Quantitative stability and numerical analysis of Markovian quadratic BSDEs with reflection*

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Abstract

We study the quantitative stability of solutions to Markovian quadratic reflected backward stochastic differential equations (BSDEs) with bounded terminal data. By virtue of bounded mean oscillation martingale and change of measure techniques, we obtain stability estimates for the variation of the solutions with different underlying forward processes. In addition, we propose a truncated discrete-time numerical scheme for quadratic reflected BSDEs and obtain the explicit rate of convergence by applying the quantitative stability result.

Keywords: Quadratic BSDE with reflection, stability of solutions, discretely reflected BSDE, rate of convergence

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1 Introduction

In this study, we are interested in the stability of solutions to the following quadratic reflected backward stochastic differential equations (BSDEs) under Markovian framework

$$\begin{aligned} &Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \\ &Y_t \geqslant g(X_t), \quad \int_0^T (Y_t - g(X_t))dK_t = 0, \end{aligned} \tag{1.1}$$

where $T > 0$ is a fixed finite time horizon, and the underlying forward process solves

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s)dW_s, \quad t \in [0, T]. \tag{1.2}$$

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Herein, $\{W_t\}_{t\geqslant 0}$ is an m-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t\geqslant 0}$ is the augmented natural filtration of W, which satisfies the usual conditions. Let $\mathcal{P}$ denote the progressively measurable σ-field on $[0,T]\times\Omega$.

We assume that all coefficients b, σ, g and f are deterministic and continuous functions and $b : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \to \mathbb{R}^{n\times m}$ satisfy, for all $t \in [0,T]$ and $x, x' \in \mathbb{R}^n$, that,

$$\begin{aligned} |b(t,0)| + |\sigma(t)| &\leqslant L, \\ |b(t,x) - b(t,x')| &\leqslant L|x - x'|, \end{aligned} \tag{HX}$$

for a positive constant L. We also assume that $g : \mathbb{R}^n \to \mathbb{R}$ satisfies Lipschitz condition $|g(x) - g(x')| \leqslant L|x - x'|$ for all $x, x' \in \mathbb{R}^n$ and is bounded by M_g, whereas $f : [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{n\times m} \to \mathbb{R}$ is Lipschitz with respect to y and locally Lipschitz with respect to both x and z, and has at most quadratic growth with respect to z, i.e., for any $t \in [0,T]$ and $(x,y,z), (x',y',z') \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{n\times m}$,

$$\begin{aligned} |f(t,x,y,z)| &\leqslant M_f(1+|y|) + \frac{\alpha}{2}|z|^2, \\ |f(t,x,y,z) - f(t,x',y,z)| &\leqslant L(1+|z|)|x - x'|, \\ |f(t,x,y,z) - f(t,x,y',z)| &\leqslant L|y - y'|, \\ |f(t,x,y,z) - f(t,x,y,z')| &\leqslant L(1+|z|+|z'|)|z - z'|, \end{aligned} \tag{HF}$$

where M_g, M_f and α are all positive constants.

Thanks to the seminal work [14], [4] and [5], the existence and uniqueness of the solution to the corresponding quadratic BSDEs (without reflection) have been well-developed. The reflected case was studied in [15] with bounded terminal value and obstacle and [1], [11] for unbounded cases. In addition to the existence and uniqueness of the solution, stability is also an essential property that focuses on the variation of the solutions under small perturbations of the coefficients. Stability is widely used to obtain continuity properties of the solutions. In this study, we apply it to the numerical analysis of quadratic reflected BSDEs.

Under the Lipschitz setting, a basic stability result was developed in [10, Proposition 3.6], which gives the variation of the solutions in terms of the suitable norms of their terminal values, generators, and obstacles. Based on this result, [16] studied the L^2-modulus regularity of the martingale integrand Z via a Feynman-Kac type formula and gave both a numerical scheme in the spirit of Bermuda options and its rate of convergence. [2] further applied the stability result to approximate (Y,Z) by its counterpart (Y^e, Z^e) constructed with the Euler scheme X^π of (1.2) and realized the convergence with the aid of a representation of the solution component Z in terms of the next reflection time, removing the uniform ellipticity condition on X in [16].

However, the counterpart of [10, Proposition 3.6] under the quadratic setting is still lacking. The existing stability results focus on the continuity of the solutions. For example, in [14] (without reflection) and [15] (with reflection), the authors showed the uniform convergence of the solutions (Y^n, Z^n) with parameters (g^n, f^n) to the solution (Y,Z) with parameters (g,f) when the obstacles g^n and generators f^n uniformly converged to g and f, respectively, by means of the comparison theorem, monotone property, and Lebesgue's theorem. However, the above continuity result does not say anything about the quantitative dependence of the variation of the solutions on those parameters, which will play a pivotal role in the numerical analysis of quadratic BSDEs with reflection.

Therefore, the main purpose of this study is to give, for the first time, *a quantitative stability result on the solutions of the Markovian quadratic BSDEs with reflection (1.1) and apply this new* *stability result to establish the convergence of a truncated discrete-time numerical scheme for (1.1).* Proceeding under the Markovian framework, we mainly focus on the perturbations of the parameters in the forward process (1.2) and study the variation of the solutions (Y, Z) to the quadratic reflected BSDE (1.1) driven by different forward processes.

Owing to the quadratic growth condition, we will work with bounded terminal data to further exploit the properties of bounded mean oscillation (BMO) martingales, which is used ubiquitously in the numerical analysis of quadratic BSDEs without reflection, see [6], [9] and [17] for example. Specifically, we first obtain some fundamental properties of the solution to the quadratic BSDE with reflection (1.1), i.e., the BMO property of the martingale integrand $Z * W$ and the L^p-integrability of $\int_0^T |Z_t|^2 dt$ and K_T. Next, working under a new equivalent probability measure induced by $Z * W$, we use the reverse Hölder inequality to estimate the variation of the solution component Y for any order in terms of the difference of underlying forward processes, followed by the estimates on the solution components (Z, K) equipped with appropriate norms. Finally, transferring back via John–Nirenberg inequality, we obtain the explicit dependence of the variation of the solutions under the original probability measure (see Theorem 3.2 for further details).

Further, we apply the stability result to the numerical analysis of quadratic reflected BSDEs. Contrary to quadratic BSDEs without reflection and Lipschitz BSDEs with reflection, where the solution component Z is typically bounded in the Markovian setup, *the solution component Z for the quadratic reflected BSDE (1.1)–(1.2) is not necessarily bounded.* This is the major difficulty to propose a numerical scheme and study its convergence. To overcome this difficulty, we resort to the discretely reflected BSDE (4.3) introduced in Section 4. Thanks to the previous work [18], we can readily extend the results therein to obtain a uniform estimate of $Z^{\mathcal{R}}$, the second component of the solution to the discretely reflected BSDE (4.3), and the convergence rate from the discretely to continuously reflected BSDEs. In turn, we truncate the generator via the bound of $Z^{\mathcal{R}}$ and propose a truncated discrete-time numerical scheme. This enables us to directly apply the existing numerical result under the Lipschitz setting (see [2]) to obtain the approximation error for the discretely reflected BSDE with quadratic growth. However, when extending the estimates to the continuously reflected case, a problematic term $\kappa|\pi|$ appears, and it will degenerate to a constant as the overall convergence rate is obtained (see Lemma 4.1). To overcome this difficulty, we introduce $Z^{\mathcal{R},e}$ as defined in (4.4), which is based on the Euler scheme for X^π, the same forward process as in our discrete-time numerical scheme. However, one needs to estimate an additional error between the solutions (Y, Z) and (Y^e, Z^e) of the continuously reflected BSDEs driven by X and X^π, respectively. It turns out this error can be controlled by applying the quantitative stability estimate (see Theorem 4.3 for further details).

The remainder of this article is organized as follows. In Section 2, we obtain some useful properties of the solution to the quadratic reflected BSDE with bounded terminal value. The quantitative stability result under the Markovian framework is derived in Section 3, exploiting techniques from BMO martingales. In the following section, we propose a truncated discrete-time numerical scheme for the quadratic reflected BSDE and apply the stability result to obtain a convergence rate for such a discrete-time approximation. Finally, Section 5 concludes this study.

2 Preliminaries

In this section, we introduce the notations of different spaces and recall some known results on quadratic reflected BSDEs with bounded terminal data.

Without loss of generality, we assume that the forward process X has dimension $n = 1$. Notably, this is merely for the sake of notational simplicity. Let $\mathbb{S}^\infty[0,T]$ denote the set of $\mathbb{R}$-valued progressively measurable bounded processes and $\mathbb{K}^2[0,T]$ denote all $\mathbb{R}$-valued continuous adapted processes $(K_t)_{0\leqslant t\leqslant T}$, which are increasing with K_0=0 and $\mathbb{E}|K_T|^2 < \infty$. For $1 \leqslant p < \infty$, $\mathbb{S}^p[0,T]$ denotes all $\mathbb{R}$-valued adapted processes $(Y_t)_{0\leqslant t\leqslant T}$ such that $\|Y\|^p_{\mathbb{S}^p} := \mathbb{E}(\sup_{0\leqslant t\leqslant T}|Y_t|^p) < \infty$, and $\mathbb{H}^p([0,T];\mathbb{R}^m)$ denotes all $\mathbb{R}^m$-valued adapted processes $(Z_t)_{0\leqslant t\leqslant T}$ satisfying $\|Z\|^p_{\mathbb{H}^p} := \mathbb{E}[\int_0^T |Z_t|^2_{\mathbb{R}^m} dt]^{p/2} < \infty$. Moreover, $\mathbb{L}^p(\mathcal{F}_t)$ denotes all $\mathbb{R}$-valued $\mathcal{F}_t$-measurable variables satisfying $\|Y\|^p_{\mathbb{L}^p} := \mathbb{E}|Y_t|^p < \infty$ for any $t \in [0,T]$, and we usually omit $(\mathcal{F}_t)$ hereafter in case there is no ambiguity.

Under the above assumptions (HX) and (HF), we know the decoupled system (1.1) and (1.2) with bounded terminal function and bounded obstacle has a unique solution $(X,Y,Z,K) \in \mathbb{S}^2[0,T] \times \mathbb{S}^\infty[0,T] \times \mathbb{H}^2([0,T];\mathbb{R}^m) \times \mathbb{K}^2[0,T]$, and we denote $\|Y\|_\infty \triangleq M$. For more details of this result, we refer the reader to [15]. In the following, unless otherwise specified, we shall use C to denote the universal constant that may depend on all given coefficients L, T, M_g, M_f, and α, and C_p further depends on an extra parameter $p \geqslant 1$.

Next, we recall the definition and some basic properties of BMO martingales, which provide the techniques for this study. For the detailed theory, we refer the reader to [12]. We say a continuous local martingale $(M_t)_{t\in[0,T]}$ is a BMO martingale if it is square-integrable with $M_0 = 0$ such that

$$\|M\|^2_{BMO} := \sup_{\tau\in\mathcal{T}[0,T]} \|\mathbb{E}[\langle M\rangle_T - \langle M\rangle_\tau|\mathcal{F}_\tau]\|_\infty < \infty,$$

where $\mathcal{T}[0,T]$ is the set of all stopping times valued in $[0,T]$.

Lemma 2.1 *Let M be a BMO martingale. Then, we have:*
1) *The stochastic exponential*

$$\mathcal{E}(M)_t := \exp\Big(M_t - \frac{1}{2}\langle M\rangle_t\Big), t \in [0,T],$$

is a uniformly integrable martingale.
2) *The energy inequality gives that*

$$\mathbb{E}[\langle M\rangle^n_T] \leqslant n!\|M\|^{2n}_{BMO}$$

for all $n \in \mathbb{N}^+$, which implies that $BMO \subset \mathbb{H}^p([0,T])$ for every $p \geqslant 1$.
3) *According to reverse Hölder inequality, there exists some $p > 1$ such that*

$$\mathbb{E}[\mathcal{E}(M)^p_T] \leqslant C_p,$$

where C_p is a constant only depending on p and the BMO norm of M. Moreover, the maximum p satisfying such property can be explicitly determined by the BMO norm of M through a decreasing function (see more details in [12, Theorem3.1]).
4) *By the John–Nirenberg inequality, we have*

$$\mathbb{E}\Big[\mathcal{E}(M)_T^{-\frac{1}{p-1}}\Big] \leqslant C_p$$

for all $p > 1$ satisfying $\|M\|_{BMO} < \sqrt{2}(\sqrt{p} - 1)$, see [12, Theorem2.4].

With the above tools at hand, we claim the following properties about the solutions to the quadratic reflected BSDE (1.1).

Proposition 2.2 *Suppose Assumptions* (HX) *and* (HF) *hold and let* (X, Y, Z, K) *be the solution of system (1.1) and (1.2). Then, the stochastic integral* $Z * W := \Big(\int_0^t Z_s dW_s\Big)_{t\in[0,T]}$ *is a BMO martingale with the BMO norm satisfying*

$$\|Z * W\|^2_{BMO} \leq \frac{\exp(4\alpha M)}{\alpha^2}[1 + 2\alpha M_f(1 + M)T]. \tag{2.1}$$

Proof. Denote $\|Y\|_\infty \triangleq M$. Making exponential change of variable $\eta_t := e^{-\alpha Y_t}$, we realize the following reflected BSDE with an upper obstacle

$$\eta_t = \theta_T + \int_t^T F(s, \eta_s, \Lambda_s)ds - \int_t^T \Lambda_s dW_s - (J_T - J_t), \quad t \in [0, T] \tag{2.2}$$

with $\theta_t = e^{-\alpha g(X_t)}, \Lambda_t = -\alpha e^{-\alpha Y_t} Z_t, dJ_t = \alpha e^{-\alpha Y_t} dK_t$ and stochastic coefficient

$$F(t, \omega, y, z) = -\alpha y\Big[f\Big(t, X_t(\omega), \frac{\ln y}{-\alpha}, \frac{z}{-\alpha y}\Big) + \frac{\alpha}{2}\Big|\frac{z}{-\alpha y}\Big|^2\Big]\mathbb{1}_{\{y\geqslant e^{-\alpha M}\}},$$

which satisfy $\eta_t \leqslant \theta_t$ and

$$\int_0^T (\theta_t - \eta_t)dJ_t = 0.$$

Moreover, by the boundedness of Y and Assumption (HF), we have

$$-\alpha M_f(1 + M)y - e^{\alpha M}|z|^2 \leqslant F(t, \omega, y, z) \leqslant \alpha M_f(1 + M)y,$$

and thus $(\eta, \Lambda, J) \in \mathbb{S}^\infty[0, T] \times \mathbb{H}^2([0, T]; \mathbb{R}^m) \times \mathbb{K}^2[0, T]$ with $e^{-\alpha M} \leqslant \eta_t \leqslant e^{\alpha M}$ for all $t \in [0, T]$.

Applying Itô's formula to $|\eta_t|^2$ gives that

$$|\eta_t|^2 = |\eta_T|^2 + \int_t^T 2\eta_s F(s, \eta_s, \Lambda_s)ds - \int_t^T 2\eta_s\Lambda_s dW_s - \int_t^T 2\eta_s dJ_s - \int_t^T |\Lambda_s|^2 ds. \tag{2.3}$$

Because $dJ_t \geqslant 0, \eta_t > 0$ and $\Lambda \in \mathbb{H}^2([0, T]; \mathbb{R}^m)$, we have

$$\begin{aligned}|\eta_t|^2 + \mathbb{E}_{\mathcal{F}_t}\Big[\int_t^T |\Lambda_s|^2 ds\Big] &\leqslant \mathbb{E}_{\mathcal{F}_t}|\eta_T|^2 + \mathbb{E}_{\mathcal{F}_t}\Big[\int_t^T 2\eta_s F(s, \eta_s, \Lambda_s)ds\Big]\\ &\leqslant \mathbb{E}_{\mathcal{F}_t}|\eta_T|^2 + 2\alpha M_f(1 + M)\mathbb{E}_{\mathcal{F}_t}\Big[\int_t^T |\eta_s|^2 ds\Big]\\ &\leqslant [1 + 2\alpha M_f(1 + M)T] \exp(2\alpha M).\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}_{\mathcal{F}_t}\Big[\int_t^T |Z_s|^2 ds\Big] &= \mathbb{E}_{\mathcal{F}_t}\Big[\int_t^T \Big|\frac{\Lambda_s}{-\alpha\eta_s}\Big|^2 ds\Big] \leqslant \frac{\exp(2\alpha M)}{\alpha^2}\mathbb{E}_{\mathcal{F}_t}\Big[\int_t^T |\Lambda_s|^2 ds\Big]\\ &\leqslant \frac{\exp(4\alpha M)}{\alpha^2}[1 + 2\alpha M_f(1 + M)T], \quad \forall t \in [0, T],\end{aligned} \tag{2.4}$$

and one can easily conclude by the definition of BMO martingales. $\square$

The above proposition implies that the BMO norm of $Z * W$ depends only on α, M_f, M_g, M, and T. Further, we have the following L^p-integrability of $\int_0^T |Z_t|^2 dt$ and K_T.

Proposition 2.3 *Suppose Assumptions* (HX) *and* (HF) *hold, and let* (X, Y, Z, K) *be the solution of system (1.1) and (1.2). Then, for any* $p \geqslant 1$,

$$\mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^p + (K_T)^p\Big] \leqslant C_p.$$

Proof. It is clear from Proposition 2.2 and assertion 2) of Lemma 2.1 to obtain the result of Z part. For the K part, rewrite the reflected equation as follows:

$$\begin{aligned} K_T &= K_0 + Y_0 - g(X_T) - \int_0^T f(s, X_s, Y_s, Z_s) ds + \int_0^T Z_s dW_s \\ &\leqslant M + M_g + \int_0^T |f(s, X_s, Y_s, Z_s)| ds + \Big| \int_0^T Z_s dW_s \Big| \\ &\leqslant M + M_g + M_f T(1 + \|Y\|_\infty) + \frac{\alpha}{2} \int_0^T |Z_s|^2 ds + \Big| \int_0^T Z_s dW_s \Big|, \end{aligned} \tag{2.5}$$

where the last line follows from the assumptions of f and g. Thus, by Burkholder–Davis–Gundy inequality and the conclusion of the Z part, we obtain

$$\begin{aligned} \mathbb{E}|K_T|^p &\leqslant C_p\Big(1 + \mathbb{E}\Big(\int_0^T |Z_s|^2 ds\Big)^p + \mathbb{E}\Big| \int_0^T Z_s dW_s \Big|^p\Big) \\ &\leqslant C_p\Big(1 + \mathbb{E}\Big(\int_0^T |Z_s|^2 ds\Big)^p + \mathbb{E}\Big(\int_0^T |Z_s|^2 ds\Big)^{p/2}\Big) \leqslant C_p. \end{aligned} \tag{2.6}$$

$\square$

3 Main stability result

Now, we are ready to deal with the variation of the solutions to quadratic reflected BSDEs driven by different forward processes. Suppose that X^j solves

$$X_t^j = x + \int_0^t b_j(s, X_s^j) ds + \int_0^t \sigma_j(s) dW_s, \quad t \in [0, T]$$

for $j = 1, 2$, where (b_j, σ_j) satisfies Assumption (HX), then we know both X^1 and X^2 are in $\mathbb{S}^2[0, T]$. Given the parameters f and g, we denote the solutions to the quadratic reflected BSDE (1.1) driven by X^1 and X^2 as (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2), respectively, which belong to $\mathbb{S}^\infty[0, T] \times \mathbb{H}^2([0, T]; \mathbb{R}^m) \times \mathbb{K}^2[0, T]$ and satisfy $\|Y^1\|_\infty \vee \|Y^2\|_\infty \leqslant M$. We further denote $\delta X = X^1 - X^2$, $\delta Y = Y^1 - Y^2$, $\delta Z = Z^1 - Z^2$ and $\delta K = K^1 - K^2$, and have the following expression

$$\begin{aligned} \delta Y_t &= \delta Y_T + \int_t^T f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^2, Z_s^2) ds - \int_t^T \delta Z_s dW_s + \int_t^T d\delta K_s \\ &= \delta Y_T + \int_t^T (\gamma_s \delta X_s + \beta_s \delta Y_s + \mu_s \delta Z_s) ds - \int_t^T \delta Z_s dW_s + \int_t^T d\delta K_s, \end{aligned} \tag{3.1}$$

where

$$\gamma_s := \frac{f(s, X_s^1, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^1, Z_s^1)}{X_s^1 - X_s^2} \mathbb{1}_{\{\delta X_s \neq 0\}},$$

$$\beta_s := \frac{f(s, X_s^2, Y_s^1, Z_s^1) - f(s, X_s^2, Y_s^2, Z_s^1)}{Y_s^1 - Y_s^2} \mathbb{1}_{\{\delta Y_s \neq 0\}},$$

and

$$\mu_s := \frac{f(s, X_s^2, Y_s^2, Z_s^1) - f(s, X_s^2, Y_s^2, Z_s^2)}{|Z_s^1 - Z_s^2|^2} (Z_s^1 - Z_s^2)^T \mathbb{1}_{\{|\delta Z_s| \neq 0\}}.$$

By the locally Lipschitz assumption of f, we have

$$|\gamma_s| \leqslant L(1 + |Z_s^1|), \quad |\beta_s| \leqslant L, \quad |\mu_s| \leqslant L(1 + |Z_s^1| + |Z_s^2|), \quad \forall s \in [0, T],$$

which further imply that $\int_0^T (|\gamma_s|^2 + |\mu_s|^2) ds$ is L^p-integrable for any $p \geqslant 1$ by Proposition 2.3 and that $\mu * W$ is a BMO martingale by Proposition 2.2.

Regarding the difficulty caused by the quadratic growth in the Z part, the BMO property of $\mu * W$ enables us to proceed under a novel equivalent probability measure $\mathbb{Q}$ defined as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}(\mu * W)_T,$$

under which $W_t^{\mathbb{Q}} = W_t - \int_0^t \mu_s ds, \ t \in [0, T]$, is a standard Brownian motion.

Moreover, because

$$\|\mu * W\|_{BMO} \leqslant L(1 + \|Z^1 * W\|_{BMO} + \|Z^2 * W\|_{BMO}),$$

we know from assertion 3) of Lemma 2.1 that there exists some $p^* > 1$, which can be determined by the BMO norm of $\mu * W$, such that $\mathcal{E}(\mu * W)_T$ is L^{p^*}-integrable, i.e., $\mathbb{E}[\mathcal{E}(\mu * W)_T^{p^*}] \leqslant C_{p^*}$. Thus, C_{p^*} depends only on the BMO norm of $\mu * W$, which essentially relies on the given coefficients L, α, M_f, M_g, and T, and we may just write it as the universal constant C hereafter. Next, we will give the L^p-estimate of the difference of solutions under such a probability measure in the following proposition.

Proposition 3.1 *For any $p > 1$ and $\|\delta X\|_{\mathbb{S}^{4pq^*}} \leq 1$,*

$$\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t \in [0,T]} |\delta Y_t|^{2p} + \Big(\int_0^T |\delta Z_t|^2 dt\Big)^p + |\delta K_T|^{2p}\Big] \leqslant C_p \|\delta X\|^p_{\mathbb{S}^{4pq^*}}, \tag{3.2}$$

where q^* is the conjugate exponent of p^*, i.e., $\frac{1}{q^*} + \frac{1}{p^*} = 1$, and the parameter p^* is determined by the BMO norm of $\mu * W$ (see assertion 3) of Lemma 2.1).

Proof. To prove this result, we first obtain the estimate of the expectation under the probability measure with an undecided parameter A but without taking the supremum. Then, choosing appropriate A gives the exact estimate under the supremum norm of the solution component Y in the second step and the components of the solution (Z, K) in the last step.

Step one. Estimates of $\mathbb{E}^{\mathbb{Q}}[|\delta Y_t|^{2p}]$ and $p(2p-1)\mathbb{E}^{\mathbb{Q}}[\int_0^T |\delta Y_t|^{2p-2} |\delta Z_t|^2 dt]$.

Applying Itô's formula to $|\delta Y_t|^{2p}$ gives that

$$
\begin{aligned}
|\delta Y_t|^{2p} =& |\delta Y_T|^{2p} + 2p\int_t^T (\delta Y_s)^{2p-1}\gamma_s \delta X_s ds + 2p\int_t^T (\delta Y_s)^{2p-1}\beta_s \delta Y_s ds - 2p\int_t^T (\delta Y_s)^{2p-1}\delta Z_s dW_s^{\mathbb{Q}} \\
&+ 2p\int_t^T (\delta Y_s)^{2p-1} d\delta K_s - p(2p-1)\int_t^T (\delta Y_s)^{2p-2}|\delta Z_s|^2 ds.
\end{aligned}
\tag{3.3}
$$

Taking expectation under probability measure $\mathbb{Q}$ and recalling the boundedness of δY, we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[|\delta Y_t|^{2p}] + p(2p-1)\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p-2}|\delta Z_s|^2 ds\Big] \\
\leqslant& \mathbb{E}^{\mathbb{Q}}[|\delta Y_T|^{2p}] + 2p\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p-1}|\gamma_s||\delta X_s| ds\Big] + 2p\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}|\beta_s| ds\Big] \\
&+ 2p\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T (\delta Y_s)^{2p-1} d\delta K_s\Big].
\end{aligned}
\tag{3.4}
$$

Recall the L^{p^*}-integrability of $\mathcal{E}(\mu * W)_T$. We have that, for any $\mathcal{F}_T$-measurable and non-negative variable $X \in L^{q^*}$,

$$
\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}[\mathcal{E}(\mu * W)_T X] \leqslant (\mathbb{E}[\mathcal{E}(\mu * W)_T^{p^*}])^{1/p^*}(\mathbb{E}[X^{q^*}])^{1/q^*} \leqslant C(\mathbb{E}[X^{q^*}])^{1/q^*}, \tag{3.5}
$$

where q^* is the conjugate exponent of p^*. Moreover, we know that both X^1 and X^2 are in the space $\mathbb{S}^p[0,T]$ under Assumption (HX) for any $p \geqslant 2$. Now, we are ready to deal with the inequality (3.4), where the first term can be estimated by the above inequality (3.5) and the Lipschitz assumption of g as

$$
\mathbb{E}^{\mathbb{Q}}[|\delta Y_T|^{2p}] \leqslant L^{2p}\mathbb{E}^{\mathbb{Q}}[|\delta X_T|^{2p}] \leqslant C_p(\mathbb{E}[|\delta X_T|^{2pq^*}])^{1/q^*} \leqslant C_p\|\delta X\|_{\mathbb{S}^{2pq^*}}^{2p}. \tag{3.6}
$$

We also list the following two estimates for later use. By (3.5), Cauchy–Schwarz inequality, and the L^p-integrability of K_T^1, K_T^2 and $\int_0^T |\gamma_t|^2 dt$ with arbitrary p, we derive that, for any $p > 1$

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta X_t|^{2p}\Big(\int_0^T |\gamma_t|^2 dt\Big)^p\Big] \leqslant C\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta X_t|^{2pq^*}\Big(\int_0^T |\gamma_t|^2 dt\Big)^{pq^*}\Big]\Big)^{1/q^*} \\
\leqslant& C\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta X_t|^{4pq^*}\Big]\Big)^{1/2q^*}\Big(\mathbb{E}\Big[\Big(\int_0^T |\gamma_t|^2 dt\Big)^{2pq^*}\Big]\Big)^{1/2q^*} \leqslant C_p\|\delta X\|_{\mathbb{S}^{4pq^*}}^{2p},
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta X_t|^p(K_T^1 + K_T^2)^p\Big] \leqslant C\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta X_t|^{pq^*}(K_T^1 + K_T^2)^{pq^*}\Big]\Big)^{1/q^*} \\
\leqslant& C\Big(\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta X_t|^{2pq^*}\Big]\Big)^{1/2q^*}\Big(\mathbb{E}[(K_T^1 + K_T^2)^{2pq^*}]\Big)^{1/2q^*} \leqslant C_p\|\delta X\|_{\mathbb{S}^{2pq^*}}^{p}.
\end{aligned}
\tag{3.8}
$$

For the second term of (3.4), we can use Hölder's inequality, Young's inequality, and (3.7) to get

$$
\begin{aligned}
&2p\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p-1}|\gamma_s||\delta X_s|ds\Big] \leqslant \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}ds\Big] + p^2\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p-2}|\gamma_s|^2|\delta X_s|^2ds\Big]\\
\leqslant&\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}ds\Big] + p^2\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p-2}\sup_{t\in[0,T]}|\delta X_t|^2\int_0^T|\gamma_s|^2ds\Big]\\
\leqslant&\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}ds\Big] + \frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{(2p-2)q}\Big] + p^{2p-1}A^{p-1}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta X_t|^{2p}\Big(\int_0^T|\gamma_s|^2ds\Big)^p\Big]\\
\leqslant&\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}ds\Big] + \frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^{2p}_{\mathbb{S}^{4pq^*}},
\end{aligned}
\tag{3.9}
$$

where q is the conjugate exponent of p and $A>1$ is a constant yet to be determined.

For the third term of (3.4), we have

$$
2p\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}|\beta_s|ds\Big] \leqslant 2pL\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p}ds\Big]. \tag{3.10}
$$

Regarding the last term of (3.4) with reflection, because $g(X^j_s)\leqslant Y^j_s$ for all $s\in[0,T]$ and K^j only increases when $Y^j=g(X^j)$, $j=1,2$, we can first derive that

$$
(Y^1_s-Y^2_s)dK^1_s=[Y^1_s-g(X^1_s)+g(X^1_s)-g(X^2_s)+g(X^2_s)-Y^2_s]dK^1_s\leqslant[g(X^1_s)-g(X^2_s)]dK^1_s,
$$

and similarly,

$$
(Y^2_s-Y^1_s)dK^2_s\leqslant[g(X^2_s)-g(X^1_s)]dK^2_s,
$$

which imply that

$$
\begin{aligned}
\delta Y_s d\delta K_s &= (Y^1_s-Y^2_s)dK^1_s+(Y^2_s-Y^1_s)dK^2_s\\
&\leqslant |g(X^1_s)-g(X^2_s)|d(K^1_s+K^2_s)\leqslant L|\delta X_s|d(K^1_s+K^2_s).
\end{aligned}
\tag{3.11}
$$

Then, we can use the same arguments as above to obtain

$$
\begin{aligned}
&2p\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T (\delta Y_s)^{2p-1}d\delta K_s\Big] \leqslant 2pL\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p-2}|\delta X_s|d(K^1_s+K^2_s)\Big]\\
\leqslant&2pL\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p-2}\sup_{t\in[0,T]}|\delta X_t|(K^1_T+K^2_T)\Big]\\
\leqslant&\frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{(2p-2)q}\Big] + \frac{(2pL)^p}{p}A^{p-1}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta X_t|^p(K^1_T+K^2_T)^p\Big]\\
\leqslant&\frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^{p}_{\mathbb{S}^{2pq^*}}.
\end{aligned}
\tag{3.12}
$$

Plugging (3.6), (3.9), (3.10), and (3.12) back into (3.4) and by Cauthy–Schwarz inequality, we get

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[|\delta Y_t|^{2p}] + p(2p-1)\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T |\delta Y_s|^{2p-2}|\delta Z_s|^2ds\Big]\\
\leqslant&(2pL+1)\int_t^T \mathbb{E}^{\mathbb{Q}}|\delta Y_s|^{2p}ds + \frac{2}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^{p}_{\mathbb{S}^{4pq^*}},\quad \forall t\in[0,T].
\end{aligned}
$$

Because δY is bounded and hence $\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]$ is finite, we obtain from Gronwall's inequality that

$$\mathbb{E}^{\mathbb{Q}}[|\delta Y_t|^{2p}] \leqslant e^{(2pL+1)T}\Big(\frac{2}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}}\Big), \quad \forall t\in[0,T], \tag{3.13}$$

and moreover,

$$\begin{aligned}
&p(2p-1)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T |\delta Y_s|^{2p-2}|\delta Z_s|^2 ds\Big]\\
\leqslant&[(2pL+1)Te^{(2pL+1)T}+1]\Big(\frac{2}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}}\Big)\\
\leqslant&\frac{C_T}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}}
\end{aligned} \tag{3.14}$$

with $C_T := 2[(2pL+1)Te^{(2pL+1)T}+1]$.

Step two. Estimate of $\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}]$.

Next, we shall go back to (3.3) in the first step of the proof and follow similar procedure to estimate $\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}]$. First, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}] + p(2p-1)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T |\delta Y_s|^{2p-2}|\delta Z_s|^2 ds\Big]\\
\leqslant&\mathbb{E}^{\mathbb{Q}}[|\delta Y_T|^{2p}] + 2p\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p-1}|\gamma_s||\delta X_s|ds\Big] + 2p\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p}|\beta_s|ds\Big]\\
&+2p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\Big|\int_t^T(\delta Y_s)^{2p-1}\delta Z_s dW_s^{\mathbb{Q}}\Big|\Big] + 2p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\int_t^T(\delta Y_s)^{2p-1}d\delta K_s\Big].
\end{aligned} \tag{3.15}$$

By (3.9), (3.10), and (3.13), we have

$$\begin{aligned}
&2p\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p-1}|\gamma_s||\delta X_s|ds\Big] + 2p\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p}|\beta_s|ds\Big]\\
\leqslant&(2pL+1)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p}ds\Big] + \frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^{2p}_{\mathbb{S}^{4pq^*}}\\
\leqslant&\frac{C_T}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}},
\end{aligned} \tag{3.16}$$

similarly by (3.11) and (3.12),

$$\begin{aligned}
&2p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\int_t^T(\delta Y_s)^{2p-1}d\delta K_s\Big] \leqslant 2p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\int_t^T L|\delta Y_s|^{2p-2}|\delta X_s|d(K_s^1+K_s^2)\Big]\\
=&2pL\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p-2}|\delta X_s|d(K_s^1+K_s^2)\Big] \leqslant \frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big] + A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{2pq^*}}.
\end{aligned} \tag{3.17}$$

As for the stochastic integral term in (3.15), we derive that

$$
\begin{aligned}
&2p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\Big|\int_t^T(\delta Y_s)^{2p-1}\delta Z_s dW_s^{\mathbb{Q}}\Big|\Big]\leqslant 2p\tilde{C}\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|\delta Y_s|^{2(2p-1)}|\delta Z_s|^2ds\Big)^{1/2}\Big]\\
\leqslant& p\mathbb{E}^{\mathbb{Q}}\Big[2\Big(\sup_{t\in[0,T]}|\delta Y_t|^{2p}\int_0^T\tilde{C}^2|\delta Y_s|^{2p-2}|\delta Z_s|^2ds\Big)^{1/2}\Big] \qquad (3.18)\\
\leqslant& p\Big(\frac{1}{2p-1}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]+(2p-1)\tilde{C}^2\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p-2}|\delta Z_s|^2ds\Big]\Big),
\end{aligned}
$$

where $\tilde{C}$ is the constant coming from the following Burkholder-Davis-Gundy inequality

$$
\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}\Big|\int_t^T\psi_s dW_s^{\mathbb{Q}}\Big|\Big]\leqslant\tilde{C}\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|\psi_s|^2ds\Big)^{1/2}\Big],
$$

which holds for all $\mathcal{F}_t$-adapted stochastic processes satisfying $\mathbb{Q}\big\{\int_0^T|\psi_s|^2ds<\infty\big\}=1$. Combining (3.15)-(3.18) and together with the results in the first step, we can finally get

$$
\begin{aligned}
\frac{p-1}{2p-1}\mathbb{E}^{\mathbb{Q}}[\sup_{t\in[0,T]}|\delta Y_t|^{2p}]\leqslant&\frac{C_T+1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]+A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}}\\
&+p(2p-1)\tilde{C}^2\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T|\delta Y_s|^{2p-2}|\delta Z_s|^2ds\Big]\Big)\\
\leqslant&[C_T\tilde{C}^2+C_T+1]\frac{1}{qA}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]+A^{p-1}C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}}.
\end{aligned}
$$

Choosing $A:=2[C_T\tilde{C}^2+C_T+1]$, we can achieve the desired result for the Y part.

Step three. Estimate of $\mathbb{E}^{\mathbb{Q}}[(\int_0^T|\delta Z_s|^2ds)^p]$ and $\mathbb{E}^{\mathbb{Q}}[|\delta K_T|^{2p}]$.

Applying Itô's formula to $|\delta Y_t|^2$ gives that

$$
\begin{aligned}
|\delta Y_t|^2=&|\delta Y_T|^2+\int_t^T2\delta Y_s(\gamma_s\delta X_s+\beta_s\delta Y_s+\mu_s\delta Z_s)ds-\int_t^T2\delta Y_s\delta Z_s dW_s\\
&+\int_t^T2\delta Y_s d\delta K_s-\int_t^T|\delta Z_s|^2ds\\
=&|\delta Y_T|^2+\int_t^T2\delta Y_s\gamma_s\delta X_s ds+\int_t^T2|\delta Y_s|^2\beta_s ds-\int_t^T2\delta Y_s\delta Z_s dW_s^{\mathbb{Q}}\\
&+\int_t^T2\delta Y_s d\delta K_s-\int_t^T|\delta Z_s|^2ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|\delta Z_s|^2ds\Big)^p\Big]\leqslant& C_p\Big\{\mathbb{E}^{\mathbb{Q}}|\delta Y_T|^{2p}+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T2\delta Y_s\gamma_s\delta X_s ds\Big|^p+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T2|\delta Y_s|^2\beta_s ds\Big|^p\\
&+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T2\delta Y_s\delta Z_s dW_s^{\mathbb{Q}}\Big|^p+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T2\delta Y_s d\delta K_s\Big|^p\Big\}.
\end{aligned}
$$

Then, by (3.7), (3.8) and similar arguments as in the first step, we obtain that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T 2\delta Y_s\gamma_s\delta X_s ds\Big|^p \leqslant C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big(\int_0^T|\gamma_s||\delta X_s|ds\Big)^p\Big]\\
\leqslant &C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]+C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta X_t|^{2p}\Big(\int_0^T|\gamma_s|^2ds\Big)^p\Big]\leqslant C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}},
\end{aligned}
$$

$$
\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T 2|\delta Y_s|^2\beta_s ds\Big|^p\leqslant C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]\leqslant C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}},
$$

and

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T 2\delta Y_s d\delta K_s\Big|^p\leqslant \mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T 2L|\delta X_s|d(K_s^1+K_s^2)\Big)^p\Big]\\
\leqslant &C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta X_t|^p(K_T^1+K_T^2)^p\Big]\leqslant C_p\|\delta X\|^p_{\mathbb{S}^{2pq^*}}.
\end{aligned}
$$

As for the martingale term, by B-D-G and Young's inequality, we derive that

$$
\begin{aligned}
C_p\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T 2\delta Y_s\delta Z_s dW_s^{\mathbb{Q}}\Big|^p &\leqslant C_p\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|\delta Y_s|^2|\delta Z_s|^2ds\Big)^{\frac{p}{2}}\Big]\leqslant C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^p\Big(\int_0^T|\delta Z_s|^2ds\Big)^{\frac{p}{2}}\Big]\\
&\leqslant C_p\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2p}\Big]+\frac{1}{2}\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|\delta Z_s|^2ds\Big)^p\Big],
\end{aligned}
$$

and then together with the result for the Y part, we get the conclusion for Z.

Regarding the increasing process K, we have the expression

$$
\begin{aligned}
\delta K_T&=\delta Y_0-[g(X_T^1)-g(X_T^2)]-\int_0^T[f(s,X_s^1,Y_s^1,Z_s^1)-f(s,X_s^2,Y_s^2,Z_s^2)]ds+\int_0^T\delta Z_s dW_s\\
&=\delta Y_0-[g(X_T^1)-g(X_T^2)]-\int_0^T\gamma_s\delta X_s ds-\int_0^T\beta_s\delta Y_s ds+\int_0^T\delta Z_s dW_s^{\mathbb{Q}}.
\end{aligned}
$$

Thus, by (3.6), (3.7) and the conclusion for the Y and Z parts, we further obtain

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}|\delta K_T|^{2p}\leqslant &C_p\Big[\mathbb{E}^{\mathbb{Q}}|\delta Y_0|^{2p}+\mathbb{E}^{\mathbb{Q}}|g(X_T^1)-g(X_T^2)|^{2p}+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T\gamma_s\delta X_s ds\Big|^{2p}\\
&+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T\beta_s\delta Y_s ds\Big|^{2p}+\mathbb{E}^{\mathbb{Q}}\Big|\int_0^T\delta Z_s dW_s^{\mathbb{Q}}\Big|^{2p}\Big]\\
\leqslant &C_p\Big[\|\delta X\|^p_{\mathbb{S}^{4pq^*}}+\|\delta X\|^{2p}_{\mathbb{S}^{2pq^*}}+\|\delta X\|^{2p}_{\mathbb{S}^{4pq^*}}+\mathbb{E}^{\mathbb{Q}}\Big[\Big(\int_0^T|\delta Z_s|^2ds\Big)^p\Big]\Big]\leqslant C_p\|\delta X\|^p_{\mathbb{S}^{4pq^*}},
\end{aligned}
$$

which concludes the proof. □

Finally, we estimate the variation of the two solutions to quadratic reflected BSDEs constructed with different forward processes, X^1 and X^2, under the original probability measure.

Theorem 3.2 *Suppose Assumptions* (HX) *and* (HF) *hold. Then, for* $\|\delta X\|_{\mathbb{S}^{4\bar{p}q^*}} \leq 1$*, we have the following conclusion:*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta Y_t|^2+\int_0^T|\delta Z_t|^2dt+|\delta K_T|^2\Big]\leqslant C\|\delta X\|_{\mathbb{S}^{4\bar{p}q^*}},$$

where q^* *is given in Proposition 3.1, and* $\bar{p}$ *is the minimum parameter corresponding to the BMO martingale* $\mu * W$ *satisfying assertion* 4) *of Lemma 2.1.*

Proof. First, let $\bar{p}>1$ be the minimum parameter such that $\|\mu * W\|_{BMO}<\sqrt{2}(\sqrt{\bar{p}}-1)$ and $\bar{q}$ be its conjugate exponent. Notably, the constant $C_{\bar{p}}$ appearing in the estimate of the assertion 4) of Lemma 2.1 can be substituted by a universal constant C, as $\bar{p}$ can be fully determined by $\|\mu * W\|_{BMO}$. Then, for any $\mathcal{F}_T$-measurable and non-negative random variable $X\in L^{\bar{p}}$, we obtain that

$$\begin{aligned}\mathbb{E}[X]=&\mathbb{E}[\mathcal{E}(\mu*W)_T^{1/\bar{p}}X\cdot\mathcal{E}(\mu*W)_T^{-1/\bar{p}}]\leqslant\Big(\mathbb{E}\big[\mathcal{E}(\mu*W)_TX^{\bar{p}}\big]\Big)^{1/\bar{p}}\Big(\mathbb{E}\big[\mathcal{E}(\mu*W)_T^{-\bar{q}/\bar{p}}\big]\Big)^{1/\bar{q}}\\=&\Big(\mathbb{E}^{\mathbb{Q}}[X^{\bar{p}}]\Big)^{1/\bar{p}}\Big(\mathbb{E}\big[\mathcal{E}(\mu*W)_T^{-\frac{1}{\bar{p}-1}}\big]\Big)^{1/\bar{q}}\leqslant C\Big(\mathbb{E}^{\mathbb{Q}}[X^{\bar{p}}]\Big)^{1/\bar{p}}.\end{aligned}\tag{3.19}$$

Thus, we can conclude the proof by applying Proposition 3.1 directly to derive

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|\delta Y_t|^2+\int_0^T|\delta Z_t|^2dt+|\delta K_T|^2\Big]\\\leqslant&C\Big(\mathbb{E}^{\mathbb{Q}}\Big[\sup_{t\in[0,T]}|\delta Y_t|^{2\bar{p}}+\Big(\int_0^T|\delta Z_t|^2dt\Big)^{\bar{p}}+|\delta K_T|^{2\bar{p}}\Big]\Big)^{1/\bar{p}}\leqslant C\|\delta X\|_{\mathbb{S}^{4\bar{p}q^*}}.\end{aligned}\tag{3.20}$$

□

4 Application to numerical scheme for quadratic reflected BSDEs

In this section, we apply the quantitative stability result to the convergence analysis for a discrete-time numerical scheme for the quadratic reflected BSDE (1.1)-(1.2) under the Markovian framework and Assumptions (HX) and (HF). For further time discretization, we need to assume in this section that b,σ and f satisfy Hölder's continuity with respect to the time variable. That is, for any $0\leqslant s\leqslant t\leqslant T$ and any $(x,y,z)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^m$,

$$|b(t,x)-b(s,x)|+|\sigma(t)-\sigma(s)|+|f(t,x,y,z)-f(s,x,y,z)|\leqslant L(t-s)^{\frac{1}{2}}.\tag{HT}$$

Different from quadratic BSDEs without reflection and Lipschitz BSDEs with reflection, where the solution component Z is typically bounded in the Markovian setup, the solution component Z for quadratic reflected BSDE (1.1)-(1.2) is not necessarily bounded. This is the major difficulty to propose a numerical scheme and study its convergence. To overcome this difficulty, we resort to the discretely reflected version of BSDE introduced in (4.3), where the reflection is only permitted to operate at specific discrete-time points. In [18], we have proven that the corresponding solution

$(Y^{\mathcal{R}}, Z^{\mathcal{R}})$ is a good approximation of its continuous counterpart (Y, Z) in (1.1) (Notably, the generator f does not involve y in [18], but one can easily extend the result therein to include y in the generator). Moreover, because the solution component $Z^{\mathcal{R}}$ is uniformly bounded, we can truncate the corresponding generator via the bound of $Z^{\mathcal{R}}$ and obtain a truncated discrete-time numerical scheme on each reflected interval. The quantitative stability result will play a pivotal role in the convergence analysis of this numerical scheme. First, we give some basic definitions, which will be used later.

4.1 Definition and notations

Given a partition $\pi := \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0, T]$, we shall first introduce the standard Euler scheme X^π for X, which has been widely studied in the literature and has the form

$$\begin{cases} X_0^\pi = x, \\ X_{t_{i+1}}^\pi = X_{t_i}^\pi + b(t_i, X_{t_i}^\pi)(t_{i+1} - t_i) + \sigma(t_i)(W_{t_{i+1}} - W_{t_i}), \quad i \leqslant N-1, \end{cases}$$

whose continuous-time version is defined correspondingly as

$$X_t^\pi = X_{t_i}^\pi + b(t_i, X_{t_i}^\pi)(t - t_i) + \sigma(t_i)(W_t - W_{t_i}), \quad t \in [t_i, t_{i+1}),\ i \leqslant N-1.$$

Denote $|\pi| := \max_{i \leqslant N-1}(t_{i+1} - t_i)$ and without loss of generality, assume that $N|\pi| \leqslant L$. Then, under Assumption (HX), we know that $X^\pi \in \mathbb{S}^{2p}[0, T]$ and

$$\mathbb{E}\Big[\sup_{0 \leqslant t \leqslant T} |X_t - X_t^\pi|^{2p}\Big] + \max_{0 \leqslant i \leqslant N-1} \mathbb{E}\Big[\sup_{t \in [t_i, t_{i+1}]} |X_t - X_{t_i}^\pi|^{2p}\Big] \leqslant C|\pi|^p, \quad p \geqslant 1, \tag{4.1}$$

(see, e.g., [13]). Further, with piecewise constant coefficients, we may regard (4.1) as a special case of (1.2) with coefficients satisfying Assumption (HX).

Next, we define (Y^e, Z^e, K^e) as the solution to the following continuously reflected BSDE driven by X^π, instead of X in (1.1),

$$\begin{aligned} &Y_t^e = g(X_T^\pi) + \int_t^T f(s, X_s^\pi, Y_s^e, Z_s^e)ds - \int_t^T Z_s^e dW_s + K_T^e - K_t^e, \\ &Y_t^e \geqslant g(X_t^\pi), \quad \int_0^T (Y_t^e - g(X_t^\pi))dK_t^e = 0. \end{aligned} \tag{4.2}$$

Because $X^\pi \in \mathbb{S}^2[0, T]$ and the system (1.1)-(1.2) is decoupled, we know that $(Y^e, Z^e, K^e) \in \mathbb{S}^\infty[0, T] \times \mathbb{H}^2([0, T]; \mathbb{R}^m) \times \mathbb{K}^2[0, T]$ and can further obtain *a priori* estimates from Propositions 2.2 and 2.3, i.e.,

$$\|Y^e\|_\infty \leqslant M, \quad \|Z^e * W\|_{BMO}^2 \leqslant \frac{\exp(4\alpha M)}{\alpha^2}[1 + 2\alpha M_f(1 + M)T]$$

and the L^p-integrability of K_T^e and $\int_0^T |Z_t^e|^2 dt$ for any $p \geqslant 1$.

Now, we introduce the abovementioned discretely reflected BSDE, which is defined recursively and only operates at specific times $\mathcal{R} = \{r_j, 0 \leqslant j \leqslant \kappa \mid 0 = r_0 < r_1 \cdots < r_{\kappa-1} < r_\kappa = T\}$.

Let $|\mathcal{R}| := \max_{j \leqslant \kappa-1}(r_{j+1} - r_j)$, and for further discussion, we assume that $\mathcal{R} \subset \pi$, which means the discrete reflection times are all included in the partition time points. The solution $(Y^{\mathcal{R}}, Z^{\mathcal{R}})$ satisfies

$$Y_T^{\mathcal{R}} = \tilde{Y}_T^{\mathcal{R}} = g(X_T),$$

and for $j \leqslant \kappa - 1$, $t \in [r_j, r_{j+1})$,

$$\begin{cases} \tilde{Y}_t^{\mathcal{R}} = Y_{r_{j+1}}^{\mathcal{R}} + \int_t^{r_{j+1}} f(s, X_s, \tilde{Y}_s^{\mathcal{R}}, Z_s^{\mathcal{R}}) ds - \int_t^{r_{j+1}} Z_s^{\mathcal{R}} dW_s, \\ Y_t^{\mathcal{R}} = \tilde{Y}_t^{\mathcal{R}} + [g(X_t) - \tilde{Y}_t^{\mathcal{R}}]^+ \mathbb{1}_{\{t \in \mathcal{R}\}}. \end{cases} \tag{4.3}$$

For later use, we also define the solution $(Y^{\mathcal{R},e}, Z^{\mathcal{R},e})$ to discretely reflected BSDE, which is the same as defined in (4.3), but with X substituted by X^π, i.e.,

$$Y_T^{\mathcal{R},e} = \tilde{Y}_T^{\mathcal{R},e} = g(X_T^\pi),$$

and for $j \leqslant \kappa - 1$, $t \in [r_j, r_{j+1})$,

$$\begin{cases} \tilde{Y}_t^{\mathcal{R},e} = Y_{r_{j+1}}^{\mathcal{R},e} + \int_t^{r_{j+1}} f(s, X_s^\pi, \tilde{Y}_s^{\mathcal{R},e}, Z_s^{\mathcal{R},e}) ds - \int_t^{r_{j+1}} Z_s^{\mathcal{R},e} dW_s, \\ Y_t^{\mathcal{R},e} = \tilde{Y}_t^{\mathcal{R},e} + [g(X_t^\pi) - \tilde{Y}_t^{\mathcal{R},e}]^+ \mathbb{1}_{\{t \in \mathcal{R}\}}. \end{cases} \tag{4.4}$$

To simplify the expression, we denote the discretely reflected BSDE systems (4.3) and (4.4) as $DR(f, g, X)$ and $DR(f, g, X^\pi)$, respectively.

Next, we apply the truncation technique to handle the locally Lipschitz and quadratic growth condition. Define $f_n(t, x, y, z) := f(t, x, y, h_n(z))$ for all $(t, x, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$, where h_n is a smooth modification of the projection on the centered ball of radius n such that $|h_n(z)| \leqslant n + 1$, $|\nabla h_n| \leqslant 1$ and satisfying that $h_n(z) = z$ when $|z| \leqslant n$, for all $n \in \mathbb{R}^+$. Thus, we can define analogously the truncated discretely reflected BSDEs $DR(f_n, g, X)$ and $DR(f_n, g, X^\pi)$ to meet the Lipschitz condition, and denote their solutions by $(Y^{\mathcal{R},n}, Z^{\mathcal{R},n})$ and $(Y^{\mathcal{R},e,n}, Z^{\mathcal{R},e,n})$, respectively.

Further, from [18, Lemma 4.5], we know that the second component $Z^{\mathcal{R}}$ of the solution to discretely reflected BSDE is uniformly bounded with regard to the discrete reflection $\mathcal{R}$, and the bound M_z only depends on the given coefficients in assumptions (HF) and (HX). One can easily check that this result also holds for $Z^{\mathcal{R},e}$. Thus, taking $n = M_z$, we know that $(Y^{\mathcal{R},M_z}, Z^{\mathcal{R},M_z})$ (resp. $(Y^{\mathcal{R},e,M_z}, Z^{\mathcal{R},e,M_z})$) coincides with $(Y^{\mathcal{R}}, Z^{\mathcal{R}})$ (resp. $(Y^{\mathcal{R},e}, Z^{\mathcal{R},e})$); therefore, we only need to focus on the discrete-time scheme for such a truncated discretely reflected BSDE with parameter M_z and generator f_{M_z}, which satisfies, for all (x, y, z), $(x', y', z') \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m$, that

$$|f_{M_z}(t, x, y, z) - f_{M_z}(t, x', y', z')| \leqslant L(M_z + 2)|x - x'| + L|y - y'| + L(2M_z + 3)|z - z'|.$$

4.2 Truncated discrete-time numerical scheme

Inspired by classical numerical schemes under Lipschitz condition (see [3] [7] and [19] for BSDEs and [2] [16] for reflected BSDEs) and the truncated discretely reflected BSDE in the above subsection, we now introduce the following truly discretized scheme with the help of the truncation function h_{M_z}. We define a pair of piecewise constant process $(\bar{Y}^\pi, \bar{Z}^\pi)$ recursively via

$$\bar{Y}_{t_N}^\pi = \tilde{Y}_{t_N}^\pi = g(X_T^\pi)$$

and

$$\begin{cases} \bar{Z}^\pi_{t_i} = (t_{i+1}-t_i)^{-1}\mathbb{E}_{t_i}\big[\bar{Y}^\pi_{t_{i+1}}(W_{t_{i+1}}-W_{t_i})\big], \\ \tilde{Y}^\pi_{t_i} = \mathbb{E}_{t_i}\big[\bar{Y}^\pi_{t_{i+1}}\big] + (t_{i+1}-t_i)f\big(t_i, X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, h_{M_z}(\bar{Z}^\pi_{t_i})\big), \qquad i \leqslant N-1, \\ \bar{Y}^\pi_{t_i} = \tilde{Y}^\pi_{t_i} + \big[g(X^\pi_{t_i}) - \tilde{Y}^\pi_{t_i}\big]^+ \mathbb{1}_{\{t_i\in\mathcal{R}\}}, \end{cases} \tag{4.5}$$

and setting

$$(\bar{Y}^\pi_t, \bar{Z}^\pi_t) = (\bar{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}) \quad \text{for } t \in [t_i, t_{i+1}),\ i \leqslant N-1.$$

For later use, we shall introduce the continuous-time scheme associated with the square integrable processes $(\bar{Y}^\pi, \bar{Z}^\pi)$. By the martingale representation theorem, we know that there exists $Z^\pi \in \mathbb{H}^2([t_i, t_{i+1}); \mathbb{R}^m)$ such that

$$\bar{Y}^\pi_{t_{i+1}} = \mathbb{E}_{t_i}\big[\bar{Y}^\pi_{t_{i+1}}\big] + \int_{t_i}^{t_{i+1}} Z^\pi_u dW_u, \quad i \leqslant N-1.$$

Then, we can define $\tilde{Y}^\pi$ and Y^π for $[t_i, t_{i+1}),\ i \leqslant N-1$ by

$$\begin{cases} \tilde{Y}^\pi_t = \bar{Y}^\pi_{t_{i+1}} + (t_{i+1}-t)f_{M_z}\big(t_i, X^\pi_{t_i}, \tilde{Y}^\pi_{t_i}, \bar{Z}^\pi_{t_i}\big) - \int_t^{t_{i+1}} Z^\pi_u dW_u, \\ Y^\pi_t = \tilde{Y}^\pi_t + \big[g(X^\pi_t) - \tilde{Y}^\pi_t\big]^+ \mathbb{1}_{\{t\in\mathcal{R}\}}. \end{cases} \tag{4.6}$$

One can check the connection between (4.5) and (4.6): $Y^\pi = \bar{Y}^\pi$ on π and $Y^\pi = \tilde{Y}^\pi$ on $[0,T]\setminus\mathcal{R}$, and by Itô's isometry,

$$\bar{Z}^\pi_t = \bar{Z}^\pi_{t_i} = (t_{i+1}-t_i)^{-1}\mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} Z^\pi_u du\Big], \quad t \in [t_i, t_{i+1}),\ i \leqslant N-1.$$

Moreover, we define the piecewise constant process for $Z^{\mathcal{R}}$ likewise by

$$\bar{Z}^{\mathcal{R}}_t := (t_{i+1}-t_i)^{-1}\mathbb{E}_{t_i}\Big[\int_{t_i}^{t_{i+1}} Z^{\mathcal{R}}_u du\Big], \quad t \in [t_i, t_{i+1}),\ i \leqslant N-1.$$

which is known as the best $\mathbb{H}^2$-approximation of $Z^{\mathcal{R}}$.

4.3 Approximation results for discretely reflected BSDEs

It has been shown in [18] that discretely reflected BSDE is a good approximation of continuously reflected BSDEs. Thus, we shall first consider the convergence from the numerical scheme (4.5) to the discretely reflected BSDE in this subsection. With the boundedness of $Z^{\mathcal{R}}$ and its truncation, we can proceed under the Lipschitz condition.

There exist results about the convergence for discretely reflected BSDEs driven by X and X^π under the Lipschitz condition, see [2, Theorem 3.1 and Corollary 3.1], wherein the authors first showed that the approximation error for the discretely reflected BSDE constructed with X (resp. X^π) is ultimately controlled by $\|Z^{\mathcal{R}} - \bar{Z}^{\mathcal{R}}\|_{\mathbb{H}^2}$ (resp. $\|Z^{\mathcal{R},e} - \bar{Z}^{\mathcal{R},e}\|_{\mathbb{H}^2}$), and then by means of the representation for $Z^{\mathcal{R}}$(resp. $Z^{\mathcal{R},e}$) in terms of the next reflection time to obtain the regularity result. We may now directly apply the result to our truncated discrete-time scheme under assumptions (HX), (HF), and (HT) and the following additional assumptions.

Assumption *g and σ further satisfy:*
(H1) $g \in C_b^1$ *with L-Lipschitz derivative.*
(H2) $g \in C_b^2$ *with L-Lipschitz first and second derivatives, σ satisfies L-Lipschitz condition with respect to time variable.*

Lemma 4.1 *Suppose* (HX), (HF), *and* (HT) *hold. Then,*

$$\max_{j\leqslant\kappa-1}\|\sup_{t\in[r_j,r_{j+1}]}|Y_t^{\mathcal{R}}-Y_t^\pi|\|_{\mathbb{L}^2}+\max_{i\leqslant N-1}\|\sup_{t\in(t_i,t_{i+1}]}|Y_t^{\mathcal{R}}-\bar{Y}_{t_{i+1}}^\pi|\|_{\mathbb{L}^2}\leqslant C\big(\alpha_1(\kappa)|\pi|^{\frac{1}{2}}+\epsilon_1(\pi)\big),$$
$$\|Z^{\mathcal{R}}-Z^\pi\|_{\mathbb{H}^2}+\|Z^{\mathcal{R}}-\bar{Z}^\pi\|_{\mathbb{H}^2}\leqslant C\big(\alpha_2(\kappa)|\pi|^{\frac{1}{2}}+\epsilon_1(\pi)\big),$$
$$\|Z^{\mathcal{R},e}-Z^\pi\|_{\mathbb{H}^2}+\|Z^{\mathcal{R},e}-\bar{Z}^\pi\|_{\mathbb{H}^2}\leqslant C\big(\alpha_1(\kappa)|\pi|^{\frac{1}{2}}+\epsilon_2(\pi)\big),$$

with $(\alpha_1(\kappa),\alpha_2(\kappa),\epsilon_1(\pi),\epsilon_2(\pi))=(\kappa^{\frac{1}{4}},\kappa^{\frac{1}{2}},|\pi|^{\frac{1}{4}},|\pi|^{\frac{1}{4}})$ *under* (H1), *and* $(\alpha_1(\kappa),\alpha_2(\kappa),\epsilon_1(\pi),\epsilon_2(\pi))=(1,\kappa^{\frac{1}{2}},|\pi|^{\frac{1}{2}},|\pi|^{\frac{1}{4}})$ *under* (H2).

Proof. Keeping $(Y^{\mathcal{R}},Z^{\mathcal{R}})=(Y^{\mathcal{R},M_z},Z^{\mathcal{R},M_z})$ and $Z^{\mathcal{R},e}=Z^{\mathcal{R},e,M_z}$ in mind and applying the main theorem in [2] under Lipschitz case to our truncated scheme (4.5) and (truncated) discretely reflected BSDE (4.3)/(4.4), we can obtain the conclusion directly. □

4.4 Approximation results for continuously reflected BSDEs

We recall our previous result about the convergence rate from discretely to continuously reflected BSDE in [18]. As mentioned before, one can readily verify that all results therein still hold when we replace X by X^π and under the general driver f involving y.

Lemma 4.2 *Let* (HX) *and* (HF) *hold. Then,*

$$\max_{j\leqslant\kappa-1}\|\sup_{t\in[r_j,r_{j+1}]}|Y_t-Y_t^{\mathcal{R}}|\|_{\mathbb{L}^2}+\|Z-Z^{\mathcal{R}}\|_{\mathbb{H}^2}\leqslant C|\mathcal{R}|^{\frac{1}{4}},$$
$$\max_{j\leqslant\kappa-1}\|\sup_{t\in[r_j,r_{j+1}]}|Y_t^e-Y_t^{\mathcal{R},e}|\|_{\mathbb{L}^2}+\|Z^e-Z^{\mathcal{R},e}\|_{\mathbb{H}^2}\leqslant C|\mathcal{R}|^{\frac{1}{4}}.$$

In addition, if Assumption (H1) *holds, the index of convergence rate will become* $\frac{1}{2}$.

Notably, the conclusion under Assumption (H1) can be obtained from [18, Theorem 4.6] using an approximation argument as usual. Finally, we present our main theorem of this section regarding the convergence result of the numerical scheme to continuously reflected BSDE with quadratic growth and deterministic σ. To ensure consistency between the two convergence criteria appearing in the above lemmas 4.1 and 4.2, we assume the reflection points and the partition points coincide, i.e., $\mathcal{R}=\pi$ (thus $\kappa=N$) in the following theorem.

Theorem 4.3 *Suppose* (HX), (HF), (HT) *and* (H1) *hold. Then, the following estimates hold with* $q=\frac{1}{4}$*:*

$$\max_{i\leqslant N-1}\|\sup_{t\in[t_i,t_{i+1}]}|Y_t-Y_t^\pi|+\sup_{t\in(t_i,t_{i+1}]}|Y_t-\bar{Y}_{t_{i+1}}^\pi|\|_{\mathbb{L}^2}\leqslant C|\pi|^q,$$
$$\|Z-Z^\pi\|_{\mathbb{H}^2}+\|Z-\bar{Z}^\pi\|_{\mathbb{H}^2}\leqslant C|\pi|^{\frac{1}{4}}.$$

Moreover, under Assumption (H2), *we have finer result for the* Y *part with* $q=\frac{1}{2}$.

Proof.
Y part: Lemma 4.1 and 4.2 lead straightforward to the result for Y with $\mathcal{R} = \pi$.

Z part: As shown in Lemma 4.1, one cannot get the final convergence with only $Z^{\mathcal{R}}$ due to the problematic term $\kappa^{\frac{1}{2}}|\pi|^{\frac{1}{2}}$ on the righthand side of the estimate. Notably, the problem cannot be resolved by simply increasing the regularity assumption on g and σ. Thus, we need to proceed with the help of $Z^{\mathcal{R},e}$.

Considering $(X^1, Y^1, Z^1) = (X, Y, Z)$, $(X^2, Y^2, Z^2) = (X^\pi, Y^e, Z^e)$ and applying the stability result in Theorem 3.2, we obtain from the estimate (4.1) that

$$\|Z - Z^e\|^2_{\mathbb{H}^2} \leqslant C\|X - X^\pi\|_{\mathbb{S}^{4\bar{p}q^*}} \leqslant C|\pi|^{\frac{1}{2}},$$

where $\bar{p}$ and q^* are given in Theorem 3.2. Then, the conclusion for the solution component Z follows from the results related to $Z^{\mathcal{R},e}$ in Lemma 4.1 and 4.2. This completes the proof. □

5 Conclusions

In this paper, we proposed a truncated discrete-time numerical scheme for quadratic reflected BSDEs. To prove the convergence, we developed a quantitative stability result for the quadratic reflected BSDE, and then adapted the numerical analysis for quadratic BSDEs without reflection and Lipschitz BSDEs with reflection. One of the critical conditions is the deterministic assumption on the volatility term σ, which was imposed to guarantee the uniform boundedness for the solution component $Z^{\mathcal{R}}$ in the corresponding discretely reflected BSDE. A natural extension is to consider the multiplicative σ by allowing it to depend on the underlying states. This is far more challenging, and the major difficulty is to obtain a uniform estimate for $Z^{\mathcal{R}}$ with respect to the discrete reflection $\mathcal{R}$. Such an extension is left for future research.

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