HIGHER TEICHMÜLLER THEORY FOR SURFACE GROUPS AND SHIFTS OF FINITE TYPE

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Abstract. The Teichmüller space of Riemann metrics on a compact oriented surface V comes equipped with a natural Riemannian metric called the Weil-Petersson metric. Bridgeman, Canary, Labourie and Sambarino generalised this to Higher Teichmüller Theory, i.e. representations of $\pi_1(V)$ in $\mathrm{SL}(d,\mathbb{R})$, and showed that their metric is analytic. In this note we will present a new equivalent definition of the Weil-Petersson metric for Higher Teichmüller Theory and also give a short proof of analyticity. Our approach involves coding $\pi_1(V)$ in terms of a symbolic dynamical system and the associated thermodynamic formalism.

1. Introduction

Given a compact oriented surface V of genus $g \geq 2$, the classical Teichmüller space $\mathcal{T}(V)$ is the space of hyperbolic structures on V, i.e. Riemannian metrics of constant curvature -1. Then $\mathcal{T}(V)$ is diffeomorphic to $\mathbb{R}^{6g-6}$ and it supports a number of natural metrics. One of the best known of these is the Weil-Petersson metric, which is negatively curved but incomplete. Let Γ denote the fundamental group $\pi_1(V)$ of V. By the uniformisation theorem, each element of $\mathcal{T}(V)$ can be realised as $\mathbb{H}^2/\rho(\Gamma)$, where $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ is a discrete co-compact representation of Γ into $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ and where the action on $\mathbb{H}^2$ is by Möbius transformations. In fact, Goldman [10] and Hitchin [11] showed that $\mathcal{T}(V)$ can be identified with a connected component in the representation space

$$\mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}),$$

where $\mathrm{PSL}(2,\mathbb{R})$ acts by conjugation. (Some modification is needed to obtain a Hausdorff quotient space, see [2] or [9] for details. However this does not affect the Teichmüller component or the Hitchin components introduced below.) Another (homeomorphic) connected component $\mathcal{T}'(V)$ is obtained through the action of an outer automorphism of $\mathrm{PSL}(2,\mathbb{R})$ (corresponding to reversing the orientation of V).

In 1958, Weil defined the Weil-Petersson metric on Teichmüller space, using the Petersson inner product on modular forms. An alternative definition of the metric was introduced by Thurston, the equivalence of which to the Weil-Petersson metric was shown by Wolpert in 1986 [31]. In 2008, McMullen gave a more thermodynamic formulation, using the pressure metric [20]. In this note, we will consider an extension of the definition of the classical Weil-Petersson metric on Teichmüller space to representations into the higher rank groups $\mathrm{PSL}(d,\mathbb{R})$, $d \geq 3$.

We begin by recalling another equivalent definition of the classical Weil-Petersson metric from [24], based on [26]. Given an analytic family of representations

$$(-\epsilon, \epsilon) \to \mathrm{Rep}(\Gamma, \mathrm{PSL}(2,\mathbb{R})) = \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}) : \lambda \mapsto \rho_\lambda,$$

with expansion

$$\rho_\lambda = \rho_0 + \rho^{(1)}\lambda + o(\lambda),$$

where $\mathrm{tr}(\rho^{(1)}) = 0$, it suffices to define the norm of the tangent $\rho^{(1)}$. (We assume that the familiy is non-trivial and thus $\epsilon > 0$ can be chosen sufficiently small that $\rho_\lambda \neq \rho_0$ for $\lambda \neq 0$.) One approach to doing this is given in Proposition 1.1 below.

For $g \in \Gamma$, let $[g]$ denote its conjugacy class and let $\mathcal{C}(\Gamma)$ denote the set of non-trivial conjugacy classes in Γ. To each conjugacy classes $[g] \in \mathcal{C}(\Gamma)$ and $\lambda \in (-\epsilon, \epsilon)$, we associate the length $l_{\rho_\lambda}(g)$ of corresponding closed geodesic in $\mathbb{H}^2/\rho_\lambda(\Gamma)$. We recall that

$$\lim_{T \to +\infty} \frac{1}{T} \log \#\{[g] \in \mathcal{C}(\Gamma) : l_{\rho_0}([g]) \le T\} = 1.$$

The next result describes how the growth rate changes if, for a given $\lambda \neq 0$, we restrict to conjugacy classes $[g]$ for which $l_{\rho_\lambda}([g])$ is close to $l_{\rho_0}([g])$.

Proposition 1.1 ([24, 26])**.** *For each $\lambda \in (-\epsilon, \epsilon) \setminus \{0\}$, there exists $0 < \alpha(\lambda) < 1$ such that*

$$\alpha(\lambda) = \lim_{\delta \to 0} \lim_{T \to +\infty} \frac{1}{T} \log \# \left\{ [g] \in \mathcal{C}(\Gamma) : l_{\rho_0}(g) \le T \text{ and } \frac{l_{\rho_\lambda}(g)}{l_{\rho_0}(g)} \in (1-\delta, 1+\delta) \right\}. \tag{1.1}$$

Furthermore, if we define $\alpha : (-\epsilon, \epsilon) \to [0,1]$ by setting $\alpha(0) = 1$ then the Weil-Petersson norm is given by

$$\|\rho^{(1)}\| = -\frac{1}{12\pi(g-1)} \frac{\partial^2 \alpha(\lambda)}{\partial \lambda^2}\bigg|_{\lambda=0}.$$

Indeed, [26] contains a stronger asymptotic result than (1.1), but the above statement suffices for our purpose of studying the Weil-Petersson metric.

It is natural to consider the generalisation of this approach to representations in the higher rank group $\mathrm{PSL}(d, \mathbb{R})$ (for $d \ge 3$). As we discuss in section 2, there is a natural representation from $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{PSL}(d, \mathbb{R})$ (induced by the action on homogeneous polynomials in two variables of degree $d-1$) and a representation $R : \Gamma \to \mathrm{PSL}(d, \mathbb{R})$ is called *Fuchsian* if it is obtained from a representation in $\mathcal{T}(V)$ or $\mathcal{T}'(V)$ by composition. Unlike the $d = 2$ case, the Fuchsian representations do not fill out a whole connected component of the representation space

$$\mathrm{Rep}(\Gamma, \mathrm{PSL}(d, \mathbb{R})) = \mathrm{Hom}(\Gamma, \mathrm{PSL}(d, \mathbb{R}))/\mathrm{PSL}(d, \mathbb{R})$$

but a component containing a Fuchsian representation is called a *Hitchin component*. Such a component is an analytic manifold diffeomorphic to an open ball of dimension $(2g-2)\dim(\mathrm{PSL}(d, \mathbb{R}))$ [12].

Let $\mathcal{H}$ be a Hitchin component. The natural problem of defining an analogue of the Weil-Petersson metric on $\mathcal{H}$ has already been considered by Bridgeman, Canary, Labourie and Sambarino (in the even more general setting of Gromov hyperbolic groups) in [3]. We start by defining a numerical characteristic called the *entropy* of a representation. Representations in the Hitchin component have the key proximality property that for $g \in \Gamma \setminus \{1_\Gamma\}$, the matrix $R(g)$ (which we can think of as lifted to $\mathrm{SL}(d, \mathbb{R})$) has a unique simple eigenvalue $\lambda(g)$ which is strictly maximal in modulus, satisfies $|\lambda(g)| > 1$, and which only depends of the conjugacy class $[g]$. This then allows us to define the entropy, $h(R)$, of a representation $R \in \mathcal{H}$ by

$$h(R) = \lim_{T \to +\infty} \frac{1}{T} \log \left(\#\{[g] \in \mathcal{C}(\Gamma) : d_R([g]) \le T\} \right),$$

where $d_R([g]) = \log |\lambda(g)|$. Bridgeman, Canary, Labourie and Sambarino have shown that the entropy is analytic on $\mathcal{H}$:

Theorem 1.2 (Bridgeman, Canary, Labourie and Sambarino [3])**.** *The map $h : \mathcal{H} \to \mathbb{R}$ is real analytic.*

In the particular case $d = 2$ then, as noted above, we always have that $h(R) = 2$ and the result is trivial. (This is because, in this case, we have $\lambda(g) = \exp(l([g])/2)$, where $l([g])$ is the length of the unique closed geodesic in the free homotopy class determined by the conjugacy class $[g]$, the claim then following from the Prime Geodesic Theorem of Huber [13]. This is closely related to the geodesic flow which has entropy one, the factor of two coming from the normalization.) In [3], Bridgeman, Canary, Labourie and Sambarino introduced a generalised Weil-Petersson form and a generalised Weil-Petersson norm on $\mathcal{H}$.

Definition 1.3. The *Weil-Petersson form* is defined on the Hitchin component by

$$I(R_0, R_1) = \lim_{T\to+\infty} \frac{\sum_{d_{R_0}([g])\le T} \frac{d_{R_1}([g])}{d_{R_0}([g])}}{\#\{[g] \in \mathcal{C}(\Gamma) : d_R([g]) \le T\}}.$$

The *normalised* Weil-Petersson form is then defined by

$$J(R_0, R_1) = \frac{h(R_1)}{h(R_0)} I(R_0, R_1).$$

Given an analytic family of representations $R_\lambda \in \mathcal{H}$, $\lambda \in (-\epsilon, \epsilon)$, with expansion

$$R_\lambda = R_0 + \lambda R^{(1)} + o(\lambda),$$

one can define the *Weil-Petersson norm* of the tangent $R^{(1)}$ by

$$\|R^{(1)}\|^2 = \left.\frac{\partial^2 J(R_0, R_\lambda)}{\partial\lambda^2}\right|_{\lambda=0}.$$

A key property of the Weil-Petersson norm is the following.

Theorem 1.4 (Bridgeman, Canary, Labourie and Sambarino [3])**.** *The form J and the norm $\|\cdot\|$ are real analytic.*

We will present short proofs of Theorem 1.2 and Theorem 1.4 in section 5.

Our main result is the following new equivalent definition of the Weil-Petersson norm, which is inspired by Proposition 1.1.

Theorem 1.5. *Let $R_\lambda \in \mathcal{H}$, $\lambda \in (-\epsilon, \epsilon)$ be a (non-constant) analytic family of representations. Then for each $\lambda \in (-\epsilon, \epsilon) \setminus \{0\}$, there exists $0 < \alpha(\lambda) < h(R_0)$ such that*

$$\alpha(\lambda) =$$

$$\lim_{\delta\to 0}\lim_{T\to+\infty} \frac{1}{T}\log\#\left\{[g] : d_{R_0}([g]) \le T \text{ and } \frac{d_{R_\lambda}([g])}{d_{R_0}([g])} \in \left(\frac{h(R_0)}{h(R_\lambda)} - \delta, \frac{h(R_0)}{h(R_\lambda)} + \delta\right)\right\}.$$

Furthermore, if we define $\alpha : (-\epsilon, \epsilon) \to [0, h(R_0)]$ by setting $\alpha(0) = h(R_0)$ then the Weil-Petersson metric is given by

$$\|R^{(1)}\|^2 = -\frac{4}{h(R_0)} \left.\frac{\partial^2 \alpha(\lambda)}{\partial\lambda^2}\right|_{\lambda=0}.$$

The approach of Bridgeman et al in [3] is to use the thermodynamic approach of McMullen [20] (involving the *pressure metric*). In the present note, we will also use the thermodynamic approach, but we introduce two new ingredients which help simplify the analysis. Firstly, we introduce the thermodynamics directly via the strongly Markov structure of Γ and an associated one-sided subshift of finite type, rather than more indirectly via the construction of a flow and the associated symbolic dynamics for that flow. Secondly, we will bypass many of the complications associated with studying the analyticity properties of pressure using Banach manifolds by the introduction of a suitable family of complex functions.

2. Representations and proximality

In this section we discuss the generalisation of the classical Teichmüller theory of representations into $\mathrm{PSL}(2, \mathbb{R})$ to $\mathrm{PSL}(d, \mathbb{R})$ (for $d \ge 3$). In particular, we discuss the Hitchin components and the associate proximality property introduced in the introduction. We then describe the key ideas that link the geometry of the representation space to a readily analysed dynamical system.

There is an irreducible representation of $\iota : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$, induced by the natural action on the space of homogeneous polynomials of degree $d-1$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X,Y) = P(aX+bY, cX+dY),$$

and representations of the form $R = \iota \circ \rho$, with $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ in $\mathcal{T}(V)$ or $\mathcal{T}'(V)$, are called *Fuchsian representations*. More generally, a representation of $R : \Gamma \to \mathrm{PSL}(d,\mathbb{R})$ is said to be in a *Hitchin component* $\mathcal{H}$ if it is in the same connected component of the representation space

$$\mathrm{Rep}(\Gamma, \mathrm{PSL}(d,\mathbb{R})) = \mathrm{Hom}(\Gamma, \mathrm{PSL}(d,\mathbb{R}))/\mathrm{PSL}(d,\mathbb{R})$$

as a Fuchsian representation. (If d is odd there is a single Hitchin component but if d is even there are two Hitchin components.)

For future use, we note that a representation in the Hitchin component can be lifted to a representation over Γ in $\mathrm{SL}(d,\mathbb{R})$. To see this, note first that since Γ is torsion free, a discrete faithful representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ can be lifted to a representation $\tilde{\rho} : \Gamma \to \mathrm{SL}(2,\mathbb{R})$ [6]. Furthermore, ι is actually obtained from a representation $\tilde{\iota} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathrm{R})$ and $\tilde{\iota} \circ \tilde{\rho}$ is a lift of $\iota \circ \rho$. Finally, Theorem 4.1 of [6] tells us that any representation in a Hitchin component also has a lift to $\mathrm{SL}(d,\mathrm{R})$. We will use the same symbol to denote both the original and lifted representations.

We next discuss the notion of a hyperconvex representation and describe how it relates the boundary of the group Γ to something akin to a limit set in $\mathbb{R}P^{d-1}$. The boundary of Γ, denoted $\partial\Gamma$, is the well-defined topological space obtained from the set of (one-sided) infinite geodesic paths in the Cayley graph of Γ by declaring that two paths are equivalent if they remain a bounded distance apart. In the case where Γ is the fundamental group of a compact surface, $\partial\Gamma$ is homeomorphic to S^1.

We recall that a flag space $\mathcal{F}$ for $\mathbb{R}^d$ is a collection of subspaces $V_1 \subset V_2 \subset \cdots \subset V_d$ of $\mathbb{R}^d$ with $\dim(V_i) = i$. There is a natural linear action of each $R(g) \in \mathrm{SL}(d,\mathbb{R})$ on $\mathbb{R}^d$ which induces a corresponding action on the vector subspaces, and thus on the flags.

Definition 2.1. A representation of $R : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ is hyperconvex if there exist Γ-equivariant (Hölder) continuous maps $(\xi,\theta) : \partial\Gamma \to \mathcal{F} \times \mathcal{F}$ such that for distinct $x, y \in \mathcal{F}$ the images $\xi(x) = (V_i(x))_{i=0}^d$ and $\theta(x) = (W_i(x))_{i=0}^d$ satisfy $V_i(x) \oplus W_{d-i}(x) = \mathbb{R}^d$, for $i = 0, \cdots, d$.

By Γ-equivariance we mean that $R(g)\xi(x) = \xi(gx)$, where $R(g)\xi(x)$ is the image under the linear action of $R(g)$ for $g \in \Gamma$.

The following fundamental result of Labourie tells us that the representations in a Hitchin component are hyperconvex.

Proposition 2.2 (Labourie [17])**.** *If $R \in \mathcal{H}$ then R hyperconvex.*

For our purposes it suffices for us to focus on one component of $\xi : \partial\Gamma \to \mathcal{F}$, say, and furthermore take the one dimensional subspace $V_1(x)$ in the flag given by $\xi_0(x) = V_1(x)$, say. This corresponds to a point in projective space and thus we have a Hölder continuous Γ-equivariant map from $\partial\Gamma$ to $\mathbb{R}P^{d-1}$.

Let $R \in \mathcal{H}$. An important consequence of the hyperconvexity of R is that, for each $g \in \Gamma\backslash\{1_\Gamma\}$, the matrix $R(g) \in \mathrm{SL}(d,\mathbb{R})$ is proximal, i.e. it has a unique simple eigenvalue $\lambda(g)$ which is strictly maximal in modulus (and which only depends on the conjugacy class $[g]$) [17], [25]. Since $\det R(g) = 1$, we have $|\lambda(g)| > 1$. As above, we will write $d_R([g]) = \log|\lambda(g)| > 0$.

It will prove important to characterise $d_R([g])$ in terms of the action that $R(g)$ induces on projective space. We can consider the projective action $\widehat{R}(g) : \mathbb{R}P^{d-1} \to$

$\mathbb{R}P^{d-1}$ of the representation $R(g) \in \mathrm{SL}(d, \mathbb{R})$ defined by $\widehat{R}(g)[v] = R(g)v/\|R(g)v\|_2$ (where $v \in \mathbb{R}^d \setminus \{0\}$ is a representative element).

The proximality of $R(g)$ ensures that $\widehat{R}(g) : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ has a unique attracting fixed point $\xi_g \in \mathbb{R}P^{d-1}$. We can use the following simple lemma to relate the weight $d_R([g])$ to the action of $R(g)$ on $\mathbb{R}P^{d-1}$.

Lemma 2.3. *If $g \in \Gamma \setminus \{1_\Gamma\}$ and $\xi_g \in \mathbb{R}P^{d-1}$ is the attracting fixed point for $\widehat{R}(g) : \mathbb{R}P^{d-1} \to \mathbb{R}P^{d-1}$ then*

$$d_R([g]) = -\frac{1}{d} \log \det(D_{\xi_g} \widehat{R}(g)).$$

Proof. We can consider the linear action of $R(g)$ on $\mathbb{R}^d$, then the fixed point corresponds to an eigenvector v and the result follows from a simple calculation using that the linear action of $R(g) \in \mathrm{SL}(d, \mathbb{R})$ preserves area in $\mathbb{R}^d$. More precisely, ξ_g corresponds to an eigenvector v for the maximal eigenvalue $\lambda(g)$, with $|\lambda(g)| > 1$, for the matrix $R(g)$. We can assume without loss of generality that $\|v\| = 1$ and then for arbitrarily small $\delta > 0$ we can consider a δ-neighbourhood of v which is the product of a $(d-1)$-dimensional neighbourhood in $\mathbb{R}P^{d-1}$ and a δ-neighbourhood in the radial direction. The effect of the linear action of $R(g)$ is to replace v by $\lambda(g)v$, and thus stretch the neighbourhood in the radial direction by a factor of $|\lambda(g)|$. Since $R(g)$ has determinant one, the volume of the $(d-1)$-dimensional neighbourhood contracts by $|\lambda(g)|^{-1}$. To calculate the effect of the projective action $\widehat{R}(g)$, we need to rescale $\lambda(g)v$ to have norm one, which corresponds to multiplication by the diagonal matrix $\mathrm{diag}(|\lambda(g)|^{-1}, \ldots, |\lambda(g)|^{-1})$. In particular, the $(d-1)$-dimensional neighbourhood in $\mathbb{R}P^{d-1}$ shrinks by a factor of approximately $|\lambda(g)|^{-d}$, giving the result. □

3. Symbolic dynamics

The structure of the group Γ allows us to code it in terms of a symbolic dynamical system, namely a subshift of finite type. We will describe this and then discuss how the geometric information given by the numbers $d_R([g])$ may also be encoded. This in turn enables use to use the machinery of thermodynamic formalism to define a form of pressure function and hence an associated metric on spaces of representations.

As the fundamental group of a compact orientable surface of genus $g \geq 2$, Γ has the standard presentation

$$\Gamma = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

We write $\Gamma_0 = \{a_1^{\pm 1}, \cdots, a_g^{\pm 1}, b_1^{\pm 1}, \cdots, b_g^{\pm 1}\}$ for the symmetrised generating set.

The surface group Γ is a particular example of a Gromov hyperbolic group and as such it is a strongly Markov group in the sense of Ghys and de la Harpe [8], i.e., they can be encoded using a directed graph and an edge labelling by elements in Γ_0. In the particular case of surface groups, the coding follows directly from the work of Adler and Flatto [1] and Series [27] on coding the action on the boundary and the associated shift of finite type is mixing.

Lemma 3.1. *We can associate to (Γ, Γ_0) a directed graph $G = (V, E)$, with a distinguished vertex $* \in V$, and an edge labelling $\rho : E \to \Gamma_0$, such that:*

1. *no edge terminates at $*$;*
2. *there is at most one directed edge between each ordered pair of vertices;*
3. *the map from the set finite paths in the graph starting at $*$ to $\Gamma \setminus \{e\}$ defined by*

$$(e_1, \ldots, e_n) \mapsto \rho(e_1) \cdots \rho(e_n)$$

is a bijection and $|\rho(e_1)\cdots\rho(e_n)| = n$*;*

(4) *the map from closed paths in* G *(modulo cyclic permutation) to* $\mathcal{C}(\Gamma)$ *induced by* ρ *is a bijection and for such a closed path* $(e_1, \ldots, e_n, e_1)$*,* n *is the minimum word length in the conjugacy class of* $\rho(e_1)\cdots\rho(e_n)$*; and*

(5) *a conjugacy class in* $\mathcal{C}(\Gamma)$ *is primitive (i.e. it does not contain an element of the form* g^n *with* $g \in \Gamma$ *and* $n \in \mathbb{Z}\setminus\{-1,1\}$*) if and only if the corresponding closed path is not a power of a shorter path.*

Furthermore, the subgraph obtained be deleting the vertex $*$ *has the aperiodicity property that there exists* $N \geq 1$ *such that, given any two* $v, v' \in V \setminus \{*\}$*, there is a directed path of length* N *from* v *to* v'*.*

We now introduce a dynamical system. We can associate to the directed graph G a subshift of finite type where the states are labelled by the edges in the graph after deleting the edges that originate in the vertex $*$. In particular, if there are k such edges then we can define a $k \times k$ matrix A by $A(e, e') = 1$ if e' follows e in the directed graph and then define a space

$$\Sigma = \{x = (x_n)_{n=0}^{\infty} \in \{1, \ldots, k\}^{\mathbb{Z}^+} : A(x_n, x_{n+1}) = 1, n \geq 0\},$$

where for convenience we have labelled the edges $1, \ldots, k$. This is a compact space with respect to the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1 - \delta(x_n, y_n)}{2^n}.$$

The shift map is the local homeomorphism $\sigma : \Sigma \to \Sigma$ defined by $(\sigma x)_n = x_{n+1}$. By Lemma 3.1, A is aperiodic (i.e. there exists $N \geq 1$ such that A^N has all entries positive) and, equivalently, the shift $\sigma : \Sigma \to \Sigma$ is mixing (i.e. for all open non-empty $U, V \subset \Sigma$, there exists $N \geq 1$ such that $\sigma^{-n}(U) \cap V \neq \varnothing$ for all $n \geq N$). The periodic orbits for σ correspond exactly to the conjugacy classes in $\mathcal{C}(\Gamma)$ and they are prime if and only if the corresponding conjugacy class is primitive.

There is a natural surjective Hölder continuous map $\pi : \Sigma \to \partial\Gamma$ defined by setting $\pi((x_n)_{n=0}^{\infty})$ to be the equivalence class of the infinite geodesic path $(\rho(x_n))_{n=0}^{\infty}$ in $\partial\Gamma$.

However, the shift $\sigma : \Sigma \to \Sigma$ only encodes information about Γ as an abstract group. In order to keep track of the additional information given by the representation of Γ in $\mathrm{PSL}(d, \mathbb{R})$ we need to introduce a Hölder continuous function $r : \Sigma \to \mathbb{R}$.

Definition 3.2. We can associate a map $r : \Sigma \to \mathbb{R}$ defined by

$$r(x) = -\frac{1}{d} \log \det(D_{\Xi(x)} \widehat{R}(g_{x_0})),$$

(i.e., the Jacobian of the derivative of the projective action) where $\Xi = \xi_0 \circ \pi$ and where $g_{x_0} = \rho(x_0)$ is the generator corresponding to the first term in $x = (x_n)_{n=0}^{\infty} \in \Sigma$.

Given $r : \Sigma \to \mathbb{R}$ and $x \in \Sigma$ we denote $r^n(x) := r(x) + r(\sigma x) + \cdots + r(\sigma^{n-1} x)$ for $n \geq 1$. We now have the following simple but key result.

Lemma 3.3. *The function* $r : \Sigma \to \mathbb{R}$ *is Hölder continuous, and if* $\sigma^n x = x$ *is a periodic point corresponding to an element* $g \in \Gamma$ *then* $r^n(x) = d_R([g])$*.*

Proof. The Hölder continuity of r follows immediately from the Hölder continuity of ξ_0, which in turn comes from Proposition 2.2. The second part of the lemma follows from the equivariance and the observation $\Xi(\sigma x) = R(g_{x_0})\Xi(x)$. Moreover, that the periodic point x has an image $\Xi(x)(= \xi_g)$ which is fixed by $\widehat{R}(g)$ and the result follows from Lemma 2.3. $\square$

The next lemma shows how the analytic dependence of the representations translates into analytic dependence of the associated function r.

Lemma 3.4. *For a C^ω family $(-\epsilon, \epsilon) \ni \lambda \mapsto R_\lambda$ of representations, the associated maps r_λ have a C^ω dependence.*

Proof. The proof is very similar to Proposition 2.2 in [14], which is in turn based on the classical approach of Mather, and the refinement of de la Llave-Marco-Moriyón [7], to showing the existence of, and analytic dependence of, a conjugating (Hölder) homeomorphism between nearby expanding maps on a manifold (i.e., structural stability). Given this similarity, it suffices to only outline the main steps in the proof. The main objective is to construct a natural family of (Hölder) continuous equivariant maps $\Xi_\lambda : \Sigma \to \mathbb{R}P^{d-1}$, that is a family of (Hölder) continuous maps satisfying $R_\lambda(g_{x_0})\Xi_\lambda(x) = \Xi_\lambda(\sigma x)$, for $x \in \Sigma$. Given any $0 < \alpha < 1$, we let $C^\alpha(\Sigma, \mathbb{R}P^{d-1})$ denote the Banach manifold of α-Hölder continuous functions on Σ taking values in the projective space $\mathbb{R}P^{d-1}$. We can now consider the family of maps $H_\lambda : C^\alpha(\Sigma, \mathbb{R}P^{d-1}) \to C^\alpha(\Sigma, \mathbb{R}P^{d-1})$ defined by $H_\lambda(\Xi)(x) = R_\lambda(g_{x_0}^{-1})\Xi(\sigma x)$, for $x \in \Sigma$ with first symbol x_0, and $\Xi \in C^\alpha(\Sigma, \mathbb{R}P^{d-1})$. In particular, providing $0 < \alpha < 1$ is sufficiently small then one can show that for each $\lambda \in (-\epsilon, \epsilon)$ there exists a unique continuous family Ξ_λ which is a fixed point (i.e., $H_\lambda(\Xi_\lambda) = \Xi_\lambda$) and, moreover, the maps $(-\epsilon, \epsilon) \in \lambda \mapsto \Xi_\lambda \in C^\alpha(\Sigma, \mathbb{R}P^{d-1})$ are analytic. This follows from an application of the Implicit Function Theorem. More precisely, in order to apply the Implicit Function Theorem we first observe that we can identify the tangent space $T_v\mathbb{R}P^{d-1}$ at $v \in \mathbb{R}P^{d-1}$ with $\mathbb{R}^{d-1}$. We can then consider the derivative $DH_\lambda : C^\alpha(\Sigma, \mathbb{R}^{d-1}) \to C^\alpha(\Sigma, \mathbb{R}^{d-1})$ which can be defined by $DH_\lambda(\Pi)(x) = Dg_{x_0}^{-1}\Pi(x)$, for $\Pi \in C^\alpha(\Sigma, \mathbb{R}^{d-1})$ and $x \in \Sigma$. For $0 < \alpha < 1$ sufficiently small the hyperbolic nature of $\widehat{R}_\lambda(g_{x_0}^{-1})$ ensures that the operator $(DH_\lambda - I) : C^\alpha(\Sigma, \mathbb{R}^{d-1}) \to C^\alpha(\Sigma, \mathbb{R}^{d-1})$ is invertible. (This is more readily seen in the case of $(DH_\lambda - I) : C(\Sigma, \mathbb{R}^{d-1}) \to C(\Sigma, \mathbb{R}^{d-1})$ on continuous functions, the setting of Mather's original proof, but then the result extends to Hölder functions providing α is sufficiently small, as in the article of de la Llave-Marco-Moriyón [7]). It then follows from the Implicit Function Theorem that there is a unique fixed point Ξ_λ and also that this depends analytically on $\lambda \in (-\epsilon, \epsilon)$. Finally, writing $r_\lambda(x) = \log\det(R_\lambda(g_{x_0}))(\Xi_\lambda(x))$ we see that this too depends analytically on $\lambda \in (-\epsilon, \epsilon)$. □

4. Thermodynamic formalism

In this section we discuss the thermodynamic formalism associated to the map $\sigma : \Sigma \to \Sigma$ and, subsequently, to an associated suspended semiflow. (We refer the reader to [22] for a more detailed account.) We say that two Hölder continuous functions $f_1, f_2 : \Sigma \to \mathbb{R}$ are cohomologous if $f_1 - f_2 = u\circ\sigma - u$, for some continuous $u : \Sigma \to \mathbb{R}$. Then f_1 and f_2 are cohomologous of and only if $f_1^n(x) = f_2^n(x)$ whenever $\sigma^n x = x$, $n \geq 1$.

Let $\mathcal{M}_\sigma$ denote the set of σ-invariant probability measure on Σ. For a Hölder continuous function $f : \Sigma \to \mathbb{R}$, its pressure $P(f)$ is defined by

$$P(f) := \sup_{\mu\in\mathcal{M}_\sigma} \left\{ h_\sigma(\mu) + \int f\, d\mu \right\},$$

where $h_\sigma(\mu)$ denotes the measure-theoretic entropy, and its equilibrium state μ_f is the unique σ-invariant probability measure for which the supremum is attained. If f is not cohomologous to a constant then

$$\mathcal{I}_\sigma(f) := \left\{ \int f\, d\mu : \mu \in \mathcal{M}_\sigma \right\}$$

is a non-trivial closed interval and, for $\xi \in \mathrm{int}(\mathcal{I}(f))$,

$$\sup\left\{ h(\mu) : \mu \in \mathcal{M}_\sigma \text{ and } \int f\, d\mu = \xi \right\} > 0.$$

The following result is standard (see [22]).

Lemma 4.1. *The map $t \mapsto P(tf_1 + f_2)$ is real analytic on $\mathbb{R}$ and satisfies*

$$\left.\frac{dP(tf_1 + f_2)}{dt}\right|_{t=0} = \int f_1 \, d\mu_{f_2}.$$

We will also need some material about suspended semi-flows over $\sigma : \Sigma \to \Sigma$. Let $f : \Sigma \to \mathbb{R}$ be strictly positive and Hölder continuous.

Definition 4.2. We define

$$\Sigma^f = \{(x,s) : x \in \Sigma,\ 0 \le s \le f(x)\}/\sim,$$

where we have quotiented by the relation $(x, f(x)) \sim (\sigma x, 0)$. The associated suspended semiflow $\sigma_t^f : \Sigma^f \to \Sigma^f$, $t \ge 0$, is defined by $\sigma_t^f(x,s) = (x, s+t)$, modulo the identifications.

Let $\mathcal{M}_{\sigma^f}$ denote the set of σ^f-invariant probability measure on Σ^f. Each $m \in \mathcal{M}_{\sigma^f}$ takes the form $dm = (d\mu \times dt)/\int f \, d\mu$, where $\mu \in \mathcal{M}_\sigma$ and their entropies are related by

$$h_{\sigma^f}(m) = \frac{h_\sigma(\mu)}{\int f \, d\mu}.$$

For a Hölder continuous function $G : \Sigma^f \to \mathbb{R}$, its equilibrium state m_G is the unique σ^f-invariant probability measure for which

$$h_{\sigma^f}(m_G) + \int G \, dm_G = P(G) := \sup_{m \in \mathcal{M}_{\sigma^f}} \left\{ h_{\sigma^f}(m) + \int G \, dm \right\}.$$

Then $dm_G = (d\mu_{g - P(G)f} \times dt)/\int f \, d\mu_{g-P(G)f}$, where $g : \Sigma \to \mathbb{R}$ is defined by

$$g(x) = \int_0^{f(x)} G(x,s) \, ds.$$

In particular, σ^f has a unique measure of measure of maximal entropy m_0 for σ^f, i.e. a unique measure m_0 such that

$$h_{\sigma^f}(m_0) = \sup_{m \in \mathcal{M}_{\sigma^f}} h_{\sigma^f}(m).$$

Furthermore, $h_{\sigma^f}(m_0)$ is equal to the topological entropy

$$h(\sigma^f) := \lim_{T \to \infty} \frac{1}{T} \log \left(\sum_{n=1}^{\infty} \# \{\sigma^n x = x : f^n(x) \le T\} \right).$$

This measure is given by $dm_0 = (d\mu_{-h(\sigma^f)f} \times dt)/\int f \, d\mu_{-h(\sigma^f)f}$ and we have

$$h(\sigma^f) = h_{\sigma^f}(m_0) = \frac{h_\sigma(\mu_{-h(\sigma^f)f})}{\int f \, d\mu_{-h(\sigma^f)f}}.$$

The topological entropy is also characterised by the equation $P(-h(\sigma^f)f) = 0$. We have the following analogue of Lemma 4.1 (see Lemma 1 of [28]).

Lemma 4.3. *The map $t \mapsto P(tG_1 + G_2)$ is real analytic on $\mathbb{R}$ and satisfies*

$$\left.\frac{dP(tG_1 + G_2)}{dt}\right|_{t=0} = \int G_1 \, d\mu_{G_2}.$$

If G is not cohomologous to a constant then

$$\mathcal{I}_{\sigma^f}(G) := \left\{ \int G \, dm : m \in \mathcal{M}_\sigma^f \right\}$$

is a non-trivial closed interval. Furthermore,

$$\left\{ \int G \, dm_{tG} : t \in \mathbb{R} \right\} = \operatorname{int}(\mathcal{I}_{\sigma^f}(G)).$$

We use the following large deviation type result.

Lemma 4.4. *Let $f_1, f_2 : \Sigma \to \mathbb{R}$ be strictly positive Hölder continuous functions such that $0 \in \mathrm{int}(\mathcal{I}_\sigma(f_1 - f_2))$. Then*

$$\beta(f_1, f_2) :=$$
$$\lim_{\delta\to 0} \limsup_{T\to\infty} \frac{1}{T} \log \left(\sum_{n=1}^{\infty} \# \left\{ \sigma^n x = x \,:\, f_1^n(x) \le T \text{ and } \frac{f_2^n(x)}{f_1^n(x)} \in (1-\delta, 1+\delta) \right\} \right)$$

satisfies

$$\beta(f_1, f_2) = \sup \left\{ \frac{h(\mu)}{\int f_1\, d\mu} \,:\, \mu \in \mathcal{M}_\sigma,\ \int f_1\, d\mu = \int f_2\, d\mu \right\}.$$

In particular, $0 < \beta(f_1, f_2) \le h := h(\sigma^{f_1})$ and $\beta(f_1, f_2) = h$ if and only if $\int f_1\, d\mu_{-hf_1} = \int f_2\, d\mu_{-hf_1}$, where μ_{-hf_1} is the equilibrium state for $-hf_1$.

Proof. We apply results about periodic orbits for hyperbolic flows, which also apply to suspended semiflows over subshifts of finite type. We have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \# \left\{ \sigma^n x = x \,:\, f_1^n(x) \le T \text{ and } \frac{f_2^n(x)}{f_1^n(x)} \in (1-\delta, 1+\delta) \right\} \\
&= \# \left\{ \tau \,:\, l(\tau) \le T \text{ and } \int F\, dm_\tau \in (1-\delta, 1+\delta) \right\},
\end{aligned}$$

where τ denotes a periodic orbit of the suspended semi-flow σ^{f_1} with least period $l(\tau)$, m_τ is the corresponding orbital measure (of total mass $l(\tau)$) and $F : \Sigma^{f_1} \to \mathrm{R}$ satisfies $\int F\, dm_\tau = f_2^n(x)$. Using Kifer's large deviations results for hyperbolic flows [15], we have

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \log \# \left\{ \tau \,:\, l(\tau) \le T \text{ and } \frac{1}{l(\tau)} \int F\, dm_\tau \in (1-\delta, 1+\delta) \right\} \\
&= \sup \left\{ h(m) \,:\, m \in \mathcal{M}_{\sigma^{f_1}} \text{ and } \int F\, dm \in (1-\delta, 1+\delta) \right\} \\
&= \sup \left\{ \frac{h(\mu)}{\int f_1\, d\mu} \,:\, \mu \in \mathcal{M}_\sigma \text{ and } \frac{\int f_2\, d\mu}{\int f_1\, d\mu} \in (1-\delta, 1+\delta) \right\} \\
&= \sup_{\xi \in (1-\delta, 1+\delta)} H(\xi),
\end{aligned}$$

where

$$H(\xi) = \sup \left\{ \frac{h(\mu)}{\int f_1\, d\mu} \,:\, \mu \in \mathcal{M}_\sigma \text{ and } \frac{\int f_2\, d\mu}{\int f_1\, d\mu} = \xi \right\}.$$

Since $H(\xi)$ is analytic, letting $\delta \to 0$ gives the required formula for $\beta(f_1, f_2)$. That $\beta(f_1, f_2) \le h$ is immediate and $\beta(f_1, f_2) > 0$ follows from $0 \in \mathrm{int}(\mathcal{I}_\sigma(f_1 - f_2))$, since $\int f_2\, d\mu / \int f_1\, d\mu = 1$ is equivalent to $\int f_1 - f_2\, d\mu = 0$. If $\int f_1\, d\mu_{-hf_1} = \int f_2\, d\mu_{-hf_1}$ then it is clear that $\beta(f_1, f_2) = h$. On the other hand, if

$$h = \beta(f_1, f_2) = \frac{h_\sigma(\mu)}{\int f_1\, d\mu},$$

for some $\mu \in \mathcal{M}_\sigma$, then

$$h_\sigma(\mu) - h \int f_1\, d\mu = 0 = P(-hf_1)$$

so uniqueness of equilibrium states gives $\mu = \mu_{-hf_1}$. This completes the proof. $\square$

5. Analyticity of the metric and the entropy

In this section we will establish the analyticity of the metric and the entropy. We will do this by considering certain complex functions, which provides a fairly direct proof avoiding the use of Lemma 3.4. We want to establish analyticity of the Weil-Petersson form and metric by using the analytic function $\eta(s, R_0, R_\lambda)$ defined below, where R_λ depends analytically on λ.

We begin by establishing the analyticity of individual weights $d_{R_\lambda}([g])$ as functions of λ.

Lemma 5.1. *For each $[g] \in \mathcal{C}(\Gamma)$, the weight $d_{R_\lambda}([g]) \in \mathbb{R}$ has a real analytic dependence on $\lambda \in (-\epsilon, \epsilon)$. Moreover, we can choose an open neighbourhood $(-\epsilon, \epsilon) \subset U \subset \mathbb{C}$ so that we have an analytic extension $U \ni \lambda \mapsto d_{R_\lambda}([g]) \in \mathbb{C}$ for each $[g] \in \mathcal{C}(\Gamma)$.*

Proof. We need only modify the approach in Proposition 1.1 of [14]. For each generator $g_0 \in \Gamma_0$ we can consider the image $X_{g_0} \subset \mathbb{R}P^{d-1}$ of the corresponding 1-cylinder $[x_0] \subset \Sigma$, say. In particular X_{g_0} is a compact set in $\mathbb{R}P^{d-1}$. Since $\mathbb{R}P^{d-1}$ is a real analytic manifold it has a (local) complexification and we can then choose a (small) neighbourhood $U_{g_0} \supset X_{g_0}$ in this complexification of $\mathbb{R}P^{d-1}$. We will still denote by $R_\lambda(g_0)^{-1}$ the unique extension of the action of $R(g_0)^{-1}$ to the neighbourhood $U_{g_0} \supset X_{g_0}$. Providing the neighbourhoods U_{g_0} are sufficiently small we have by continuity of the extension $R_\lambda(g_0)^{-1}$ that $R_\lambda(g_0)^{-1}U_{g_0} \supset \overline{U}_{g_1}$, for $\lambda \in (-\epsilon, \epsilon)$, where $g_1 \in \Gamma_0$ satisfies $R_\lambda(g_0)^{-1}X_{g_0} \supset X_{g_1}$, since we know that the restriction $R_\lambda(g_0)^{-1}|X_{g_0}$ is a contraction. Moreover, by continuity and by choosing U_{g_0} smaller, if necessary, we can assume that the inclusion $R_\lambda(g_0)^{-1}U_{g_0} \supset \overline{U}_{g_1}$ also holds for each g_0 for the complexification of R_λ for λ lying in a suitably small open subset $\mathbb{C} \supset V \supset (-\epsilon, \epsilon)$, say.

The key observation now is that when we extend these inclusions to conjugacy classes of more general elements $g \in \Gamma \setminus \{1_\Gamma\}$ without further reducing the neighbourhood $(-\epsilon, \epsilon) \subset V \subset \mathbb{C}$. More precisely, for each reduced word $g = g_{i_0} \cdots g_{i_{n-1}}$ (where $g_{i_0}, \ldots, g_{i_{n-1}} \in \Gamma_0$) we have from the above construction that $R_\lambda(g)^{-1}U_{g_0} \supset \overline{U}_{g_{n-1}}$ for $\lambda \in V$. Moreover, writing $\xi_g^\lambda \in \mathbb{R}P^{d-1}$ for the fixed point for $R_\lambda(g_\lambda)^{-1}$, we see that $V \ni \lambda \mapsto \xi_g^\lambda$ is analytic and $V \ni \lambda \mapsto d_{R_\lambda}([g]) = -\frac{1}{2}\log\det(D_{\xi_g^\lambda}\widehat{R}_\lambda(g)) \in \mathbb{C}$ is analytic as the sum of analytic terms. In particular, these functions are analytic on the region V. □

We now define a complex function using these weights.

Definition 5.2. We can associate to the two representations $R_0, R_\lambda \in \mathcal{H}$ a complex function

$$\eta(s, R_0, R_\lambda) = \sum_{[g]} d_{R_\lambda}([g]) e^{-s d_{R_0}([g])}$$

which converges for $\mathrm{Re}(s)$ sufficiently large.

From now on, we shall write $h(R_0) = h$.

Lemma 5.3. *The function $\eta(s, R_0, R_\lambda)$ is analytic for $\mathrm{Re}(s) > h$. Moreover, $s = h$ is a simple pole with residue equal to*

$$\frac{\int r_\lambda \, d\mu_{-hr_0}}{\int r_0 \, d\mu_{-hr_0}},$$

where r_0, r_λ correspond to R_0, R_λ using Lemma 3.3. In particular, $\eta(s, R_\lambda, R_0)$ has a simple pole at $h(R_\lambda)$.

Proof. We will write $\mathcal{C}'(\Gamma) \subset \mathcal{C}(\Gamma)$ for the set of primitive conjugacy classes in Γ. We can associate to R_0 and R_λ a zeta function formally defined by

$$\zeta(s,z,R_0,R_\lambda) = \prod_{[g]\in\mathcal{C}'(\Gamma)} \left(1 - e^{-sd_{R_0}([g])+zd_{R_\lambda}([g])}\right)^{-1}, \text{ for } s \in \mathbb{C}, z \in \mathbb{R},$$

which converges for $\mathrm{Re}(s)$ sufficiently large and $|z|$ sufficiently small (depending on s). We can rewrite this in terms of the shift $\sigma : \Sigma \to \Sigma$ and the functions r_0, r_λ as

$$\zeta(s,z,R_0,R_\lambda) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma^n x = x} e^{-sr_0^n(x) - zr_\lambda^n(x)}\right).$$

(Here we use the fact that primitive conjugacy classes correspond to prime periodic orbits for the shift map and then there is convergence to an analytic function $P(-\mathrm{Re}(s)r_0 - zr_\lambda) < 0$ [22].) Using the analysis of [22], we see that $\zeta(s,z,R_0,R_\lambda)$ converges for $\mathrm{Re}(s) > h$. Furthermore, for s close to h and z close to zero,

$$\zeta(s,z,R_0,R_\lambda) = \frac{A(s,z)}{1 - e^{P(-sr_0+zr_\lambda)}},$$

where $A(s,z)$ is non-zero and analytic and $e^{P(-sr_0+zr_\lambda)}$ is the standard analytic extension of the exponential of the pressure function to complex arguments (obtained via perturbation theory applied to the maximal eigenvalue of the associated transfer operator cf. [22]).

It is easy to show that

$$\eta(s,R_0,R_\lambda) = \frac{\partial}{\partial z} \log \zeta(s,z,R_0,R_\lambda)\Big|_{z=0} + \phi(s),$$

where $\phi(s)$ is analytic for $\mathrm{Re}(s) > h/2$, while, for s close to h,

$$\begin{aligned} \frac{\partial}{\partial z} \log \zeta(s,z,R_0,R_\lambda)\Big|_{z=0} &= \frac{\partial A(s,z)/\partial z|_{z=0}}{A(s,0)} + \frac{\frac{\partial P(-sr_0+zr_\lambda)}{\partial z}\Big|_{z=0}}{1 - e^{P(-sr_0)}} \\ &= \frac{\int r_\lambda \, d\mu_{-hr_0}}{\int r_0 \, d\mu_{-hr_0}} \frac{1}{s-h} + B(s) \end{aligned}$$

where $B(s)$ is analytic in a neighbourhood of $s = h$. The final statement follows by reversing the roles of R_0 and R_λ. □

We have the following result (which implies Theorem 1.2)

Lemma 5.4. *The function $(-\epsilon,\epsilon) \ni \lambda \mapsto h(R_\lambda)$ is real analytic.*

Proof. By Lemma 5.1, the function $1/\zeta(s,\lambda)$, where $\zeta(s,\lambda) := \zeta(s,0,R_\lambda,R_\lambda)$, has an analytic dependence on $\lambda \in (-\epsilon,\epsilon)$ for $\mathrm{Re}(s)$ sufficiently large. It follows from [23] that, for each $\lambda \in (-\epsilon,\epsilon)$, $1/\zeta(s,\lambda)$ has an analytic extension to a half plane $\mathrm{Re}(s) > \nu(\lambda)$, where $\nu(\lambda) < h(R_\lambda)$ depends continuously on λ. We can therefore find a common domain $\mathcal{D}$, containing $\bigcup_{-\epsilon<\lambda<\epsilon}\{s \in \mathbb{C} : \mathrm{Re}(s) \geq h(R_\lambda)\}$, such that $1/\zeta(s,\lambda)$ is separately analytic for $s \in \mathcal{D}$ and $\lambda \in (-\epsilon,\epsilon)$. We may then apply Theorem 1 of [30] to conclude that $(s,\lambda) \mapsto 1/\zeta(s,\lambda)$ is real analytic on $\mathcal{D} \times (-\epsilon,\epsilon)$. Finally, we can use the Implicit Function Theorem to show that $\lambda \mapsto h(R_\lambda)$ is real analytic. □

In order to establish further analyticity results, we need to show that the intersection $I(R_0,R_\lambda)$ is equal to the residue of $\eta(s,R_0,R_\lambda)$ at $s = h$. To do this, it will be convenient to use the following technical result.

Lemma 5.5. *Let $R \in \mathcal{H}$. Then there does not exist $\alpha > 0$ such that $\{d_R([g]) : g \in \Gamma \setminus \{1_\Gamma\}\} \subset \alpha\mathbb{Z}$.*

Proof. Let $g, h \in \Gamma \setminus \{1_\Gamma\}$ be two distinct elements of the group. For any $N > 0$ we can consider $g^N, h^N \in \Gamma$. The linear maps on $\mathbb{R}^d$ for the associated matrices $R(g^N), R(h^N) \in \mathrm{SL}(d, \mathbb{R})$ can be written in the form $\lambda(g)^N \pi_g + U_{g^N}$ and $\lambda(h)^N \pi_h + U_{h^N}$, respectively, where $\lambda(g), \lambda(h)$ are the largest simple eigenvalues, $\pi_g, \pi_h : \mathbb{R}^d \to \mathbb{R}^d$ are the eigenprojections onto their one dimensional eigenspaces, $\limsup_{N\to+\infty} \|U_{g^N}\|^{1/N} < \lambda(g)$ and $\limsup_{N\to+\infty} \|U_{h^N}\|^{1/N} < \lambda(h)$.

Let us now consider $g^N h^N \in \Gamma$ and and associated matrix $R(g^N h^N)$. The associated linear map will be of the form $\lambda(g^N h^N)\pi_{g^N h^N} + U_{g^N h^N}$ where $\lambda(g^N h^N)$ is the largest simple eigenvalue, $\pi_{g^N h^N} : \mathbb{R}^d \to \mathbb{R}^d$ is the eigenprojection onto their one dimensional eigenspaces, and $\limsup_{N\to+\infty} \|U_{g^N h^N}\|^{1/N} < \lambda(g^N h^N)$. However, since we have the identity $R(g^N h^N) = R(g^N)R(g^N)$ for the matrix representations we can also write the corresponding relationship for the linear maps:

$$\lambda(g^N h^N)\pi_{g^N h^N} + U_{g^N h^N} = \left(\lambda(g^N)\pi_{g^N} + U_{g^N}\right)\left(\lambda(h^N)\pi_{h^N} + U_{h^N}\right). \tag{5.1}$$

In particular, we see that as N becomes larger

$$\lim_{N\to+\infty} \exp\left((d_R([g^N h^N]) - d_R([g^N]) - d_R([h^N])\right) = \lim_{N\to+\infty} \frac{\lambda(g^N h^N)}{\lambda(g^N)\lambda(h^N)} = \langle \pi_h, \pi_g \rangle$$

where $\langle \pi_h, \pi_g \rangle$ is simply the cosine of the angle between the eigenvectors associated to $\lambda(g)$ and $\lambda(h)$, respectively. However, if we assume for a contradiction that the conclusion of the lemma does not hold, then the right hand side of (5.1) must be of the form $e^{n\alpha}$, for some $n \in \mathbb{Z}$. However, the directions for the associated eigenprojections form an infinite set in $\mathbb{R}P^{d-1}$ and have an accumulation point. Thus for suitable choices of g, h we can arrange that $0 < \langle \pi_h, \pi_g \rangle < e^\alpha$, leading to a contradiction. This completes the proof of the lemma. □

Corollary 5.6. *Apart from the simple pole at $s = h$, $\eta(s, R_0, R_\lambda)$ has an analytic extension to a neighbourhood of* $\mathrm{Re}(s) \geq h$.

Proof. Given Lemma 5.5, it follows from the analysis of [22] that $\zeta(s, z, R_0, R_\lambda)$ has an analytic and non-zero extension to a neighbourhood of each point $s = h + it$, $t \neq 0$, for $|z|$ sufficiently small depending on s. Using again that

$$\eta(s, R_0, R_\lambda) = \frac{\partial}{\partial z} \log \zeta(s, z, R_0, R_\lambda)\bigg|_{z=0} + \phi(s),$$

where $\phi(s)$ is analytic for $\mathrm{Re}(s) > h/2$, we obtain the result. □

We now have the following result which characterises the intersection number of $I(R_0, R_\lambda)$.

Lemma 5.7. *We can write*

$$I(R_0, R_\lambda) = \frac{\int r_\lambda \, d\mu_{-hr_0}}{\int r_0 \, d\mu_{-hr_0}}$$

Proof. Recall from Lemma 5.3 that the right hand side in the statement is the residue of $\eta(s, R_0, R_\lambda)$ at $s = h$. In view of Corollary 5.6, we can apply the Ikehara Tauberian theorem to $\eta(s, R_0, R_\lambda)$ to deduce that

$$\sum_{d_{R_0}([g]) \leq T} d_{R_\lambda}([g]) \sim \frac{\int r_\lambda \, d\mu_{-hr_0}}{\int r_0 \, d\mu_{-hr_0}} e^{hT}, \text{ as } T \to +\infty.$$

Moreover, upon taking $R_\lambda = R_0$, we deduce that

$$\sum_{d_{R_0}([g]) \leq T} d_{R_0}([g]) \sim e^{hT}, \text{ as } T \to +\infty.$$

An elementary argument given in [21] shows that

$$\lim_{T\to+\infty} \frac{\sum_{d_{R_0}([g])\leq T} \frac{d_{R_\lambda}([g])}{d_{R_0}([g])}}{\sum_{d_{R_0}([g])\leq T} 1} = \lim_{T\to+\infty} \frac{\sum_{d_{R_0}([g])\leq T} d_{R_\lambda}([g])}{\sum_{d_{R_0}([g])\leq T} d_{R_0}([g])},$$

so that

$$I(R_0, R_\lambda) = \frac{\int r_\lambda \, d\mu_{-hr_0}}{\int r_0 \, d\mu_{-hr_0}},$$

as required. □

We can now use the characterisation of $I(R_0, R_\lambda)$ in terms of a complex function to deduce the following.

Lemma 5.8. *The function $(-\epsilon, \epsilon) \to \mathbb{R} : \lambda \mapsto I(R_0, R_\lambda)$ is real analytic.*

Proof. By Lemma 5.1, $\eta(s, R_0, R_\lambda)$ has an analytic dependence on $\lambda \in U$. More precisely, it is a uniformly convergent series with individually analytic terms in $\lambda \in U$ for $\mathrm{Re}(s) > h$ and thus bianalytic for $\lambda \in U$. Moreover, by Hartogs' Theorem for functions of several complex variables [16], $1/\eta(s, R_0, R_\lambda)$ is bi-analytic for s in a neighbourhood of h and $\lambda \in U$. Thus the residue of $\eta(s, R_0, R_\lambda)$ at $s = h$ is analytic. Thus, using the residue theorem, $I(R_0, R_\lambda)$, which is the residue of $\eta(s, R_0, R_\lambda)$, depends analytically on λ. □

Since $h(R_\lambda)$ and $I(R_0, R_\lambda)$ both depend analytically on λ, we have the following.

Corollary 5.9. *The function $(-\epsilon, \epsilon) \to \mathbb{R} : \lambda \mapsto J(R_0, R_\lambda)$ is real analytic.*

By differentiating twice and using that $\|R^{(1)}\|^2 = \left.\frac{\partial^2 J(R_0, R_\lambda)}{\partial \lambda^2}\right|_{\lambda=0}$ we have the following result.

Corollary 5.10. *The function $(-\epsilon, \epsilon) \to \mathbb{R} : \lambda \mapsto \|R^{(1)}\|$ is real analytic.*

6. Proof of Theorem 1.5

The first part of Theorem 1.5 will follow from Lemma 4.4 once we formulate things appropriately. Given an analytic family of representations $\lambda \mapsto R_\lambda$, we define strictly positive Hölder continuous functions $r_\lambda : \Sigma \to \mathbb{R}$ as in section 3 so that if $\sigma^n x = x$ corresponds to a conjugacy class $[g]$ then $d_{R_\lambda}([g]) = r_\lambda^n(x)$, using Lemma 3.3. By Lemma 3.4, r_λ depends analytically on λ. We then have $h(\sigma^{r_0}) = h(R_0)$ and $h(\sigma^{r_\lambda}) = h(R_\lambda)$. We now define $f_0 = h(R_0)r_0$ and $f_\lambda = h(R_\lambda)r_\lambda$, so that, in particular, $P(-f_0) = P(-f_\lambda) = 0$. Since periodic point measures are dense in $\mathcal{M}_\sigma$, it is clear that $0 \in \mathrm{int}(\mathcal{I}_\sigma(f_0 - f_\lambda))$ if and only if there exist two conjugacy classes $[g]$ and $[g']$ such that $h(R_0)d_{R_0}([g]) < h(R_\lambda)d_{R_\lambda}([g])$ and $h(R_0)d_{R_0}([g']) > h(R_\lambda)d_{R_\lambda}([g'])$. We will show that this latter condition holds provided the representations R_0 and R_λ are not equal up to conjugacy.

Lemma 6.1. *If R_0 and R_λ are not conjugate then there exist two conjugacy classes $[g]$ and $[g']$ such that we have $h(R_0)d_{R_0}([g]) < h(R_\lambda)d_{R_\lambda}([g])$ and $h(R_0)d_{R_0}([g']) > h(R_\lambda)d_{R_\lambda}([g'])$.*

Proof. We will prove the contrapositive. Without loss of generality, suppose that $h(R_0)d_{R_0}([g]) \leq h(R_1)d_{R_\lambda}([g])$ for all $[g] \in \mathcal{C}(\Gamma)$, i.e. that $(f_0 - f_\lambda)^n(x) \leq 0$ whenever $\sigma^n x = x$. Then $\int (f_0 - f_\lambda)\, d\mu \leq 0$ for every $\mu \in \mathcal{M}_\sigma$.

Now consider the real analytic map $Q : [0,1] \to \mathbb{R}$ defined by $Q(t) = P(-f_0 + t(f_0 - f_\lambda)$. This has derivative $Q'(t) = \int (f_0 - f_\lambda)\, d\mu_t \leq 0$, where μ_t is the equilibrium state for $-f_0 + t(f_0 - f_\lambda)$. Since $Q(0) = Q(1) = 0$ we deduce that $Q(t) = 0$ for all $t \in [0,1]$ and then the strict convexity of pressure implies that $f_0 - f_\lambda$ is cohomologous to a constant. Since $P(f_0) = P(-f_\lambda)$, the constant must be zero and so $f_0^n(x) = f_\lambda^n(x)$, whenever $\sigma^n x = x$. This implies that $h(R_0)d_{R_0}([g]) =$

$h(R_\lambda)d_{R_\lambda}([g])$ for all g and hence that $J(R_0, R_\lambda) = 1$. It then follows by Corollary 1.5 of [3] that the representations are equal up to conjugacy. □

Write $h = h(R_0)$. We may now apply Lemma 4.4 to show that, for each $\lambda \in (-\epsilon, \epsilon)$, the limit

$$\begin{aligned}
&\alpha(\lambda) \\
&= \lim_{\delta\to 0}\lim_{T\to+\infty}\frac{1}{T}\log\#\left\{[g] : d_{R_0}([g]) \le T \text{ and } \frac{d_{R_\lambda}([g])}{d_{R_0}([g])} \in \left(\frac{h(R_0)}{h(R_\lambda)} - \delta, \frac{h(R_0)}{h(R_\lambda)} + \delta\right)\right\} \\
&= \lim_{\delta\to 0}\limsup_{T\to\infty}\frac{1}{T}\log\left(\sum_{n=1}^{\infty}\#\left\{\sigma^n x = x : \frac{f_0^n(x)}{h} \le T \text{ and } \frac{f_\lambda^n(x)}{f_0^n(x)} \in (1-\delta, 1+\delta)\right\}\right) \\
&= h\beta(f_0, f_\lambda)
\end{aligned}$$

exists and satisfies $0 < \alpha(\lambda) \le h$. (Here we have used that $h(\sigma^{f_0}) = 1$.) The next result shows that we have a strict inequality when $\lambda \neq 0$.

Lemma 6.2. *For $\lambda \in (-\epsilon, \epsilon) \setminus \{0\}$, $\alpha(\lambda) < h$.*

Proof. By Proposition 4.4, we will have $\alpha(\lambda) < h$ unless $\int f_0\, d\mu_{-f_0} = \int f_\lambda\, d\mu_{-f_0}$. The latter condition may be rewritten as

$$\frac{\int f_\lambda\, d\mu_{-f_0}}{\int f_0\, d\mu_{-f_0}} = 1 = \frac{h(\sigma^{f_0})}{h(\sigma^{f_\lambda})} = \frac{h_\sigma(\mu_{-f_0})}{\int f_0\, d\mu_{-f_0}}\frac{\int f_\lambda\, d\mu_{-f_\lambda}}{h_\sigma(\mu_{-f_\lambda})}.$$

Rearranging, this becomes

$$\frac{h_\sigma(\mu_{-f_\lambda})}{\int f_\lambda\, d\mu_{-f_\lambda}} = \frac{h_\sigma(\mu_{-f_0})}{\int f_\lambda\, d\mu_{-f_0}},$$

which, by uniqueness of the measure of maximal entropy for σ^{f_λ}, forces $\mu_{-f_0} = \mu_{-f_\lambda}$. The latter equality implies that $f_0 - f_\lambda$ is cohomologous to a constant and, since $P(-f_0) = P(-f_\lambda)$, the constant is necessarily zero. This means that $h(R_0)d_{R_0}([g]) = h(R_\lambda)d_{R_\lambda}([g])$ for all $[g] \in \mathcal{C}(\Gamma)$, contradicting Lemma 6.1. □

We now complete the proof of Theorem 1.5 by establishing the characterisation of the Weil-Petersson metric in terms of the growth rate $\alpha(\lambda)$. It is more convenient to work with $\beta(\lambda) := \beta(f_0, f_\lambda) = \alpha(\lambda)/h$. For $t \in \mathbb{R}$, consider the pressure $P(-tf_0 - f_\lambda)$ and define $\chi_\lambda(t)$ by the equation $P(-tf_0 - \chi_\lambda(t)f_\lambda) = 0$. We trivially have $\chi_0(t) = 1 - t$ but we are interested in the function when $\lambda \neq 0$.

Lemma 6.3. *For each $\lambda \in (-\epsilon, \epsilon) \setminus \{0\}$, the function $\chi_\lambda(t)$ is well-defined and real analytic. Furthermore,*

$$\lim_{t\to\pm\infty}\chi_\lambda(t) = \mp\infty.$$

Proof. That $\chi_\lambda(t)$ is well-defined and real analytic follows from the Implicit Function Theorem. Suppose $\lim_{t\to+\infty}\chi_\lambda(t) \neq -\infty$. Then there exists a sequence $t_n \to +\infty$ and a constant $A \ge 0$ such that $\chi_\lambda(t_n) \ge -A$ for all n. We have

$$-t_n f_0 - \chi_\lambda(t_n)f_\lambda \le -t_n f_0 + A\|f_\lambda\|_\infty$$

and so

$$0 = P(-t_n f_0 - \chi_\lambda(t_n)f_\lambda) \le P(-t_n f_0 + A\|f_\lambda\|_\infty) = P(-t_n f_0) + A\|f_\lambda\| \to -\infty,$$

as $n \to \infty$, a contradiction. A similar argument show that $\lim_{t\to-\infty}\chi_\lambda(t) = +\infty$. □

We want to show that there is a unique number $0 < t_\lambda < 1$ for which $\chi_\lambda'(t_\lambda) = -1$. To do this, it is convenient to use the following alternative characterisation of $\chi_\lambda(t)$ in terms of the semiflow σ^{f_0}. Let $F_\lambda : \Sigma^{f_0} \to \mathbb{R}$ be a Hölder continuous function such that, for a periodic σ^{f_0}-orbit τ corresponding to a periodic σ-orbit $\sigma^n x = x$, $\int F_\lambda\, dm_\tau = f_\lambda^n(x)$. Here, as above, m_τ is the associated periodic orbit measure of total mass $l(\tau) = f_0^n(x)$. It is easy to construct such an F_λ and this also satisfies $\int F_\lambda \, dm > 0$, for every $m \in \mathcal{M}_{\sigma^{f_0}}$. We then have that $\chi_\lambda(t)$ is defined by

$$P(-\chi_\lambda(t)F_\lambda) = t.$$

It is then easy to calculate that

$$\chi_\lambda'(t) = \frac{-1}{\int F_\lambda \, dm_{-\chi_\lambda(t)F_\lambda}}.$$

In particular, $\chi_\lambda(t)$ is strictly decreasing. By Lemma 6.3, χ_λ takes all real values and so

$$\left\{ \int F_\lambda \, dm_{-\chi_\lambda(t)F_\lambda} \, : \, t \in \mathbb{R} \right\} = \left\{ \int F_\lambda \, dm_{tF_\lambda} \, : \, t \in \mathbb{R} \right\} = \mathrm{int}(\mathcal{I}_{\sigma^{f_0}}).$$

However, by Lemma 6.1, we can find periodic σ^{f_0}-orbits τ and τ' (corresponding to conjugacy classes $[g]$ and $[g']$) such that

$$\frac{1}{l(\tau)} \int F_\lambda \, dm_\tau > 1 \quad \text{and} \quad \frac{1}{l(\tau')} \int F_\lambda \, dm_{\tau'} < 1.$$

Hence, in particular, for $\lambda \neq 0$, there exists a unique t_λ such that $\chi_\lambda'(t_\lambda) = -1$.

Lemma 6.4. *We have $\beta(\lambda) = t_\lambda + \chi_\lambda(t)$.*

Proof. By Proposition 4.4, we have

$$\beta(\lambda) = \sup \left\{ \frac{h_\sigma(\mu)}{\int f_0 \, d\mu} \, : \, \mu \in \mathcal{M}_\sigma \text{ and } \int f_0 \, d\mu = \int f_\lambda \, d\mu \right\}.$$

Let ν denote the equilibrium state of $-t_\lambda f_0 - \chi_\lambda(t_\lambda) f_\lambda$. By the definition of t_λ,

$$\frac{\int f_\lambda \, d\nu}{\int f_0 \, d\nu} = \int F_\lambda \, dm_{-\chi_\lambda(t_\lambda)F_\lambda} = 1.$$

Thus

$$\begin{aligned} 0 = P(-t_\lambda f_0 - \chi_\lambda(t_\lambda) f_\lambda) &= h_\sigma(\nu) - t_\lambda \int f_0 \, d\nu - \chi_\lambda(t_\lambda) \int f_\lambda \, d\nu \\ &= h_\sigma(\nu) - (t_\lambda + \chi_\lambda(t_\lambda)) \int f_0 \, d\nu, \end{aligned}$$

so that

$$t_\lambda + \chi_\lambda(t_\lambda) = \frac{h_\sigma(\nu)}{\int f_0 \, d\nu}$$

and $\int f_0 \, d\nu = \int f_\lambda \, d\nu$. On the other hand, if $\mu \in \mathcal{M}_\sigma$, $\mu \neq \nu$ satisfies $\int f_0 \, d\mu = \int f_\lambda \, d\mu$ then

$$\begin{aligned} 0 = P(-t_\lambda f_0 - \chi_\lambda(t_\lambda) f_\lambda) &> h_\sigma(\mu) - t_\lambda \int f_0 \, d\mu - \chi_\lambda(t_\lambda) \int f_\lambda \, d\mu \\ &= h_\sigma(\mu) - (t_\lambda + \chi_\lambda(t_\lambda)) \int f_0 \, d\mu, \end{aligned}$$

so that

$$t_\lambda + \chi_\lambda(t_\lambda) > \frac{h_\sigma(\nu)}{\int f_0 \, d\nu}.$$

Combining these two observations shows that $t_\lambda + \chi_\lambda(t_\lambda) = \beta(\lambda)$. $\square$

Since $\lambda \mapsto r_\lambda$ and $\lambda \mapsto h(R_\lambda)$ are analytic, we can write $f_\lambda = f_0 + f_0^{(1)}\lambda + f_0^{(2)}\lambda^2/2 + o(\lambda^2)$. It follows from the definition of the Weil-Petersson metric in terms of $J(R_0, R_\lambda)$ and Lemma 5.7 that

$$\|R^{(1)}\|^2 = \frac{\int f_0^{(2)} \, d\mu_{-f_0}}{\int f_0 \, d\mu_{-f_0}}.$$

We may then use the calculation in the proof of Lemma 4.2 of [24] to show that

$$\frac{\partial^2 \chi_\lambda}{\partial \lambda^2}(t)\bigg|_{\lambda=0} = t(t-1)\|R^{(1)}\|^2.$$

The next lemma establishes the final part of Theorem 1.5.

Lemma 6.5. *The function* $\alpha : (-\epsilon, \epsilon) \to (0, h(R_0)]$ *satisfies*

$$\|R^{(1)}\|^2 = -4\, \frac{\partial^2 \beta(\lambda)}{\partial \lambda^2}\bigg|_{\lambda=0} = -\frac{4}{h(R_0)}\, \frac{\partial^2 \alpha(\lambda)}{\partial \lambda^2}\bigg|_{\lambda=0}.$$

Proof. This follows from the calculations in the proof of Theorem 4.3 of [24], once one replaces the function D_λ there with $\beta(\lambda)$, combined with Lemma 6.4. □

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