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Plus/minus p-adic L-functions for GL_{2n}

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Abstract
We generalise Pollack's construction of plus/minus L-functions to certain cuspidal automorphic representations of $\mathrm{GL}_{2\mathrm{n}}$ using the p-adic L-functions constructed in work of Barrera Salazar et al. (On p-adic l-functions for GL_{2n} in finite slope shalika families, 2021). We use these to prove that the complex L-functions of such representations vanish at at most finitely many twists by characters of p-power conductor.

Keywords P-adic L-functions · Automorphic forms · Iwasawa theory

Résumé
Nous généralisons la construction des fonctions L plus/moins de Pollack à certaines représentations automorphes cuspidales de $\mathrm{GL}_{2\mathrm{n}}$ en utilisant les fonctions L p-adiques construites dans les travaux de Barrera Salazar et al. [2]. Nous les utilisons pour démontrer que les fonctions L complexes de telles représentations disparaissent pour au plus un nombre fini de torsions par des caractères de conducteur une puissance de p.

Mathematics Subject Classification 11G40 · 11F67 · 11R23

1 Introduction

Let $f = \sum_{n=0}^{\infty} a_n q^n$ be a normalized cuspidal newform of weight k and level N with character ε, and let p be a prime such that $p \nmid N$. Let α be a root of the Hecke polynomial $X^2 - a_p X + p^{k-1}\varepsilon(p)$ which, after fixing an isomorphism $\bar{\mathbb{Q}}_p \cong \mathbb{C}$, satisfies $r := v_p(a_p) < k-1$, where v_p is the p-adic valuation on $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. From this data we can construct an order r locally analytic distribution $L_p^{(\alpha)}$ on $\mathbb{Z}_p^\times$ whose values at special characters interpolate the critical values of the complex L-function of f and its twists. The arithmetic of $L_p^{(\alpha)}$ is well understood in the case that f is ordinary at p i.e. when $r = 0$, but is more mysterious in the non-ordinary case, since the unbounded growth of $L_p^{(\alpha)}$ means that it does not lie in the Iwasawa algebra, and hence cannot be the characteristic element of an Iwasawa module.

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In [16] Pollack provides a solution to this problem in the case that $a_p = 0$ by constructing bounded distributions L_p^+, L_p^- each of which interpolate half the values of the complex L-function of f and its twists. Kobayashi [11] and Lei [13] have formulated Iwasawa main conjectures using these 'plus/minus p-adic L-functions', shown them to be equivalent to Kato's main conjecture and proved one inclusion in these conjectures using Kato's Euler system. The converse inclusion has been proved in many cases by Wan [19].

Now let Π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_\mathbb{Q})$. Suppose that Π is cohomological with respect to some pure dominant integral weight μ, and that it is the transfer of a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$. Let p be a prime at which Π is unramified, and let $\alpha_1, \dots, \alpha_{2n}$ be the Satake parameters at p. We call a choice of $\alpha = \prod_{i=1}^n \alpha_{j_i}$ for $\{j_1, \dots, j_n\} \subset \{1, \dots, 2n\}$ a p-stabilisation of Π. When a p-stabilisation α is *non-critical* and under some further auxiliary technical assumptions Dimitrov, Januszewski and Raghuram [7] (ordinary case) and Barrera, Dimitrov and Williams [2] construct a locally analytic distribution $L_p^{(\alpha)}$ on $\mathbb{Z}_p^\times$ interpolating the L-values of Π. If we assume α satisfies a non-critical slope condition then this p-stabilisation is non-critical, although this is a stronger condition. We show that that there are at most two choices of α satisfying the non-critical slope condition and thus at most two non-critical slope $L_p^{(\alpha)}$ can be constructed from a given Π.

There is an increasing filtration $\mathscr{D}^r(\mathbb{Z}_p^\times, \mathbb{C}_p)$ on the space $\mathscr{D}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ of $\mathbb{C}_p$-valued distributions on $\mathbb{Z}_p^\times$ which measures the 'growth' of the distribution in a precise way (Definition 4). The 0th part of this filtration is the space of measures on $\mathbb{Z}_p^\times$. The construction of [2] shows that

$$L_p^{(\alpha)} \in \mathscr{D}^{v_p(\alpha)}(\mathbb{Z}_p^\times, \mathbb{C}_p).$$

Suppose we have two non-critical slope p-adic L-functions for a given Π and suppose the following condition, which we dub the 'Pollack condition', holds:

$$\textbf{Pollack condition: } \alpha_n + \alpha_{n+1} = 0. \tag{1}$$

We prove the following theorem, stated for an odd prime p:

Theorem 1 *Let α be a p-stabilisation satisfying the non-critical slope condition and let $\mathrm{Crit}(\Pi)$ be the set of critical integers for Π defined in Definition 7. There exist a pair of distributions $L_p^\pm \in \mathscr{D}^{v_p(\alpha)-\#\mathrm{Crit}(\Pi)/2}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ satisfying*

$$L_p^{(\alpha)} = \log_\Pi^+ L_p^+ + \log_\Pi^- L_p^-,$$

where $\log_\Pi^\pm \in \mathscr{D}^{\#\mathrm{Crit}(\Pi)/2}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ are distributions depending only on $\mathrm{Crit}(\Pi)$ of order $\#\mathrm{Crit}(\Pi)/2$. If the valuation of $\prod_{i=1}^{n-1} \alpha_i$ is minimal (see Proposition 1) the distributions $L_p^\pm$ are contained in $\mathscr{D}^0(\mathbb{Z}_p^\times, \mathbb{C}_p)$. These distributions satisfy the following interpolation property for $j \in \mathrm{Crit}(\Pi)$:

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) L_p^+(x) = (*) \frac{L(\Pi \otimes \theta, j + 1/2)}{\log_\Pi^+(x^j \theta)}$$

for θ a Dirichlet character of conductor an even power of p, and

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) L_p^-(x) = (*) \frac{L(\Pi \otimes \theta, j + 1/2)}{\log_\Pi^-(x^j \theta)}$$

for θ a Dirichlet character of conductor an odd power of p, where the $(*)$ are non-zero constants.

When $p = 2$ the result holds with the signs of the distributions $\log_{\Pi}^{\pm}$ swapped.

Remark 1 Since we assume that μ is pure, the condition that $\prod_{i=1}^{n-1} \alpha_i$ be minimal is equivalent to the statement that this p-stabilisation is $\mathscr{P}$-ordinary (See [9, Sect. 6.2]) where $\mathscr{P} \subset \mathrm{GL}_{2n}$ is the parabolic subgroup given by the partition $2n = (n-1) + 2 + (n-1)$.

As an application we prove the following extension of the main result of [7]:

Theorem 2 *In the case that $L_p^{\pm}$ are bounded distributions, the purity weight w is even, and* $\mathrm{Crit}(\Pi) \neq \{w/2\}$, *we have*

$$L(\Pi \otimes \theta, (w+1)/2) \neq 0$$

for all but finitely many characters θ of p-power conductor.

Remark 2 The assumption on the purity weight is to ensure that the central L-value is critical.

Relation to other work: Since this paper first appeared in preprint form, Lei and Ray [14] have used the results of this paper to formulate an Iwasawa main conjecture for Π, relating the signed p-adic L-functions of Theorem 1.0.1 to signed Selmer groups. They have also generalised the construction of the signed p-adic L-functions to allow certain cases with $\alpha_n + \alpha_{n+1} \neq 0$, using the theory of Wach modules.

2 Preliminaries

2.1 *p*-adic Fourier theory

We lay out the relevant theory of continuous functions on $\mathbb{Z}_p^\times$. The main reference for this section is [5, Sect. I.5].

Let L be a complete extension of $\mathbb{Q}_p$, denote by $\mathscr{C}(\mathbb{Z}_p, L)$ the Banach space of continuous functions on $\mathbb{Z}_p$ taking values in L. Write $\mathrm{LA}(\mathbb{Z}_p, L)$ for the subspace of L-valued locally analytic functions and $LA_h(\mathbb{Z}_p, L)$ for the locally h-analytic functions. We give these spaces a valuation v_{LA_h} in the following way: Let $u_h = (p^h(1-p))^{-1}$ and let $v_{\bar{\mathbb{B}}(a,u_h)}$ be the valuation on power series which converge on $\bar{\mathbb{B}}(a, u_h) = \{z \in \mathbb{C}_p : v(z-a) \geq u_h\}$ given by

$$v_{\bar{\mathbb{B}}(a,u_h)}(f) = \inf_m \{v_p(a_m) + nu_h : f(X) = \sum_{i=0}^{\infty} a_i (X-a)^i\}.$$

An element $f \in LA_h(\mathbb{Z}_p, L)$ locally extends to such a power series and we define

$$v_{\mathrm{LA}_h}(f) = \inf_{a \in \mathbb{Z}_p} v_{\mathbb{B}(a,u_h)}(f).$$

This gives $LA(\mathbb{Z}_p, L)$ the structure of a Fréchet space.

Definition 1 Let $r \in \mathbb{R}_{\geq 0}$. Let $f : \mathbb{Z}_p \to L \in \mathscr{C}(\mathbb{Z}_p, L)$. We say that f is of *order* r if there are functions $f^{(i)} : \mathbb{Z}_p \to L$ such that if we define

$$\varepsilon_h(f) = \inf_{\substack{x \in \mathbb{Z}_p \\ y \in p^h \mathbb{Z}_p}} v_p \left(f(x+y) - \sum_{i=0}^{\lfloor r \rfloor} f^{(i)}(x) y^i / i! \right),$$

then

$$\varepsilon_h(f) - rh \to \infty \text{ as } h \to \infty.$$

We denote the set of such functions by $\mathscr{C}^r(\mathbb{Z}_p, L)$.

The space $\mathscr{C}^r(\mathbb{Z}_p, L)$ is a Banach space with valuation given by

$$v_{\mathscr{C}^r}(f) = \inf\left(\inf_{0\le j\le \lfloor r\rfloor, x\in\mathbb{Z}_p}(\frac{f^{(i)}(x)}{i!}), \inf_{x,y\in\mathbb{Z}_p}\left(\varepsilon_n(f) - rv_p(y)\right)\right).$$

Definition 2
- Define the space of *locally analytic distributions* on $\mathbb{Z}_p$ to be the continuous dual of $\mathrm{LA}(\mathbb{Z}_p, L)$, denoted $\mathscr{D}(\mathbb{Z}_p, L)$. This space has the structure of a Fréchet space via the family of valuations given by restricting to $LA_h(\mathbb{Z}_p, L)$ and taking the dual of v_{LA_h}.
- For $r \in \mathbb{R}_{\ge 0}$ define the subspace $\mathscr{D}^r(\mathbb{Z}_p, L)$ of $\mathscr{D}(\mathbb{Z}_p, L)$ of *order r distributions* to be the continuous dual of $\mathscr{C}^r(\mathbb{Z}_p, L)$. The space $\mathscr{D}^0(\mathbb{Z}_p, L)$ of bounded distributions is often referred to as the space of *measures* on $\mathbb{Z}_p$. We equip each $\mathscr{D}^r(\mathbb{Z}_p, L)$ with the valuation

$$v_{\mathscr{D}^r}(\mu) = \inf_{f\in\mathscr{C}^r(\mathbb{Z}_p,L)\setminus\{0\}}\left(v_p(\mu(f)) - v_{\mathscr{C}_r}(f)\right).$$

For $\mu \in \mathscr{D}^r(\mathbb{Z}_p, L)$, $f \in \mathscr{C}^r(\mathbb{Z}_p, L)$ we write

$$\mu(f) =: \int_{\mathbb{Z}_p} f(x)\mu(x).$$

The space $\mathscr{D}(\mathbb{Z}_p, L)$ is given the structure of an L-algebra via convolution of distributions:

$$\int_{\mathbb{Z}_p} f(x)(\mu * \lambda)(x) := \int_{\mathbb{Z}_p}\left(\int_{\mathbb{Z}_p} f(x+y)\mu(x)\right)\lambda(y).$$

These distribution spaces will be our main object of our study. Though rather inscrutable by themselves, they become more amenable to study by identifying them with spaces of power series.

For $x \in \mathbb{C}_p, a \in \mathbb{R}$, let $\mathbb{B}(x, a) = \{y \in \mathbb{C}_p : v_p(y - x) > a\}$. We define

$$\mathscr{R}^+ = \left\{f = \sum_{n=0}^{\infty} a_n(f)X^n \in L[[X]] : f \text{ converges on } \mathbb{B}(0, 0)\right\}$$

and we give this space the structure of a Fréchet space via the family of valuations $v_{\bar{\mathbb{B}}(0,u_h)}$.

Let $\ell(n) = \inf\{m : n < p^m\}$, and for $r \in \mathbb{R}_{\ge 0}$ define

$$\mathscr{R}_r^+ = \{f = \sum_{n=0}^{\infty} a_n X^n \in L[[X]] : v_p(a_n) + r\ell(n) \text{ is bounded below as } n \to \infty\}.$$

We can put a valuation on these spaces

$$v_r(f) = \inf_h b_h + r\ell(h),$$

where $\ell(h)$ is the smallest integer satisfying $p^{\ell(h)} > h$. However, a different valuation will be useful for our purposes.

Lemma 1 *A power series $f \in L[[X]]$ is in $\mathscr{R}_r^+$ if and only if* $\inf_{h \in \mathbb{Z}_{\geq 0}}(v_{\bar{\mathbb{B}}(0,u_h)}(f) + rh) \neq -\infty$. *Furthermore, the spaces $\mathscr{R}_r^+$ are Banach spaces when equipped with the valuation*

$$v_r(f) = \inf_{h \in \mathbb{Z}_{\geq 0}}(v_{\bar{\mathbb{B}}(0,u_h)}(f) + rh).$$

Moreover, $v_r(f)$ is equivalent to $v'_r(f)$.

Proof [5, Lemme II.1.1]. □

Lemma 2 *If $f \in \mathscr{R}_r^+$, $g \in \mathscr{R}_s^+$, then $fg \in \mathscr{R}_{r+s}^+$.*

Proof [5, Corollaire II.1.2]. □

Theorem 3 *Define the Amice transform:*

$$\mathscr{A} : \mathscr{D}(\mathbb{Z}_p, L) \cong \mathscr{R}^+$$
$$\mu \mapsto \int_{\mathbb{Z}_p} (1+X)^x \mu(x).$$

The Amice transform is an isomorphism of L-algebras under which the spaces $\mathscr{D}^r(\mathbb{Z}_p, L)$ and $\mathscr{R}_r^+$ are identified isometrically with respect to the valuations $v_{\mathscr{D}^r}$ and v'_r.

Proof [5, Théorème II.2.2] and [5, Proposition II.3.1]. □

We now consider the multiplicative topological group $\mathbb{Z}_p^\times$. Let

$$q = \begin{cases} p & \text{if } p \text{ odd} \\ 4 & \text{otherwise.} \end{cases}$$

We have the well-known isomorphism

$$\mathbb{Z}_p^\times \cong (\mathbb{Z}/q\mathbb{Z})^\times \times 1 + q\mathbb{Z}_p,$$

the second factor of which is topologically cyclic. Let γ be a topological generator of $1+q\mathbb{Z}_p$. Such a choice allows us to write any $x \in 1 + q\mathbb{Z}_p$ in the form $x = \gamma^s$ for a unique $s \in \mathbb{Z}_p$, giving us an isomorphism of topological groups

$$1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$$
$$\gamma^s \mapsto s.$$

Thus $\mathbb{Z}_p^\times$ is homeomorphic to $p - 1$ (resp. 2 when $p = 2$) copies of $\mathbb{Z}_p$, and we can use the above theory of $\mathbb{Z}_p$ in this context, defining $\mathrm{LA}(\mathbb{Z}_p^\times, L)$, $\mathscr{D}(\mathbb{Z}_p^\times, L)$ in the obvious way; each space decomposes as a direct sum over their restrictions to each $\mathbb{Z}_p$ component and we take the infimum of the valuations on each summand.

Definition 3 Define *weight space* to be the rigid analytic space $\mathcal{W}$ over $\mathbb{Q}_p$ representing

$$L \mapsto \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, L^\times).$$

Integrating characters gives a canonical identification

$$\mathscr{D}(\mathbb{Z}_p^\times, \mathbb{Q}_p) = H^0(\mathcal{W}, \mathcal{O}_{\mathcal{W}}),$$

where $\mathcal{O}_{\mathcal{W}}$ is the structure sheaf of $\mathcal{W}$. This is isomorphism commutes with base change in the sense that for a finite extension $L/\mathbb{Q}_p$ we have

$$\mathscr{D}(\mathbb{Z}_p^\times, L) = \mathscr{D}(\mathbb{Z}_p^\times, \mathbb{Q}_p) \hat{\otimes}_{\mathbb{Q}_p} L = H^0(\mathcal{W}_L, \mathcal{O}_{\mathcal{W}_L}),$$

where $\mathcal{W}_L = \mathcal{W} \times_{\mathbb{Q}_p} \mathrm{Sp}(L)$ and $\mathrm{Sp}(L)$ is the affinoid space associated to L. We identify $\mathcal{W}(\mathbb{C}_p)$ with the set $\sqcup_\psi \mathbb{B}_\psi$, where $\mathbb{B}_\psi = \mathbb{B}(0, 0)$ and the disjoint union runs over characters of $\mathbb{Z}_p^\times$ which factor through $(\mathbb{Z}/q\mathbb{Z})^\times$, for details see [4, Remark B1.1]. We can thus identify $\mathscr{D}(\mathbb{Z}_p^\times, L)$ with functions on $\sqcup_\psi \mathbb{B}_\psi$ which are described by elements of $\mathscr{R}^+$ on each $\mathbb{B}_\psi$. Given a distribution $\mu \in \mathscr{D}(\mathbb{Z}_p^\times, L)$ we write the corresponding rigid function on $\mathcal{W}$ as $\mathscr{M}(\mu)$.

On each $\mathbb{B}_\psi$ the global sections $\mathcal{O}_\mathcal{W}(\mathbb{B}_\psi)$ are given (after choosing a coordinate X) by precisely $\mathscr{R}^+$. As these are quasi-Stein spaces, the topology on $\mathcal{O}_\mathcal{W}(\mathbb{B}_\psi)$ is that of a Fréchet space induced by an increasing chain of affinoids

$$Y_1 \subset Y_2 \subset \cdots,$$

which we can choose to be the closed discs of radius u_h, whence the topology as global sections over a rigid space coincides with the topology on $\mathscr{R}^+$ given by the family of valuations $v_{\bar{\mathbb{B}}(0,u_h)}$ (see [17, 1C]).

Definition 4 For $r \in \mathbb{R}_{\geq 0}$ we define a subspace $\mathscr{D}^r(\mathbb{Z}_p^\times, L) \subset \mathscr{D}(\mathbb{Z}_p^\times, L)$ by

$$\mathscr{D}^r(\mathbb{Z}_p^\times, L) = \{\mu \in \mathscr{D}(\mathbb{Z}_p^\times, L) : \mathscr{M}(\mu)|_{\mathbb{B}_\psi} \in \mathscr{R}_r^+ \text{ for all } \psi\}.$$

These spaces decompose as a direct sum

$$\mathscr{D}^r(\mathbb{Z}_p^\times, L) = \oplus_\psi \mathscr{D}^r(\mathbb{Z}_p, L)$$

and we equip it with the valuation given by

$$v_{\mathscr{D}_r}(\mu) := \inf_\psi v_{\mathscr{D}^r}(\mu_\psi)$$

where μ_ψ is the projection of μ to the ψ component.

2.2 Automorphic representations

Fix $n \geq 1$ and set $G = \mathrm{GL}_{2n}$. Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$. Let $T \subset G$ be the maximal diagonal torus and let

$$\mu = (\mu_1, \ldots, \mu_{2n}) \in \mathbb{Z}^{2g}$$

be an integral weight. We say μ is *dominant* if $\mu_1 \geq \cdots \geq \mu_{2n}$, and we say μ is *pure* if there is $\omega \in \mathbb{Z}$, the *purity weight* of μ, such that

$$\mu_i + \mu_{2n+1-i} = \omega$$

for all $i = 1, \ldots, n$.

Definition 5 We say that Π is *cohomological* with respect to a dominant integral weight μ if the $(\mathfrak{g}_\infty, K_\infty)$-cohomology

$$H^q(\mathfrak{g}_\infty, K_\infty, \Pi \otimes V_\mathbb{C}^\mu)$$

is non-vanishing for some q. Here $\mathfrak{g}_\infty$ is the Lie algebra of $G(\mathbb{R})$, $K_\infty^\circ \subset G(\mathbb{R})$ is the identity component of the maximal open compact subgroup and $V_\mathbb{C}^\mu$ is the irreducible $\mathbb{C}$-linear G-representation of highest weight μ.

Purity of μ is a necessary condition for Π to be cohomological.

Definition 6 We say that Π is the transfer of a globally generic cuspidal automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_{\mathbb{Q}})$ if

$$\Pi_\ell \cong \pi_\ell.$$

for all primes ℓ at which π is unramified.

Remark 3
- For a given globally generic automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ the existence of such a transfer was proved by Asgari-Shahidi [1, Theorem 1.1].
- A necessary and sufficient condition for Π to be the transfer of a globally generic cuspidal automorphic representation of GSpin_{2n+1} is that it admits a *Shalika model* which realises Π in a certain space of functions $W : G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$, see [2, Sect. 2.6] for details.

2.3 *p*-stabilisations

Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ which is cohomological with respect to a pure dominant integral weight μ and suppose that Π is the transfer of a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_{\mathbb{Q}})$. Let B denote the upper triangular Borel subgroup of G.

Given a prime p at which Π is unramified, define the *Hodge-Tate weights* of Π at p to be the integers

$$h_i = \mu_i + 2n - i, \ i = 1, \ldots, 2n. \tag{2}$$

Remark 4 These weights coincide with the Hodge-Tate weights of the Galois representation associated to Π when the Hodge-Tate weight of the cyclotomic character is taken to be 1.

Definition 7 Define a set

$$\mathrm{Crit}(\Pi) = \{j \in \mathbb{Z} : \mu_n \geq j \geq \mu_{n+1}\}.$$

Remark 5 It is shown in [8, Proposition 6.1.1] that the half integers $j + 1/2$ for $j \in \mathrm{Crit}(\Pi)$ are precisely the critical points of the L-function $L(s, \Pi)$ in the sense of Deligne [6, Definition 1.3].

Let p be a prime at which Π is unramified. There is an unramified character

$$\lambda_p : T(\mathbb{Q}_p) \to \mathbb{C}^\times$$

such that Π_p is isomorphic to the normalised parabolic induction module $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(|\cdot|^{\frac{2n-1}{2}}\lambda_p)$. We define the Satake parameters at p to be the values $\alpha_i = \lambda_{p,i}(p)$, where $\lambda_{p,i}$ denotes the projection to the ith diagonal entry. After choosing an isomorphism $\bar{\mathbb{Q}}_p \cong \mathbb{C}$, we reorder the α_i so that they are ordered with respect to decreasing p-adic valuation and such that $\alpha_i \alpha_{2n+1-i} = \lambda$ for a fixed λ with p-adic valuation $2n - 1 + w$. That we can do this is a result of the transfer from GSpin_{2n+1}, see [1, (64)].

We define the Hodge polygon of Π to be the piecewise linear curve joining the following points in $\mathbb{R}^n$:

$$\left\{(0, 0), \left(j, \sum_{i=1}^{j} h_{2n+1-i}\right) : j = 1, \ldots, 2n\right\}$$

and define the Newton polygon on Π at p to be the piecewise linear curve joining the points

$$\left\{(0,0),\left(j,\sum_{i=1}^{j} v_p(\alpha_{2n+1-i})\right) : j=1,\ldots,2n\right\}.$$

The following result is due in this form to Hida [9, Theorem 8.1].

Proposition 1 *The Newton polygon lies on or above the Hodge polygon and the end points coincide.*

Definition 8 Let $I=(i_1,\ldots,i_n)\subset\mathbb{Z}^n$ satisfy $1\le i_1<\cdots<i_n\le 2n$, and set

$$\alpha_I := \alpha_{i_1}\cdots\alpha_{i_n}.$$

We call α_I *p-stabilisation data* for Π.

Let $Q\subset \mathrm{GL}_{2n}$ be the parabolic subgroup given by the partition $2n=n+n$. The following conditions are translations of the conditions of the same name given in [2, Sect. 2.7].

Definition 9 Let I be as above.

- We say that the product α_I is of *Shalika type* if I contains precisely one element of each pair $\{i, 2n+1-i\}$ for $i=1,\ldots,n$, see [7, Definition 3.5(ii)].
- We say that α_I is *Q-regular* if it is of Shalika type and if for any other choice of $J\subset\mathbb{Z}^n$ satisfying the above properties $a_J\neq a_I$. This amounts to choosing a simple Hecke eigenvalue for the U_p-operator associated to Q acting on the Q-parahoric invariants of Π_p, see [7, Definition 3.5(i)] and [2, Sect. 2.7].
- Set $r_I=v_p(\alpha_I)-\sum_{i=n+1}^{2n} h_i$. We say that α_I is *non-critical slope* if it satisfies

$$r_I < \#\mathrm{Crit}(\Pi).$$

- We say that α_I is *minimal slope* if

$$r_I = \#\mathrm{Crit}(\Pi)/2.$$

Remark 6 The conditions in Definition 9 are used to control a certain local twisted integral at p, attached to a choice of parahoric-invariant vector W in the Shalika model. In [7, Proposition 3.4], the authors show that this local zeta integral is an explicit multiple of $W(1)$. In [7, Lem. 3.6], they use the Shalika-type and Q-regular conditions to exhibit an explicit vector W in the Shalika model attached to α_I with $W(1)=1$, and hence deduce non-vanishing of the local zeta integral. We note that Π_p always admits p-stabilisations of Shalika type. The terminology is justified by [2, Remark 2.5], which explains that the refinements of Shalika type are exactly those that arise from refinements of GSpin_{2n+1}. Finally, if the Satake parameter of Π_p is regular semisimple, then all stabilisations of Π_p are Q-regular.

In [2] the authors construct [1] a locally analytic distribution $L_p^{(\alpha_I)}\in\mathscr{D}(\mathbb{Z}_p^\times,\mathbb{C}_p^\times)$ with respect to a choice of non-critical slope Q-regular p-stabilisation data α_I. The distribution $L_p^{(\alpha_I)}$ is of order r_I and by [18, Lemma 2.10] is uniquely defined by the following interpolation property: Let $\theta:\mathbb{Z}_p^\times\to\bar{\mathbb{Q}}_p$ be a finite-order character of conductor p^m, then for $m\ge 1$ we have

$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)L_p^{(\alpha_I)}(x) = \xi_{\infty,j}\frac{c_{x^j\theta}}{\alpha_I^m}L(\Pi\otimes\theta, j+1/2),\ \ j\in\mathrm{Crit}(\Pi), \tag{3}$$

[1] The authors actually construct p-adic L-functions for the wider class of *non-critical* p-stabilisations, but we only work with non-critical slope p-stabilisations in this paper.

where $c_{x^j\theta}$ is a constant depending only on $x^j\theta$ and the infinite factor $\xi_{\infty,j}$ is the product of a choice of period and a zeta integral at infinity. We call such a $L_p^{(\alpha_I)}$ a 'non-critical slope p-adic L-function'.

3 Plus/Minus *p*-adic *L*-functions

We construct the titular plus/minus L-functions. We first note that the condition of non-critial slope imposes strong restrictions on the number of p-adic L-functions we can construct.

Theorem 4 *There are at most two choices of p-stabilisation α_I for which $L_p^{(\alpha_I)}$ is non-critical slope.*

Proof Without loss of generality we may assume that $\mu_{2n} = 0$, forcing $w = \mu_1$. The end points of the Newton and Hodge polygons coinciding implies that

$$v_p(\lambda) = h_i + h_{2n+1-i}, \ i = 1, \ldots, n. \tag{$\dagger$}$$

The 'non-critical slope' condition for $I = (i_1, \ldots, i_n)$ is equivalent to

$$v_p(\alpha_I) - \sum_{i=n+1}^{2n} h_i < h_n - h_{n+1}.$$

We observe that any I that includes a a 2-tuple of integers the form $(i, 2n+1-i)$ is not non-critical slope. Indeed, we can find an explicit I containing some $(i, 2n+1-i)$ with minimal valuation, namely $(n, n+1, n+3, \ldots, 2n)$, amongst all I containing some $(i, 2n+1-i)$. For such an I we have

$$\begin{aligned}
&v_p(\alpha_I) - (h_{n+1} + h_{n+2} \cdots + h_{2n}) \\
&\geq h_n + h_{n+1} + h_{n+3} + \cdots + h_{2n} - (h_{n+1} + \cdots + h_{2n}) \\
&= h_n - h_{n+2} \\
&> h_n - h_{n+1},
\end{aligned}$$

where the first inequality is a consequence of the Newton polygon lying above the Hodge polygon and ($\dagger$), and the strict inequality is due to dominance. Thus any I containing a pair of integers (i, j) with $i < j$ and $j \leq 2n+1-i$ cannot be non-critical slope, since any such I has greater valuation than $(n, n+1, n+3, \ldots, 2n)$. This leaves us with two choices of potential non-critical slope n-tuples:

$$I_{n+1} = (n+1, n+2, \ldots, 2n),$$

and

$$I_n = (n, n+2, \ldots, 2n).$$

□

In light of Theorem 4 it is clear that the only two choices of p-stabilisation data which can give a non-critical slope distribution are

$$\alpha = \alpha_{n+1}\alpha_{n+2} \ldots \alpha_{2n}, \ \beta = \alpha_n\alpha_{n+2} \ldots \alpha_{2n}.$$

Definition 10 We say that Π satisfies the 'Pollack condition' if

$$\alpha_n + \alpha_{n+1} = 0.$$

Corollary 1 *For a non-critical slope p-stabilisation α_I we have the following:*

- *The p-stabilisation α_I is of Shalika type.*
- *If we assume the Pollack condition, then α_I is Q-regular.*

Proof The first part is immediate from Theorem 4.

For the second claim recall that a p-stabilisation α_I is Q-regular if

$$\alpha_I \neq \alpha_J$$

for all $I \neq J$. Suppose α_I is a non-critical slope p-stabilisation. Then by Theorem 4

$$\alpha_I = \alpha_n \alpha_{n+2} \cdots \alpha_{2n} \text{ or } \alpha_{n+1}\alpha_{n+2} \cdots \alpha_{n+1},$$

and by the Pollack condition these are clearly not equal since they are non-vanishing by non-critical slope. Finally, for a critical slope p-stabilisation α_J we must have $v_p(\alpha_J) > v_p(\alpha_I)$ so $\alpha_J \neq \alpha_I$. □

3.1 Pollack ±-*L*-functions

Let Π be as in the previous section.

As in the previous section, set

$$\alpha = \alpha_{n+1}\alpha_{n+2} \ldots \alpha_{2n}, \ \beta = \alpha_n \alpha_{n+2} \ldots \alpha_{2n},$$

and let $r = v_p(\alpha) - \sum_{i=n+1}^{2n} h_i = v_p(\beta) - \sum_{i=n+1}^{2n} h_i$. The Pollack condition forces

$$r \geq \#\mathrm{Crit}(\Pi)/2$$

since

$$\begin{aligned} r = v_p(\alpha) - \sum_{i=n+1}^{2n} h_i \geq v_p(\alpha_{n+1}) - h_{n+1} \\ &= \frac{h_n + h_{n+1}}{2} - h_{n+1} \\ &= \frac{h_n - h_{n+1}}{2} \\ &= \#\mathrm{Crit}(\Pi)/2, \end{aligned}$$

where the first inequality comes from Newton-above-Hodge, and the lower bound given is tight, with the bound being achieved when the end point of the segment of the Newton polygon corresponding to $\alpha_{n+2} \cdots \alpha_{2n}$ touches the Hodge polygon. This justifies the use of the term 'minimal slope' in Definition 9.

We assume that

$$r < \#\mathrm{Crit}(\Pi)$$

so that we can construct precisely two non-critical slope p-adic L-functions $L_p^{(\alpha)}, L_p^{(\beta)} \in \mathscr{D}^r(\mathbb{Z}_p^\times, \mathbb{C}_p)$.

Remark 7 Unlike in the case of GL_2, for $n > 1$ the non-critical slope condition for α, β is not necessarily implied by the Pollack condition. Indeed, suppose one has a cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{Q}})$ satisfying the Pollack condition at a prime p for which $v_p(\alpha_i) = v_p(\alpha_j)$ for $1 \leq i, j \leq 2n$. The value r is then the same for any choice of p-stabilisation, so there are either $\binom{2n}{n}$ non-critical slope p-adic L-functions or there are none. But Theorem 4 says there can be at most two choices of non-critical slope p-stabilisation.

Following Pollack, we define

$$G^{\pm} = \frac{L_p^{(\alpha)} \pm L_p^{(\beta)}}{2},$$

so that

$$L_p^{(\alpha)} = G^+ + G^-$$
$$L_p^{(\beta)} = G^+ - G^-.$$

We note that in the case of $L_p^{(\beta)}$, the interpolation formula is given by

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) L_p^{(\beta)}(x) = (-1)^m \xi_{\infty,j} \frac{c_{x^j\theta}}{\alpha^m} L(\Pi \otimes \theta, j + 1/2), \ \ j \in \mathrm{Crit}(\Pi),$$

from which it follows that

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) G^+(x) = 0, \ \text{if the conductor of } \theta \text{ is } p^m, m \text{ odd}$$
$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) G^-(x) = 0, \ \text{if the conductor of } \theta \text{ is } p^m, m \text{ even}.$$

Equivalently (noting that characters of conductor m correspond to $(m-1)$th roots of unity), if ζ_{p^m} is any p^mth root of unity and p is odd,

$$\mathscr{M}(G^+)(\gamma^j \zeta_{p^m} - 1) = 0 \text{ for } m \text{ even}$$
$$\mathscr{M}(G^-)(\gamma^j \zeta_{p^m} - 1) = 0 \text{ for } m \text{ odd}$$

on each of the connected components[2] of $\mathcal{W}(\mathbb{C}_p)$ (which we recall we are identifying with $p-1$ copies of $\mathbb{B}(0, 0)$). When $p = 2$ the sign flips and the above vanishing is equivalent to.

$$\mathscr{M}(G^+)(\gamma^j \zeta_{p^m} - 1) = 0 \text{ for } m \text{ odd}$$
$$\mathscr{M}(G^-)(\gamma^j \zeta_{p^m} - 1) = 0 \text{ for } m \text{ even}$$

For any $j \in \mathbb{Z}$, Pollack defines the following power series

$$\log^+_{p,j}(X) := \frac{1}{p} \prod_{m=1}^{\infty} \frac{\Phi_{2m}(\gamma^{-j}(1+X))}{p}$$
$$\log^-_{p,j}(X) := \frac{1}{p} \prod_{m=1}^{\infty} \frac{\Phi_{2m-1}(\gamma^{-j}(1+X))}{p},$$

in $\mathbb{Q}_p[[X]]$, where Φ_m is the p^mth cyclotomic polynomial.

[2] We are referring to the connected components as a rigid space as opposed to those of the topology on $\mathbb{C}_p$ induced by v_p.

Lemma 3 *The power series* $\log^+_{p,j}(X)$ *(resp.* $\log^-_{p,j}(X)$*) is contained in* $\mathscr{R}^+_{1/2}$ *and vanishes at precisely the points* $\gamma^j\zeta_{p^m}-1$ *for every* p^m*th root of unity* ζ_{p^m} *with* m *even (resp. odd).*

Proof The statements in the lemma are proved in [16, Lemma 4.1] and [16, Lemma 4.5]. We reprove that $\log^+_{p,j}$ is contained in $\mathscr{R}^+_{1/2}$ using the setup of Sect. 2, the result for $\log^-_{p,j}$ being similar.

An analysis of the Newton copolygon of the Eisenstein polynomial Φ_n gives us that

$$v_{\bar{\mathbb{B}}(0,u_h)}(\Phi_n((\gamma^{-j}(1+X))/p) = \begin{cases} 0 & \text{if } h \le n-1 \\ p^{n-h-1}-1 & \text{otherwise,} \end{cases}$$

and thus

$$\begin{aligned} v_{\bar{\mathbb{B}}(0,u_h)}(\log^+_{p,j}) &= \sum_{m=1}^{\frac{h+1}{2}} \left(p^{2m-h-1}-1\right) \\ &= \frac{p^{-(h+1)}-1}{1-p^2} - \frac{1}{2} - \frac{h}{2}, \end{aligned}$$

whence

$$\inf_h \left(v_{\bar{\mathbb{B}}(0,u_h)}(\log^+_{p,j}) + \frac{h}{2} \right) = \frac{1}{p^2-1} - \frac{1}{2} < \infty,$$

so $\log^+_{p,j}(X) \in \mathscr{R}^+_{1/2}$ by Lemma 1. ◻

We define

$$\log^{\pm}_{\Pi}(X) = \prod_{j \in \#\mathrm{Crit}(\Pi)} \log^{\pm}_{p,j}(X) \in \mathscr{R}^+_{\mathrm{Crit}(\Pi)/2}.$$

By abuse of notation we will write $\log^{\pm}_{\Pi}(X)$ for the element of $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$ given by $\log^{\pm}_{\Pi}(X)$ on each connected component of $\mathcal{W}$.

Lemma 4 *We have*

$$\limsup_h \left(v_{\bar{\mathbb{B}}(0,u_h)}(\log^{\pm}_{\Pi}) + \frac{\#\mathrm{Crit}(\Pi)}{2} h \right) < \infty.$$

Proof It follows from the proof of Lemma 3 and the multiplicativity of $v_{\bar{\mathbb{B}}(0,u_h)}$ that

$$v_{\bar{\mathbb{B}}(0,u_h)}(\log^+_{\Pi}) + \frac{\#\mathrm{Crit}(\Pi)}{2} h = \#\mathrm{Crit}(\Pi) \left(\frac{p^{-(h+1)}-1}{1-p^2} - \frac{1}{2} \right).$$

The right side converges as $h \to \infty$ so the $\limsup$ is finite. A similar argument works for $\log^-_{\Pi}$. ◻

It follows from the above discussion and [12, 4.7] that for odd p the rigid function $\log^{\pm}_{\Pi}(X)$ divides $\mathscr{M}(G^{\pm})$ in $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$, and for $p=2$ we have that $\log^{\mp}_{\Pi}(X)$ divides $\mathscr{M}(G^{\pm})$ in $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$. Define plus/minus p-adic L-functions $L^{\pm}_p(X)$ to be the elements of $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$ satisfying

$$\mathscr{M}(G^{\pm}) = \log^{\pm}_{\Pi}(X) \cdot L^{\pm}_p(X)$$

for p odd, and

$$\mathscr{M}(G^{\pm}) = \log_{\Pi}^{\mp}(X) \cdot L_p^{\pm}(X)$$

for $p = 2$. We write $L_p^{\pm}$ for the distribution $\mathscr{M}^{-1}(L_p^{\pm}(X))$.

Proposition 2 *We have*

$$L_p^{\pm} \in \mathscr{D}^{r-\#\mathrm{Crit}(\Pi)/2}(\mathbb{Z}_p^{\times}, \mathbb{C}_p).$$

Proof We note that

$$\liminf_h \left(-v_{\bar{\mathbb{B}}(0,u_h)}(\log_{\Pi}^{\pm}) - \tfrac{\#\mathrm{Crit}(\Pi)}{2}h\right) = -\limsup_h \left(v_{\bar{\mathbb{B}}(0,u_h)}(\log_{\Pi}^{\pm}) + \tfrac{\#\mathrm{Crit}(\Pi)}{2}h\right) > -\infty.$$

By the additivity of $v_{\bar{\mathbb{B}}(0,u_h)}$ ([5, Proposition I.4.2]) we have

$$v_{\bar{\mathbb{B}}(0,u_h)}(L_p^{\pm}) + (r - \tfrac{\#\mathrm{Crit}(\Pi)}{2})h = v_{\bar{\mathbb{B}}(0,u_h)}(G^{\pm}) + rh - v_{\bar{\mathbb{B}}(0,u_h)}(\log_{\Pi}^{\pm}) - \tfrac{\#\mathrm{Crit}(\Pi)}{2}h,$$

and so since $G^{\pm} \in \mathscr{D}^r(\mathbb{Z}_p^{\times}, \mathbb{C}_p)$ (and thus $\liminf \left(v_{\bar{\mathbb{B}}(0,u_h)}(G^{\pm}) + rh\right) > -\infty$) we have

$$\liminf_h \left(v_{\bar{\mathbb{B}}(0,u_h)}(L_p^{\pm}) + (r - \tfrac{\#\mathrm{Crit}(\Pi)}{2})h\right) > -\infty$$

and so by Lemma 1 we are done. □

In particular, in the minimal slope case $r = \frac{\#\mathrm{Crit}(\Pi)}{2}$ we get two bounded distributions.

Remark 8
- One might ask if there is an analogue of the plus/minus theory for p-adic L-functions for GL_{2n+1}. Beyond the exact methods used in the present paper, there is an immediate stumbling block: in general, for $n \geq 3$ odd even the usual theory of p-adic L-functions is very poorly developed. For non-ordinary Π on GL_{2n+1}, the only constructions of p-adic L-functions are for $n = 1$ and Π a symmetric square lift from GL_2; in this case, a study of signed Iwasawa theory has been considered in [3].
- The proofs above show that if we relax the minimal slope hypothesis we still obtain a pair of plus/minus L-functions which are unfortunately not bounded. Since the subsets of weight space on which these functions interpolate L-values are disjoint it seems that there is no hope in attempting a similar construction for these functions.
- In the case that we have a $\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{Q}})$ representation admitting p-stabilisations which are critical but not non-critical slope Theorem 4 no longer holds. As a result, for each such p-stabilisation we can construct a p-adic L-function, giving us at most $\binom{2n}{n}$ p-adic L-functions. It's possible that one could generalise the methods of this paper and utilise all of these p-adic L-functions to construct bounded functions analogous to $L_p^{\pm}$, but this is not something we have explored.

3.2 An example of a $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ representation satisfying the Pollack condition

We give an example of a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ satsifying the Pollack condition and having minimal slope using the theory of twisted Yoshida lifts (see [15, Sect. 6] for an overview).

Let $F = \mathbb{Q}(\sqrt{5})$ and let $\sigma_i : F \to \mathbb{R}$, $i = 1, 2$ be the embeddings of F into $\mathbb{R}$. The prime 41 splits in F and we write M for one of its prime factors. Using Magma we see that there is a weight $(4, 2)$ cuspidal Hilbert newform f over F of level $N = M^2$ and of trivial character and with complex multiplication by the unique extension E/F such that $E/\mathbb{Q}$ is not Galois and in which M ramifies. Thus there is a Hecke character ψ over E with infinity type $z \mapsto \varepsilon_1(z)^3\varepsilon_2(z)^2\bar{\varepsilon}_2(z)$, where $\varepsilon_i, \bar{\varepsilon}_i : E \to \mathbb{C}$, $i = 1, 2$, are the pairs of conjugate embeddings of E and such that the Hecke eigenvalue of f at a prime $\wp$ of F is given by

$$a_\wp = \begin{cases} \psi(\mathfrak{q}_1) + \psi(\mathfrak{q}_2) & \text{if } \wp = \mathfrak{q}_1\mathfrak{q}_2 \text{ in } E \\ \psi(\mathfrak{q}) & \text{if } \wp = \mathfrak{q}^2 \text{ in } E \\ 0 & \text{otherwise.} \end{cases}$$

Let L be the number field generated by the Hecke eigenvalues of f. Since the weight is not parallel we have $F \subset L$. Fix a rational prime $p \nmid N$ and let L_v be the completion of L at a prime v over p. Suppose now that p splits in F and write $p = \wp_1 \cdot \wp_2$, labelled such that $\sigma_i(\wp_i)$ is below v.

Let $\psi_{\mathrm{Gal},v} : G_E \to L_v$ be the v-adic character of $G_F = \mathrm{Gal}(\bar{F}/F)$ associated to ψ by class field theory, so that $V_{f,v} := \mathrm{Ind}_{G_E}^{G_F}\psi_{\mathrm{Gal},v}$ is the G_F-representation associated to f. This representation is crystalline at primes not dividing N. The Hodge-Tate weights of $V_{f,v}$ are given by (0, 3) at σ_1 and (1, 2) at σ_2.

The restriction of $V_{f,v}$ to G_E splits as a direct sum of characters:

$$V_{f,v}|_{G_E} = \psi_{\mathrm{Gal},v} \oplus \psi^c_{\mathrm{Gal},v},$$

where $c \in \mathrm{Gal}(E/F)$ is the non-trivial element. If a prime $\wp$ of F above p splits in E then a decomposition group $D_\wp \subset G_F$ at a prime over $\wp$ is contained in G_E and we have $\mathbb{D}_{\mathrm{cris}}(V_{f,v}|_{D_\wp}) = \mathbb{D}_{\mathrm{cris}}(\psi_{\mathrm{Gal},v}|_{D_\wp}) \oplus \mathbb{D}_{\mathrm{cris}}(\psi^c_{\mathrm{Gal},v}|_{D_\wp})$ whence f is ordinary at $\wp$ as the image of the functor $\mathbb{D}_{\mathrm{cris}}$ is weakly admissible. Taking v to be above $\sigma_1(\wp)$ and writing the prime decomposition of $\wp$ in E as $\wp = \mathfrak{q}_1\mathfrak{q}_2$ we have

$$v_v(\psi(\mathfrak{q}_1)) = 0$$
$$v_v(\psi(\mathfrak{q}_2)) = 3$$

up to reordering of the $\mathfrak{q}_i$.

Theorem 5 *Suppose π is a cuspidal automorphic representation generated by a holomorphic Hilbert modular form of weight (k_1, k_2) over a totally real field F. Suppose that:*

- *For $1 \neq \theta \in \mathrm{Gal}(F/\mathbb{Q})$ we have $\pi^\theta \not\approx \pi$,*
- *There is a Hecke character ε over $\mathbb{Q}$ such that the central character ω_π of π satisfies*

$$\omega_\pi = \varepsilon \circ \mathrm{Norm}_{F/\mathbb{Q}}.$$

Then there is a unique globally generic cuspidal automorphic representation $\Theta(\pi, \omega_\pi)$ of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ (a twisted Yoshida lift) of weight $(\frac{k_1+k_2}{2}, \frac{|k_1-k_2|}{2} - 2)$ with central character ε satisfying

$$L(\Pi, s) = L\left(\pi, s + \frac{\max\{k_1, k_2\} - 1}{2}\right).$$

Proof See [15, Theorem 6.1.1 and Proposition 6.1.4]. □

Let π be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ generated by f. Since f has non-parallel weight we see that $\pi \not\approx \pi^\theta$ for non-trivial $\theta \in \mathrm{Gal}(F/\mathbb{Q})$.

Set $\Pi = \Theta(\pi, 1)$. Then Π a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ of weight (3, 3) with trivial central character. This weight lies in the cohomological range and thus Π is cohomological. The Hodge-Tate weights of Π are (0, 1, 2, 3).

Recall that we have a rational prime p such that p splits in F:

$$p\mathcal{O}_F = \wp_1\wp_2.$$

We assume further that $\wp_2$ is inert in E and $\wp_1$ splits, and we write the factorisation of $\wp_1$ as

$$\wp_1 = \mathfrak{P}_1\mathfrak{P}_2.$$

Primes satisfying the above conditions are not uncommon, for example, the primes 11 and 19 admit this splitting phenomena in the tower $E/F/\mathbb{Q}$. We remark that a necessary condition for such a splitting is that E is non-Galois over $\mathbb{Q}$. The local L-factor of Π at such a p is given by

$$L_p(\Pi, s)^{-1} = (1 - \psi(\mathfrak{P}_1)p^{-s})(1 - \psi(\mathfrak{P}_2)p^{-s})(1 - \psi(\wp_2)p^{-2s}).$$

Choosing a prime v of L lying above p such that v lies above $\sigma_i(\wp_i)$, we deduce that Π satisfies the Pollack condition and has two minimal slope p-stabilisations. By Corollary 1 these p-stabilisations are Q-regular and Shalika. There is an exceptional isomorphism $\mathrm{GSp}_4 \cong \mathrm{GSpin}_5$ and so we are done by applying the functorial lift from GSpin_5 to GL_4.

4 Non-vanishing of twists

We use $L_p^{\pm}$ to show non-vanishing of the complex L-function of Π at the central value, extending work of Dimitrov, Januszewski, Raghuram [7] to a non-ordinary setting.

Proposition 3 *In the case that* $\mathrm{Crit}(\Pi) \neq \{w/2\}$, *we have*

$$L_p^{\pm} \neq 0.$$

Proof We consider L_p^+, the case of L_p^- being essentially identical and we further assume p is odd for brevity of notation (although the same argument works in this case). Note that a character $\theta : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ of conductor p^{m+1} corresponds to a choice of primitive p^mth root of unity ζ_θ in a disc $\mathcal{W}$ determined by the restriction ψ_θ of θ to $(\mathbb{Z}/q\mathbb{Z})^\times$, which gives us the identification

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) \log_\Pi^+(x) = \log_\Pi^+(\psi_\theta, \gamma^j\zeta_\theta - 1),$$

where on the left hand side we use the description of $\log_\Pi^+$ as a distribution and on the right hand side $\log_\Pi^+(\psi_\theta, -)$ is the restriction of $\log_\Pi^+$ to the disc in $\mathcal{W}$ corresponding to ψ_θ. We adopt the analogous notation for L_p. It follows from Lemma 3 that $\log_\Pi^+(\psi_\theta, \gamma^j\zeta_\theta - 1) \neq 0$ if m is odd (resp. even for $p = 2$). Thus for characters θ of odd p-power conductor we have the interpolation property

$$L_p(\psi_\theta, \gamma^j\zeta_\theta - 1) \sim \frac{L(\Pi \otimes \theta, j + 1/2)}{\log_\Pi^+(\psi_\theta, \gamma^j\zeta_\theta - 1)}, \quad j \in \mathrm{Crit}(\Pi),$$

where $\sim$ is used here to mean ‘up to non-zero constant’. By Jacquet-Shalika [10, 1.3] we have

$$L(\Pi \otimes \theta, s) \neq 0 \text{ for } \mathfrak{Re}(s) \geq w/2 + 1$$

for finite order characters θ, and by applying the functional equation we get non-vanishing for $\mathfrak{Re}(s) \leq w/2$. Since $\mathrm{Crit}(\Pi)$ contains an integer k not equal to $w/2$, the above discussion gives us

$$L(\Pi, k + 1/2) \neq 0,$$

and thus $L_p^+ \neq 0$. $\square$

Remark 9 Proposition 3 actually proves the stronger result that the power series $\mathscr{M}(L_p^{\pm})|_{\mathbb{B}_\psi}$ is non-zero for each choice of ψ.

We can turn this back on itself and use $L_p^{\pm}$ to say something about nonvanishing of $L(\Pi \otimes \theta, (\omega + 1)/2)$ in the case when $L_p^{\pm} \in \mathscr{D}^0(\mathbb{Z}_p^\times, \mathbb{C}_p)$.

Theorem 6 *In the case that $L_p^{\pm}$ are bounded distributions, w is even, and* $\mathrm{Crit}(\Pi) \neq \{w/2\}$, *we have*

$$L(\Pi \otimes \theta, (w + 1)/2) \neq 0$$

for all but finitely many characters θ of p-power conductor.

Proof Assume p odd for brevity, again noting that the argument works fine in this case. For any character ψ of $(\mathbb{Z}/p\mathbb{Z})^\times$ we can write $\mathscr{M}(L_p^{\pm})|_{\mathbb{B}_\psi} = L_p^{\pm}(\psi, T) \in \mathcal{O}_L[[T]] \otimes_{\mathcal{O}_L} L$ for some finite extension $L/\mathbb{Q}_p$. We note that $\mathscr{M}(L_p^{\pm}) = \sum_\psi 1_{\mathbb{B}_\psi} L_p^{\pm}$ where $1_{\mathbb{B}_\psi}$ denotes the indicator function on $\mathbb{B}_\psi$. This power series is non-zero by Proposition 3 and Remark 9, and so Weierstrass preparation tells us that each $L_p^{\pm}(\psi, T)$, and thus $L_p^{\pm}$, has only finitely many zeroes. Given any character θ of p-power conductor, we have

$$\int_{\mathbb{Z}_p^\times} x^{w/2} \theta(x) L_p^{?}(x) \sim L(\Pi \otimes \theta, (w + 1)/2),$$

where

$$? = \begin{cases} + & \text{if the conductor of } \theta \text{ is odd } p\text{ -power} \\ - & \text{otherwise} \end{cases}.$$

Thus, for all but finitely many θ, we have

$$L(\Pi \otimes \theta, (w + 1)/2) \neq 0.$$

$\square$

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