled over $(\mathfrak{se}(n), \mathfrak{so}(n))$, the model space is given by the corresponding quotient $\mathbb{R}^n \simeq SE(n)/SO(n)$, and P is the orthonormal frame bundle $\mathcal{O}(M)$ viewed as a $SO(n)$-principle bundle. The connection ω constructed above then indeed satisfies the properties of a Cartan connection, see Sharpe [30]. Note that the Levi–Civita connection is just a particular instance of a Cartan connection which is characterised by the vanishing of the torsion part of the curvature two-form.

The usefulness of Cartan geometries for our work arises from the property that they possess a good notion of development of curves. Let us fix $q \in M$. A curve $\gamma_{G/H}\colon [0,1] \to G/H$ on the model space G/H is developed via a Cartan connection ω to a curve $\gamma_M\colon [0,1] \to M$ with $\gamma_M(0) = q$ on the manifold M as follows.

1. The curve $\gamma_{G/H}$ is lifted to a curve $\gamma_G\colon [0,1] \to G$.
2. Fixing some $p \in P$ such that $\pi(p) = q$, we define the development $\gamma_P\colon [0,1] \to P$ of the lift γ_G by requiring
   $$\gamma_P^*\omega = \gamma_G^*\omega_G \tag{2.8}$$
   subject to $\gamma_P(0) = p$.

3. The development γ_M of $\gamma_{G/H}$ is given by $\gamma_M = \pi(\gamma_P)$.

The relation (2.8) defines the development γ_P of the lift γ_G uniquely once an initial point for γ_P is specified. Moreover, according to Sharpe [30, Proposition 5.4.13] we have the following property.

Theorem 2.3. *In a Cartan geometry (P, ω) on a manifold M modelled on $(\mathfrak{g}, \mathfrak{h})$, the development γ_M of a curve $\gamma_{G/H}$ depends neither on the choice of a lift γ_G nor on the choice of a lift $p = \gamma_P(0)$ of $q = \gamma_M(0)$.*

The same scheme works in the opposite way, where the relation (2.8) is used to define an anti-development on the model space G/H of a curve $\gamma_M\colon [0,1] \to M$ on the manifold.

2.1 Cartan geometry on a sub-Riemannian manifold

As discussed in the Introduction, a local model for an equinilpotentisable sub-Riemannian manifold $(M, \mathcal{D}, g)$ is the nilpotent Carnot group N with the Lie algebra $\mathfrak{n} = \mathrm{gr}(T_qM)$. It inherits the natural grading

$$\mathfrak{n} = \mathfrak{n}_{-1} \oplus \cdots \oplus \mathfrak{n}_{-m}$$

and a scalar product g_{-1} on $\mathfrak{n}_{-1}$. Thus, N is itself a sub-Riemannian manifold. Throughout, we denote the dimension of the manifold M by n and the constant rank of the distribution $\mathcal{D}$ by k_1.

In order to define a Cartan geometry on the sub-Riemannian manifold $(M, \mathcal{D}, g)$, we need to consider the Lie algebra

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \ ,$$

where $\mathfrak{h}$ is the Lie algebra associated with the Lie group H of all automorphisms of $\mathfrak{n}$ which preserve g_{-1}, that is,

$$\begin{aligned} \mathfrak{h} = \{\varphi\colon \mathfrak{n} \to \mathfrak{n} \text{ such that } & g_{-1}(\varphi(\cdot), \cdot) + g_{-1}(\cdot, \varphi(\cdot)) = 0 \\ & \text{and } \varphi([X,Y]) = [\varphi(X), Y] + [X, \varphi(Y)] \text{ for all } X, Y \in \mathfrak{n}\} \ . \end{aligned} \tag{2.9}$$

In particular, the Lie algebra $\mathfrak{h}$ is isomorphic to a sub-algebra of $\mathfrak{so}(\mathfrak{n}_{-1})$. Equivalently, we could have asked H to preserve the metric on all of $\mathfrak{n}$. Indeed, since the metric on $\mathfrak{n}$ is constructed by identifying each $\mathfrak{n}_{-l}$ with a subspace of l tensor products of $\mathfrak{n}_{-1}$, the metric on $\mathfrak{n}_{-l}$ is simply given by the induced metric from the tensor product, and as H preserves the structure of the brackets, $\mathfrak{n}_{-l}$ corresponds to an orthogonal sub-representation of H in $\otimes_{i=1}^{l} \mathfrak{n}_{-1}$.

We emphasise that the Lie algebra $\mathfrak{g}$ is a graded space with elements of $\mathfrak{h}$ having degree zero and elements of $\mathfrak{n}_{-i}$ having degree $-i$. Similarly, we define the degree of elements of the dual spaces $\mathfrak{n}_{-i}^*$ to be i. This endows any tensor product of those spaces with a grading. For example, we use later that all elements from $\mathfrak{n}_{-i} \otimes \mathfrak{n}_{-j}^* \wedge \mathfrak{n}_{-k}^*$ have degree $j + k - i$. In order to make calculations consistent, we further treat the zero element as an element which can take any degree. We denote by $+$ in the subscript the subspace spanned by the elements of positive degree.

The second ingredient needed to define a Cartan geometry is a right principle bundle P. Similar with Riemannian geometry, P is a bundle of graded frames which is formed by all Lie algebra morphisms

$$f\colon \mathfrak{n} \to \mathrm{gr}(TM)$$

compatible with the metric. This means that for any orthonormal basis $\{e_1, \ldots, e_{k_1}\}$ of $\mathfrak{n}_{-1}$ the elements $X_i = f(e_i)$, for $i \in \{1, \ldots, k_1\}$, should form an orthonormal basis of $\mathcal{D}$. The bundle P has H as its structure group, and it is a reduction of the orthonormal frame bundle $\mathcal{O}(\mathcal{D})$ to the group H. In what follows, we use the notation $\mathcal{O}_H(\mathcal{D})$ for this bundle P.

The graded frame bundle possess the canonical soldering form $\omega_{\mathfrak{n}}$ which is defined as follows. For all $v \in T\mathcal{O}_H(\mathcal{D})$ and all $f \in \mathcal{O}_H(\mathcal{D})$, we set

$$(\omega_{\mathfrak{n}})_f(v) = f^{-1}\,\mathrm{gr}(\mathrm{d}\pi(v)) ,$$

where $\mathrm{gr}\colon TM \to \mathrm{gr}(TM)$ is defined by (1.3).

Even though $\mathrm{gr}(T_{\pi(f)}M)$ is isomorphic to $T_{\pi(f)}M$, it is not a canonical isomorphism. Indeed, any element of an adapted frame is defined only modulo terms of higher degree. This means that, for any $q \in M$, we have to choose an additional isomorphism $T_qM \to \mathrm{gr}(T_qM)$ which would allow us to decompose a vector field $X \in \Gamma(TM)$ according to the filtration

$$\mathrm{gr}(X) = X_{-1} \oplus \cdots \oplus X_{-m} .$$

Any choice of a soldering gauge provides us with a needed automorphism. The canonical soldering form gives us only the degree one components of the needed isomorphism.

Remark 2.4. In the study of filtered manifolds, one usually has to apply a procedure known as the Tanaka prolongation, which consists of adding higher derivations of $\mathfrak{g}$. However, the fact that we study the metric geometry on equinilpotentisable sub-Riemannian manifolds forces the Tanaka prolongation to be trivial, see [1, Section 2]. Hence, all information we need is already contained in $\mathfrak{g}$. □

For a generic sub-Riemannian manifold, a torsion-free connection does not exist. However, we describe below how to construct linear conditions on the curvature function which guarantees the existence of a unique Cartan connection for a given pair $(\mathfrak{n}, g_{-1})$.

Definition 2.5. *The curvature function* $\kappa\colon P \to \mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})$ *of a Cartan connection* ω *is defined as*

$$\kappa(p)(\cdot,\cdot) = \Omega_p\left(\omega_p^{-1}(\cdot), \omega_p^{-1}(\cdot)\right) .$$

We recall that the Lie algebra differential $\partial\alpha$ of $\alpha \in \mathrm{Hom}(\wedge^k\mathfrak{n}, \mathfrak{g})$ is defined, for any vectors $X_0, \ldots, X_k \in \mathfrak{n}$, by

$$\begin{aligned}\partial\alpha(X_0, X_1, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i \cdot \alpha\left(X_0, \ldots, \hat{X}_i, \ldots, X_k\right) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j} \alpha\left([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k\right) ,\end{aligned}$$

where hat means the omission of the corresponding vector and where $X_i\,\cdot$ denotes the adjoint action ad of X_i. For a basis $\{e_1, \ldots, e_{\dim\mathfrak{g}}\}$ of $\mathfrak{g}$ and the corresponding dual basis $\{e^1, \ldots, e^{\dim\mathfrak{g}}\}$, the Lie algebra differential ∂ satisfies

$$\partial\left(e_i \otimes e^j\right) = \partial e_i \wedge e^j + e_i \otimes \partial e^j ,$$

where $\partial e_i = -\operatorname{ad} e_i$ for $e_i \in \mathfrak{n}$.

The construction of Cartan connections on sub-Riemannian manifolds satisfying a type of torsion-free-like condition relies on finding suitable normal modules of $\mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})_+$. This is also the heart of Section 4.

Definition 2.6. *A subspace* $\mathcal{N} \subset \mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})_+$ *is called a normal module if*

1. $\mathcal{N}$ *is a* H*-module with respect to the adjoint action of* H *on* $\mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})$*, and*

2. *we have* $\mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})_+ = \mathcal{N} \oplus \mathrm{im}\,\partial(\mathrm{Hom}(\mathfrak{n}, \mathfrak{g})_+)$.

To relate Cartan connections and normal modules, we further need the notion of an adapted Cartan connection.

Definition 2.7. *A Cartan connection ω on a sub-Riemannian manifold $(M, \mathcal{D}, g)$ is called adapted if for any arbitrary section $s\colon M \to \mathcal{O}_H(\mathcal{D})$ the $\mathfrak{n}$-valued part of the Cartan gauge $s^*\omega$ forms a coframe dual to an adapted frame.*

We then have the following theorem, see Morimoto [25, Theorem 3.10.1].

Theorem 2.8. *Given an equinilpotentisable sub-Riemannian manifold $(M, \mathcal{D}, g)$ and a normal module $\mathcal{N} \subset \mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})_+$ there exists a unique Cartan geometry $(\mathcal{O}_H(\mathcal{D}), \omega)$ on M modelled on $(\mathfrak{g}, \mathfrak{h})$ such that the Cartan connection ω is adapted and the corresponding curvature function κ takes values in $\mathcal{N}$.*

The problem that we face is how to choose such a normal module $\mathcal{N}$. One of the possibilities is according to the following construction due to Morimoto [25], see Grong [17] for an alternative description. In the Introduction, we discuss how the induced metric g_{-1} on $\mathfrak{n}_{-1}$ defines a metric on all of $\mathfrak{n}$. We extend it to $\mathfrak{g}$ by assuming that $\mathfrak{h}$ is endowed with a bi-invariant metric orthogonal to $\mathfrak{n}$, which further gives rise to a metric on any tensor product of $\mathfrak{g}$ and its subspaces. Thus, we can also define the duals of linear operators acting on these products. In particular, we can define the adjoint map ∂^*, and by the usual linear algebra arguments $\ker \partial^*$ is the orthogonal complement to $\operatorname{im} \partial$. Hence, the subspace $\mathcal{N} = \ker \partial^*$ gives a natural choice for a H-module. However, this module does not always give rise to the sub-Riemannian diffusion associated with the sub-Laplacian defined with respect to the Popp volume.

In Section 4, we construct Cartan connections yielding the desired sub-Riemannian diffusion by choosing the normal module $\mathcal{N}$ to be orthogonal to a different module $\mathcal{S}$, which depends on the structure of the Lie algebra $\mathfrak{n}$.

We end this section by discussing some natural associated bundles related to equiregular sub-Riemannian manifolds. They are used in Section 4 to give invariant necessary and sufficient conditions for the existence of Cartan connections which develop the canonical sub-Riemannian diffusion process on the nilpotent group N to the stochastic process with generator $\frac{1}{2}\Delta_{\mathcal{P}}$ on M.

Given a right principle H-bundle $\pi\colon P \to M$ and a representation $\rho\colon H \to V$ for some vector space V, we can construct the associated bundle $P \times_H V$. The following proposition, see Sharpe [30, Section 1.3], is needed in the analysis in Section 4.

Proposition 2.9. *There exists a bijection between sections of $P \times_H V$ and the space of all equivariant functions, that is, the space*

$$\left\{ f\colon P \to V \text{ such that } f \in C^\infty(P, V) \text{ and } f(ph) = \rho\left(h^{-1}\right) f(p) \text{ for all } p \in P, h \in H \right\} .$$

For example, let us consider the curvature function κ. From the equivariant properties of the Cartan connection ω, it follows that, see [30, Lemma 5.3.23],

$$\kappa(ph)(v_1, v_2) = \mathrm{Ad}\left(h^{-1}\right)\left(\kappa(p)(\mathrm{Ad}(h)v_1, \mathrm{Ad}(h)v_2)\right) \text{ for all } v_1, v_2 \in \mathfrak{n} , \tag{2.10}$$

which implies that the function $\kappa\colon P \to \mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})$ is actually a section of the associated bundle $P \times_H \mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})$. Another example is given by the tangent bundle TM, which according to [30, Theorem 5.3.15] is isomorphic to $P \times_H \mathfrak{n}$. Similarly, as mentioned after the definition of $\mathfrak{h}$, each $\mathfrak{n}_{-l}$ is an orthogonal representation of H, and therefore, for $l \in \{2, \dots, m\}$, we can consider the associated bundle $P \times_H (\mathfrak{n}_{-1} \oplus \cdots \oplus \mathfrak{n}_{-l})$, which is isomorphic to $\mathcal{D}^{-l}$.

3 Stochastic development in sub-Riemannian geometry

The goal of this section is to introduce a notion of stochastic development on sub-Riemannian manifolds, and to determine the generator of the stochastic process obtained as the stochastic development of a canonical sub-Riemannian diffusion on the associated nilpotent model. To motivate our definition, we start by discussing how horizontal differentiable curves on the model space are developed.

We consider an arbitrary Cartan geometry $(\mathcal{O}_H(\mathcal{D}), \omega)$ with an adapted Cartan connection ω on an equinilpotentisable sub-Riemannian manifold $(M, \mathcal{D}, g)$. Let $(X_1, \ldots, X_n)$ be an adapted frame on M and let $e_i = \operatorname{gr}(X_i)$, for $i \in \{1, \ldots, n\}$, be the elements of the corresponding adapted basis of the Lie algebra $\mathfrak{n}$. We denote by $\{e^1, \ldots, e^n\}$ the dual basis of $\{e_1, \ldots, e_n\}$. For $h \in H$, we write ρ_h for the corresponding action of H on, depending on the context, all of TM or all of T^*M, and h for the action on subspaces isomorphic to $\mathbb{R}^k$, for example, on $\mathcal{D}^{-1}$ or $\mathfrak{n}_{-1}$. The Maurer–Cartan form ω_G on G is given by

$$\omega_G = \sum_{i=1}^{\dim G} e_i \otimes e^i \; .$$

Suppose we are given a model horizontal curve $\gamma_N \colon [0,1] \to N$ which we wish to develop to our sub-Riemannian manifold. Then it must satisfy

$$\dot{\gamma}_N = \sum_{i=1}^{k_1} u_i e_i(\gamma_N)$$

for some smooth functions $u_i \colon [0,1] \to \mathbb{R}$. The assumption that γ_N is horizontal means that $\gamma_N^* e^i = 0$ for all $e^i \notin \mathfrak{n}_{-1}^*$. Since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, any lift γ_G of γ_N to the Lie group G is uniquely defined by its projections γ_N and γ_H to N and H, respectively, and we write $\gamma_G = (\gamma_H, \gamma_N)$. Let us choose $\gamma_H \equiv \mathrm{id}$. This choice, by Theorem 2.3, does not affect the development of γ_N, but it greatly simplifies the subsequent computations. We obtain that

$$\gamma_G^* \omega_G = \sum_{i=1}^{k_1} e_i \otimes \left(\gamma_N^* e^i\right) = \begin{pmatrix} u_1 \\ \vdots \\ u_{k_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathrm{d}t \; . \tag{3.1}$$

Note that the $\mathfrak{h}$-valued part of the Maurer–Cartan form ω_G is zero. Following the scheme of the development of curves discussed in Section 2, we now want to compute $\gamma_{\mathcal{O}_H(\mathcal{D})}^* \omega$ of a Cartan connection ω for any curve $\gamma_{\mathcal{O}_H(\mathcal{D})} \colon [0,1] \to \mathcal{O}_H(\mathcal{D})$ in the adapted orthonormal frame bundle. Performing the computation in a local trivialisation of the adapted frame bundle, we can write $\gamma_{\mathcal{O}_H(\mathcal{D})} = (h, \gamma_M)$. For $i \in \{1, \ldots, n\}$ fixed, let $j \in \{1, \ldots, m\}$ be such that $e_i \in \mathfrak{n}_{-j}$. Then the bundle map $f \colon \mathfrak{n} \to \operatorname{gr}(T_{\pi(f)} M)$ is given by

$$f(e_i) = X_i + \sum_{l=1}^{k_1 + \cdots + k_{j-1}} b_i^l X_l \; ,$$

where the b_i^l are smooth functions on M. Hence, we can identify the bundle map f with a block lower-triangular matrix all of whose diagonal blocks are identity matrices. We denote the transpose inverse of this matrix by F.

As before, let $(\theta^1, \dots, \theta^n)$ be the dual frame to $(X_1, \dots, X_n)$, and consider the $\mathfrak{n}$-valued vector form

$$\theta = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^n \end{pmatrix} .$$

The soldering gauge $\theta_{\mathfrak{n}}$ is then written as

$$\theta_{\mathfrak{n}} = F\theta \ . \tag{3.2}$$

We start by analysing the $\mathfrak{n}$-part $\omega_{\mathfrak{n}}$ of the Cartan connection ω. Due to (3.1) and the defining relation (2.8) of the development, we have the following two equalities

$$\gamma^*_{\mathcal{O}_H(\mathcal{D})}\omega_{\mathfrak{n}} = \begin{pmatrix} u_1 \\ \vdots \\ u_{k_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathrm{d}t \ , \tag{3.3}$$

$$\gamma^*_{\mathcal{O}_H(\mathcal{D})}\omega_{\mathfrak{h}} = 0 \tag{3.4}$$

for the pull-backs of the $\mathfrak{n}$-valued part $\omega_{\mathfrak{n}}$ and of the $\mathfrak{h}$-valued part $\omega_{\mathfrak{h}}$, respectively, of the considered Cartan connection ω.

Let us first derive an explicit description for γ_M from the relation (3.3). We define functions $a_i \in C^\infty([0,1])$ as $\gamma_M^*\theta^i = a_i\,\mathrm{d}t$ for each $i \in \{1, \dots, n\}$. By explicitly writing down the left hand side of (3.3), we see that

$$\gamma^*_{\mathcal{O}_H(\mathcal{D})}\omega_{\mathfrak{n}} = \rho_h^{-1}\gamma_M^*\theta_{\mathfrak{n}} = \rho_h^{-1}\left(\gamma_M^*F\right)\gamma_M^*\theta = \rho_h^{-1}\left(\gamma_M^*F\right)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mathrm{d}t \ .$$

Comparing this with (3.3) yields

$$\rho_h^{-1}\left(\gamma_M^*F\right)\begin{pmatrix} a_1 \\ \vdots \\ a_{k_1} \\ a_{k_1+1} \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_{k_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{3.5}$$

It follows that we can solve the system (3.5) for the functions a_i for $i \in \{1, \dots, n\}$. Indeed, each $\mathfrak{n}_{-j}$ is an orthogonal sub-representation of H. Thus, each matrix ρ_h^{-1} is block-diagonal. The matrix F is block upper-triangular unipotent. Hence, we can solve the above system of equations block by block starting from the lowest rows, which give us $a_{k_m+1} = \cdots = a_n = 0$. Continuing iteratively, we find that $a_i = 0$ for all $i \in \{k_1+1, \dots, n\}$. Since by construction the first $k_1 \times k_1$ minor of F is the identity matrix, we finally obtain

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_{k_1} \end{pmatrix} = h \begin{pmatrix} u_1 \\ \vdots \\ u_{k_1} \end{pmatrix} = hu \ , \tag{3.6}$$

and, due to $\gamma_M^*\theta^i = a_i\,\mathrm{d}t$, the curve γ_M satisfies

$$\dot{\gamma}_M = \sum_{i=1}^{k_1} a_i X_i(\gamma_M)\ .$$

It remains to determine h from the $\mathfrak{h}$-part $\omega_{\mathfrak{h}}$ of the connection ω. Let $\theta_{\mathfrak{h}}$ be a Cartan gauge of $\omega_{\mathfrak{h}}$. If we take a basis $\{A_\alpha\colon 1 \le \alpha \le \dim H\}$ of $\mathfrak{h}$ then $\theta_{\mathfrak{h}}$ can be written as

$$\theta_{\mathfrak{h}} = \sum_{\alpha=1}^{\dim H} A_\alpha \left((\tilde{\Gamma}^\alpha)^T \theta_{\mathfrak{n}}\right), \tag{3.7}$$

where $\tilde{\Gamma}^\alpha\colon C^\infty(M) \to \mathbb{R}^n$ are called Christoffel symbols. Since $\gamma_H \equiv \mathrm{id}$ is constant by assumption, we deduce from (3.4) that

$$h^{-1}\dot{h} + h^{-1}\left(\gamma_M^*\theta_{\mathfrak{h}}\right) h = 0\ ,$$

which, using the change of variables $\tilde{h} = h^{-1}$, simplifies to

$$\dot{\tilde{h}} = \tilde{h}\left(\gamma_M^*\theta_{\mathfrak{h}}\right).$$

Let Γ^α denote the reduced vector

$$\Gamma^\alpha = \begin{pmatrix} \Gamma_1^\alpha \\ \vdots \\ \Gamma_{k_1}^\alpha \end{pmatrix}\ ,$$

and similarly we define the differential operator X on M with values in $\mathcal{D}$ given by

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{k_1} \end{pmatrix}.$$

Since $a_i = 0$ for all $i \in \{k_1+1, \ldots, n\}$, that is,

$$\gamma_M^*\theta_{\mathfrak{n}} = \begin{pmatrix} a_1 \\ \vdots \\ a_{k_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}\ ,$$

the expression (3.7) simplifies to

$$\gamma_M^*\theta_{\mathfrak{h}} = \sum_{\alpha=1}^{\dim H} A_\alpha \left((\tilde{\Gamma}^\alpha)^T(\gamma_M)\left(\gamma_M^*\theta_{\mathfrak{n}}\right)\right) = \sum_{\alpha=1}^{\dim H} A_\alpha \left(a^T \Gamma^\alpha(\gamma_M)\right)\,\mathrm{d}t\ .$$

Let $\{Y_\alpha\colon 1 \le \alpha \le \dim H\}$ be the family of left-invariant vector fields on H corresponding to the basis $\{A_\alpha\colon 1 \le \alpha \le \dim H\}$. In particular, we have $Y_\alpha(\tilde{h}) = \tilde{h}A_\alpha$. Using (3.6) and the fact that $H \subset SO(k_1)$, which gives $a^T = u^T\tilde{h}$, the proof of the proposition stated below follows.

Proposition 3.1. *Let $(M, \mathcal{D}, g)$ be an equinilpotentisable sub-Riemannian manifold with a symmetry group H and a model space $N = G/H$. Let $(\mathcal{O}_H(\mathcal{D}), \omega)$ be a Cartan geometry on M modelled on $(\mathfrak{g}, \mathfrak{h})$. Choose an adapted frame $(X_1, \dots, X_n)$ of TM and let $e_i = \operatorname{gr}(X_i)$ be the corresponding basis of $\mathfrak{n}_{-1}$. Then any development $\gamma_M \colon [0,1] \to M$ of a horizontal curve $\gamma_N \colon [0,1] \to N$ is a projection of a curve $\gamma_{\mathcal{O}_H(\mathcal{D})} \colon [0,1] \to \mathcal{O}_H(\mathcal{D})$ which in the chosen basis satisfies the following system of differential equations, written in a local trivialisation of the bundle $\mathcal{O}_H(\mathcal{D})$,*

$$\begin{aligned} \dot{\gamma}_M &= u^T \tilde{h} X(\gamma_M) , \\ \dot{\tilde{h}} &= \sum_{\alpha=1}^{\dim H} \left(u^T \tilde{h} \Gamma^\alpha(\gamma_M) \right) Y_\alpha(\tilde{h}) , \end{aligned} \tag{3.8}$$

where u_i are defined by $\gamma_N^* e^i = u_i \, \mathrm{d}t$, for $i \in \{1, \dots, k_1\}$ and $\{e^1, \dots, e^{k_1}\}$ the dual basis of $\{e_1, \dots, e_{k_1}\}$.

Remark 3.2. We recall that Theorem 2.3 says that the development γ_M only depends on a choice of initial point $\gamma_M(0) \in M$. In particular, once $\gamma_M(0)$ has been chosen, the solution of the smooth system (3.8) of ordinary differential equations always projects to the same curve on M irrespective of the initial condition for h. □

This motivates the definition of stochastic development we provide below for which we need one last ingredient taking the place of the horizontal curve γ_N we develop in the deterministic setting. Any semimartingale $(w_t)_{t \geq 0}$ on $\mathbb{R}^{k_1}$ lifts uniquely to a semimartingale $(\tilde{w}_t)_{t \geq 0}$ on the Carnot group N. In particular, the lift of Brownian motion $(b_t)_{t \geq 0}$ on $\mathbb{R}^{k_1}$ is the stochastic process $(\tilde{b}_t)_{t \geq 0}$ on N whose generator $\frac{1}{2}\Delta$ is given by

$$\Delta = \sum_{i=1}^{k_1} V_i^2 ,$$

where V_i is the left-invariant vector field on N corresponding to e_i for $i \in \{1, \dots, k_1\}$. As the operator Δ on the nilpotent Lie group N is the sub-Laplacian with respect to the Popp volume, see Vigneron [31], and since in this setting the Popp volume further coincides with the right Haar measure, the left Haar measure and the Lebesgue measure, the lift $(\tilde{b}_t)_{t \geq 0}$ on N can be considered as a canonical sub-Riemannian diffusion on N.

By formally replacing time derivatives with Stratonovich differentials and the control u by the differential of the driving stochastic process, we obtain the following definition.

Definition 3.3. *Let $(\tilde{w}_t)_{t \geq 0}$ be a semimartingale on N which is the lift of $(w_t)_{t \geq 0}$ on $\mathbb{R}^{k_1}$. Then the stochastic development of $(\tilde{w}_t)_{t \geq 0}$ is the stochastic process on the manifold M which arises as the projection to M of the unique solution to the system of Stratonovich stochastic differential equations, written in a local trivialisation,*

$$\begin{aligned} \partial \gamma_M &= \partial w^T \tilde{h} X(\gamma_M) , \\ \partial \tilde{h} &= \sum_{\alpha=1}^{\dim H} \left(\partial w^T \tilde{h} \Gamma^\alpha(\gamma_M) \right) Y_\alpha(\tilde{h}) , \end{aligned}$$

subject to a choice of initial condition.

Some remarks from the stochastic analysis side are needed at this point. The definition above does not only use the notion of stochastic differential equations, see e.g. Øksendal [27] and Rogers, Williams [28, 29], but also relies on the extension of Stratonovich differentials to manifolds, see Norris [26]. To avoid delving too deeply into the theory of stochastic calculus, we simply provide a brief overview. A semimartingale is the right kind of stochastic process needed when wanting to consider a stochastic differential of such a process. While the Stratonovich differential is not the only stochastic differential available in stochastic calculus, it is the one which, unlike the Itô differential, is invariant under coordinate transformations and is therefore more suited to differential geometry. Moreover, as discussed in [18, 26], it is also important to note that Stratonovich differentials are only symbolic and need to be understood as part of an integral equation. As a result of this and due to the rotational invariance of Brownian motion, even though Definition 3.3 is stated in a local trivialisation, the stochastic development of the canonical sub-Riemannian diffusion $(\tilde{b}_t)_{t\geq 0}$ does not depend on this choice.

While we are mainly interested in the geometric and algebraic picture, we need the relation from stochastic analysis that, for sufficiently nice vector fields $Z_1, \dots, Z_k$ on $\mathbb{R}^N$ and Brownian motion $(b_t)_{t\geq 0}$ on $\mathbb{R}^k$, the unique solution $(z_t)_{t\geq 0}$ in $\mathbb{R}^N$ to the Stratonovich stochastic differential equation

$$\partial z_t = \sum_{i=1}^{k} Z_i(z_t)\,\partial b_t^i$$

is the stochastic process whose generator is the sum of squares operator $\frac{1}{2}\sum_{i=1}^{k} Z_i^2$.

For completeness, we remark that in deriving the formula (1.5) given in Theorem 1.1, we extensively use the fact that the symmetry group H is a subgroup of the orthogonal group. Without this feature the resulting stochastic process would not project well to the base manifold. For example, in the Lorentzian setting the stochastic process is indeed studied as a stochastic process on the pseudo-orthonormal frame bundle, see Franchi and Le Jan [14].

We are now ready to prove Theorem 1.1 stated in the Introduction.

Proof of Theorem 1.1. We rewrite the system of Stratonovich stochastic differential equations from Definition 3.3 for the development of the canonical sub-Riemannian diffusion $(\tilde{b}_t)_{t\geq 0}$ as

$$\partial \gamma_{\mathcal{O}_H(\mathcal{D})} = \partial b^T Z\left(\gamma_{\mathcal{O}_H(\mathcal{D})}\right)\ ,$$

where in a local trivialisation we write $\gamma_{\mathcal{O}_H(\mathcal{D})} = (h, \gamma_M)$ and, with $X_1, \dots, X_{k_1}$ and the Y_α for $1 \leq \alpha \leq \dim H$ understood as vector fields on $\mathcal{O}_H(\mathcal{D})$,

$$Z = \tilde{h}X + \sum_{\alpha=1}^{\dim H}\left(\tilde{h}\Gamma^\alpha\right) Y_\alpha\ .$$

The generator of the stochastic process on $\mathcal{O}_H(\mathcal{D})$ is then given by a sum of squares operator, which in our notation can be compactly written as

$$\frac{1}{2}\Delta_{\mathcal{O}_H(\mathcal{D})} = \frac{1}{2}Z^T Z\ .$$

Thus, the generator $\frac{1}{2}\Delta$ of the stochastic development of the canonical sub-Riemannian diffusion is given, for a function $f \in C^\infty(M)$, by

$$\begin{aligned}\Delta(f) = \Delta_{\mathcal{O}_H(\mathcal{D})}(\pi^* f) = Z^T Z(\pi^* f) &= X^T \tilde{h}^T \tilde{h} X(f) + \sum_{\alpha=1}^{\dim H} (\Gamma^\alpha)^T \tilde{h}^T Y_\alpha(\tilde{h}) X(f) \\ &= X^T X(f) + \sum_{\alpha=1}^{\dim H} (\Gamma^\alpha)^T A_\alpha X(f)\ ,\end{aligned}$$

where we used $Y_\alpha(\tilde{h}) = \tilde{h}A_\alpha$ and the orthogonality of $\tilde{h}$. $\square$

As a straightforward consequence of Theorem 1.1, we recover the Riemannian case. For a Riemannian manifold of dimension n, we have $N = \mathbb{R}^n$ and $H = SO(n)$. In particular, the basis elements of $\mathfrak{so}(\mathfrak{n})$ are the skew-symmetric matrices $A_{ij} = E_{ij} - E_{ji}$, for $i, j \in \{1, \ldots, n\}$, where E_{ij} is the matrix whose only non-vanishing element is the $(i,j)^{\text{th}}$ entry, which is equal to one. We denote the corresponding vectors of Christoffel symbols by

$$\Gamma^i_j = \begin{pmatrix} \Gamma^i_{1j} \\ \vdots \\ \Gamma^i_{nj} \end{pmatrix} .$$

We compute

$$\begin{aligned}
\Delta &= \sum_{i=1}^n X_i^2 + \sum_{1\le j<k\le n} \left(\Gamma^j_k\right)^T (E_{jk} - E_{kj})\, X = \\
&= \sum_{i=1}^n X_i^2 + \sum_{1\le j<k\le n} \left(\Gamma^j_{jk} X_k - \Gamma^j_{kk} X_j\right) = \sum_{i=1}^n X_i^2 - \sum_{1\le j<k\le n} \left(\Gamma^k_{jj} X_k + \Gamma^j_{kk} X_j\right) = \\
&= \sum_{i=1}^n X_i^2 - \sum_{1\le j<k\le n} \Gamma^k_{jj} X_k - \sum_{1\le k<j\le n} \Gamma^k_{jj} X_k = \sum_{i=1}^n X_i^2 - \sum_{j\ne k} \Gamma^k_{jj} X_k = \\
&= \sum_{i=1}^n X_i^2 - \sum_{j,k=1}^n \Gamma^k_{jj} X_k \ ,
\end{aligned}$$

where we used $\Gamma^i_{jk} = -\Gamma^k_{ji}$ in the second row, which is a consequence of the skew-symmetry of the matrices A_{ij}. On the other hand, we similarly have

$$\operatorname{div}(X_i) = \sum_{j=1}^n g(\nabla_{X_j} X_i, X_j) = \sum_{j=1}^n \Gamma^j_{ji} = -\sum_{j=1}^n \Gamma^i_{jj} \ ,$$

which yields

$$\Delta = \sum_{i=1}^n X_i^2 + \sum_{i=1}^n \operatorname{div}(X_i) X_i = \sum_{i=1}^n \left(X_i^2 - \sum_{j=1}^n \Gamma^i_{jj} X_i \right) \ ,$$

agreeing with the expression obtained above with the Cartan geometry approach. This formula holds for any metric connection. If we now use the Levi–Civita connection, whose Christoffel symbols can be computed in terms of the structure constants of an orthonormal frame as

$$\Gamma^i_{jk} = \frac{1}{2}\left(c^k_{ij} - c^i_{jk} + c^j_{ki}\right) \ ,$$

we exactly recover (1.4).

4 Cartan connections for sub-Riemannian diffusions

We prove Theorem 1.2 by explicitly constructing a Cartan connection for which the stochastic development of the canonical sub-Riemannian diffusion on the nilpotent model has generator $\frac{1}{2}\Delta_{\mathcal{P}}$ for $\Delta_{\mathcal{P}}$ the sub-Laplacian defined with respect to the Popp volume $\mathcal{P}$. We further illustrate the construction for manifolds modelled by free nilpotent structures with two generators.

For a given Cartan geometry $(\mathcal{O}_H(\mathcal{D}), \omega)$ with an adapted Cartan connection ω on an equinilpotentisable sub-Riemannian manifold $(M, \mathcal{D}, g)$, let Δ be twice the generator of the stochastic development using the Cartan connection ω of the canonical sub-Riemannian diffusion on the nilpotent model N. The first goal is to give an invariant description of $\Delta - \Delta_{\mathcal{P}}$. Since the second order partial differential operators Δ and $\Delta_{\mathcal{P}}$ have the same principal symbol, the difference $\Delta - \Delta_{\mathcal{P}}$ can be understood as a vector field, which, as we see below, is in fact a horizontal vector field on $(M, \mathcal{D}, g)$. We start by providing an invariant description of this object, followed by giving an expression in local coordinates.

Consider the curvature function $\kappa\colon \mathcal{O}_H(\mathcal{D}) \to \mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})$ of the Cartan connection ω and let $\kappa_{\mathfrak{n}}$ be its $\mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{n})$-valued part. As discussed in Subsection 2.1, the curvature function κ is equivariant, and thus so is $\kappa_{\mathfrak{n}}$ with the same law (2.10) of transformation. Note that due to the semi-direct product structure of G, the adjoint action Ad of H on $\mathfrak{n}$ coincides with the usual action of H. This implies that the adjoint action of H on $\mathrm{Hom}(\wedge^2\mathfrak{n}, \mathfrak{n})$ coincides with the standard action of H on the isomorphic space $\mathfrak{n} \otimes \mathfrak{n}^* \wedge \mathfrak{n}^*$. Hence, $\kappa_{\mathfrak{n}}$ is a section of the associated bundle $\mathcal{O}_H(\mathcal{D}) \times_H (\mathfrak{n} \otimes \mathfrak{n}^* \wedge \mathfrak{n}^*)$.

Let $R\colon \mathfrak{n} \otimes \mathfrak{n}^* \wedge \mathfrak{n}^* \to \mathfrak{n}_{-1}$ be the map that is defined as follows as a composition of maps

$$R\colon \mathfrak{n} \otimes \mathfrak{n}^* \wedge \mathfrak{n}^* \xrightarrow{\mathrm{tr}} \mathfrak{n}^* \xrightarrow{\flat} \mathfrak{n} \xrightarrow{\pi_{-1}} \mathfrak{n}_{-1} \ . \tag{4.1}$$

Here, tr is the contraction map, $\flat$ is the operation of lowering of indices, and π_{-1} denotes the orthogonal projection to $\mathfrak{n}_{-1}$. By Proposition 2.9, we can associate with $\kappa_{\mathfrak{n}}$ an equivariant function $K_{\mathfrak{n}}$ with values in $\mathfrak{n}\otimes\mathfrak{n}^*\wedge\mathfrak{n}^*$. It follows that $R\circ K_{\mathfrak{n}}$ is a function with values in $\mathfrak{n}_{-1}$ which is also equivariant since H preserves the metric on $\mathfrak{n}$ and $\mathfrak{n}_{-1}$ is an orthogonal sub-representation. In particular, it is possible to associate with $R \circ K_{\mathfrak{n}}$ a section of $\mathcal{O}_H(\mathcal{D}) \times_H \mathfrak{n}_{-1} \simeq \mathcal{D}$, which we denote by $R \circ \kappa_{\mathfrak{n}}$.

The object $R \circ \kappa_{\mathfrak{n}}$ has a simple description in a local trivialisation. Let $\{e_1, \dots, e_n\}$ be an adapted orthonormal basis of $\mathfrak{n}$, let $\{e^1, \dots, e^n\}$ be the dual basis of $\mathfrak{n}^*$ and let $(X_1, \dots, X_{k_1})$ be the corresponding orthonormal frame of $\mathcal{D}$. In this trivialisation, we can write

$$\kappa_{\mathfrak{n}} = \sum_{j,k,l=1}^{n} \Omega^l_{jk} e_l \otimes e^j \wedge e^k \ .$$

Taking the trace, lowering the indices and using the isomorphism between $\mathcal{D}$ and $\mathfrak{n}_{-1}$, we obtain

$$R \circ \kappa_{\mathfrak{n}} = \sum_{i=1}^{k_1} \sum_{j=1}^{n} \Omega^j_{ji} X_i \ . \tag{4.2}$$

Proposition 4.1. *Suppose $(M, \mathcal{D}, g)$ is an equinilpotentisable sub-Riemannian manifold with a symmetry group H and a model space $N = G/H$ that admits a Cartan geometry $(\mathcal{O}_H(\mathcal{D}), \omega)$ modelled on $(\mathfrak{g}, \mathfrak{h})$. Then, we have*

$$\Delta - \Delta_{\mathcal{P}} = R \circ \kappa_{\mathfrak{n}} \ .$$

Proof. Throughout, we use summation convention over repeated indices for $i \in \{1, \dots, k_1\}$ and all other lowercase Latin indices ranging from 1 to n, and for $\alpha \in \{n+1, \dots, n+\dim H\}$. The computations are performed in an arbitrary trivialisation. As before, let $(X_1, \dots, X_n)$ be the adapted orthonormal frame of TM corresponding to $\{e_1, \dots, e_n\}$, let $\{e_{n+1}, \dots, e_{n+\dim H}\}$ be a basis of $\mathfrak{h}$ and let $(\theta^1, \dots, \theta^n)$ be the coframe dual to $(X_1, \dots, X_n)$.

From the formula for the curvature two-form Ω, we see that

$$\Omega^j_{ji} = (\mathrm{d}\theta_{\mathfrak{n}})^j_{ji} + \frac{1}{2}[\theta, \theta]^j_{ji} \ . \tag{4.3}$$

Let us first consider the second term. We observe that $[\theta_{\mathfrak{h}}, \theta_{\mathfrak{h}}]$ takes values in $\mathfrak{h}$ and that $[\theta_{\mathfrak{n}}, \theta_{\mathfrak{n}}]$ consists of elements that have degree zero. Since the terms $\Omega^j_{ji}(e_j \otimes \theta^j_{\mathfrak{n}} \wedge \theta^i_{\mathfrak{n}})$ have degree one, it follows that only the mixed commutators between $\theta_{\mathfrak{h}}$ and $\theta_{\mathfrak{n}}$ contribute to the expressions. By using (2.5) and (2.6) as well as

$$\theta_{\mathfrak{h}} = e_\alpha \otimes \theta^\alpha_{\mathfrak{h}} = e_\alpha \otimes \Gamma^\alpha_j \theta^j_{\mathfrak{n}} \ ,$$

we obtain, for A_α the matrix of the action of e_α on $\mathfrak{n}$, that

$$\begin{aligned}\frac{1}{2}[\theta,\theta]^j_{ji} &= \frac{1}{2}[\theta_{\mathfrak{h}},\theta_{\mathfrak{n}}]^j_{ji} + \frac{1}{2}[\theta_{\mathfrak{n}},\theta_{\mathfrak{h}}]^j_{ji} = [\theta_{\mathfrak{h}},\theta_{\mathfrak{n}}]^j_{ji} = \\ &= \left[e_\alpha \otimes \Gamma^\alpha_k \theta^k_{\mathfrak{n}}, e_l \otimes \theta^l_{\mathfrak{n}}\right]^j_{ji} = \left([e_\alpha \Gamma^\alpha_k, e_l] \otimes \theta^k_{\mathfrak{n}} \wedge \theta^l_{\mathfrak{n}}\right)^j_{ji} = \Gamma^\alpha_j [e_\alpha, e_i]^j = \Gamma^\alpha_j (A_\alpha)^j_i \ .\end{aligned} \tag{4.4}$$

To deal with the first term, we recall that the soldering gauge $\theta_{\mathfrak{n}}$ is given by (3.2). The important thing to remember about F is that it is an upper-block triangular matrix whose diagonal blocks are identity matrices. Let f^j_k and $\tilde{f}^j_k$ be the components of the matrices F and F^{-1}, respectively. Then we can rewrite (3.2) as

$$\theta^j_{\mathfrak{n}} = f^j_k \theta^k \ .$$

Thus, we have

$$\begin{aligned}\mathrm{d}\theta^j_{\mathfrak{n}} &= \mathrm{d}f^j_k \wedge \theta^k + f^j_k\, \mathrm{d}\theta^k = X_l\left(f^j_k\right)\theta^l \wedge \theta^k - \frac{1}{2} f^j_k c^k_{ls} \theta^l \wedge \theta^s = \\ &= \left(\tilde{f}^l_p \tilde{f}^k_q X_l\left(f^j_k\right) - \frac{1}{2} f^j_k \tilde{f}^l_p \tilde{f}^s_q c^k_{ls}\right)\theta^p_{\mathfrak{n}} \wedge \theta^q_{\mathfrak{n}}\end{aligned}$$

and we deduce that

$$(\mathrm{d}\theta_{\mathfrak{n}})^j_{ji} = \tilde{f}^l_j \tilde{f}^k_i X_l\left(f^j_k\right) - \tilde{f}^l_i \tilde{f}^k_j X_l\left(f^j_k\right) - \frac{1}{2} f^j_k \tilde{f}^l_j \tilde{f}^s_i c^k_{ls} + \frac{1}{2} f^j_k \tilde{f}^l_i \tilde{f}^s_j c^k_{ls} \ .$$

The expression on the right hand side simplifies significantly due to F and F^{-1} being block upper-triangular matrices with identity matrices as diagonal blocks. For the first term, we use that the elements of the i^{th} column of F and F^{-1} satisfy $f^k_i = \tilde{f}^k_i = \delta^k_i$ to deduce that

$$\tilde{f}^l_j \tilde{f}^k_i X_l\left(f^j_k\right) = \tilde{f}^l_j \delta^k_i X_l\left(f^j_k\right) = \tilde{f}^l_j X_l\left(f^j_i\right) = \tilde{f}^l_j X_l\left(\delta^j_i\right) = 0 \ ,$$

whereas for the second term, the property $f^j_k = \tilde{f}^j_k = 0$ for $k < j$ yields

$$\tilde{f}^l_i \tilde{f}^k_j X_l\left(f^j_k\right) = \tilde{f}^l_i X_l\left(f^j_j\right) = \tilde{f}^l_i X_l\left(n\right) = 0 \ .$$

For the last two terms, we exploit $f^j_k \tilde{f}^l_j = \delta^l_k$ and the antisymmetry of the structure constants to obtain that

$$(\mathrm{d}\theta_{\mathfrak{n}})^j_{ji} = -\frac{1}{2} f^j_k \tilde{f}^l_j \tilde{f}^s_i c^k_{ls} + \frac{1}{2} f^j_k \tilde{f}^l_i \tilde{f}^s_j c^k_{ls} = -\frac{1}{2}\tilde{f}^s_i c^l_{ls} + \frac{1}{2}\tilde{f}^l_i c^s_{ls} = -\tilde{f}^s_i c^l_{ls} \ .$$

Using once more that $\tilde{f}^k_i = \delta^k_i$ as observed above, we find

$$(\mathrm{d}\theta_{\mathfrak{n}})^j_{ji} = -c^l_{li} \ . \tag{4.5}$$

Inserting (4.4) and (4.5) into (4.3) gives

$$\Omega^j_{ji} = \Gamma^\alpha_j (A_\alpha)^j_i - c^l_{li} \ ,$$

and the claimed result follows from Theorem 1.1 as well as formulas (1.4) and (4.2). $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let us first prove the necessary part for the existence. The soldering form $\omega_{\mathfrak{n}}$ gives an isomorphism between $\mathcal{D}$ and $\mathfrak{n}_{-1}$. If $H \subset SO(k_1)$ has a one-dimensional representation, then there exists $v \in \mathfrak{n}_{-1}$ such that $Hv = \pm v$, and consequently $Av = 0$ for all $A \in \mathfrak{h}$. Take $X \in \Gamma(\mathcal{D})$ such that $\omega_{\mathfrak{n}}(X) = v$ and complete it to an orthonormal adapted frame. Then $\omega_{\mathfrak{n}}^{-1}(v) = X$, $AX = 0$ and the vector field X does not appear in the local divergence term of the formula (1.5).

It remains to prove that this condition is also sufficient. We fix an orthonormal adapted frame $\{e_1, \dots, e_{n+\dim H}\}$ in $\mathfrak{g}$ where the first n elements form a basis of $\mathfrak{n}$ and the last $\dim H$ elements a basis of $\mathfrak{h}$. As before, we assume that $\{e_1, \dots, e_{k_1}\}$ forms an orthonormal basis of the space $\mathfrak{n}_{-1}$. Let

$$\ker \mathfrak{h} = \{v \in \mathfrak{n}_{-1} \text{ such that } Av = 0 \text{ for all } A \in \mathfrak{h}\}.$$

We can suppose that $\ker \mathfrak{h} = \operatorname{span}\{e_{k_0+1}, \dots, e_{k_1}\}$, for some $k_0 \in \{0, \dots, k_1\}$, and that all vector fields corresponding to $\ker \mathfrak{h}$ are divergence-free. In the following, we abuse notation and we use g to refer to the extended metric on $\mathfrak{g}$ with g_{ij} and g^{ij}, for $i, j \in \{1, \dots, n + \dim H\}$, denoting the components of g and of the corresponding metric on the dual space $\mathfrak{g}^*$, respectively.

Since the spaces $\mathfrak{n}_{-l}$ are pairwise orthogonal and using the orthonormality of $\{e_1, \dots, e_{k_1}\}$, we obtain, for $i \in \{1, \dots, k_1\}$ and $k, l, s \in \{1, \dots, n\}$,

$$g\left(\sum_{j=1}^{n} e_j \otimes e^j \wedge e^i, e_k \otimes e^l \wedge e^s\right) = \frac{1}{2}\sum_{j=1}^{n} g_{jk}\left(g^{jl}g^{is} - g^{js}g^{il}\right) = \frac{1}{2}\left(\delta_k^l \delta^{is} - \delta_k^s \delta^{il}\right) . \tag{4.6}$$

It follows that, for $k \neq i$,

$$g\left(\sum_{j=1}^{n} e_j \otimes e^j \wedge e^i, e_k \otimes e^k \wedge e^i\right) = -g\left(\sum_{j=1}^{n} e_j \otimes e^j \wedge e^i, e_k \otimes e^i \wedge e^k\right) = \frac{1}{2} .$$

and that the scalar product of $\sum_{j=1}^{n} e_j \otimes e^j \wedge e^i$ with any other element of $\operatorname{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})_+$ is zero. Therefore, we obtain

$$g\left(\sum_{j=1}^{n} e_j \otimes e^j \wedge e^i, \kappa\right) = \frac{1}{2}\sum_{j=1}^{n} \Omega_{ji}^j .$$

Proposition 4.1 shows that the coefficients in the difference between the local divergence terms of Δ and $\Delta_{\mathcal{P}}$ are given exactly by $\sum_{j=1}^{n} \Omega_{ji}^j$. Hence from $\ker \mathfrak{h} = \operatorname{span}\{e_{k_0+1}, \dots, e_{k_1}\}$ and the divergence-free assumption, we obtain $\sum_{j=1}^{n} \Omega_{ji}^j = 0$ for all $i \in \{k_0 + 1, \dots, k_1\}$. Thus, if we consider the adapted Cartan connection associated with a normal module $\mathcal{N}$ which lies in the orthogonal complement of the module

$$\mathcal{S} = \operatorname{span}\left\{\sum_{j=1}^{n} e_j \otimes e^j \wedge e^i \text{ for } 1 \leq i \leq k_0\right\}$$

then, due to the curvature function κ taking values in $\mathcal{N}$, we have

$$\sum_{j=1}^{n} \Omega_{ji}^j = 0 . \tag{4.7}$$

The desired result would then follow from Proposition 4.1.

To complete the proof, we need to show that we can find such a normal module $\mathcal{N}$. We recall that a module is called normal if it is complement to $\partial(\operatorname{Hom}(\mathfrak{n}, \mathfrak{g})_+)$ in $\operatorname{Hom}(\wedge^2\mathfrak{n}, \mathfrak{g})_+$. We are looking for a normal module $\mathcal{N} \subset \mathcal{S}^\perp$ such that $\mathcal{N} \oplus \operatorname{im} \partial_+ = \operatorname{Hom}(\wedge^2 \mathfrak{n}, \mathfrak{g})_+$ or, equivalently, we need $\mathcal{N}$ to satisfy

$$\mathcal{S} \subset \mathcal{N}^\perp \quad \text{and} \quad \mathcal{N}^\perp \oplus (\operatorname{im} \partial_+)^\perp = \operatorname{Hom}(\wedge^2 \mathfrak{n}, \mathfrak{g})_+ \ .$$

If $\mathcal{S} \cap (\operatorname{im} \partial_+)^\perp = \{0\}$, then we can take

$$\mathcal{N}^\perp = \mathcal{S} \oplus \left(\mathcal{S} \oplus (\operatorname{im} \partial_+)^\perp\right)^\perp .$$

Indeed, under this assumption, by construction, we have $\mathcal{N}^\perp \cap (\operatorname{im} \partial_+)^\perp = \{0\}$ and a dimension count shows that

$$\dim \mathcal{N}^\perp = \dim \operatorname{Hom}(\wedge^2 \mathfrak{n}, \mathfrak{g})_+ - \dim(\operatorname{im} \partial_+)^\perp.$$

Due to $\mathcal{S}$ and $(\operatorname{im} \partial_+)^\perp$ being $\mathfrak{h}$-submodules, $\mathcal{N}$ is a $\mathfrak{h}$-submodule as well.

Let us now prove that $\mathcal{S} \cap (\operatorname{im} \partial_+)^\perp = \{0\}$. Since we can relabel the basis vectors e_i for $i \in \{1, \dots, k_0\}$, without loss of generality, it suffices to prove that there exists some $v \in \operatorname{im} \partial_+$ such that

$$g\left(\sum_{j=1}^n e_j \otimes e^j \wedge e^1, v\right) \neq 0 \ .$$

As $\ker \mathfrak{h} = \operatorname{span}\{e_{k_0+1}, \dots, e_{k_1}\}$, there exists some $h_1 \in \mathfrak{h}$ where $h_1 \cdot e_1 = \sum_{j=1}^n \alpha^j e_j$ has at least one non-zero coefficient α^i for $i \in \{2, \dots, k_0\}$. Recall that due to the filtered algebra structure, we have $\partial e^1 = 0$, and by using formula (4.6), we obtain

$$g\left(\sum_{j=1}^n e_j \otimes e^j \wedge e^1, \partial(h_1 \otimes e^i)\right) = g\left(\sum_{j=1}^n e_j \otimes e^j \wedge e^1, \sum_{l=1}^n \alpha_l e_l \otimes e^1 \wedge e^i\right) = -\alpha^i \neq 0 \ ,$$

as required. $\square$

A natural question that comes to mind is whether the Cartan connection constructed by Morimoto allows us to obtain the sub-Laplacian $\Delta_\mathcal{P}$ defined with respect to the Popp volume via the discussed stochastic development procedure. As we have seen in the above proof, for the stochastic development of the canonical sub-Riemannian diffusion on the model space to have generator $\frac{1}{2}\Delta_\mathcal{P}$, the curvature function needs to satisfy

$$g\left(\sum_{j=1}^n e_j \otimes e^j \wedge e^i, \kappa\right) = 0 \ . \tag{4.8}$$

This means that if we impose Morimoto's normalisation and the difference $\Delta - \Delta_\mathcal{P}$ vanishes, then $\sum_{j=1}^n e_j \otimes e^j \wedge e^i \in \operatorname{im} \partial$. Since $\partial e^i = 0$ for all $i \in \{1, \dots k_1\}$, a necessary condition for the latter is

$$\partial\left(\sum_{j=1}^n e_j \otimes e^j\right) \wedge e^i = 0 \quad \text{for all } i \in \{1, \dots, k_1\} \ , \tag{4.9}$$

which is easy to check in practice. In Section 4.2, we use condition (4.9) to establish that if Morimoto's normalisation condition is chosen for a free structure with two generators which is not a 3D contact structure then we have $\Delta - \Delta_\mathcal{P} \neq 0$. Nevertheless, Theorem 1.2 applies in this situation and in the proof of Theorem 1.2 we construct an explicit Cartan connection which gives rise to $\Delta = \Delta_\mathcal{P}$.

Examples where a stochastic development of the canonical sub-Riemannian diffusion on the model space never gives rise to the stochastic process with generator $\frac{1}{2}\Delta_\mathcal{P}$ are easy to find.

Let us consider a Goursat manifold, that is, a sub-Riemannian manifold $(M, \mathcal{D}, g)$ with growth vector $(2, 3, \dots, n-1, n)$ for some $n \in \mathbb{N}$ with $n \geq 4$. The associated Levy form $\mathcal{L}$ is a map

$$\mathcal{L}\colon \mathcal{D}^{-2} \times \mathcal{D}^{-2} \to \mathcal{D}^{-3}/\mathcal{D}^{-2}$$

defined pointwise as follows. For vectors $v, w \in \mathcal{D}_q^{-2}$ and vector fields $X_v, X_w \in \mathcal{D}^{-2}$ extending v and w, respectively, we set

$$\mathcal{L}_q(v, w) = [X_v, X_w](q) \mod \mathcal{D}_q^{-2} \ .$$

The forms $\mathcal{L}_q$ are skew-symmetric bilinear forms on odd-dimensional spaces and hence, they must have non-trivial kernels L_q which form a line field L. The non-integrability condition implies that $L \subset \mathcal{D}$, see [9]. The presence of the characteristic line field L breaks the $SO(2)$ symmetry and leaves us with $H = \{\mathrm{id}\}$. This implies that $\mathfrak{h} = \{0\}$ and therefore, the $\mathfrak{h}$-part $\omega_{\mathfrak{h}}$ of the Cartan connection is trivial. Thus, no matter what Cartan connection we choose, the generator of the developed stochastic process always ends up having vanishing first order term and we are left with the sum of squares term.

There is a simple way to generate many explicit examples of this kind, because Goursat distributions often arise as Cartan distributions in jet bundles. For instance, let us consider a two-dimensional manifold M with a global frame (X_1, X_2) of vector fields. We define a contact distribution $\mathcal{D}_1$ on the direct product $M \times S^1$ which is the span of the two vector fields

$$Y_1 = \frac{\partial}{\partial \theta_1} \ , \qquad Y_2 = \cos\theta_1 X_1 + \sin\theta_1 X_2 \ ,$$

where θ_1 is a coordinate on S^1. We then apply the same procedure a second time but this time to the pair (Y_1, Y_2) of vector fields to obtain an Engel distribution $\mathcal{D}_2$ on $M \times S^1 \times S^1$ spanned by the vector fields

$$Z_1 = \frac{\partial}{\partial \theta_2} \ , \qquad Z_2 = \cos\theta_2 Y_1 + \sin\theta_2 Y_2 \ ,$$

where θ_2 is a coordinate on the newly added circle S^1. We can carry on with this prolongation procedure and at each iteration it takes a Goursat manifold of step n and gives us a Goursat manifold of step $n+1$.

If we consider the upper half-plane $\mathbb{R}^2_+$ with coordinates (x, y) for $y > 0$ and the vector fields

$$X_1 = y\frac{\partial}{\partial x} \ , \qquad X_2 = y\frac{\partial}{\partial y} \ ,$$

then after applying the prolongation procedure twice, we find the two vector fields Z_1, Z_2 that span $\mathcal{D}_2$ of $\mathbb{R}^2_+ \times \mathbb{T}^2$. Assuming that Z_1 and Z_2 are orthonormal, we obtain a sub-Riemannian structure on $(\mathbb{R}^2_+ \times \mathbb{T}^2, \mathcal{D}_2)$. Setting

$$Z_3 = [Z_1, Z_2] \quad \text{and} \quad Z_4 = [Z_2, Z_3] \ ,$$

we find

$$[Z_2, Z_4] = -\left(\cos\theta_1 \sin\theta_2 + \cos\theta_2\right)\left(\sin\theta_2 Z_2 + \cos\theta_2 Z_3\right) + \sin\theta_1 \sin\theta_2 Z_4 \ .$$

In particular, we see that $c_{21}^1 = c_{22}^2 = c_{23}^3 = 0$ whereas $c_{24}^4 \neq 0$ and hence, by formula (1.4), the sub-Laplacian $\Delta_{\mathcal{P}}$ defined with respect to the Popp volume is not a sum of squares operator.

Remark 4.2. It should be underlined again that the local divergence terms appearing in (1.2) and (1.5) are frame dependent. If in the previous example we take a rotated frame

$$U_1 = \cos\varphi Z_1 - \sin\varphi Z_2 \ ,$$
$$U_2 = \sin\varphi Z_1 + \cos\varphi Z_2 \ ,$$

then the generator $\frac{1}{2}\Delta$ written in this frame would contain some non-trivial first order differential terms. However, notice that this new frame is not adapted. Indeed, if we write $U_3 = [U_1, U_2]$ then

$$[U_1, U_3] = \cos\varphi\, U_4 + \text{smaller order terms} ,$$
$$[U_2, U_3] = \sin\varphi\, U_4 + \text{smaller order terms} ,$$

where smaller order terms are understood with respect to the underlying grading. It is important to emphasise that the statement of Theorem 1.1 holds for adapted frames only and not just any orthonormal frame. □

Despite this example, we want to stress that there are plenty of geometrically interesting structures where all sub-representations of H on $\mathfrak{n}_{-1}$ have dimension strictly greater than one. In particular, the existence of a Cartan connection which develops the canonical sub-Riemannian diffusion on the model space to the stochastic process with generator $\frac{1}{2}\Delta_{\mathcal{P}}$ is guaranteed by Theorem 1.2, and we can follow the proof of Theorem 1.2 to construct such a Cartan connection. This is demonstrated in Section 4.2 for free sub-Riemannian structures with two generators.

Before that, we illustrate the full Cartan machinery in Section 4.1 and show that in fact any adapted Cartan connection on a 3D contact structure satisfies condition (4.8). As a result each stochastic development of the canonical sub-Riemannian diffusion on the Heisenberg group has generator $\frac{1}{2}\Delta_{\mathcal{P}}$.

It is worth noting that the Cartan connection constructed by Doubrov and Slovák in [12] for step two free structures also satisfies condition (4.8).

4.1 The three-dimensional contact case

The discussions in this subsection establish the following result.

Proposition 4.3. *For any Cartan connection on a 3D contact structure which is constructed by using any normal module $\mathcal{N}$, the stochastic development of the canonical sub-Riemannian diffusion process on the 3D Heisenberg group has generator $\frac{1}{2}\Delta_{\mathcal{P}}$.*

For 3D contact manifolds, $\mathfrak{n}$ is the 3D Heisenberg Lie algebra, $\mathfrak{h}$ is isomorphic to $\mathfrak{so}(2)$ and $\mathfrak{g}$ is a semi-direct product of the two. The Lie algebra $\mathfrak{n}$ admits a basis $\{e_1, e_2, e_3\}$ which satisfies

$$[e_1, e_2] = e_3 ,$$

with the other commutators in $\mathfrak{n}$ being zero. The first step is to determine the commutation relations in the Lie algebra $\mathfrak{g}$. Let e_4 be the only non-trivial element of $\mathfrak{h}$. Its action on $\mathfrak{n}$ is characterised by the Lie algebra derivation condition in (2.9), that is, we need to have

$$e_4([X, Y]) = [e_4(X), Y] + [X, e_4(Y)]$$

for all $X, Y \in \mathfrak{n}$. To determine the infinitesimal action of e_4, we start with its action on $\mathfrak{n}_{-1}$ which is simply given by the infinitesimal rotation such that $e_4(e_1) = -e_2$ and $e_4(e_2) = e_1$. We then apply the formula above to obtain the action of e_4 on e_3 which yields

$$e_4(e_3) = e_4([e_1, e_2]) = [e_4(e_1), e_2] + [e_1, e_4(e_2)] = 0 .$$

In particular, we can view the action of e_4 as the adjoint action on the Lie algebra $\mathfrak{n} \subset \mathfrak{g}$. This gives rise to the following structure equations on $\mathfrak{g}$

$$[e_1, e_2] = e_3 , \qquad [e_4, e_1] = -e_2 , \qquad [e_4, e_2] = e_1 , \tag{4.10}$$

with the remaining commutators being zero.

If we take X_3 to be the Reeb field and if (X_1, X_2) is an orthonormal frame of the contact distribution, then we have the structure equations

$$\begin{aligned}
[X_1, X_2] &= X_3 + c^1_{12}X_1 + c^2_{12}X_2 \ , \\
[X_3, X_1] &= c^1_{31}X_1 + c^2_{31}X_2 \ , \\
[X_3, X_2] &= c^1_{32}X_1 + c^2_{32}X_2 \ ,
\end{aligned}$$

or, by duality, for the coframe $(\theta^1, \theta^2, \theta^3)$ dual to (X_1, X_2, X_3), we obtain

$$\begin{aligned}
\mathrm{d}\theta^1 &= -c^1_{12}\theta^1 \wedge \theta^2 - c^1_{13}\theta^1 \wedge \theta^3 - c^1_{23}\theta^2 \wedge \theta^3 \ , \\
\mathrm{d}\theta^2 &= -c^2_{12}\theta^1 \wedge \theta^2 - c^2_{13}\theta^1 \wedge \theta^3 - c^2_{23}\theta^2 \wedge \theta^3 \ , \\
\mathrm{d}\theta^3 &= -\theta^1 \wedge \theta^2 \ .
\end{aligned}$$

We now first construct one particular Cartan connection which gives rise to $\Delta = \Delta_{\mathcal{P}}$ in order to illustrate the whole machinery as well as the idea of the proof of Theorem 1.2. A Cartan connection ω on $\mathcal{O}_H(\mathcal{D})$ can be defined as

$$\omega = \sum_{i=1}^{4} e_i \otimes \omega^i \ .$$

The first three one-forms $\omega^1, \omega^2, \omega^3$ are components of the the soldering form $\omega_{\mathfrak{n}}$, while ω^4 is $\omega_{\mathfrak{h}}$. We can express these forms in terms of the coframe $(\theta^1, \theta^2, \theta^3)$, some coefficients f^1_3, f^2_3 and the Christoffel symbols defined in Section 3 as follows

$$\begin{aligned}
\theta^i_{\mathfrak{n}} &= \theta^i + f^i_3\theta^3 \ , \quad \text{for } i \in \{1, 2\} \ , \\
\theta^3_{\mathfrak{n}} &= \theta^3 \ , \\
\theta^4_{\mathfrak{h}} &= \Gamma^4_1\theta^1_{\mathfrak{n}} + \Gamma^4_2\theta^2_{\mathfrak{n}} + \Gamma^4_3\theta^3_{\mathfrak{n}} \ .
\end{aligned}$$

Note that the coefficients f^1_3 and f^2_3 are nothing but the non-trivial off-diagonal components of the matrix-valued function F introduced in Section 3. They arise because the soldering form is not canonically defined. The curvature two-form Ω corresponding to ω is given by

$$\Omega = \sum_{i=1}^{4} e_i \otimes \Omega^i \ ,$$

where

$$\begin{aligned}
\Omega^1 &= \mathrm{d}\theta^1_{\mathfrak{n}} + \theta^4_{\mathfrak{h}} \wedge \theta^2_{\mathfrak{n}} \ , \\
\Omega^2 &= \mathrm{d}\theta^2_{\mathfrak{n}} - \theta^4_{\mathfrak{h}} \wedge \theta^1_{\mathfrak{n}} \ , \\
\Omega^3 &= \mathrm{d}\theta^3_{\mathfrak{n}} + \theta^1_{\mathfrak{n}} \wedge \theta^2_{\mathfrak{n}} \ , \\
\Omega^4 &= \mathrm{d}\theta^4_{\mathfrak{h}} \ ,
\end{aligned}$$

follows from (2.4). We determine the components Ω^i_{jk} by putting the formulae for the exterior differentials $\mathrm{d}\theta^1$, $\mathrm{d}\theta^2$ and $\mathrm{d}\theta^3$ into the above expressions and by reading off the coefficients in front of $\theta^j_{\mathfrak{n}} \wedge \theta^k_{\mathfrak{n}}$. In particular, we find

$$\begin{aligned}
\Omega^2_{12} &= -c^1_{12} - f^1_3 + \Gamma^1_4 \ , & \Omega^2_{32} &= f^1_3 \ , \\
\Omega^1_{21} &= c^2_{12} + f^2_3 - \Gamma^2_4 \ , & \Omega^1_{31} &= -f^2_3 \ .
\end{aligned}$$

The condition (4.7) implies that the Christoffel symbols of the Cartan connection constructed in the proof of Theorem 1.2 satisfy $\Gamma^1_4 = c^1_{12}$ and $\Gamma^4_2 = c^2_{12}$. Therefore, the difference $\Delta - \Delta_{\mathcal{P}}$ does indeed vanish.

An interesting property of 3D contact structures is that we have $\Delta - \Delta_{\mathcal{P}} = 0$ for any choice of normal module $\mathcal{N}$. We argue as follows. As a consequence of the relations (4.10), the basic differentials are given by

$$\begin{aligned}
\partial e_1 &= -e_3 \otimes e^2 \ , \\
\partial e_2 &= e_3 \otimes e^1 \ , \\
\partial e_3 &= 0 \ , \\
\partial e_4 &= e_2 \otimes e^1 - e_1 \otimes e^2 \ , \\
\partial e^3 &= -e^1 \wedge e^2 \ , \\
\partial e^1 &= \partial e^2 = 0 \ .
\end{aligned}$$

Recall that $\deg e_1 = \deg e_2 = -1$, $\deg e_3 = -2$, $\deg e_4 = 0$ and the opposite signs for the upper index. Therefore, the differential preserves the grading of the spaces $\mathrm{Hom}(\wedge^k \mathfrak{n}, \mathfrak{g})$.

To verify condition (4.8), we only need to compute the components of degree one. We find that

$$\begin{aligned}
\partial \left(e_4 \otimes e^1\right) &= -e_1 \otimes e^2 \wedge e^1 \ , \\
\partial \left(e_4 \otimes e^2\right) &= e_2 \otimes e^1 \wedge e^2 \ , \\
\partial \left(e_1 \otimes e^3\right) &= -e_3 \otimes e^2 \wedge e^3 \ , \\
\partial \left(e_2 \otimes e^3\right) &= e_3 \otimes e^1 \wedge e^3 \ ,
\end{aligned}$$

which shows that on degree one forms the Lie algebra differential ∂ is a bijection. Hence, any normal module has no degree one components as it is transversal to $\mathrm{im}\,\partial$. Since the curvature function κ takes values in the normal module, it follows that also κ has no degree one components, and condition (4.8) is satisfied automatically.

In particular, the Cartan connection built using Morimoto's normalisation agrees with the Cartan connection constructed in the proof of Theorem 1.2.

4.2 Free sub-Riemannian structures with two generators

The Cartan connection that we construct in the proof of Theorem 1.2 looks particularly simple in the case of free structures with two generators. For these structures, the nilpotent Lie algebra $\mathfrak{n}$ is a free nilpotent Lie algebra with generators e_1, e_2, and the Lie algebra $\mathfrak{h}$ is generated by a single element e_0.

We recall that in the proof of Theorem 1.2 we construct the normal module $\mathcal{N}$ by considering $\partial(\mathrm{Hom}(\mathfrak{n}, \mathfrak{g})_+)$ and by replacing certain k_1 elements from $\partial\,\mathrm{Hom}(\mathfrak{n}_{-1}, \mathfrak{h})$ with $\sum_{j=1}^n e_j \otimes e^j \wedge e^i$ for $i \in \{1, \ldots, k_1\}$. If we have only two generators, that is $k_1 = 2$, then

$$\mathcal{S} = \mathrm{span}\left\{\sum_{j=1}^n e_j \otimes e^j \wedge e^1, \sum_{j=1}^n e_j \otimes e^j \wedge e^2\right\}$$

and

$$\dim\left(\partial\,\mathrm{Hom}(\mathfrak{n}_{-1}, \mathfrak{h})\right) = \dim \mathcal{S} = 2 \ .$$

Following the construction in the proof of Theorem 1.2, we take the normal module $\mathcal{N}$ to be orthogonal to

$$\mathrm{span}\left\{\sum_{j=1}^n e_j \otimes e^j \wedge e^1, \sum_{j=1}^n e_j \otimes e^j \wedge e^2, \partial\,\mathrm{Hom}\left(\mathfrak{n}_{-i-1}, \mathfrak{n}_{-i}\right) \text{ for } 1 \leq i \leq m-1\right\} .$$

This module is not the only possible choice for generating the operator $\Delta_{\mathcal{P}}$, as the following example shows. Let us consider structures with growth vector $(2,3,5)$, that is, the space $\mathfrak{n}$ is spanned by e_1, e_2, e_3, e_4, e_5 and we have the following structure equations on $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$

$$
\begin{aligned}
&[e_1, e_2] = e_3 \ , && [e_1, e_3] = e_4 \ , && [e_2, e_3] = e_5 \ , \\
&[e_0, e_1] = -e_2 \ , && [e_0, e_4] = -e_5 \ , && \\
&[e_0, e_2] = e_1 \ , && [e_0, e_5] = e_4 \ . &&
\end{aligned}
$$

The differentials of degree one components are given, for $e_i \otimes e^j \wedge e^k$ abbreviated to e_i^{jk}, by

$$
\begin{aligned}
\partial\left(e_0 \otimes e^1\right) &= -e_1^{21} + e_5^{41} - e_4^{51} \ , \\
\partial\left(e_0 \otimes e^2\right) &= e_2^{12} + e_5^{42} - e_4^{52} \ , \\
\partial\left(e_1 \otimes e^3\right) &= -e_3^{23} - e_1^{12} \ , \\
\partial\left(e_2 \otimes e^3\right) &= e_3^{13} - e_2^{12} \ , \\
\partial\left(e_3 \otimes e^4\right) &= e_4^{14} + e_5^{24} - e_3^{13} \ , \\
\partial\left(e_3 \otimes e^5\right) &= e_4^{15} + e_5^{25} - e_3^{23} \ .
\end{aligned}
$$

Then we can take the normal module $\mathcal{N}$ to be orthogonal to

$$
\operatorname{span}\left\{e_1^{12}, e_2^{12}, e_3^{13}, e_3^{23}, e_4^{i4} + e_5^{i5} \text{ for } i = 1 \text{ and } i = 2\right\}
$$

because this span contains $\mathcal{S}$ as a submodule.

We close by noting that a simple calculation establishes

$$
\partial\left(\sum_{j=1}^{5} e_j \otimes e^j\right) \wedge e^1 = -e_5 \otimes e^3 \wedge e^2 \wedge e^1 \neq 0 \ . \tag{4.11}
$$

Thus, the condition (4.9) is not satisfied, which implies that under Morimoto's normalisation the operator Δ does not coincide with $\Delta_{\mathcal{P}}$. Moreover, we can use the observation (4.11) to show that the normalisation of Morimoto gives $\Delta - \Delta_{\mathcal{P}} \neq 0$ for any free structure with two generators and step strictly greater than two. Indeed, for higher step structures, for $i \in \{1, \ldots, 5\}$, the expressions for the ∂e^i agree with the ones in the $(2,3,5)$ case, whereas the expressions for the ∂e_i only differ from the ones in the $(2,3,5)$ case by higher order terms. Hence, the differential

$$
\partial\left(\sum_{j=1}^{n} e_j \otimes e^j\right) \wedge e^1
$$

agrees with (4.11) modulo some terms involving elements from higher steps. In particular, it does not vanish.

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