Cartel Stability and Product Di¤erentiation: How Much Do the Size of the Cartel and the Size of the Industry Matter?

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Abstract

This article analyses how the degree of product di¤erentiation, the size of the cartel and the size of the industry a¤ect the stability of a cartel formed by any number of …rms in an industry of any size. The paper considers a supergame-theoretic model to de…ne stability. After a non-loyal member leaves the cartel, two possible reactions by the remaining members of the cartel are assumed. The …rst one is a trigger strategy where the cartel dissolves after one member has left and the second is one where the cartel keeps acting as a cartel with one member less. The work also extends the analysis to investigate the stability of the remaining cartel. The results indicate that the relation between the cartel stability and the degree of di¤erentiation of the products depends considerably on the size of the cartel, the size of the industry and the reaction of the loyal members of the cartel.

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I. INTRODUCTION

Related issues to cartel stability and product di¤erentiation have been studied for many years. In a seminal article, Stigler (1964) suggested that the ability for …rms to collude would be reached easily amongst …rms whose products were relatively more homogeneous. Recent work, such as that done by Deneckere (1983), Chang (1991), Ross (1992), Rothschild (1992), Hackner (1994), Lambertini (1995) and Rothschild (1997), has addressed the question of how the degree of product di¤erentiation a¤ects the ability for …rms to collude. The

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basic framework used in this work is mainly models of horizontally differentiated products in both non-spatial (Chamberlin) and spatial (Hotelling) senses, as well as models of vertical differentiation in the case of Hackner (1994). The authors have basically applied the tools provided by the supergame-theoretic models, in which a trigger strategy is implemented to sustain collusion. They have mainly examined industries competing in price although, in some cases such as Deneckere (1983), Rothschild (1992) and Lambertini (1995), competition in quantities is also investigated. Nevertheless, this work has been restricted to the particular case of collusion of a duopoly, in the case of Chamberlinian differentiation and collusion of the whole industry, in the case of Hotelling differentiation.

The framework provided by the supergame-theoretic models to sustain stability has not been only applied to industries with product differentiation. The first article in this research line goes back to the work of Friedman (1971), who shows how an industry may settle at a cooperative price when static games are infinitely repeated. Green and Porter (1984) and Rotemberg and Saloner (1986) explain temporary stable cartels in markets with uncertain demand. Lambertini (1996) analyses the effect of the curvature of the demand function on the stability of a cartel composed of the whole industry. Rothschild (1999), also using this tool, analyses cartel stability for an industry with heterogeneous costs competing in quantities and for the particular case of collusion of the whole industry.

A second research line on cartel stability was initiated by D'Aspremont et al. (1983). They concentrate on the existence of stable cartels in static models of markets without uncertainty. A cartel is defined to be stable if neither members of the cartel, nor members of the fringe have incentives to move to the fringe or to the cartel, respectively. Stability is sustained, not by a trigger strategy and its consequent punishment, but by the market structure itself[2]. Much work has been done in this line for different market structures. D'Aspremont et al. (1983) and Donsimoni et al. (1986) prove the existence of a stable cartel in a price-leadership model. Donsimoni (1985) and Bockem (1998) analyse cartel stability for firms with heterogeneous cost structures. Shaffer (1995) shows the existence of a stable cartel competing with a Cournot fringe, in which the cartel plays the role of a Stackelberg leader.

This paper aims to analyse cartel stability in an industry that produces horizontally differentiated products in a Chamberlinian sense. The objective is to extend the work done by Deneckere (1983), Ross (1992) and Rothschild (1997) for the particular case of a duopoly, to a general case of an industry with $n \geq 2$ firms, in which k ($2 \leq k \leq n$) of them collude to form a cartel. The

[2] D'Aspremont et al. (1983) define a cartel to be internally stable if $\pi_f(k-1) \leq \pi_c(k)$ and to be externally stable if $\pi_c(k+1) \leq \pi_f(k)$; where $\pi_c(k)$ and $\pi_f(k)$ represent the profit of each of the members of the cartel and each of the members of the fringe when there are k members in the cartel, respectively.

analysis tries to capture how the stability of collusion is affected not only by the degree of product differentiation, but also by the size of the cartel, the size of the industry and the kind of strategy implemented to sustain collusion. The basic definition of stability is that provided by a supergame-theoretic model, in which firms compete during an infinite number of periods, there is a lag in the price adjustment process after one member has deviated from the cartel and there exists a discount factor in the industry. In this case, not only a trigger strategy is considered to sustain collusion but also the strategy of keeping the cartel once a member has deviated. Although the concept of stability is mainly viewed in a dynamic sense, its equivalence with the static definition of stability, as given by D'Aspremont et al. (1983), is also examined.

As far as I am aware, two closely related papers have been developed so far. Firstly, Hirth (1999), using numerical calculations, found the conditions under which a cartel is stable in a static sense for the general case of a cartel of size k in an industry with n firms supplying differentiated products. This article differs from Hirth's work because in this case a supergame-theoretic model is implemented to sustain stability and the analysis is not done numerically. Secondly, Eaton et al. (1998) assume a supergame-theoretic model to sustain stability in an industry with n firms, with k of them forming a cartel. Their article considers a trigger strategy as well as the strategy where the cartel keeps acting as a cartel. The authors found some general results regarding the effect of changes in the cartel size on the prices and profits of the industry. Nevertheless, explicit expressions for the prices, the profits of the firms and the critical discount factor in terms of the exogenous parameters of the model are not presented. They did not carry out a deep analysis on the effect of changes of the parameters of the model on the critical discount factor needed to sustain stability. Instead, some particular cases are presented for specific values of the different parameters, such as cartel size, size of the industry, size of the demand and the parameter of product differentiation, to exemplify some results. This paper is more general since it involves a general analysis with no specific value of the parameters.

The article is structured as follows. The second section describes the model and shows some general results regarding the behaviour of the prices and the profits. Section three examines the particular case of a collusion of the whole industry for a trigger strategy and for the case where the cartel keeps acting as a cartel with one member less. The next section analyses the general case of collusion by one part of the industry for the two different strategies. The credibility of the threat is also investigated and augmented, in section five, to analyse to what extent defection of a non-loyal member can originate future defection of other members in the remaining cartel. The final section summarizes the findings and suggests feasible future research lines.

II. MODEL

Consider an industry composed of $n \geq 2$ firms competing in prices during an infinite number of periods. Assume that the industry produces horizontally differentiated products such that the degree of differentiation between the products of any two firms is the same. Hence, the demand function exhibits a Chamberlinian symmetry where the price of product i in any period is given by

$$p_i = a - bq_i - c\sum_{j \neq i} q_j; \qquad a > 0; \qquad b > 0; \qquad 0 \leq c \leq b. \tag{2.1}$$

The value range for c implies that the products are viewed as substitutes rather than complements and that the price of each product is more susceptible to changes on its own demand than changes on other product demands.

Expression (2.1) implies a demand function for each period given by

$$q_i = \frac{a}{b - c + nc} - \frac{b + nc - 2c}{(b-c)^2 + nc(b-c)} p_i + \frac{c}{(b-c)^2 + nc(b-c)} \sum_{j \neq i} p_j \equiv$$

$$\alpha - \beta p_i + \gamma \sum_{j \neq i} p_j, \tag{2.2}$$

where

$$\alpha = \frac{a}{b - c + nc}, \quad \beta = \frac{b + nc - 2c}{(b-c)^2 + nc(b-c)}, \quad \gamma = \frac{c}{(b-c)^2 + nc(b-c)}. \tag{2.3}$$

Let $d \equiv c/b$ be the parameter to measure the degree of differentiation between any two products in the industry. Hence, $d = 0$ implies that the products are completely heterogeneous and $d = 1$ indicates that they are perfect substitutes. In general, $0 \leq d \leq 1$. A different parameter of product differentiation can also be defined in terms of the parameters of the demand function

$$\delta \equiv \frac{\gamma}{\beta} = \frac{d}{1 + nd - 2d}, \qquad \delta \in [0, 1/(n-1)]. \tag{2.4}$$

Both parameters, d and δ, measure the degree of differentiation between any two products. However, d will be preferentially used, since this implies a linear measure of differentiation, besides that δ is limited to take values that depend on the size of the industry. Nevertheless, a few expressions are more easily written in terms of δ.

Assume that, in each period, k of the n firms in the industry collude to form a cartel. The remaining $n - k$ firms in the industry, called the fringe, act independently. For simplicity, and without loss of generality, it is assumed that the total production costs for every firm are equal to zero and that there are no capacity constrains. Hence, the cartel does not play any lead role and every entity in the industry can have an influence on the market demand.

II.1. One period solution

In the first period of the game, it is assumed that k firms precommit to form a cartel. The cartel aims to maximise its joint profit and to share it among its members. This maximisation problem is equivalent to that where a member of the cartel maximizes its own profit subject to the restriction of having its price equalled to the price of every other member of the cartel[3]. Hence, the problem confronting a firm in the cartel can be established as

$$\max_{p_i} p_i\Big(\alpha - \beta p_i + \gamma \sum_{j \neq i} p_j\Big) = \max_{p_i} p_i\Big[\alpha - \beta p_i + \gamma \sum_{j \in F} p_j + \gamma \sum_{j \neq i \in C} p_j\Big] =$$

$$\max_{p_i} p_i\Big[\alpha - \beta p_i + \gamma \sum_{j \in F} p_j + (k-1)\gamma p_i\Big], \tag{2.5}$$

where $j \in F, C$ denotes the sum over the members of the fringe and over the members of the cartel, respectively. The first order condition implies

$$\alpha - 2\beta p_i + \gamma \sum_{j \in F} p_j + 2(k-1)\gamma p_i = 0. \tag{2.6}$$

By symmetry, $p_j = p_f$ for every $j \in F$. Therefore, (2.6) becomes

$$\alpha - 2\beta p_c + (n-k)\gamma p_f + 2(k-1)\gamma p_c = 0, \tag{2.7}$$

where p_c and p_f already denote the price of the cartel and the price of the fringe, respectively.

In contrast, each member of the fringe faces the problem of maximising its profits without any kind of restriction. Hence, the optimization problem confronting one fringe member is

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[3] The profit of the cartel is k times the profit of each one of its members, even before the maximisation process, since they have precommitted to set the same price. Therefore, each member of the cartel maximise $1/k$ times the profit of the cartel, which does not change the first order condition at all.

$$\max_{p_i} p_i\Big(\alpha - \beta p_i + \gamma \sum_{j \neq i} p_j\Big) = \max_{p_i} p_i\Big[\alpha - \beta p_i + \gamma \sum_{j \neq i \in F} p_j + \gamma \sum_{j \in C} p_j\Big]. \quad (2.8)$$

The first order condition implies

$$\alpha - 2\beta p_i + \gamma \sum_{j \neq i \in F} p_j + \gamma \sum_{j \in C} p_j = 0. \quad (2.9)$$

Using again symmetry, this becomes

$$\alpha - 2\beta p_f + (n - k - 1)\gamma p_f + k\gamma p_c = 0. \quad (2.10)$$

Solving for the price of the cartel and the price of the fringe from equations (2.7) and (2.10) there arises

$$p_c(k,n) = \frac{\alpha(2+\delta)}{\beta A} \quad (2.11) \qquad \text{and} \qquad p_f(k,n) = \frac{\alpha(2 - k\delta + 2\delta)}{\beta A}, \quad (2.12)$$

where $A = (2 - n\delta)(2 - k\delta + 2\delta) + \delta(2 - k^2\delta + 2\delta)$.

(2.11) and (2.12) imply a profit for each member of the cartel and each member of the fringe equal to

$$\pi_c(k,n) = \frac{\alpha^2(2+\delta)^2(1 + \delta - k\delta)}{\beta A^2} \quad (2.13)$$

and

$$\pi_f(k,n) = \frac{\alpha^2(2 - k\delta + 2\delta)^2}{\beta A^2}, \quad (2.14)$$

respectively.

(2.11), (2.12), (2.13) and (2.14) can easily be written in terms of the original parameters a, b, c and d by direct substitution of (2.3) and (2.4).

Proposition 1 (see appendix for proof). The price of the cartel, the price of the fringe, as well as the profit of each member of the cartel and the profit of each member of the fringe are increasing functions of k.

Therefore, every firm is better off with the existence of a cartel. Moreover, the larger the cartel the larger the profit of every firm in the industry.

Proposition 2 (see appendix for proof). $\pi_f(k;n) \geq \pi_c(k;n)$:

There is the usual problem of free riding of the members of the fringe in an industry with a cartel.

II.2. Supergame solution

As Friedman (1971) has shown, it is possible for firms to sustain cooperation in a infinitely repeated game, in which it would not be possible for the corresponding static case. In order to sustain cooperation, every firm in the cartel plays a trigger strategy, i.e., they set a price $p_c(k;n)$ as long as every other member in the cartel has done so in previous periods. When one member deviates to any other price, the remaining members revert to the non-cooperative case ($p_f(0;n)$) for ever, but with one lag period. Cooperation can be sustained if there exists a discount factor in the industry large enough to prevent a firm from deviating. In other words, the extra profit that this non-loyal member earns in the deviating period is offset by the lowered profit the firm gets once every firm has reverted to the non-cooperative case.

If a non-loyal member deviates from the cartel it will do it to a price that offers it the greatest possible benefit. Therefore, it will charge the price that maximises its profits given that the other members of the cartel has charged $p_c(k;n)$: Hence, the maximisation problem that this non-loyal member confronts is

$$\max_p \; p[\alpha - \beta p + (n - k)\gamma p_f(k;n) + \gamma(k - 1)p_c(k;n)]; \qquad (2.15)$$

where it has also been assumed that the fringe adjusts its price with one period lag after one member has deviated from the cartel.

The first order condition implies

$$p_{ch} = \frac{\alpha(2+\delta)(2+\delta-\delta k)}{2\beta A} \quad (2{:}16) \quad \text{and} \quad \pi_{ch} = \frac{\alpha^2(2+\delta)^2(2+\delta-\delta k)^2}{4\beta A^2}: \quad (2{:}17)$$

Proposition 3 (see appendix for proof). The price to which a non-loyal member deviates as well as the profits it gets in the deviating period are increasing functions of k.

The condition to maintain stability is that the present discounted value of remaining a member of the cartel must exceed the present discounted value of deviating, i.e.

$$\sum_{t=0} \pi_c(k;n)\delta^t \geq \pi_{ch} + \sum_{t=1} \pi_f(0;n)\delta^t; \qquad (2.18)$$

where δ is the discount factor of the industry. Evaluating this condition in term of the interest rate, $r = (1-\delta)/\delta$, results in

$$r \leq r^* \equiv \frac{\pi_c(k;n) - \pi_f(0;n)}{\pi_{ch} - \pi_c(k;n)}. \qquad (2.19)$$

r^* is the critical value below which a member of a cartel does not have incentives to deviate. A large value of r^* implies that it is more likely that the corresponding interest rate of the industry is below this critical value. On the other hand, a low value of r^* makes it less likely for this interest rate to be below r^*. Therefore r^* can be seen as a measure of cartel stability. A large value of it implies that the cartel is very likely to be stable and low values indicate that the cartel is very likely to be unstable. Negative values imply complete instability.

Although the trigger strategy ensures a certain degree of stability, the threat of reverting to the non-cooperative is not collectively credible, since the cartel punishes itself when it punishes the non-loyal member. A further possible reaction by the remaining members of the cartel can be considered. This strategy is simply to assume that the remaining members of the cartel will keep acting as a cartel after a member has deviated. They will only adjust their price to $p_c(k-1;n)$. Hence, the non-loyal member gets a profit equal to $\pi_f(k-1;n)$ from the second period on. The condition to maintain stability becomes

$$\sum_{t=0} \pi_c(k;n)\delta^t \geq \pi_{ch} + \sum_{t=1} \pi_f(k-1;n)\delta^t; \qquad (2.20)$$

which implies

$$r \leq r^* \equiv \frac{\pi_c(k;n) - \pi_f(k-1;n)}{\pi_{ch} - \pi_c(k;n)}. \qquad (2.21)$$

It is worth noting that the sign of the critical interest is given by the sign of $\pi_c(k;n) - \pi_f(k-1;n)$, since $\pi_{ch} - \pi_c(k;n) > 0$. However, a positive value of this amount corresponds to the static internal stability defined by D'Aspremont et al.(1983). Therefore, the qualitative behaviour of the stability as defined here is equivalent to the static internal stability concept.

III. INDUSTRY-WIDE CARTELS

III.1. Trigger strategy

The first case of the analysis is that in which all the firms in the industry form a cartel ($k = n$) and a trigger strategy is implemented. (2.16) and (2.17) for $k = n$ imply that the price and the profit of the non-loyal member in the deviating period are

$$p_{ch} = \frac{\alpha(\delta - n\delta + 2)}{4\beta(\delta - n\delta + 1)} \qquad (3.1) \qquad \text{and} \qquad \pi_{ch} = \frac{\alpha^2(\delta - n\delta + 2)^2}{16\beta(\delta - n\delta + 1)^2}; \qquad (3.2)$$

respectively. On the other hand, the condition to sustain stability (2.19) becomes

$$r \leq r^* \equiv \frac{\pi_c(n;n) - \pi_f(0;n)}{\pi_{ch} - \pi_c(n;n)}; \qquad (3.3)$$

and, in terms of the original parameters of the model, in

$$r \leq r^* \equiv \frac{4(1 - d)(1 + nd - 2d)}{(2 + nd - 3d)^2}. \qquad (3.4)$$

Before proceeding with the analysis, it is important to note that when the non-loyal member deviates to p_{ch} the remaining demand of the loyal members of the cartel may become negative for large values of δ. For the values of δ where the demand would become negative the non-loyal firm must reduce its price until the demand of the remaining members of the cartel is equal to zero, i.e.

$$\pi_{lc} = p_c(n;n)[\alpha - \beta p_c(n;n) + (n - 2)\gamma p_c(n;n) + \gamma p_{ch})] = 0; \qquad (3.5)$$

where π_{lc} denotes the profit of the loyal members of the cartel in the deviating period. Solving for p_{ch} results in

$$p_{ch} = \frac{\alpha(n\delta - 1)}{2\gamma(\delta - n\delta + 1)} \qquad (3.6) \qquad \text{and} \qquad \pi_{ch} = \frac{\alpha^2(\delta + 1)(n\delta - 1)}{4\delta\gamma(\delta - n\delta + 1)}. \qquad (3.7)$$

The critical value for δ (δ^*) is found by equalling expressions (3.1) and (3.6)

$$\delta^* = \sqrt{\frac{n + 1}{n - 1}} - 1; \qquad (3.8) \qquad \text{or, equivalently} \qquad d^* = \frac{n - 3 + \sqrt{n^2 - 1}}{3n - 5}. \qquad (3.9)$$

Therefore, the non-loyal member deviates according to (3.1) as long as $0 \leq d \leq d^*$ and according to (3.6) as long as $d^* \leq d \leq 1$.

For values of $d \geq d^*$, the condition to sustain stability (3.3) becomes

$$r \leq r^* \equiv \frac{d^4(n-1)^2}{(2+nd-3d)^2(-1+3d-3d^2-nd+2nd^2)}. \tag{3.10}$$

Hence, the measure of stability, r^*, can be calculated according to

$$r^*(n;d) = \begin{cases} \frac{4(1-d)(1+nd-2d)}{(2+nd-3d)^2} & \text{for } 0 \leq d \leq d^*; & (3.11a) \\ \frac{d^4(n-1)^2}{(2+nd-3d)^2(-1+3d-3d^2-nd+2nd^2)} & \text{for } d^* \leq d \leq 1. & (3.11b) \end{cases}$$

Proposition 4 (see appendix for proof). The critical interest rate is always positive. It takes a value of 1 at $d = 0$ and a value of $1/(1-n)$ at $d = 1$ for every n. For $n \in \{2, 3, 4\}$ r^* reaches a global minimum at

$$d_{min} = \frac{12-4n-2\sqrt{2(3+n)(n-1)}}{21-14n+n^2}. \tag{3.12}$$

For $n \geq 5$ the critical interest rate is always a decreasing function of d.

Figs 1, 2 and 3 show the critical interest rate as a function of d for values of n equal to 2, 3, and 7. The economic intuition behind this result can be explained as follows. When the products are very heterogeneous, every firm acts basically as a monopoly within its own market. The incentive for a non-loyal member to deviate is low and, by the same reasoning, so is its punishment. Therefore, stability is neither very low, nor very high. As the degree of homogeneity increases, the incentives to deviate also increases, because the non-loyal member can now supply its product to other consumer firms and steal their markets. Nevertheless the punishment is also larger, the first effect is always more important and the stability decreases. This would keep decreasing until reaching no stability at all. However, when d^* is reached, the profit of the non-loyal member is restricted to avoid the rest of the industry having a negative demand. Therefore, the incentive to deviate diminishes and the stability fall reverts.

Regarding the behaviour of the critical interest rate as a function of n, for fixed values of d, the following results were found:

Proposition 5 (see appendix for proof). r^* is a decreasing function of n.

This result derives from the fact that a cartel composed of the whole industry always has larger prices, the larger the size of the industry. Therefore, a non-loyal member has stronger incentives to deviate in large industries, because the market gains it gets once it has deviated are very large, due to the high price of its new competitor.

By evaluating d^* in $n = 2$ and by taking its limit when n goes to infinity it is directly shown that $d^* \in (2/3, \sqrt{3} - 1]$. Therefore, for values of $d \leq 2/3$ the non-loyal member deviates according to 3.1 and the critical interest rate is given by $3.11a$. If $d \geq \sqrt{3} - 1$ the non-loyal member deviates according to 3.6 and the critical interest rate is given by $3.11b$. If $2/3 \leq d \leq \sqrt{3} - 1$ the critical interest rate is given by $3.11a$ in the interval $2 \leq n \leq n^*$ and by $3.11b$ in the interval $n \geq n^*$. Where n^* is defined by the inverse function of d^*, i.e.

$$n^* = \frac{2 - 6d + 5d^2}{3d^2 - 2d}. \tag{3.13}$$

III.2. The cartel keeps acting as a cartel

When the cartel keeps acting as a cartel its members only adjust their price to $p_c(n-1; n)$, after a non-loyal member has deviated. Therefore, the non-loyal member get a profit equal to $\pi_f(n-1; n)$ from the second period on. The condition to maintain stability (2.20) becomes

$$\sum_{t=0} \pi_c(n; n)\delta^t \geq \pi_{ch} + \sum_{t=1} \pi_f(n-1; n)\delta^t; \tag{3.14}$$

which implies

$$r \leq r^* \equiv \frac{\pi_c(n; n) - \pi_f(n-1; n)}{\pi_{ch} - \pi_c(n; n)}; \tag{3.15}$$

and, in terms of the original parameters

$$r \leq r^* \equiv \begin{cases} \frac{4(1-d)(1+nd-2d)B}{(n-1)(4-8d+d^2+4dn-d^2n)^2} & \text{for } 0 \leq d \leq d^*; \quad (3.16a) \\ \frac{d^4(n-1)B}{(-4+8d-d^2-4dn+d^2n)^2(-1+3d-3d^2-dn+2d^2n)} & \text{for } d^* \leq d \leq 1; \quad (3.16b) \end{cases}$$

where $B = 12 - 28d + 7d^2 - 4n + 24dn - 11d^2n - 4dn^2 + 4d^2n^2$.

Conjecture 1[4]. For $n = 3$ the critical interest rate takes a value of 0 at $d = 0$. It has its global minimum at $d = 0$; a local maximum at $d = 0.59$; a local

[4] By direct substitution, it can be easily proved the case $n = 3$. Nevertheless for $n \geq 4$ it was not possible for this author to prove formally that r^* is an increasing function of d. However, numerical calculations and informal proofs suggest strongly the validity of this conjecture.

minimum at $d = 0.74$ and its global maximum (0.5) at $d = 1$. For $n \geq 4$ the critical interest rate is an increasing function of d. At $d = 0$ the critical interest rate has a negative value equal to $(3 - n)/(n - 1)$. $r^* = 0$ at

$$d_o = \frac{2[7 - 6n + n^2 + (n - 2)\sqrt{n^2 - 4n + 7}]}{4n^2 - 11n + 7} \qquad (3.17)$$

and it reaches its maximum at $d = 1$, with a positive value equal to $1/(n - 1)$.

Figs 4, 5, 6 show the critical interest rate as a function of d for values of n equal to 3, 4 and 7. The fact that the critical interest rate takes negative values and no cartel is stable for low values of d, is explained because there is no punishment against the non-loyal member. However, as d increases the stability increases and can reach positive values. At first glance, this result could seem contradictory since, as was just mentioned, there is no punishment at all against the non-loyal member. In this case, the punishment comes from the market itself. When the degree of homogeneity is large, so is the degree of competition between the cartel and a non-loyal member acting now as the fringe. The prices in the industry can fall substantially, nevertheless there are only two entities competing in the industry. Although the market share is larger for the non-loyal member, the price fall has reduced his profits. This effect is not observed for low values of homogeneity, because competition is not present when the products are heterogeneous. The stability is even strengthened when d is close to 1, because the profits that the non-loyal member gets in the deviating period is, as is known, restricted for $d \geq d^*$.

Proposition 6 (see appendix for proof). r^* is a decreasing function of n.

A graph of r^* as function of n, for fixed values of d, must be plotted in the following way: If $d \leq 2/3$, the non–loyal member deviates according to 3.1 and the critical interest rate is given by $3.16a$. If $d \geq \sqrt{3} - 1$, the non-loyal firm deviates according to 3.6 and the critical interest rate is given by $3.16b$. If $2/3 \leq d \leq \sqrt{3} - 1$ the optimal interest rate is given by $3.16a$ in the interval $2 \leq n \leq n^*$ and by $3.16b$ in the interval $n \geq n^*$.

The threat of keeping the cartel is always more credible than that of reverting to the non-cooperative case[5].

It is also direct to prove that expression (3.11) is larger than (3.16)[6]. Therefore, stability decreases when the strategy of keeping the cartel is carried out instead of the trigger strategy. This result is obvious because the trigger strategy implies a more severe punishment against the non-loyal member.

[5] This result comes from the fact that every member in the industry is better off with the existence of a cartel, in this case a cartel of size $k = n - 1$.

[6] By comparing 3.3 and 3.15 and the fact that $\pi_{if}(n - 1; n) > \pi_{if}(0; n)$, from proposition 1 for the particular case $k = n - 1$.

IV. CARTELS SMALLER THAN THE WHOLE INDUSTRY

This section analyses the case of a cartel composed of only a proportion of the industry, i.e., $2 \leq k < n$.

IV.1. Trigger strategy

Lemma 1 (see appendix for proof). For $k < n$, the price reduction of the non-loyal member according to (2.16) always keeps the profits of the remaining members of the cartel positive, for every value of the parameter d. Consequently, every price reduction is carried out according to (2.16).

The condition to maintain stability (2.19) becomes, in terms of the original parameters of the model

$$r \leq r^* \equiv \frac{4(1 - 2d + dn)C}{(k - 1)(2 - 3d + dn)^2(2 - 3d + 2dn)^2}, \tag{4.1}$$

where

$$C = -4 + 16d - 21d^2 + 9d^3 + 4k - 16dk + 19d^2k - 6d^3k - 4dk^2 + 15d^2k^2 - 14d^3k^2 - d^2k^3 + 2d^3k^3 - 8dn + 22d^2n - 15d^3n + 12dkn - 34d^2kn + 23d^3kn - 6d^2k^2n + 11d^3k^2n - d^3k^3n - 5d^2n^2 + 7d^3n^2 + 11d^2kn^2 - 16d^3kn^2 - 2d^3k^2n^2 - d^3n^3 + 3d^3kn^3.$$

It can be firstly observed that $r^* > 0$[7]. Thus, every cartel in every industry can exist if the interest rate is small enough.

Proposition 7 (see appendix for proof). r^* is a decreasing function of k.

Therefore, the most likely cartels to exist are small cartels for any size of the industry. To understand this, it is important to remember that a large cartel implies a high price in the industry. Therefore, a non-loyal member gets high profits by deviating from large cartels, since the high prices of its new competitor, the low numbers of competitors and in general the high prices in all the industry, will permit him to get a larger market share and a large gain margin.

Conjecture 2[8]. For cartels of size $n - 1, n - 2, ..., k^*$ the critical interest rate starts at a value of 1 at $d = 0$. Its value then increases with the

[7] This result derives from the fact that $\pi_c(k; n) > \pi_f(0; n)$ $(= \pi_c(1; n))$ from proposition 1 and $\pi_{ch} > \pi_c(k; n)$ which, by definition of π_{ch}, is clearly true.

[8] The validity of this conjecture is based on numerical calculations for particular cases since it was not possible for this author to find a formal proof. However, no exception to this conjecture was found.

degree of homogeneity and it reaches its maximum at a point between 0 and 1. Subsequently, it decreases.

In this case the maximum is reached at a higher value of d; the smaller the size of the cartel. Therefore, k^* can be found by computing the last k that permits $\frac{\partial r^*}{\partial d} = 0$; for the largest possible value of d. $\frac{\partial r^*}{\partial d}(d = 1) = 0$ implies

$$-1 + 6k - 14k^2 + 6k^3 + n - 4kn + 24k^2n - 8k^3n - 10kn^2 - 8k^2n^2 + 8kn^3 = 0. \tag{4.2}$$

The explicit expression for the roots of this equation can be easily found, because of its third degree nature. However it is not presented here. Instead, a table is given below, where k_o indicates the root of this equation.

n	3	4	5	6	7	8	9	10	15	25
k_o	2.23	2.85	3.47	4.09	4.71	5.33	5.95	6.57	9.66	15.85

Thus, k^* is the smallest integer greater or equal to k_o.

It is worth noting that $\lim_{n \to \infty} k_o/n = (\sqrt{5} - 1)/2 \approx 0.62$. Therefore, for large industries the highest degree of stability is reached for an intermediate value of the product differentiation parameter as long as the cartel is composed by more than 62% of the firms in the industry.

Conjecture 3[9]. For cartels of size $k^* - 1$; $k^* - 2$; ...; 2; r^*starts at 1 at $d = 1$ but it is always an increasing function of the degree of homogeneity.

The behaviour of the stability as a function of d can be understood as follows. When the degree of homogeneity increases, but it still has low values, the punishment that the non-loyal member receives is large enough to diminish its incentives to deviate, therefore the stability increases. However, as d increases it is possible to appropriate other firm's markets, since the products have reached a large value of substitutability. This incentive starts to be more important than the punishment and the stability decreases, after having reached a maximum. However, as is known from proposition 3, the incentive to deviate is larger, the larger the size of the cartel. If the cartel is smaller than k^* the gain from reducing the price is never more important than the punishment and the stability will always increase reaching its maximum at $d = 1$.

Figs 7, 8 and 9 show the pattern described below. Fig 7 corresponds to a cartel of size 6 in an industry consisting of 7 firms. In this case the maximum is reached at $d = 0.11$. Fig 8 is a cartel of size 5 in an industry with 7 firms.

[9] The validity of this conjecture is based on numerical calculations for particular cases since it was not possible for this author to find a formal proof. However, no exception to this conjecture was found.

The maximum is reached at $d = 0.53$. Fig 9 is a cartel of size 4 in the same industry. In this case the maximum is reached at a value of $d = 1$ since $k^*(7) = 5$ ($k_o(7) = 4.71$).

IV.2. The cartel keeps acting as a cartel

If the cartel keeps acting as a cartel the condition to maintain (2.21) becomes, in terms of the original parameters

$$r \le r^* \equiv \frac{4(1-2d+dn)D}{(k-1)(2-3d+2dn)^2E^2}, \tag{4.3}$$

where

$$\begin{aligned} D = {} & 48 - 304d + 728d^2 - 808d^3 + 399d^4 - 63d^5 - 16k + 32dk + 152d^2k \\ & - 544d^3k + 571d^4k - 186d^5k + 16dk^2 - 92d^2k^2 + 160d^3k^2 - 59d^4k^2 \\ & - 44d^5k^2 + 4d^2k^3 + 4d^3k^3 - 51d^4k^3 + 58d^5k^3 - 4d^3k^4 + 17d^4k^4 - 19d^5k^4 \\ & - d^4k^5 + 2d^5k^5 + 224dn - 1128d^2n + 2020d^3n - 1494d^4n + 369d^5n \\ & - 64dkn + 80d^2kn + 468d^3kn - 1024d^4kn + 533d^5kn + 48d^2k^2n - 216d^3k^2n \\ & + 254d^4k^2n - 49d^5k^2n + 12d^3k^3n + 2d^4k^3n - 37d^5k^3n - 6d^4k^4n + 13d^5k^4n \\ & - d^5k^5n + 412d^2n^2 - 1552d^3n^2 + 1851d^4n^2 - 685d^5n^2 - 100d^2kn^2 + 64d^3kn^2 \\ & + 465d^4kn^2 - 476d^5kn^2 + 52d^3k^2n^2 - 167d^4k^2n^2 + 99d^5k^2n^2 + 11d^4k^3n^2 \\ & - 2d^5k^4n^2 + 372d^3n^3 - 936d^4n^3 + 559d^5n^3 - 76d^3kn^3 + 16d^4kn^3 + 149d^5kn^3 \\ & + 24d^4k^2n^3 - 43d^5k^2n^3 + 3d^5k^3n^3 + 164d^4n^4 - 208d^5n^4 - 28d^4kn^4 + 4d^5k^2n^4 \\ & + 28d^5n^5 - 4d^5kn^5 \end{aligned}$$

and

$$E = 4 - 8d + d^2 - 2dk + 6d^2k - d^2k^2 + 6dn - 7d^2n - d^2kn + 2d^2n^2$$

Cojecture 4[10]**.** r^* is an increasing function of d and takes a value of $(3-k)/(k-1)$ at $d = 0$.

This proposition implies that for cartels of size 2 and 3 the critical interest rate is always positive.

The behaviour of the stability of the cartel as a function of d can be explained because the incentives to deviate and to become a new member of the fringe are high when the degree of homogeneity is low. The low level of competition will lead to a small fall in prices that will be compensated by the larger market share. However, as the degree of homogeneity increases this incentive disappears

[10] The second part of the conjecture is directly shown by substitution. However, it was not possible for this author to prove formally the first part, although numerical calculations and informal proofs suggest its validity.

since the price of the industry can fall more drastically due to the higher degree of competition.

The general behaviour of r^* as a function of d for a cartel of size $k < n$ in an industry with n firms follows three different patterns, which depend on the size of the industry.

Proposition 8[11]. For an industry of up to 9 firms the critical interest rate starts with a negative value (except for $k = 2, 3$), as d increases, the critical interest rate increases and ends with a positive value at $d = 1$.

Therefore, every cartel has a positive probability[12] to exist as long as the degree of homogeneity is large enough. Unfortunately, there does not exist an explicit expression to calculate the value of d, above which the cartel has a positive probability to exist. This value is one of the roots of D but, because of its fifth degree nature, calculations must be carried out numerically.

Proposition 9[13]. For $n = 10$ the same behaviour described in proposition 12 occurs, with the exception of $k = 8$, in which case the critical interest rate is always negative.

Proposition 10 (see outline proof ahead). For $n \geq 11$ the critical interest rate is always negative for $k \in [7, n - 1]$. Therefore, no cartel greater than 6 can exist in an industry with 11 or more firms. For cartels smaller than 7 the behaviour is the same as in proposition 8.

Due to the fact that $\frac{\partial r^*}{\partial d} > 0$ it is possible to predict analytically which cartel has no possibility of existing. The condition for non-existence is that $r^*(d = 1) < 0$. In other words, the last cartel that has a possibility to exist is that of size $k < k^{**}$, where k^{**} is the solution of $r^*(d = 1) = 0$. This equation implies

$$\begin{aligned} &-9 + 19k - 15k^2 + 6k^3 - k^4 + 26n - 52kn + 29k^2n - 8k^3n + k^4n - 5n^2 \\ &+45kn^2 - 19k^2n^2 + 3k^3n^2 - 44n^3 + 28n^4 - 4kn^4 = 0. \end{aligned} \tag{4.4}$$

The next table shows the only roots of (4.4) in the valid interval for k, for different values of n.

n	11	12	13	14	15	25	50	100	500	1000
k^{**}	6.93	6.80	6.74	6.71	6.69	6.71	6.82	6.90	6.98	6.99

[11] The proof of this proposition is based on numerical computations.

[12] Positive probability must be understood here as a cartel that is likely to exist. That is, a cartel in an industry in which the interest rate is below r^*.

[13] The proof of this proposition is based on numerical computations.

It is obvious that the last series suggests that $\lim_{n \to \infty} k^{**} = 7$. This result can easily be found by evaluating $\lim_{n \to \infty} r^{*}(d = 1) = (7 - k)/(k - 1)$, which clearly has a root at $k^{**} = 7$.

Figs 10 and 11 show the critical interest rate as a function of d for an industry with 12 firms and 6 and 8 firms in the cartel, respectively. It can be seen that a cartel of size 6 can be stable for $d > 0.49$, but a cartel with 8 firms is never stable.

Regarding the stability of a cartel as a function of the cartel's size, for fixed values of d, the following results were found.

Proposition 11[14]**.** For $n \leq 10$ the stability basically decreases with the size of the cartel[15].

Only small cartels can be stable for small values of d since the large level of competition (there are many entities in the industry) will bring about a fall in prices that will never be compensated by a larger market share. However, for larger values of d cartels of any size can be stable, with the exception of a cartel of size 8 in an industry with 10 firms. This result is explained by the fact that a non-loyal member will originate a drastic fall in prices by deviating from the cartel but, due this time, to a large degree of homogeneity of the products.

Conjecture 5[16]**.** For $n \geq 11$ the same pattern described as in proposition 11 occurs for small values of d, i.e., only small cartels can exist. However, for large values of d a kind of convex parabola with two roots in the interval $(3, n)$ is obtained. The first root is always lower than 7 and the second is always in the interval $(n - 1, n)$.

This implies that the critical interest rate is always negative in the interval $[7, n - 1]$ for any value of d. Therefore, no cartel greater than 6 can exist in an industry with 11 or more firms, with the exception of a collusion of the whole industry.

To understand why no cartels between 7 and $n - 1$ can exist in large industries, the following can be observed. If the industry is very large (more than 11 firms) and the cartel is also very large (almost all the firms in it), a member of the cartel will always have incentives to become a part of a really small fringe[17].

[14] The validity of this proposition is based on numerical calculations.

[15] A few exceptions were found for $n = 9$, $n = 10$ and large degrees of homogeneity. In this case, the function reaches a local minimum for values of k close to n, although the general tendency keeps as a decreasing function of k.

[16] The validity of this conjecture is based on numerical calculations for particular cases since it was not possible for this author to find a formal proof. However, no exception to this conjecture was found.

[17] In this case a really small fringe is understood as a very small fringe relative to the size of the industry and not necessarily according to its absolute size.

The fall in prices will always be compensated by a larger market share even for very homogeneous products. When a cartel has a medium size relative to the industry, for instance 10 in an industry of 20 firms, the fact that the fringe is already very large has lead to a high level of competition even before one member deviates from the cartel. The profit of the cartel is rather low and has to be shared among many members. If a member deflects from the cartel it will not drastically increase the level of competition, this was already presented even before it has deviated but, on the other hand, it will not have to share its profits any more. Therefore, the only possible stable cartels will be those where the profits do not have to be shared among many members. Figs 12, 13 and 14 show the critical interest rate as a function of the size of the cartel for three different values of d, 0.25, 0.50 and 0.99 and for an industry with 12 firms.

The threat of keeping the cartel with one member less is always more credible than that of breaking the cartel up[18].

It is directly shown that the stability of the cartel diminishes when a trigger strategy is implemented instead of that where the cartel keeps acting as a cartel[19].

V. DYNAMIC STABILITY

Finally, in this section a new concept of stability can be introduced. This concept aims to find out to what extent the threat of keeping the cartel is credible, since after a non-loyal member has deflected from the cartel a second member could also have an incentive to follow it in a subsequent period.

For this analysis the critical interest rate of a cartel of size k was compared with the critical interest rate of a cartel of size $k-1$. If the critical interest rate for k is larger than that for $k-1$ if a first member had an incentive to leave the cartel a second one will have stronger incentives to follow him. If $r^*(k-1) > r^*(k)$ it is likely that no other member will leave the cartel. Actually, as long as $r^*(k-1) > r > r^*(k)$ the cartel will be stable with one member less.

The analysis was first carried out for $k = n$ where it is known that the critical interest rate is an increasing function in d, with the exception of $n = 3$. It starts at a negative value and reaches its maximum at $d = 1$ where it takes positive values. The analysis focuses only on cartels that have a positive probability to exist, i.e., those in the interval $(d_o, 1]$, where d_o is the root of expression 3.16 and it is given by 3.17.

Calculations for $n \in [3, 10]$ were carried out with the following results

[18] This result derives from proposition 1, which establishes that every firm in the industry (inside and outside the cartel) is always better off with the existence of a cartel.

[19] From the fact that $\pi_f(k-1, n) > \pi_f(0, n)$ (particular case of proposition 1) and by comparing 2.19 with 2.21.

Proposition 12. For industry-wide cartels of size $n \in \{3; 4; 5; 6; 7\}$, and with a positive probability to exist $r^*(n) < r^*(k = n - 1)$.

Therefore, if a member of a cartel composed by the whole industry of size 3 to 7 and that can exist with some positive probability leaves the cartel, the remaining cartel will be stable as long $r^*(n) < r < r^*(k = n - 1)$. In other words, the threat of keeping the cartel can be credible.

Proposition 13. For industry-wide cartels of size $n \geq 8$, and with a positive probability to exist $r^*(n) > r^*(k = n - 1)$[20].

Therefore, if a member of a cartel which can exist with some positive probability and which is composed of the whole industry of size $n \geq 8$ leaves the cartel, the remaining cartel will be unstable and a second member will also have incentives to leave the cartel. Hence, the threat of keeping the cartel with one member less is not credible.

For $k < n$; it has been shown that in an industry with 11 or more firms no cartel between 7 and $n - 1$ firms can exist. It has also been mentioned that for $n < 10$ the stability basically decreases with the size of the cartel. This last result can be extended for $n = 10$, with a few exceptions, and for $n > 10$ for cartels of sizes between 2 and 6. The general behaviour can be established as follows:

Proposition 14. For $n \geq 11$ and cartels with positive probability to exist; $r^*(k) < r^*(k - 1)$ for $k \in \{3; 4; 5; 6\}$. For $n < 11$ and cartels with positive probability to exist; $r^*(k) < r^*(k - 1)$ with two exceptions: $r^*(9) > r^*(8)$ for $n = 10$ and $d \in (0.60; 1]$ and; $r^*(8) > r^*(7)$ for $n = 9$ and $d \in (0.98; 1]$.

Hence, with two very restrictive exceptions, the threat of keeping the cartel with one member less can be credible.

VI. CONCLUSION

The crucial point in this paper is that parameter values k, n, considerably affect cartel stability as a function of the degree of product differentiation. Therefore, the results found by Deneckere (1983), Ross (1992) and Rothschild (1997) are specific to the duopoly case ($k = n = 2$).

For a cartel that involves the whole industry and uses a trigger strategy the general result is that a cartel is more likely to be stable the larger the degree of heterogeneity of the products and the smaller the size of the industry. On the other hand, if a member deviates from the cartel it is more credible that

[20] For $n \geq 11$, it is known that $r^*(k = n - 1) < 0$. Therefore, $r^*(n) > r^*(k = n - 1)$ for $n \geq 11$.

the remaining members will only adjust their price and keep acting as a cartel with one fewer member. However, this will diminish the degree of stability and actually it will break it completely for industries with heterogeneous products or large number of …rms. Therefore, the stability of the cartel can only be sustained in small industries or industries with very homogeneous products.

For cartels that involve only a proportion of the industry and uses a trigger strategy, small cartels are more likely to exist. The probability of sustaining small cartels is even strengthened the larger the degree of homogeneity of the products. However, cartels that involve almost every …rm in the industry are more likely to be stable the larger the degree of heterogeneity of the products.

When the cartel keeps acting as a cartel, every cartel of size three can exist with positive probability, moreover its stability increases with the degree of homogeneity. When the industry is composed of less than 10 …rms no cartel larger than four can exist when the degree of homogeneity is low enough, actually for large degrees of heterogeneity only small cartels can be stable. As the degree of homogeneity increases every cartel has a positive probability to exist. However, for industries greater than 10, although the stability always increases with the degree of homogeneity, no cartels larger than six exist. For small values of d only small cartels can be stable. As d increases, greater cartels can be stable but never greater than 6.

In the case of cartels that involve all members of an industry of sizes 3 to 7 and that have a positive probability to exist (large values of homogeneity), it was found that if one of its members leaves the cartel it is likely that no other members will have an incentive to follow it in subsequent periods. In contrast, for cartels involving all members of an industry of size greater than 7 and with a positive probability to exist if one member leaves the cartel a second member will have stronger incentives to follow it in subsequent periods.

For cartels smaller than n and a positive probability to exist, with very restricted exceptions it was found that if a member leaves the cartel it is likely that the remaining members will not have any incentive to follow it.

Finally, a trigger strategy always implies less stable cartels than that where the cartel keeps acting as a cartel.

This work opens up to a great number of direct extensions. The …rst one is to analyse the stability of cartels for the same industry competing in quantities and not in prices. A second one could be a study of complementary goods. An analysis of di¤erent strategies to prevent non-loyal members from deviating is also possible. The incentives of the members of the fringe to join the cartel must also be taken into account[21]. Moreover, a non-loyal member could have incentives to rejoin the cartel in subsequent periods. Here, an exogenous mechanism

[21] That is, a study closely related to the concept of static internal stability.

to expel a non-loyal member forever from the cartel has been assumed, but it was never clear which form this could take. The symmetry can also be broken assuming that firms have differentiated products but the degree of heterogeneity is different among the different products in the industry. Other market structures such as the Hotteling product differentiation model, the price-leadership model and the Stackelberg leader-follower model can be analysed. Finally, it could also be worth carrying out a welfare analysis considering industry profits and consumer surplus.

VII. APPENDIX

VII.1. Proof Proposition 1

Writing p_c in terms of d and taking the derivative respect to k it is found that $\frac{\partial p_c}{\partial k} > 0$ as long as $2 - 4d + 2dk + dn = 2 + d[n + 2(k - 2)] > 0$; which is clearly true for $k \geq 1$ as long as $n \geq 2$.

Writing p_f in terms of d and taking the derivative respect to k it is found that $\frac{\partial p_f}{\partial k} > 0$ as long as $-2 + 2d + 4k - 4dk - dk^2 - 2dn + 4dkn > 0$. This expression is a concave parabola in k. Evaluating in $k = 1$ it results in $2 + d(2n - 3)$, which is clearly positive for $n \geq 2$. Evaluating in $k = n$ it results in $3dn(n - 2) + 4n + 2(d - 1)$, which is also clearly positive for $n \geq 2$ since the first term is always larger or equal to zero and the second term is always larger than the third one, provided that $0 \leq d \leq 1$. Therefore, the expression is positive for $k = 1$ and for $k = n$. Due to the fact that it is a concave parabola it is also positive for any intermediate value of k.

$\frac{\partial \pi_c}{\partial k} > 0$ implies, in terms of d,

$$\frac{2 - 2d - 2k + 3dk^2 + 2dn - 3dkn}{-4 + 10d - 6d^2 + 2dk - 4d^2k + d^2k^2 - 6dn + 8d^2n + d^2kn - 2d^2n^2} > 0.$$

The denominator of this expression takes a value of $d(1 - n) < 0$ at $k = 1$ and a value of $2(n - 1)(d - 1) < 0$ at $k = n$. Due to the fact that it is a convex parabola this term takes always negative values for any $k \in [1, n]$. The similar argument can be applied to the denominator, since it is also a convex parabola that takes values of $-[2 + d(2n - 3)][2 + d(n - 3)] < 0$ at $k = 1$ and $2(d - 1)[2 + d(2n - 3)] < 0$ at $k = n$. Therefore, since the numerator and the denominator are always negative, the ratio is always positive for $k \in [1, n]$.

That π_f is an increasing function of k is a direct result of the fact that π_f can be written as βp_f^2, i.e., a monotone transformation of an increasing function.

VII.2. Proof Proposition 2

By direct substitution it is found that $\pi_f(k,n) \geq \pi_c(k,n)$ as long as $0 \leq k + 1 + \delta$, which is clearly true.

VII.3. Proof Proposition 3

$$\frac{\partial p_{ch}}{\partial k} > 0 \Rightarrow -4 + 6d + 4k - 6dk - dk^2 - 3dn + 4dkn > 0.$$

Evaluating this expression at $k = 1$, $d(n-1) > 0$ is obtained. Evaluating at $k = n$ results in $(n-1)[4 + 3d(n-2)] > 0$. Due to the fact that this expression in a concave parabola in k, every $k \in [1, n]$ also takes positive values.

Since $\pi_{ch} = \beta p_{ch}^2$, i.e., a monotone transformation of an increasing function, π_{ch} is also a increasing function of k.

VII.4. Proof Proposition 4

It is known that $\pi_c(n,n) > \pi_f(0,n)$, since it is always better for every firm to have the existence of a cartel. On the other hand, it is also known that $\pi_{ch} > \pi_c(n,n)$, by definition of π_{ch}. These two results imply then

$$r^* \equiv \frac{\pi_c(n,n) - \pi_f(0,n)}{\pi_{ch} - \pi_c(n,n)} > 0.$$

By substitution, it is directly shown that the critical interest rate takes a value of 1 at $d = 0$ and a value of $1/(1-n)$ at $d = 1$ for every n.

$\frac{\partial r^*}{\partial d} = -\frac{4d(n-1)^2}{[2+d(n-3)]^3}$ for $0 \leq d \leq d^*$, which is clearly negative for every $n \geq 2$.

At this point it is useful to consider that, by construction, the critical interest rate is a continuous function. It is also directly shown, by evaluating the first derivative of 3.11a and 3.11b at d^*, that continuity is also a property of the first derivative. Therefore 3.11b is also a decreasing function at d^*.

By calculating $\frac{\partial r^*}{\partial d}$ for the interval $d^* \leq d \leq 1$ it is direct to show that r^* has a minimum at

$$d_{min} = \frac{12 - 4n - 2\sqrt{2(3+n)(n-1)}}{21 - 14n + n^2},$$

where $d^* < d_{min} < 1$ only for $n = \{2, 3, 4\}$. Since r^* has a negative derivative at d^* and, it does not have any minimum for $n \geq 5$, the function is decreasing in d for $n \geq 5$.

VII.5. Proof Proposition 5

$\frac{\partial r^*}{\partial n} = -\frac{4(1-d)d^2(n-1)}{[2+d(n-3)]^3}$ in the interval $0 \le d \le d^*$, which is clearly negative for every $n \ge 2$.

$\frac{\partial r^*}{\partial n} < 0$ in the interval $d^* \le d \le 1$ implies

$$4 - 14d + 17d^2 - 6d^3 + 2dn - 4d^2n - d^2n^2 + 2d^3n^2 \equiv y > 0$$

To prove that y is positive in this interval it is sufficient to show that this is the case for $d > 2/3$, since $d^* > 2/3$ for every $n \ge 2$[22]. Thus, the proof can be reduced to show that $y(d = 2/3)$ is positive and its first derivative is positive. To prove that $\frac{\partial y}{\partial d}$ is positive it is sufficient to show that $\frac{\partial y}{\partial d}(d = 2/3)$ is positive and $\frac{\partial^2 y}{\partial d^2}$ is positive but, to prove that $\frac{\partial^2 y}{\partial k^2}$ is positive it is sufficient to show that $\frac{\partial^2 y}{\partial d^2}(d = 2/3)$ is positive and $\frac{\partial^3 y}{\partial k^3}$ is positive. Therefore the proof can be reduced to show that y, $\frac{\partial y}{\partial d}$, and $\frac{\partial^2 y}{\partial d^2}$, evaluated at $2/3$, are positive and $\frac{\partial^3 y}{\partial d^3} > 0$.

$$y(d = 2/3) = \frac{4(n^2 - 3n + 3)}{27} > 0; \qquad \frac{\partial y}{\partial d}(d = 2/3) = \frac{2(2n^2 - 5n + 1)}{3} > 0;$$

$$\frac{\partial^2 y}{\partial d^2}(d = 2/3) = 2(3n^2 - 4n + 5) > 0; \qquad \frac{\partial^3 y}{\partial d^3} = 12(n^2 - 3) > 0.$$

VII.6. Proof Proposition 6

$$\frac{\partial r^*}{\partial n} < 0 \text{ in the interval } 0 \le d \le d^* \Rightarrow \frac{F}{-4 + 8d - d^2 - 4dn + d^2n} < 0$$

where,

$$F = 32 - 192d + 416d^2 - 364d^3 + 94d^4 - 13d^5 + 96dn - 424d^2n + 560d^3n - 184d^4n$$
$$+ 33d^5n + 104d^2n^2 - 268d^3n^2 + 110d^4n^2 - 27d^5n^2 + 40d^3n^3 - 20d^4n^3 + 7d^5n^3.$$

$-4 + 8d - d^2 - 4dn + d^2n$ is always negative for $n \ge 1$, since this expression takes a value of $-4(1-d) < 0$ at $n = 1$ and its first derivative, $-(4-d)d$, is always negative. Thus, it is sufficient to prove that $F > 0$. Using the same argument as in the proof of proposition 5, the proof can be reduced to show that F, $\frac{\partial F}{\partial n}$ and $\frac{\partial^2 F}{\partial n^2}$ at $n = 1$ and $\frac{\partial^3 F}{\partial n^3}$ are positive.

[22] By evaluating d^* in $n = 2$ and by taking its limit when n goes to infinity it is directly shown that $d^* \in (2/3, \sqrt{3} - 1)$.

$F(n=1) = 32(1-d)^3 > 0. \qquad \frac{\partial F}{\partial n}(n=1) = 24(4-d)(d-1)^2 d > 0.$

$\frac{\partial^2 F}{\partial n^2}(n=1) = 4(1-d)d^2(52-22d+3d^2) > 0.$

$\frac{\partial^3 F}{\partial n^3} = 6d^3(40-20d+7d^2) > 0.$

$\frac{\partial r^*}{\partial n} < 0$ in the interval $d^* \leq d \leq 1$ implies $\frac{G}{-4+8d-d^2-4dn+d^2n} < 0,$

where,

$$G = 64-384d+864d^2-892d^3+412d^4-89d^5-2d^6-32n+320dn-976d^2n$$
$$+1240d^3n-672d^4n+195d^5n+6d^6n-64dn^2+336d^2n^2-556d^3n^2$$
$$+348d^4n^2-139d^5n^2-6d^6n^2-32d^2n^3+80d^3n^3-56d^4n^3+33d^5n^3+2d^6n^3.$$

It has already been shown that $-4+8d-d^2-4dn+d^2n$ is always negative. Therefore, the proof can be reduced to show that $G > 0$. Using again the same argument as in the proof of proposition 5, it is sufficient to show that G; $\frac{\partial G}{\partial d}$; $\frac{\partial^2 G}{\partial d^2}$; $\frac{\partial^3 G}{\partial d^3}$; $\frac{\partial^4 G}{\partial d^4}$ and $\frac{\partial^5 G}{\partial d^5}$ at $2/3$ are positive and; $\frac{\partial^6 G}{\partial d^6} > 0.$

$G(d=2/3) = \frac{32[5n^3+62n^2(n-3)+64(3n-1)]}{729} > 0$ for $n \geq 3.$

$\frac{\partial G}{\partial d}(d=2/3) = \frac{16[161n^2(n-3)+43n^2+114(3n-1)+27n]}{81} > 0$ for $n \geq 3.$

$\frac{\partial^2 G}{\partial d^2}(d=2/3) = \frac{16[278n^2(n-4)+175n^2+307(3n-1)+117n]}{27}$. Numerical calculations show that this expression is positive for $n=3$ and it is clearly positive for $n \geq 4.$

$\frac{\partial^3 G}{\partial d^3}(d=2/3) = \frac{8[602n^2(n-4)+509n^2+1355(n-1)+1009n]}{9}$. Numerical calculations show that this expression is positive for $n=3$ and it is clearly positive for $n \geq 4.$

$\frac{\partial^4 G}{\partial d^4}(d=2/3) = 16[101n^2(n-3)+70n^2+27(n-6)+9]$. Numerical calculations show that this expression is positive for $n = \{3;4;5\}$ and it is clearly positive for $n \geq 6.$

$\frac{\partial^5 G}{\partial d^5}(d=2/3) = 120[n-1][41n(n-3)+97] > 0$ for $n \geq 3.$

$\frac{\partial^6 G}{\partial d^6} = 1440(n-1)^3 > 0$ for $n \geq 3.$

VII.7. Proof Lemma 1

The profit of a loyal member of the cartel is given by

$$\pi_{lc} = p_{ic}(k;n)[\alpha - \beta p_{ic}(k;n) + \gamma(n-k)p_{if}(k;n) + \gamma(k-2)p_{ic}(k;n) + \gamma p_{ch}];$$

where $p_{ch} = \frac{\alpha(2+\delta)(2+\delta-\delta k)}{2\beta A}$.

Substituting $p_c(k;n)$; $p_f(k;n)$ and p_{ch} results, in terms of d, in

$$[2 + d(2n-3)][2 - 6d + 5d^2 - 2dk + 3d^2k + 4dn - 6d^2n - 2d^2kn + 2d^2n^2]$$

The first term of this expression is clearly positive for $n \geq 3$. The proof can be reduced to show that the second term is also positive

$$2 - 6d + 5d^2 - 2dk + 3d^2k + 4dn - 6d^2n - 2d^2kn + 2d^2n^2 > 0 \Rightarrow$$

$$d^2(2n^2 - 6n + 5 + 3k - 2kn) + 2d(2n - k - 3) + 2 > 0 \Rightarrow$$

$$d^2[n^2 + (n-5)(n-1)] - d^2k(2n-3) + 2d(2n-3) - 2dk + 2 > 0 \Rightarrow$$

The first, third and fifth terms of this expression are always positive and the second and fourth are always negative for $n \geq 3$. Hence, this expression could take negative values when the second and fourth terms take their lowest possible value, i.e., when k takes its highest possible value $(n-1)$. If it is shown that this expression is positive for the lowest possible value of the second and third terms then it will be positive for any other value of $d; n$ and $k < n$. Substituting in the last expression, k for $n-1$; $d^2(2-n) + 2d(n-2) + 2$ is obtained. Since $2d(n-2) > d^2(n-2)$; provided that $2 > d$, then $d^2(2-n) + 2d(n-2) > 0$; which implies that $d^2(2-n) + 2d(n-2) + 2 > 0$.

VII.8. Proof Proposition 7

$$\frac{\partial r^*}{\partial k} < 0 \Rightarrow 2d - 3d^2 + 8k - 30dk + 28d^2k - 4k^2 + 18dk^2 - 20d^2k^2 - 2dk^3 + 4d^2k^3 - 4n + 12dn - 8d^2n + 12dkn - 22d^2kn - 6dk^2n + 14d^2k^2n - 2d^2k^3n - 6dn^2 + 9d^2n^2 + 4d^2kn^2 - 2d^2k^2n^2 - 2d^3n^3 \equiv z < 0.$$

Applying the same argument as in the proof of proposition 5, it is sufficient to show that z; $\frac{\partial z}{\partial k}$; and $\frac{\partial^2 z}{\partial k^2}$ at $k = 1$ and; $\frac{\partial^3 z}{\partial k^3}$ are negative.

$$z(k=1) = \frac{\partial z}{\partial k}(k=1) = \frac{\partial^2 z}{\partial k^2}(k=1) = -(n-1)[2+d(2n-3)][2+d(n-3)] < 0$$

for $n \geq 2$:

$$\frac{\partial^3 z}{\partial k^3} = -12[1+d(n-2)] < 0 \text{ for } n \geq 2.$$

FIGURES

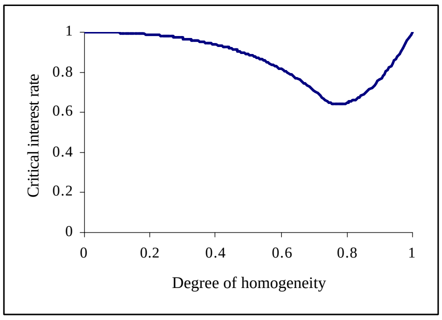

Fig 1: Critical interest rate as a function of the degree of homogeneity for a wide-industry cartel of size 2 when a trigger strategy is implemented.

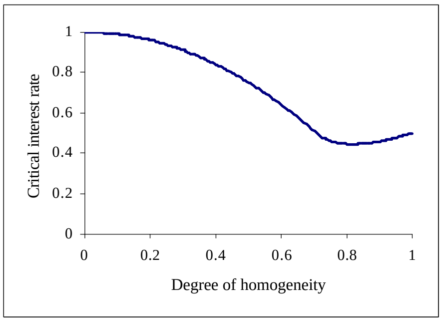

Fig 2: Critical interest rate as a function of the degree of homogeneity for a wide-industry cartel of size 3 when a trigger strategy is implemented.

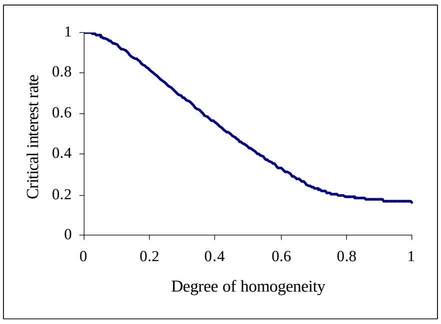

Fig 3: Critical interest rate as a function of the degree of homogeneity for a wide-industry cartel of size 7 when a trigger strategy is implemented.

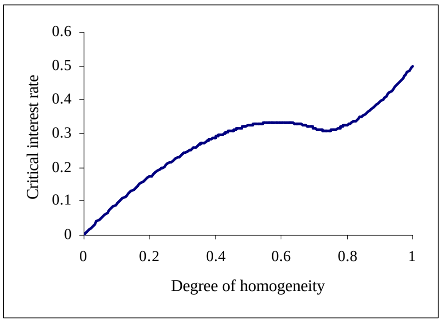

Fig 4: Critical interest rate as a function of the degree of homogeneity for a wide-industry cartel of size 3 when the cartel keeps acting as a cartel once a member has defected.

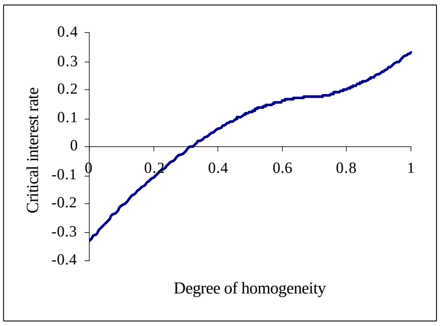

Fig 5: Critical interest rate as a function of the degree of homogeneity for a wide-industry cartel of size 4 when the cartel keeps acting as a cartel once a member has de‡ected.

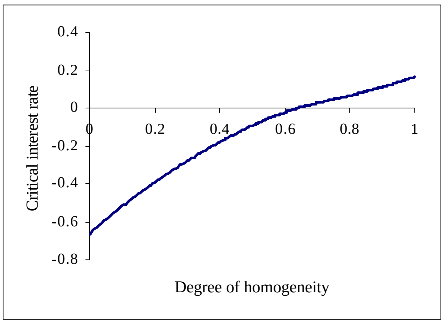

Fig 6: Critical interest rate as a function of the degree of homogeneity for a wide-industry cartel of size 7 when the cartel keeps acting as a cartel once a member has de‡ected.

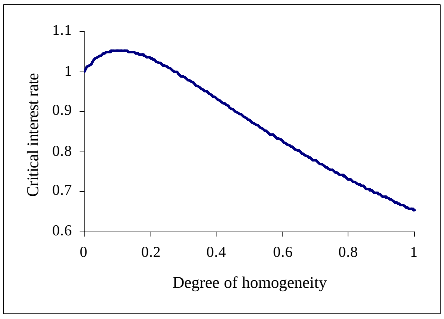

Fig 7: Critical interest rate as a function of the degree of homogeneity for a cartel of size 6 in an industry with 7 …rms when a trigger strategy is implemented.

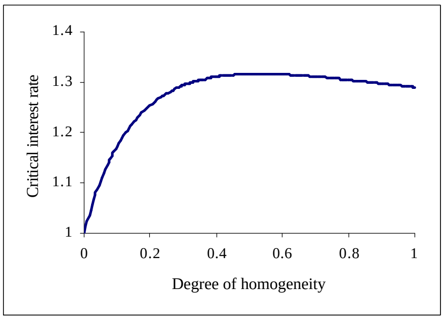

Fig 8: Critical interest rate as a function of the degree of homogeneity for a cartel of size 5 in an industry with 7 …rms when a trigger strategy is implemented.

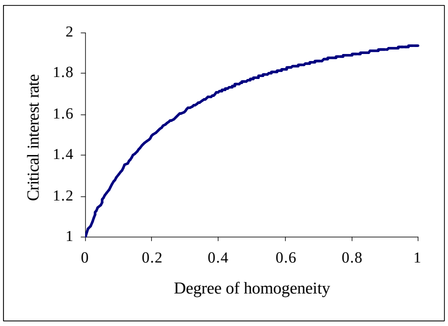

Fig 9: Critical interest rate as a function of the degree of homogeneity for a cartel of size 4 in an industry with 7 …rms when a trigger strategy is implemented.

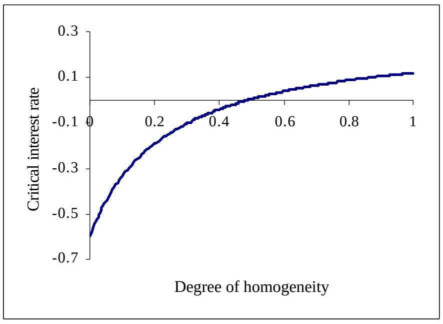

Fig 10: Critical interest rate as a function of the degree of homogeneity for a cartel of size 6 in an industry with 12 …rms when the cartel keeps acting as a cartel once a member has de‡ected.

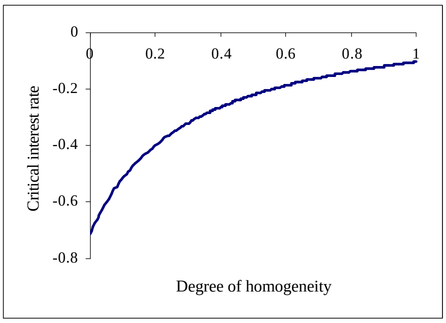

Fig 11: Critical interest rate as a function of the degree of homogeneity for a cartel of size 8 in an industry with 12 …rms when the cartel keeps acting as a cartel once a member has de‡ected.

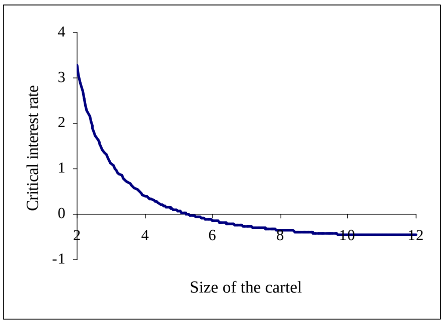

Fig 12: Critical interest rate as a function of the size of the cartel for an industry with 12 …rms, a degree of homogeneity of d=0.25 and when the cartel keeps acting as a cartel once a member has de‡ected.

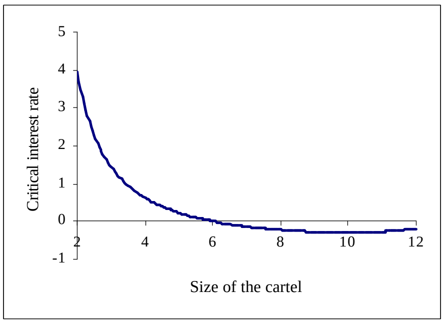

Fig 13: Critical interest rate as a function of the size of the cartel for an industry with 12 …rms, a degree of homogeneity of d=0.50 and when the cartel keeps acting as a cartel once a member has de‡ected.

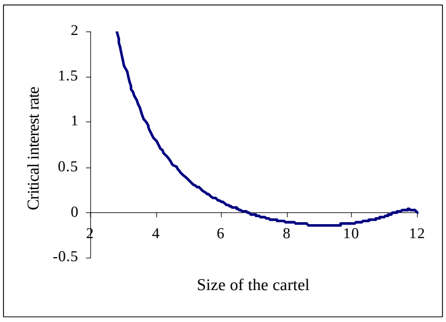

Fig 14: Critical interest rate as a function of the size of the cartel for an industry with 12 …rms, a degree of homogeneity of d=0.99 when the cartel keeps acting as a cartel once a member has de‡ected.

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