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Continuum Random Cluster and Potts models with Delaunay interactions

by

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Contents

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Declarations

I hereby declare that this thesis is a product of my own work and the ideas discussed with my supervisor, Dr Stefan Adams, unless specifically referenced or cited otherwise and it is in accordance with the university's guidelines on plagiarism. I also declare that it has not been submitted for any other degree or qualification at this or any other university.

Abstract

In this thesis we study a class of marked Gibbsian point processes with geometry dependent interactions known as Delaunay Potts models. We use a random cluster representation to show that a phase transition occurs in one such model for which the interactions depend on the geometry of the triangles which make up the Delaunay triangulation. The random cluster representation relates the finite volume Gibbs distribution to a hyperedge percolation model called the Delaunay random cluster model. We subsequently show that an infinite volume Delaunay random cluster model, as defined by the standard DLR formalism, exists when the potential satisfies two hard-core conditions and the edge weights are uniformly bounded away from 0 and 1.

Chapter 1

Introduction

The goal of equilibrium statistical mechanics is to explain the macroscopic behaviour of physical systems in thermodynamic equilibrium in terms of the interactions between their microscopic elements. A central topic is the phenomenon of *phase transitions*: the abrupt changing of a macroscopic property of a physical system as the result of a relevant parameter, such as temperature or porosity, passing a critical value. Examples of phase transitions include the transition of a real gas from liquid to vapour and the magnetisation of a ferromagnetic metal such as iron. In the case of a ferromagnet, magnetisation is lost once it is heated beyond a critical temperature, called the Curie temperature.

There are two types of mathematical models which we consider in this thesis - *spin models* and *percolation models.* Although we are mainly interested in continuum models, we first discuss lattice models in which particle positions are fixed as this is where some key ideas and concepts originate. The first model of a system of locally interacting particles in which a phase transition was shown to occur is a spin model known the Ising model. The Ising model is arguably the most famous model in statistical mechanics, it is a relatively simple model but its theory is simultaneously incredibly rich. It was introduced in 1920 by Wilhelm Lenz [Len20] with the hope of obtaining an understanding of ferromagnetic behaviour and was studied by his student Ising in his PhD thesis. In the Ising model each vertex of the lattice $\mathbb{Z}^d$ is envisaged as a particle and randomly assigned a spin value of -1 or 1 representing its magnetic moment, with interactions between particles which favour the agreement of neighbouring spins. In dimensions $d \geq 2$ there is a critical temperature below which the interactions are strong enough that the magnetic moments can align and one of the two spin types dominates the other. This behaviour is called *spontaneous magnetisation*, and the phase transition is often called an *order-disorder* transition.

The q-state Potts model for $q \in \mathbb{N}$ is a generalisation of the Ising model where the spin value assigned to a particle (now called its 'type') can take any value in the set $\{2, ..., q\}$, with $q = 2$ corresponding to the Ising model. This model exhibits a similar break of symmetry as the Ising model; there is a critical temperature below which particles of one type dominate the others. The proof of this relies on the relationship between the Potts model and a lattice percolation model known as the random-cluster model, which we shall discuss shortly.

The first model for percolation in a disordered medium was introduced in 1957 by Broadbent and Hammersley [BH57]. The question they posed was the following: "If a lump of porous material is put in a bucket of water, will the interior of the lump get wet and, if so, to what extent?" The material was visualised as a network of interconnecting pores, only some of which are large enough to allow the passage of water. The situation was idealised by assuming that the structure of the pores is that of a cubic lattice: the pores (now called *bonds*) connect together points in $\mathbb{Z}^3$ (now called *sites*) which are unit distance apart from one another. Each bond, independently of all other bonds, is said to be *open* with some prescribed probability p, representing the situation in which it is large enough to allow the passage of water from one of its connection points to the other. With probability $1 - p$ each bond is too small and unable to transmit water; such bonds are termed *closed*. Of interest to us is the structure of the random subgraph generated by the removal of the closed bonds. The connected components of this subgraph are known as *open clusters*, and the large-scale penetration of the porous material by water is related to the existence of infinitely large open clusters.

This model can of course be generalised to the case where the sites belong to $\mathbb{Z}^d$ for any $d \in \mathbb{N}^+$. The resulting process is often referred to as *Bernoulli bond percolation on* $\mathbb{Z}^d$. A principal quantity of interest is the *percolation probability*, $\theta(p)$, which is the probability that the origin belongs to an infinite open cluster. The case where $d = 1$ is of no interest as $\theta(p) = 1$ if $p = 1$ and $\theta(p) = 0$ if $p < 1$. When $d \geq 2$, an interesting phase transition occurs: there is a non trivial *critical probability* $p_c(d) \in (0, 1)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$. This was shown by Broadbent and Hammersley [BH57] and Hammersley [Ham57, Ham59]. A corollary of this, shown using an appropriate zero-one law, is that above the critical probability there exists an infinite open cluster almost surely, and below the critical probability no such cluster exists. The famous calculation that $p_c(2) = 1/2$ is due to Kesten [Kes80], who also showed that there is no infinite open cluster in two dimensions case when $p = 1/2$. The exact values of $p_c(d)$ for $d \geq 3$ are currently unknown, with the fact that $p_c(1) = 1$ being a trivial calculation.

The random cluster model is a family of more complex bond percolation models on $\mathbb{Z}^d$ in which the states of the bonds are no longer independent of one another. It was introduced by Fortuin and Kasteleyn in [FK72] partially in an attempt to unify percolation with the Ising and Potts models. The model has two parameters, $p \in [0, 1]$ and $q > 0$. The former to some extent controls the density of open bonds and the latter impacts the number of open clusters. When $q < 1$ configurations with fewer clusters are favoured, and when $q > 1$ large numbers of clusters are favoured. When $q = 1$ the model coincides with Bernoulli bond percolation with parameter p. Fortuin and Kasteleyn discovered that there is a intimate relationship between the random cluster model with $q \in \{2, 3, ...\}$ and the Potts model with q states. For these values of q the correlation functions of the Potts model may be expressed as connectivity functions of the random cluster model. This relationship makes the random cluster model a useful tool for studying the Ising and Potts models using geometrical techniques originating in the study of percolation. The connection between these models is so strong that the random cluster model is sometimes referred to as the 'FK representation' of the Ising and Potts models. In [ACCN88], Aizenmann et al used the random cluster model to show that the Potts model exhibits a phase transition for all values of q. A modern approach to studying the relationship between these models is by constructing them on a common probability space using the Edwards-Sokal coupling, discovered in [ES88].

The nature of the phase transitions in these models is a subject of intense study, and many recent advances have been made. Of particular interest is whether the phase transitions are *sharp*, and whether they are *continuous*. For the Ising and Potts models, the phase transition is sharp if there is an exponential decay of correlations whenever the temperature is above critical temperature, and continuous if there is no spontaneous magnetisation when the temperature is precisely equal to the critical temperature. For the random cluster and Bernoulli bond percolation models, the phase transition is sharp if there is exponential decay in the radius of open clusters and continuous if there is no infinite open cluster when $p = p_c$. The sharpness of the phase transition for Bernoulli bond percolation for all d was proven by Aizenmann and Barkley [AB87] and Menshikov [Men86]. In [DCRT19] the authors used a novel approach utilising randomised algorithms to show that the phase transition in the random cluster and Potts models is sharp for $q \geq 1$. The critical probability for the random cluster model with $q \geq 1$ on $\mathbb{Z}^2$ was computed in [BDC12], and as a corollary the authors were able to compute the critical temperature for the Potts model on $\mathbb{Z}^2$. Even for Bernoulli bond percolation, little is known about what occurs at the critical probability. We have mentioned previously that there is no infinite open cluster in two dimensions when $p = p_c$ (so the phase transition is continuous.) The absence of an infinite cluster at criticality has been shown for $d \geq 19$ [HS90] and was recently improved to $d \geq 11$ [FvdH17]. It is conjectured that for all $d \geq 2$ there is no infinite open cluster at criticality. The phase transition for the Ising model is known to be continuous for all $d \geq 2$. The proof for $d \geq 3$ was accomplished using a random-current representation [ADCS15]. As for the random cluster model, it is known that in two dimensions the phase transition is continuous for $q \in \{2, 3, 4\}$ [DCST17] and discontinuous for $q \geq 5$ [DCGH$^+$16], and as a corollary the same is true for the q-state Potts model. It is conjectured that the phase transition in the Potts model is discontinuous whenever $d \geq 3$ and $q \geq 3$. Partial progress has been made here: it has been shown that for a fixed dimension, the phase transition is discontinuous for large enough q [KS82], and for fixed q, the phase transition is discontinuous for sufficiently large d [BC03].

Although we have thus far only discussed results for models on the cubic lattice $\mathbb{Z}^d$, it is possible to study statistical mechanics models on many graphs.

Common examples include complete graphs (resulting in *mean-field models*), trees, and non-amenable graphs, of which Cayley graphs are a particular example. There is also a plethora of other spin models such as $O(n)$ models, the discrete Gaussian free field and ϕ^4 models which we shall not discuss here.

Although the study of phase transitions is one of the main subjects of statistical mechanics, most models which are known to exhibit a phase transition are discrete like those discussed above. From here on we are interested in the existence of phase transitions in continuum particle systems, where the position of each particle is random. Notable examples of phase transitions in the continuous setting include a gas-liquid phase transition [LMP99] and the spontaneous breaking of rotational symmetry in a simple model of a two dimensional crystal without defects for small temperatures [MR09].

One approach to continuum particle systems is through the study of *germ-grain* models. A prominent model of this kind is the Widom-Rowlinson model, introduced by Widom and Rowlinson [WR70] as a model for the study of a liquid-vapour phase transition. In this model each particle has an associated radius and is modelled as a closed ball. There are two species of particles with deterministic radii and a hardcore exclusion interaction forcing particles of differing types to not overlap. This was the first continuum model for which a phase transition was rigorously proved. The proof was accomplished by Ruelle [Rue71] using a variant of Peierls' argument. More recently, this phase transition was proven using percolation arguments by finding a representation of the Widom-Rowlinson model analogous to the Fortuin-Kasteleyn representation of the Ising and Potts models [CCK95, GH96].

Recently, a generalised version of the Widom-Rowlinson model has been studied where there are $q \geq 2$ types of particle, with each particle of type i having a random radius following a distribution Q_i on $\mathbb{R}^+$. This can be thought of as a collection of q Poisson Boolean models which are conditioned to not overlap each other. The case where the distributions Q_i are the same is called the *symmetric* case. The existence of this model in the symmetric case with unbounded radii was shown in [DH15] by constructing an FK representation known as the *continuum random cluster model* which is analogous to the FK representation from [GH96] but for infinite volume. The continuum random cluster model was constructed as a Gibbsian modification of the Poisson Boolean model. Subsequently it was shown [DH19] that in the symmetric integrable case ($\int r^d Q_i(dr) < \infty$) the Widom-Rowlinson model exhibits a standard phase transition in which there is a unique phase when the activity parameter is small, and q ordered phases when the activity is large. In the same paper it was shown that in the nonintegrable case there exists at least $q + 1$ phases for small activities. The random cluster representation between the Widom-Rowlinson model and the continuum random cluster model plays a crucial role in the proof of the uniqueness result in the symmetric integrable case. In this case it allows insights to be gained from (disagreement-)percolation results which apply to the continuum random cluster model [Hou17, HTH19]. Another variant of the Widom-Rowlinson model in which the particles are permitted to overlap has been shown to exhibit a sharp phase transition [DH21] by utilising a randomised algorithm approach similar to that used for the lattice models in [DCRT19].

The results from [GH96] do not only apply to the Widom-Rowlinson model. In fact, the authors established phase transitions and FK representations for a class of models, termed continuum Potts models, in dimensions $d \geq 2$ with $q \geq 2$ different species of particle and finite range repulsive pair interactions between particles of different types. Their approach involves a random cluster representation analogous to the Edwards-Sokal coupling of the discrete Potts model and its Fortuin-Kasteleyn representation. The percolation transition in the FK representation was sufficient to prove that occurance of phase transitions. In [DDG12], the class of models considered was expanded and a framework for studying Gibbsian point processes whose interactions depend on the geometry of an underlying hypergraph was set up. This new class includes interactions which depend on Delaunay edges or triangles, cliques of Voronoi cells or clusters of k-nearest neighbours. Sufficient conditions for the existence of infinite volume measures were laid out. In [AE16] the authors used this framework to study a model in which the interaction is based on the geometry of the Delaunay graph with a hard-core background potential depending on the lengths of the Delaunay edges. Both edge and triangle-dependent interactions were considered, and percolation in an FK representation analogous to that used in [GH96] was used to show that a phase transition occurs. The percolation proof was accomplished using a coarse-graining method. In a continuation [AE19] the authors also obtained a phase transition result with no background potential and finite range interaction depending on the lengths of the Delaunay edges using the same method. Other models incorporating Delaunay triangulations can be found in [BBD99b, BBD99a, Der08, DG09].

In this thesis we continue the study of Gibbsian point processes with interactions based on the geometry of the Delaunay triangulation, which we refer to as Delaunay Potts models. In chapter 3 we use the same approach as in [AE16] and [AE19] to prove that a phase transition occurs in a model in which both the background interaction and the interaction between unlike particles depends on the geometry of the Delaunay triangles. The random cluster representation we use couples the finite volume Gibbs distribution in $\Delta \subset \mathbb{R}^2$ (which we call the Delaunay continuum Potts distribution in Δ) with the Delaunay random cluster distribution in Δ. The coupling is restricted to the case when the former has a boundary condition made up of particles of the same type, which corresponds to a Delaunay random cluster distribution with a 'wired' boundary condition in which all hyperedges sufficiently far away from Δ are open.

We have seen for the Widom-Rowlinson model that the extension of the random cluster representation to infinite volume facilitates the proof that the symmetric, integrable Widom-Rowlinson measures are unique for small activities. We take this as inspiration in chapter 4 and propose the study of a set of measures we call infinite volume Delaunay random cluster measures, which are infinite volume extensions of the Delaunay random cluster distributions, defined using the standard DLR formalism. We hope that ultimately it will be possible to uncover a random cluster representation connecting these Delaunay random cluster measures and the Gibbs measures arising from the Delaunay continuum Potts distributions, analogous to the representation we use in chapter 3, and that analysis of these random cluster measures will enable us to show that the Delaunay Potts measures are unique for small activities. We start by extending the definition of the finite volume Delaunay random cluster distributions to allow for arbitrary boundary conditions, which is non-trivial since it is possible that the number of infinite connected components is infinite. The rest of the chapter is dedicated to proving the existence of Delaunay random cluster measures.

This thesis will proceed as follows. In chapter 2 we introduce the necessary definitions and notation that will be used throughout, and in particular we introduce Delaunay continuum Potts measures and the finite volume Delaunay random cluster distributions. We recap the existence result from [DDG12] and the construction of the random cluster representation, which mostly follows [Eye14]. We then explore how the random cluster representation provides a connection between phase transitions and connectivity, before proving a short technical result regarding mixed site-bond percolation on $\mathbb{Z}^2$ and summarising the literature on Delaunay continuum Potts models. In chapter 3 we apply these results to a model in which the interactions depend on the geometry of the Delaunay triangles. We prove that Gibbs measures exist for large values of the activity parameter and that a phase transition occurs if both the activity and the parameter β, which controls the strength of the interaction, are sufficiently large. We then move on in chapter 4 and show that an infinite volume Delaunay random cluster measure exists if the background potential satisfies two hard-core conditions and the edge weights are uniformly bounded away from 0 and 1. Finally, in chapter 5 we discuss possibilities for future research.

Chapter 2

Preliminaries

2.1 Gibbsian point processes with geometry-dependent interactions

We begin with a presentation of the framework, introduced in [DDG12], for studying Gibbsian point processes with hypergraph interactions. The framework allows for the study of a wide class of interactions which depend on the local geometry of configurations. The interactions we are interested in fall into this class as they depend on the local geometry of the Delaunay triangulation associated to each configuration. Furthermore, we present the main result from [DDG12], which gives sufficient conditions under which a Gibbs measure exists.

2.1.1 Point configurations

We consider systems of particles in $\mathbb{R}^d$, both in the case where the particles are described by their spatial location only, and where the particles possess a *mark* describing their type or internal degrees of freedom. In the latter case the marks belong to a *mark space*, Q, and each marked point lies in the set $\mathbb{R}^d \times Q$. We shall focus on marked point processes with mark space $Q = [q] := \{1, ..., q\}$ for some $q \in \mathbb{N}$, which are broadly known as *Potts models*. It is worth noting however that the existence results of [DDG12] do in fact hold for any mark space which is standard Borel. The set of *point configurations* Ω is defined to be the set of locally finite subsets of $\mathbb{R}^d$, that is

$$\Omega := \left\{\omega \subset \mathbb{R}^d : |\omega \cap \Delta| < \infty \text{ for all } \Delta \Subset \mathbb{R}^d\right\}, \tag{2.1.1}$$

where $\Delta \Subset \mathbb{R}^d$ denotes the fact that Δ is a bounded Borel subset of $\mathbb{R}^d$. A *marked point configuration* is a subset $\boldsymbol{\omega} \subset \mathbb{R}^d \times [q]$ whose projection ω onto the spatial coordinate is locally finite. The set of marked point configurations is therefore

$$\boldsymbol{\Omega} := \left\{\boldsymbol{\omega} \subset \mathbb{R}^2 \times [q] : \omega \in \Omega\right\}. \tag{2.1.2}$$

Each $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ has an associated *mark function* $\sigma_\omega \in [q]^\omega$ which retrieves the mark of a point given its position, i.e $\sigma_\omega(x) = i$ if $(x,i) \in \boldsymbol{\omega}$. A marked configuration $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ can therefore be represented as a pair (ω, σ_ω).

For each Borel set $B \subset \mathbb{R}^d \times [q]$, the *counting variable* $N_B : \boldsymbol{\omega} \mapsto |\boldsymbol{\omega} \cap B|$ on $\boldsymbol{\Omega}$ gives the number of marked points in B. Similarly, for each Borel set $\Delta \subset \mathbb{R}^d$, we define the counting variable $N_\Delta : \omega \mapsto |\omega \cap \Delta|$ on Ω. The spaces Ω and $\boldsymbol{\Omega}$ are equipped with the σ-algebras $\mathcal{F} := \sigma(N_\Delta : \Delta \Subset \mathbb{R}^d)$ and $\boldsymbol{\mathcal{F}} := \sigma(N_B : B \Subset \mathbb{R}^d \times [q])$ respectively. $\boldsymbol{\mathcal{F}}$ is the Borel σ-algebra for the Polish topology which is generated by the variables $\boldsymbol{\omega} \mapsto \int g \mathrm{d}\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is interpreted as a counting measure on $\mathbb{R}^d \times [q]$ and $g : \mathbb{R}^d \times [q] \to \mathbb{R}$ is bounded and continuous with spatially bounded support (see [MKM78]). The analogous statement is true for $\mathcal{F}$, and so $(\Omega, \mathcal{F})$ and $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ are standard Borel spaces. The reference measures on the spaces $(\Omega, \mathcal{F})$ and $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ are the Poisson point processes Π^z and $\boldsymbol{\Pi}^z$ on $\mathbb{R}^d$ and $\mathbb{R}^d \times [q]$ with intensity measures $z\,\mathrm{Leb}$ and $z\,\mathrm{Leb} \otimes U$ respectively, where $z > 0$ is a parameter known as the *activity*, Leb denotes the Lebesgue measure on $\mathbb{R}^d$ and U denotes the uniform measure on $[q]$. In particular, for $B \Subset \mathbb{R}^d \times [q]$ and $n \in \mathbb{N}$ we have

$$\boldsymbol{\Pi}^z(N_B = n) = \frac{(z\,\mathrm{Leb} \otimes U(B))^n}{n!} e^{-z\,\mathrm{Leb} \otimes U(B)}.$$

Note that $\boldsymbol{\Pi}^z$ is also the law of the random variable $\boldsymbol{\omega}$ obtained by sampling an unmarked configuration ω according to Π^z and attaching marks to each point independently according to the distribution U (see for instance [Kin92]).

For each $\Delta \Subset \mathbb{R}^d$ we shall consider the set of marked point configurations which are located in Δ, namely $\boldsymbol{\Omega}_\Delta := \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : \boldsymbol{\omega} \subset \Delta \times [q]\}$. The sigma algebra of events that happen in Δ only is $\boldsymbol{\mathcal{F}}_\Delta := \mathbf{pr}_\Delta^{-1} \boldsymbol{\mathcal{F}}|_{\boldsymbol{\Omega}_\Delta}$, where $\boldsymbol{\mathcal{F}}|_{\boldsymbol{\Omega}_\Delta} := \{A \cap \boldsymbol{\Omega}_\Delta : A \in \boldsymbol{\mathcal{F}}\}$ is the trace sigma algebra and $\mathbf{pr}_\Delta : \boldsymbol{\omega} \to \boldsymbol{\omega}_\Delta := \boldsymbol{\omega} \cap (\Delta \times [q])$ denotes the projection onto $\boldsymbol{\Omega}_\Delta$. We take the reference measure on $(\boldsymbol{\Omega}_\Delta, \boldsymbol{\mathcal{F}}|_{\boldsymbol{\Omega}_\Delta})$ to be the pushforward measure $\boldsymbol{\Pi}^z_\Delta := \boldsymbol{\Pi}^z \circ \mathbf{pr}_\Delta^{-1}$. These objects are defined in a similar way for unmarked configurations: $\Omega_\Delta := \{\omega \in \Omega : \omega \subset \Delta\}$, $\mathrm{pr}_\Delta : \omega \to \omega_\Delta := \omega \cap \Delta$, and $\mathcal{F}_\Delta := \mathrm{pr}_\Delta^{-1} \mathcal{F}|_{\Omega_\Delta}$. The reference measure on Ω_Δ is $\Pi^z_\Delta := \Pi^z \circ \mathrm{pr}_\Delta^{-1}$. If Δ belongs to the set of open cubes,

$$\mathscr{C} := \left\{ \prod_{i=1}^{d} (x_i, x_i + p) : x = (x_1, ..., x_d) \in \mathbb{R}^d, p > 0 \right\}, \tag{2.1.3}$$

then Ω_Δ and $\boldsymbol{\Omega}_\Delta$ are G_δ-sets and are therefore Polish spaces when equipped with their subspace topologies. In this case, $\mathcal{F}|_{\Omega_\Delta}$ and $\boldsymbol{\mathcal{F}}|_{\boldsymbol{\Omega}_\Delta}$ are their associated Borel σ-algebras. The set of marked and unmarked configurations with finitely many points are denoted $\boldsymbol{\Omega}_f$ and Ω_f, with sigma algebras $\mathcal{F}|_{\Omega_f}$ and $\boldsymbol{\mathcal{F}}|_{\boldsymbol{\Omega}_f}$ respectively.

From now on we shall focus mostly on marked configurations, but the concepts we introduce have natural analogues when the particles do not carry marks.

2.1.2 Hypergraphs and hyperedge interactions

The interaction between points considered here will depend on the geometry of their location, which is described in terms of a hypergraph. The interaction potential, which describes the strength of the interaction between points, will be defined on the hyperedges of the hypergraph. We now restrict our exposition to the case where we have marked point configurations, but all of the concepts described here can be defined analogously for unmarked point configurations.

Definition 2.1.1. A *hypergraph structure* is a measurable subset $\boldsymbol{\mathcal{H}}$ of $\boldsymbol{\Omega}_f \times \boldsymbol{\Omega}$ (i.e $\boldsymbol{\mathcal{H}} \in \boldsymbol{\mathcal{F}}|_{\boldsymbol{\Omega}_f} \otimes \boldsymbol{\mathcal{F}}$) such that $\boldsymbol{\tau} \subset \boldsymbol{\omega}$ for all $(\boldsymbol{\tau}, \boldsymbol{\omega}) \in \boldsymbol{\mathcal{H}}$. A hypergraph structure $\boldsymbol{\mathcal{H}}$ gives us a way of assigning a hypergraph to each configuration $\boldsymbol{\omega}$. The pair $(\boldsymbol{\omega}, \boldsymbol{\mathcal{H}}(\boldsymbol{\omega}))$ is called a *hypergraph* where $\boldsymbol{\omega}$ is the set of *vertices* and $\boldsymbol{\mathcal{H}}(\boldsymbol{\omega}) := \{\boldsymbol{\tau} \in \boldsymbol{\Omega}_f : (\boldsymbol{\tau}, \boldsymbol{\omega}) \in \boldsymbol{\mathcal{H}}\}$ is the set of *hyperedges*.

A *hyperedge potential* is a measurable function φ from a hypergraph structure $\boldsymbol{\mathcal{H}}$ to $\mathbb{R} \cup \{\infty\}$.

We only consider hyperedge potentials which are shift-invariant and exhibit the finite-horizon property, defined below. Shift-invariance requires that $\boldsymbol{\mathcal{H}}$ and φ are not affected by translations and the finite-horizon property requires that the value $\varphi(\boldsymbol{\tau}, \boldsymbol{\omega})$ can be determined by looking at $\boldsymbol{\omega}$ in a bounded neighbourhood of the hyperedge $\boldsymbol{\tau}$ (although the size of this neighbourhood depends on $\boldsymbol{\tau}$ and $\boldsymbol{\omega}$.)

Definition 2.1.2. Let $\varphi : \boldsymbol{\mathcal{H}} \to \mathbb{R} \cup \{\infty\}$ be a hyperedge potential and $\boldsymbol{\theta}_x : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$ denote the function which translates the location of points by the vector $-x$. Then the pair $(\boldsymbol{\mathcal{H}}, \varphi)$ is called *shift-invariant* if

$$(\boldsymbol{\theta}_x\boldsymbol{\tau}, \boldsymbol{\theta}_x\boldsymbol{\omega}) \in \boldsymbol{\mathcal{H}} \text{ and } \varphi(\boldsymbol{\theta}_x\boldsymbol{\tau}, \boldsymbol{\theta}_x\boldsymbol{\omega}) = \varphi(\boldsymbol{\tau}, \boldsymbol{\omega})$$

for all $(\boldsymbol{\tau}, \boldsymbol{\omega}) \in \boldsymbol{\mathcal{H}}$ and $x \in \mathbb{R}^d$.

Furthermore, the pair $(\boldsymbol{\mathcal{H}}, \varphi)$ is said to satisfy the *finite-horizon property* if for each hyperedge $(\boldsymbol{\tau}, \boldsymbol{\omega}) \in \boldsymbol{\mathcal{H}}$ there exists a set $\Delta \Subset \mathbb{R}^2$ called a *horizon* such that:

$$(\boldsymbol{\tau}, \boldsymbol{\omega}') \in \boldsymbol{\mathcal{H}} \text{ and } \varphi(\boldsymbol{\tau}, \boldsymbol{\omega}') = \varphi(\boldsymbol{\tau}, \boldsymbol{\omega}) \text{ when } \boldsymbol{\omega}' = \boldsymbol{\omega} \text{ on } \Delta. \tag{2.1.4}$$

2.1.3 Gibbs measures

Before defining the concept of a Gibbs measure in the context of a hyperedge potential on a hypergraph structure, we need to introduce the Hamiltonian for a fixed region $\Delta \Subset \mathbb{R}^2$ with boundary condition $\boldsymbol{\omega}$. The Hamiltonian is defined as a sum over the hyperedges for which either $\boldsymbol{\tau}$ itself or $\varphi(\boldsymbol{\tau}, \boldsymbol{\omega})$ depends on $\boldsymbol{\omega}_\Delta$. We extend the domain of φ to $\boldsymbol{\Omega}_f \times \boldsymbol{\Omega}$ by adopting the convention that $\varphi \equiv 0$ on $(\boldsymbol{\Omega}_f \times \boldsymbol{\Omega}) \setminus \boldsymbol{\mathcal{H}}$ and define

$$\boldsymbol{\mathcal{H}}_\Delta(\boldsymbol{\omega}) := \{\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}(\boldsymbol{\omega}) : \varphi(\boldsymbol{\tau}, \boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c}) \neq \varphi(\boldsymbol{\tau}, \boldsymbol{\omega}) \text{ for some } \boldsymbol{\omega}' \in \boldsymbol{\Omega}_\Delta\}. \tag{2.1.5}$$

The *Hamiltonian in* Δ *with boundary condition* $\boldsymbol{\omega}$ is given by the formula

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') := \sum_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{H}}_\Delta(\boldsymbol{\omega}'\cup\boldsymbol{\omega}_{\Delta^c})} \varphi(\boldsymbol{\tau}, \boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c}) \tag{2.1.6}$$

for $\boldsymbol{\omega}' \in \boldsymbol{\Omega}_\Delta$, provided it is well-defined, and

$$Z_\Delta^z(\boldsymbol{\omega}) := \int e^{-H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}')}\boldsymbol{\Pi}_\Delta^z(\mathrm{d}\boldsymbol{\omega}')$$

is the *partition function.* In order to define the finite volume Gibbs distribution in $\Delta \Subset \mathbb{R}^d$ we require that the Hamiltonian is well defined and the partition function is finite and non-zero. A configuration is *admissible for* $\Delta \Subset \mathbb{R}^d$, $z > 0$ *and* φ if $Z_\Delta^z(\boldsymbol{\omega}) \in (0, \infty)$ and

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') := \sum_{\boldsymbol{\tau}\in\boldsymbol{\mathcal{H}}_\Delta(\boldsymbol{\omega}'\cup\boldsymbol{\omega}_{\Delta^c})} \varphi^-(\boldsymbol{\tau}, \boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c}) < \infty \tag{2.1.7}$$

for $\boldsymbol{\Pi}_\Delta^z$-almost all $\boldsymbol{\omega}'$, where $\varphi^- := (-\varphi) \vee 0$ is the negative part of φ. The set of such configurations is denoted $\boldsymbol{\Omega}_{\Delta,z}^\varphi$. If $z' \geq z$ then $\boldsymbol{\Omega}_{\Delta,z'}^\varphi \subset \boldsymbol{\Omega}_{\Delta,z}^\varphi$ since $\boldsymbol{\Pi}_\Delta^{z'}$ and $\boldsymbol{\Pi}_\Delta^z$ are mutually absolutely continuous with

$$\frac{\mathrm{d}\boldsymbol{\Pi}_\Delta^{z'}(\mathrm{d}\boldsymbol{\omega}')}{\mathrm{d}\boldsymbol{\Pi}_\Delta^z(\mathrm{d}\boldsymbol{\omega}')} \propto (z'z^{-1})^{|\omega'|} \in [1, \infty).$$

The *finite-volume Gibbs distribution* (alternatively the *finite-volume geometric continuum Potts distribution*) in $\Delta \Subset \mathbb{R}^d$ for φ, z and admissible boundary condition $\boldsymbol{\omega} \in \boldsymbol{\Omega}_{\Delta,z}^\varphi$ is the probability measure on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ defined by

$$\gamma_{\Delta,\boldsymbol{\omega}}^z(A) := \frac{1}{Z_\Delta^z(\boldsymbol{\omega})} \int_{\boldsymbol{\Omega}_\Delta} \mathbb{1}_A(\boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c}) e^{-H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}')}\ \boldsymbol{\Pi}_\Delta^z(\mathrm{d}\boldsymbol{\omega}') \tag{2.1.8}$$

for $A \in \boldsymbol{\mathcal{F}}$. Gibbs measures are defined by prescribing the conditional distributions when the configuration outside a region $\Delta \Subset \mathbb{R}^d$ is known:

Definition 2.1.3. A probability measure P on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ is called a *Gibbs measure* (alternatively a *geometric continuum Potts measure*) for $\boldsymbol{\mathcal{H}}, \varphi$ and z if for every $\Delta \Subset \mathbb{R}^d$,

- $P(\boldsymbol{\Omega}_{\Delta,z}^\varphi) = 1$, and

· For every measurable function $f : \mathbf{\Omega} \to [0, \infty)$,

$$P(f) = \int \frac{1}{Z_\Delta^z(\boldsymbol{\omega})} \int_{\mathbf{\Omega}_\Delta} f(\boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c}) e^{-H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}')} \, \mathbf{\Pi}_\Delta^z(\mathrm{d}\boldsymbol{\omega}') P(\mathrm{d}\boldsymbol{\omega}). \quad (2.1.9)$$

The equations (2.1.9) are known as the *DLR equations*, after Dobrushin, Lanford and Ruelle. They express that $\gamma^z_{\Delta,\boldsymbol{\omega}}(\cdot)$ is a version of the conditional probability $P(\cdot|\mathcal{F}_{\Delta^c})(\boldsymbol{\omega})$. We are particularly interested in Gibbs measures which are invariant under the translation group $\mathbf{\Theta} := (\boldsymbol{\theta}_x)_{x \in \mathbb{R}^d}$, the set of which is denoted $\mathscr{G}_{\mathbf{\Theta}}(\varphi, z)$. The main problem we investigate here is the existence of *phase transitions* - the existence of multiple distinct translation-invariant Gibbs measures. We first must address the problem of proving that $\mathscr{G}_{\mathbf{\Theta}}(\varphi, z)$ is non-empty.

Remark 2.1.4. The quantities we have introduced thus far are not in fact measurable with respect to the σ-algebras we have defined, but are measurable with respect to their universal completions. We therefore identify all sigma algebras considered here with their universal completions and all probability measures with their complete extensions. For more details regarding this see [DDG12, Remark 2.1 and Appendix]. The proofs can be easily adapted to the case of marked particles.

2.1.4 Pseudo-periodic configurations

Here we introduce the pseudo-periodic configurations. They will play a crucial role in our forthcoming arguments.

Definition 2.1.5. Let $M \in \mathbb{R}^{d \times d}$ be an invertible $d \times d$ matrix and define for $k \in \mathbb{Z}^d$ the cells

$$C(k) := \left\{ Mx \in \mathbb{R}^d : x - k \in [-1/2, 1/2)^2 \right\}, \quad (2.1.10)$$

which form a partition of $\mathbb{R}^d$ into parallelotopes. We write $C = C(0)$ and let $\mathbf{\Gamma}$ be a measurable subset of $\mathbf{\Omega}_C \setminus \{\emptyset\}$. The configurations that belong to the set

$$\overline{\mathbf{\Gamma}} := \{\boldsymbol{\omega} \in \mathbf{\Omega} : \theta_{Mk}(\boldsymbol{\omega}_{C(k)}) \in \mathbf{\Gamma} \text{ for all } k \in \mathbb{Z}^d\}. \quad (2.1.11)$$

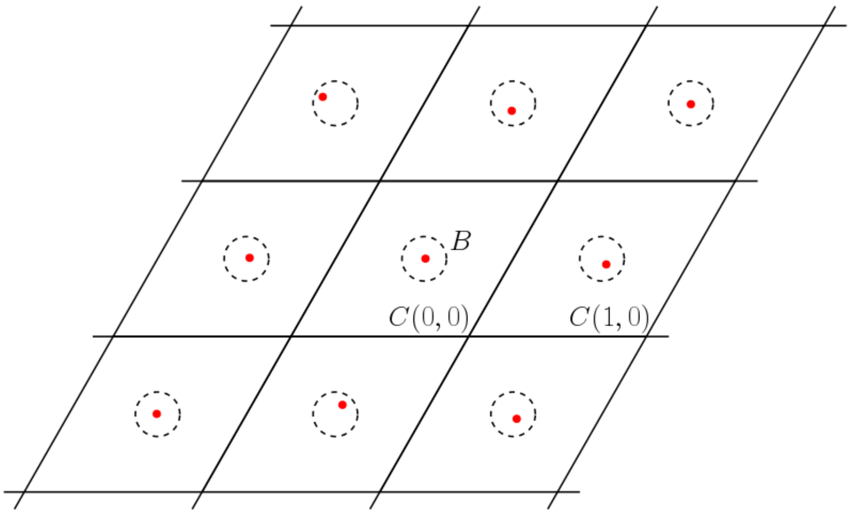


Figure 2.1: An example of a pseudo periodic configuration $\boldsymbol{\omega} \in \overline{\boldsymbol{\Gamma}}$ where $d = 2, \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^B := \{\boldsymbol{\omega} \in \boldsymbol{\Omega}_C : \omega = \{x\}$ for some $x \in B\}$ and B is a ball centred at $(0, 0)$.

are called *pseudo-periodic.*

2.1.5 Existence

Here we present a sufficient criteria under which a translation-invariant Gibbs measure can be shown to exist. We begin by defining a set of configurations for which the Hamiltonian is localised in the sense that dependence of $H_{\Delta,\boldsymbol{\omega}}$ on $\boldsymbol{\omega}$ is restricted to the points of $\boldsymbol{\omega}$ which lie in a bounded region.

Definition 2.1.6. Let $\Delta \Subset \mathbb{R}^d$. A configuration $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ is said to *confine the range of* φ *from* Δ if there exists a set $\partial\Delta(\boldsymbol{\omega}) \Subset \mathbb{R}^d$ such that $\varphi(\boldsymbol{\tau}, \boldsymbol{\zeta} \cup \boldsymbol{\omega}_{\Delta^c}) = \varphi(\boldsymbol{\tau}, \boldsymbol{\zeta} \cup \boldsymbol{\omega}'_{\Delta^c})$ whenever $\boldsymbol{\omega} = \boldsymbol{\omega}'$ on $\partial\Delta(\boldsymbol{\omega}), \boldsymbol{\zeta} \in \boldsymbol{\Omega}_\Delta$ and $\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}_\Delta(\boldsymbol{\zeta} \cup \boldsymbol{\omega}_{\Delta^c})$. If this is true we write $\boldsymbol{\omega} \in \boldsymbol{\Omega}^{\text{cr}}_\Delta$. We use the abbreviation $\partial_\Delta\boldsymbol{\omega} = \boldsymbol{\omega}_{\partial\Delta(\boldsymbol{\omega})}$. For $\boldsymbol{\omega} \in \boldsymbol{\Omega}^{\text{cr}}_\Delta$, from now on we will choose to take $\partial\Delta(\boldsymbol{\omega}) = \Delta \oplus r \setminus \Delta$, where $\Delta \oplus r := \{x + y \in \mathbb{R}^d : x \in \Delta, y \in \mathbb{R}^d$ and $|y| \leq r\}$ and $r = r_{\Delta,\boldsymbol{\omega}}$ is chosen to be as small as possible.

Clearly if $\boldsymbol{\omega} \in \boldsymbol{\Omega}^{\text{cr}}_\Delta$ then

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') = \sum_{\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}_\Delta(\boldsymbol{\omega}' \cup \partial_\Delta \boldsymbol{\omega})} \varphi(\boldsymbol{\tau}, \boldsymbol{\omega}' \cup \partial_\Delta \boldsymbol{\omega}), \tag{2.1.12}$$

which is a finite sum.

Our first condition is the range condition. It ensures that the Hamiltonian is localised in the above sense almost surely. More concretely, if the range condition is satisfied and P is a translation-invariant probability measure on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ with finite intensity such that $P(\emptyset) = 0$, then $P(\boldsymbol{\Omega}_\Delta^{\mathrm{cr}}) = 1$ (see [DDG12, Proposition 3.1] for the unmarked case. The same proof is applicable here.) Therefore if $\boldsymbol{\omega} \in \boldsymbol{\Omega}_\Delta^{\mathrm{cr}}$ then the second half of the definition of admissibility (2.1.7) is satisfied.

(R) *The range condition.* There exist constants $l_R, n_R \in \mathbb{N}$ and $\delta_R < \infty$ such that for all $(\boldsymbol{\tau}, \boldsymbol{\omega}) \in \boldsymbol{\mathcal{H}}$ one can find a horizon (as in (2.1.4)) Δ for φ satisfying the following: For every $x, y \in \Delta$ there exist l open balls $B_1, ..., B_l$ (with $l \leq l_R$) such that

- $\cup_{i=1}^{l} \bar{B}_i$ is connected and contains x and y, and
- for each i, either diam $B_i \leq \delta_R$ or $N_{B_i}(\omega) \leq n_R$.

The second condition is stability, which ensures that the partition function is finite.

(S) *Stability.* There exists a constant $c_S \geq 0$ such that

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') \geq -c_S |\boldsymbol{\omega}' \cup \partial_\Delta \boldsymbol{\omega}|$$

for all $\Delta \Subset \mathbb{R}^d, \boldsymbol{\omega}' \in \boldsymbol{\Omega}_\Delta$ and $\boldsymbol{\omega} \in \boldsymbol{\Omega}_\Delta^{\mathrm{cr}}$.

The third condition is upper regularity, which is split into three parts: uniform confinement, uniform summability and strong non-rigidity. Uniform confinement states that the configurations in $\overline{\boldsymbol{\Gamma}}$ confine the range of φ in a uniform way, uniform summability ensures that the local Hamiltonians $H_{\Delta,(\cdot)}$ when restricted to $\overline{\boldsymbol{\Gamma}}$, admit an upper bound which scales appropriately with $|\Delta|$, and strong non-rigidity ensures that the pseudo-periodic configurations appear with enough weight to counterbalance the interactions.

(U) *Upper Regularity.* M and $\boldsymbol{\Gamma}$ can be chosen so that the following conditions hold.

(U1) *Uniform confinement:* $\mathbf{\Gamma} \subset \mathbf{\Omega}^{\mathrm{cr}}_{\Delta}$ for all $\Delta \Subset \mathbb{R}^d$, and

$$r_{\mathbf{\Gamma}} := \sup_{\Delta \Subset \mathbb{R}^d} \sup_{\boldsymbol{\omega} \in \overline{\mathbf{\Gamma}}} r_{\Delta,\boldsymbol{\omega}} < \infty.$$

(U2) *Uniform summability:*

$$c^+_{\mathbf{\Gamma}} := \sup_{\boldsymbol{\omega} \in \overline{\mathbf{\Gamma}}} \sum_{\substack{\boldsymbol{\tau} \in \boldsymbol{\mathcal{H}}(\boldsymbol{\omega}): \\ \tau \cap C \neq \emptyset}} \frac{\varphi^+(\boldsymbol{\tau}, \boldsymbol{\omega})}{|\tilde{\boldsymbol{\tau}}|} < \infty,$$

where $\tilde{\boldsymbol{\tau}} := \{k \in \mathbb{Z}^d : \tau \cap C(k) \neq \emptyset\}$ and $\varphi^+ := \max(\varphi, 0)$.

(U3) *Strong non-rigidity:*

$$e^{z|C|}\mathbf{\Pi}^z_C(\mathbf{\Gamma}) > e^{c_{\mathbf{\Gamma}}},$$

where $c_{\mathbf{\Gamma}}$ is defined as in **(U2)** except with φ replacing φ^+.

Now we come to the existence theorem, which is an extension of [DDG12, Theorem 3.2] to allow for marked particles. For the extended proof see [Nol13, Theorem 2.1], but it is essentially the same as the original.

Theorem 2.1.7. (Existence). *Let $\varphi : \boldsymbol{\mathcal{H}} \to \mathbb{R} \cup \{\infty\}$ be a hyperedge potential and $z > 0$. If the hypotheses **(R)**, **(S)** and **(U)** are satisfied then there exists a translation-invariant Gibbs measure $P \in \mathscr{G}_{\mathbf{\Theta}}(\varphi, z)$.*

Remark 2.1.8. During the course of the proof it is shown that if **(R)**, **(S)** are satisfied and **(U)** holds for M and $\mathbf{\Gamma}$ then

$$\left\{ \boldsymbol{\omega} \in \overline{\mathbf{\Gamma}} : \sup_{k \in \mathbb{R}^d} |\boldsymbol{\omega}_{C(k)}| < \infty \right\} \subset \mathbf{\Omega}^{\varphi}_{\Lambda_n, z}$$

for all $n \in \mathbb{N}$ and $z > 0$, where $\Lambda_n := \bigcup_{k \in \{-n,..,n\}^d} C(k)$.

2.2 Delaunay continuum Potts measures

We are interested in two-dimensional ($d = 2$) Gibbsian point processes whose interactions depend on the local geometry of the Delaunay triangulation. More explicitly, we want to investigate the structure of the set of geometric continuum Potts measures when the hypergraph under consideration is based on the Delaunay triangulation.

If $\tau = \{x, y, z\} \subset \mathbb{R}^2$ is non-collinear, then $B(\tau)$ denotes the *circumball* of τ, which is the unique open ball whose boundary has τ as a subset. Its boundary $\partial B(\tau)$ is called the *circumcircle* of τ and its radius, called the *circumradius* of τ, is denoted $\delta(\tau)$. The following hypergraph structures (where $i \in \{1, 2\}$ and $\Delta \subset \mathbb{R}^2$) are of interest to us:

$$\begin{aligned}
\mathbf{Del}_3 &:= \left\{ (\boldsymbol{\tau}, \boldsymbol{\omega}) \in \boldsymbol{\Omega}_f \times \boldsymbol{\Omega} \;\middle|\; \begin{array}{c} \boldsymbol{\tau} \subset \boldsymbol{\omega}, |\tau| = 3, \tau \text{ non-collinear,} \\ \partial B(\tau) \cap \omega = \tau, \text{ and } B(\tau) \cap \omega = \emptyset. \end{array} \right\}, \\
\mathbf{Del}_{3,\Delta} &:= \left\{ (\boldsymbol{\tau}, \boldsymbol{\omega}) \in \mathbf{Del}_3 \;\middle|\; \overline{B(\tau)} \cap \Delta \neq \emptyset \right\}, \\
\mathbf{Del}_i &:= \{ (\boldsymbol{\eta}, \boldsymbol{\omega}) \in \boldsymbol{\Omega}_f \times \boldsymbol{\Omega} \mid |\eta| = i \text{ and } \exists \boldsymbol{\tau} \in \mathbf{Del}_3(\boldsymbol{\omega}) \text{ such that } \boldsymbol{\eta} \subset \boldsymbol{\tau} \}, \\
\mathbf{Del}_{i,\Delta} &:= \left\{ (\boldsymbol{\eta}, \boldsymbol{\omega}) \in \mathbf{Del}_i \;\middle|\; \begin{array}{c} \exists \tau \in \mathrm{Del}_3(\omega) \text{ such that} \\ \eta \subset \tau \text{ and } \overline{B(\tau)} \cap \Delta \neq \emptyset. \end{array} \right\}.
\end{aligned}$$

We denote the unmarked analogues $\mathrm{Del}_3, \mathrm{Del}_{3,\Delta}, \mathrm{Del}_i$ and $\mathrm{Del}_{i,\Delta}$. We will often refer to the elements of $\mathrm{Del}_2(\omega)$ and $\mathbf{Del}_2(\boldsymbol{\omega})$ as *edges* and those of $\mathrm{Del}_3(\omega)$ and $\mathbf{Del}_3(\boldsymbol{\omega})$ as *triangles* or *tiles*. Edges and triangles will typically be denoted by $\boldsymbol{\eta}$ and $\boldsymbol{\tau}$ respectively.

From now on we assume that the point locations ω are in *general quadratic position*, which is to say that

(a) no 3 points lie on a single line,

(b) no 4 points lie on the boundary of a circle.

If in addition to (a) and (b) every half plane contains at least one point, then the elements of ω, $\mathrm{Del}_2(\omega)$ and $\mathrm{Del}_3(\omega)$ form a triangulation of the whole plane, called the *Delaunay triangulation*. By this we mean that the set containing the elements of ω and the convex hulls of the elements of $\mathrm{Del}_2(\omega)$ and $\mathrm{Del}_3(\omega)$ is a simplicial complex covering the whole plane. These conditions are satisfied almost surely with respect to the reference measure Π^z [Ml94, Proposition 4.1.2]. Furthermore, the sets $\mathrm{Del}_{i,\Delta}(\omega)$ for $i = 1, 2, 3$ are almost surely finite (Proposition A.0.1).

The set $\mathbf{Del}_{3,\Delta}(\boldsymbol{\omega})$ contains the triangles in the Delaunay triangulation whose circumcircle intersects Δ. These are the triangles which can be removed

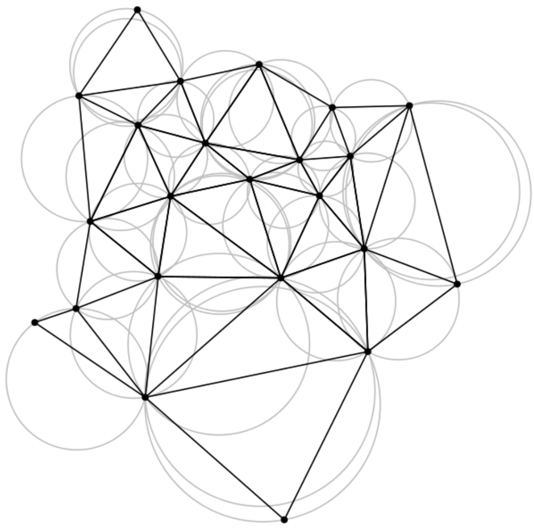


Figure 2.2: A Delaunay triangulation. The grey circles represent the open balls $B(\tau, \omega)$.

from the triangulation by changing the configuration inside Δ, that is to say

$$\boldsymbol{\tau} \in \mathbf{Del}_{3,\Delta}(\boldsymbol{\omega}) \iff \boldsymbol{\tau} \in \mathbf{Del}_3(\boldsymbol{\omega}) \text{ and } \exists \boldsymbol{\omega}' \in \boldsymbol{\Omega}_\Delta \text{ s.t } \boldsymbol{\tau} \notin \mathbf{Del}_3(\boldsymbol{\omega}_{\Delta^c} \cup \boldsymbol{\omega}'). \tag{2.2.1}$$

To see this, notice that if $\boldsymbol{\tau} \in \mathbf{Del}_{3,\Delta}(\boldsymbol{\omega})$ and $\boldsymbol{x} \in (\Delta \cap B(\tau)) \times [q]$ then $\boldsymbol{\tau} \notin \mathbf{Del}_3(\boldsymbol{\omega} \cup \boldsymbol{x})$. On the other hand, if $\boldsymbol{\tau}$ belongs to the right hand side then $\boldsymbol{\tau} \in \mathbf{Del}_3(\boldsymbol{\omega})$ and either $\tau \cap \omega_\Delta \neq \emptyset$ or $\exists \boldsymbol{\omega}' \in \boldsymbol{\Omega}_\Delta$ such that $\overline{B(\tau)} \cap \omega' \neq \emptyset$. In both cases $\overline{B(\tau)} \cap \Delta \neq \emptyset$, and so $\boldsymbol{\tau} \in \mathbf{Del}_{3,\Delta}(\boldsymbol{\omega})$.

Definition 2.2.1. If $\boldsymbol{\mathcal{H}} \in \{\mathbf{Del}_2, \mathbf{Del}_3, \mathbf{Del}_2 \cup \mathbf{Del}_3\}$ then the finite-volume geometric continuum Potts distribution in Δ (2.1.8) is known as the *Delaunay continuum Potts distribution in* Δ and geometric continuum Potts measures (Definition 2.1.3) are known as *Delaunay continuum Potts measures.*

If $\boldsymbol{\mathcal{H}} = \mathbf{Del}_3$ and φ is a hypergraph potential on $\mathbf{Del}_3$ of the form

$$\varphi(\boldsymbol{\tau}, \boldsymbol{\omega}) = \varphi'(\boldsymbol{\tau}) \tag{2.2.2}$$

for some measurable function $\varphi' : \{\boldsymbol{\tau} \in \boldsymbol{\Omega}_f : |\boldsymbol{\tau}| = 3\} \to \mathbb{R} \cup \{\infty\}$ then by

(2.2.1),

$$\boldsymbol{\mathcal{H}}_\Delta(\boldsymbol{\omega}) \setminus \{\boldsymbol{\tau} : \varphi'(\boldsymbol{\tau}) = 0\} = \mathbf{Del}_{3,\Delta}(\boldsymbol{\omega}) \setminus \{\boldsymbol{\tau} : \varphi'(\boldsymbol{\tau}) = 0\}, \tag{2.2.3}$$

and therefore

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') = \sum_{\boldsymbol{\tau}\in\mathbf{Del}_{3,\Delta}(\boldsymbol{\omega}'\cup\boldsymbol{\omega}_{\Delta^c})} \varphi'(\boldsymbol{\tau}), \tag{2.2.4}$$

where $H_{\Delta,\boldsymbol{\omega}}$ is the Hamiltonian in Δ with boundary condition $\boldsymbol{\omega}$ defined in (2.1.6). $H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}')$ is therefore a finite sum, so the first criterion of admissibility, (2.1.7), is always satisfied.

2.3 The random cluster representation

The purpose of this section is to obtain a joint construction of the finite-volume Gibbs distribution and a related continuum hyperedge percolation model called the *geometric continuum random cluster model.* The construction is analogous to the joint construction of the discrete Potts model and its Fortuin-Kastelyn representation. This representation will allow us to analyse the phase transition behaviour of geometric continuum Potts models, much in the same way as the Fortuin-Kastelyn random cluster model has proven to be of great value in analysing the phase transition behaviour of the Ising and Potts models. Random cluster representations for continuum particle systems were first discovered independently by Georgii and Häggström [GH96] and Chayes et al [CCK95]. Chayes et al considered a hard-core repulsion between particles of different type, whereas Georgii and Häggström considered finite range repulsive pair interactions more generally. These representations were generalised in [Eye14] to Gibbsian point processes with hypergraph interactions, although the author's definition of $\boldsymbol{\mathcal{H}}_\Delta$ differs to the one used here and in [DDG12]. Phase transitions for Potts models on Delaunay graphs with a variety of interactions have been shown in [AE16] and [AE19] using this representation. We present a similar representation here although we only consider the hypergraph $\mathbf{Del}_3$.

2.3.1 Hyperedge percolation models

Hyperedge percolation models are created by taking random unmarked point configurations and declaring hyperedges of the associated hypergraph to be either 'open' or 'closed' according to some hyperedge process. A notion of connectedness is obtained by declaring that two points are connected if one can travel between them via open hyperedges. We shall limit ourselves to unmarked hypergraphs for which each hyperedge contains the same number of points, that is $\exists k \in \mathbb{N}$ such that $|\tau| = k$ for all $(\tau, \omega) \in \mathcal{H}$. The geometric continuum random cluster model is an example of such a model.

One could view a configuration in this kind of model as a pair $(\omega, \sigma_{\mathcal{H}(\omega)})$ where $\omega \in \Omega$ and $\sigma_{\mathcal{H}(\omega)} \in \{0,1\}^{\mathcal{H}(\omega)}$, with $\sigma_{\mathcal{H}(\omega)}(\tau) = 1$ signifying that τ is open and $\sigma_{\mathcal{H}(\omega)}(\tau) = 0$ signifying that τ is closed. However, we prefer to represent a configuration as a pair $G = (\omega, E)$ where $\omega \in \Omega$ and E is a locally finite set of hyperedges. More precisely, $E \in \mathcal{E}$, where

$$\mathcal{E} := \left\{ E \subset \mathcal{E}_{\mathbb{R}^d,k} : \bigcup_{e \in E} e \text{ is locally finite} \right\},$$

and

$$\mathcal{E}_{\mathbb{R}^d,k} := \{ e \subset \mathbb{R}^d : |e| = k \}. \tag{2.3.1}$$

The sample space is therefore $\mathcal{G} = \Omega \times \mathcal{E}$. In this formulation E represents the set of hyperedges which are considered to be open. Note that $\mathcal{G}$ also contains elements which do not belong to $\mathcal{H}$ since not every $(\omega, E) \in \mathcal{G}$ satisfies $e \in E \implies e \subset \omega$. We equip $\mathcal{E}$ with the sigma algebra Σ generated by the counting variables $N_\Delta : E \mapsto |E_\Delta|$ for $\Delta \Subset \mathbb{R}^{dk}$, similar to how we defined the sigma algebra $\mathcal{F}$ on Ω. As before, Σ is the Borel σ-algebra for the Polish topology on $\mathcal{E}$. Therefore the σ-algebra $\mathcal{A} := \mathcal{F} \otimes \Sigma$ turns $\mathcal{G}$ into a standard Borel space.

Let $G = (\omega, E) \in \mathcal{G}$. Two points $x, y \in \omega$ are *adjacent* if there exists $\tau \in E$ such that $x, y \in \tau$. A *path* connecting x and y is a sequence of points $(x_i)_{i=1}^n \subset \omega$ with $n \in \mathbb{N}$, $x_1 = x$ and $x_n = y$ such that for all $i \in \{1, ..., n-1\}$ there exists $\tau_i \in E$ such that $x_i, x_{i+1} \in \tau_i$. We say that $x \in \omega$ and $y \in \omega$

belong to the same connected component of G if there is a path connecting them.

2.3.2 Background and type interactions

We consider the scenario where there are two interactions, one of which only applies to hyperedges containing particles of different type. The strength of both interactions is irrespective of the particles' type.

Suppose there exist measurable functions $\psi, \phi : \mathcal{E}_{\mathbb{R}^2,3} \to \mathbb{R} \cup \{\infty\}$, such that the hyperedge potential φ on $\mathbf{Del}_3$ can be written as follows:

$$\varphi(\boldsymbol{\tau}, \boldsymbol{\omega}) = \psi(\tau) + \phi(\tau)(1 - \delta_{\sigma_\omega}(\tau)), \tag{2.3.2}$$

where

$$\delta_{\sigma_\omega}(\tau) := \begin{cases} 1 & \text{if } \sigma_\omega(x) = \sigma_\omega(y) \text{ for all } \{x, y\} \subset \tau. \\ 0 & \text{otherwise.} \end{cases}$$

Notice that each $\boldsymbol{\tau} \in \mathbf{Del}_3(\boldsymbol{\omega})$ can be considered as a configuration $\boldsymbol{\tau} = (\tau, \sigma_\tau)$ in its own right, satisfying $\delta_{\sigma_\tau}(\tau) = \delta_{\sigma_\omega}(\tau)$. We assume that ϕ is *repulsive,* which is to say that $\phi \geq 0$. ψ and ϕ are known as the *background* and *type* interactions repectively. Recall that since (2.2.2) is satisfied, we write the Hamiltonian as in (2.2.4):

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') = \sum_{\boldsymbol{\tau} \in \mathbf{Del}_{3,\Delta}(\boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c})} \psi(\tau) + \phi(\tau)(1 - \delta_{\sigma_\tau}(\tau)).$$

For notational convenience we make the definitions

$$\begin{aligned} H^{\psi}_{\Delta,\omega}(\omega') &:= \sum_{\tau \in \mathrm{Del}_{3,\Delta}(\omega' \cup \omega_{\Delta^c})} \psi(\tau), \\ H^{\phi}_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') &:= \sum_{\boldsymbol{\tau} \in \mathbf{Del}_{3,\Delta}(\boldsymbol{\omega}' \cup \boldsymbol{\omega}_{\Delta^c})} \phi(\tau)(1 - \delta_{\sigma_\tau}(\tau)). \end{aligned} \tag{2.3.3}$$

The Hamiltonian can then be split into two terms:

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') = H^{\psi}_{\Delta,\omega}(\omega') + H^{\phi}_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}').$$

Notice that (2.3.3) is the Hamiltonian in Δ with boundary condition ω in the unmarked regime for the potential $\psi' : \mathrm{Del}_3 \to \mathbb{R} \cup \{\infty\}$ defined by $\psi'(\tau, \omega) := \psi(\tau)$. This is to say that if $\mathcal{H} = \mathrm{Del}_3$ then

$$\sum_{\tau \in \mathcal{H}_\Delta(\omega' \cup \omega_{\Delta^c})} \psi(\tau) = \sum_{\tau \in \mathrm{Del}_{3,\Delta}(\omega' \cup \omega_{\Delta^c})} \psi(\tau).$$

In a slight abuse of terminology we will at times refer to ψ itself as a hyperedge potential and write $\Omega^{\psi}_{\Delta,z}$ in place of $\Omega^{\psi'}_{\Delta,z}$.

2.3.3 The Delaunay continuum random cluster distribution

The Delaunay continuum random cluster distribution is an example of a hyperedge percolation model as described in section 2.3.1. The Delaunay continuum random cluster distribution in $\Delta \Subset \mathbb{R}^d$ with boundary condition ω is constructed by sampling a collection of points according to the finite volume Gibbs distribution with interaction ψ and opening each hyperedge independently with some probability which depends on ϕ. Each configuration is then weighted according to the number of connected components that are present.

Let $\omega \in \Omega$ be a configuration which is admissible with respect to $\Delta \Subset \mathbb{R}^d$, hyperedge potential ψ and activity $z > 0$. The distribution of particle positions is given by the (unmarked) finite volume Gibbs distribution in Δ with potential ψ, that is the measure defined by

$$P^z_{\Delta,\omega}(A) := \frac{1}{Z^z_\Delta(\omega)} \int_{\Omega_\Delta} \mathbb{1}_A(\omega' \cup \omega_{\Delta^c}) e^{-H^{\psi}_{\Delta,\omega}(\omega')} \, \Pi^z_\Delta(\mathrm{d}\omega') \tag{2.3.4}$$

for $A \in \mathcal{F}$, where $Z^z_\Delta(\omega)$ is the normalising constant.

For $\omega \in \Omega$, let $\mu_{\omega,\Delta}$ denote the distribution of the random hyperedge configuration $\{\tau \in \mathrm{Del}_3(\omega) : \xi_\tau = 1\} \in \mathcal{E}$, where $(\xi_\tau)_{\tau \in \mathrm{Del}_3(\omega)}$ are independent Bernoulli random variables such that $\xi_\tau = 1$ with probability

$$p_\Delta(\tau) := \begin{cases} 1 - e^{-\phi(\tau)} & \text{if } \tau \in \mathrm{Del}_{3,\Delta}(\omega), \\ 1 & \text{if } \tau \in \mathrm{Del}_3(\omega) \setminus \mathrm{Del}_{3,\Delta}(\omega). \end{cases} \tag{2.3.5}$$

Notice that $0 \le 1 - e^{-\phi(\tau)} \le 1$ because it was assumed that ϕ is repulsive. The measure $\mu_{\omega,\Delta}$ is called the *hyperedge drawing mechanism*. The fact that the map $\Omega \times \Sigma \ni (\omega, A) \mapsto \mu_{\omega,\Delta}(A)$ is a stochastic kernel is established in [Eye14, Lemma 2.1.3].

Let $N_{cc}(\omega, E)$ denote the number of connected components in the hypergraph (ω, E). If

$$\mathsf{Z}_\Delta^z(\omega) := \iint q^{N_{cc}(\omega_{\Delta^c}\cup\omega',E)} \mu_{\omega_{\Delta^c}\cup\omega',\Delta}(\mathrm{d}E) e^{-H^\psi_{\Delta,\omega}(\omega')}\ \Pi^z_\Delta(\mathrm{d}\omega') \in (0,\infty) \quad (2.3.6)$$

then $Z_\Delta^z(\omega) \in (0,\infty)$ also, since $q^{N_{cc}(\omega_{\Delta^c}\cup\omega',E)} \in (1,\infty)$. We can therefore make the following definition.

Definition 2.3.1. If $\mathsf{Z}_\Delta^z(\omega) \in (0,\infty)$ the *Delaunay continuum random cluster distribution in* $\Delta \Subset \mathbb{R}^d$ for ψ, ϕ, z, and boundary condition ω is the probability measure on $(\mathcal{G}, \mathcal{A})$ defined by

$$C^z_{\Delta,\omega}(A) := \frac{Z^z_\Delta(\omega)}{\mathsf{Z}^z_\Delta(\omega)} \iint \mathbb{1}_A(\omega', E) q^{N_{cc}(\omega',E)} \mu_{\omega',\Delta}(\mathrm{d}E) P^z_{\Delta,\omega}(\mathrm{d}\omega'),$$

where $N_{cc}(\omega', E)$ is the number of connected components in the hypergraph (ω', E). Since $\mu_{\omega',\Delta}$ leaves the hyperedges outside of $\mathrm{Del}_{3,\Delta}(\omega')$ open we say that $C^z_{\Delta,\omega}$ has a *wired* boundary condition.

We claim that if $\omega'_{\Delta^c} = \omega_{\Delta^c}$ then

$$1 \le N_{cc}(\omega', E) \le |\,\mathrm{Del}_{1,\Delta}(\omega_{\Delta^c})| + |\omega'_\Delta| + 1. \quad (2.3.7)$$

for $\mu_{\omega',\Delta}$-almost all E. We start by assuming that E is empty, which results in the upper bound

$$N_{cc}(\omega', E) \le K(\omega') + |\omega'_\Delta| + 1,$$

where

$$K(\omega') := \left| \left\{ x \in \omega'_{\Delta^c} \;\middle|\; \begin{matrix} \tau \in \mathrm{Del}_{3,\Delta}(\omega') \text{ for every} \\ \tau \in \mathrm{Del}_3(\omega') \text{ with } x \in \tau \end{matrix} \right\} \right|$$
$$\le |(\mathrm{Del}_{1,\Delta}(\omega'))_{\Delta^c}|.$$

To prove (2.3.7) we now show that $|(\mathrm{Del}_{1,\Delta}(\omega'))_{\Delta^c}| = |\,\mathrm{Del}_{1,\Delta}(\omega_{\Delta^c})|$.

Suppose $x \in (\mathrm{Del}_{1,\Delta}(\omega'))_{\Delta^c}$. Let τ denote one of the triangles in $\mathrm{Del}_{3,\Delta}(\omega')$ to which x belongs. If $\tau \in \mathrm{Del}_{3,\Delta}(\omega_{\Delta^c})$ then clearly $x \in \mathrm{Del}_{1,\Delta}(\omega_{\Delta^c})$. Otherwise $\tau \in \mathrm{Del}_3(\omega') \setminus \mathrm{Del}_3(\omega_{\Delta^c})$, which implies that there is a triangle $\tau' \in \mathrm{Del}_3(\omega_{\Delta^c})$ containing x for which $\overline{B(\tau')} \cap \omega'_\Delta \neq \emptyset$. In this case $x \in \tau' \in \mathrm{Del}_{3,\Delta}(\omega_{\Delta^c})$, and so $x \in \mathrm{Del}_{1,\Delta}(\omega_{\Delta^c})$.

Conversely, if $x \in \mathrm{Del}_{1,\Delta}(\omega_{\Delta^c})$, then $\exists \tau \in \mathrm{Del}_{3,\Delta}(\omega_{\Delta^c})$ containing x. If $\tau \in \mathrm{Del}_{3,\Delta}(\omega')$ then clearly $x \in (\mathrm{Del}_{1,\Delta}(\omega'))_{\Delta^c}$. Otherwise, $\overline{B(\tau)} \cap \omega_\Delta \neq \emptyset$, and so $\exists \tau' \in \mathrm{Del}_{3,\Delta}(\omega')$ containing x. Thus $x \in (\mathrm{Del}_{1,\Delta}(\omega'))_{\Delta^c}$, which concludes the proof.

From (2.3.7) we obtain the following bounds on Z^z_Δ :

$$Z^z_\Delta(\omega) \leq \mathsf{Z}^z_\Delta(\omega) \leq e^{(q-1)z|\Delta|} q^{1+|\,\mathrm{Del}_{1,\Delta}(\omega_{\Delta^c})|} Z^{zq}_\Delta(\omega).$$

Therefore $\mathsf{Z}^z_\Delta(\omega) \in (0,\infty)$ when ω is admissible with respect to ψ, zq and Δ.

2.3.4 The random cluster representation measure

To obtain the representation measure, particles are positioned according to the measure $P^z_{\Delta,\omega}$ (defined in (2.3.4)), given marks independently with a uniform distribution and hyperedges connecting them are opened according to the hyperedge drawing mechanism. The resulting measure is then conditioned on the event that two points in the same connected component must have the same mark.

For a fixed set of particle positions ω, let $\lambda_{\omega,\Delta}$ denote the distribution of the mark vector $\sigma_\omega \in [q]^\omega$ where $(\sigma_\omega(x))_{x\in\omega_\Delta}$ are independent and uniformly distributed on $[q]$ and $\sigma_\omega(x) = 1$ for all $x \in \omega_{\Delta^c}$. $\lambda_{\omega,\Delta}$ is called the *type picking mechanism.* We say that $\lambda_{\omega,\Delta}$ has a *monochromatic* boundary condition since all points outside of Δ have the same mark.

Let $B \in \boldsymbol{\mathcal{F}} \times \Sigma$ denote the event that any two points can only belong to the same connected component if they have the same mark, that is

$$B = \left\{ (\boldsymbol{\omega}, E) \in \boldsymbol{\Omega} \times \mathcal{E} : \sum_{\tau \in E} (1 - \delta_{\sigma_\omega}(\tau)) = 0 \right\}.$$

If $\omega \in \Omega^{\psi}_{\Delta,z}$ we can define $m^z_{\Delta,\omega}$ on $(\boldsymbol{\Omega} \times \mathcal{E}, \boldsymbol{\mathcal{F}} \times \Sigma)$ as the measure satisfying

$$m^z_{\Delta,\omega}(A) := \int \mathbb{1}_A((\omega', \sigma_{\omega'}), E)\mu_{\omega',\Delta}(\mathrm{d}E)\lambda_{\omega',\Delta}(\mathrm{d}\sigma_{\omega'})P^z_{\Delta,\omega}(\mathrm{d}\omega').$$

There exists $k \in \mathbb{Z}$ such that $P^z_{\Delta,\omega}(\omega' : \omega'_{\Delta^c} = k) > 0$, so

$$m^z_{\Delta,\omega}(B) \geq P^z_{\Delta,\omega}(\omega' : \omega'_{\Delta^c} = k)\; q^{-k} > 0.$$

Definition 2.3.2. The *Delaunay random cluster representation measure* in $\Delta \Subset \mathbb{R}^2$ for ψ, z, and boundary condition $\omega \in \Omega^{\psi}_{\Delta,z}$ is the probability measure on $(\boldsymbol{\Omega} \times \mathcal{E}, \boldsymbol{\mathcal{F}} \times \Sigma)$ defined by

$$\mathsf{P}^z_{\Delta,\omega}(\cdot) := m^z_{\Delta,\omega}(\cdot|B).$$

The random cluster representation measure is a joint construction of the Delaunay continuum Potts distribution and the Delaunay continuum random cluster distribution. The former can be obtained if one only looks at the particle positions and their types and disregards the hyperedges. Alternatively, the latter can be obtained by are ignoring the type of each particle. These statements are formalised in the next proposition. They are very similar to Propositions 2.14 and 2.15 from [Eye14] (which are in turn are very similar to Propositions 2.1 and 2.2 from [GH96]) but the author uses a different definition of the set $\mathcal{H}_\Delta$. The proofs in our context do not differ in any notable way so they are omitted.

Let ρ_1 and ρ_2 denote the projections from $\boldsymbol{\Omega} \times \mathcal{E}$ to $\boldsymbol{\Omega}$ and $\Omega \times \mathcal{E}$ respectively.

Proposition 2.3.3. (The random cluster representation). *Let* $\Delta \Subset \mathbb{R}^2, z > 0, \omega \in \Omega^{\psi}_{\Delta,zq}$ *and* $\boldsymbol{\omega} = (\omega, \sigma_\omega)$ *where* $\sigma_\omega \equiv 1$. *Then* $\boldsymbol{\omega} \in \boldsymbol{\Omega}^{\varphi}_{\Delta,zq}$ *and*

1. $\mathsf{P}^{zq}_{\Delta,\omega} \circ \rho_1^{-1} = \gamma^{zq}_{\Delta,\boldsymbol{\omega}}$,
2. $\mathsf{P}^{zq}_{\Delta,\omega} \circ \rho_2^{-1} = C^z_{\Delta,\omega}$.

2.3.5 Phase transitions, symmetry breaking and connectivity

The significance of the random cluster representation is that it allows us to relate the influence the boundary condition of the Delaunay continuum Potts distribution has on the mark of each particle to the percolative properties of the Delaunay random cluster distribution. Consider the Delaunay continuum Potts distribution in some region $\Delta \Subset \mathbb{R}^2$. Since ϕ is repulsive, it is clear that there is an incentive for particles of the same hyperedge to have the same mark, and so particles whose mark is the same as the monochromatic boundary condition will be more prevalent in Δ than the others. This effect will be diminished as the size of the region Δ increases, but thanks to the random cluster representation we shall deduce later that this effect does not disappear entirely in the limit $\Delta \nearrow \mathbb{R}^2$, so long as the probability with respect to the Delaunay random cluster distribution that a fixed region $\Lambda \Subset \mathbb{R}^2$ is connected to Δ^c is bounded away from 0. There will remain one mark which is preferred over the others, a phenomenon which we call *breaking the symmetry of the mark distribution*. This leads us to the discovery of q distinct phases (Delaunay continuum Potts measures), which each have a preferred mark.

For $\Lambda \subset \Delta \Subset \mathbb{R}^2$, let $N_{\Lambda,i}(\boldsymbol{\omega})$ denote the number of particles located in Λ with mark $i \in [q]$ and $N_{\Lambda \leftrightarrow \Delta^c}(\omega, E)$ denote the number of particles in Λ which are connected to Δ^c:

$$\begin{aligned} N_{\Lambda,i}(\boldsymbol{\omega}) &:= |\{x \in \omega_\Lambda : \sigma_\omega(x) = i\}|, \\ N_{\Lambda \leftrightarrow \Delta^c}(\omega, E) &:= |\{x \in \omega_\Lambda : \exists \text{ a path in } (\omega, E) \text{ from } x \text{ to some } y \in \omega_{\Delta^c}\}|. \end{aligned}$$

The following result can be shown using the random cluster representation (Proposition 2.3.3). For details see [Eye14, Proposition 2.17] as it is proven in the same way.

Proposition 2.3.4. *If* $\Lambda \subset \Delta \Subset \mathbb{R}^2, \omega \in \Omega^{\psi}_{\Delta,zq}$ *and* $\boldsymbol{\omega} = (\omega, \sigma_\omega)$ *where* $\sigma_\omega \equiv 1$. *Then*

$$\int (qN_{\Lambda,1} - N_\Lambda) \mathrm{d}\gamma^{zq}_{\Delta,\boldsymbol{\omega}} = (q-1) \int N_{\Lambda \leftrightarrow \Delta^c} \mathrm{d}C^z_{\Delta,\omega}.$$

Using this result one can prove that if there is a lower bound $\int N_{\Lambda\leftrightarrow\Delta^c}\mathrm{d}C^z_{\Delta,\omega} \geq c > 0$ which is uniform over all Δ then there is a phase transition:

Theorem 2.3.5. (Phase Transition). *Let $\Lambda_n := \bigcup_{k\in\{-n,\dots,n\}^2} C(k)$. Suppose **(R)**, **(S)** & **(U)** are satisfied by $\varphi, M, \mathbf{\Gamma}$ & z. If there exists $\omega \in \cap_{n=1}^{\infty}\Omega^{\psi}_{\Lambda_n,zq}$ such that $\boldsymbol{\omega} = (\omega, \sigma_\omega) \in \mathbf{\Gamma}$ for all $\sigma_\omega : \omega \to [q]$ and there exists $c > 0$ such that*

$$\int N_{C(k)\leftrightarrow\Lambda_n^c}\mathrm{d}C^z_{\Lambda_n,\omega} \geq c \tag{2.3.8}$$

for all n and all $k \in \{-n, ..., n\}^2$, then there exists at least q translation-invariant Delaunay continuum Potts measures with interaction φ and activity zq.

The full proof of this statement mimics the proof of [Eye14, Section 2.7]. We just give a brief overview here. The proof begins by showing that constructing a sequence of probability measures $(P^1_n)_{n\in\mathbb{N}}$ on $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}})$ such that

1. P^1_n are invariant under the skewed lattice translations $(\theta_x)_{x\in M\mathbb{Z}^2}$,
2. For $i \in [q]$ and $\Delta \subset \Lambda_n$,

$$\int (qN_{\Delta,i} - N_\Delta)\mathrm{d}P^1_n = \int (qN_{\Delta,i} - N_\Delta)\mathrm{d}\gamma^{zq}_{\Lambda_n,\boldsymbol{\omega}},$$

   where $\sigma_\omega \equiv 1$,

3. P^1_n has a subsequence which converges *locally*[1] to some measure P^1, and after spatially averaging the measure $P^1(\cdot|\{\emptyset\}^c)$, one obtains a translation-invariant Gibbs measure $\tilde{P}^1$.

The local convergence in conjunction with the uniform lower bound and Proposition 2.3.4 then implies that for all $\Delta \Subset \mathbb{R}^2$,

$$\int (qN_{\Delta,1} - N_\Delta)\mathrm{d}\tilde{P}^1 \geq (q-1)c > 0,$$

[1] In this instance the local convergence topology is the weak* topology generated by the set of local and tame real-valued functions on $\mathbf{\Omega}$. These are the functions which which are $\boldsymbol{\mathcal{F}}_\Delta$-measurable and satisfy $|f(\boldsymbol{\omega})| \leq a|\omega_\Delta| + b$ for some $\Delta \Subset \mathbb{R}^2$ and $a, b \in \mathbb{R}$. You could alternatively use the coarser weak* topology generated by the set of local bounded functions here, but the finer topology is preferred in most settings (see for instance [GZ93]) as it has additional useful properties including the fact that it makes the intensity functionals and the Palm mappings continuous.

and since $\gamma^{zq}_{\Lambda_n,\boldsymbol{\omega}}$ is invariant under permutations of $\{2,...,q\}$, we have

$$\int N_{\Delta,1}\,\mathrm{d}\tilde{P}^1 > \int N_{\Delta,2}\,\mathrm{d}\tilde{P}^1 = ... = \int N_{\Delta,q}\,\mathrm{d}\tilde{P}^1,$$

and the symmetry of the mark distribution is broken. Finally, in the same way, for each $i \in [q] \setminus \{1\}$ it is possible to obtain a measure $\tilde{P}^i$ in which the mark i is preferred. This concludes the proof.

2.3.6 Hyperedge percolation to site percolation

We now focus on how to show that condition (2.3.8) is satisfied. We will construct a continuum site percolation model $\hat{C}^{z,\mathrm{site}}_{\Delta,\omega}$ in which the event that Λ is connected to Δ^c is smaller that it is with respect to $C^z_{\Delta,\omega}$. We can then use a coarse graining argument to bound this event from below. The new percolation model will share the same particle distribution as $C^z_{\Delta,\omega}$.

First we must introduce the notion of stochastic dominance between probability measures. A function $f : \mathcal{E} \to \mathbb{R}$ is said to be *increasing* if $f(A) \leq f(B)$ whenever $A \subset B$. For two probability measures μ_1, μ_2 on $(\mathcal{E}, \Sigma)$, we say that μ_1 *stochastically dominates* μ_2 and write $\mu_1 \succcurlyeq \mu_2$ if $\mu_1(f) \geq \mu_2(f)$ for all increasing functions $f : \mathcal{E} \to \mathbb{R}$.

The measure $\hat{C}^{z,\mathrm{site}}_{\Delta,\omega}$ will be defined as a measure on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ where $q = 2$, although instead of the mark space $\{1, 2\}$ we will use the mark space $\{0, 1\}$. Points with mark 1 are considered to be 'open' and points with mark 0 are 'closed.' A *path* in $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$ is a sequence of points $(\boldsymbol{x}_i)_{i=1}^n \subset \boldsymbol{\omega}$ with $n \in \mathbb{N}, \boldsymbol{x}_1 = \boldsymbol{x}$ and $\boldsymbol{x}_n = \boldsymbol{y}$ such that $\sigma_\omega(x_i) = 1$ for all $i \in [n]$ and there exists $\tau_j \in \mathbf{Del}_3(\boldsymbol{\omega})$ such that $\boldsymbol{x}_j, \boldsymbol{x}_{j+1} \in \boldsymbol{\tau}_j$ for all $j \in [n-1]$. The event that Λ is connected to Δ^c is the following:

$$\{\Lambda \leftrightarrow \Delta^c\} := \left\{ \boldsymbol{\omega} \in \boldsymbol{\Omega} \;\middle|\; \begin{array}{c} \text{There exists a path } (\boldsymbol{x}_i)_{i=1}^n \\ \text{in } \boldsymbol{\omega} \text{ with } x_1 \in \Lambda \text{ and } x_n \in \Delta^c. \end{array} \right\}$$

Let $M^z_{\Delta,\omega}$ denote the marginal distribution $C^z_{\Delta,\omega}(\cdot, \mathcal{E})$. We can then write

$$C^z_{\Delta,\omega}(d\omega', \mathrm{d}E) = \mu^q_{\omega',\Delta}(\mathrm{d}E) M^z_{\Delta,\omega}(\mathrm{d}\omega'),$$

where

$$\mu^q_{\omega',\Delta}(\mathrm{d}E) := \frac{q^{N_{cc}(\omega',E)}\mu_{\omega',\Delta}(\mathrm{d}E)}{\int q^{N_{cc}(\omega',E)}\mu_{\omega',\Delta}(\mathrm{d}E)}. \tag{2.3.9}$$

Let $\hat{\mathcal{H}} \subset \mathrm{Del}_3$ denote a fixed but arbitrary unmarked hypergraph structure, and $\hat{p} \in [0,1]$. For $\omega \in \Omega$, let $\hat{\mu}_\omega$ denote the distribution of the random hyperedge configuration $\{\tau \in \mathrm{Del}_3(\omega) : \xi_\tau = 1\} \in \mathcal{E}$, where $(\xi_\tau)_{\tau\in\mathcal{H}(\omega)}$ are independent Bernoulli random variables such that $\xi_\tau = 1$ with probability $\hat{p}\mathbb{1}_{\hat{\mathcal{H}}(\omega)}(\tau)$. In other words, each hyperedge $\tau \in \mathrm{Del}_3(\omega)$ is declared open independently with probability $\hat{p}\mathbb{1}_{\hat{\mathcal{H}}(\omega)}(\tau)$, and closed otherwise.

In addition, define $\hat{\omega} := \{x \in \omega : \exists \tau \in \hat{\mathcal{H}}(\omega) \text{ with } x \in \tau\}$ and let $\hat{\lambda}_\omega$ denote the distribution of the mark vector $\sigma_\omega \in \{0,1\}^\omega$ where $(\sigma_\omega(x))_{x\in\omega}$ are independent and identically distributed such that $\sigma_\omega(x) = 1$ with probability $\hat{p}\mathbb{1}_{\hat{\omega}}(x)$.

We can now define our site percolation measure on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}})$ to be

$$\hat{C}^{z,\mathrm{site}}_{\Delta,\omega}(\mathrm{d}\omega', \mathrm{d}\sigma_\omega) = \hat{\lambda}_\omega(\mathrm{d}\sigma_{\omega'})M^z_{\Delta,\omega}(\mathrm{d}\omega').$$

The result regarding connectivity is the following. For the full proof see [Eye14, Proposition 2.18].

Proposition 2.3.6. *If $\mu^q_{\omega,\Delta} \succcurlyeq \hat{\mu}_\omega$ then for all $\Lambda \subset \Delta \Subset \mathbb{R}^2$,*

$$\int N_{\Lambda\leftrightarrow\Delta^c}\mathrm{d}C^z_{\Delta,\omega} \geq \hat{C}^{z,\mathrm{site}}_{\Delta,\omega}(\Lambda \leftrightarrow \Delta^c).$$

Let $\hat{\mu}_{\omega,\Delta}$ denote the measure for which each hyperedge $\tau \in \mathrm{Del}_3(\omega)$ is opened independently with probability

$$\hat{p}\mathbb{1}_{\hat{\mathcal{H}}(\omega)\cap\mathrm{Del}_{3,\Delta}(\omega)}(\tau) + \mathbb{1}_{\mathrm{Del}_3(\omega)\setminus\mathrm{Del}_{3,\Delta}(\omega)}(\tau),$$

With respect to both $\mu^q_{\omega,\Delta}$ and $\hat{\mu}_{\omega,\Delta}$, the status of all but finitely many hyperedges are fixed. In this case, one can show that $\mu^q_{\omega,\Delta} \succcurlyeq \hat{\mu}_{\omega,\Delta}$ if for all $\tau \in \mathrm{Del}_3(\omega)$, the *comparison inequalities*

$$\frac{p_\Delta(\tau)}{q^2(1-p_\Delta(\tau))} \geq \frac{\hat{p}}{1-\hat{p}} \tag{2.3.10}$$

are satisfied (with the convention that $\frac{p}{1-p} = \infty$ when $p = 1$) by applying the same method as in the case where the hypergraph is finite. For the proof in the case of a finite hypergraph, see [Eye14, Proposition 2.3] (which generalises a result originally proven in [For72]). Employing a coupling argument similar to that used in Lemma 2.4.1 we see that $\hat{\mu}_{\omega,\Delta} \succcurlyeq \hat{\mu}_\omega$. Therefore, to show that $\mu^q_{\omega,\Delta} \succcurlyeq \hat{\mu}_\omega$ we need only verify (2.3.10).

2.4 Mixed site-bond percolation on $\mathbb{Z}^d$

Here we take a small detour to prove a technical result about mixed site-bond percolation on $\mathbb{Z}^d$ for $d \geq 1$ which will prove useful when carrying out the aforementioned coarse-graining procedure. This result is not strictly necessary, and in fact [Rus82, Lemma 1] would suffice for our purposes since we only need to consider site percolation in chapter 3. We choose to include this result here because there are situations (for instance the model considered in [AE19]) when one needs to use mixed site-bond percolation to accomplish the coarse-graining argument. The result states that if all conditional probabilities are uniformly bounded from below then the percolation probability is greater than the percolation probability of Bernoulli site-bond percolation. For more results regarding comparisons of site percolation measures with product measures see [LSS97].

Let $\mathcal{B} = \{\{x, y\} \subset \mathbb{Z}^d : |x - y| = 1\}$ denote the set of edges (or *bonds*) between neighbouring vertices in $\mathbb{Z}^d$, and $\Omega := \{0, 1\}^{\mathbb{Z}^d \cup \mathcal{B}}$. For $\omega \in \Omega$ we say that a site or bond $x \in \mathbb{Z}^d \cup \mathcal{B}$ is *open* if $\omega(x) = 1$ and *closed* otherwise. We will use the shorthand ω_x and $\omega_{\{x,y\}}$ in place of $\omega(x)$ and $\omega(\{x, y\})$ respectively. Let

$$C(\omega) := \left\{ x \in \mathbb{Z}^d \,\middle|\, \begin{array}{c} \text{There exists a path } 0 = x_1, ..., x_n = x \text{ s.t } \omega_{x_i} = 1 \\ \text{for } i \in [n] \text{ and } \omega_{\{x_i, x_{i+1}\}} = 1 \text{ for } i \in [n-1]. \end{array} \right\}$$

This is the *open cluster* around the origin. Similarly, if $\omega \in \Omega$ or $\omega \in \tilde{\Omega} :=$

$\{0,1\}^{\mathbb{Z}^d}$ we define the *open site cluster* as follows:

$$C_s(\omega) := \left\{ x \in \mathbb{Z}^d \;\middle|\; \begin{array}{c} \text{There exists a path } 0 = x_0, , .., x_n = x \\ \text{such that } \omega_{x_i} = 1 \text{ for } i \in [n] \end{array} \right\}.$$

The event that 0 is connected to the set A is

$$\{0 \leftrightarrow A\} = \{\omega \in \Omega \mid \text{There exists } x \in A \cap C(\omega)\},$$

and the event that percolation occurs is

$$\{0 \leftrightarrow \infty\} := \{\omega \in \Omega \mid |C(\omega)| = \infty\} = \bigcap_{n \in \mathbb{N}} \{0 \leftrightarrow \Lambda_n^c\},$$

where $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$.

Let $\mu_{p,p'}$ denote the measure for which each site is opened independently with probability p and each bond is opened independently with probability p'.

Lemma 2.4.1. *If $\mathbb{P}$ is a measure on Ω satisfying the conditions:*

1. *For all $x \in \mathbb{Z}^d$ and $\omega' \in \Omega$,*

$$\mathbb{P}\left(\omega_x = 1 \mid \omega_z = \omega'_z \textit{ for all } z \in (\mathbb{Z}^d \setminus x) \cup \mathcal{B}\right) \geq p, \tag{2.4.1}$$

2. *For all $\{x, y\} \in \mathcal{B}$ and $\omega' \in \Omega$ satisfying $\omega'_x = \omega'_y = 1$,*

$$\mathbb{P}\left(\omega_{\{x,y\}} = 1 \mid \omega_z = \omega'_z \textit{ for all } z \in \mathbb{Z}^d \cup (\mathcal{B} \setminus \{x, y\})\right) \geq p'. \tag{2.4.2}$$

Then $\mathbb{P}(0 \leftrightarrow \infty) \geq \mu_{p,p'}(0 \leftrightarrow \infty)$.

Proof. Let $\tilde{\mathbb{P}}$ and $\tilde{\mu}_{p,p'}$ denote the marginal measures of $\mathbb{P}$ and $\mu_{p,p'}$ on $\tilde{\Omega}$. Inequality (2.4.1) will allow us to couple these measures together. We start by identifying $\mathbb{Z}^d$ with the natural numbers via an arbitrary ordering. Let E_k^+ (E_k^-) be the event that the site k is open (closed) and let $\omega^{(k)} = \{\omega' \in \tilde{\Omega} \mid \omega'_i = \omega_i \text{ for all } i < k\}$. We define the measure m on $\tilde{\Omega} \times \tilde{\Omega}$ inductively by setting (as was done in [Rus82])

$$m(E_1^+ \times E_1^+) = p, \qquad m(E_1^+ \times E_1^-) = \mathbb{P}(E_1^+) - p,$$
$$m(E_1^- \times E_1^+) = 0, \qquad m(E_1^- \times E_1^-) = 1 - \mathbb{P}(E_1^+),$$

and then for $k \geq 2$ and $\zeta, \omega \in \tilde{\Omega}$,

$$\begin{aligned}
m(E_k^+ \times E_k^+ \mid \zeta^{(k)} \times \omega^{(k)}) &= p, \\
m(E_k^- \times E_k^+ \mid \zeta^{(k)} \times \omega^{(k)}) &= 0, \\
m(E_k^+ \times E_k^- \mid \zeta^{(k)} \times \omega^{(k)}) &= \mathbb{P}(E_k^+ \mid \omega^{(k)}) - p, \\
m(E_k^- \times E_k^- \mid \zeta^{(k)} \times \omega^{(k)}) &= 1 - \mathbb{P}(E_k^+ \mid \omega^{(k)}).
\end{aligned}$$

This measure satisfies the following:

$$\begin{aligned}
\mathbb{P}(0 \leftrightarrow \Lambda_n^c) &= \int \mathbb{P}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega))\ \tilde{\mathbb{P}}(\mathrm{d}\omega) \\
&= \int \mathbb{P}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega))\ m(\mathrm{d}\omega, \mathrm{d}\omega'),
\end{aligned}$$

and similarly

$$\mu_{p,p'}(0 \leftrightarrow \Lambda_n^c) = \int \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega'))\ m(\mathrm{d}\omega, \mathrm{d}\omega').$$

Therefore if

$$\int \mathbb{P}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)) - \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega'))\ m(\mathrm{d}\omega, \mathrm{d}\omega') \geq 0$$

then we have

$$\mathbb{P}(0 \leftrightarrow \Lambda_n^c) \geq \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c),$$

and so to complete the proof we need only to show the former inequality. First we note that if $\omega \geq \omega'$, then we have

$$\mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)) \geq \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega')), \tag{2.4.3}$$

since the event $0 \leftrightarrow \Lambda_n^c$ only depends on the bonds between sites in C_s. Secondly, since all sites in C_s are open, we can conclude using property (2.4.2) that for all $\omega \in \tilde{\Omega}$

$$\mathbb{P}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)) \geq \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)). \tag{2.4.4}$$

This can be shown by another coupling on $\{0,1\}^{B(\omega)}$, where $B(\omega)$ is the set of bonds between sites in $C_s(\omega)$. Since $\omega \geq \omega'$ almost surely with respect to m we can now conclude using (2.4.3) and (2.4.4) that:

$$\begin{aligned}
&\int \mathbb{P}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)) \; - \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega')) \; m(\mathrm{d}\omega, \mathrm{d}\omega') \\
\geq &\int \mathbb{P}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)) \; - \mu_{p,p'}(0 \leftrightarrow \Lambda_n^c \mid C_s = C_s(\omega)) \; m(\mathrm{d}\omega, \mathrm{d}\omega') \\
\geq &\; 0.
\end{aligned}$$

□

Corollary 2.4.2. *Let p_c denote the critical probability for site percolation on $\mathbb{Z}^d$. If $\mathbb{P}$ is a measure on Ω such that*

1. *For all $x \in \mathbb{Z}^d$ and $\omega' \in \Omega$,*

$$\mathbb{P}(\omega_x = 1 \mid \omega_y = \omega'_y \textit{ for } y \neq x) > \sqrt{p_c},$$

2. *For all $\{x,y\} \in \mathcal{B}$ and all $\omega' \in \Omega$ satisfying $\omega'_x = \omega'_y = 1$,*

$$\mathbb{P}(\omega_{\{x,y\}} = 1 \mid \omega_{\{w,z\}} = \omega'_{\{w,z\}} \textit{ for } \{w,z\} \neq \{x,y\}) > \sqrt{p_c}.$$

Then $\mathbb{P}(0 \leftrightarrow \infty) > 0$.

Proof. By Lemma 2.4.1 there exists $\epsilon > 0$ such that

$$\mathbb{P}(0 \leftrightarrow \infty) \geq \mu_{\sqrt{p_c}+\epsilon, \sqrt{p_c}+\epsilon}(0 \leftrightarrow \infty).$$

By applying inequality (4) of [Ham80] we can see that the right hand side is greater than $\mu_{(\sqrt{p_c}+\epsilon)^2,1}(0 \leftrightarrow \infty)$, which is greater than 0 since $(\sqrt{p_c}+\epsilon)^2 > p_c$. □

2.5 Previous results for Delaunay interactions

To close this chapter we will review previous results for Delaunay continuum Potts measures. In [AE16] Adams and Eyers considered three models in which the Hamiltonian takes the form

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') = \sum_{\eta \in \mathrm{Del}_{2,\Delta}(\omega_{\Delta^c} \cup \omega')} \psi(\eta) + \sum_{\boldsymbol{\tau} \in \mathbf{Del}_{m,\Delta}(\boldsymbol{\omega}_{\Delta^c} \cup \boldsymbol{\omega}')} \phi(\tau)(1 - \delta_{\sigma_\tau}(\tau))$$

for $m \in \{2, 3\}$. In all three ψ was chosen to be the hard-core potential

$$\psi(\{x, y\}) := \begin{cases} \infty & \text{if } |x - y| < \delta_0 \\ 0 & \text{otherwise.} \end{cases}$$

Since the Delaunay graph is a nearest neighbour graph, ψ ensures that no two particles are within δ_0 of each other. The three choices of m and ϕ were as follows, where $\beta > 0$ is a parameter which adjusts the strength of the interaction.

1. $m = 2$ and

$$\phi_1(\{x, y\}) := \log\left(1 + \beta\left(\frac{\delta_0}{|x - y|}\right)^3\right), \tag{2.5.1}$$

2. $m = 3$ and

$$\phi_2(\tau) := \log\left(1 + \beta\alpha(\tau)^3\right),$$

   where $\alpha(\tau)$ is the smallest interior angle of τ,

3. $m = 3$ and

$$\phi_3(\tau) := \begin{cases} \beta & \text{if } \alpha(\tau) \geq \alpha_0 \\ 0 & \text{otherwise,} \end{cases}$$

   where $\alpha_0 \in (0, \pi/3)$.

Interaction 1 discourages short edges and interactions 2 and 3 encourage the presence of triangles with small angles. All three models were shown to ex-

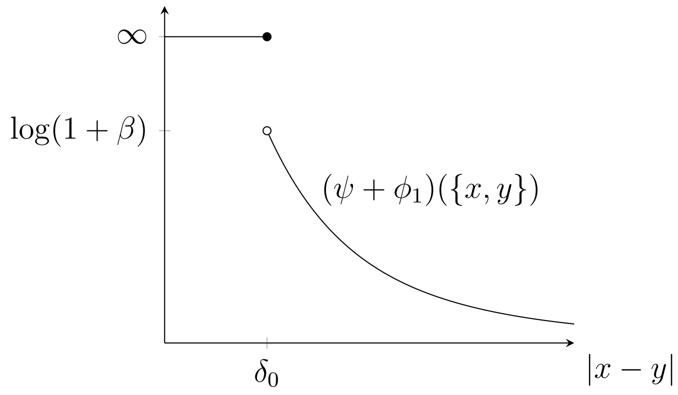


Figure 2.3: A graph of $\psi + \phi_1$.

hibit a phase transition when the parameters are chosen appropriately. More precisely, for models 1 and 3 it was shown that for all δ_0 there are functions $\beta_0 = \beta_0(\delta_0)$ and $z_0(\beta, \delta_0)$ such that for all $\beta > \beta_0$ and $z > z_0$ there are at least q distinct Gibbs measures. For model 2, it was shown that for all $\delta_0 > 0$ and α_0 sufficiently small there exists $\beta_0 = \beta_0(\delta_0, \alpha_0)$ and $z_0 = z_0(\delta_0, \alpha_0)$ such that for all $z > z_0$ and $\beta > \beta_0$ there exists at least q distinct Gibbs measures.

Adams and Eyers also studied a system with no background interaction and a finite range type interaction in [AE19]. They considered the Hamiltonian

$$H_{\Delta,\boldsymbol{\omega}}(\boldsymbol{\omega}') = \sum_{\boldsymbol{\eta}=\{\boldsymbol{x},\boldsymbol{y}\}\in \mathbf{Del}_{2,\Delta}(\boldsymbol{\omega}_{\Delta^c}\cup\boldsymbol{\omega}')} \log(1+\beta|x-y|^{-3-\epsilon})\mathbb{1}_{\{|x-y|\leq R\}}(1-\delta_{\sigma_\eta}(\eta))$$

where $\beta, R > 0$. It was shown that for any $R, \epsilon > 0$, there are functions $z_0(R, q)$ and $\beta_0(q, R, z)$ such that for all for all $z > z_0$ and $\beta > \beta_0$ there exists at least q distinct Gibbs measures [AE19][Theorem 1.4 and Remark 1.5(c)]).

Chapter 3

Delaunay Potts models with triangle interactions

Here we consider a Delaunay Potts model with interactions between the triples of points which are the triangles (2-simplices) of the Delaunay triangulation. This is in contrast to the models in section 2.5 which have interactions based at least in part on the relationship between pairs of points (the 1-simplices). We consider a hardcore background interaction which places upper and lower bounds on the circumradius of each triangle and a lower bound on the interior angles of each triangle. The edge lengths can therefore be bounded from above *and* below, so the restrictions here are stronger than those imposed in the models from [AE16]. We will show that a translation invariant Delaunay continuum Potts measure exists when the activity z is large enough and that there is a phase transition for certain values of the model parameters.

3.1 Definitions

For a triangle $\tau \in \mathcal{E}_{\mathbb{R}^2,3}$ whose points are non-collinear, let $A(\tau)$ denote its area, $\delta(\tau)$ its circumradius and $\alpha(\tau)$ the size of its smallest interior angle. The background and type potentials we consider in this chapter are

$$\psi(\tau) := \begin{cases} 0 & \text{if } \delta(\tau) \in (r, R) \text{ and } \alpha(\tau) > \alpha_0 \\ \infty & \text{otherwise.} \end{cases}$$

and

$$\phi(\tau) := \log\left(1 + \beta A(\tau)^{-1}\right)$$

respectively, where $\beta > 0, \alpha_0 \in (0, \pi/3)$ and $0 < r < R < \infty$. Recall from (2.3.2) that the hypergraph potential $\varphi : \mathbf{Del}_3 \to \mathbb{R} \cup \{\infty\}$ is given by

$$\varphi(\boldsymbol{\tau}, \boldsymbol{\omega}) := \psi(\tau) + \phi(\tau)(1 - \delta_{\sigma_\omega}(\tau)).$$

3.2 Results

Theorem 3.2.1. (Existence.) *If $\beta > 0$ and*

$$\begin{aligned} z &> z_0^{\mathrm{ex}}(\beta, R, r, \alpha_0) \\ &:= \frac{(1+6\rho_0)^6}{3\pi R^2 \rho_0^2 (1-6\rho_0)^4} \left(1 + \frac{\beta(1+6\rho_0)^6}{3^{3/2} R^2 (1-6\rho_0)^{11/2} (1-2\rho_0)^{1/2}}\right)^2, \end{aligned} \tag{3.2.1}$$

where

$$\rho_0(R, r, \alpha_0) := \frac{R^{1/3} - r^{1/3}}{6\,(R^{1/3} + r^{1/3})} \wedge \frac{(1 - (\frac{1}{2} + \cos(\alpha_0))^{\frac{1}{2}})^2}{2\cos(\alpha_0) - 1}, \tag{3.2.2}$$

then there exists a translation-invariant Delaunay continuum Potts measure for $\mathbf{Del}_3, z$ and φ.

Theorem 3.2.2. (Phase Transition.) *Let $\alpha_0 < \sin^{-1}(3/64)$, and $64r < 3R$. There exists $\beta_0(q, R, r, \alpha_0), z_0(\beta, q, R, r, \alpha_0) > 0$ such that for all $\beta > \beta_0(q, R, r, \alpha_0)$ and $z > z_0(\beta, q, R, r, \alpha_0)$ there exist at least q translation-invariant Delaunay continuum Potts measures for $\mathbf{Del}_3, z$ and φ.*

Remark 3.2.3. The dependence of z_0 on β here comes from the existence proof, there is no additional dependence on β required to show that a phase transition occurs. In fact, it is possible to remove the dependence of z on β entirely by using the alternative criteria for existence given in [DDG12, Theorem 3.3] which replaces uniform confinement **(U1)** and strong non-rigidity **(U3)** with a lower density bound and weak non-rigidity. The lower density bound can be obtained since our hardcore potential ensures that the distance between adjacent points in the Delaunay triangulation is no more than $2R$.

The hardcore potential ψ imposes very strict restrictions on the particle configurations, and so we believe that these results could be replicated for a large variety of background potentials. A mild generalisation is to the type potentials

$$\phi(\tau) := \log\left(1 + \beta A(\tau)^{-k}\right)$$

for $k > 0$, although one can go much further than this. The main requirement on ϕ is that one can find $\hat{p} \in (0, 1)$ such that whenever a configuration ω satisfies the hardcore condition, $\frac{\hat{p}}{1-\hat{p}} \leq \frac{e^{\phi(\tau)}-1}{q^2}$ for all $\tau \in \mathrm{Del}_3(\omega)$. This ensures that the comparison inequalities (2.3.10) are satisfied and our coarse graining method results in a measure which stochastically dominates a Bernoulli product measure. The coarse graining argument begins in section 3.4.2 and the value of $\hat{p}$ we use is shown in (3.4.2).

In addition to this, we believe that it should be possible using our method to extend these results to higher dimensional Delaunay tessellations, assuming that an analogous background potential is used. In contrast, the model (without a hardcore background potential) considered in [AE19] required an intricate argument estimating the expected number of connected components in a Delaunay graph, and it is hard to see how this could be replicated in higher dimensions.

3.3 Existence proof

To prove that a translation-invariant Delaunay continuum Potts measure exists we will verify that the hypotheses of Theorem 2.1.7 are met. For $(\boldsymbol{\tau}, \boldsymbol{\omega}) \in \mathbf{Del}_3$ the ball $B(\tau)$ is a horizon for φ satisfying the requirements of **(R)** since $\omega \cap \overline{B(\tau)} = \tau$. **(S)** is trivially satisfied since $\varphi \geq 0$. We will show that **(U)** is satisfied when

$$M = \begin{pmatrix} \ell & \ell/2 \\ 0 & \sqrt{3}\ell/2 \end{pmatrix} \tag{3.3.1}$$

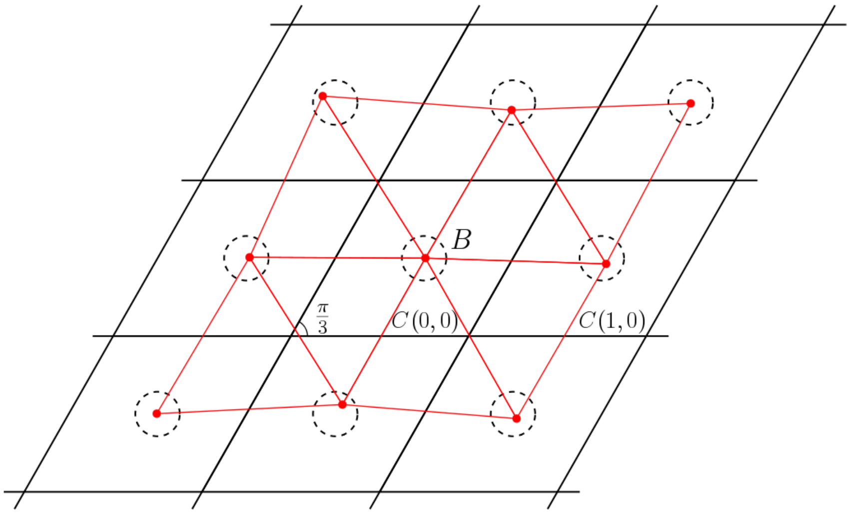


Figure 3.1: A pseudo-periodic configuration with $\rho < \frac{1}{2\sqrt{3}}$.

and

$$\mathbf{\Gamma} = \mathbf{\Gamma}^B := \{\boldsymbol{\omega} \in \mathbf{\Omega}_C : \omega = \{x\} \text{ for some } x \in B\}, \tag{3.3.2}$$

where $B = B(0, \rho\ell)$ is the ball of radius $\rho\ell$ around the origin and the parameters $\rho, \ell > 0$ are chosen appropriately. We assume that $\rho < \frac{1}{2\sqrt{3}}$, in which case for every pseudo-periodic configuration $\boldsymbol{\omega} \in \overline{\mathbf{\Gamma}^B}$, each point $\boldsymbol{x} \in \boldsymbol{\omega}$ has 6 neighbours and the Delaunay triangulation becomes a perturbed triangular lattice ([Nol13, Remark 2.5]) as shown in figure 3.1. In this case the length of each edge lies in the interval $(\ell(1-2\rho), \ell(1+2\rho))$. Thus by the law of cosines

$$\begin{aligned}\cos(\alpha(\tau)) &\leq \frac{2\ell^2(1+2\rho)^2 - \ell^2(1-2\rho)^2}{2\ell^2(1-2\rho)^2} \\ &= \left(\frac{1+2\rho}{1-2\rho}\right)^2 - \frac{1}{2}\end{aligned}$$

for all $\tau \in \mathrm{Del}_3(\omega)$. The roots of the quadratic $(1+2\rho)^2 - (\frac{1}{2} + \cos(\alpha_0))(1-2\rho)^2$ are $\frac{(1 \pm (\frac{1}{2}+\cos(\alpha_0))^{\frac{1}{2}})^2}{2\cos(\alpha_0)-1}$, and so since $\cos(\alpha_0) > \frac{1}{2}$ we have the following result.

Lemma 3.3.1. *If* $\rho < \frac{(1-(\frac{1}{2}+\cos(\alpha_0))^{\frac{1}{2}})^2}{2\cos(\alpha_0)-1} \wedge \frac{1}{2\sqrt{3}}$ *then for any pseudo-periodic* *configuration* $\boldsymbol{\omega} \in \overline{\boldsymbol{\Gamma}^B}$*,*

$$\alpha(\tau) > \alpha_0 \text{ for all } \tau \in \mathrm{Del}_3(\omega).$$

By Lemma A.0.2,

$$\frac{\ell(1-2\rho)}{\sqrt{3}} \le \delta(\tau)$$

for all $\tau \in \mathrm{Del}(\omega)$. If we further assume that $\rho < \frac{1}{6}$, then

$$0 < L(\rho) := \frac{1-6\rho}{\sqrt{3}} \le \frac{\delta(\tau)}{\ell}. \tag{3.3.3}$$

On the other hand, the circumradius δ of a triangle with area A and edge lengths a, b and c is $\frac{abc}{4A}$, so we have

$$\delta(\tau) \le \frac{\ell^3(1+2\rho)^3}{4A(\tau)}.$$

The lower bound

$$\begin{aligned} A(\tau) &\ge \sqrt{3\ell(1-2\rho)(2\ell(1-2\rho)-\ell(1+2\rho))^3} \\ &= \ell^2\sqrt{3(1-2\rho)(1-6\rho)^3} \end{aligned} \tag{3.3.4}$$

can be obtained using Heron's formula, which implies

$$U(\rho) := \frac{(1+6\rho)^3}{\sqrt{3}(1-6\rho)^2} \ge \frac{(1+2\rho)^3}{\sqrt{3(1-2\rho)(1-6\rho)^3}} \ge \frac{\delta(\tau)}{\ell}. \tag{3.3.5}$$

These inequalities are used to prove the following result.

Proposition 3.3.2. *If* $\rho < \rho_0(r, R, \alpha_0) := \frac{R^{1/3}-r^{1/3}}{6\left(R^{1/3}+r^{1/3}\right)} \wedge \frac{(1-(\frac{1}{2}+\cos(\alpha_0))^{\frac{1}{2}})^2}{2\cos(\alpha_0)-1}$ *then* $\frac{r}{L(\rho)} < \frac{R}{U(\rho)}$*. Furthermore, if* $\ell \in \left(\frac{r}{L(\rho)}, \frac{R}{U(\rho)}\right)$ *then all pseudo-periodic configurations* $\boldsymbol{\omega} \in \overline{\boldsymbol{\Gamma}^B}$ *satisfy*

$$\delta(\tau) \in (r, R) \text{ and } \alpha(\tau) > \alpha_0 \text{ for all } \tau \in \mathrm{Del}_3(\omega). \tag{3.3.6}$$

Proof. The first part is just a simple rearrangement:

$$\begin{aligned}\rho &< \frac{R^{1/3} - r^{1/3}}{6\left(R^{1/3} + r^{1/3}\right)}, \\ &\implies (1 + 6\rho)r^{1/3} < (1 - 6\rho)R^{1/3}, \\ &\implies \frac{r}{R} < \left(\frac{1 - 6\rho}{1 + 6\rho}\right)^3 = \frac{L(\rho)}{U(\rho)}, \\ &\implies \frac{r}{L(\rho)} < \frac{R}{U(\rho)}.\end{aligned}$$

Since $\rho < \frac{R^{1/3} - r^{1/3}}{6\left(R^{1/3} + r^{1/3}\right)} < \frac{1}{6} < \frac{1}{2\sqrt{3}}$ we can apply Lemma 3.3.1 and inequalities (3.3.3) & (3.3.5) to obtain

$$r < \ell L(\rho) \leq \delta(\tau) \leq \ell U(\rho) < R$$

and $\alpha(\tau) > \alpha_0$ for all $\tau \in \mathrm{Del}_3(\omega)$ when $\ell \in \left(\frac{r}{L(\rho)}, \frac{R}{U(\rho)}\right)$. □

Suppose $\boldsymbol{\omega} \in \overline{\boldsymbol{\Gamma}}, \boldsymbol{\zeta} \in \boldsymbol{\Omega}_\Delta$ and $\boldsymbol{\tau} \in \mathbf{Del}_{3,\Delta}(\boldsymbol{\zeta} \cup \boldsymbol{\omega}_{\Delta^c})$. There exists $k \in \mathbb{R}$ (independent of $\boldsymbol{\omega}, \Delta$ and $\boldsymbol{\zeta}$) such that $B(\tau) \subset \Delta \oplus k$, since if $B(\tau)$ protrudes too far outside of Δ then $B(\tau) \cap \omega_{\Delta^c} \neq \emptyset$. Therefore if $\boldsymbol{\omega}'_{\Delta\oplus k} = \boldsymbol{\omega}_{\Delta\oplus k}$ then $\boldsymbol{\tau} \in \mathbf{Del}_3(\boldsymbol{\zeta} \cup \boldsymbol{\omega}'_{\Delta^c})$, which implies $\varphi(\boldsymbol{\tau}, \boldsymbol{\zeta} \cup \boldsymbol{\omega}_{\Delta^c}) = \varphi(\boldsymbol{\tau}, \boldsymbol{\zeta} \cup \boldsymbol{\omega}'_{\Delta^c})$. **(U1)** is then satisfied with $r_{\boldsymbol{\Gamma}} \leq k < \infty$.

To prove **(U2)**, we need an upper bound on $c_{\boldsymbol{\Gamma}}$. If (3.3.6) holds then $\psi(\tau) = 0$, and so utilising the lower bound (3.3.4) we have

$$c_{\boldsymbol{\Gamma}} \leq \frac{6}{3} \log\left(1 + \frac{\beta}{\ell^2\sqrt{3(1 - 2\rho)(1 - 6\rho)^3}}\right) < \infty. \tag{3.3.7}$$

Finally, $e^{z|C|}\boldsymbol{\Pi}^z_C(\boldsymbol{\Gamma}) = z|B|$, so if

$$z > \frac{1}{\pi\rho^2\ell^2}\left(1 + \frac{\beta}{\ell^2\sqrt{3(1 - 2\rho)(1 - 6\rho)^3}}\right)^2$$

then **(U3)** is satisfied. Thus far we have proved the following:

Proposition 3.3.3. *If* $\beta > 0$,

$$\rho < \rho_0(R, r, \alpha_0) := \frac{R^{1/3} - r^{1/3}}{6\,(R^{1/3} + r^{1/3})} \wedge \frac{(1 - (\frac{1}{2} + \cos(\alpha_0))^{\frac{1}{2}})^2}{2\cos(\alpha_0) - 1},$$

$$\ell \in \left(\frac{r}{L(\rho)}, \frac{R}{U(\rho)}\right),$$

and

$$z > z_0'(\beta, \rho, \ell) := \frac{1}{\pi \rho^2 \ell^2} \left(1 + \frac{\beta}{\ell^2 \sqrt{3(1-2\rho)(1-6\rho)^3}}\right)^2, \qquad (3.3.8)$$

then there exists a translation-invariant Delaunay continuum Potts measure for $\mathbf{Del}_3, z$ *and* φ.

Since the lower bound on z is continuous at $(\rho, \ell) = (\rho_0, \frac{R}{U(\rho)})$, we can replace ρ with ρ_0 and ℓ with $\frac{R}{U(\rho)}$. The condition on z then becomes

$$z > z_0^{\text{ex}}(\beta, R, r, \alpha_0) := \frac{(1+6\rho_0)^6}{3\pi R^2 \rho_0^2 (1-6\rho_0)^4} \left(1 + \frac{\beta(1+6\rho_0)^6}{3^{3/2} R^2 (1-6\rho_0)^{11/2} (1-2\rho_0)^{1/2}}\right)^2,$$

and therefore Theorem 3.2.1 can be seen as a consequence of Proposition 3.3.3.

Remark 3.3.4. $z_0^{\text{ex}}(\beta, R, r, \alpha_0) \to \infty$ as $\alpha_0 \to \pi/3$ or $\frac{R^{1/3} - r^{1/3}}{6\left(R^{1/3} + r^{1/3}\right)} \to 0$.

3.4 Phase transition proof

To prove Theorem 3.2.2 we use a coarse graining procedure to obtain a lower bound on $\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}(C(k) \leftrightarrow \Lambda_n^c)$, and then apply Proposition 2.3.6 and Theorem 2.3.5. Before specifying the unmarked hypergraph structure $\hat{\mathcal{H}}$ and the parameter $\hat{p}$, we will first prove some statements about the measure $M^z_{\Delta,\omega}$. In particular we consider the effect of augmenting a configuration by an additional particle

3.4.1 Augmenting a configuration by a particle

The situation here differs from the case of a classical many-body interaction since adding a point x does not merely result in additional interaction terms representing the interaction between x and the other particles. Instead, when the particle x is added, some hyperedges are created and others are destroyed. The result of this is that for $\omega' \in \Omega_\Delta$ and $x \in \Delta$, both $H^{\psi}_{\Delta,\omega}(\omega')$ and $H^{\psi}_{\Delta,\omega}(\omega' \cup \{x\})$ may contain terms which are not present in the other. It is thus possible (depending on ψ) that $H^{\psi}_{\Delta,\omega}(\omega') = \infty$ and $H^{\psi}_{\Delta,\omega}(\omega' \cup \{x\}) < \infty$. In this case the function $e^{-H^{\psi}_{\Delta,\omega}(\cdot)}$ is called *non-hereditary* [DVJ08, Definition 10.4.IV]. We start with the *point insertion lemma* which expresses that the circumradius of each 'new' triangle (when x is added) is less than the circumradius of particular 'old' triangles. The point insertion lemma formalises an argument found in section 12.2.6 (page 462) of [Lis99], with some details filled in.

The following sets contain the tiles that remain intact, the ones that are created, and those that are destroyed when adding the point x_0 to an unmarked configuration ω. From now on we will write $\omega \cup x_0$ rather than $\omega \cup \{x_0\}$.

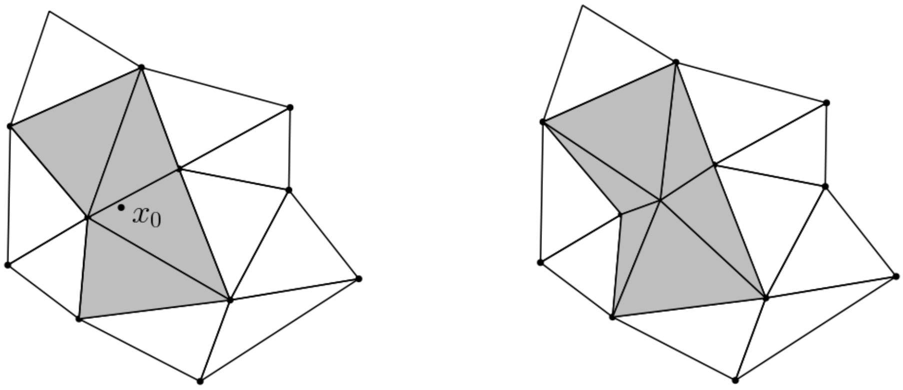
$$\begin{aligned}
T^{\mathrm{ext}}_{x_0,\omega} &:= \mathrm{Del}_3(\omega) \cap \mathrm{Del}_3(\omega \cup x_0) = \{\tau \in \mathrm{Del}_3(\omega) : x_0 \notin \overline{B(\tau)}\},\\
T^{+}_{x_0,\omega} &:= \mathrm{Del}_3(\omega \cup x_0) \setminus \mathrm{Del}_3(\omega) = \{\tau \in \mathrm{Del}_3(\omega \cup x_0) : x_0 \in \tau\},\\
T^{-}_{x_0,\omega} &:= \mathrm{Del}_3(\omega) \setminus \mathrm{Del}_3(\omega \cup x_0) = \{\tau \in \mathrm{Del}_3(\omega) : x_0 \in \overline{B(\tau)}\}.
\end{aligned}$$

An example is shown in figure 3.2. The area covered by the triangles in $T^{-}_{x_0,\omega}$ (or $T^{+}_{x_0,\omega}$) is shown in grey and referred to as the *Delaunay cavity* created by x_0.

Lemma 3.4.1. (Point insertion lemma.) *Suppose* $\tau = \{x_0, y, z\} \in \mathrm{Del}_3(\omega \cup x_0)$ *and let* τ_1, τ_2 *denote the two triangles in* $\mathrm{Del}_3(\omega)$ *which have* $\{y, z\}$ *as a subset. Then*

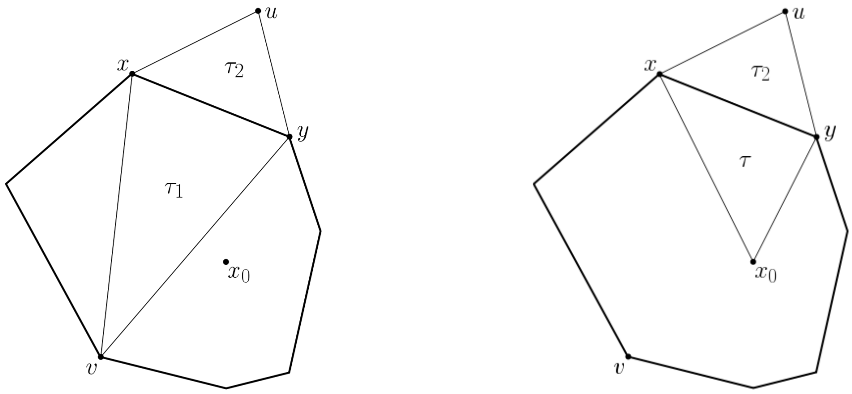
$$\delta(\tau) \leq \max(\delta(\tau_1), \delta(\tau_2)).$$

Proof. Without loss of generality, let $\tau_1 = \{v, y, z\} \in T^{-}_{x_0,\omega}$ and $\tau_2 = \{u, y, z\} \in T^{\mathrm{ext}}_{x_0,\omega}$ (see figure 3.3). Let C_τ denote the circumcentre of τ. Consider the two



(a) $T^-_{x_0,\omega} = \text{Del}_3(\omega) \setminus \text{Del}_3(\omega \cup x_0)$ (b) $T^+_{x_0,\omega} = \text{Del}_3(\omega \cup x_0) \setminus \text{Del}_3(\omega)$

Figure 3.2: Augmenting the configuration ω by a point x_0.

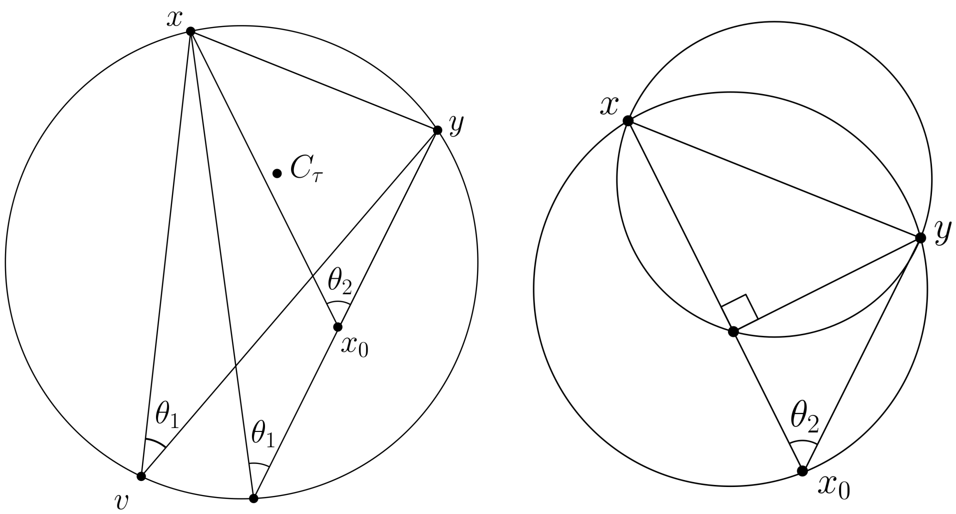


(a) (b)

Figure 3.3

half planes separated by the line $\overleftrightarrow{xy}$ passing through x and y. We will show that if C_τ is in the same half-plane as v then $\delta(\tau) \leq \delta(\tau_1)$ and if C_τ is in the same half plane as u then $\delta(\tau) \leq \delta(\tau_2)$.

In the former case, the angle θ_1 subtended by the chord $\overline{xy}$ at v is less than the angle θ_2 subtended by $\overline{xy}$ at x_0. This can be seen by extending the line $\overline{yx_0}$ until it intersects the circumcircle of τ_1 (figure 3.4a), which is possible since x_0 lies inside the circumcircle of τ_1. Since C_τ lies on the same side of $\overline{xy}$ as v, there is a right angle subtended by $\overline{xy}$ at some point along $\overline{xx_0}$ or $\overline{yx_0}$ (figure 3.4b), so $\theta_2 \leq \frac{\pi}{2}$. From the relationship $\theta_1 \leq \theta_2 \leq \frac{\pi}{2}$ we can conclude



(a) The angle θ_1 subtended by the chord $\overline{xy}$ at v is less than the angle θ_2 subtended by $\overline{xy}$ at x_0.

(b) There is a right angle subtended by $\overline{xy}$ at some point along $\overline{xx_0}$ or $\overline{yx_0}$.

Figure 3.4

that

$$\delta(\tau) = \frac{\mathrm{d}(x,y)}{2\sin(\theta_2)} \leq \frac{\mathrm{d}(x,y)}{2\sin(\theta_1)} = \delta(\tau_1).$$

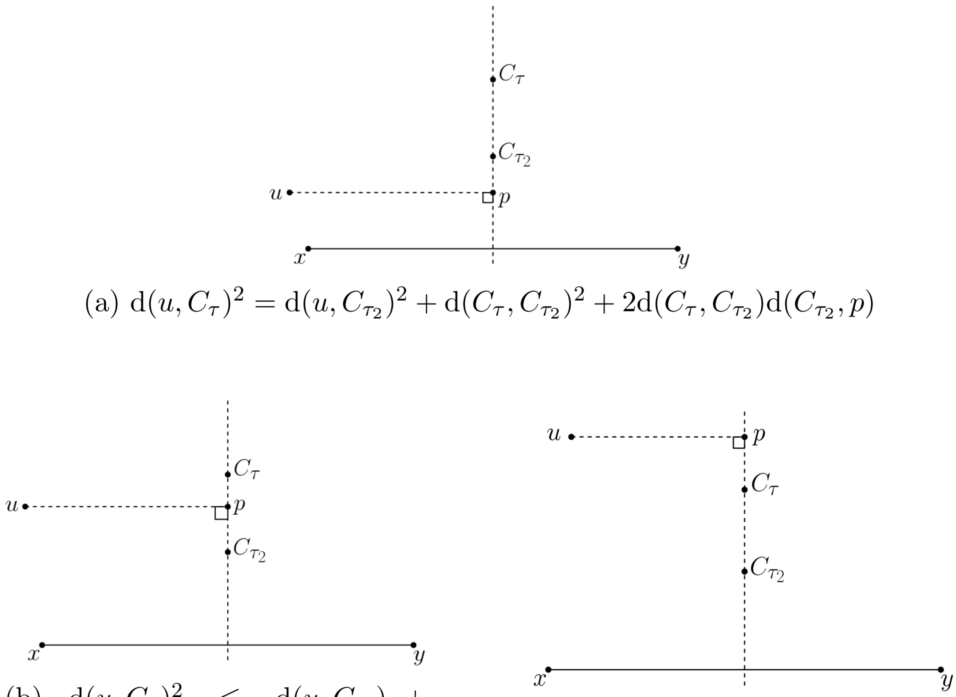
Now suppose that C_τ is in the same half plane as u. If C_{τ_2} is farther away from $\overline{xy}$ than C_τ, then it is clear that $\delta(\tau) = \mathrm{d}(x, C_\tau) \leq \mathrm{d}(x, C_{\tau_2}) = \delta(\tau_2)$, as required.

Conversely, if C_{τ_2} is between $\overline{xy}$ and C_τ then $\delta(\tau_2) \leq \delta(\tau)$. In fact

$$\delta(\tau_2) \leq \delta(\tau) < \mathrm{d}(u, C_\tau) \tag{3.4.1}$$

since the point u is outside of $\overline{B(\tau)}$. But we also have (see figure 3.5)

$$\begin{aligned}
\delta(\tau)^2 &= \mathrm{d}(x, C_\tau)^2 \\
&= \mathrm{d}(x, C_{\tau_2})^2 + \mathrm{d}(C_\tau, C_{\tau_2})^2 + 2\mathrm{d}(C_\tau, C_{\tau_2})\mathrm{d}\left(C_{\tau_2}, \frac{x+y}{2}\right) \\
&= \mathrm{d}(u, C_{\tau_2})^2 + \mathrm{d}(C_\tau, C_{\tau_2})^2 + 2\mathrm{d}(C_\tau, C_{\tau_2})\mathrm{d}\left(C_{\tau_2}, \frac{x+y}{2}\right) \\
&\geq \mathrm{d}(u, C_\tau)^2.
\end{aligned}$$



(a) $\mathrm{d}(u, C_\tau)^2 = \mathrm{d}(u, C_{\tau_2})^2 + \mathrm{d}(C_\tau, C_{\tau_2})^2 + 2\mathrm{d}(C_\tau, C_{\tau_2})\mathrm{d}(C_{\tau_2}, p)$

(b) $\mathrm{d}(u, C_\tau)^2 \leq \mathrm{d}(u, C_{\tau_2}) + \mathrm{d}(C_\tau, C_{\tau_2})^2$.

(c) $\mathrm{d}(u, C_\tau) \leq \mathrm{d}(u, C_{\tau_2})$.

Figure 3.5

which contradicts (3.4.1). □

Let $\omega \in \Omega^{\psi}_{\Delta,zq}$. Recall that $M^z_{\Delta,\omega}$ denotes the marginal distribution $C^z_{\Delta,\omega}(\cdot, \mathcal{E})$. The Radon-Nikodym density of $M^z_{\Delta,\omega}$ with respect to $P^z_{\Delta,\omega}$ is

$$h^z_{\Delta,\omega}(\omega') := \mathbb{1}_{\{\omega_{\Delta^c} = \omega'_{\Delta^c}\}} \frac{Z^z_\Delta(\omega)}{\mathsf{Z}^z_\Delta(\omega)} \int q^{N_{cc}(\omega',T)} \mu_{\omega',\Delta}(\mathrm{d}T).$$

The next lemma gives a lower bound on the ratio

$$\frac{h^z_{\Delta,\omega}(\omega' \cup x_0)}{h^z_{\Delta,\omega}(\omega')},$$

which is known as the *Papangelou conditional intensity.* Recall that according to the hyperedge drawing mechanism $\mu_{\omega,\Delta}$ each edge in $\mathrm{Del}_3(\omega)$ is opened independently according to the probabilities given in (2.3.5). Let $\mu^{\mathrm{ext}}_{x_0,\omega}, \mu^+_{x_o,\omega}$ and $\mu^-_{x_0,\omega}$ denote the measures which open the edges in $T^{\mathrm{ext}}_{x_0,\omega}, T^+_{x_0,\omega}$ and $T^-_{x_0,\omega}$

respectively with the same probabilities. Then

$$\mu^{\text{ext}}_{x_0,\omega} \otimes \mu^{+}_{x_0,\omega} = \mu_{\omega\cup x_0,\Delta} \qquad \text{and} \qquad \mu^{\text{ext}}_{x_0,\omega} \otimes \mu^{-}_{x_0,\omega} = \mu_{\omega,\Delta}.$$

Lemma 3.4.2. *Suppose that $\omega \in \Omega^{\psi}_{\Delta,zq}, \omega' \in \Omega$ with $\omega_{\Delta^c} = \omega'_{\Delta^c}$, and $x_0 \in \Delta \setminus \omega'$. If $H^{\psi}_{\Delta,\omega}(\omega'_\Delta), H^{\psi}_{\Delta,\omega}(\omega'_\Delta \cup x_0) < \infty$. Then*

$$\frac{h^z_{\Delta,\omega}(\omega' \cup x_0)}{h^z_{\Delta,\omega}(\omega')} \geq q^{1-\frac{2\pi}{\alpha_0}}.$$

Proof. The structure of this proof is the same as [AE19, Lemma 2.3] but the details are slightly different since we are dealing with triangle interactions rather than edge interactions.

$$\begin{aligned}\frac{h^z_{\Delta,\omega}(\omega' \cup x_0)}{h^z_{\Delta,\omega}(\omega')} &= \frac{\int q^{N_{cc}(\omega'\cup x_0,T)}\mu_{\omega'\cup x_0,\Delta}(\mathrm{d}T)}{\int q^{N_{cc}(\omega',T)}\mu_{\omega',\Delta}(\mathrm{d}T)} \\ &= \frac{\int q^{N_{cc}(\omega'\cup x_0,T_1\cup T_2)-N_{cc}(\omega',T_1)}q^{N_{cc}(\omega',T_1)}\mu^{\text{ext}}_{x_0,\omega'}(\mathrm{d}T_1)\mu^{+}_{x_0,\omega'}(\mathrm{d}T_2)}{\int q^{N_{cc}(\omega',T_3\cup T_4)-N_{cc}(\omega',T_3)}q^{N_{cc}(\omega',T_3)}\mu^{\text{ext}}_{x_0,\omega'}(\mathrm{d}T_3)\mu^{-}_{x_0,\omega'}(\mathrm{d}T_4)}.\end{aligned}$$

Opening more triangles can only reduce the number of connected components, so

$$N_{cc}(\omega', T_3 \cup T_4) \leq N_{cc}(\omega', T_3).$$

Furthermore, since $H^{\psi}_{\Delta,\omega}(\omega'_\Delta \cup x_0) < \infty$ the point x_0 is connected to at most $\frac{2\pi}{\alpha_0}$ other points in the graph $(\omega' \cup x_0, \mathrm{Del}_2(\omega' \cup x_0))$, so

$$N_{cc}(\omega' \cup x_0, T_1 \cup T_2) - N_{cc}(\omega', T_1) \geq 1 - \frac{2\pi}{\alpha_0}.$$

Therefore

$$\begin{aligned}\frac{h^z_{\Delta,\omega}(\omega' \cup x_0)}{h^z_{\Delta,\omega}(\omega')} &\geq \frac{\int q^{1-\frac{2\pi}{\alpha_0}}q^{N_{cc}(\omega',T_1)}\mu^{\text{ext}}_{x_0,\omega'}(\mathrm{d}T_1)\mu^{+}_{x_0,\omega'}(\mathrm{d}T_2)}{\int q^{N_{cc}(\omega',T_3)}\mu^{\text{ext}}_{x_0,\omega'}(\mathrm{d}T_3)\mu^{-}_{x_0,\omega'}(\mathrm{d}T_4)} \\ &= q^{1-\frac{2\pi}{\alpha_0}}.\end{aligned}$$

□

3.4.2 Coarse graining

In order to prove the existence of a uniform lower bound on $\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}(C(k,m) \leftrightarrow \Lambda_n^c)$, we will devise a criterion by which, according to the underlying configuration $\boldsymbol{\omega}$, each cell $C(k,m)$ (defined in 2.1.10) is declared open or closed. This criterion will be devised in such a way that there exists an infinite connected component containing a point in $C(k,m)$ if $C(k,m)$ belongs to an infinite connected component of open cells. We call this procedure of moving from points to cells *coarse graining.* Formally, for each n we will construct a map $X_n : \boldsymbol{\Omega} \to \{0,1\}^{\mathbb{Z}^2}$ where $\boldsymbol{\omega} \in \{C(k,m) \leftrightarrow \Lambda_n^c\}$ if (k,m) belongs to an infinite open cluster in $X_n(\boldsymbol{\omega})$. The cell $C(k,m)$ is considered to be open if $X_n(\boldsymbol{\omega})(k,m) = 1$. The desired lower bound will then be obtained by making a stochastic comparison between the law of X_n and a Bernoulli product measure using Corollary 2.4.2.

Let M and $\boldsymbol{\Gamma}$ be as in (3.3.1) and (3.3.2) respectively, and let the parameters ρ, ℓ and z satisfy the requirements of Proposition 3.3.3 with zq in place of z. Then **(U)** is satisfied (in addition to **(R)** and **(S)**). The same argument as in the previous section can be used to show that these conditions are also satisfied in the unmarked regime with respect to ψ instead of φ, and $\Gamma = \Gamma^B := \{\omega \in \Omega_C : \omega = \{x\} \text{ for some } x \in B\}$ instead of $\boldsymbol{\Gamma}$. The situation is simpler since in this case $c_\Gamma = 0$. By Remark 2.1.8, $\boldsymbol{\Gamma} \subset \boldsymbol{\Omega}^{\varphi}_{\Lambda_n,zq}$ and $\Gamma \subset \Omega^{\psi}_{\Lambda_n,zq}$ for all $n \in \mathbb{N}$, where $\Lambda_n = \bigcup_{k,m\in\{-n\dots,n\}} C(k,m)$.

The cells $(C(k,m))_{k,m\in\mathbb{Z}}$ form a partition of the plane into rhombi of length ℓ. Let us split each cell into 64 smaller sub-cells of length $\ell/8$ denoted $(C^{i,j}_{k,m})_{0\le i,j\le 7}$, where

$$C^{i,j}_{k,m} := \left\{ Mx \in \mathbb{R}^2 : x - (k,m) \in \left[\frac{i-4}{8}, \frac{i-3}{8}\right) \times \left[\frac{j-4}{8}, \frac{j-3}{8}\right) \right\}.$$

Let $F_{k,m}$ denote the event that there is least one particle in each sub-cell of $C(k,m)$ and $O_{k,m}$ denote the event that additionally all points in $C(k,m)$ are open:

$$F_{k,m} := \bigcap_{0\le i,j\le 7} \left\{\boldsymbol{\omega} \in \Omega : |\omega \cap C_{k,m}^{i,j}| \ge 1\right\}.$$

$$O_{k,m} := \{\boldsymbol{\omega} \in \boldsymbol{\Omega} : \sigma_\omega(x) = 1 \text{ for all } x \in \omega_{C_{k,m}}\}.$$

The map $X_n : \boldsymbol{\Omega} \to \{0,1\}^{\mathbb{Z}^2}$ is constructed by opening the sites (k,m) inside $\{-n,...,n\}^2$ for which $F_{k,m} \cap O_{k,m}$ occurs, and opening the sites outside $\{-n,...,n\}^2$ for which $O_{k,m}$ occurs, i.e

$$X_n(\boldsymbol{\omega})(k,m) := \begin{cases} \mathbb{1}_{F_{k,m}\cap O_{k,m}}(\boldsymbol{\omega}) & \text{if } |k|,|m| \le n \\ \mathbb{1}_{O_{k,m}}(\boldsymbol{\omega}) & \text{otherwise.} \end{cases}$$

X_n is therefore a stochastically decreasing sequence. To complete the definition of $\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}$, let $\hat{\mathcal{H}} = \text{Del}_3$ and

$$\hat{p} = \frac{1}{\frac{3\sqrt{3}}{4\beta} q^2 R^2 + 1}. \tag{3.4.2}$$

For a given circumradius, the triangle τ with maximal area is the equilateral triangle, for which $A(\tau) = \frac{3\sqrt{3}}{4}\delta(\tau)^2$. Therefore, if $H^{\psi}_{\Lambda_n,\omega}(\omega') < \infty$ then

$$\hat{p} \le \frac{1}{q^2\beta^{-1}A(\tau) + 1}$$
$$\implies \frac{\hat{p}}{1-\hat{p}} \le \frac{\beta}{q^2 A(\tau)}$$

for all $\tau \in \text{Del}_{3,\Delta}(\omega_{\Lambda_n^c} \cup \omega')$. The comparison inequalities (2.3.10) are satisfied since

$$\frac{p_\Delta(\tau)}{q^2(1-p_\Delta(\tau))} = \frac{1-e^{-\phi(\tau)}}{q^2 e^{-\phi(\tau)}} = \frac{e^{\phi(\tau)}-1}{q^2} = \frac{\beta}{q^2 A(\tau)},$$

for all $\tau \in \text{Del}_{3,\Delta}(\omega_{\Lambda_n^c} \cup \omega')$, and therefore $\mu^q_{\omega',\Lambda_n} \succcurlyeq \hat{\mu}_{\omega'}$ almost surely with respect to $M^z_{\Delta,\omega}$. Hence the premises of Theorem 2.3.5 and Proposition 2.3.6 are satisfied, so it remains to show that there exists $c > 0$ such that

$$\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}(C(k,m) \leftrightarrow \Lambda_n^c) \ge c \quad \text{for all } n \text{ and } \omega \in \Gamma.$$

The following lemma shows that a uniform lower bound on the percolation probability of the law of X_n is sufficient.

Lemma 3.4.3. *If* $\omega \in \Gamma, \sigma_\omega(x) \equiv 1$, $H^{\psi}_{\Lambda_n,\omega}(\omega') < \infty$ *and* $X_n(\boldsymbol{\omega}_{\Lambda_n{}^c} \cup \boldsymbol{\omega}') \in \{(k,m) \leftrightarrow \infty\}$, *then* $\boldsymbol{\omega}_{\Lambda_n{}^c} \cup \boldsymbol{\omega}' \in \{C(k,m) \leftrightarrow \Lambda_n^c\}$. *Therefore*

$$\hat{C}^{z,\mathrm{site}}_{\Lambda_n,\omega}(C(k,m) \leftrightarrow \Lambda_n^c) \geq \mathcal{L}_{X_n}((k,m) \leftrightarrow \infty)$$

where $\mathcal{L}_{X_n}$ *is the law of* X_n*.*

Proof. For $k \in \{-n, ..., n-1\}, |m| \leq n$ let $x_{k,m}, x_{k+1,m} \in \omega_{\Lambda^c} \cup \omega'$ denote the points whose Voronoi cells contain the centers of $C(k,m)$ and $C(k+1,m)$. The Voronoi cell associated to x is given by

$$\mathrm{Vor}_{\omega_{\Lambda^c} \cup \omega'}(x) := \{z \in \mathbb{R}^2 : |x - z| \leq |w - z| \text{ for all } w \in \omega_{\Lambda^c} \cup \omega'\}.$$

If $X_n(\boldsymbol{\omega}_{\Lambda_n^c} \cup \boldsymbol{\omega}')(k,m) = X_n(\boldsymbol{\omega}_{\Lambda_n^c} \cup \boldsymbol{\omega}')(k+1,m) = 1$ then $\boldsymbol{x}_{k,m}$ is connected to $\boldsymbol{x}_{k+1,m}$ via a path whose points are located in

$$\left\{ \bigcup_{2\leq i\leq 7, 2\leq j\leq 5} C^{i,j}_{k,m} \right\} \cup \left\{ \bigcup_{0\leq i\leq 5, 2\leq j\leq 5} C^{i,j}_{k+1,m} \right\}.$$

This can be seen via the same argument as [AE19, Lemma 2.7, step (iv)]. In fact, the points can be joined via a path whose points all have Voronoi cells intersecting the line segment between the centers of the cells $C(k,m)$ and $C(k+1,m)$.

Furthermore, the same applies if $|m| \leq n$ and $X_n(\boldsymbol{\omega}_{\Lambda_n^c} \cup \boldsymbol{\omega}')(n,m) = X_n(\boldsymbol{\omega}_{\Lambda_n^c} \cup \boldsymbol{\omega}')(n+1,m) = 1$; there is a path between $\boldsymbol{x}_{n,m}$ and the point in $\boldsymbol{\omega}_{C(n+1,m)}$ (recall that $\rho < \frac{1}{6}$) via a path whose points all have Voronoi cells intersecting the line segment between the centers of the cells $C(n,m)$ and $C(n+1,m)$. Figure 3.6 shows an example of a path passing through two open cells and across the boundary of Λ_n.

Note that we have only discussed horizontal crossings between cells $C(k,m)$ and $C(k+1,m)$. The proof for vertical crossings can be performed similarly. It is now clear that if there is a path $(k_r, m_r)_{r=1}^s$ in $X_n(\boldsymbol{\omega}_{\Lambda_n{}^c} \cup \boldsymbol{\omega}')$ with $(k_1, m_1) = (k,m)$ and $|k_s|, |m_s| > n$ then there is a path in $\boldsymbol{\omega}_{\Lambda_n{}^c} \cup \boldsymbol{\omega}'$ connecting $\boldsymbol{x}_{k,m} \in \boldsymbol{\omega}'_{C(k,m)}$ to the point in $\boldsymbol{\omega}_{C(k_s,m_s)}$. □

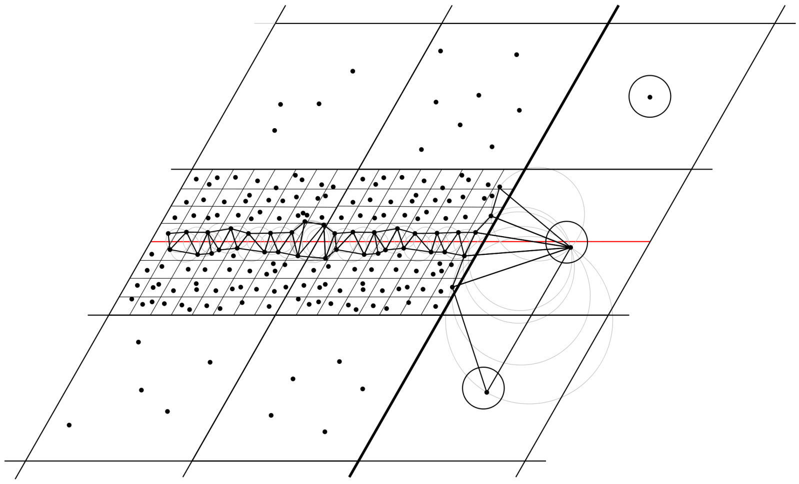


Figure 3.6: An illustration of two open cells meeting the boundary of Λ_n, which is represented by the bold line. The open cells in Λ_n have at least one point in each of their 64 sub-cells.

3.4.3 Percolation of $\mathcal{L}_{X_n}$.

It remains to show that the assumptions of Corollary 2.4.2 are satisfied for the measures $(\mathcal{L}_{X_n})_{n=1}^{\infty}$. The measures $\mathcal{L}_{X_n}$ can be considered as measures on $\{0,1\}^{\mathbb{Z}^2\cup\mathcal{B}}$ where all bonds are opened with probability 1. For any $(k,m) \in \{-n,...,n\}^2$ and $X \in \{0,1\}^{\mathbb{Z}^2}$ satisfying $X(i,j)=1$ for all $(i,j) \notin \{-n,...,n\}^2$,

$$\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}\left(X_n(k,m)=1\middle|X_n(i,j)=X(i,j) \text{ for } (i,j)\neq(k,m)\right) \tag{3.4.3}$$

$$= \hat{C}^{z,\text{site}}_{\Lambda_n,\omega}\left(\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}(F_{k,m}\cap O_{k,m}|\boldsymbol{\mathcal{F}}_{C(k,m)^c})\middle|X_n(i,j)=X(i,j) \text{ for } (i,j)\neq(k,m)\right)$$

$$\geq \operatorname*{ess\,inf}_{\boldsymbol{\omega}'\in\boldsymbol{\Omega}_{C(k,m)^c}} \hat{C}^{z,\text{site}}_{\Lambda_n,\omega}\left(F_{k,m}\cap O_{k,m}\middle|\mathbf{pr}_{C(k,m)^c}=\boldsymbol{\omega}'\right), \tag{3.4.4}$$

where the essential infimum is taken with respect to $\hat{C}^{z,\text{site}}_{\Lambda_n,\omega}\circ\mathbf{pr}_{C(k,m)^c}{}^{-1}$. Since we are dealing with standard Borel spaces, the regular conditional probability in (3.4.4) is guaranteed to exist. For $(k,m)\notin\{-n,...,n\}^2$, there is only one point in each cell, so the expression (3.4.3) is equal to $\hat{p}$, which is in turn greater than (3.4.4) since the latter is at most $\hat{p}^{64}$. Therefore it is sufficient to show that (3.4.4) is greater than the critical probability for site percolation on $\mathbb{Z}^2$, denoted $p_c^{\text{site}}(\mathbb{Z}^2)$.

First we will bound the probability of the event $F_{k,m}$ from below. For

$\Delta \subset \Lambda_n$ the regular conditional probability

$$\Omega_{\Delta^c} \times \mathcal{F} \ni (\omega', B) \mapsto M^z_{\Lambda_n,\omega}(B | \operatorname{pr}_{\Delta^c} = \omega')$$

is given $M^z_{\Lambda_n,\omega} \circ \operatorname{pr}^{-1}_{\Delta^c}$-almost everywhere by the function

$$(\omega', B) \mapsto \frac{\int \mathbb{1}_B(\omega' \cup \omega'') h^z_{\Lambda_n,\omega}(\omega' \cup \omega'') e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'' \cup \omega'_{\Lambda_n})} \Pi^z_{\Delta}(\mathrm{d}\omega'')}{\int h^z_{\Lambda_n,\omega}(\omega_{(\Lambda_n)^c} \cup \omega') e^{-H^{\psi}_{\Lambda_n,\omega}(\omega')} \Pi^z_{\Lambda_n}(\mathrm{d}\omega')}.$$

The proof of this follows that of the analogous case in [Eye14, page 40-41]. For the rest of this section let $\epsilon = \frac{1}{2}(1 - p_c^{\mathrm{site}}(\mathbb{Z}^2))$.

Lemma 3.4.4. *Suppose* $\alpha_0 < \sin^{-1}\left(\frac{3}{64}\right)$, $64r < 3R$, $\ell \in \left(\frac{64}{\sqrt{3}}(r \ \vee \ R\sin(\alpha_0)), \sqrt{3}R\right)$ *and*

$$z > z_0''(\ell, q, r, R, \alpha_0) := \frac{64 q^{\frac{2\pi}{\alpha_0} - 1}}{\epsilon\left(\frac{\ell}{8} - \frac{8}{\sqrt{3}}(r \vee R\sin(\alpha_0))\right)^2}. \tag{3.4.5}$$

Then for any pseudo-periodic boundary condition $\omega \in \Gamma$ *and any sub-cell* $C^{i,j}_{k,m}$ *with* $|k|, |m| \le n$,

$$M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} \ge 1 | \operatorname{pr}_{(C^{i,j}_{k,m})^c} = \omega') > 1 - \frac{\epsilon}{64}.$$

for $M^z_{\Lambda_n,\omega} \circ \operatorname{pr}^{-1}_{(C^{i,j}_{k,m})^c}$*-almost all* ω'.

Proof. Assume $M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 0 | \operatorname{pr}_{(C^{i,j}_{k,m})^c} = \omega') > 0$, else the result is trivial. This implies that

$$\psi(\tau) = 0 \text{ for all } \tau \in \operatorname{Del}_{3,\Lambda_n}(\omega'). \tag{3.4.6}$$

Define $\nabla^{i,j}_{k,m}$ to be the rhombus of side length $d = \frac{\ell}{8} - \frac{8}{\sqrt{3}}(r \vee R\sin(\alpha_0))$ which is a contraction of $C^{i,j}_{k,m}$ about its center point (see figure 3.7). We first claim that for $M^z_{\Lambda_n,\omega} \circ \operatorname{pr}^{-1}_{(C^{i,j}_{k,m})^c}$-almost all ω' and $x \in \nabla^{i,j}_{k,m}$,

$$\psi(\tau) = 0 \text{ for all } \tau \in \operatorname{Del}_{3,\Lambda_n}(\omega' \cup x). \tag{3.4.7}$$

It suffices to only consider the triangles $\tau \in T^+_{x,\omega'}$. Any edge $\{x,y\} \in \mathrm{Del}_2(\omega' \cup x)$ must satisfy $|x-y| > \frac{\sqrt{3}}{4}(\frac{1}{8}\ell - d) = 2(r \vee R\sin(\alpha_0)) > 2r$. Thus $\delta(\tau) > r$ for all $\tau \in T^+_{x,\omega'}$. By Lemma 3.4.1 and (3.4.6) we also have $\delta(\tau) < R$.
By the same argument used to compare the angles θ_1 and θ_2 in Lemma 3.4.1, if θ is an angle belonging to a triangle $\tau \in T^+_{x,\omega'}$ which is subtended at x then $\theta \geq \alpha_0$. All edges $\{x,y\} \in \mathrm{Del}_2(\omega' \cup x)$ must have length at least $\frac{\sqrt{3}}{4}(\frac{1}{8}\ell - d) \geq 2R\sin(\alpha_0)$, so by the law of sines if $\tau \in T^+_{x,\omega'}$ and θ is an angle of τ not subtended at x_0 then

$$\sin(\theta) \geq \frac{2R\sin(\alpha_0)}{2\delta(\tau)} \geq \sin(\alpha_0),$$

and so $\theta \geq \alpha_0$ since we know that $\alpha_0 \leq \pi/3$. This completes the proof of (3.4.7). Together with (3.4.6) this implies that

$$e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'_{\Lambda_n} \cup x)} = e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'_{\Lambda_n})} = 1 \qquad \text{for } x \in \nabla^{i,j}_{k,m}. \tag{3.4.8}$$

We now move on to the computation of the lower bound. We start by applying the following formula for the Poisson point process:

$$\int f(\omega'')\Pi^z_{C^{i,j}_{k,m}}(\mathrm{d}\omega'') = e^{-z|C^{i,j}_{k,m}|}\sum_{n=0}^{\infty}\frac{z^n}{n!}\int_{(C^{i,j}_{k,m})^n} f(\{x_1,...,x_n\})\mathrm{d}x_1,..,\mathrm{d}x_n$$

(which is valid for bounded measurable functions $f : \Omega_{C^{i,j}_{k,m}} \to [0,\infty)$.) This gives us

$$\begin{aligned}
&\frac{M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 1|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega')}{M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 0|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega')} \\
&= \frac{\int \mathbb{1}_{\{N_{C^{i,j}_{k,m}}=1\}}(\omega'')h^z_{\Lambda_n,\omega}(\omega' \cup \omega'')e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'_{\Lambda_n} \cup \omega'')}\Pi^z_{C^{i,j}_{k,m}}(\mathrm{d}\omega'')}{\int \mathbb{1}_{\{N_{C^{i,j}_{k,m}}=0\}}(\omega'')h^z_{\Lambda_n,\omega}(\omega')e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'_{\Lambda_n})}\Pi^z_{C^{i,j}_{k,m}}(\mathrm{d}\omega'')} \\
&= \frac{ze^{-z|C^{i,j}_{k,m}|}\int_{C^{i,j}_{k,m}} h^z_{\Lambda_n,\omega}(\omega' \cup x)e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'_{\Lambda_n} \cup x)}\mathrm{d}x}{e^{-z|C^{i,j}_{k,m}|}h^z_{\Lambda_n,\omega}(\omega')e^{-H^{\psi}_{\Lambda_n,\omega}(\omega'_{\Lambda_n})}}
\end{aligned}$$

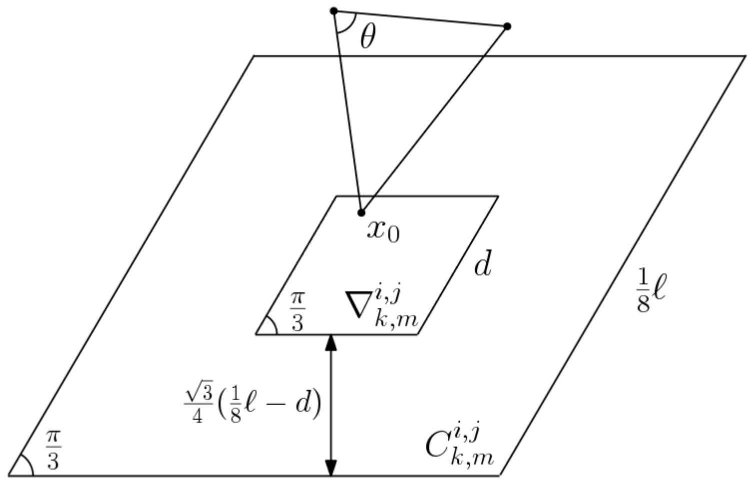


Figure 3.7

Now restricting the domain of integration to $\nabla^{i,j}_{k,m}$ and applying (3.4.8) yields

$$\frac{M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 1|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega')}{M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 0|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega')} \geq z \int_{\nabla^{i,j}_{k,m}} \frac{h^z_{\Lambda_n,\omega}(\omega' \cup x)}{h^z_{\Lambda_n,\omega}(\omega')} \mathrm{d}x.$$

By applying Lemma 3.4.2 we can see that the right hand side is greater than

$$zq^{1-\frac{2\pi}{\alpha_0}} |\nabla^{i,j}_{k,m}| = zq^{1-\frac{2\pi}{\alpha_0}} \left(\frac{\ell}{8} - \frac{8}{\sqrt{3}}(r \vee R\sin(\alpha_0))\right)^2,$$

and therefore

$$\begin{aligned} M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} \geq 1|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega') &\geq 1 - \frac{M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 0|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega')}{M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 1|\,\mathrm{pr}_{(C^{i,j}_{k,m})^c} = \omega')} \\ &\geq 1 - \frac{q^{\frac{2\pi}{\alpha_0}-1}}{z\left(\frac{\ell}{8} - \frac{8}{\sqrt{3}}(r \vee R\sin(\alpha_0))\right)^2} \\ &> 1 - \frac{\epsilon}{64}, \end{aligned}$$

where the last inequality is due to assumption (3.4.5). □

Corollary 3.4.5. *If the assumptions of Lemma 3.4.4 are satisfied, then*

$$M^z_{\Lambda_n,\omega}(F_{k,m}|\operatorname{pr}_{C(k,m)^c} = \omega') > 1 - \epsilon.$$

for $M^z_{\Lambda_n,\omega} \circ \operatorname{pr}^{-1}_{C(k,m)^c}$*-almost all* ω'.

Proof.

$$\begin{aligned}
&M^z_{\Lambda_n,\omega}(F_{k,m}|\operatorname{pr}_{C(k,m)^c} = \omega') \\
&\geq 1 - \sum_{0\leq i,j\leq 7} M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 0|\operatorname{pr}_{C(k,m)^c} = \omega') \\
&\geq 1 - \sum_{0\leq i,j\leq 7} \int M^z_{\Lambda_n,\omega}(N_{C^{i,j}_{k,m}} = 0|\operatorname{pr}_{(C^{i,j}_{k,m})^c} = \omega''_{(C^{i,j}_{k,m})^c}) \\
&\qquad\qquad\qquad\qquad M^z_{\Lambda_n,\omega}(\mathrm{d}\omega''|\operatorname{pr}_{C(k,m)^c}(\omega'') = \omega') \\
&> 1 - \epsilon.
\end{aligned}$$

□

The final component we need to finish the proof is an upper bound on the number of particles in a cell $C(k,l)$. If $H_{\Lambda_n,\omega}(\omega') < \infty$ then $\{x,y\} \in \operatorname{Del}_{2,\Lambda_n}(\omega_{\Lambda_n^c} \cup \omega') \implies |x-y| \geq 2r\sin(\alpha_0)$ by the law of sines. Since the Delaunay graph is a nearest neighbour graph, this means that no two particles are within a distance of $2r\sin(\alpha_0)$ of one another. Therefore

$$m(\ell, r, \alpha_0) := \left(\frac{\ell + 2r\sin(\alpha_0)}{r\sin(\alpha_0)}\right)^2 = \left(\frac{\ell}{r\sin(\alpha_0)} + 2\right)^2, \tag{3.4.9}$$

which is an upper bound on the number of non-overlapping circles with radius $r\sin\alpha_0$ that can fit inside a rhombus with side length $\ell + 2r\sin\alpha_0$, is an upper bound for $|\omega'|$.

We can now prove the existence of the required lower bound on (3.4.4).

Proposition 3.4.6. *Suppose* $\alpha_0 < \sin^{-1}(\frac{3}{64})$, $64r < 3R$ *and* $\ell \in (\frac{64}{\sqrt{3}}(r \vee R\sin(\alpha_0)), \sqrt{3}R)$. *If*

$$\beta > \beta_0'(\ell, q, R, r, \alpha_0) := \frac{\frac{3\sqrt{3}}{4}q^2R^2}{(1-\epsilon)^{-1/m(\ell,r,\alpha_0)} - 1}$$

and $z > z_0''(\ell, q, R, r, \alpha_0)$ *then there exists* $c > 0$ *such that for any* $n \in \mathbb{N}$, *any*

$|k|, |m| \leq n$ *and any pseudo-periodic boundary condition* $\omega \in \Gamma$,

$$\hat{C}^{z,\mathrm{site}}_{\Lambda_n,\omega}\left(F_{k,m} \cap O_{k,m}\middle| \mathbf{pr}_{C(k,m)^c} = \boldsymbol{\omega}'\right) \geq c > 0$$

for $\hat{C}^{z,\mathrm{site}}_{\Lambda_n,\omega} \circ \mathbf{pr}^{-1}_{C(k,m)^c}$*-almost all* $\boldsymbol{\omega}'$.

Proof. Using the lower bound from Corollary 3.4.5 and the lower bound β_0 we have

$$\begin{aligned}
\hat{C}^{z,\mathrm{site}}_{\Lambda_n,\omega}(F_{k,m} \cap O_{k,m}| \mathbf{pr}_{C(k,m)^c} = \boldsymbol{\omega}') &\geq \hat{p}^m M^z_{\Lambda_n,\omega}(F_{k,m}| \mathrm{pr}_{C(k,m)^c} = \omega') \\
&\geq \left(\frac{1}{\frac{3\sqrt{3}}{4\beta}q^2R^2 + 1}\right)^m (1-\epsilon) \\
&\geq \left(\frac{1}{(1-\epsilon)^{-\frac{1}{m}}}\right)^m (1-\epsilon) \\
&= (1-\epsilon)^2 \\
&> 1 - 2\epsilon = p_c^{site}(\mathbb{Z}^2) > 0.
\end{aligned}$$

□

Recall that at the beginning of section 3.4.2 we assumed that the parameters ρ, ℓ and zq satisfied the requirements of Proposition 3.3.3. We need to check that these requirements can be satisfied simultaneously with those of the previous proposition. The only possible conflict relates to the parameter ℓ. We require that $\ell \in \left(\frac{64}{\sqrt{3}}(r \ \vee \ R\sin(\alpha_0)), \sqrt{3}R\right) \cap \left(\frac{r}{L(\rho)}, \frac{R}{U(\rho)}\right) := I_0(R, r, \alpha_0, \rho)$. This set is non-empty for small enough ρ since

$$\frac{64}{\sqrt{3}}(r \ \vee \ R\sin(\alpha_0)) < \sqrt{3}R = \lim_{\rho\to 0}\frac{R}{U(\rho)}$$

and

$$\sqrt{3}R > \sqrt{3}r = \lim_{\rho\to 0}\frac{r}{L(\rho)}.$$

We therefore have the following, from which Theorem 3.2.2 is derived by selecting particular values of ρ and ℓ.

Corollary 3.4.7. *There exists* $\rho_0'(R, r, \alpha_0) > 0$ *such that* $I_0 \neq \emptyset$ *if* $\rho < \rho_0'$.

Moreover, if

$$\begin{aligned} \alpha_0 &< \sin^{-1}(3/64), \\ 64r &< 3R, \\ \rho &< \rho_0'(R, r, \alpha_0), \\ \ell &\in I_0(R, r, \alpha_0, \rho), \\ \beta &> \beta_0(\ell, q, R, r, \alpha_0), \\ \text{and} \qquad z &> z_0''(\ell, q, R, r, \alpha_0) \ \vee \ (1/q) z_0'(\beta, \rho, \ell) \end{aligned}$$

then there exists at least q *translation-invariant Delaunay continuum Potts measures for* $\mathbf{Del}_3, z$ *and* φ.

Chapter 4

Infinite Volume Delaunay Random Cluster Measures

The method that we employed in the previous chapter to prove that a phase transition occurs centers around the random cluster representation (Theorem 2.3.3), which provides a connection between the Delaunay continuum Potts distribution and the Delaunay continuum random cluster distribution in a bounded region $\Delta \Subset \mathbb{R}^2$. This connection was enough to prove the phase transition in the previous chapter. In this chapter we are interested in an infinite volume analogue of the Delaunay continuum random cluster distribution. This choice of direction is inspired by work on a similar model introduced by Dereudre and Houdebert [DH15] called the *infinite volume continuum random cluster model*, which is a Gibbs modification of the stationary Poisson Boolean model, defined using the DLR equations. The authors proved the existence of such a model and discovered a Fortuin-Kastelyn representation relating it to the Widom-Rowlinson model. This FK representation was previously known for finite volume measures but the authors extended it to infinite volume measures, which facilitated the extension of the phase transition results of [CCK95] and [GH96] for the Widom-Rowlinson model to the case of nondeterministic radii [Hou17]. In [HTH19], the authors used a continuum extension of the classical disagreement percolation technique introduced in [vdBM94] to show that there exists a unique Gibbs measure for a variety of germ-grain models including the infinite volume continuum random cluster model. The FK representation could then be used to conclude that the Widom-Rowlinson model is unique for small activities. We hope that an investigation of Delaunay continuum random cluster measures in infinite volume might lead to similar insights for Delaunay continuum Potts measures, as we expect these measures to be related via a random cluster representation.

In contrast to previous chapters we use the hypergraph structure Del_2 instead of Del_3 here. There is still a random cluster representation in this case (see [AE16]) so the motivations of the last paragraph are still relevant. Our configurations therefore consist of a locally finite set of points and a locally finite set of edges ($k = 2$ in equation (2.3.1)). Thus far we have only defined the finite volume Delaunay continuum random cluster model for the wired boundary condition (Definition 2.3.1). In this case the number of connected components N_{cc} is always finite, and so the definition is straightforward. In this chapter we want to expand this definition to include arbitrary boundary conditions $G = (\omega, E) \in \mathcal{G}$ and so we need to define a notion of the 'local' number of connected components.

The main result of this chapter is the existence of an infinite volume Delaunay continuum random cluster measure (Theorem 4.3.1). We require that the background potential satisfies two hardcore constraints and the edge probabilities (those assigned by $\mu_{\omega,\Delta}$) are bounded away from 0 and 1. The proof follows an analogous approach to [DH15]. We begin by constructing a sequence of probability measures from the finite volume distributions and showing that it has an accumulation point. This is done using a compactness result relating to the specific entropy. Then we prove that the limit, after conditioning out the empty configuration, satisfies the DLR equations. In the course of the proof we show that the infinite volume measure has at most one infinite connected component. The question of whether there is a unique infinite volume measure is still open.

4.1 Preliminaries

4.1.1 Sample spaces and sigma algebras

We use the definitions given in section 2.3.1, with $d = 2$ and $\mathcal{H} = \mathrm{Del}_2$ (and so $k = 2$), but some additional definitions are needed. For $\Delta \Subset \mathbb{R}^2$, $\Lambda \Subset \mathbb{R}^4$, let

$$\begin{aligned}\Omega_\Delta &:= \{\omega \in \Omega : \omega \subset \Delta\},\\ \mathcal{E}_\Lambda &:= \{E \in \mathcal{E} : E \subset \Lambda\},\\ \mathcal{G}_{\Delta,\Lambda} &:= \Omega_\Delta \times \mathcal{E}_\Lambda.\end{aligned}$$

The σ-algebras we equip these spaces with are $\mathcal{F}|_{\Omega_\Delta} := \{A \cap \Omega_\Delta : A \in \mathcal{F}\}$, $\Sigma|_{\mathcal{E}_\Lambda} := \{A \cap \mathcal{E}_\Lambda : A \in \Sigma\}$ and $\mathcal{A}|_{\mathcal{G}_{\Delta,\Lambda}} = \mathcal{F}|_{\Omega_\Delta} \otimes \Sigma|_{\mathcal{E}_\Lambda}$ respectively. The projections onto Ω_Δ and $\mathcal{E}_\Lambda$ are denoted $\mathrm{pr}^1_\Delta : \Omega \to \Omega_\Delta,\ \omega \mapsto \omega_\Delta$, and $\mathrm{pr}^2_\Lambda : \mathcal{E} \to \mathcal{E}_\Lambda,\ E \mapsto E_\Lambda$.

If Δ and Λ are 2 and 4-dimensional open cubes respectively (the set of d-dimensional cubes is defined in (2.1.3),) then $\Omega_\Delta, \mathcal{E}_\Lambda$ and $\mathcal{G}_{\Delta,\Lambda}$ are G_δ sets and are therefore Polish spaces when equipped with their subspace topologies. Furthermore, in this case $\mathcal{F}|_{\Omega_\Delta}$, $\Sigma|_{\mathcal{E}_\Lambda}$ and $\mathcal{A}|_{\mathcal{G}_{\Delta,\Lambda}}$ are the associated Borel σ-algebras for these spaces. If $\Delta_n \nearrow \mathbb{R}^2$ and $\Lambda_n \nearrow \mathbb{R}^4$ are sequences of cubes then

$$\begin{aligned}\mathcal{F} &= \sigma\left(\bigcup_{n\geq 1} \mathcal{F}_{\Delta_n}\right), \qquad \Sigma = \sigma\left(\bigcup_{n\geq 1} \Sigma_{\Lambda_n}\right), \text{ and}\\ \mathcal{A} &= \sigma\left(\bigcup_{n\geq 1} \mathcal{F}_{\Delta_n} \times \bigcup_{n\geq 1} \Sigma_{\Lambda_n}\right) = \sigma\left(\bigcup_{n\geq 1} \mathcal{A}_{\Delta_n,\Lambda_n}\right),\end{aligned}$$

where $\mathcal{F}_\Delta := (\mathrm{pr}^1_\Delta)^{-1}\mathcal{F}|_{\Omega_\Delta}$, $\Sigma_\Lambda := (\mathrm{pr}^2_\Lambda)^{-1}\Sigma|_{\mathcal{E}_\Lambda}$ and $\mathcal{A}_{\Delta,\Lambda} := \mathcal{F}_\Delta \otimes \Sigma_\Lambda$.

Hereafter we only consider subregions of $\mathbb{R}^4$ of the form Δ^2 for some $\Delta \Subset \mathbb{R}^2$, and so we will write E_Δ, $\mathcal{E}_\Delta$ and Σ_Δ in place of E_{Δ^2}, $\mathcal{E}_{\Delta^2}$ and Σ_{Δ^2} respectively. We also define $\mathcal{G}_\Delta := \mathcal{G}_{\Delta,\Delta^2}$, $\mathcal{A}_\Delta := \mathcal{A}_{\Delta,\Delta^2}$, and the projection $\mathrm{pr}_\Delta : \mathcal{G} \to \mathcal{G}_\Delta$, $G \mapsto G_\Delta := (\omega_\Delta, E_\Delta)$.

4.1.2 Properties of Delaunay triangulations

The following lemmas describe the effect on the Delaunay triangulation when a configuration is changed in a bounded region. They will be used throughout this chapter. In particular, Lemma 4.1.5 is used to prove the additivity property (4.2.4), which is crucial for verifying the consistency of the finite volume distributions (Proposition 4.2.6). As discussed in section 2.2, we assume that all configurations $\omega \in \Omega$ are in general quadratic position.

Proposition 4.1.1. *For $\Lambda \subset \Delta \subset \mathbb{R}^2$ and $\omega \in \Omega$,*

$$\mathrm{Del}_{3,\Delta}(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega) = \mathrm{Del}_{3,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{3,\Lambda}(\omega_{\Lambda^c}).$$

Proof. $\tau \in \mathrm{Del}_{3,\Delta}(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega)$ if and only if τ satisfies

1. $|\tau| = 3$,
2. $\tau \subset \omega$,
3. $B(\tau) \cap \omega = \emptyset$,
4. $\overline{B(\tau)} \cap \Delta \neq \emptyset$,
5. $\overline{B(\tau)} \cap \Lambda = \emptyset$,

and $\tau \in \mathrm{Del}_{3,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{3,\Lambda}(\omega_{\Lambda^c})$ if and only if τ satisfies 1, 4, 5 and

2'. $\tau \subset \omega_{\Lambda^c}$

3'. $B(\tau) \cap \omega_{\Lambda^c} = \emptyset$.

But $2 \wedge 5 \iff 2' \wedge 5$ and $3 \wedge 5 \iff 3' \wedge 5$. □

Corollary 4.1.2. *For $\Lambda \subset \Delta \subset \mathbb{R}^2$ and $\omega \in \Omega$,*

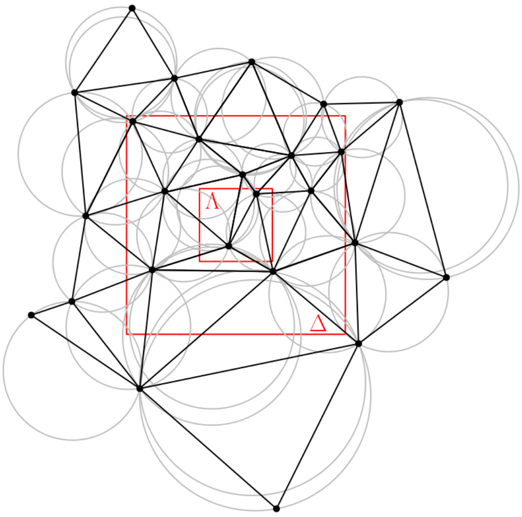
$$\mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega) = \mathrm{Del}_{2,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{2,\Lambda}(\omega_{\Lambda^c}).$$

Proof. $\eta \in \mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)$ if and only if $\exists \tau_1, \tau_2 \supset \eta$ where either both τ_1 and τ_2 belong to $\mathrm{Del}_{3,\Delta}(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega)$, or one belongs to $\mathrm{Del}_{3,\Delta}(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega)$ and the other belongs to $\mathrm{Del}_3(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega)$. But by Proposition 4.1.1, $\mathrm{Del}_{3,\Delta}(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega) = \mathrm{Del}_{3,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{3,\Lambda}(\omega_{\Lambda^c})$ and $\mathrm{Del}_3(\omega) \setminus \mathrm{Del}_{3,\Lambda}(\omega) = \mathrm{Del}_3(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{3,\Lambda}(\omega_{\Lambda^c})$, which gives the result. □

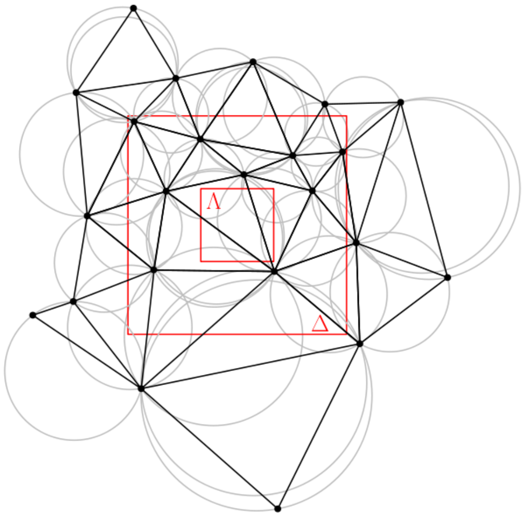
For $G = (\omega, E) \in \mathcal{G}$, and $\Lambda \subset \Delta \subset \mathbb{R}^2$ we make the following definitions:

- $G^{\Delta} := (\omega^{\Delta}, E^{\Delta})$, where $\omega^{\Delta} := \mathrm{Del}_{1,\Delta}(\omega)$ and $E^{\Delta} := E \cap \mathrm{Del}_{2,\Delta}(\omega)$,
- $G^{\Delta,\Lambda} := (\omega^{\Delta,\Lambda}, E^{\Delta,\Lambda})$, where $\omega^{\Delta,\Lambda} := (\omega^{\Delta})_{\Lambda^c} = (\mathrm{Del}_{1,\Delta}(\omega))_{\Lambda^c}$ and $E^{\Delta,\Lambda} := E^{\Delta} \setminus \mathrm{Del}_{2,\Lambda}(\omega) = E \cap \mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)$.

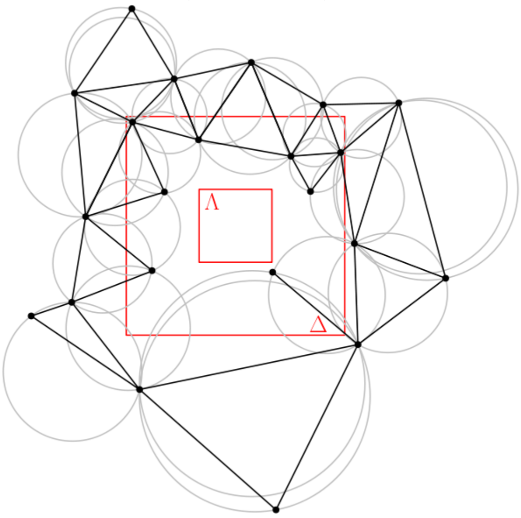
As we shall see later, our model is defined in such a way that if G is chosen as a boundary condition for the finite volume distribution in Δ then $G^{\mathbb{R}^2,\Delta}$ remains fixed. We now present a few more lemmas which will be used implicitly throughout this chapter.



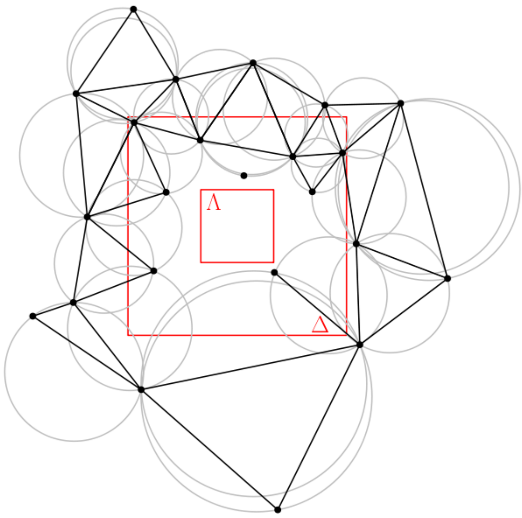
(a) $(\omega, \mathrm{Del}_2(\omega))^{\Delta} = (\mathrm{Del}_{1,\Delta}(\omega), \mathrm{Del}_{2,\Delta}(\omega))$



(b) $(\omega_{\Lambda^c}, \mathrm{Del}_2(\omega_{\Lambda^c}))^{\Delta}$



(c) The graph induced by the edge set $\mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)$ $= \mathrm{Del}_{2,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{2,\Lambda}(\omega_{\Lambda^c})$.



(d) $(\omega, \mathrm{Del}_2(\omega))^{\Delta,\Lambda}$.

Figure 4.1: The graph (c) is obtained by taking either (a) or (b) and removing the edges belonging to triangles whose circumcircles intersect Λ (see Corollary 4.1.2). Notice that there may be points in $\omega^{\Delta,\Lambda}$ for which all incident edges are in $\mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)$. These points will belong to the graph (d) but not in (c).

Lemma 4.1.3. *For $\Lambda \subset \Delta \subset \mathbb{R}^2$ and $G = (\omega, E) \in \mathcal{G}$,*

$$(G^{\mathbb{R}^2,\Lambda})^{\Delta,\Lambda} = G^{\Delta,\Lambda}.$$

Proof. Let $(\omega', E') = (G^{\mathbb{R}^2,\Lambda})^{\Delta,\Lambda}$. By definition $\omega' = ((\omega_{\Lambda^c})^{\Delta})_{\Lambda^c} = (\omega_{\Lambda^c})^{\Delta}$.
If $x \in \omega^{\Delta,\Lambda}$ then x belongs to a triangle $\tau \in \mathrm{Del}_{3,\Delta}(\omega)$. If $\tau \notin \mathrm{Del}_{3,\Delta}(\omega_{\Lambda^c})$ then $\exists \tau' \in \mathrm{Del}_3(\omega_{\Lambda^c})$ with $x \in \tau'$ whose circumcircle intersects Λ. But $\Lambda \subset \Delta$, so $\tau' \in \mathrm{Del}_{3,\Delta}(\omega_{\Lambda^c})$, and therefore $x \in (\omega_{\Lambda^c})^{\Delta} = \omega'$.
Conversely, suppose $x \in \omega'$. Then x belongs to a triangle $\tau \in \mathrm{Del}_{3,\Delta}(\omega_{\Lambda^c})$. If $\tau \notin \mathrm{Del}_{3,\Delta}(\omega)$ then $\overline{B(\tau)} \cap \omega_\Lambda \neq \emptyset$, and so $\exists \tau' \in \mathrm{Del}_{3,\Delta}(\omega)$ containing x. Thus $x \in \omega^{\Delta,\Lambda}$.
To prove that the edge sets are the same we use Corollary 4.1.2:

$$\begin{aligned}
E' &= (E \cap \mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)) \cap (\mathrm{Del}_{2,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{2,\Lambda}(\omega_{\Lambda^c})) \\
&= E \cap (\mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)) \cap (\mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)) \\
&= E \cap \mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega) \\
&= E^{\Delta,\Lambda}.
\end{aligned}$$

□

Lemma 4.1.4. *For $\Lambda \subset \Delta \subset \mathbb{R}^2$ and $G = (\omega, E) \in \mathcal{G}$,*

$$(G^{\Delta,\Lambda})_{\Lambda^c} = G^{\Delta,\Lambda} = (G_{\Lambda^c})^{\Delta,\Lambda}.$$

Proof. The first equality is clear. For the second equality the proof of Lemma 4.1.3 gives us $\omega^{\Delta,\Lambda} = (\omega_{\Lambda^c})^{\Delta}$, which is clearly equal to $((\omega_{\Lambda^c})^{\Delta})_{\Lambda^c}$ as required. Showing that the edge sets are the same is another application of Corollary 4.1.2:

$$\begin{aligned}
E^{\Delta,\Lambda} &= E \cap \mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega) \\
&= E_{\Lambda^c} \cap \mathrm{Del}_{2,\Delta}(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega) \\
&= E_{\Lambda^c} \cap \mathrm{Del}_{2,\Delta}(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{2,\Lambda}(\omega_{\Lambda^c})
\end{aligned}$$

□

Lemma 4.1.5. *For $\Lambda \subset \Delta \subset \mathbb{R}^2$ and $G = (\omega, E) \in \mathcal{G}$,*

$$(G^{\mathbb{R}^2,\Lambda})^{\mathbb{R}^2,\Delta} = G^{\mathbb{R}^2,\Delta}.$$

Proof. The vertex sets are clearly the same since $(\omega_{\Lambda^c})_{\Delta^c} = \omega_{\Delta^c}$.
To show the equivalence of the edge sets, we apply Corollary 4.1.2:

$$\begin{aligned}
&(E \cap \mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)) \cap (\mathrm{Del}_2(\omega_{\Lambda^c}) \setminus \mathrm{Del}_{2,\Delta}(\omega_{\Lambda^c}))\\
=&(E \cap \mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Lambda}(\omega)) \cap (\mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Delta}(\omega))\\
=&E \cap \mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Delta}(\omega)\\
=&E^{\mathbb{R}^2,\Delta}.
\end{aligned}$$

□

4.2 Definitions

4.2.1 The reference measure

To define the reference measures we begin with a function p which assigns to each possible edge $e \in \mathcal{E}_{\mathbb{R}^2,2}$ an *edge weight* $p(e) \in [0,1]$. These will represent the probabilities that each edge of Del_2 is open with respect to the reference measure. For $\omega \in \Omega$, let $\{\xi_e \mid e \in \mathrm{Del}_2(\omega)\}$ be a set of independent Bernoulli random variables where $\xi_e = 1$ with probability $p(e)$. Let μ_ω denote the law of the random set $\{e \in \mathrm{Del}_2(\omega) | \ \xi_e = 1\}$. This measure is called the *edge drawing mechanism*. For any subset $A \subset \mathrm{Del}_2(\omega)$, the law of the random set $\{e \in A | \ \xi_e = 1\}$ is called the *edge drawing mechanism on A*.

For $\Lambda \subset \Delta \Subset \mathbb{R}^2$, $\omega \in \Omega$ and $\omega' \in \Omega_\Delta$ let $\mu_{\Delta,\omega',\omega}$ denote the edge drawing mechanism on $\mathrm{Del}_{2,\Delta}(\omega' \cup \omega_{\Delta^c})$, and let $\mu^{\Lambda}_{\Delta,\omega',\omega}$ denote the edge drawing mechanism on $\mathrm{Del}_{2,\Delta}(\omega' \cup \omega_{\Delta^c}) \setminus \mathrm{Del}_{2,\Lambda}(\omega' \cup \omega_{\Delta^c})$. Note that $\mu_{\Delta,\omega',\omega,} = \mu_{\Delta,\omega',\omega_{\Delta^c}}$ and $\mu^{\Lambda}_{\Delta,\omega',\omega} = \mu^{\Lambda}_{\Delta,\omega'_{\Delta\setminus\Lambda},\omega_{\Delta^c}}$ by Corollary 4.1.2, and furthermore if $\omega'' \in \Omega_\Delta$ with $(\omega'')_{\Delta\setminus\Lambda} = (\omega')_{\Delta\setminus\Lambda}$ then

$$\mu_{\Delta,\omega',\omega} = \mu_{\Lambda,\omega'_\Lambda,\omega''\cup\omega_{\Delta^c}} \otimes \mu^{\Lambda}_{\Delta,\omega'',\omega}. \tag{4.2.1}$$

Intersections and unions are defined as follows for $G_1 = (\omega_1, E_1)$,
$G_2 = (\omega_2, E_2) \in \mathcal{G}$:

$$\begin{aligned}
G_1 \cup G_2 :=&(\omega_1 \cup \omega_2, E_1 \cup E_2),\\
G_1 \cap G_2 :=&(\omega_1 \cap \omega_2, E_1 \cap E_2).
\end{aligned}$$

Definition 4.2.1. For $G = (\omega, E) \in \mathcal{G}$ and $\Delta \Subset \mathbb{R}^2$, let $\nu_{\Delta,G}$ be the measure on $(\mathcal{G}, \mathcal{A})$ given by

$$\nu_{\Delta,G}(A) := \int \mathbb{1}_A(G^{\mathbb{R}^2,\Delta} \cup (\omega', E'))\mu_{\Delta,\omega',\omega}(\mathrm{d}E')\Pi^z_\Delta(\mathrm{d}\omega')$$

for any $A \in \mathcal{A}$.

Notice that the graph $G^{\mathbb{R}^2,\Delta} \cup (\omega', E')$ is a subgraph of $(\omega_{\Delta^c} \cup \omega', \mathrm{Del}_2((\omega_{\Delta^c} \cup \omega'))$. The kernels $(\nu_\Delta : \mathcal{G} \times \mathcal{A} \to [0,1])_{\Delta \Subset \mathbb{R}^2}$ where $\nu_\Delta(G, A) := \nu_{\Delta,G}(A)$ satisfy the *Gibbs consistency condition* $\nu_{\Delta,G}(A) = \int \nu_{\Lambda,G'}(A)\nu_{\Delta,G}(\mathrm{d}G')$ for $\Lambda \subset \Delta \Subset \mathbb{R}^2$. Indeed by (4.2.1) and Corollary 4.1.2,

$$\begin{aligned}
\nu_{\Delta,G}(A) &= \int \mathbb{1}_A(\omega_1 \cup \omega_2 \cup \omega_{\Delta^c}, E_1 \cup E_2 \cup E^{\mathbb{R}^2,\Delta}) \\
&\qquad \mu_{\Lambda,\omega_1,\omega_2 \cup \omega_{\Delta^c}}(\mathrm{d}E_1)\Pi^z_\Lambda(\mathrm{d}\omega_1)\mu^\Lambda_{\Delta,\omega_2,\omega}(\mathrm{d}E_2)\Pi^z_{\Delta\setminus\Lambda}(\mathrm{d}\omega_2) \\
&= \int \mathbb{1}_A(\omega_1 \cup (\omega_2)_{\Delta\setminus\Lambda} \cup \omega_{\Delta^c}, E_1 \cup (E_2)^{\Delta,\Lambda} \cup E^{\mathbb{R}^2,\Delta}) \\
&\qquad \mu_{\Lambda,\omega_1,\omega_2 \cup \omega_{\Delta^c}}(\mathrm{d}E_1)\Pi^z_\Lambda(\mathrm{d}\omega_1)\mu_{\Delta,\omega_2,\omega}(\mathrm{d}E_2)\Pi^z_\Delta(\mathrm{d}\omega_2) \\
&= \int \mathbb{1}_A(G_1)\nu_{\Lambda,G_2}(\mathrm{d}G_1)\nu_{\Delta,G}(\mathrm{d}G_2). \qquad (4.2.2)
\end{aligned}$$

It this case it is also said that the kernels $(\nu_\Delta)_{\Delta \Subset \mathbb{R}^2}$ form a *specification.* The measure

$$\nu(A) := \int \mathbb{1}_A(\omega, E)\mu_\omega(\mathrm{d}E)\Pi^z(\mathrm{d}\omega) \qquad (4.2.3)$$

for $A \in \mathcal{A}$ also satisfies equation (4.2.2) in place of $\nu_{\Delta,G}$:

$$\nu(A) = \int \mathbb{1}_A(G_1)\nu_{\Lambda,G_2}(G_1)\nu(\mathrm{d}G_2).$$

4.2.2 The local number of connected components

Proposition 4.2.2. *If* $G = (\omega, E) \in \mathcal{G}$ *with* $|\omega^\Delta| < \infty$ *for all* $\Delta \Subset \mathbb{R}^2$, *then for* $\Lambda \Subset \mathbb{R}^2$ *the limit*

$$N^\Lambda_{cc}(G) := \lim_{\Delta \nearrow \mathbb{R}^2} \left(N_{cc}(G^\Delta) - N_{cc}(G^{\Delta,\Lambda})\right)$$

exists, and is called the local number of connected components in Λ. In addition, $N_{cc}^{\Lambda}(G) \leq |\omega_{\Lambda}|$.

Proof. Let Δ_n be an increasing sequence of sets whose limit is $\mathbb{R}^2$. We will show that the sequence

$$a_n := N_{cc}(G^{\Delta_n}) - N_{cc}(G^{\Delta_n,\Lambda})$$

converges. Clearly a_n is maximised when $E^{\Lambda} = \emptyset$, in which case $a_n = |\omega_{\Lambda}|$. We will now show that a_n is increasing and therefore convergent.
Let $\omega^{\Delta_n} \setminus \omega^{\Delta_{n-1}} = \{x_1, ..., x_{n_m}\}$. If n is large enough that $\Lambda \subset \Delta_{n-1}$, then for any $k \in \{1, .., n_m\}$, $x_k \notin \omega^{\Lambda}$ and so

$$\begin{aligned} E_{n,k} :&= \left\{\{x_k, y\} \in E^{\Delta_n} : y \in \omega^{\Delta_{n-1}} \cup \{x_1, , ..., x_{k-1}\}\right\} \\ &= \left\{\{x_k, y\} \in E^{\Delta_n,\Lambda} : y \in \omega^{\Delta_{n-1}} \cup \{x_1, , ..., x_{k-1}\}\right\}. \end{aligned}$$

Removing the points ω^{Λ} and the edges E^{Λ} from the graph can only increase the number of connected components which are adjacent to x_k, so

$$\begin{aligned} &N_{cc}((\omega^{\Delta_{n-1}} \cup \{x_1, ..., x_k\}, E^{\Delta_{n-1}} \cup_{i=1}^{k} E_{n,i})) \\ &-N_{cc}((\omega^{\Delta_{n-1}} \cup \{x_1, ..., x_{k-1}\}, E^{\Delta_{n-1}} \cup_{i=1}^{k-1} E_{n,i})) \\ &\geq N_{cc}((\omega^{\Delta_{n-1},\Lambda} \cup \{x_1, ..., x_k\}, E^{\Delta_{n-1},\Lambda} \cup_{i=1}^{k} E_{n,i})) \\ &-N_{cc}((\omega^{\Delta_{n-1},\Lambda} \cup \{x_1, ..., x_{k-1}\}, E^{\Delta_{n-1},\Lambda} \cup_{i=1}^{k-1} E_{n,i})). \end{aligned}$$

By writing $a_n - a_{n-1}$ as a telescoping sum over k we conclude that
$a_n - a_{n-1} \geq 0$. □

Proposition 4.2.3. *If $G = (\omega, E) \in \mathcal{G}$ and $|\omega^{\Delta}| < \infty$ for all $\Delta \Subset \mathbb{R}^2$, then for $\Lambda \Subset \mathbb{R}^2$*

$$N_{cc}^{\Lambda}(G) \geq 1 - |(\omega_{\Lambda^c})^{\Lambda}|.$$

Proof. N_{cc}^{Λ} is minimised when $E^{\Lambda} = \text{Del}_{2,\Lambda}(\omega)$, in which case each point in ω^{Λ} belongs to the same connected component. Thus adding the points and edges of G^{Λ} to the graph $G^{\Delta,\Lambda}$ can connect at most $|(\omega^{\Lambda})_{\Lambda^c}|$ components. By Lemma 4.1.3, $|(\omega^{\Lambda})_{\Lambda^c}| = |(\omega_{\Lambda^c})^{\Lambda}|$. □

For $\Lambda \subset \Delta \Subset \mathbb{R}^2$, the function $X_{\Delta,\Lambda}(G) := N_{cc}^{\Delta}(G) - N_{cc}^{\Lambda}(G)$ depends only on $G^{\mathbb{R}^2,\Lambda}$, i.e

$$X_{\Delta,\Lambda}(G) = X_{\Delta,\Lambda}(G^{\mathbb{R}^2,\Lambda}). \tag{4.2.4}$$

To see this we apply Lemmas 4.1.5 and 4.1.3:

$$\begin{aligned}
X_{\Delta,\Lambda}(G) &= \lim_{\nabla \nearrow \mathbb{R}^2} \left(N_{cc}\left(G^{\nabla,\Lambda}\right) - N_{cc}\left(G^{\nabla,\Delta}\right)\right) \\
&= \lim_{\nabla \nearrow \mathbb{R}^2} \Big(N_{cc}((G^{\mathbb{R}^2,\Lambda})^{\nabla,\Lambda}) - N_{cc}((G^{\mathbb{R}^2,\Delta})^{\nabla,\Delta})\Big) \\
&= \lim_{\nabla \nearrow \mathbb{R}^2} \Big(N_{cc}((G^{\mathbb{R}^2,\Lambda})^{\nabla,\Lambda}) - N_{cc}(((G^{\mathbb{R}^2,\Lambda})^{\mathbb{R}^2,\Delta})^{\nabla,\Delta})\Big) \\
&= \lim_{\nabla \nearrow \mathbb{R}^2} \Big(N_{cc}((G^{\mathbb{R}^2,\Lambda})^{\nabla,\Lambda}) - N_{cc}((G^{\mathbb{R}^2,\Lambda})^{\nabla,\Delta})\Big) \\
&= X_{\Delta,\Lambda}(G^{\mathbb{R}^2,\Lambda}).
\end{aligned}$$

This property will be useful when showing that the finite volume distributions form a specification (Proposition 4.2.6).

4.2.3 Delaunay random cluster measures

Let ψ' be a hyperedge potential on Del_3 which can be written $\psi'(\tau,\omega) = \psi(\tau)$ for some measurable function $\psi : \mathcal{E}_{\mathbb{R}^2,3} \to \mathbb{R} \cup \{\infty\}$. Recall from (2.2.4) that the Hamiltonian in $\Delta \Subset \mathbb{R}^2$ with boundary condition $\omega \in \Omega$ satisfies

$$H_{\Delta,\omega}(\omega') = \sum_{\tau \in \mathrm{Del}_{3,\Delta}(\omega_{\Delta^c} \cup \omega')} \psi(\tau).$$

For $q \geq 1$ and $G, G' \in \mathcal{G}$ such that $(G')^{\mathbb{R}^2,\Delta} = G^{\mathbb{R}^2,\Delta}$, define

$$Q_{\Delta,G}(G') := q^{N_{cc}^{\Delta}(G')} e^{-H_{\Delta,\omega}(\omega'_{\Delta})}.$$

This function (up to multiplication by a constant) is the density of the finite volume distribution in Δ with boundary condition G with respect to $\nu_{\Delta,G}$.

We say that $G = (\omega, E) \in \mathcal{G}$ is *admissible* for $\Delta \Subset \mathbb{R}^2$ and ψ if the *partition function*

$$\begin{aligned}\mathsf{Z}_\Delta(G) &:= \int Q_{\Delta,G}(G')\ \nu_{\Delta,G}(\mathrm{d}G') \\ &= \int q^{N_{cc}^{\Delta}(G^{\mathbb{R}^2,\Delta}\cup(\omega',E'))} e^{-H_{\Delta,\omega}(\omega')} \mu_{\Delta,\omega',\omega}(\mathrm{d}E')\Pi_\Delta^z(\mathrm{d}\omega')\end{aligned}$$

is finite and non-zero. By Lemma 4.1.5, $\mathsf{Z}_\Delta(G) = \mathsf{Z}_\Delta(G^{\mathbb{R}^2,\Delta})$. The set of admissible boundary conditions is denoted $\mathcal{G}_*^\Delta$.

Definition 4.2.4. Let $G \in \mathcal{G}_*^\Delta$. The *(finite volume) Delaunay continuum random cluster distribution in* $\Delta \Subset \mathbb{R}^d$ for $\psi, z > 0, q \geq 1$, boundary condition ω and edge weights $p(e)_{\mathcal{E}_{\mathbb{R}^2,2}}$ is the probability measure on $(\mathcal{G}, \mathcal{A})$ defined by

$$C_{\Delta,G}(A) := \mathsf{Z}_\Delta(G)^{-1} \int \mathbb{1}_A(G') Q_{\Delta,G}(G') \nu_{\Delta,G}(\mathrm{d}G').$$

Remark 4.2.5. Let $G = (\omega, E) \in \mathcal{G}_*^\Delta$ where $\mathrm{Del}_2(\omega) \setminus \mathrm{Del}_{2,\Delta}(\omega) \subset E$. This is called a *wired* boundary condition. If $\Lambda \Subset \mathbb{R}^2$ is large enough and $(G')^{\mathbb{R}^2,\Delta} = G^{\mathbb{R}^2,\Delta}$, then

$$\begin{aligned}N_{cc}^\Lambda(G') &= N_{cc}((G')^\Lambda) - N_{cc}((G')^{\Lambda,\Delta}) \\ &= N_{cc}(G') - N_{cc}(G^{\mathbb{R}^2,\Delta}).\end{aligned}$$

Therefore

$$C_{\Delta,G}(A) = \frac{q^{-N_{cc}(G^{\mathbb{R}^2,\Delta})}}{\mathsf{Z}_\Delta(G)} \int q^{N_{cc}(G^{\mathbb{R}^2,\Delta}\cup(\omega',E'))} e^{-H_{\Delta,\omega}(\omega')} \mu_{\Delta,\omega',\omega}(\mathrm{d}E')\Pi_\Delta^z(\mathrm{d}\omega'),$$

which is the analogue to the Delaunay random cluster distribution from Definition 2.3.1 in the case where q is a natural number and edges rather than triangles are declared open or closed.

Similar to Delaunay continuum Potts measures, Delaunay random cluster measures are defined by prescribing conditional probabilities according to the DLR equations. To do this we first need to check that the kernels $(C_\Delta : \mathcal{G}_*^\Delta \times \mathcal{A} \to [0,1])_{\Delta \Subset \mathbb{R}^2}$ defined by $C_\Delta(G, A) := C_{\Delta,G}(A)$ form a specification.

Proposition 4.2.6. *For $\Lambda \subset \Delta \Subset \mathbb{R}^2$ and $G \in \mathcal{G}_*^\Delta$,*

$$C_\Delta(G, \mathcal{G}_*^\Lambda) = 1 \quad \textit{and} \quad C_\Delta(G, f) = \int C_\Lambda(G', f)\ C_\Delta(G, \mathrm{d}G'). \tag{4.2.5}$$

for all measurable functions $f : \mathcal{G} \to [0, \infty)$.

Proof. If $G, G_1, G_2 \in \mathcal{G}$ with $G_1^{\mathbb{R}^2,\Lambda} = G_2^{\mathbb{R}^2,\Lambda}$ and $G^{\mathbb{R}^2,\Delta} = G_1^{\mathbb{R}^2,\Delta} = G_2^{\mathbb{R}^2,\Delta}$ then using (4.2.4) we can deduce that

$$\begin{aligned} N_{cc}^\Delta(G_1) + N_{cc}^\Lambda(G_2) &= N_{cc}^\Lambda(G_1) + X_{\Delta,\Lambda}(G_1^{\mathbb{R}^2,\Lambda}) + N_{cc}^\Lambda(G_2) \\ &= N_{cc}^\Lambda(G_1) + N_{cc}^\Delta(G_2), \end{aligned} \tag{4.2.6}$$

and courtesy of Corollary 4.1.2,

$$H_{\Delta,\omega}((\omega_1)_\Delta) + H_{\Lambda,\omega_1}((\omega_2)_\Lambda) = H_{\Delta,\omega}((\omega_2)_\Delta) + H_{\Lambda,\omega_1}((\omega_1)_\Lambda). \tag{4.2.7}$$

Combining (4.2.6) and (4.2.7) we obtain

$$Q_{\Delta,G}(G_1) Q_{\Lambda,G_1}(G_2) = Q_{\Delta,G}(G_2) Q_{\Lambda,G_1}(G_1)$$

(so the family of functions $(G \mapsto Q_{\Delta,G}(G))_{\Delta \Subset \mathbb{R}^2}$ is a *pre-modification*). After integrating with respect to G_2 we have

$$\begin{aligned} Q_{\Delta,G}(G_1) \int Q_{\Lambda,G_1}(G') \nu_{\Lambda,G_1}(\mathrm{d}G') &= Q_{\Lambda,G_1}(G_1) \int Q_{\Delta,G}(G') \nu_{\Lambda,G_1}(\mathrm{d}G') \\ \implies Q_{\Delta,G}(G_1) \mathsf{Z}_\Lambda(G_1) &= Q_{\Lambda,G_1}(G_1) \int Q_{\Delta,G}(G') \nu_{\Lambda,G_1}(\mathrm{d}G'). \end{aligned} \tag{4.2.8}$$

By the bounds in Propositions 4.2.2 and 4.2.3, $q^{N_{cc}^\Delta}$ and $q^{N_{cc}^\Lambda}$ are both finite and non-zero almost surely with respect to $\nu_{\Delta,G}$. Therefore $Q_{\Delta,G}, Q_{\Lambda,G_1} < \infty$,

$$\begin{aligned} Q_{\Delta,G}(G_1) = 0 &\iff H_{\Delta,\omega}((\omega_1)_\Delta) = \infty, \text{ and} \\ Q_{\Lambda,G_1}(G_1) = 0 &\iff H_{\Lambda,\omega_1}((\omega_1)_\Lambda) = \infty \end{aligned}$$

for $\nu_{\Delta,G}$-almost all G_1. Furthermore, taking this into account along with the inclusion $\mathrm{Del}_{3,\Lambda}(\omega_1) \subset \mathrm{Del}_{3,\Delta}(\omega_1)$ we have

$$C_{\Delta,G}\left(G' \in \mathcal{G} : Q_{\Lambda,G'}(G'), Q_{\Delta,G}(G') \in (0,\infty)\right) = 1. \tag{4.2.9}$$

Now combining (4.2.9) with the relationship (4.2.8) and the consistency condition (4.2.2), we see that

$$\begin{aligned} C_{\Delta,G}\left(\mathsf{Z}_\Lambda(\cdot) = 0\right) &= C_{\Delta,G}\left(\nu_{\Lambda,(\cdot)}(Q_{\Delta,G}) = 0\right) \\ &= \int_{[\nu_{\Lambda,(\cdot)}(Q_{\Delta,G})=0]} Q_{\Delta,G}(G')\nu_{\Delta,G}(\mathrm{d}G') \\ &= \iint_{[\nu_{\Lambda,(\cdot)}(Q_{\Delta,G})=0]} Q_{\Delta,G}(G')\nu_{\Lambda,G''}(\mathrm{d}G')\nu_{\Delta,G}(\mathrm{d}G'') \\ &= 0, \end{aligned}$$

and

$$C_{\Delta,G}\left(\mathsf{Z}_\Lambda(\cdot) = \infty\right) = C_{\Delta,G}\left(\int Q_{\Delta,G}(G')\nu_{\Lambda,(\cdot)}(\mathrm{d}G') = \infty\right) = 0,$$

since

$$\begin{aligned} &\iint Q_{\Delta,G}(G')\nu_{\Lambda,G''}(\mathrm{d}G')\nu_{\Delta,G}(\mathrm{d}G'') = \mathsf{Z}_\Delta(G) < \infty \\ \Longrightarrow\ &\nu_{\Delta,G}\left(\int Q_{\Delta,G}(G')\nu_{\Lambda,(\cdot)}(\mathrm{d}G') = \infty\right) = 0 \\ \Longrightarrow\ &C_{\Delta,G}\left(\int Q_{\Delta,G}(G')\nu_{\Lambda,(\cdot)}(\mathrm{d}G') = \infty\right) = 0. \end{aligned}$$

This finishes the proof of the statement $C_\Delta(G, \mathcal{G}_*^\Lambda) = 1$.

For the second statement, we start by multiplying both sides of (4.2.8) by $\frac{f(G_1)}{\mathsf{Z}_\Delta(G)\mathsf{Z}_\Lambda(G_1)}$, which results in the equation

$$f(G_1)\frac{Q_{\Delta,G}(G_1)}{\mathsf{Z}_\Delta(G)} = f(G_1)\frac{Q_{\Lambda,G_1}(G_1)}{\mathsf{Z}_\Lambda(G_1)}\int \frac{Q_{\Delta,G}(G')}{\mathsf{Z}_\Delta(G)}\nu_{\Lambda,G_1}(\mathrm{d}G').$$

Furthermore, by integrating with respect to G_1, and using the consistency equation (4.2.2) we obtain:

$$\begin{aligned}
&\int f(G')\frac{Q_{\Delta,G}(G')}{\mathsf{Z}_\Delta(G)}\nu_{\Delta,G}(\mathrm{d}G')\\
=&\iiint f(G''')\frac{Q_{\Lambda,G'}(G''')}{\mathsf{Z}_\Lambda(G')}\frac{Q_{\Delta,G}(G'')}{\mathsf{Z}_\Delta(G)}\nu_{\Lambda,G'}(\mathrm{d}G'')\nu_{\Lambda,G'}(\mathrm{d}G''')\nu_{\Delta,G}(\mathrm{d}G').
\end{aligned}$$

The former expression is equal to $C_{\Delta,G}(f)$ and the latter is equal to

$$\begin{aligned}
&\iint C_{\Lambda,G'}(f)\frac{Q_{\Delta,G}(G'')}{\mathsf{Z}_\Delta(G)}\nu_{\Lambda,G'}(\mathrm{d}G'')\nu_{\Delta,G}(\mathrm{d}G')\\
=&\int C_{\Lambda,G'}(f)\frac{Q_{\Delta,G}(G')}{\mathsf{Z}_\Delta(G)}\nu_{\Delta,G}(\mathrm{d}G')\\
=&\int C_{\Lambda,G'}(f)C_{\Delta,G}(\mathrm{d}G'),
\end{aligned}$$

as required. □

The Gibbs measures specified by the above set of kernels are called Delaunay random cluster measures:

Definition 4.2.7. A probability measure P on $(\mathcal{G},\mathcal{A})$ is called a *Delaunay random cluster measure (DRCM)* for parameters $z>0, q\geq 1$, the hyperedge potential ψ and edge weights $(p(e))_{e\in\mathcal{E}_{\mathbb{R}^2,2}}$ if $P(\mathcal{G}_*^\Delta)=1$ and

$$\begin{aligned}
P(f)&=\int_{\mathcal{G}_*^\Delta} C_{\Delta,G}(f)P(\mathrm{d}G)\\
&=\int_{\mathcal{G}_*^\Delta}\mathsf{Z}_\Delta(G)^{-1}\int f(G')Q_{\Delta,G}(G')\nu_{\Delta,G}(\mathrm{d}G')P(\mathrm{d}G)
\end{aligned}$$

for every $\Delta\Subset\mathbb{R}^2$ and every bounded measurable function $f:\mathcal{G}\to\mathbb{R}\cup\{\infty\}$.

The set of DRCMs which are invariant under the translation group $\Theta=(\theta_x)_{x\in\mathbb{R}^2}$ is denoted $\mathscr{G}_\Theta$, and more generally the sets of probability measures and translation-invariant probability measures on $(\mathcal{G},\mathcal{A})$ are denoted $\mathscr{P}$ and $\mathscr{P}_\Theta$ respectively. Note that these notations had different meanings in the previous chapters.

4.3 Existence result

Theorem 4.3.1. *Let $z > 0, q \geq 1$, and suppose that the edge weights are uniformly bounded away from* 0 *and* 1*, i.e there exists $p_-, p_+ \in (0,1)$ such that*

$$p_- \leq p(e) \leq p_+ \text{ for all } e \in \mathcal{E}_{\mathbb{R}^2,2}. \tag{4.3.1}$$

Furthermore, suppose that the hyperedge potential ψ satisfies (R), (S) and (U) in addition to the hardcore conditions

(HCC) $\exists R_0 > 0$ such that

$$\delta(\tau) \geq R_0 \implies \psi(\tau) = \infty, \text{ and} \tag{4.3.2}$$

(HCL) $\exists \ell_0 > 0$ such that

$$\exists x, y \in \tau, x \neq y \text{ with } |x - y| \leq \ell_0 \implies \psi(\tau) = \infty. \tag{4.3.3}$$

Then there exists a translation-invariant Delaunay random cluster measure $P \in \mathscr{G}_\Theta$.

Remark 4.3.2. Note that the potential ψ considered in Chapter 3 satisfies both (HCC) and (HCL) with $R_0 = r$ and $\ell_0 = 2r\sin(\alpha_0)$. If $p(e) := 1 - e^{-\phi(e)}$ for some function $\phi : \mathcal{E}_{\mathbb{R}^2,2} \to \mathbb{R}$ then we recover a model similar to the finite volume Delaunay random cluster distribution from definition 2.3.1 except for that fact that edges rather than triangles are declared to be open or closed.

4.4 Proof

The rest of this chapter is devoted to the proof of this theorem. In section 4.4.1 we introduce two relevant topologies and notion of specific entropy along with a useful convergence result. In section 4.4.2 we construct a sequence $\tilde{P}_n$ of probability measures from the finite volume distributions and use this result to find a subsequence which converges to some measure $\tilde{P}$. We then set about proving that $P := \tilde{P}(\cdot|\{\emptyset\}^c) \in \mathscr{G}_\Theta$. To this end, in section 4.4.3 we define, for each $\Delta \Subset \mathbb{R}^2$, another sequence of measures C_n^Δ which converges to $\tilde{P}$. These sequences are more useful than $\tilde{P}_n$ because the measures C_n^Δ satisfy the DLR equation for Δ. In section 4.4.4 we use these new sequences to prove that $\tilde{P}$ has a unique infinite connected component almost surely, a fact which is utilised in section 4.4.5 to prove that P satisfies the DLR equations.

4.4.1 Local convergence and the specific entropy

In this section we introduce the topology of local convergence and develop a useful tool for determining whether a sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ in $\mathscr{P}_\Theta$ has an accumulation point with respect to this topology. When proving the existence of Gibbsian point processes with geometry dependent interactions [DDG12] or infinite volume random cluster measures (defined as Gibbsian modifications of the Poisson Boolean model) [DH15], the relevant tool ([Geo94, Lemma 3.4]) is derived from the sequential compactness of the level sets of the specific entropy [GZ93, Proposition 2.6]. If the specific entropy is defined in the analogous way here (using the sub-σ-algebras of local events), it becomes difficult to compute in the cases that we are interested in. We therefore define the entropy slightly differently, using the sub-σ-algebra of *strictly local* events. We take advantage of analogues of standard results regarding convergence in the topology of local convergence (Lemmas 4.4.9 and 4.4.10) in the course of proving our main convergence tool, Proposition 4.4.5, which is used in the next section.

Definitions

For $\Delta \Subset \mathbb{R}^2$, let us define the following subset of $\mathcal{G}_\Delta$:

$$\mathcal{G}'_\Delta := \{(\omega, E) \in \mathcal{G}_\Delta : \forall e \in E \ \exists \tau \in \mathrm{Del}_3(\omega) \text{ such that } e \subset \tau \text{ and } B(\tau) \subset \Delta\},$$

and the projection $\rho_\Delta : \mathcal{G} \to \mathcal{G}'_\Delta$ given by $\rho_\Delta(\omega, E) := (\omega_\Delta, E \cap \rho^{\mathcal{E}}_\Delta(\omega))$ where

$$\rho^{\mathcal{E}}_\Delta(\omega) := \{e \in \mathrm{Del}_2(\omega) : \exists \tau \in \mathrm{Del}_3(\omega) \text{ such that } e \subset \tau \text{ and } B(\tau) \subset \Delta\}.$$

$\rho^{\mathcal{E}}_{\Delta}(\omega)$ represents the subset of $\mathrm{Del}_2(\omega)$ which is unaffected by changes to the configuration ω which occur outside Δ. More precisely,

$$\rho^{\mathcal{E}}_{\Delta}(\omega_{\Delta} \cup \omega') = \rho^{\mathcal{E}}_{\Delta}(\omega) \text{ for all } \omega' \in \Omega_{\Delta^c}. \tag{4.4.1}$$

Definition 4.4.1. · Events in $\mathcal{A}'_{\Delta} := \rho_{\Delta}^{-1}\mathcal{A}|_{\mathcal{G}'_{\Delta}}$ and $\mathcal{A}_{\Delta}$ are called *strictly local* and *local* respectively.

- A function $f : \mathcal{G} \to \mathbb{R}$ is *strictly local* (resp. *local*) is if it is measurable with respect to the sigma algebra $\mathcal{A}'_{\Delta}$ (resp. $\mathcal{A}_{\Delta}$) for some $\Delta \Subset \mathbb{R}^2$. The sets of strictly local and local functions are denoted $\mathcal{L}'$ and $\mathcal{L}$ respectively. Since $\mathcal{A}'_{\Delta} \subset \mathcal{A}_{\Delta}$ we have $\mathcal{L}' \subset \mathcal{L}$.
- The *topology of local convergence*, or $\mathcal{L}$-topology, on $\mathscr{P}_{\Theta}$ is defined to be the weak* topology induced by the set of bounded local functions. This is the coarsest topology for which the mappings $P \mapsto P(f)$ for $f \in \mathcal{L}$ are continuous.

If P and Q are probability measures on the same measurable space, the *relative entropy* (also known as the *Kullback-Leibler divergence*) of P relative to Q is defined to be

$$I(P|Q) := \begin{cases} \int f \log f \mathrm{d}Q & \text{if } P \ll Q \text{ with density } f, \\ \infty & \text{otherwise.} \end{cases}$$

An alternative characterisation is

$$I(P|Q) = \sup_{g \in \mathcal{B}} P(g) - \log Q(\exp(g)), \tag{4.4.2}$$

where $\mathcal{B}$ is the set of bounded measurable functions ([Var88, Theorem 4.1]).

Let M be a fixed invertible 2×2 matrix with entries in $\mathbb{R}$, and let $\Lambda_n = \bigcup_{i,j=-n}^{n} C(i,j)$, where $C(i,j)$ is given by (2.1.10).

Definition 4.4.2. The *specific entropy* or *mean entropy* of a measure $P \in \mathscr{P}_{\Theta}$ (relative to ν defined in (4.2.3)) is defined to be

$$I^z(P) := \lim_{n\to\infty} |\Lambda_n|^{-1} I(P_{\Lambda_n}|\nu^z_{\Lambda_n}), \tag{4.4.3}$$

where $P_{\Lambda_n} := P \circ \rho_{\Lambda_n}^{-1}$ and $\nu_{\Lambda_n}^z := \nu \circ \rho_{\Lambda_n}^{-1}$ (the intensity parameter z is made explicit in the notation here for clarity). Using the characterisation of I in (4.4.2) we have

$$I^z(P) = \lim_{n\to\infty} |\Lambda_n|^{-1} \sup_{g\in\mathcal{B}(\mathcal{G}'_\Delta,\mathcal{A}'_\Delta)} P_\Delta(g) - \log \nu_\Delta^z(\exp(g)),$$

where $\mathcal{B}(\mathcal{G}'_\Delta, \mathcal{A}'_\Delta)$ is the set of bounded $\mathcal{A}'_\Delta$-measurable functions $g : \mathcal{G}'_\Delta \to \mathbb{R}$.

The existence of $I^z(P)$ and the fact that it is independent of the matrix M can be inferred from Proposition 4.4.3. For $A \in \mathcal{A}|_{\mathcal{G}'_\Delta}$, (4.4.1) and the independence properties of Π^z and μ_ω give rise to the formula

$$\nu_\Delta^z(A) = \int \mathbb{1}_A(\omega, E)\mu_{\Delta,\omega}(\mathrm{d}E)\Pi_\Delta^z(\mathrm{d}\omega),$$

where $\mu_{\Delta,\omega}$ denotes the tile drawing mechanism on $\rho_\Delta^{\mathcal{E}}(\omega)$. Furthermore, when $\Delta, \Lambda \Subset \mathbb{R}^2$ are disjoint and $A \in \mathcal{A}|_{\mathcal{G}'_{\Delta\cup\Lambda}}$,

$$\nu_{\Delta\cup\Lambda}^z(A) = \int \mathbb{1}_A(G \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega\cup\omega'}(\mathrm{d}E'')\nu_\Lambda^z(\mathrm{d}G')\nu_\Delta^z(\mathrm{d}G),$$

where $\mu_{\Delta,\Lambda,\omega}$ denotes the tile drawing mechanism on $\rho_{\Delta\cup\Lambda}^{\mathcal{E}}(\omega)\backslash\left(\rho_\Delta^{\mathcal{E}}(\omega) \cup \rho_\Lambda^{\mathcal{E}}(\omega)\right)$.

Existence of the specific entropy

The existence of the specific entropy is a consequence of the following proposition.

Proposition 4.4.3. *For all $P \in \mathscr{P}_\Theta$,*

$$I^z(P) = \sup_{\Delta\in\mathscr{C}} |\Delta|^{-1} I(P_\Delta|\nu_\Delta^z),$$

where $\mathscr{C}$ denotes the set of open cubes in $\mathbb{R}^2$, defined in (2.1.3).

The first step in proving Proposition 4.4.3 is to prove that the specific entropy is *sub-additive* in the following sense. The proof mimics that of [Geo11, Proposition 15.10].

Lemma 4.4.4. *For each $P \in \mathscr{P}$ and all disjoint $\Delta, \Lambda \in \mathscr{C}$,*

$$I(P_{\Delta\cup\Lambda}|\nu^z_{\Delta\cup\Lambda}) \geq I(P_\Delta|\nu^z_\Delta) + I(P_\Lambda|\nu^z_\Lambda).$$

Proof. We can assume that $I(P_{\Delta\cup\Lambda}|\nu^z_{\Delta\cup\Lambda}) < \infty$, in which case the density $f_{\Delta\cup\Lambda} := \frac{\mathrm{d}P_{\Delta\cup\Lambda}}{\mathrm{d}\nu^z_{\Delta\cup\Lambda}}$ exists. Let us also define the functions

$$\begin{aligned} f_\Delta(G) &:= \nu^z_{\Delta\cup\Lambda}\big(f_{\Delta\cup\Lambda} \bigm| \rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}} = G\big) \\ &= \int f_{\Delta\cup\Lambda}(G \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega_\Delta\cup\omega'}(\mathrm{d}E'')\nu^z_\Lambda(\mathrm{d}G'), \end{aligned}$$

and

$$\begin{aligned} f_\Lambda(G) &:= \nu^z_{\Delta\cup\Lambda}\big(f_{\Delta\cup\Lambda} \bigm| \rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}} = G\big) \\ &= \int f_{\Delta\cup\Lambda}(G \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega_\Lambda\cup\omega'}(\mathrm{d}E'')\nu^z_\Delta(\mathrm{d}G'), \end{aligned}$$

where $\rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}}$ denotes the restriction of ρ_Δ to $\mathcal{G}'_{\Delta\cup\Lambda}$. It is then true that $f_\Delta = \frac{\mathrm{d}P_\Delta}{\mathrm{d}\nu^z_\Delta}$ and $f_\Lambda = \frac{\mathrm{d}P_\Lambda}{\mathrm{d}\nu^z_\Lambda}$. We consider the measure λ on $\mathcal{A}|_{\mathcal{G}'_{\Delta\cup\Lambda}}$ defined by

$$\lambda(A) := \int \mathbb{1}_A(\rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}}(G) \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega_\Lambda\cup\omega'}(\mathrm{d}E'')\nu^z_\Delta(\mathrm{d}G')P_{\Delta\cup\Lambda}(\mathrm{d}G).$$

Now $\lambda(f_\Lambda \circ \rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}} > 0) = P_{\Delta\cup\Lambda}(f_\Lambda \circ \rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}} > 0) = 1$, so the quotient $f_{\Delta\cup\Lambda|\Lambda} := \frac{f_{\Delta\cup\Lambda}}{f_\Lambda \circ \rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}}}$ is well-defined λ-almost surely. Moreover, $\frac{\mathrm{d}P_{\Delta\cup\Lambda}}{\mathrm{d}\lambda} = f_{\Delta\cup\Lambda|\Lambda}$ since for $A \in \mathcal{A}|_{\mathcal{G}'_{\Delta\cup\Lambda}}$,

$$\begin{aligned} &\lambda(f_{\Delta\cup\Lambda|\Lambda}\mathbb{1}_A) \\ &= \int \frac{\mathbb{1}_A f_{\Delta\cup\Lambda}(\rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}}(G) \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega_\Lambda\cup\omega'}(\mathrm{d}E'')\nu^z_\Delta(\mathrm{d}G')}{\int f_{\Delta\cup\Lambda}(\rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}}(G) \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega_\Lambda\cup\omega'}(\mathrm{d}E'')\nu^z_\Delta(\mathrm{d}G')} P_{\Delta\cup\Lambda}(\mathrm{d}G) \\ &= \int P_{\Delta\cup\Lambda}\left(A|\mathcal{A}'_\Lambda\right)(G)P_{\Delta\cup\Lambda}(\mathrm{d}G) \\ &= P_{\Delta\cup\Lambda}(A) \end{aligned}$$

by the tower property. Similarly, for $A \in \mathcal{A}|_{\mathcal{G}'_\Delta}$,

$$\begin{aligned}
&\lambda \circ (\rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}})^{-1}(\mathbb{1}_A f_\Delta) \\
&= \lambda((\mathbb{1}_A f_\Delta) \circ \rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}}) \\
&= \int \mathbb{1}_A f_\Delta(G')\mu_{\Delta,\Lambda,\omega_\Lambda\cup\omega'}(\mathrm{d}E'')\nu^z_\Delta(\mathrm{d}G')P_{\Delta\cup\Lambda}(\mathrm{d}G) \\
&= \int \mathbb{1}_A f_\Delta(G)\nu^z_\Delta(\mathrm{d}G) \\
&= \int \mathbb{1}_A(G) f_{\Delta\cup\Lambda}(G \cup G' \cup (\emptyset, E''))\mu_{\Delta,\Lambda,\omega_\Delta\cup\omega'}(\mathrm{d}E'')\nu^z_\Lambda(\mathrm{d}G')\nu^z_\Delta(\mathrm{d}G) \\
&= \int \mathbb{1}_A(\rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}}(G)) f_{\Delta\cup\Lambda}(G)\nu^z_{\Delta\cup\Lambda}(\mathrm{d}G) \\
&= P_{\Delta\cup\Lambda}(\mathbb{1}_A \circ \rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}}) \\
&= P_{\Delta\cup\Lambda} \circ (\rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}})^{-1}(A),
\end{aligned}$$

and so $\frac{\mathrm{d}P_\Delta}{\mathrm{d}\lambda\circ(\rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}})^{-1}} = f_\Delta$. Therefore we can write, using the fact that the function $\Delta \mapsto I(P_\Delta|\nu^z_\Delta)$ is increasing (see [Geo11, Proposition 15.5(c)]),

$$\begin{aligned}
I(P_\Delta|\nu^z_\Delta) &= P_\Delta(\log f_\Delta) \\
&= I(P_\Delta \mid \lambda \circ (\rho_\Delta|_{\mathcal{G}'_{\Delta\cup\Lambda}})^{-1}) \\
&\le I(P_{\Delta\cup\Lambda}|\lambda) \\
&= P_{\Delta\cup\Lambda}(\log f_{\Delta\cup\Lambda|\Lambda}) \\
&= P_{\Delta\cup\Lambda}(\log f_{\Delta\cup\Lambda} - \log f_\Lambda \circ \rho_\Lambda|_{\mathcal{G}'_{\Delta\cup\Lambda}}) \\
&= I(P_{\Delta\cup\Lambda}|\nu^z_{\Delta\cup\Lambda}) - I(P_\Lambda|\nu^z_\Lambda).
\end{aligned}$$

□

Proof of Proposition 4.4.3. Let us use the shorthand I_Δ in place of $I^z(P_\Delta|\nu^z_\Delta)$. The following three properties are satisfied, and justifications are given below.

(i) $I_{\Delta+x} = I_\Delta$ for all $\Delta \in \mathscr{C}$ and $x \in \mathbb{R}^2$,

(ii) $I_\Delta + I_\Lambda \le I_{\Lambda\cup\Delta}$ whenever $\Delta, \Lambda \in \mathscr{C}$ and $\Delta \cap \Lambda = \emptyset$, and

(iii) $I_\Lambda \le I_\Delta$ whenever $\Lambda \subset \Delta$.

(i) follows from the translation invariance of P and ν, (ii) follows from the previous lemma, and (iii) is [Geo11, Proposition 15.5(c)]. We can then follow the argument of [Geo11, Lemma 15.11]. Choose any real $\epsilon > 0$ and a cube $\Delta \in \mathscr{C}$ satisfying

$$|\Delta|^{-1} I_\Delta > \sup_{\Delta \in \mathscr{C}} |\Delta|^{-1} I_\Delta - \epsilon.$$

If N_n denotes the largest number of disjoint translates of Δ which can fit inside Λ_n, then $\frac{|\Lambda_n|}{N_n |\Delta_n|} \to 1$, so our assumptions allow us to obtain the estimate

$$I_{\Lambda_n} \geq N_n I_\Delta, \tag{4.4.4}$$

and therefore

$$\limsup_{n \to \infty} |\Lambda_n|^{-1} I_{\Lambda_n} \geq |\Delta|^{-1} I_\Delta > \sup_{\Delta \in \mathscr{C}} |\Delta|^{-1} I_\Delta - \epsilon.$$

Since ϵ is arbitrary, we obtain the desired result. □

Convergence results

For any bounded measurable function $g : \mathcal{G}'_\Delta \to \mathbb{R}$, we have $g \circ \rho_\Delta \in \mathcal{L}' \subset \mathcal{L}$, and so the map $\mathscr{P}_\Theta \ni P \mapsto P_\Delta(g) = P(g \circ \rho_\Delta)$ is continuous with respect to the $\mathcal{L}$-topology. Therefore the function

$$\mathscr{P}_\Theta \ni P \mapsto I(P_\Delta | \nu^z_\Delta) = \sup_{g \in \mathcal{B}(\mathcal{G}'_\Delta, \mathcal{A}'_\Delta)} P_\Delta(g) - \log \nu^z_\Delta(\exp(g))$$

(where $\mathcal{B}(\mathcal{G}'_\Delta, \mathcal{A}'_\Delta)$ is the set of bounded $\mathcal{A}'_\Delta$-measurable functions $g : \mathcal{G}'_\Delta \to \mathbb{R}$) is a supremum of continuous functions, making it lower semi-continuous. Furthermore, the characterisation of I^z in Proposition 4.4.3 shows that I^z is also lower semi-continuous, and hence the level sets

$$L^z(c) := \{P \in \mathscr{P}_\Theta : I^z(P) \leq c\} \tag{4.4.5}$$

for $z > 0$ and $c \in \mathbb{R}$ are closed in the $\mathcal{L}$-topology.

This section is devoted to proving the following proposition, which will be accomplished through a series of lemmas. For $P \in \mathscr{P}_\Theta$, the *point intensity* of P is $i(P) := \int |\omega_{[0,1]^2}| P(\mathrm{d}(\omega, E))$. For any $\Delta \Subset \mathbb{R}^2$, $i(P) = |\Delta|^{-1} \int |\omega_\Delta| P(\mathrm{d}(\omega, E))$.

Proposition 4.4.5. *Let $z > 0$ and $c_1, c_2 \in \mathbb{R}$. If $(P_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{P}_\Theta$ such that $I^z(P_n) \le c_1 i(P_n) + c_2$, and there exists $k > 0$ such that*

$$\mathrm{pr}_\Delta(G) = \mathrm{pr}_\Delta \circ \rho_{\Delta\oplus k}(G) \text{ for } P_n\text{-almost all } G \tag{4.4.6}$$

for all $n \in \mathbb{N}$, then P_n has a subsequence (P_{n_i}) which converges in the $\mathcal{L}$-topology and whose limit $P \in \mathscr{P}_\Theta$ satisfies $I^z(P) \le c_1 i(P) + c_2$.

The following notions of equicontinuity are useful for establishing convergence.

Definition 4.4.6. A net $(P_i)_{i\in I}$ in $\mathscr{P}$ is *locally equicontinuous* if for each $\Delta \Subset \mathbb{R}^2$ and each sequence $(A_n)_{n\ge1} \subset \mathcal{A}_\Delta$ with $A_n \searrow \emptyset$,

$$\lim_{n\to\infty} \limsup_{i\in I} P_i(A_n) = 0, \tag{4.4.7}$$

and *strictly locally equicontinuous* if (4.4.7) holds for each $\Delta \Subset \mathbb{R}^2$ and each sequence $(A_n)_{n\ge1} \subset \mathcal{A}'_\Delta$ with $A_n \searrow \emptyset$.

The two notions of equicontinuity are equivalent when (4.4.6) is satisfied. This property holds for the sequences we consider in the next chapter with $k = 2R_0$, due to (HCC).

Lemma 4.4.7. *Let $(P_i)_{i\in I}$ be a net in $\mathscr{P}$ such that for all $i \in I, \Delta \subset \mathbb{R}^2$,*

$$\mathrm{pr}_\Delta(G) = \mathrm{pr}_\Delta \circ \rho_{\Delta\oplus k}(G) \text{ for } P_i\text{-almost all } G.$$

Then $(P_i)_{i\in I}$ is locally equicontinuous if and only if $(P_i)_{i\in I}$ is strictly locally equicontinuous.

Proof. Since $\mathcal{A}'_\Delta \subset \mathcal{A}_\Delta$ for all $\Delta \Subset \mathbb{R}^2$, local continuity always implies strict local equicontinuity. For the converse, let $(A_n)_{n\ge1}$ be a sequence in $\mathcal{A}_\Delta$ such that $A_n \searrow \emptyset$. Then $A_n = (\mathrm{pr}_\Delta)^{-1}(B_n \cap \mathcal{G}_\Delta)$ for some $B_n \in \mathcal{A}$ and $\rho^{-1}_{\Delta\oplus k} \circ (\mathrm{pr}_\Delta|_{\mathcal{G}'_{\Delta\oplus k}})^{-1}(B_n \cap \mathcal{G}_\Delta) \searrow \emptyset$. Then by the assumptions we have

$$\lim_{n\to\infty} \limsup_{i\in I} P_i(A_n) = \lim_{n\to\infty} \limsup_{i\in I} P_i\left(\rho^{-1}_{\Delta\oplus k} \circ (\mathrm{pr}_\Delta|_{\mathcal{G}'_{\Delta\oplus k}})^{-1}(B_n \cap \mathcal{G}_\Delta)\right) = 0.$$

□

To prove the following lemma we follow the same method as [Geo11][Proposition 15.14].

Lemma 4.4.8. *Let $z > 0$ and $c \in \mathbb{R}$. Every net in $L^z(c)$ is strictly locally equicontinuous.*

Proof. We assume without loss of generality that $c \geq 0$, since $I^z(P) \geq 0$ for all $P \in \mathscr{P}_\Theta$. Let $\Delta \in \mathscr{C}$ and $(\rho_\Delta^{-1} A_m)_{m\geq 1}$ be a sequence in $\mathcal{A}'_\Delta$ with $\rho_\Delta^{-1} A_m \searrow \emptyset$. For every $P \in L^z(c)$ we have $I(P_\Delta|\nu_\Delta^z) \leq c|\Delta|$ by Proposition 4.4.3. In particular each P_Δ is absolutely continuous with respect to ν_Δ^z with density f_Δ^P. For a given $\epsilon > 0$, let $\delta > 0$ satisfy $\epsilon \log \epsilon/\delta \geq 1 + c|\Delta|$. If m is large enough that $\nu_\Delta^z(A_m) = \nu^z(\rho_\Delta^{-1} A_m) \leq \delta$ then

$$\begin{aligned}
P(\rho_\Delta^{-1} A_m) &= \nu_\Delta^z(\mathbb{1}_{A_m \cup \{f_\Delta^P \leq \epsilon/\delta\}} f_\Delta^P) + \nu_\Delta^z(\mathbb{1}_{A_m \cup \{f_\Delta^P > \epsilon/\delta\}} f_\Delta^P) \\
&\leq \epsilon + (\log \epsilon/\delta)^{-1} \nu_\Delta^z(\mathbb{1}_{A_m \cup \{f_\Delta^P > \epsilon/\delta\}} f_\Delta^P \log f_\Delta^P) \\
&\leq \epsilon + (\log \epsilon/\delta)^{-1} \Big(I(P_\Delta|\nu_\Delta^z) - \nu_\Delta^z(\mathbb{1}_{A_m^c \cap \{f_\Delta^P \leq \epsilon/\delta\}} f_\Delta^P \log f_\Delta^P) \Big) \\
&\leq \epsilon + (\log \epsilon/\delta)^{-1} \left(I(P_\Delta|\nu_\Delta^z) + 1 \right) \\
&\leq 2\epsilon
\end{aligned}$$

for all $P \in L^z(c)$. The second to last line follows from the inequality
$x \log x \geq -1$. □

The analogue of the next lemma for random fields on $\mathbb{Z}^d$ is [Geo11, Proposition 4.9]. We use a similar argument here.

Lemma 4.4.9. *Every locally equicontinuous net in $\mathscr{P}$ has at least one cluster point in $\mathscr{P}$ with respect to the $\mathcal{L}$-topology.*

Proof. Let $(P_i)_{i\in I}$ be a net in $\mathscr{P}$. Its restriction $(P_i^0)_{i\in I}$ to $\mathcal{A}^0 := \cup_{\Delta \in \mathscr{C}} \mathcal{A}_\Delta$ is a net in the compact Hausdorff space $[0,1]^{\mathcal{A}^0}$, and so there is a subnet $(P_{i_j})_{j\in I'}$ with the property that $(P_{i_j}^0)_{j\in I'}$ converges setwise to some $P^0 \in [0,1]^{\mathcal{A}^0}$. P^0 is additive, and is in fact σ-additive on each $\mathcal{A}_\Delta$. To see this note that if $(A_n)_{n=1}^\infty$ is a sequence of events in $\mathcal{A}_\Delta$ such that $A_n \searrow \emptyset$ then

$$\lim_{n\to\infty} P^0(A_n) = \lim_{n\to\infty} \lim_{j\in I'} P_{i_j}(A_n) \leq \lim_{n\to\infty} \limsup_{i\in I} P_i(A_n) = 0.$$

Since each $(\mathcal{G}_\Delta, \mathcal{A}|_{\mathcal{G}_\Delta})$ is a standard Borel space and $\mathcal{A} = \sigma(\cup_{\Delta\mathscr{C}}\mathcal{A}_\Delta)$, by Kolmogorov's extension theorem there exists a unique measure $P \in \mathscr{P}$ which extends P^0. $\square$

The proof of the following lemma is omitted but it can be proven in the same way as [Geo11][Theorem 4.15].

Lemma 4.4.10. *If $P \in \mathscr{P}$ is a cluster point of a locally equicontinuous sequence $(P_n)_{n\geq 1}$ in $\mathscr{P}$ then there is a subsequence $(P_{n_k})_{k\geq 1}$ which converges to P in the $\mathcal{L}$-topology.*

Proof of Proposition 4.4.5. Since $\frac{\mathrm{d}\Pi^z_{\Lambda_n}}{\mathrm{d}\Pi^1_{\Lambda_n}} = z^{|\omega_{\Lambda_n}|}e^{(1-z)|\Lambda_n|}$ it can be easily shown that for any $P \in \mathscr{P}_\Theta$,

$$I^z(P) = I^1(P) - i(P)\log(z) + z - 1,$$

and therefore if $z' = ze^{c_1}$ for $c_1 \in \mathbb{R}$,

$$\begin{aligned} I^z(P) - I^{z'}(P) &= i(P)(\log(z') - \log(z)) + z - z', \\ \implies I^z(P) &= I^{z'}(P) + c_1 i(P) + z - z'. \end{aligned}$$

Consequently, for any $c_1, c_2 \in \mathbb{R}$,

$$\begin{aligned} \{P \in \mathscr{P}_\Theta : I^z(P) \leq c_1 i(P) + c_2\} &= \left\{P \in \mathscr{P}_\Theta : I^{z'}(P) \leq c_2 - z + z'\right\}. \\ &= L^{z'}(c_2 - z + z'). \end{aligned} \tag{4.4.8}$$

Together with Lemmas 4.4.7 and 4.4.8, this implies that P_n is a locally equicontinuous sequence in $L^{z'}(c_2 - z + z')$. By Lemmas 4.4.9 and 4.4.10 P_n therefore has a subsequence which converges with respect to the $\mathcal{L}$-topology. Since the level sets are closed, the limit P is in $L^{z'}(c_2 - z + z')$ as required. $\square$

4.4.2 Existence of an accumulation point

In this section we construct a sequence $(\tilde{P}_n)_{n\in\mathbb{N}}$ of measures from the finite volume Delaunay random cluster distributions. We show that the assumptions of Proposition 4.4.5 are satisfied, ensuring that $\tilde{P}_n$ has a convergent subsequence.

In subsequent sections we will show that the limit, after conditioning out the empty configuration, is a Delaunay random cluster measure.

From here on, we assume that the premises of Theorem 4.3.1 are satisfied. Suppose M and Γ are chosen so that (U1) - (U3) are satisfied. Let $\omega \in \overline{\Gamma}$ be a fixed pseudo-periodic configuration with

$$\sup_{i,j\in\mathbb{Z}} N_{C(i,j)}(\omega) < \infty \tag{4.4.9}$$

(for instance ω could be periodic), and $G = (\omega, E) \in \mathcal{G}$ for some $E \in \mathcal{E}$. We first show that G is admissible with respect to Λ_n (i.e $G \in \mathcal{G}_*^{\Lambda_n}$). By (U1), $\omega \in \Omega_{cr}^{\Lambda_n}$, and so by (S)

$$H_{\Lambda_n,\omega}(\omega') \geq -c_S|\omega' \cup \partial_{\Lambda_n}\omega| \tag{4.4.10}$$

for all $\omega' \in \Omega_{\Lambda_n}$. This, in conjunction with Proposition 4.2.2, yields

$$\mathsf{Z}_{\Lambda_n}(G) \leq e^{c_S|\partial^{\Gamma}_{\Lambda_n}\omega|} \int q^{|\omega'|} e^{c_S|\omega'|} \Pi^z_{\Lambda_n}(d\omega') < \infty.$$

Conversely, it can be shown that under assumptions (U1), (U2) and shift-invariance of ψ that if $\omega' \in \Omega_{\Lambda_n}$ and $\omega \cup \omega' \in \overline{\Gamma}$ then

$$H_{\Lambda_n,\omega}(\omega') \leq c_{\Gamma}(2n+1) + o(|\Lambda_n|),$$

where the error term is uniform in ω' (see [DDG12, equation (5.8)] and its proof). Together with Proposition 4.2.3 this gives the estimate

$$\mathsf{Z}_{\Lambda_n}(G) \geq q^{1-|(\omega_{(\Lambda_n)^c})^{\Lambda_n}|} e^{-c_{\Gamma}(2n+1)-o(|\Lambda_n|)} \Pi^z_C(\Gamma)^{(2n+1)} > 0, \tag{4.4.11}$$

where we recall that $\Pi^z_C(\Gamma) > 0$ by (U3). Therefore $G \in \mathcal{G}_*^{\Lambda_n}$, so we can define the Gibbs distribution

$$C_n := C_{\Lambda_n,G} \circ \rho^{-1}_{\Lambda_n}.$$

in Λ_n with boundary condition G, projected to $\mathcal{G}'_{\Lambda_n}$. In order to construct a shift invariant Gibbs measure we spatially average this measure. Let P_n denote the probability measure on $(\mathcal{G}, \mathcal{A})$ relative to which the configuration in each parallelogram $\Lambda_n + (2n+1)Mk,\ k \in \mathbb{Z}^2$ is independent with distribution C_n. The spatially averaged measure is

$$\tilde{P}_n := |\Lambda_n|^{-1} \int_{\Lambda_n} P_n \circ \theta_x^{-1} \mathrm{d}x.$$

Let $x \in \Lambda_n$. There exist $k_1, k_2, k_3, k_4 \in \mathbb{Z}^2$ and $\Lambda_n^i \subset \Lambda_n + (2n+1)Mk_i$ such that $\Lambda_n - x = \cup_{i=1}^4 \Lambda_n^i$. Then $\Lambda_n = \cup_{i=1}^4 \theta_{-(2n+1)Mk_i}^{-1}(\Lambda_n^i)$. Since $P_n \circ \theta_x^{-1}$ is invariant under the shifts $\theta_{(2n+1)Mk}$ for $k \in \mathbb{Z}^2$, we have

$$\begin{aligned}
\int |\omega_{\Lambda_n}| P_n \circ \theta_x^{-1}(\mathrm{d}G) &= \int |\omega_{\Lambda_n - x}| P_n(\mathrm{d}G) \\
&= \sum_{i=1}^4 \int |\omega_{\Lambda_n^i}| P_n(\mathrm{d}G) \\
&= \int |\omega_{\Lambda_n}| P_n(\mathrm{d}G) = \int |\omega| C_n(\mathrm{d}G).
\end{aligned}$$

One can therefore show that $i(\tilde{P}_n)$ finite using stability (S) and Proposition 4.2.2,

$$\begin{aligned}
i(\tilde{P}_n) &= |\Lambda_n|^{-1} \int |\omega| C_n(\mathrm{d}G) \\
&= |\Lambda_n|^{-1} \int |\omega'_{\Lambda_n}| q^{N_{cc}^{\Lambda_n}(G')} e^{-H_{\Lambda_n,\omega}(\omega'_{\Lambda_n})} C_{\Lambda_n,G}(\mathrm{d}G') \\
&\leq |\Lambda_n|^{-1} e^{c_S |\partial_{\Lambda_n} \omega|} \int |\omega'| e^{(c_S + \log(q))|\omega'|} \Pi_{\Lambda_n}(\mathrm{d}\omega') < \infty.
\end{aligned}$$

For convenience, if $\Delta \Subset \mathbb{R}^2$ we let $\mu_{\Delta,\omega'}$ and $\mu^+_{\Delta,\omega',\omega}$ denote the edge drawing mechanisms on the sets $\rho^{\mathcal{E}}_\Delta(\omega')$ and $\mathrm{Del}_{2,\Delta}(\omega' \cup \omega_{\Delta^c}) \setminus \rho^{\mathcal{E}}_\Delta(\omega')$ respectively. Then for $A \in \mathcal{A}'_{\Lambda_n}$,

$$\begin{aligned} C_n(A) &= \mathsf{Z}_{\Lambda_n}(G)^{-1} \int \mathbb{1}_A \circ \rho_{\Lambda_n}(G') Q_{\Lambda_n,G}(G') \nu_{\Lambda_n,G}(\mathrm{d}G') \\ &= \mathsf{Z}_{\Lambda_n}(G)^{-1} \int \mathbb{1}_A(\omega', E') Q_{\Lambda_n,G}(G^{\mathbb{R}^2,\Lambda_n} \cup (\omega', E' \cup E'')) \\ &\qquad\qquad \mu^+_{\Lambda_n,\omega',\omega}(\mathrm{d}E'') \mu_{\Lambda_n,\omega'}(\mathrm{d}E') \Pi^z_{\Lambda_n}(\mathrm{d}\omega') \\ &= \mathsf{Z}_{\Lambda_n}(G)^{-1} \int \mathbb{1}_A(G') Q_{\Lambda_n,G}(G^{\mathbb{R}^2,\Lambda_n} \cup G' \cup (\emptyset, E'')) \\ &\qquad\qquad \mu^+_{\Lambda_n,\omega',\omega}(\mathrm{d}E'') \nu^z_{\Lambda_n}(\mathrm{d}G'), \end{aligned}$$

so

$$\frac{\mathrm{d}C_n}{\mathrm{d}\nu^z_{\Lambda_n}}(G') = \mathsf{Z}_{\Lambda_n}(G)^{-1} \int Q_{\Lambda_n,G}(G^{\mathbb{R}^2,\Lambda_n} \cup G' \cup (\emptyset, E'')) \mu^+_{\Lambda_n,\omega',\omega}(\mathrm{d}E'').$$

Lemma 4.4.11. $I^z(\tilde{P}_n) \le |\Lambda_n|^{-1} I(C_n | \nu^z_{\Lambda_n})$.

Proof. Suppose that for a fixed $x \in \Lambda_n$, Δ can be written as a finite union $\Delta = \cup_{i=1}^m \Lambda_n^{k_i}$ where $\Lambda_n^k := \Lambda_n + x + (2n+1)Mk$ for $k \in \mathbb{Z}^2$. Let

$$A := \left\{ (\omega, E) \in \mathcal{G} \;\middle|\; \begin{array}{l} e \in E \implies \forall \tau \in \mathrm{Del}_3(\omega) \text{ with } e \subset \tau \\ \exists i \in \{1, ..., m\} \text{ such that } B(\tau) \subset \Lambda_n^{k_i} \end{array} \right\}.$$

Then

$$\frac{\mathrm{d}(P_n \circ \theta_x^{-1})_\Delta}{\mathrm{d}\nu^z_\Delta}(G) = \mathbb{1}_A(G) \prod_{i=1}^m \frac{\mathrm{d}C_n \circ \theta^{-1}_{-x-(2n+1)Mk_i}}{\mathrm{d}\nu^z_{\Lambda_n^{k_i}}}(\rho_{\Lambda_n^{k_i}}(G)),$$

and therefore, since $0\log 0 = 0$,

$$\begin{aligned}
&I((P_n\circ\theta_x^{-1})_\Delta|\nu_\Delta^z)\\
&\le \int\prod_{i=1}^m \frac{\mathrm{d}C_n\circ\theta^{-1}_{-x-(2n+1)Mk_i}}{\mathrm{d}\nu_{\Lambda_n^{k_i}}}(G_i)\\
&\qquad \log\left(\prod_{j=1}^m \frac{\mathrm{d}C_n\circ\theta^{-1}_{-x-(2n+1)Mk_j}}{\mathrm{d}\nu_{\Lambda_n^{k_j}}}(G_j)\right)\mathrm{d}\nu_{\Lambda_n^{k_1}}(\mathrm{d}G_1)\ldots\mathrm{d}\nu_{\Lambda_n^{k_m}}(\mathrm{d}G_m)\\
&=\sum_{j=1}^m\int\log\left(\frac{\mathrm{d}C_n\circ\theta^{-1}_{-x-(2n+1)Mk_j}}{\mathrm{d}\nu_{\Lambda_n^{k_j}}}(G_j)\right)\\
&\qquad \prod_{i=1}^m \frac{\mathrm{d}C_n\circ\theta^{-1}_{-x-(2n+1)Mk_i}}{\mathrm{d}\nu_{\Lambda_n^{k_i}}}(G_i)\mathrm{d}\nu_{\Lambda_n^{k_1}}(\mathrm{d}G_1)\ldots\mathrm{d}\nu_{\Lambda_n^{k_m}}(\mathrm{d}G_m)\\
&= mI(C_n|\nu^z_{\Lambda_n}).
\end{aligned}$$

Dividing by $|\Delta|$ we obtain

$$|\Delta|^{-1}I^z((P_n\circ\theta_x^{-1})_\Delta|\nu_\Delta^z)\le|\Lambda_n|^{-1}I(C_n|\nu^z_{\Lambda_n}). \tag{4.4.12}$$

Now for an arbitrary cube $\Delta\in\mathscr{C}$ and a fixed n, let Δ^+ denote the union of all blocks Λ_n^k that meet Δ. Let Δ_m be a sequence in $\mathscr{C}$ such that

$$I^z(P_n\circ\theta_x^{-1})=\lim_{m\to\infty}|\Delta_m|^{-1}I^z((P_n\circ\theta_x^{-1})_{\Delta_m}|\nu^z_{\Delta_m}).$$

Then by (4.4.12) we have

$$\begin{aligned}
I^z(P_n\circ\theta_x^{-1}) &\le \lim_{m\to\infty}|\Delta_m|^{-1}I^z((P_n\circ\theta_x^{-1})_{\Delta_m^+}|\nu^z_{\Delta_m^+})\\
&\le \lim_{m\to\infty}\frac{|\Delta_m^+|}{|\Delta_m||\Lambda_n|}I^z(C_n|\nu^z_{\Lambda_n})\\
&= |\Lambda_n|^{-1}I^z(C_n|\nu^z_{\Lambda_n}).
\end{aligned}$$

Since I^z is lower semi-continuous and affine, [DS89, Lemma 5.4.24] implies that

$$\begin{aligned}I^z(\tilde{P}_n) &= |\Lambda_n|^{-1}\int_{\Lambda_n} I^z(P_n\circ\theta_x^{-1})\mathrm{d}x\\ &\le |\Lambda_n|^{-1}I^z(C_n|\nu^z_{\Lambda_n}),\end{aligned}$$

which finishes the proof. □

Proposition 4.4.12. *The sequence $\tilde{P}_n$ has a convergent subsequence, and its limit $\tilde{P}\in\mathscr{P}_\Theta$ is non-degenerate ($\tilde{P}\neq\delta_\emptyset$).*

Proof. We seek to apply Proposition 4.4.5 to obtain the convergent subsequence. Since ψ satisfies (HCC), (4.4.6) is satisfied with $k=2R_0$ and $\tilde{P}_n$ in place of P_n. To show the other assumption of Proposition 4.4.5 holds we will prove that

$$I^z(\tilde{P}_n)-(c_S+\log q)i(\tilde{P}_n)\le |C|^{-1}(c_\Gamma-\log\Pi_C(\Gamma))+o(1). \tag{4.4.13}$$

Applying Lemma 4.4.11 we have

$$\begin{aligned}I^z(C_n) &\le |\Lambda_n|^{-1}I(C_n|\nu^z_{\Lambda_n})\\ &= |\Lambda_n|^{-1}\int\log\Big(\mathsf{Z}_\Delta(G)^{-1}Q_{\Delta,G}(G^{\mathbb{R}^2,\Delta}\cup G'\cup(\emptyset,E''))\mu^+_{\Delta,\omega',\omega}(\mathrm{d}E'')\Big)\\ &\qquad C_n(\mathrm{d}G'),\end{aligned}$$

and applying the upper bound from Proposition 4.2.2 yields

$$\begin{aligned}I^z(\tilde{P}_n) &\le -|\Lambda_n|^{-1}\log\mathsf{Z}_{\Lambda_n}(G)+|\Lambda_n|^{-1}\int\log\Big(q^{|\omega'|}e^{-H_{\Lambda_n,\omega}(\omega')}\Big)C_n(\mathrm{d}G')\\ &= -|\Lambda_n|^{-1}\log\mathsf{Z}_{\Lambda_n}(G)+\log(q)i(\tilde{P}_n)-|\Lambda_n|^{-1}\int H_{\Lambda_n,\omega}(\omega')C_n(\mathrm{d}G').\end{aligned} \tag{4.4.14}$$

By assumption (4.4.9), $|\partial_{\Lambda_n}\omega|=o(|\Lambda_n|)$, so we can deduce from (4.4.10) that

$$-|\Lambda_n|^{-1}\int H_{\Lambda_n,\omega}(\omega')C_n(\mathrm{d}G')\le c_S i(\tilde{P}_n)+o(1).$$

By (HCC) and Lemma 4.1.4,

$$H_{\Lambda_n,\omega}(\omega') < \infty \implies (\omega_{(\Lambda_n)^c})^{\Lambda_n} = ((\omega_{(\Lambda_n)^c} \cup \omega')^{\Lambda_n})_{(\Lambda_n)^c} \subset \omega_{\Lambda_n \oplus 2R_0 \setminus \Lambda_n},$$

and by (4.4.9), $|\omega_{\Lambda_n \oplus 2R_0 \setminus \Lambda_n}| = o(|\Lambda_n|)$, so similar to (4.4.11) we have the estimate

$$\begin{aligned} \mathsf{Z}_{\Lambda_n}(G) &\geq q^{-o(|\Lambda_n|)} e^{-c_\Gamma (2n+1)^2 - o(|\Lambda_n|)} \Pi_C(\Gamma)^{(2n+1)^2} \\ \implies -|\Lambda_n|^{-1} \log \mathsf{Z}_{\Lambda_n}(G) &\leq c_\Gamma |C|^{-1} + o(1) - |C|^{-1} \log \Pi_C(\Gamma). \end{aligned}$$

Combining these inequalities with (4.4.14) gives (4.4.13), as required.

Finally, it is left to check that the limit $\tilde{P}$ is non-degenerate. Let $c_1 := c_S + \log(q)$, $c_2 := |C|^{-1}(c_\Gamma - \log(\Pi_C(\Gamma)))$. By (4.4.13) and (4.4.8) we have $I^{ze^{c_1}}(\tilde{P}_n) \leq c_2 - z + ze^{c_1} + o(1)$. Since $I^{ze^{c_1}}$ is lower semi-continuous, we also have $I^{ze^{c_1}}(\tilde{P}) \leq c_2 - z + ze^{c_1}$. By assumption (U3), $c_2 < z$, and so

$$I^{ze^{c_1}}(\tilde{P}) < ze^{c_1} = I^{ze^{c_1}}(\delta_\emptyset),$$

which implies $\tilde{P}_n \neq \delta_\emptyset$. □

The remainder of this Chapter will be spent showing that the conditioned measure $P := \tilde{P}(\cdot|\{\emptyset\}^c)$ is a Delaunay random cluster measure. For ease of notation, henceforth we identify $\tilde{P}_n$ with its subsequence that converges to $\tilde{P}$.

4.4.3 A second converging sequence

In this short section we construct, for each $\Delta \Subset \mathbb{R}^2$, a sequence $(C_n^\Delta)_{n\in\mathbb{N}}$ of sub-probability measures which also has a subsequence converging locally to the limit $\tilde{P}$. We do this by showing that $\lim_{n\to\infty} |C_n^\Delta(f) - \tilde{P}_n(f)| = 0$ for each local function f. The measures C_n^Δ satisfy the DLR equation for $\Delta \Subset \mathbb{R}^2$, making them useful for proving that the same is true for P. If n is large enough that $\Delta \subset \Lambda_n$ then the Δ-*interior* of Λ_n,

$$\Lambda_n^\circ := \{x \in \mathbb{R}^2 : \Delta + x \subset \Lambda_n\},$$

is non-empty. We can then define the measure

$$C_n^\Delta := |\Lambda_n|^{-1} \int_{\Lambda_n^\circ} C_{\Lambda_n,G} \circ \theta_x^{-1} \mathrm{d}x = |\Lambda_n|^{-1} \int_{\Lambda_n^\circ} C_{\Lambda_n - x, \theta_x G} \, \mathrm{d}x,$$

where the equality is due to the fact that Q is shift-invariant.

Proposition 4.4.13. *If n is large enough that $\Delta \subset \Lambda_n$, then C_n^Δ satisfies the DLR equation for Δ, i.e for any local bounded function f,*

$$C_n^\Delta(f) = \int C_{\Delta,G}(f) C_n^\Delta(\mathrm{d}G).$$

In addition,

$$\lim_{n\to\infty} |C_n^\Delta(f) - \tilde{P}_n(f)| = 0, \tag{4.4.15}$$

and so C_n^Δ has a subsequence which converges locally to $\tilde{P}$.

Proof. The DLR equations follow from the fact that the kernels $(C_\Delta)_{\Delta \Subset \mathbb{R}^2}$ form a specification (Proposition 4.2.6). To prove (4.4.15), let f be an $\mathcal{A}_\Lambda$-measurable function with $|f(G)| \le U$ for all $G \in \mathcal{G}$. Since $P_n \circ \theta_x^{-1}$ satisfies (4.4.6) for $k = 2R_0$, if $(\Lambda \oplus 2R_0 \cup \Delta) + x \subset \Lambda_n$,

$$C_{\Lambda_n - x, \theta_x G}(f) = P_n \circ \theta_x^{-1}(f). \tag{4.4.16}$$

Hence, if n is large enough that $\Lambda \oplus 2R_0, \Delta \subset \Lambda_n$,

$$\begin{aligned}
&|C_n^\Delta(f) - \tilde{P}_n(f)| \\
&= |\Lambda_n|^{-1} \left| \int_{\Lambda_n^\circ} C_{\Lambda_n - x, \theta_x G}(f) \mathrm{d}x - \int_{\Lambda_n} P_n \circ \theta_x^{-1}(f) \mathrm{d}x \right| \\
&= |\Lambda_n|^{-1} \Bigg| \int_{\{x : \Delta + x \subset \Lambda_n, \Lambda \oplus 2R_0 + x \not\subset \Lambda_n\}} C_{\Lambda_n - x, \theta_x G}(f) \mathrm{d}x \\
&\qquad\qquad - \int_{\{x : (\Lambda \oplus 2R_0 \cup \Delta) + x \not\subset \Lambda_n\}} P_n \circ \theta_x^{-1}(f) \mathrm{d}x \Bigg| \\
&\le 2U \; |\Lambda_n|^{-1} |\{x \in \mathbb{R}^2 : (\Lambda \oplus 2R_0 \cup \Delta) + x \not\subset \Lambda_n\}| \xrightarrow{n\to\infty} 0.
\end{aligned}$$

□

4.4.4 Uniqueness of the infinite connected component

This section is devoted to showing that with respect to $\tilde{P}$ (and therefore P), there is at most one infinite connected component almost surely. This fact is subsequently used to bound the range of N_{cc}^{Δ}, which will allow us in the next section to use the local convergence of C_n^{Δ} to $\tilde{P}$ to show that $\tilde{P}$ satisfies the DLR equation in Δ. This section largely follows the approach taken in [DH15, section 4.2]. We start by proving some basic properties of $\tilde{P}$ and some preliminary lemmas.

Suppose $\Delta \Subset \mathbb{R}^2$, $n \in \mathbb{R}^+$ and $G = (\omega, E) \in \mathcal{G}$. Define $\Delta_n := \Delta \oplus n$ and let $f_n^{\Delta} : \mathcal{G} \to \mathcal{G}'_{\Delta_n}$ be the function defined by $f_n^{\Delta}(G) = (\omega_n, E_n)$, where

$$\omega_n := \{x \in \omega : \exists \tau \in \mathrm{Del}_{3,\Delta}(\omega) \text{ with } x \in \tau \text{ and } B(\tau) \subset \Delta_n\},$$

and

$$E_n := \{e \in E : \exists \tau \in \mathrm{Del}_{3,\Delta}(\omega) \text{ with } e \subset \tau \text{ and } B(\tau) \subset \Delta_n\}.$$

See figure 4.2 for a diagram of $f_n^{\Delta}(\omega, (\mathrm{Del}_2(\omega)))$. The following properties are satisfied by f_n^{Δ}.

1. $f_n^{\Delta}(G) = f_n^{\Delta}(G_{\Delta_n})$ (f_n^{Δ} is local), (4.4.17)
2. $f_n^{\Delta}(G)^{\Delta} = f_n^{\Delta}(G)$, (4.4.18)
3. for $n \leq m$, $f_n^{\Delta}(G) \subset f_m^{\Delta}(G)$, with $\cup_{n \in \mathbb{N}} f_n^{\Delta}(G) = G^{\Delta}$. (4.4.19)

Lemma 4.4.14. *For any $\Delta \Subset \mathbb{R}^2$, $\tilde{P}(f_{2R_0}^{\Delta}(G) = G^{\Delta}) = 1$.*

Proof. Since f_n^{Δ} is local we can apply the local convergence of C_m^{Δ} to $\tilde{P}$ (Proposition 4.4.13):

$$\begin{aligned}\tilde{P}(f_{2R_0}^{\Delta}(G) = G^{\Delta}) &= \lim_{n\to\infty} \tilde{P}(f_{2R_0}^{\Delta}(G) = f_n^{\Delta}(G)) \\ &= \lim_{n\to\infty} \lim_{m\to\infty} C_m^{\Delta}(f_{2R_0}^{\Delta}(G) = f_n^{\Delta}(G)).\end{aligned}$$

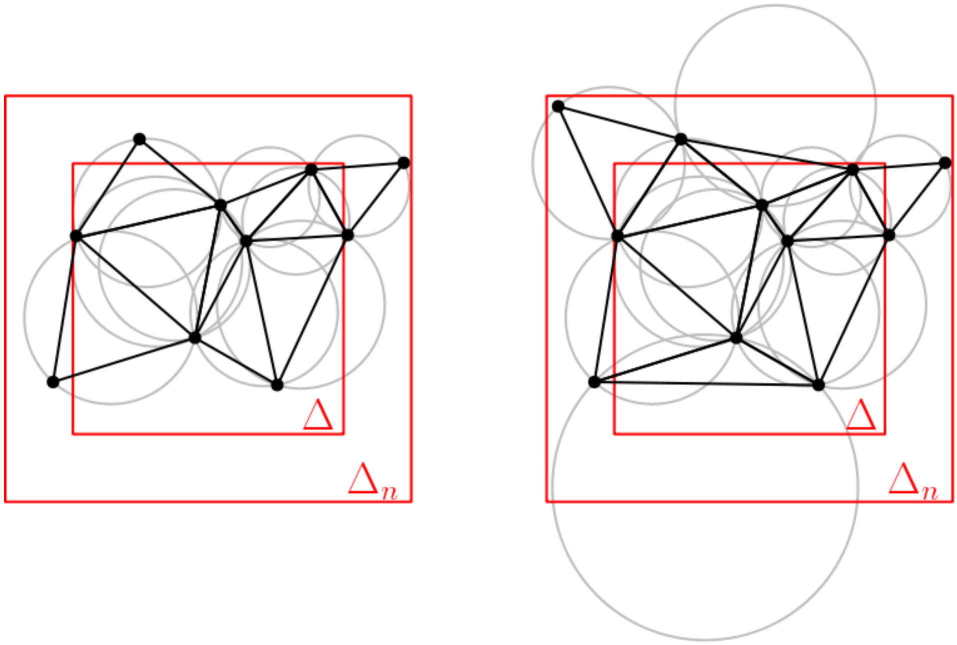


Figure 4.2: A comparison of $f_n^\Delta(\omega, (\mathrm{Del}_2(\omega)))$ (left) with $(\omega, (\mathrm{Del}_2(\omega)))_{\Delta_n}$ (right) for a fixed configuration ω.

The last expression is equal to 1 since (HCC) implies that

$$C_m^\Delta(f_n^\Delta(G) = G^\Delta) = \frac{|\Lambda_m^\circ|}{|\Lambda_m|}$$

for all $n \geq 2R_0$. $\square$

Lemma 4.4.15. *Let A_+ and A_- denote events that $E^\Delta = \mathrm{Del}_{2,\Delta}(\omega)$ and $E^\Delta = \emptyset$ respectively, where $\Delta \Subset \mathbb{R}^2$. If $G \in \mathcal{G}_*^\Delta$ there exists $K(\Delta, \ell_0) \in \mathbb{R}$ such that*

$$C_{\Delta,G}\left(A_+\right) \geq q p_-^{-6} \left(\frac{p_-^3}{q}\right)^{K(\Delta,\ell_0)+|\omega_{\Delta_{2R_0}} \setminus \Delta|}$$

and

$$C_{\Delta,G}\left(A_-\right) \geq q (1-p_+)^{-6} \left(\frac{(1-p_+)^3}{q}\right)^{K(\Delta,\ell_0)+|\omega_{\Delta_{2R_0}} \setminus \Delta|}.$$

Proof. Using the bounds from Propositions 4.2.2 and 4.2.3 along with (HCC) we have

$$|\omega'_\Delta| \geq N_{cc}^\Delta(G') \geq 1 - |((\omega')^\Delta)_{\Delta^c}| \geq 1 - |\omega'_{\Delta_{2R_0}\setminus\Delta}|, \tag{4.4.20}$$

which implies

$$\frac{Q_{\Delta,G}(G')}{\mathsf{Z}_\Delta(G)} \geq \frac{q^{1-|\omega'_{\Delta_{2R_0}\setminus\Delta}|}e^{-H_{\Delta,\omega}(\omega'_\Delta)}}{\int q^{|\omega'|}e^{-H_{\Delta,\omega}(\omega')}\Pi^z_\Delta(\mathrm{d}\omega')}. \tag{4.4.21}$$

Now since the Delaunay graph is a nearest neighbour graph and there is a lower bound ℓ_0 on the edge lengths enforced by (HCL), the number of points in Δ is bounded above by some number $K(\Delta, \ell_0)$. It is a simple consequence of Euler's formula for planar graphs that the maximum number of edges in a planar graph with V vertices is $3V - 6$, so using (4.3.1) we have

$$\begin{aligned} C_{\Delta,G}(A_+) &\geq \frac{\int p_-^{3(K(\Delta,\ell_0)+|\omega_{\Delta_{2R_0}\setminus\Delta}|)-6} q^{1-|\omega_{\Delta_{2R_0}\setminus\Delta}|}e^{-H_{\Delta,\omega}(\omega')}\Pi^z_\Delta(\mathrm{d}\omega')}{\int q^{K(\Delta,\ell_0)}e^{-H_{\Delta,\omega}(\omega')}\Pi^z_\Delta(\mathrm{d}\omega')} \\ &= p_-^{3(K(\Delta,\ell_0)+|\omega_{\Delta_{2R_0}\setminus\Delta}|)-6} q^{1-|\omega_{\Delta_{2R_0}\setminus\Delta}|-K(\Delta,\ell_0)}. \end{aligned}$$

The proof of the second inequality is carried out in the same way. □

The proof that there is at most one infinite connected component is based on the following property called the *local modification property.* It states that all of the edges in G^Δ are all open or all closed with positive probability.

Proposition 4.4.16. (Local modification property.) *Let $\Delta \Subset \mathbb{R}^2$. For any event B satisfying $\tilde{P}(B) > 0$ and $\mathbb{1}_B(G) = \mathbb{1}_B(G^{\mathbb{R}^2,\Delta})$ for all G,*

$$\min(\tilde{P}(A_+ \cap B), \tilde{P}(A_- \cap B)) > 0. \tag{4.4.22}$$

Proof. We will prove that $\tilde{P}(A_+ \cap B) > 0$. The proof that $\tilde{P}(A_- \cap B) > 0$ is the same except that $1 - p_+$ replaces p_-.
By Lemma 4.4.14, we have $\mathbb{1}_{A_+}(f^\Delta_{2R_0}(G)) = \mathbb{1}_{A_+}(G^\Delta) = \mathbb{1}_{A_+}(G)$ $\tilde{P}-$a.s, with the leftmost expression being a local function due to (4.4.17). By Levy's upwards theorem, we can write $\mathbb{1}_B$ as a limit of local functions (which only

depend on $G^{\mathbb{R}^2,\Delta}$):

$$\mathbb{1}_B(G) = \lim_{\Lambda \nearrow \mathbb{R}^2} \tilde{P}(B|\mathcal{A}_\Lambda)(G_\Lambda) = \lim_{\Lambda \nearrow \mathbb{R}^2} \tilde{P}(B|\mathcal{A}_\Lambda)(G^{\mathbb{R}^2,\Delta}) \qquad \tilde{P}\text{-a.s.}$$

Therefore the local convergence of C_m^Δ to $\tilde{P}$ gives us the following:

$$\begin{aligned}
\tilde{P}(A_+ \cap B) &= \lim_{\Lambda \nearrow \mathbb{R}^2} \int \mathbb{1}_{A_+}(f_{2R_0}^\Delta(G))\tilde{P}(B|\mathcal{A}_\Lambda)(G_\Lambda)\tilde{P}(\mathrm{d}G) \\
&= \lim_{\Lambda \nearrow \mathbb{R}^2} \lim_{m\to\infty} \int \mathbb{1}_{A_+}(f_{2R_0}^\Delta(G))\tilde{P}(B|\mathcal{A}_\Lambda)(G_\Lambda)C_m^\Delta(\mathrm{d}G) \\
&= \lim_{\Lambda \nearrow \mathbb{R}^2} \lim_{m\to\infty} \int \mathbb{1}_{A_+}(G)\tilde{P}(B|\mathcal{A}_\Lambda)(G_\Lambda)C_m^\Delta(\mathrm{d}G) \\
&= \lim_{\Lambda \nearrow \mathbb{R}^2} \lim_{m\to\infty} \int C_{\Delta,G}(A_+)\tilde{P}(B|\mathcal{A}_\Lambda)(G_\Lambda)C_m^\Delta(\mathrm{d}G). \qquad (4.4.23)
\end{aligned}$$

Now applying the result of Lemma 4.4.15 and the local convergence of C_m^Δ to $\tilde{P}$ again we have

$$\begin{aligned}
\tilde{P}(A_+ \cap B) &= \lim_{\Lambda \nearrow \mathbb{R}^2} \lim_{m\to\infty} \int p_-^{3(K(\Delta,\ell_0)+|\omega_{\Delta_{2R_0}}\setminus\Delta|)-6} q^{1-|\omega_{\Delta_{2R_0}}\setminus\Delta|-K(\Delta,\ell_0)} \\
&\qquad\qquad \tilde{P}(B|\mathcal{A}_\Lambda)(G_\Lambda)C_m^\Delta(\mathrm{d}G) \\
&= qp_-^{-6}\left(\frac{p_-^3}{q}\right)^{K(\Delta,\ell_0)} \int_B \left(\frac{p_-^3}{q}\right)^{|\omega_{\Delta_{2R_0}}\setminus\Delta|} \tilde{P}(\mathrm{d}G) > 0. \qquad \square
\end{aligned}$$

Now since $\tilde{P}$ is translation invariant, it can be written as a mixture

$$\tilde{P} = \int \tilde{P}_H \tilde{P}(\mathrm{d}H), \qquad (4.4.24)$$

where $\tilde{P}_H$ are ergodic probability measures [MR96, Proposition 7.2]. Like $\tilde{P}$, the measures $\tilde{P}_H$ also satisfy the local modification property. The event that the number of infinite connected components is exactly k is translation invariant, so by ergodicity the number of infinite connected components is $\tilde{P}_H$-almost surely constant for all H. Notice that a consequence of Lemma 4.4.14 is that

$$\tilde{P}_H(f_{2R_0}^\Delta = G^\Delta) = 1 \qquad (4.4.25)$$

for $\tilde{P}$-almost all H.

Proposition 4.4.17. *If P is an ergodic measure satisfying the local modification property then for all integers $k > 1$*

$$P(N_{cc}^{\infty} = k) = 0,$$

where N_{cc}^{∞} denoted the number of infinite connected components.

Proof. Since P is ergodic and the event $N_{cc}^{\infty} = k$ is translation invariant it is enough to show that $P(N_{cc}^{\infty} = k) < 1$. Suppose $P(N_{cc}^{\infty} = k) = 1$ and let $\Delta \Subset \mathbb{R}^2$ be large enough that with positive probability all k infinite connected components intersect ω_{Δ}. Any path from a point in ω_{Δ^c} to a point in ω_{Δ} must contain a point in $(\omega^{\Delta})_{\Delta^c}$, so with positive probability all k infinite connected components intersect $(\omega^{\Delta})_{\Delta^c}$. This happens if and only if $(\omega^{\Delta})_{\Delta^c}$ intersects all (possibly more than k but still finite) infinite connected components of $G^{\mathbb{R}^2,\Delta}$, which only depends on $G^{\mathbb{R}^2,\Delta}$. By applying the local modification property we conclude that with positive probability this occurs simultaneously with $E^{\Delta} = \mathrm{Del}_{2,\Delta}(\omega)$. But in this case all of the infinite connected components of $G^{\mathbb{R}^2,\Delta}$ are connected via edges in E^{Δ}, so $\tilde{P}(N_{cc}^{\infty}(G) = 1) > 0$, which is a contradiction. □

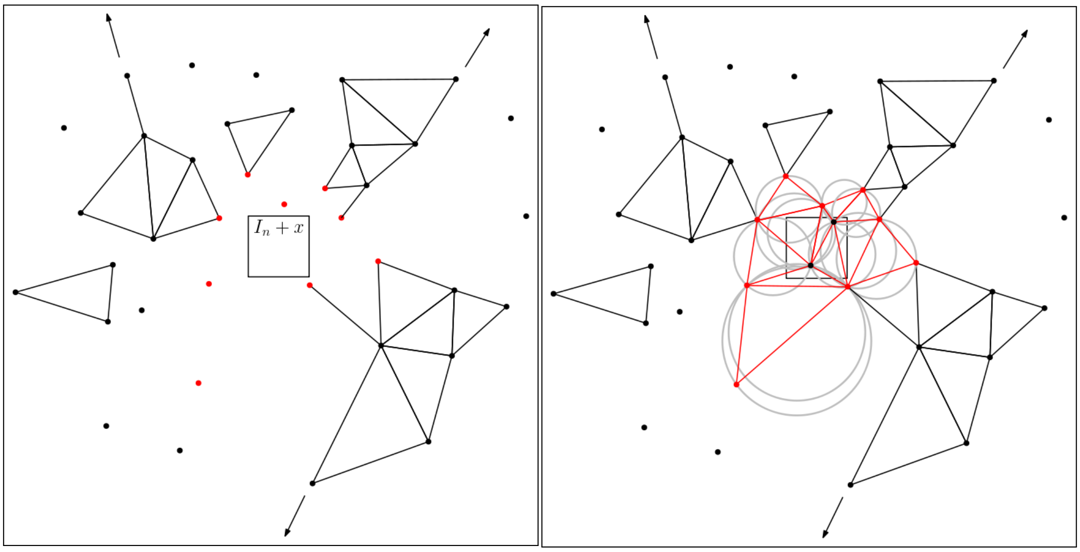
Proposition 4.4.18. (Uniqueness of the infinite connected component)

$$\tilde{P}(N_{cc}^{\infty} \leq 1) = 1.$$

Proof. By Proposition 4.4.17 and the ergodicity of $\tilde{P}_H$, it is enough to show that $\tilde{P}_H(N_{cc}^{\infty} = \infty) < 1$ for $\tilde{P}$-almost all H. Suppose that $\tilde{P}_H(N_{cc}^{\infty} = \infty) = 1$ and let $I_n := [-n, n]^2$. A point $x \in (3n + 6R_0)\mathbb{Z}^2$ is called a *trifurcation point* (pictured in figure 4.3b) if

- There are at least 3 infinite connected components of $G^{\mathbb{R}^2, I_n+x}$ intersecting $(\omega^{I_n+x})_{(I_n+x)^c}$, and
- $E^{I_n+x} = \mathrm{Del}_{2,I_n+x}(\omega)$.

By the local modification property, one can choose n large enough that 0 is a trifurcation point with positive probability. By the translation invariance of



(a) $G^{\mathbb{R}^2, I_n+x}$ (b) G

Figure 4.3: A configuration G with a trifurcation point at x. The vertices of $(\omega^{I_n+x})_{(I_n+x)^c}$ and the edges of E^{I_n+x} are coloured in red.

$\tilde{P}_H$, the probability that any point $x \in (3n + 6R_0)\mathbb{Z}^2$ is a trifurcation point is the same, so by the linearity of the expectation the expected number of trifurcation points inside the box I_m is in $\mathcal{O}(m^2) \setminus o(m^2)$.

Now we apply [BR06, Lemma 3, p. 121] in a similar fashion to how it was applied in [BR06, Theorem 4, p.121]. Let $\mathcal{O}$ denote the union of all infinite connected components of G which intersect I_{m+n+2R_0}. Let T_m denote the set of trifurcation points in I_m. If for every $x \in T_m$ the edges in E^{I_n+x} are removed, $\mathcal{O}$ is split into several components. Let $L_1, ..., L_t$ denote the infinite ones and $F_1, ..., F_u$ denote the finite ones. Now we consider the graph J obtained from G by contracting ω^{I_n+x} to a single point v_x for each $x \in T_m$ and contracting the components L_i and F_i to single points l_i and f_i. By the definition of a trifurcation point, the removal of any point v_x from J disconnects a connected component into at least three components containing at least one vertex of $L = \{l_1, ..., l_t\}$. Therefore, by Lemma 3 of [BR06, p. 121],

$$|L| \geq 2 + |T_m|. \tag{4.4.26}$$

Every member of $\{L_1, ..., L_t\}$ must contain an element of $(\omega^{I_{m+n+2R_0}})_{(I_{m+n+2R_0})^c}$, so by (4.4.25)

$$|\omega_{(I_{m+n+4R_0})\setminus(I_{m+n+2R_0})}| \geq |(\omega^{I_{m+n+2R_0}})_{I_{m+n+2R_0}{}^c}| \geq 2 + |T_m|.$$

Since $\tilde{P}_H$ is translation invariant, the expected number of points in a region is proportional to its Lebesgue measure, so the left hand side is proportional to $2R_0(2m + 2n + 4R_0)$. This is a contradiction because $T_m \in \mathcal{O}(m^2) \setminus o(m^2)$ as we found earlier. □

4.4.5 The DLR equations

We can now prove that $P = \tilde{P}(\cdot|\{\emptyset\}^c)$ is a Delaunay random cluster measure. We use the uniqueness of the infinite connected component to construct events which localise N_{cc}^{Δ} and whose probability approaches 1. We also make use of the events $\hat{\Omega}_{\Delta,n}^{\mathrm{cr}}$ introduced in [DDG12], which localises the Hamiltonian. Combining these events allows us to use the DLR equations for C_m^{Δ} and the local convergence to show that P also satisfies the DLR equations.

The following statement summarises [DDG12, Proposition 5.4] and the discussion that precedes it. The precise definition of $\hat{\Omega}_{\Delta,n}^{\mathrm{cr}}$ and the values of n_Δ and m are given in Appendix A for reference.

Proposition 4.4.19. *There exist constants n_Δ and m, depending on the matrix M and the constants given in (R), such that for $n \geq n_\Delta$ there are events $\hat{\Omega}_{\Delta,n}^{\mathrm{cr}} \in \mathcal{F}_{\hat{\Lambda}_n \setminus \Delta}$ satisfying*

$$\hat{\Omega}_{\Delta}^{\mathrm{cr}} := \bigcup_{n \geq n_\Delta} \hat{\Omega}_{\Delta,n}^{\mathrm{cr}} \subset \Omega_{\Delta}^{\mathrm{cr}} \text{ with } \partial\Delta(\omega) \subset \hat{\Lambda}_n \text{ when } \omega \in \hat{\Omega}_{\Delta,n}^{\mathrm{cr}}, \tag{4.4.27}$$

for $\hat{\Lambda}_n := \Lambda_{n+(2n+1)m}$. Moreover, $P(\hat{\Omega}_{\Delta}^{\mathrm{cr}}) = 1$ for all $P \in \mathscr{P}_\Theta$ with $P(\{\emptyset\}) = 0$.

It is implied by (4.4.27) that

$$H_{\Delta,\omega}(\omega') = H_{\Delta,\omega_{\hat{\Lambda}_n \setminus \Delta}}(\omega') \text{ when } \omega \in \hat{\Omega}_{\Delta,n}^{\mathrm{cr}}. \tag{4.4.28}$$

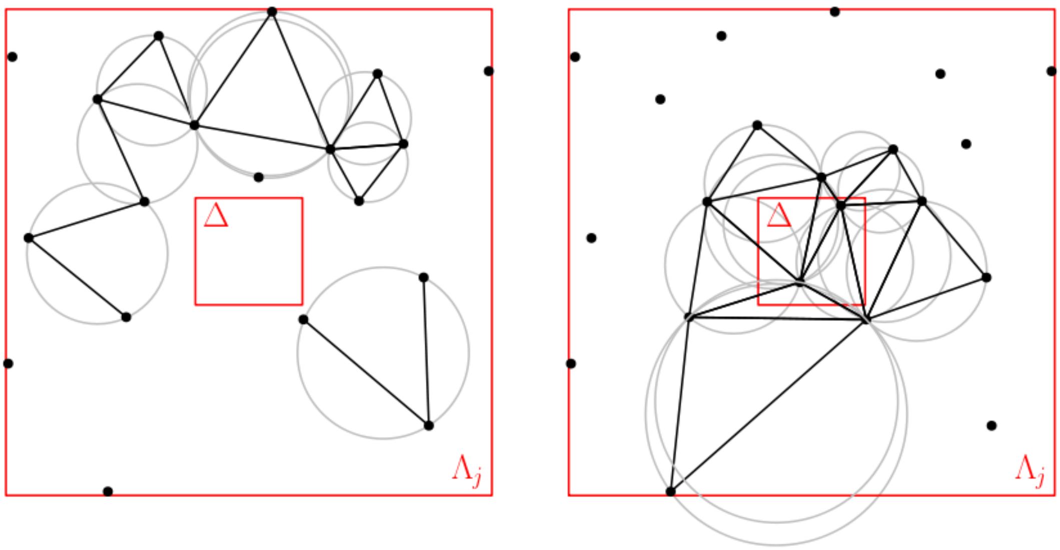


Figure 4.4: $h_j(\omega, \mathrm{Del}_2(\omega))$ (left) and $(\omega_{\Lambda_j}, \mathrm{Del}_{2,\Delta}(\omega_{\Lambda_j}))$ (right).

For $G \in \mathcal{G}$ we define $h_j(G) := (\omega_{\Lambda_j \setminus \Delta}, E_j)$ where

$$E_j := \left\{ e \in E \cap \mathrm{Del}_2(\omega_{\Lambda_j}) \setminus \mathrm{Del}_{2,\Delta}(\omega_{\Lambda_j}) \,\middle|\, \begin{array}{c} \exists \tau \in \mathrm{Del}_3(\omega_{\Lambda_j \setminus \Delta}) \text{ with } e \subset \tau \\ \text{and } B(\tau) \subset \Lambda_j \setminus \Delta. \end{array} \right\}$$

This definition ensures that $h_j(G) = h_j(G_{\Lambda_j}) = h_j(G^{\mathbb{R}^2,\Delta})$ and $h_j(G) \nearrow G^{\mathbb{R}^2,\Delta}$. We can then define the events (when $\Delta_{2R_0} \subset \hat{\Lambda}_p \subset \Lambda_j$)

$$W_{p,j} := \left\{ G = (\omega, E) \in \mathcal{G} \,\middle|\, \begin{array}{c} \text{At most one connected component of} \\ h_j(G) \text{ intersects } \Delta_{2R_0} \text{ and } \hat{\Lambda}_p^c. \end{array} \right\}$$

and

$$A_p := \left\{ G = (\omega, E) \in \mathcal{G} \mid \delta(\tau) < R_0 \text{ for all } \tau \in \mathrm{Del}_{3,\hat{\Lambda}_p}(\omega) \setminus \mathrm{Del}_{3,\Delta}(\omega) \right\}.$$

Notice that $\mathbb{1}_{W_{p,j}}(G) = \mathbb{1}_{W_{p,j}}(G_{\Lambda_j}) = \mathbb{1}_{W_{p,j}}(G^{\mathbb{R}^2,\Delta})$, and by Corollary 4.1.2 $\mathbb{1}_{A_p}(G) = \mathbb{1}_{A_p}(G^{\mathbb{R}^2,\Delta})$.

Lemma 4.4.20. $\lim_{p\to\infty} \lim_{j\to\infty} \tilde{P}(W_{p,j}) = 1$.

Proof.

$$\begin{aligned}\lim_{p\to\infty}\lim_{j\to\infty}\tilde{P}(W_{p,j}) &= \tilde{P}\left(\bigcup_{p\in\mathbb{N}}\bigcap_{j>p}W_{p,j}\right)\\ &= \tilde{P}(G^{\mathbb{R}^2,\Delta} \text{ has at most one infinite cc intersecting } \Delta_{2R_0})\\ &\geq \tilde{P}(G^{\mathbb{R}^2,\Delta} \text{ has at most one infinite cc}).\end{aligned}$$

If the last expression is not equal to one, then with positive probability there are at least two infinite connected components of $G^{\mathbb{R}^2,\Delta}$. If so, then there exists a region Δ' such that with positive probability $G^{\mathbb{R}^2,\Delta'}$ has at least two infinite connected components which contain points in $\omega^{\Delta'}$. By the local modification property (Proposition 4.4.16), $E^{\Delta'} = \emptyset$ occurs simultaneously with positive probability, which means that there would be at least two infinite connected components in G, which contradicts Proposition 4.4.18. □

Lemma 4.4.21. *$\tilde{P}(A_p) = 1$ for all p, and $\lim_{n\to\infty} C_n^{\Delta}(A_p) = 1$.*

Proof. $\tilde{P}(A_p) = 1$ is a consequence of Lemma 4.4.14 with $\hat{\Lambda}_p$ replacing Δ. For the second statement note that

$$\Delta \subset \hat{\Lambda}_p \implies \{x \in \mathbb{R}^2 : \hat{\Lambda}_p + x \subset \Lambda_n\} \subset \{x \in \mathbb{R}^2 : \Delta + x \subset \Lambda_n\},$$

and so

$$C_n^{\Delta}(A_p) \geq C_n^{\hat{\Lambda}_p}(A_p) \xrightarrow{n\to\infty} 1.$$

by (HCC). □

The following proposition expresses how the events $\hat{\Omega}^{\mathrm{cr}}_{\Delta,p}, W_{p,j}$ and A_p localise the dependency of the finite volume distributions $C_{\Delta,G}$ on G.

Proposition 4.4.22. *Suppose p and j are such that $\Delta \subset \hat{\Lambda}_p$ and $\hat{\Lambda}_p \oplus 2R_0 \subset \Lambda_j$. Then there exists an $\mathcal{A}_{\Lambda_j}$-measurable function $\mathsf{Z}_{\Delta,p,j} : \mathcal{G} \to [0,\infty]$ such that $\mathsf{Z}_{\Delta}(G) = \mathsf{Z}_{\Delta,p,j}(G)$ for all $G \in W_{p,j} \cap (\hat{\Omega}^{\Delta,p}_{cr} \times \mathcal{E}) \cap A_p$.*
Furthermore, there exists a kernel $C_{\Delta,p,j} : \mathcal{G} \times M(\mathcal{G}) \to [0,\infty]$ such that for

all $G \in W_{p,j} \cap (\hat{\Omega}_{cr}^{\Delta,p} \times \mathcal{E}) \cap A_p$ *and all* $\mathcal{A}_{\hat{\Lambda}_p}$*-measurable functions* f*,*

$$C_{\Delta,p,j}(G,f) = C_{\Delta,p,j}(G_{\Lambda_j}, f) = C_{\Delta,G}(f). \tag{4.4.29}$$

Proof. By (4.4.28), $H_{\Delta,\omega}(\omega') = H_{\Delta,\omega_{\hat{\Lambda}_p\setminus\Delta}}(\omega')$ for Π_Δ-almost all ω'. Since ψ satisfies (HCC), if $H_{\Delta,\omega}(\omega') < \infty$, then $\mathrm{Del}_{2,\Delta}(\omega_{\Delta^c} \cup \omega')$ is equal to

$$\left\{ e \in \mathrm{Del}_{2,\Delta}(\omega_{\Delta_{2R_0}\setminus\Delta} \cup \omega') \,\middle|\, \begin{array}{c} \exists \tau \in \mathrm{Del}_{3,\Delta}(\omega_{\Delta_{2R_0}\setminus\Delta} \cup \omega') \\ \text{with } e \subset \tau \text{ and } B(\tau) \subset \Delta_{2R_0} \end{array} \right\}, \tag{4.4.30}$$

and therefore if $\tilde{\mu}_{\omega_{\Delta_{2R_0}},\omega'}$ is the type drawing mechanism on the set (4.4.30) then

$$\mathsf{Z}_\Delta(G) = \int q^{N_{cc}^\Delta(G^{\mathbb{R}^2,\Delta} \cup (\omega',E'))} \tilde{\mu}_{\omega_{\Delta_{2R_0}},\omega'}(\mathrm{d}E') e^{-H_{\Delta,\omega_{\hat{\Lambda}_p}}(\omega')} \Pi_\Delta^z(\mathrm{d}\omega').$$

Now let

$$\hat{E}_p = \left\{ e \in E_{\hat{\Lambda}_p} \,\middle|\, \begin{array}{c} B(\tau) \subset \hat{\Lambda}_p \oplus 2R_0 \setminus \Delta \text{ for all} \\ \tau \in \mathrm{Del}_3(\omega_{\hat{\Lambda}_p \oplus 2R_0} \cup \omega') \text{ satisfying } e \subset \tau \end{array} \right\}.$$

We claim that

$$\mathsf{Z}_\Delta(G) = \int q^{N_{cc}^\Delta((\omega_{\hat{\Lambda}_p\setminus\Delta}, \hat{E}_p) \cup (\omega',E'))} \tilde{\mu}_{\omega_{\Delta_{2R_0}},\omega'}(\mathrm{d}E') e^{-H_{\Delta,\omega_{\hat{\Lambda}_p}}(\omega')} \Pi_\Delta^z(\mathrm{d}\omega'), \tag{4.4.31}$$

in which case the right hand side is a suitable definition of $\mathsf{Z}_{\Delta,p,j}(G)$. If $H_{\Delta,\omega}(\omega') < \infty$ and $G \in A_p$, then by (HCC)

$$\delta(\tau) < R_0 \text{ for all } \tau \in \mathrm{Del}_{3,\hat{\Lambda}_p}(\omega_{\Delta^c} \cup \omega'). \tag{4.4.32}$$

From (4.4.32) and the fact that $G \in W_{p,j}$ we can conclude that

$$N_{cc}^\Delta(G^{\mathbb{R}^2,\Delta} \cup (\omega',E')) = N_{cc}((\omega_{\hat{\Lambda}_p\setminus\Delta}, \hat{E}_p) \cup (\omega',E')) - N_{cc}((\omega_{\hat{\Lambda}_p\setminus\Delta}, \hat{E}_p)),$$

since

1. $\omega_{\hat{\Lambda}_p\setminus\Delta}$ contains all points which are adjacent to $(\omega_{\Delta^c} \cup \omega')^\Delta$ in $G^{\mathbb{R}^2,\Delta} \cup (\omega',E')$, and

2. Adding any combination of points and edges from $G^{\mathbb{R}^2,\Delta} \cup (\omega', E')$ to $(\omega_{\hat{\Lambda}_p \setminus \Delta}, \hat{E}_p) \cup (\omega', E')$ cannot join two connected components which intersect $(\omega_{\Delta^c} \cup \omega')^\Delta = (\omega_{\hat{\Lambda}_p \setminus \Delta} \cup \omega')^\Delta$.

Note that this is a similar argument to that made in [DH15, Proposition 4.3.1]. Finally, when (4.4.32) is true and f is $\hat{\Lambda}_p$-measurable,

$$f(G^{\mathbb{R}^2,\Delta} \cup (\omega', E')) = f((\omega_{\hat{\Lambda}_p \setminus \Delta}, \hat{E}_p) \cup (\omega', E')),$$

and so the function

$$C_{\Delta,p,j}(G, f) := \mathsf{Z}_{\Delta,p,j}(G)^{-1} \int f((\omega_{\hat{\Lambda}_p \setminus \Delta}, \hat{E}_p) \cup (\omega', E')) q^{N_{cc}^{\Delta}((\omega_{\hat{\Lambda}_p \setminus \Delta}, \hat{E}_p) \cup (\omega', E'))} \tilde{\mu}_{\omega_{\Delta_{2R_0}}, \omega'}(\mathrm{d}E') e^{-H_{\Delta,\omega_{\hat{\Lambda}_p}}(\omega')} \Pi^z_\Delta(\mathrm{d}\omega'),$$

satisfies the requirements. □

Corollary 4.4.23. *Let* $\mathcal{G}^\Delta_{*,p,j} := \{G \in \mathcal{G} \mid \mathsf{Z}_{\Delta,p,j}(G) \in (0, \infty)\} \in \mathcal{A}_{\Lambda_j}$*. Then*

$$\mathcal{G}^\Delta_{*,p,j} \cap W_{p,j} \cap (\hat{\Omega}^{\Delta,p}_{cr} \times \mathcal{E}) \cap A_p = \mathcal{G}^\Delta_* \cap W_{p,j} \cap (\hat{\Omega}^{\Delta,p}_{cr} \times \mathcal{E}) \cap A_p.$$

Proposition 4.4.24. $P := \tilde{P}(\cdot | \{\emptyset\}^c) \in \mathscr{P}_\Theta$ *is a Delaunay random cluster measure.*

Proof. Let $f : \mathcal{G} \to \mathbb{R}$ be a bounded local function and $\Delta \Subset \mathbb{R}^2$. Choose $p \geq n_\Delta$ and j large enough that f is $\mathcal{A}_{\hat{\Lambda}_p}$-measurable, $\Delta \subset \hat{\Lambda}_p$, and $\hat{\Lambda}_p \oplus 2R_0 \subset \Lambda_j$. Let $\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} = \bigcup_{n=n_\Delta}^p \hat{\Omega}^{\mathrm{cr}}_{\Delta,n} \in \mathcal{F}_{\hat{\Lambda}_p \setminus \Delta}$. It is sufficient to show that

$$\int_{(\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} f \mathrm{d}\tilde{P} = \int_{\mathcal{G}^\Delta_* \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} C_{\Delta,G'}(f) \tilde{P}(\mathrm{d}G'). \tag{4.4.33}$$

To see this, we note that $\emptyset \notin \hat{\Omega}^{\mathrm{cr}}_\Delta$, and so (4.4.33) is equivalent to

$$\int_{(\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} f \mathrm{d}P = \int_{\mathcal{G}^\Delta_* \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} C_{\Delta,G'}(f) P(\mathrm{d}G'),$$

and by Proposition 4.4.19 and Lemma 4.4.20,

$$\lim_{p\to\infty} \lim_{j\to\infty} P\left((\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}\right) = 1,$$

which implies

$$\int f \mathrm{d}P = \int_{\mathcal{G}_*^\Delta} C_{\Delta,G'}(f) P(\mathrm{d}G').$$

Taking $f \equiv 1$ yields $P(\mathcal{G}_*^\Delta) = 1$, which finishes the proof.

To prove (4.4.33), let n be large enough that $\hat{\Lambda}_p \subset \Lambda_n$. Then $x \in \Lambda_n^\circ = \{x \in \mathbb{R}^2 : \hat{\Lambda}_p + x \subset \Lambda_n\} \implies \Delta \subset \Lambda_n - x$ and so by Proposition 4.2.6,

$$\begin{aligned} &\int_{(\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} f \mathrm{d}C_{\Lambda_n - x, \theta_x G} = \int_{\mathcal{G}_*^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} C_{\Delta,G'}(f) C_{\Lambda_n - x, \theta_x G}(\mathrm{d}G') \\ \implies &\int_{(\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} f \mathrm{d}C_n^{\hat{\Lambda}_p} = \int_{\mathcal{G}_*^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} C_{\Delta,G'}(f) C_n^{\hat{\Lambda}_p}(\mathrm{d}G') \end{aligned}$$

The integrand on the left hand side is $\mathcal{A}_{\Lambda_j}$-measurable, so applying the local convergence in Proposition 4.4.13 yields

$$\int_{(\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} f \mathrm{d}\tilde{P} = \lim_{n \to \infty} \int_{\mathcal{G}_*^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} \gamma_{\Delta,G'}(f) C_n^{\hat{\Lambda}_p}(\mathrm{d}G').$$

Now we apply the local convergence to the right hand side as well, after manipulating the expression with the help of Lemma 4.4.21, Proposition 4.4.22 and Corollary 4.4.23.

$$\begin{aligned} \int_{(\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} f \mathrm{d}\tilde{P} &= \lim_{n \to \infty} \int_{\mathcal{G}_*^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j} \cap A_p} C_{\Delta,G'}(f) C_n^{\hat{\Lambda}_p}(\mathrm{d}G'). \\ &= \lim_{n \to \infty} \int_{\mathcal{G}_{*,p,j}^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j} \cap A_p} C_{\Delta,p,j}(G'_{\Lambda_j}, f) C_n^{\hat{\Lambda}_p}(\mathrm{d}G'). \\ &= \lim_{n \to \infty} \int_{\mathcal{G}_{*,p,j}^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E})) \cap W_{p,j}} C_{\Delta,p,j}(G'_{\Lambda_j}, f) C_n^{\hat{\Lambda}_p}(\mathrm{d}G'). \\ &= \int_{\mathcal{G}_{*,p,j}^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} C_{\Delta,p,j}(G'_{\Lambda_j}, f) \tilde{P}(\mathrm{d}G'). \\ &= \int_{\mathcal{G}_{*,p,j}^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j} \cap A_p} C_{\Delta,p,j}(G'_{\Lambda_j}, f) \tilde{P}(\mathrm{d}G'). \\ &= \int_{\mathcal{G}_*^\Delta \cap (\hat{\Omega}^{\mathrm{cr}}_{\Delta,\leq p} \times \mathcal{E}) \cap W_{p,j}} C_{\Delta,G'}(f) \tilde{P}(\mathrm{d}G'). \qquad \square \end{aligned}$$

Chapter 5

Outlook

To conclude, we proved in chapter 3 that a particular Delaunay Potts model with interactions between the triples of points which make up the triangles of the Delaunay triangulation exhibits a phase transition for large activities and large values of β (which plays a similar role to the inverse temperature). In chapter 4 we proved that an infinite volume Delaunay random cluster model exists when the background potential satisfies two hard-core conditions and the edge weights are bounded away from 0 and 1. We briefly outline here some future research directions.

As mentioned in the introduction and at the beginning of chapter 4, the main motivation for introducing the infinite volume Delaunay random cluster model is to eventually obtain a uniqueness result for the corresponding Delaunay continuum Potts model via a random cluster representation. The analogous result for the Widom-Rowlinson model was obtained using a random cluster representation and the method of disagreement percolation [HTH19]. The next step in our case is to prove that such a random cluster representation exists, which would likely provide a connection between Delaunay continuum Potts measures with background and type potentials ψ and ϕ and Delaunay random cluster measures with the hyperedge potential ψ and edge weights $p(e) = e^{1-\phi(e)}$. To show uniqueness we would then have to investigate the percolative properties of Delaunay random cluster measures. Uniqueness would likely correspond to a lack of percolation in these measures.

The Delaunay random cluster measures in chapter 4 were defined as edge percolation models; each edge in the Delaunay graph is declared to be either open or closed. It should be straightforward to extend the existence result to a hyperedge percolation model in which each triangle in Del_3 is declared to be either open or closed instead. Such a model would be the infinite volume analogue of the finite volume distribution given in definition 2.3.1. The set of wired boundary conditions in this case are the elements $G = (\omega, E)$ for which $\mathrm{Del}_3(\omega) \setminus \mathrm{Del}_{3,\Delta}(\omega) \subset E$. Wired boundary conditions are also discussed in remark 4.3.2.

Another direction would be to investigate the existence of phase transitions for other geometric continuum Potts models. One could consider for instance models in which Voronoi cells interact with one another. In these models the hyperedge potential acts on pairs of points $\{\boldsymbol{x}, \boldsymbol{y}\} \subset \boldsymbol{\omega}$ whose Voronoi cells share a face. The interaction could depend on the marks of $\boldsymbol{x}$ and $\boldsymbol{y}$ as well as the attributes of their Voronoi cells $\mathrm{Vor}_\omega(x)$ and $\mathrm{Vor}_\omega(y)$ such as geometry of the face they share or their relative volumes. One could also consider models in which the hyperedge potential acts on neighbouring pairs of Voronoi cells or Delaunay triangles.

Finally, as discussed in section 3.2, it should be possible to relax the choice of type potential used and potentially extend the phase transition result to higher dimensions.

Appendix A

Proposition A.0.1. *For* $\Delta \Subset \mathbb{R}^2$*,* $|\omega^\Delta| < \infty$ *for* Π^z*-almost all* ω*, where* $\omega^\Delta := \mathrm{Del}_{1,\Delta}(\omega)$.

Proof. Since ω is locally finite Π^z-almost surely, if $|\omega^\Delta| = \infty$ then there exists a sequence $(x_n)_{n\geq 1} \subset \mathrm{Del}_{1,\Delta}(\omega)$ with $|x_n| \to \infty$. In this case

$$\omega \in A := \left\{ \omega \in \Omega : \begin{array}{c} \forall n \in \mathbb{N}\ \exists \tau \in \mathrm{Del}_{3,\Delta}(\omega) \text{ with } \delta(\tau) \geq n, \\ \mathring{B}(\tau) \cap \omega = \emptyset \text{ and } B(\tau) \cap \Delta \neq \emptyset \end{array} \right\}$$

$$\subset \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \bigcup_{x \in \mathbb{Q}^2 \cap \Delta \oplus (m+1)} \left\{ \mathring{B}(x,m) \cap \omega = \emptyset \right\}.$$

By the first Borel-Cantelli Lemma, $\Pi^z(|\omega^\Delta| = \infty) \leq \Pi^z(A) = 0$ since

$$\Pi^z \left(\bigcup_{x \in \mathbb{Q}^2 \cap \Delta \oplus (m+1)} \left\{ \mathring{B}(x,m) \cap \omega = \emptyset \right\} \right) \leq \sum_{x \in \mathbb{Q}^2 \cap \Delta \oplus (m+1)} e^{-z\pi m^2} < \infty.$$

□

Lemma A.0.2. *Let* $\delta(\tau)$ *and* $p(\tau)$ *denote the circumradius and perimeter of the triangle* τ*. Then*

$$\delta(\tau) \geq \frac{p(\tau)}{3\sqrt{3}},$$

with equality when τ *is equilateral.*

Proof. Let a, b and c denote the side lengths of τ which are opposite to the angles A, B and C respectively. By the law of sines,

$$2\delta(\tau) = \frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
$$\implies \sin(A) + \sin(B) + \sin(C) = \frac{a+b+c}{2\delta(\tau)} = \frac{p(\tau)}{2\delta(\tau)} \tag{A.0.1}$$

Now by Jensen's inequality,

$$\frac{1}{3}(\sin(A) + \sin(B) + \sin(C)) \leq \sin\left(\frac{A+B+C}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},$$

with equality only when $A = B = C$, in which case the triangle is equilateral. Combining this with equation (A.0.1) yields the result. □

Definitions used in Section 4.4.5

The following excerpt is from [DDG12, page 664-665]. In our case $d = 2$, and we use the notation $\hat{\Omega}^{\text{cr}}_{\Lambda,n}$ instead of $\hat{\Omega}^{\Lambda,n}_{\text{cr}}$. The authors define $L_m := \{-n, ..., n\}^d$.

> Let ℓ_R, n_R, δ_R be the constants introduced in condition (R). Also, let δ_- and δ_+ the diameters of the largest open ball in C and of the smallest closed ball containing C, respectively. Fix an integer $m \geq 6\ell_R\delta_+/\delta_-$. For each $n \geq 1$, we decompose the parallelotope $\hat{\Lambda}_n := \Lambda_{n+(n+1)m}$ into the $(2m+1)^d$ translates $\Lambda_n^k := \Lambda_n + (2n+1)Mk$ of Λ_n , where $k \in L_m$. For any $\Lambda \Subset \mathbb{R}^d$, let $n_\Lambda \geq 1$ be the smallest number with $\Lambda_{n_\Lambda} \supset \Lambda$ and $n_\Lambda \geq \delta_R/6\delta^+$. For all $n \geq n_\Lambda$ we consider the events
>
> $$\hat{\Omega}^{\Lambda,n}_{\text{cr}} = \left\{ \min_{0 \neq k \in L_m} N_{\Lambda_n^k} > n_R \right\} \in \mathcal{F}_{\hat{\Lambda}_n \setminus \Lambda}$$
>
> as well as $\hat{\Omega}^{\Lambda}_{\text{cr}} = \bigcup_{n \geq n_\Lambda} \hat{\Omega}^{\Lambda,n}_{\text{cr}} \in \mathcal{F}_{\Lambda^c}$.

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