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Gibbs measures for hyperbolic attractors defined by densities

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February 14, 2022

Abstract

In this article we will describe a new construction for Gibbs measures for hyperbolic attractors generalizing the original construction of Sinai, Bowen and Ruelle of SRB measures. The classical construction of the SRB measure is based on pushing forward the normalized volume on a piece of unstable manifold. By modifying the density at each step appropriately we show that the resulting measure is a prescribed Gibbs measure. This contrasts with, and complements, the construction of Climenhaga-Pesin-Zelerowicz who replace the volume on the unstable manifold by a fixed reference measure. Moreover, the simplicity of our proof, which uses only explicit properties on the growth rate of unstable manifold and entropy estimates, has the additional advantage that it applies in more general settings.

1 Introduction

In broad terms, smooth ergodic theory describes the study of invariant measures for diffeomorphisms of compact manifolds. Gibbs measures form an especially natural family of invariant probability measures which have played an important role, particularly in the study of hyperbolic dynamical systems, for over 50 years. The best known examples of such measures include the well known Sinai-Ruelle-Bowen measures and the measure of maximal entropy, also known as the Bowen-Margulis measure. More generally, Gibbs measures are invariant probability measures μ_G associated to Hölder continuous potential functions G . In this article we will describe a simple construction based on the push forward of the induced volume on local unstable manifolds. This generalizes the original method used by Sinai and Ruelle to construct the SRB measure.

*The second author is partly supported by ERC-Advanced Grant 833802-Resonances and EPSRC grant EP/T001674/1

1.1 Gibbs measures for hyperbolic diffeomorphisms

Given Hölder continuous function G there are various ways to describe the associated Gibbs measures. One standard approach due to Ruelle and Walters is to use the variational principle, which characterizes the Gibbs measures as *equilibrium states*. More generally, given a continuous map $f : X \rightarrow X$ and a continuous function $G : X \rightarrow \mathbb{R}$ we say that an f -invariant probability measure μ_G is an equilibrium state for G if

$$h(\mu_G) + \int G d\mu_G = \sup \left\{ h(\mu) + \int G d\mu : \mu \in \mathcal{M}_f(X) \right\} \quad (1.1)$$

where $\mathcal{M}_f(X)$ is the space of f -invariant probability measures i.e., μ_G is a measure which maximizes the sum of the entropy and the integral of G , over all f -invariant measures [23], [28]. The value attained by the supremum in (1.1) is called the pressure and denoted $P(G)$.

It is known that for every continuous function $G : X \rightarrow \mathbb{R}$ and expansive map f there exists at least one equilibrium state. Under the additional assumptions that G is Hölder continuous and f is uniformly hyperbolic then there is a unique measure $\mu_G \in \mathcal{M}_f(X)$ satisfying (1.1) which is the Gibbs measure for G . [3]. One approach to constructing Gibbs measures is to use weighted Dirac measures on periodic orbits, which was developed by Bowen [2] and Parry [20], [21]. We refer the reader to the recent comprehensive survey of Climenhaga et al for a detailed discussion of constructing Gibbs measures, including their own original construction [7]. For other perspectives on the construction of Gibbs measures we refer the reader to [25], [15], [16] and [30].

To explain the viewpoint we take in this article we begin by recalling there is a well known construction of the classical Sinai-Ruelle-Bowen measure μ_{SRB} using the push forward of the normalized volume λ on any piece of local unstable manifold $W_\delta^u(x)$. Of course it follows from the classical Krylov–Bogolyubov theorem on the existence of invariant measures that the weak star limit points of

$$\frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda, \quad n \geq 1,$$

are f -invariant. (Here we denote by $f_*^k \lambda(A) = \lambda(f^{-k}A)$ the push forward measure supported on $f^k W_\delta^u(x)$.) However, in the context of hyperbolic attractors much more is true. More precisely, there is the following famous result due to Sinai (for the particular case of Anosov diffeomorphisms) and Ruelle (in the general setting of hyperbolic attractors).

Theorem 1.1 (Sinai [24], Ruelle [23]). *Let $f : X \rightarrow X$ be a $C^{1+\alpha}$ topologically mixing hyperbolic attractor. Given $x \in X$ and $\delta > 0$ consider a (normalized) volume measure $\lambda = \lambda_{W_\delta^u(x)}$ on a piece of local unstable manifold $W_\delta^u(x)$. Then the averages*

$$\mu_n^{SRB} = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda, \quad n \geq 1,$$

converge in the weak star topology to μ_{SRB} as $n \rightarrow +\infty$ ¹.

¹Sinai and Ruelle actually show the stronger result that the measures $f_*^k \lambda$ converge to μ_{SRB} in the weak star topology without the need to average. Moreover, the topological mixing hypothesis is not restrictive because of the Smale spectral decomposition theorem [26]

In the case of the SRB measures, it seems particularly appropriate to use λ as a starting measure since the limiting measure μ_{SRB} is an f -invariant measure which is supported on X , but induces an absolutely continuous probability measure on all of the unstable manifolds.

The SRB measure can also be viewed as a particular example of a Gibbs measure associated to the Hölder continuous function $\Phi : X \rightarrow \mathbb{R}$ defined by

$$\Phi(x) = -\log |\det(Df|E_x^u)|,$$

which is usually called the unstable expansion coefficient, where E_x^u is the unstable bundle for the diffeomorphism. More generally, given any Hölder continuous function $G : X \rightarrow \mathbb{R}$ we present an analogous construction of the Gibbs measure μ_G .

Theorem 1.2. *Let $f : X \rightarrow X$ be a $C^{1+\alpha}$ topologically mixing hyperbolic attracting diffeomorphism and let $G : X \rightarrow \mathbb{R}$ be a Hölder continuous function. Given $x \in X$ and $\delta > 0$ consider the sequence of probability measures $(\lambda_n)_{n=1}^\infty$ supported on $W_\delta^u(x)$ and absolutely continuous with respect to the induced volume $\lambda = \lambda_{W_\delta^u(x)}$ with densities*

$$\frac{d\lambda_n}{d\lambda}(y) := \frac{\exp\left(\sum_{i=0}^{n-1} (G - \Phi)(f^i y)\right)}{\int_{W_\delta^u(x)} \exp\left(\sum_{i=0}^{n-1} (G - \Phi)(f^i z)\right) d\lambda(z)} \quad \text{for } y \in W_\delta^u(x). \quad (1.2)$$

Then the averages

$$\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda_n, \quad n \geq 1, \quad (1.3)$$

converge in the weak star topology to μ_G .

It is clear from the statement that the construction is independent of the choice of x and $\delta > 0$.

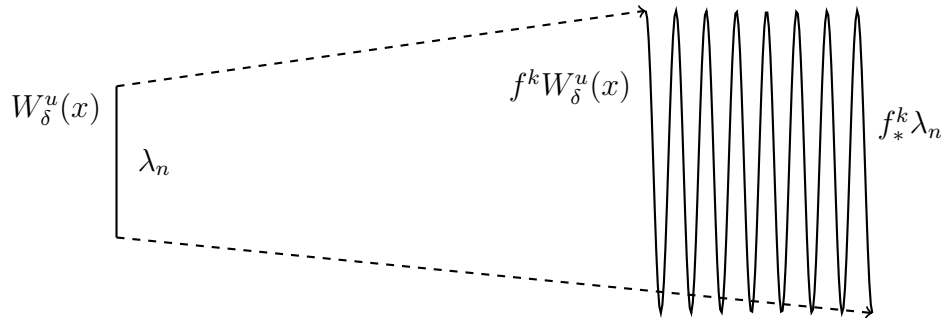


Figure 1: The push forward of the measure λ_n on $W_\delta^u(x)$ by f^k .

The proof of Theorem 1.2 actually gives the following result.

Theorem 1.3. *If $G : X \rightarrow \mathbb{R}$ is continuous then the weak star accumulation points for the measures (1.3) are equilibrium states for G .*

The term Φ appearing in (1.2) comes from the change of variables for $f^n : W_\delta^u(x) \rightarrow f^n(W_\delta^u(x))$ for which the Jacobian is

$$|\det(Df^n|E_y^u)| = \prod_{i=1}^n |\det(Df|E_{f^{i-1}y}^u)| = \exp\left(-\sum_{i=0}^{n-1} \Phi(f^i y)\right)$$

by the chain rule. Thus we can reformulate (1.2) as

$$\lambda_n(A) = \frac{\int_{f^n(A)} \exp\left(\sum_{i=1}^n G(f^{-i}y)\right) d\lambda_{f^n W_\delta^u}(y)}{\int_{f^n W_\delta^u(x)} \exp\left(\sum_{i=1}^n G(f^{-i}z)\right) d\lambda_{f^n W_\delta^u}(z)} \text{ for Borel } A \subset W_\delta^u(x)$$

which is more convenient in the proofs.

The proof of Theorem 1.2 will be given in §5. The simple nature of the proof suggests a number of potential generalizations, which we describe in §7.

An interesting aspect of this approach is that all of the reference measures λ_n on $W_\delta^u(x)$ are absolutely continuous with respect to the volume λ , although they now typically depend on n (except, of course, in the original case of SRB measures in Theorem 1.1 where the same measure λ is used throughout). This is reminiscent of the weighted standard pair construction in (5.12) of [9] which was used to prove stronger statistical properties in a different setting. In contrast to this, in a recent very interesting article, Climenhaga-Pesin-Zelerowicz use a different method to construct Gibbs measures by pushing forward a fixed reference measure on $W_\delta^u(x)$ (dependent on G and typically singular with respect to the leaf volume) constructed via a Carathéodory type construction [7].

Remark 1.4. *It is clear from the proof that the conclusion of Theorem 1.2 remains true if $W_\delta^u(x)$ is replaced by an embedded disk $\mathcal{D} \subset M$ of dimension E^u provided it is transverse to the strong stable manifolds.*

Remark 1.5. *The examples of attractors from [29] can be seen to have an exponential rate of convergence of $\int G d\mu_n^{SRB}$ to $\int G d\mu_{SRB}$ (for G Hölder continuous) in Theorem 1.1, but this rate can be made arbitrarily small.*

Remark 1.6. *One of the advantages of considering attractors is that one can naturally study the induced volume on the unstable manifolds as a reference measure. There is a prospect that these results can be extended to basic sets (see [13], §6.4.d) using another ambient measure, which will be pursued elsewhere.*

1.2 Gibbs measures for hyperbolic flows

The results for diffeomorphisms in the previous subsection have analogues for hyperbolic attracting flows $\phi_t : X \rightarrow X$. Given a Hölder continuous function $G : X \rightarrow \mathbb{R}$ we can again use a variational principle, to first characterize the Gibbs measure μ_G as an equilibrium state. More generally, given a continuous flow $\phi_t : X \rightarrow X$ and a continuous

function $G : X \rightarrow \mathbb{R}$ we say a ϕ -invariant probability measure μ_G is an equilibrium state for G if

$$h(\mu_G) + \int G d\mu_G = \sup \left\{ h(\mu) + \int G d\mu : \mu \in \mathcal{M}_\phi(X) \right\} \quad (1.4)$$

where $\mathcal{M}_\phi(X)$ is the space of ϕ -invariant probability measures and $h(\mu)$ is the entropy of the time-one flow $\phi_{t=1} : X \rightarrow X$ and $\mu \in \mathcal{M}_\phi(X)$, i.e., μ_G is a measure which maximizes the sum of the entropy and the integral over all ϕ -invariant measures [23], [28]. Furthermore, the value of the supremum is again called the pressure and denoted $P(G)$.

It is known that for every continuous function $G : X \rightarrow \mathbb{R}$ there exists at least one equilibrium state. Under the additional assumption that G is Hölder continuous then there is a unique measure $\mu_G \in \mathcal{M}_\phi(X)$ satisfying (1.4) which is the Gibbs measure for G .

The SRB measure for a hyperbolic attracting flow $\phi_t : X \rightarrow X$ is a Gibbs measure associated to the Hölder continuous function $\Phi : X \rightarrow \mathbb{R}$ defined by

$$\Phi(x) = - \lim_{t \rightarrow 0^+} \frac{1}{t} \log |\det(D\phi_t|E_x^u)|,$$

which is again called the unstable expansion coefficient, where E_x^u is the unstable bundle for the flow. The analogue of Theorem 1.1 for flows was proved by Bowen and Ruelle [4].

More generally, given any Hölder continuous function $G : X \rightarrow \mathbb{R}$ we give a construction of the Gibbs measure μ_G which is analogous to that for diffeomorphisms in Theorem 1.2:

Theorem 1.7. *Let $\phi_t : X \rightarrow X$ be a $C^{1+\alpha}$ mixing hyperbolic attracting flow and let $G : X \rightarrow \mathbb{R}$ be a Hölder continuous function. Given $x \in X$ and $\delta > 0$ consider the family of probability measures $(\lambda_T)_{T>0}$ supported on a local unstable manifold $W_\delta^u(x)$ and absolutely continuous with respect to the induced volume $\lambda = \lambda_{W_\delta^u(x)}$ with densities*

$$\frac{d\lambda_T}{d\lambda}(y) := \frac{\exp\left(\int_0^T (G - \Phi)(\phi_v y) dv\right)}{\int_{W_\delta^u(x)} \exp\left(\int_0^T (G - \Phi)(\phi_v y) dv\right) d\lambda_{W_\delta^u(x)}(z)} \quad \text{for } y \in W_\delta^u(x).$$

Then the averages

$$\mu_T := \frac{1}{T} \int_0^T (\phi_t)^* \lambda_T dt, \quad T > 0,$$

converge in the weak star topology to μ_G as $T \rightarrow +\infty$.

As in the case of hyperbolic diffeomorphisms each of the absolutely continuous reference measures λ_T depends on $T > 0$. More generally if we only assume that $G : X \rightarrow \mathbb{R}$ is continuous then the limit points of μ_T are equilibrium states for G . Similarly, the contribution of the term involving Φ comes from the change of variables for the time T flow diffeomorphism $\phi_T : W_\delta^u(x) \rightarrow \phi_T(W_\delta^u(x))$ for which the Jacobian is

$$|\det(D\phi_T|E_y^u)| = \exp\left(-\int_0^T \Phi(\phi_v y) dv\right)$$

and we can rewrite

$$\lambda_T(A) = \frac{\int_{\phi_T(A)} \exp\left(\int_0^T G(\phi_{-v}y)dv\right) d\lambda_{\phi_T W_\delta^u(x)}(y)}{\int_{\phi_T(W_\delta^u(x))} \exp\left(\int_0^T G(\phi_{-v}z)dv\right) d\lambda_{\phi_T W_\delta^u(x)}(z)} \text{ for Borel } A \subset W_\delta^u(x).$$

Remark 1.8. *In Theorem 1.7 it is possible to replace the local unstable manifold $W_\delta^u(x)$ by an embedded curve $\mathcal{C} \subset X$ of dimension $\dim E^u$ provided it does not wholly lie inside a single stable manifold. This can again be done by uniformly approximating \mathcal{C} by pieces of unstable manifold.*

Whereas the original proof(s) of Theorem 1.1 used symbolic dynamics, we use a more geometric approach. The main ingredients are estimates on the rate of growth of the volume of the local unstable manifolds (i.e., $\text{vol}(f^n W_\delta^u(x))$ as $n \rightarrow +\infty$ for hyperbolic attracting diffeomorphisms and $\text{vol}(\phi_T W_\delta^u(x))$ as $T \rightarrow +\infty$ for hyperbolic attracting flows) and entropy bounds adapted from work of Misiurewicz. In particular, this gives a new proof of Theorem 1.1.

2 SRB measures and measures of maximal entropy

In this section we will concentrate on two particularly important examples of Gibbs measures for each of the cases of diffeomorphisms and flows. These are the SRB measures (which we already encountered in Theorem 1.1) and the measure of maximal entropy, also known as the Bowen-Margulis measure.

2.1 SRB measures and measures of maximal entropy for hyperbolic diffeomorphisms

The importance of the SRB measure is that it describes the behaviour of orbits for typical points in the basin of attraction [23]. On the other hand, the measure of maximal entropy describes the distribution of various quantities which grow at a rate corresponding to the topological entropy, such as the periodic points [2], [20].

Example 2.1 (SRB-measures). *In the special case that the Gibbs measure is the SRB measure we can take $G = \Phi = -\log |\det(Df|E_x^u)|$ and thus the weights reduce to*

$$\frac{d\lambda_n}{d\lambda}(y) = \exp\left(\sum_{i=0}^{n-1} (G - \Phi)(f^i y)\right) = 1 \text{ for all } y \in W_\delta^u(x)$$

and then by definition $\lambda_n = \lambda$, for all $n \geq 1$. In particular,

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda,$$

for $n \geq 1$, converges to μ_{SRB} , and so Theorem 1.2 reduces to Theorem 1.1.

Example 2.2 (Measure of Maximal Entropy). *In the special case that the Gibbs measure is the measure of maximal entropy we can take $G = 0$. In particular, (1.1) now reduces to $P(0) = h(f)$, the topological entropy. Furthermore, the sequence of densities $(\lambda_n)_{n=1}^\infty$ is given by*

$$\begin{aligned} \frac{d\lambda_n}{d\lambda}(y) &:= \frac{\exp\left(-\sum_{i=0}^{n-1} \Phi(f^i y)\right)}{\int_{W_\delta^u(x)} \exp\left(-\sum_{i=0}^{n-1} \Phi(f^i z)\right) d\lambda_{W_\delta^u(x)}(z)} \\ &= \frac{|\det(Df^n|E_y^u)|}{\int_{W_\delta^u(x)} |\det(Df^n|E_z^u)| d\lambda_{W_\delta^u(x)}(z)} \text{ for } y \in W_\delta^u(x). \end{aligned}$$

In particular, the averages

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \left(\frac{|\det(Df^n|E_x^u)|}{\int_{W_\delta^u(x)} |\det(Df^n|E_x^u)| d\lambda_{W_\delta^u(x)}(x)} \lambda \right), \quad n \geq 1,$$

converge to the measure of maximal entropy μ_{BM} .

For typical hyperbolic attracting diffeomorphisms the measure of maximal entropy and the SRB measure are singular with respect to each other.

Example 2.3 (Toral automorphism). *In the particular case that $M = \mathbb{T}^2$ then any Anosov diffeomorphism $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is topologically conjugate to a linear hyperbolic diffeomorphism. In the special case that f is C^1 conjugate to a linear hyperbolic toral automorphism then the function $\Phi : \mathbb{T}^2 \rightarrow \mathbb{R}$ is (cohomologous to a) constant and the measure of maximal entropy coincides with the SRB measure and is equal to the normalized Haar measure. Otherwise, the measure of maximal entropy and the SRB measure are singular with respect to each other.*

2.2 SRB measures and measures of maximal entropy for hyperbolic attracting flows

As in the case of hyperbolic attracting diffeomorphisms we can consider the two special cases of Gibbs measures for hyperbolic attracting flows, i.e., the SRB measure and the measure of maximal entropy.

Example 2.4 (SRB-measures). *In the special case that the Gibbs measure is the SRB measure we can take $G(x) = \Phi(x) = -\lim_{t \rightarrow 0} \frac{1}{t} \log |\det(D\phi_t|E_x^u)|$ and thus the densities $(\lambda_T)_{T>0}$ reduce to*

$$\frac{d\lambda_T}{d\lambda}(y) = \exp\left(\int_0^T (G - \Phi)(\phi_u y) du\right) = 1 \text{ for all } y \in W_\delta^u(x)$$

and then by definition $\lambda_T = \lambda$, for all $T > 0$. In particular,

$$\mu_T = \frac{1}{T} \int_0^T (\phi_t)^* \lambda_T dt = \frac{1}{T} \int_0^T (\phi_t)^* \lambda dt$$

for $T > 0$ converges to μ_{SRB} and Theorem 1.7 reduces to the analogue of Theorem 1.1 for flows, which was originally proved by [4].

The second example is the measure of maximal entropy, or Bowen-Margulis measure.

Example 2.5 (Measure of Maximal Entropy). *In the special case that the Gibbs measure is the measure of maximal entropy we can take $G = 0$. Thus the reference measures $(\lambda_T)_{T>0}$ given by*

$$\begin{aligned} \frac{d\lambda_T}{d\lambda}(y) &:= \frac{\exp\left(-\int_0^T \Phi(\phi_u y) du\right)}{\int_{W_\delta^u(x)} \exp\left(-\int_0^T \Phi(\phi_u y) du\right) d\lambda_{W_\delta^u(x)}(z)} \\ &= \frac{|\det(D\phi_T|E_y^u)|}{\int_{W_\delta^u(x)} |\det(D\phi_T|E_z^u)| d\lambda_{W_\delta^u(x)}(z)} \text{ for } y \in W_\delta^u(x). \end{aligned}$$

In particular,

$$\mu_T = \frac{1}{T} \int_0^T (\phi_t)^* \left(\frac{|\det(D\phi_T|E_y^u)|}{\int_{W_\delta^u(x)} |\det(D\phi_T|E_z^u)| d\lambda_{W_\delta^u(x)}(z)} \lambda \right) dt, \quad T > 0,$$

converges to the measure of maximal entropy μ_{BM} .

Example 2.6 (Geodesic flows). *The classical example of an attracting hyperbolic flow (or even an Anosov flow) is the geodesic flow on a compact surface with negative curvatures [1]. In the special case of the curvatures being constant the function Φ is constant and the measure of maximal entropy coincides with the SRB measure and is equal to the normalized volume. For surfaces of negative non-constant curvature the SRB measure is still equivalent to the volume but the measure of maximal entropy is singular with respect to the volume.*

3 Definitions and simple examples

We now recall the formal definitions of hyperbolic attracting diffeomorphisms and Anosov diffeomorphisms. We then give some simple examples which illustrate Theorems 1.2 and 1.7

3.1 Definition of hyperbolic diffeomorphisms

We recall the general definition of a hyperbolic attractor. Let $f : M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism on a compact Riemannian manifold, and let $X \subset M$ be a closed f -invariant set.

Definition 3.1. *The map $f : X \rightarrow X$ is called a mixing hyperbolic attracting diffeomorphism if:*

1. *there exists a continuous splitting $T_X M = E^s \oplus E^u$ and $C > 0$ and $0 < \lambda < 1$ such that*

$$\|Df^n|E^s\| \leq C\lambda^n \text{ and } \|Df^{-n}|E^u\| \leq C\lambda^n$$

for $n \geq 0$;

2. there exists an open set $X \subset U \subset M$ such that $X = \bigcap_{n=0}^{\infty} f^n U$;
3. $f : X \rightarrow X$ is topologically mixing; and
4. the periodic orbits for $f : X \rightarrow X$ are dense in X .

In the particular case $X = M$ the diffeomorphism f is an Anosov diffeomorphism:

Definition 3.2. A mixing diffeomorphism $f : M \rightarrow M$ is called Anosov if:

1. there exists a continuous splitting $TM = E^s \oplus E^u$ and $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|E^s\| \leq C\lambda^n \text{ and } \|Df^{-n}|E^u\| \leq C\lambda^n$$

for $n \geq 0$; and

2. $f : X \rightarrow X$ is topologically mixing.

We next recall some classic examples of hyperbolic diffeomorphisms.

Example 3.3 (Axiom A). An Axiom A diffeomorphism is a diffeomorphism for which the non-wandering set Ω is a finite union of hyperbolic sets and attracting or repelling periodic points [26].

A more concrete example is the following.

Example 3.4 (Solenoid). One of the simplest examples of a hyperbolic attractor (which is not Anosov) is the solenoid constructed in terms of a solid torus mapped inside itself (whose interior plays the role of the neighbourhood U), see [29] and [13], §7.1. More concretely, let $M = \{(\theta, x, y) \in \mathbb{R}/\mathbb{Z} : x^2 + y^2 \leq 1\}$ be a solid torus and let $f : M \rightarrow M$ be defined by

$$f(\theta, x, y) = \left(2\theta, \frac{x}{10} + \frac{\cos \theta}{2}, \frac{x}{10} + \frac{\sin \theta}{2} \right).$$

It is easily seen that $f(M) \subset \text{int}(M)$ and the f -invariant set $\Lambda = \bigcap_{n=0}^{\infty} f^n \text{int}(M)$ is an attractor (see [13], Proposition 17.1.2). In this example the unstable manifold is one dimensional and the stable manifold is two dimensional. The SRB induces the usual Haar measure on the unstable manifolds, which are locally parameterized by the θ -coordinate. In this case the measure of maximal entropy coincides with the SRB-measure.

3.2 Example illustrating Theorem 1.2

It is illuminating to consider the following simple example.

Example 3.5 (Arnol'd CAT map). Let $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be a linear hyperbolic toral automorphism of the form $f((x, y) + \mathbb{Z}^2) = (ax + by, cx + dy) + \mathbb{Z}^2$ where $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ and $|a + d| \neq 2$. The unstable manifolds are line segments of irrational slope α , say, and Φ is a constant function. Thus, for any Hölder function $G : \mathbb{T}^2 \rightarrow \mathbb{R}$ we can write

$$\frac{d\lambda_n}{d\lambda}(y) := \frac{\exp\left(\sum_{i=0}^{n-1} G(f^i y)\right)}{\int_{W_\delta^u(x)} \exp\left(\sum_{i=0}^{n-1} G(f^i z)\right) d\lambda(z)} \text{ for } y \in W_\delta^u(x), \quad (3.1)$$

for $n \geq 1$. If we take the specific choice that

$$W_\delta^u(x) = \{t(\ell, \alpha\ell) : 0 \leq t \leq 1\} + \mathbb{Z}^2$$

is a line segment from $(0,0)$ to $(\ell, \alpha\ell) + \mathbb{Z}^2$, for some $\ell > 0$, then we see that

$$f^n W_\delta^u(x) = \left\{ t(e^{hn}\ell, e^{hn}\alpha\ell) : 0 \leq t \leq 1 \right\} + \mathbb{Z}^2$$

is the line segment from $(0,0)$ to $(e^{hn}\ell, e^{hn}\alpha\ell) + \mathbb{Z}^2$, where $h > 0$ is the topological entropy of f .

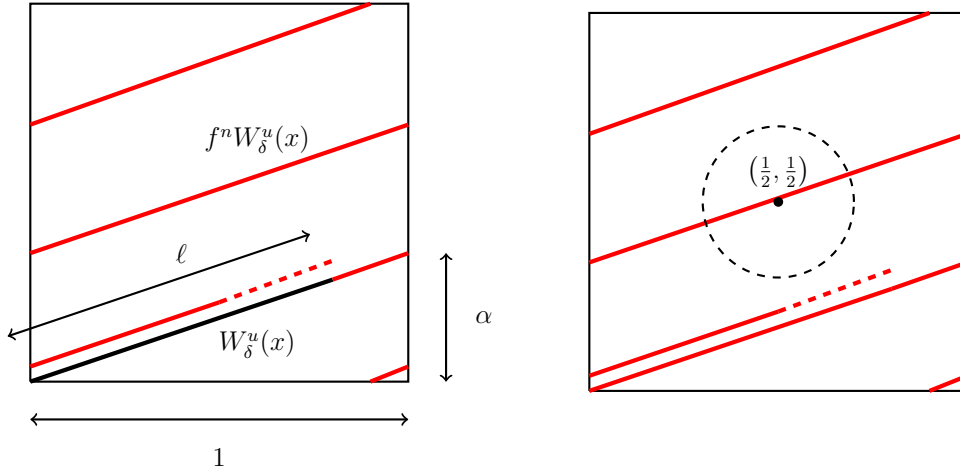


Figure 2: A piece of local unstable manifold $W_\delta^u(x)$ of length ℓ for the Arnol'd CAT map and its image $f^n W_\delta^u(x)$

In the case that $G = 0$ the SRB measure and the measure of maximal entropy coincide, and are both equal to the Haar measure. We can consider two balls:

1. $B_1 = B\left((0,0), \frac{1}{3}\right)$ of radius $\epsilon = \frac{1}{3}$ centred at $(0,0)$; and
2. $B_2 = B\left(\left(\frac{1}{2}, \frac{1}{2}\right), \frac{1}{3}\right)$ of radius $\epsilon = \frac{1}{3}$ centred at $\left(\frac{1}{2}, \frac{1}{2}\right)$,

which both have λ -measure $\pi^2/9$. Figure 3 shows the values of $\mu_n(B_1)$ and $\mu_n(B_2)$ for each of the two potentials $G(x, y) = 0$ and $G(x, y) = \frac{1}{10} \sin(2\pi x)$.

3.3 Example illustrating Theorem 1.7

We return to the important example of geodesic flows.

Example 3.6 (Geodesic flow). Let $\phi_t : M \rightarrow M$ be the geodesic flow on the three dimensional unit tangent bundle $M = SV$ of a compact surface V of negative curvature. In this case the unstable manifolds correspond to horocycles. Given $x \in V$ we let $\mathcal{C} = S_x V$ be the fibre above $x \in V$. We define a family of densities $(\lambda_T)_{T>0}$ on \mathcal{C} by

$$\lambda_T(A) = \frac{\int_{\phi_T(A)} \exp\left(\int_0^T G(\phi_{-v}x) dv\right) d\lambda_{\mathcal{C}}(x)}{\int_{\phi_T(\mathcal{C})} \exp\left(\int_0^T G(\phi_{-v}x) dv\right) d\lambda_{\mathcal{C}}(x)} \text{ for Borel } A \subset \mathcal{C}.$$

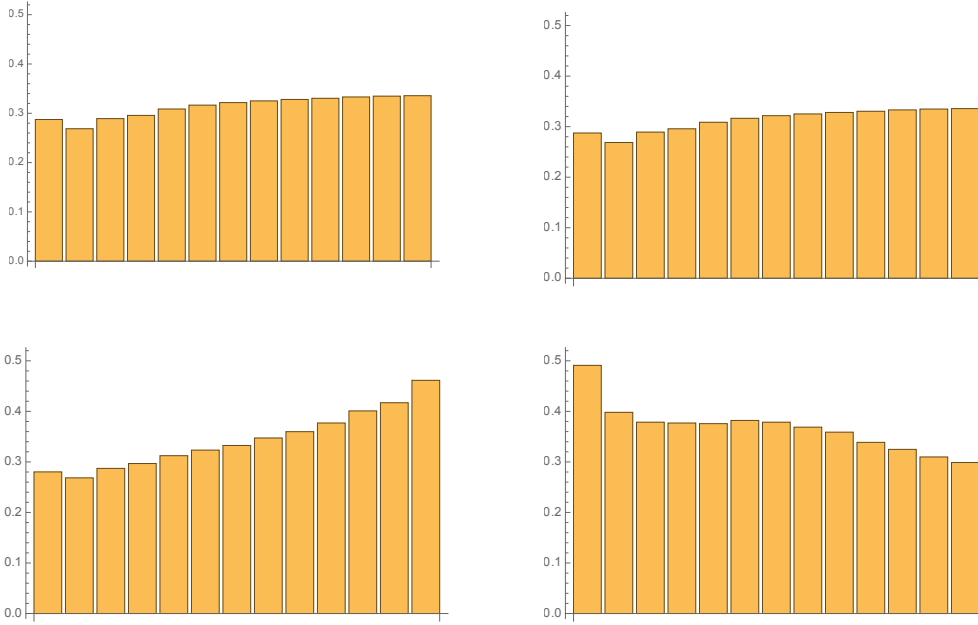


Figure 3: (a) When $G = 0$ the two top bar charts represent $\mu_n(B_1)$ and $\mu_n(B_2)$ for $n = 1, \dots, 12$ and the convergence to $\frac{\pi^2}{9}$ can be seen. (b) When $G(x, y) = \frac{1}{10} \sin(2\pi x)$ the two lower bar charts represent $\mu_n(B_1)$ and $\mu_n(B_2)$ for $n = 1, \dots, 12$. The sequences $\mu_n(B_1)$ and $\mu_n(B_2)$ converge to the measures of the balls B_1 and B_2 with respect to the corresponding Gibbs measure. It is empirically clear these values are different, but the precise values are unknown to us.

Then by Theorem 1.7 the averages of the push forwards

$$\mu_T := \frac{1}{T} \int_0^T (\phi_t)^* \lambda_T dt, \quad T > 0,$$

converge in the weak star topology to μ_G as $T \rightarrow +\infty$.

1. If $G = 0$ then the limiting measure μ_G is the Bowen-Margulis measure.
2. If $G = \Phi$ then the limiting measure μ_G is the Liouville measure (or, equivalently, the SRB measure).

If V is a surface of constant curvature $\kappa = -1$ then $\Phi = 1$ is a constant function and the Bowen-Margulis measure is equal to the Liouville measure.

4 Preliminaries for Theorem 1.2

In this section we collect together some results for hyperbolic attracting diffeomorphisms in anticipation of the proof of Theorem 1.2 in the next section.

4.1 Stable and unstable manifolds

We next recall the definition of the stable and unstable manifolds.

Definition 4.1. We associate to each $x \in X$ the stable manifold defined by

$$W^s(x) = \{y \in M : d(f^n x, f^n y) \rightarrow 0 \text{ as } n \rightarrow +\infty\}$$

and unstable manifold defined by

$$W^u(x) = \{y \in M : d(f^{-n} x, f^{-n} y) \rightarrow 0 \text{ as } n \rightarrow +\infty\}.$$

For sufficiently small $\delta > 0$ we define local versions $W_\delta^s(x) \subset W^s(x)$ and $W_\delta^u(x) \subset W^u(x)$ defined by

$$W_\delta^s(x) = \{y \in M : d(f^n x, f^n y) \leq \delta, \forall n \geq 0\}$$

and

$$W_\delta^u(x) = \{y \in M : d(f^{-n} x, f^{-n} y) \leq \delta, \forall n \geq 0\}.$$

The basic properties of the (local) stable and unstable manifolds are the following.

Lemma 4.2 (Hirsch-Pugh [12]). For each $x \in X$ and sufficiently small $\delta > 0$, the sets:

1. $W^s(x)$ and $W^u(x)$ are C^1 immersed submanifolds of dimension $\dim(E^s)$ and $\dim(E^u)$, respectively, with $T_x W^s = E_x^s$ and $T_x W^u = E_x^u$; and
2. $W_\delta^s(x)$ and $W_\delta^u(x)$ are C^1 embedded disks of dimension $\dim(E^s)$ and $\dim(E^u)$, respectively, with $T_x W_\delta^s(x) = E_x^s$ and $T_x W_\delta^u(x) = E_x^u$.

4.2 Entropy

We now describe some results on the entropy of invariant probability measures. For the duration of this subsection we will only require the weaker assumption that $f : X \rightarrow X$ is a homeomorphism.

We begin with some standard definitions.

Definition 4.3. Given a finite measurable partition $\mathcal{P} = \{P_1, \dots, P_k\}$ and a probability measure ν we can associate the entropy of the partition defined by

$$H_\nu(\mathcal{P}) = - \sum_{i=1}^k \nu(P_i) \log \nu(P_i).$$

We let $\bigvee_{i=0}^{n-1} f^{-i} \mathcal{P} = \{P_{i_0} \cap f^{-1} P_{i_1} \cap \dots \cap f^{-(n-1)} P_{i_{n-1}} : 1 \leq i_0, \dots, i_{n-1} \leq k\}$ be the refinement of the partitions $\mathcal{P}, f^{-1} \mathcal{P}, \dots, f^{-(n-1)} \mathcal{P}$.

Definition 4.4. We can then define the entropy associated to the partition \mathcal{P} by

$$h_\nu(\mathcal{P}) = \lim_{n \rightarrow +\infty} \frac{1}{n} H_\nu \left(\bigvee_{i=0}^{n-1} f^{-i} \mathcal{P} \right)$$

Lastly, the entropy with respect to the measure is defined by

$$h(\nu) = \sup_{\mathcal{P}} \{h_\nu(\mathcal{P}) : \mathcal{P} \text{ is a countable partition with } H_\nu(\mathcal{P}) < \infty\}.$$

In the case that \mathcal{P} is a generating partition we have that $h(\nu) = h_\nu(\mathcal{P})$ is the entropy of the measure ν .

The following lemma will be useful later.

Lemma 4.5.

$$qH_{\lambda_n} \left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P} \right) \leq nH_{\mu_n} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right) + 2q^2 \log \text{Card}(\mathcal{P}).$$

Proof. Following a construction of Misiurewicz (see [28], p.220) we proceed as follows. Given $0 < q < n$ choose $0 \leq k < q$ and then let $a = a(k) = \lfloor (n-k)/q \rfloor$. We can then partition $\{0, 1, 2, \dots, n-1\}$ by

$$\{0, \dots, k\} \cup \bigcup_{l=0}^{a-1} \{lq+k, \dots, (l+1)q+k-1\} \cup \{aq+k, \dots, n-1\}.$$

We can use this to rewrite the refinement as

$$\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P} = \bigvee_{t=0}^k f^{-t}\mathcal{P} \vee \left(\bigvee_{r=0}^{a-1} f^{-(rq+k)} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right) \right) \vee \bigvee_{u=aq+k}^{n-1} f^{-u}\mathcal{P}.$$

Thus we can write

$$\begin{aligned} H_\nu \left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P} \right) &\leq H_\nu \left(\bigvee_{t=0}^k f^{-t}\mathcal{P} \right) + \sum_{r=0}^{a-1} H_\nu \left(f^{-(rq+k)} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right) \right) \\ &\quad + H_\nu \left(\bigvee_{u=aq+k}^{n-1} f^{-u}\mathcal{P} \right) \end{aligned} \tag{4.1}$$

(cf. Walters [28]) as $k < q$ and $aq+k \geq n-q$ we have the bounds

$$H_\nu \left(\bigvee_{t=0}^k f^{-t}\mathcal{P} \right), H_\nu \left(\bigvee_{u=aq+k}^{n-1} f^{-u}\mathcal{P} \right) \leq q \log \text{Card}(\mathcal{P}).$$

Summing both sides of (4.1) over k , we can write

$$\begin{aligned} qH_\nu \left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P} \right) &\leq \sum_{k=0}^{q-1} \left(\sum_{r=0}^{a-1} H_\nu \left(f^{-(rq+k)} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right) \right) \right) + 2q \log \text{Card}(\mathcal{P}) \\ &\leq \sum_{m=0}^{n-1} H_{f_*^m \nu} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right) + 2q^2 \log \text{Card}(\mathcal{P}), \end{aligned} \tag{4.2}$$

using [28], Remark 2 (iii) §8.2 and that $H_\nu(f^{-s}\xi) = H_{f_*^s\nu}(\xi)$ ($s \geq 0$) for any partition ξ . If we let $\nu = \lambda_n$ and $\mu_n := \frac{1}{n} \sum_{m=0}^{n-1} f_*^m \lambda_n$ then by concavity of the entropy

$$H_{\mu_n} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right) \geq \frac{1}{n} \sum_{m=0}^{n-1} H_{f_*^m \lambda_n} \left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P} \right). \quad (4.3)$$

(cf. [28], Remark 1, §8.2). Comparing (4.2) and (4.3) concludes the proof. \square

4.3 Pressure and growth

The pressure $P(G)$ has various alternative interpretations in terms of the growth of different quantities. We will need the following particular variant on these.

Proposition 4.6. *Let $f : X \rightarrow X$ be a mixing hyperbolic attracting diffeomorphism. For any continuous function $G : X \rightarrow \mathbb{R}$ we have*

$$P(G) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{W_\delta^u(x)} \exp \left(\sum_{k=0}^{n-1} (G - \Phi)(f^k y) \right) d\lambda_{W_\delta^u(x)}(y). \quad (4.4)$$

Again, we can use the change of variables formula to rewrite this in the equivalent form

$$P(G) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{f^n W_\delta^u(x)} \exp \left(\sum_{k=1}^n G(f^{-k} y) \right) d\lambda_{f^n W_\delta^u(x)}(y).$$

Proof of Proposition 4.6. We begin with the following standard result.

Lemma 4.7. *For any $\epsilon > 0$ there exists an $m > 0$ such that $f^m W_\delta^u(x)$ is ϵ -dense in X . In particular, we can assume that $X = \cup_{y \in f^m W_\delta^u(x)} W_\epsilon^s(y)$.*

Proof. This is a consequence of the minimality of the unstable foliation for mixing hyperbolic attracting diffeomorphisms and the local product structure. \square

We recall that the pressure $P(G)$ can also be written in terms of the growth rates of spanning sets and separated sets. Recall that given $\epsilon > 0$ and $n \geq 1$ an (n, ϵ) -spanning set $S \subset X$ is such that $\cup_{x \in S} B(x, n, \epsilon)$ covers X , where $B(x, n, \epsilon) := \cap_{k=0}^{n-1} f^{-k} B(f^k x, \epsilon)$ is a Bowen ball. On the other hand an (n, ϵ) -separated set $\Sigma \subset X$ is such that $d_n(x, y) > \epsilon$ for $x, y \in \Sigma$ (and, in particular, $B(x, n, \epsilon/2)$, $x \in \Sigma$, are disjoint in X).

Lemma 4.8. *Given $n \geq 1$ and $\epsilon > 0$, let*

$$Z_0(n, \epsilon) = \inf \left\{ \sum_{y \in S} \exp(G^n(y)) : S \text{ is an } (n, \epsilon)\text{-spanning set} \right\}$$

and

$$Z_1(n, \epsilon) = \sup \left\{ \sum_{y \in \Sigma} \exp(G^n(y)) : \Sigma \text{ is an } (n, \epsilon)\text{-separated set} \right\}.$$

where $G^n(x) = \sum_{j=0}^{n-1} G(f^j x)$. Then

$$P(G) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \log Z_0(n, \epsilon) = \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \log Z_1(n, \epsilon).$$

(See [28], chapter 9, and also [13], Propositions 20.2.7 and 20.3.2)

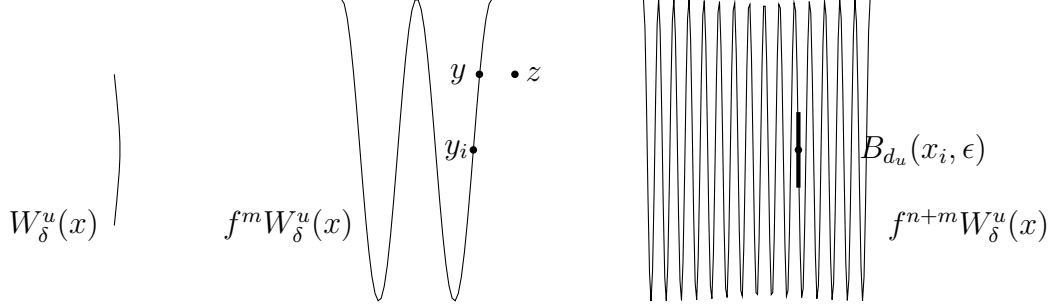


Figure 4: The push forward $f^m W_\delta^u(x)$ is ϵ -dense.

To get a lower bound on the growth rate in Proposition 4.6 given $\epsilon > 0$ and $n \geq 1$ we want to construct an $(n, 2\epsilon)$ -spanning set. We begin by choosing a covering of $f^{n+m} W_\delta^u(x)$ by ϵ -balls

$$B_{d_u}(x_i, \epsilon) : i = 1, \dots, N := N(n+m, \epsilon)$$

contained within the unstable manifold with respect to the induced metric d_u and let $A_\epsilon := f^{n+m} W_\delta^u(x) \setminus \bigcup_{y \in \partial f^{n+m} W_\delta^u(x)} B_{d_u}(y, \epsilon/2)$. We can choose a maximal set $S = \{x_1, \dots, x_{N(n+m, \epsilon)}\}$ with the additional properties that $d_u(x_i, x_j) > \epsilon/2$ for $i \neq j$ and $x_i \in A_\epsilon$. By our choice of S we have that

$$A_\epsilon \subset \bigcup_{i=1}^{N(n+m, \epsilon)} B_{d_u}(x_i, \epsilon/2).$$

By the triangle inequality we have that

$$f^{n+m} W_\delta^u(x) \subset \bigcup_{i=1}^{N(n+m, \epsilon)} B_{d_u}(x_i, \epsilon).$$

Since $B_{d_u}(x_i, \epsilon/4) \cap B_{d_u}(x_j, \epsilon/4) = \emptyset$ for $i \neq j$ we have that the disjoint union satisfies

$$\bigcup_{i=1}^{N(n+m, \epsilon)} B_{d_u}(x_i, \epsilon/4) \subset f^{n+m} W_\delta^u(x).$$

We can assume without loss of generality that

$$f^n : f^m W_\delta^u(x) \rightarrow f^{n+m} W_\delta^u(x)$$

locally expands distance along the unstable manifold (otherwise we can achieve this by a suitable choice of Riemannian metric, e.g., the Mather metric). In particular, this means that the primages $y_i := f^{-n}x_i \in f^m(W_\delta^u(x))$ ($i = 1, \dots, N$) form an $(n, 2\epsilon)$ -spanning set. By Lemma 4.7, for any point $z \in X$ we can choose a point $y \in f^m(W_\delta^u(x))$ with $z \in W_\epsilon^s(y)$ and observe that $d(f^j z, f^j y) < \epsilon$ for $0 \leq j \leq n$. We can then choose a y_i such that $d_n(y, y_i) < \epsilon$ since f^n is locally expanding along unstable manifolds. In particular, by the triangle inequality

$$d(f^j z, f^j y_i) \leq d(f^j z, f^j y) + d(f^j y, f^j y_i) \leq 2\epsilon$$

for $0 \leq j \leq n$.

Since G is continuous we have the following bound.

Lemma 4.9. *For all $\tau > 0$ there exists $\epsilon > 0$ sufficiently small such that for all $n \geq 1$ and points $y_i, z \in X$ satisfying $d(f^j y_i, f^j z) \leq \epsilon$ for $0 \leq j \leq n-1$ we have $|G^n(y) - G^n(z)| \leq n\tau$.*

It remains to relate $Z_0(n, 2\epsilon)$ to an integral over $f^{n+m}W_\delta^u(x)$. Because of the properties of our choice of ϵ -cover for $f^{n+m}W_\delta^u(x)$ we have that for all $n \geq 1$

$$\begin{aligned} Z_0(n, 2\epsilon) &\leq \sum_{i=1}^N \exp(G^n(y_i)) \\ &\leq \sum_{i=1}^N \frac{1}{\lambda(B_{d_u}(x_i, \epsilon/4))} \int_{B_{d_u}(x_i, \epsilon/4)} \exp(G^n(f^{-n}x_i)) d\lambda(z) \\ &\leq \frac{1}{M} e^{n\tau} \int_{f^{n+m}W_\delta^u(x)} e^{G^n(f^{-n}z)} d\lambda(z) \end{aligned}$$

where $M = M(\epsilon) = \inf_z \lambda(B_{d_u}(z, \epsilon/4)) > 0$. Finally, we can bound

$$\int_{f^{n+m}W_\delta^u(x)} e^{G^n(f^{-n}z)} d\lambda(z) \leq e^{m\|G\|_\infty} \int_{f^{n+m}W_\delta^u(x)} e^{G^{n+m}(f^{-(n+m)}z)} d\lambda(z).$$

Comparing these inequalities we see that

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \log Z_0(n, 2\epsilon) \leq \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{f^n W_\delta^u(x)} e^{G^n(f^{-n}z)} d\lambda(z) + \tau.$$

Since $\tau > 0$ can be chosen arbitrarily small the lower bound follows.

To get an upper bound on the growth rate in Proposition 4.6, given $\epsilon > 0$ and $n \geq 1$ then we want to create an $(n, \kappa\epsilon)$ -separated set. To this end, we can choose a maximal number of points $x_i \in f^n W_\delta^u(x)$ ($i = 1, \dots, N = N(n, \epsilon)$) so that $d_u(x_i, x_j) > \epsilon$ whenever $i \neq j$. We can again assume without loss of generality that $f^n : W_\delta^u(x) \rightarrow f^n W_\delta^u(x)$ is locally distance expanding and thus, in particular, the points $y_i = f^{-n}x_i$ ($i = 1, \dots, N = N(n, \epsilon)$) form an $(n, \kappa\epsilon)$ -separated set, for some $\kappa > 0$ independent of n and ϵ .

The balls $B_{d_u}(x_i, \epsilon)$ ($i = 1, \dots, N = N(n, \epsilon)$) form a cover for $f^n W_\delta^u(x)$, since otherwise we could choose an extra point $z \in f^n W_\delta^u(x)$ with $\inf_i \{d(z, x_i)\} \geq \epsilon$

contradicting the maximality of the previous family. We can therefore use Lemma 4.9 to bound

$$\begin{aligned} Z_1(n, \kappa\epsilon) &\geq \sum_{i=1}^N \frac{e^{-n\tau}}{\lambda(B_{d_u}(x_i, \epsilon))} \int_{B_{d_u}(x_i, \epsilon)} \exp(G^n(f^{-n}z)) d\lambda(z) \\ &\geq \frac{e^{-n\tau}}{L} \int_{f^n W_\delta^u(x)} e^{G^n(f^{-n}z)} d\lambda(z). \end{aligned}$$

where $L = L(\epsilon) = \sup_z \lambda(B_{d_u}(z, \epsilon)) > 0$. In particular, we see that

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow +\infty} \frac{1}{n} \log Z_1(n, \kappa\epsilon) \geq \lim_{n \rightarrow +\infty} \frac{1}{n} \log \int_{f^n W_\delta^u(x)} e^{G^n(f^{-n}z)} d\lambda(z) - \tau.$$

Since $\tau > 0$ is arbitrary this completes the proof. \square

Remark 4.10. *Under the stronger assumption that G is Hölder continuous then in the above argument we can replace $n\tau$ by a constant $C > 0$ independent of n by using a telescoping argument. This leads to some simplifications.*

We can consider Proposition 4.6 in the special case of the two distinguished examples of the SRB measure and the measure of maximal entropy.

Example 4.11. *In the particular case that $G = \Phi$ then we see that*

$$\int_{W_\delta^u(x)} \underbrace{\exp\left(\sum_{k=0}^{n-1} (G - \Phi)(f^k z)\right)}_{=1} d\lambda(z) = \lambda(W_\delta^u(x))$$

which is independent of n and thus we recover from Proposition 4.6 that $P(\Phi) = 0$ (see [4]).

Example 4.12. *In particular, when $G = 0$ then we see that the topological entropy can be written in the form*

$$h(f) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log \left(\lambda_{f^n W_\delta^u} (f^n W_\delta^u(x)) \right).$$

There have been previous results related to growth rates of the lengths of curves. For general C^∞ diffeomorphisms, for example, Newhouse has an expression for the entropy in terms of the growth of pieces of unstable manifolds [18].

5 Proof of Theorem 1.2

By Alaoglu's theorem on the weak star compactness of the space of f -invariant probability measures we can find an f -invariant probability measure, which we denote by μ , and a

subsequence n_k such that the measures (1.3) have a weak star convergent subsequence $\lim_{k \rightarrow \infty} \mu_{n_k} = \mu$. Moreover, for any continuous $F : X \rightarrow \mathbb{R}$ we can compare

$$\begin{aligned} \left| \int F d\mu_n - \int F \circ f d\mu_n \right| &= \left| \frac{1}{n} \sum_{k=0}^{n-1} \int F \circ f^k d\lambda_n - \frac{1}{n} \sum_{k=0}^{n-1} \int F \circ f^{k+1} d\lambda_n \right| \\ &\leq \frac{2\|F\|_\infty}{n} \rightarrow 0 \text{ as } n \rightarrow +\infty \end{aligned}$$

and, in particular, one easily sees that μ is f -invariant.

For convenience we denote

$$\begin{aligned} Z_n^G &= \int_{W_\delta^u(x)} \exp \left(\sum_{k=0}^{n-1} (G - \Phi)(f^k y) \right) d\lambda_{W_\delta^u(x)}(y) \\ &= \int_{f^n W_\delta^u(x)} \exp \left(\sum_{k=1}^n G(f^{-k} y) \right) d\lambda_{f^n W_\delta^u(x)}(y) \end{aligned}$$

using the change of variables on $f^n : W_\delta^u(x) \rightarrow f^n W_\delta^u(x)$. We want to show that μ is the Gibbs measure for G . As G is Hölder continuous then by uniqueness of the Gibbs measure μ_G we would have that $\mu_n \rightarrow \mu_G$ as $n \rightarrow +\infty$.

Definition 5.1. Given a finite partition $\mathcal{P} = \{P_i\}_{i=1}^N$ we say that it has size $\epsilon > 0$ if $\sup_i \{\text{diam}(P_i)\} < \epsilon$.

Since unstable manifolds are C^1 , there exists a constant $C > 0$ such that for all $y, z \in f^{n-1}W_\delta^u(x)$ and for $\epsilon > 0$ sufficiently small, if $d_u(y, z) > \epsilon$ then there exists an $0 \leq i \leq n-1$ such that $d(f^{-i}y, f^{-i}z) > C\epsilon$. In particular, $d_n(f^{-(n-1)}y, f^{-(n-1)}z) > C\epsilon$. By Lemma 4.9, for any $\tau > 0$ we can choose a partition \mathcal{P} of size $C\epsilon$ such that for all $x, y \in A \in \bigvee_{i=0}^{n-1} f^{-i}\mathcal{P}$ we have that

$$\left| \sum_{k=0}^{n-1} G(f^k x) - \sum_{k=0}^{n-1} G(f^k y) \right| \leq n\tau. \quad (5.1)$$

Under the assumption that G is Hölder continuous the upper bound in (5.1) can be replaced by τ , say. However, if we only assume that G is continuous then the above bound suffices and then the proof also applies to give Theorem 1.3.

Proceeding with the proof of Theorem 1.2, for each $A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}$ we can fix an $x_A \in A$. By definition of λ_n ,

$$\int_{W_\delta^u(x)} G(x) d\lambda_n(x) = \frac{1}{Z_n^G} \int_{f^n W_\delta^u(x)} \exp \left(\sum_{k=1}^n G(f^{-k} y) \right) G(f^{-n} y) d\lambda_{f^n W_\delta^u(x)}(y)$$

and so we can write for each $0 \leq j \leq n-1$:

$$\begin{aligned} \int_{f^j W_\delta^u(x)} G d(f_*^j \lambda_n) &= \int_{W_\delta^u(x)} G(f^j y) d\lambda_n(y) \\ &= \frac{1}{Z_n^G} \int_{f^n(W_\delta^u(x))} \exp \left(\sum_{k=1}^n G(f^{-k} y) \right) G(f^{-(n-j)} y) d\lambda_{f^n W_\delta^u(x)}(y). \end{aligned}$$

Hence, using the definition of μ_n in (1.3) we can write

$$\begin{aligned}
& \int G(y) d\mu_n(y) \\
&= \frac{1}{nZ_n^G} \int_{f^n W_\delta^u(x)} \exp\left(\sum_{k=1}^n G(f^{-k}y)\right) \left(\sum_{j=0}^{n-1} G(f^{-(n-j)}y)\right) d\lambda_{f^n W_\delta^u}(y) \\
&= \frac{1}{nZ_n^G} \sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \int_{f^n(A \cap W_\delta^u(x))} \exp\left(\sum_{k=1}^n G(f^{-k}y)\right) \left(\sum_{j=0}^{n-1} G(f^{-(n-j)}y)\right) d\lambda_{f^n W_\delta^u}(y) \\
&\geq \frac{1}{nZ_n^G} \sum_{A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}} \left(\sum_{j=0}^{n-1} G(f^j x_A) - n\tau\right) \int_{f^n(A \cap W_\delta^u(x))} \exp\left(\sum_{k=1}^n G(f^{-k}y)\right) d\lambda_{f^n W_\delta^u}(y)
\end{aligned} \tag{5.2}$$

the last inequality coming from equation (5.1). We can write for a set $A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}$:

$$\lambda_n(A) = \int_{f^n W_\delta^u(x)} \exp\left(\sum_{k=1}^n G(f^{-k}y)\right) \chi_{f^n A}(y) d\lambda_{f^n W_\delta^u(x)}(y),$$

then writing $\lambda_{f^n(A \cap W_\delta^u(x))}$ for the restriction of $\lambda_{f^n(W_\delta^u(x))}$ to $f^n A$ we have that

$$\begin{aligned}
\log \lambda_n(A) &= \log \int_{f^n W_\delta^u(x)} \exp\left(\sum_{k=1}^n G(f^{-k}y)\right) d\lambda_{f^n(A \cap W_\delta^u(x))} \\
&\leq \sum_{k=0}^{n-1} G(f^k x_A) + n\tau + \log \lambda_{f^n W_\delta^u(x)}(f^n(A)) \\
&\leq \sum_{k=0}^{n-1} G(f^k x_A) + n\tau
\end{aligned} \tag{5.3}$$

The last inequality requires the following justification.

First note that as $A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}$ and since \mathcal{P} is of size $C\epsilon$ then by definition, there is a $z \in X$ such that $A \in B_n(z, C\epsilon)$. We will now show, for any $\epsilon' > 0$, there is a $p \in f^n W_\delta^u(x)$ and $\epsilon > 0$ small enough such that $f^n A \cap f^n W_\delta^u(x) \subset B_{d_u}(p, \epsilon')$ and in particular, the intersection is contained within one small piece of local unstable manifold, $f^n A \cap f^n W_\delta^u(x) \subset W_{\epsilon'}^u(p)$.

By continuity, for $\epsilon > 0$ small enough, if $f^{n-1}A \cap f^{n-1}W_\delta^u(x) \subset B_{d_u}(f^{-1}p, \epsilon)$ then $f^n A \cap f^n W_\delta^u(x) \subset B_{d_u}(p, \epsilon')$. Therefore, we only need to check $f^{n-1}A \cap f^{n-1}W_\delta^u(x) \subset B_{d_u}(f^{-1}p, \epsilon)$ for some $f^{-1}p \in f^{n-1}W_\delta^u(x)$. If this were not the case then there exist $z_1, z_2 \in f^{n-1}A \cap f^{n-1}W_\delta^u(x)$ such that $B_{d_u}(z_1, \frac{\epsilon}{2}) \cap B_{d_u}(z_2, \frac{\epsilon}{2}) = \emptyset$ (since $A \in \bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}$ and \mathcal{P} is of size $C\epsilon$). Then $d_u(z_1, z_2) > \epsilon$ and so $d_n(f^{-(n-1)}z_1, f^{-(n-1)}z_2) > C\epsilon$ (by the choice of C above equation (5.1)), but this contradicts that $A \in B_n(z, C\epsilon)$ since $f^{-(n-1)}z_1, f^{-(n-1)}z_2 \in A$.

Therefore, as $f^n A \cap f^n W_\delta^u(x) \subset B_{d_u}(p, \epsilon')$, we can take ϵ' sufficiently small (by taking ϵ small enough) such that $\lambda_{f^n W_\delta^u(x)}(f^n(A)) < 1$ and so $\log \lambda_{f^n W_\delta^u(x)}(f^n(A))$ is negative as required.

Letting $K_{n,A} = \int_{f^n W_\delta^u(x)} \exp\left(\sum_{k=1}^n G(f^{-k}y)\right) d\lambda_{f^n(A \cap W_\delta^u(x))}$ we can consider the entropy

$$\begin{aligned} H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right) &= - \sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} \lambda_n(A) \log \lambda_n(A), \\ &= - \sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} \log \frac{K_{n,A}}{Z_n^G}, \\ &= \log Z_n^G - \sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} \log K_{n,A} \end{aligned} \quad (5.4)$$

where the last equality uses $\sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} K_{n,A} = Z_n^G$. Therefore, comparing (5.3) and (5.4) gives

$$H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right) \geq \log Z_n^G - \sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} \left(\sum_{k=0}^{n-1} G(f^k x_A) + n\tau\right). \quad (5.5)$$

By (5.2) we can also bound

$$n \int_X G d\mu_n \geq \frac{1}{Z_n^G} \sum_{A \in \mathcal{V}_{h=1}^n f^{-h}\mathcal{P}} \left(\sum_{k=1}^n G(f^k x_A) - n\tau\right) K_{n,A}. \quad (5.6)$$

Summing (5.5) and (5.6) we can write

$$\begin{aligned} &H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right) + n \int_X G d\mu_n \\ &\geq \log Z_n^G - \sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} \frac{K_{n,A}}{Z_n^G} \left(\sum_{k=0}^{n-1} G(f^k x_A) + n\tau\right) \\ &\quad + \frac{1}{Z_n^G} \sum_{A \in \mathcal{V}_{h=0}^{n-1} f^{-h}\mathcal{P}} \left(\sum_{k=0}^{n-1} G(f^k x_A) - n\tau\right) K_{n,A}, \\ &\geq \log Z_n^G - 2n\tau. \end{aligned} \quad (5.7)$$

We can use (5.7) and Lemma 4.5 to write,

$$\begin{aligned} q \log Z_n^G - qn \int_X G d\mu_n - 2qn\tau &\leq q H_{\lambda_n}\left(\bigvee_{h=0}^{n-1} f^{-h}\mathcal{P}\right), \\ &\leq n H_{\mu_n}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right) + 2q^2 \log \text{Card}(\mathcal{P}), \end{aligned}$$

then dividing by q and n , and rearranging the terms gives

$$\frac{\log Z_n^G}{n} - \frac{2n\tau}{n} - \frac{2q \log |\mathcal{P}|}{n} \leq \frac{H_{\mu_n}\left(\bigvee_{i=0}^{q-1} f^{-i}\mathcal{P}\right)}{q} + \int_X G d\mu_n.$$

Letting $n_k \rightarrow +\infty$

$$\begin{aligned}
P(G) &= \lim_{k \rightarrow \infty} \frac{\log Z_{n_k}^G}{n_k} \\
&\leq \lim_{k \rightarrow \infty} \left(\frac{H_{\mu_{n_k}} \left(\bigvee_{i=0}^{q-1} f^{-i} \mathcal{P} \right)}{q} + \int_X G d\mu_{n_k} \right) + 2\tau \\
&= \frac{H_{\mu} \left(\bigvee_{i=0}^{q-1} f^{-i} \mathcal{P} \right)}{q} + \int_X G d\mu + 2\tau,
\end{aligned}$$

where we assume without loss of generality that the boundaries of the partition have zero measure. Letting $q \rightarrow \infty$,

$$P(G) \leq h_{\mu}(\mathcal{P}) + \int_X G d\mu + 2\tau. \quad (5.8)$$

Finally, we recall that τ is arbitrary. Therefore, since μ is a f -invariant probability measure we see from the variational principle (1.1) that the inequalities in (5.8) are actually equalities (since $h_{\mu}(\mathcal{P}) \leq h(\mu)$) and therefore we conclude that the measure μ is the Gibbs measure for G , μ_G .

6 Proof of Theorem 1.7

The proof of Theorem 1.7 for Gibbs measures for flows is completely analogous to that for diffeomorphisms in Theorem 1.2.

6.1 Hyperbolic attracting flows

We begin by recalling the definition of a hyperbolic attracting flow. Let $\phi_t : M \rightarrow M$ ($t \in \mathbb{R}$) be a $C^{1+\alpha}$ flow on a compact Riemannian manifold, and let $X \subset M$ be a closed ϕ -invariant set.

Definition 6.1. *The flow $\phi_t : X \rightarrow X$ is called a mixing hyperbolic attractor if:*

1. *there exists a continuous splitting $T_X M = E^0 \oplus E^s \oplus E^u$ where E^0 is a one dimensional subbundle tangent to the flow orbits and there exist $C > 0$ and $0 < \lambda < 1$ such that*

$$\|D\phi_t|E^s\| \leq C\lambda^t \text{ and } \|D\phi^{-t}|E^u\| \leq C\lambda^t$$

for $t \geq 0$;

2. *there exists an open set $X \subset U \subset M$ such that $X = \bigcap_{t=0}^{\infty} \phi_t U$;*
3. *$\phi_t : X \rightarrow X$ is topologically mixing; and*
4. *the periodic orbits for $\phi_t : X \rightarrow X$ are dense in X .*

In the particular case that the entire manifold is hyperbolic, i.e., $X = M$ then we call the flow Anosov.

Given any Hölder continuous function $G : X \rightarrow \mathbb{R}$ we want to consider an analogous construction of a Gibbs measure.

Let us first define a Hölder continuous function $\Phi : X \rightarrow \mathbb{R}$ defined by

$$\Phi(y) = -\lim_{t \rightarrow 0} \frac{1}{t} \log |\det(D\phi_t|E_y^u)|.$$

Example 6.2. 1. In the particular case that $G = 0$ we have that the limit is the measure of maximal entropy.

2. In the particular case that $G = \Phi(x)$ we have that the limit is μ_{SRB} , the SRB measure.

The proof of Theorem 1.7 follows the same lines as Theorem 1.2. We will therefore only need to explain the main ideas.

6.2 Basic properties

We next recall the definition of the stable and unstable manifolds for the flow.

Definition 6.3. We associate to each $x \in X$ the stable manifold defined by

$$W^s(x) = \{y \in M : d(\phi_t x, \phi_t y) \rightarrow 0 \text{ as } t \rightarrow +\infty\}$$

and unstable manifold defined by

$$W^u(x) = \{y \in M : d(\phi_{-t} x, \phi_{-t} y) \rightarrow 0 \text{ as } t \rightarrow +\infty\}.$$

For sufficiently small $\delta > 0$ we define local versions $W_\delta^s(x) \subset W^s(x)$ and $W_\delta^u(x) \subset W^u(x)$ defined by

$$W_\delta^s(x) = \{y \in M : d(\phi_t x, \phi_t y) \rightarrow 0 \text{ as } t \rightarrow +\infty \text{ and } d(\phi_t x, \phi_t y) \leq \delta, \forall t \geq 0\}$$

and

$$W_\delta^u(x) = \{y \in M : d(\phi_{-t} x, \phi_{-t} y) \rightarrow 0 \text{ as } t \rightarrow +\infty \text{ and } d(\phi_{-t} x, \phi_{-t} y) \leq \delta, \forall t \geq 0\}.$$

The basic properties of the (local) stable and unstable manifolds is the following analogue of Lemma 4.2

Lemma 6.4. For each $x \in \Lambda$ and $\delta > 0$ sufficiently small:

1. the sets $W^s(x)$ and $W^u(x)$ are C^1 immersed submanifolds of dimension $\dim(E^s)$ and $\dim(E^u)$, respectively, with $T_x W^s = E_x^s$ and $T_x W^u = E_x^u$.
2. the sets $W_\delta^s(x)$ and $W_\delta^u(x)$ are C^1 embedded disks of dimension $\dim(E^s)$ and $\dim(E^u)$, respectively, with $T_x W_\delta^s(x) = E_x^s$ and $T_x W_\delta^u(x) = E_x^u$.

6.3 Pressure and growth rates for hyperbolic attracting flows

We will need the following useful characterization of the pressure from the flow, which is analogous to the corresponding result for diffeomorphisms (i.e., Proposition 4.6).

Proposition 6.5. *Let $\phi_t : X \rightarrow X$ be a mixing attracting hyperbolic flow. For any continuous function $G : X \rightarrow \mathbb{R}$*

$$P(G) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \int_{W_\delta^u(x)} \exp \left(\int_0^t (G - \Phi)(\phi_v x) dv \right) d\lambda_{W_\delta^u(x)}(x).$$

Again, we can use the change of variables to rewrite this as

$$P(G) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \int_{\phi_t W_\delta^u(x)} \exp \left(\int_0^t G(\phi_{-v} x) dv \right) d\lambda_{\phi_t W_\delta^u(x)}(x).$$

Example 6.6. *In particular, when $G = \Phi$ then we see that*

$$\int_{W_\delta^u(x)} \underbrace{\exp \left(\int_0^t (G - \Phi)(\phi_v x) dv \right)}_{=1} d\lambda(x) = \lambda(W_\delta^u(x))$$

which is independent of t and thus we recover from Proposition 6.5 that $P(\Phi) = 0$.

The proof of Proposition 6.5 is analogous to the corresponding result for hyperbolic diffeomorphisms so we will only provide details where the proof differs.

Proof. We begin with the following analogue of Lemma 4.7.

Lemma 6.7. *Let $\phi_t : X \rightarrow X$ be a mixing hyperbolic flow. For any $\epsilon > 0$ there exists $T_0 > 0$ such that $\phi_{T_0} W_\delta^u(x)$ is 2ϵ -dense. In particular, we can assume that*

$$X = \phi_{[-\epsilon, \epsilon]} \left(\bigcup_{y \in \phi_{T_0} W_\delta^u(x)} W_\epsilon^s(y) \right).$$

Proof. This is a consequence of the minimality of the unstable foliation. \square

The pressure $P(G)$ for flows can be written in terms of the growth rates of sets of spanning sets and separated sets. Recall that a (T, ϵ) -spanning set $S \subset X$ is such that $\bigcup_{x \in S} B(x, T, \epsilon)$ covers X , where $B(x, T, \epsilon) := \bigcap_{0 \leq t \leq T} \phi_{-t} B(\phi_t x, \epsilon)$ is a Bowen ball. On the other hand a (T, ϵ) -separated set $\Sigma \subset X$ is one such that $\max_{0 \leq t \leq T} d(\phi_t x, \phi_t y) > \epsilon$ for $x, y \in \Sigma$ ($x \neq y$). (In particular, the sets $B(x, T, \epsilon/2)$ are disjoint in X).

Lemma 6.8. *Given $n \geq 1$ and $\epsilon > 0$ we denote*

$$Z_0(T, \epsilon) = \inf \left\{ \sum_{x \in S} \exp(G^T(x)) : S \text{ is a } (T, \epsilon)\text{-spanning set} \right\}$$

and

$$Z_1(T, \epsilon) = \sup \left\{ \sum_{x \in \Sigma} \exp(G^T(x)) : \Sigma \text{ is a } (T, \epsilon)\text{-separated set} \right\}.$$

where $G^T(x) = \int_0^T G(\phi_t x) dt$. Then

$$P(G) = \lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow +\infty} \frac{1}{T} \log Z_0(T, \epsilon) = \lim_{\epsilon \rightarrow 0} \lim_{T \rightarrow +\infty} \frac{1}{T} \log Z_1(T, \epsilon).$$

To get a lower bound on the growth rates in the statement of the Proposition, given $\epsilon > 0$ and $T > 0$ then we want to construct a $(T, 3\epsilon)$ -spanning set.

The construction is analogous with the diffeomorphism case. We begin by choosing a covering of $\phi_{T+T_0} W_\delta^u(x)$ by ϵ -balls

$$B_{d_u}(x_i, \epsilon) : i = 1, \dots, N := N(n + m, \epsilon)$$

contained within the unstable manifold with respect to the induced metric d_u and let $A_\epsilon := \phi_{T+T_0} W_\delta^u(x) \setminus \bigcup_{y \in \partial \phi_{T+T_0} W_\delta^u(x)} B_{d_u}(y, \epsilon/2)$. We can choose a maximal set $S = \{x_1, \dots, x_{N(n+m, \epsilon)}\}$ with the additional properties that $d_u(x_i, x_j) > \epsilon/2$ for $i \neq j$ and $x_i \in A_\epsilon$. By our choice of S we have that

$$A_\epsilon \subset \bigcup_{i=1}^{N(n+m, \epsilon)} B_{d_u}(x_i, \epsilon/2).$$

By the triangle inequality we have that

$$\phi_{T+T_0} W_\delta^u(x) \subset \bigcup_{i=1}^{N(n+m, \epsilon)} B_{d_u}(x_i, \epsilon).$$

Since $B_{d_u}(x_i, \epsilon/4) \cap B_{d_u}(x_j, \epsilon/4) = \emptyset$ for $i \neq j$ we have that the disjoint union satisfies

$$\bigcup_{i=1}^{N(n+m, \epsilon)} B_{d_u}(x_i, \epsilon/4) \subset \phi_{T+T_0} W_\delta^u(x).$$

We can assume without loss of generality that

$$\phi_T : \phi_{T_0} W_\delta^u(x) \rightarrow \phi_{T_0+T} W_\delta^u(x)$$

locally expands distance along the unstable manifold (otherwise we can achieve this by a suitable choice of Riemannian metric, eg, the Mather metric).

We claim that the primages $y_i := \phi_{-T} x_i \in \phi_{T_0}(W_\delta^u(x))$ ($i = 1, \dots, N$) form a $(T, 3\epsilon)$ -spanning set. By Lemma 6.7 for any point $\hat{z} \in X$ we can choose a point $z = \phi_{\hat{t}} \hat{z}$ for $\hat{t} \in [-\epsilon, \epsilon]$ and a point $y \in \phi_{T_0}(W_\delta^u(x))$ with $z \in W_\epsilon^s(y)$ and observe that $d(\phi_t z, \phi_t y) < \epsilon$ for $0 \leq t \leq T$. We can then choose a y_j such that $d(\phi_t y, \phi_t y_j) < \epsilon$ for $0 \leq t \leq T$ since ϕ_t is locally expanding along unstable manifolds. In particular, by the triangle inequality

$$d(\phi_t \hat{z}, \phi_t y_j) \leq d(\phi_t \hat{z}, \phi_t z) + d(\phi_t z, \phi_t y) + d(\phi_t y, \phi_t y_j) \leq 3\epsilon$$

for $0 \leq t \leq T$.

Since G is continuous we have the following bound.

Lemma 6.9. *For all $\tau > 0$ there exists $\epsilon > 0$ sufficiently small such that for all $T \geq 1$ and points $y, z \in X$ satisfying $d(\phi_t y, \phi_t z) \leq \epsilon$ for $0 \leq t \leq T$ we have $|G^T(y) - G^T(z)| \leq T\tau$.*

It remains to relate $Z_0(T, 3\epsilon)$ to an integral over $\phi_{T+T_0} W_\delta^u(x)$, which is analogous to the diffeomorphism case.

To get an upper bound on the growth rate in Proposition 6.5 we proceed analogously to the diffeomorphism case to conclude the proof. □

Remark 6.10. *One can see from the proof that in the statements of Proposition 6.5 and Theorem 1.7 extend to the case that $W_\delta^u(x)$ is replaced by any embedded submanifold in dimension $\dim E^u$, provided it is not contained in a weak stable manifold.*

6.3.1 Growth rates and geodesic flows on surfaces of variable negative curvature

We want to relate the results on hyperbolic attracting flows to classical results on geodesic flows.

Given a compact manifold V with negative sectional curvatures we can associate the unit tangent bundle $M = SV$. Let $\phi_t : M \rightarrow M$ be the geodesic flow. Given a point $x \in V$ we can denote by $S_x V$ the fibre above x . We can consider the image $\phi_t(S_x V)$ under the geodesic flow for time $t > 0$.

We begin with the results on entropy and pressure due to Manning and Ruelle.

Theorem 6.11 (after Manning). *We can write*

$$h(\phi) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \lambda(\phi_t(S_x V))$$

where λ denotes the induced volume.

The original statement of Manning was for the rate of growth of the volume of a ball in the universal cover. However, because of the hypothesis of negative curvature this corresponds to the rate of growth of the induced volume of the boundary.

Remark 6.12. *It is easy to see that the above result is closely related to Remark 6.10 and Proposition 6.5. In particular, if we partition $S_x V$ into a finite number of pieces then we see that the images of these pieces under the geodesic flow can be uniformly approximated by longer pieces of unstable manifold. This point of view is similar to the classical approach of Marcus to horocycles [14].*

Given a continuous function $G : M \rightarrow \mathbb{R}$ there is a natural generalization which takes the following form. Let $G_{(x,y)} = \int_{[x,y]} G$ be the integral of G along the canonical geodesic segment from x to y on V .

Theorem 6.13 (after Ruelle). *We can write*

$$P(G) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \int_{\phi_t(S_x V)} \exp(G_{(x,y)}) d\lambda_{\phi_t(S_x V)}(y)$$

where $\lambda_{\phi_t(S_x V)}$ denotes the induced volume on $\phi_t(S_x V)$.

The original statement of Ruelle was a generalization of the result of Manning and formulated on the universal cover \widetilde{M} . The above version can be deduced as before.

6.4 Entropy and the end of the proof of Theorem 1.7

The arguments in §3 did not require that the transformation is hyperbolic merely a homeomorphism. In particular, we can apply the same reasoning where we let $f = \phi_{t=1}$ be a time one map of a hyperbolic flow. In particular, we can deduce the following analogue of (5.8). Let μ denote a limit point of μ_T ($T > 0$).

Proposition 6.14. *Let $\phi_{t=1} : X \rightarrow X$ be a time one map of a hyperbolic attracting flow and let $G : X \rightarrow \mathbb{R}$ be a continuous function. Then any limit point μ of the measures μ_T satisfies*

$$P(G) \leq h_\mu(\phi_{t=1}, \mathcal{P}) + \int_X G d\mu$$

We recall that

$$h_\mu(\phi_{t=1}, \mathcal{P}) \leq h_\mu(\phi_{t=1}) = h_\mu(\phi)$$

and deduce from the variational principle that μ is the Gibbs measure for G , μ_G . In particular, if G is Hölder continuous then by uniqueness of the Gibbs measure we see that $\mu_T \rightarrow \mu_G$ as $T \rightarrow +\infty$.

7 Generalizations and questions

We can consider some generalizations of these results by examining the proof of Theorem 1.2 and it is possible to formulate the result in a more general setting.

Proposition 7.1. *Let $f : M \rightarrow M$ be a C^2 diffeomorphism and let $W \subset M$ be a C^2 embedded disk. Let $G : M \rightarrow \mathbb{R}$ be a continuous function and assume that the analogue of condition (4.4) holds for W . If we define λ_n as above then any weak star accumulation point of*

$$\mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \lambda_n$$

will be an equilibrium state for G .

If we further assume that G is Hölder continuous and f satisfies expansion and specification hypotheses then there will be a unique Gibbs measure [11].

We can now describe some future direction where this theorem could be applied.

1. In the case of general C^∞ surface diffeomorphisms of positive topological entropy assume that there exists at least one strong unstable manifold for which (4.4) holds. Then this can be used to construct the Gibbs measure. In the case of the measure of maximal entropy this hypothesis holds by an observation of Newhouse and Pignataro [19].

2. For a partially hyperbolic diffeomorphism $f : X \rightarrow X$ the analogue of the SRB measures are naturally replaced by u -Gibbs measures, as introduced by Pesin and Sinai [22] (see also [10]). More generally, if $G : X \rightarrow \mathbb{R}$ is (Hölder) continuous then there still exists Gibbs measures defined using (1.1) provided that f is C^∞ or that f is entropy expansive (for example, when $\dim E^0 = 1$). For partially hyperbolic systems uniqueness is more of an issue and often requires some additional hypotheses. Interesting examples include Quasi hyperbolic toral automorphisms and frame flows.
3. For two sided subshifts of finite type $\sigma : \Sigma_A \rightarrow \Sigma_A$ we can consider a copy $\{x\} \times \Sigma_A^+$ of the one sided subshift, equipped with a suitable reference measure, and Σ_A^+ in place of $W_\delta^u(x)$.

It is a natural question to ask about the speed of convergence of $\int f \mu_n$ and $\int f d\mu_T$ for hyperbolic diffeomorphisms and flows, respectively. This should be related to the speed of mixing or decay of correlations,

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