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The Singularity and Cosingularity Categories of C^*BG for Groups with Cyclic Sylow p-subgroups

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Abstract

We construct a differential graded algebra (DGA) modelling certain A_∞ algebras associated with a finite group G with cyclic Sylow subgroups, namely H^*BG and $H_*\Omega BG^\wedge_p$. We use our construction to investigate the singularity and cosingularity categories of these algebras. We give a complete classification of the indecomposables in these categories, and describe the Auslander–Reiten quiver. The theory applies to Brauer tree algebras in arbitrary characteristic, and we end with an example in characteristic zero coming from the Hecke algebras of symmetric groups.

Keywords A_∞ algebras · Auslander–Reiten quiver · Auslander–Reiten triangles · Brauer trees · Cyclic Sylow subgroups · Cohomology of groups · Cosingularity categories · Derived categories · DG Hopf algebras · Hecke algebras · Hochschild cohomology · Loop spaces · Massey products · p-completed classifying spaces · Singularity categories · Spectral sequences

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1 Introduction

Our purpose is to study the singularity categories of certain A_∞ algebras over a field k. We were led to these examples from the representation theory in characteristic p of finite groups

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with cyclic Sylow p-subgroups, but our earlier work [9], on which we build, showed these examples were members of a more general family of examples: the general case illuminates those we first considered, and the other examples also occur elsewhere.

In fact the examples occur in Koszul dual pairs A and B. The BGG correspondence shows that it is illuminating to consider both members of the pair together: the classical example occurs with A an exterior algebra and B the Koszul dual polynomial algebra. The singularity category of A is equivalent to the cosingularity category of B, which by a theorem of Serre is the bounded derived category of quasicoherent sheaves on $\mathsf{Proj}(B)$. Since A is finite dimensional, its cosingularity category is trivial; since B is regular, its singularity category is trivial. In this case both A and B are formal as k-algebras.

We consider here a family of the next simplest cases consisting of a non-formal A_∞ k-algebra A, usually with homology

$$H_*(A) = k[\tau] \otimes \Lambda(\xi)$$

where τ has even degree $2b$ and ξ has odd degree $2a - 1$. The family of examples we study is determined by a, b and two further parameters $h, \ell \geq 2$ related by $ah - b\ell = 1$. The parameter h is the length of the shortest non-trivial Massey product (when $h = 2$ the homology ring is a little different to that above, since $\xi^2 = -\tau^\ell$). We give a full description of A in Section 4. It is shown in [23] that the BGG correspondence extends to a more general A_∞ context, and it is again natural to consider the Koszul dual B. In fact B is of exactly the same form as A but with different degrees, and the parameters h and ℓ exchanged. In most cases it again has homology of form

$$H_*(B) = \Lambda(t) \otimes k[x]$$

where x is of even degree $-2a$ and t is of odd degree $-2b - 1$. The parameter ℓ is the length of the shortest non-trivial Massey product (when $\ell = 2$ the homology is a little different to that above, since $t^2 = -x^h$). We give a full description of B in Section 9. In this case both A and B have singularity categories that are non-trivial and we are able to give a complete description: they each have finitely many indecomposable objects and we describe their Auslander–Reiten quivers. This behaviour is rather special. In general, even when the singularity category of A has finitely many indecomposables, the singularity category of B can have infinitely many. The behaviour of A and B is also quite different to the behaviour of the formal algebras $H_*(A)$ and $H_*(B)$, whose singularity and cosingularity categories all have infinitely many indecomposable objects.

Our first task is to describe a small and explicit DG algebra Q in the same quasi-isomorphism class as A, with some good properties that make it suitable for both theoretical and computational work. As a step towards Q, we first introduce an auxiliary DG algebra R in Section 2, which embodies the algebra of an odd element all of whose Massey powers vanish: it is generated by elements $\xi_1, \xi_2, \ldots$, and has homology $H_*(R) = \Lambda(\xi)$. The algebra Q can be viewed as a universal object for an algebra with an h-fold Massey power of an element of odd degree which is an ℓth power of an element of even degree. It is generated by elements $\tau, \xi_1, \ldots, \xi_{h-1}$, with τ and ξ_1 representing elements τ, ξ in $H_*Q \cong A$. Explicit formulas for the relations and the action of the differential d are given in Section 3. The element τ of Q is central, so Q may be regarded as an algebra over $k[\tau]$. Our principal goal is to determine the structure of the singularity and cosingularity categories of A and B (see Section 6 for definitions). Our main theorems classify the indecomposable objects in these categories, see Theorems 9.3 and 15.3. The technical hypotheses for this theorem are spelled out in Section 4, and we repeat them in condensed form here for convenience.

Context 1.1 Let a, b, h and ℓ be integers with $h, \ell \geq 2$ and $ah - b\ell = 1$. Let A be the A_∞ algebra with a second (internal) $\mathbb{Z}$-grading, such that m_2 is the strictly associative multiplication gives the ring structure of $k[\tau] \otimes \Lambda(\xi)$ if $h \geq 3$ and $k[\tau, \xi]/(\xi^2 + \tau^\ell)$ if $h = 2$, with $|\xi| = (2a - 1, \ell)$ and $|\tau| = (2b, h)$. All m_i are zero except for m_2 and m_h, which is determined by $m_h(\xi, \dots, \xi) = (-1)^{h(h-1)/2}\,\tau^\ell$.

Let B be the Koszul dual A_∞ algebra with the roles of a and b replaced by $-b$ and $-a$, and the roles of h and ℓ replaced by $-\ell$ and $-h$. Thus the ring structure is given by $k[x] \otimes \Lambda(t)$ if $\ell \geq 3$ and $k[x, t]/(t^2 + x^h)$ if $\ell = 2$, with $|t| = (-2b - 1, -h)$ and $|x| = (-2a, -\ell)$. All m_i are zero except m_2 and m_ℓ, which is determined by $m_\ell(t, \dots, t) = (-1)^{\ell(\ell-1)/2}\,x^h$.

Theorem 1.2 *Suppose that A and B are the Koszul dual A_∞ algebras as in Context 1.1. The equivalence of triangulated categories $\mathsf{D}^{\mathsf{b}}(A) \simeq \mathsf{D}^{\mathsf{b}}(B)$ induces equivalences*

$$\mathsf{D}_{\mathsf{csg}}(A) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}]) \simeq \mathsf{D}_{\mathsf{sg}}(B).$$

The latter categories satisfy the Krull–Schmidt theorem, and have $|b|(h-1)$ isomorphism classes of indecomposable objects, in $[h/2]$ orbits of the shift functor Σ. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{h-1}/T^{|b|}$, where T is the translation functor Σ^{-2a}. This is a cylinder of height $h-1$ and circumference $|b|$. The functor Σ switches the two ends of the cylinder.

We give explicit descriptions of the indecomposable objects, both as elements of $\mathsf{D}_{\mathsf{csg}}(A) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$ and as elements of $\mathsf{D}_{\mathsf{sg}}(B)$. Reversing the roles of A and B, swapping h and ℓ, and replacing a by $-b$ and b by $-a$ gives us the structure of $\mathsf{D}_{\mathsf{sg}}(A) \simeq \mathsf{D}_{\mathsf{csg}}(B)$, with $|a|(\ell - 1)$ isomorphism classes of indecomposable objects, coming in $[\ell/2]$ orbits of Σ. We also give an explicit description of the Auslander–Reiten quivers of these categories, and explain their position in the Amiot's classification of finite triangulated categories. This leads to some explicit models such as $\mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}^{\mathsf{b}}(A_{h-1})/\mathsf{T}^{|b|}$ in Section 16.

Of course the simplicity of the A_∞ structure is essential for explicit calculations, but key structural features making a complete analysis possible are the τ-periodicity and the Tate duality of Theorem 10.8. The key ingredient for Tate duality is the fact that finitely generated modules are automatically dualizable, which we proved in this case using a Hochschild cohomology calculation.

We are especially interested in the following occurrence of the A_∞ algebras A and B. Let p be an odd prime, let G be a finite group with cyclic Sylow p-subgroups P of order p^n and inertial index $q > 1$, and let k be a field of characteristic p. Omitting notation for coefficients in the field k, and writing $\Omega BG^\wedge_p$ for the loop space on the Bousfield–Kan mod p completion of BG, we showed in [9] that the A_∞ algebra structures on $A = H_*\Omega BG^\wedge_p$ and $B = H^*BG$ gave an instance of Context 1.1 with $a = q$, $b = q - 1$, $h = p^n - (p^n - 1)/q$, $\ell = p^n$. Then the DG algebra Q describes a model for the DG algebra $C_*\Omega BG^\wedge_p$ up to quasi-isomorphism, and by reversing the roles of A and B we obtain a model for C^*BG.

Theorem 1.3 *Let $h = p^n - (p^n - 1)/q$, $\ell = p^n$. Then the equivalence of triangulated categories $\mathsf{D}^{\mathsf{b}}(C^*BG) \simeq \mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_p)$ induces equivalences*

$$\mathsf{D}_{\mathsf{csg}}(C_*\Omega BG^\wedge_p) \simeq \mathsf{D}^{\mathsf{b}}(C_*\Omega BG^\wedge_p)[\tau^{-1}] \simeq \mathsf{D}_{\mathsf{sg}}(C^*BG).$$

The latter categories are finite Krull–Schmidt triangulated categories with $(q-1)(h-1)$ *indecomposable objects in* $[h/2]$ *orbits of the shift functor. It also induces equivalences*

$$\mathsf{D}_{\mathsf{csg}}(C^*BG) \simeq \mathsf{D}^{\mathsf{b}}(C^*BG)[x^{-1}] \simeq \mathsf{D}_{\mathsf{sg}}(C_*\Omega BG^\wedge_p).$$

The latter categories are finite Krull–Schmidt triangulated categories with $q(p^n-1)$ *indecomposable objects in* $[p^n/2]$ *orbits of the shift functor.*

Other examples of the A_∞ algebras A and B occur every time an algebra is described by a Brauer tree of finite representation type. These occur throughout representation theory, both in characteristic zero and in prime characteristic. For the sake of describing an example in characteristic zero, we discuss the Hecke algebras of symmetric groups. Let $\mathcal{H} = \mathcal{H}(n, q)$ be the Hecke algebra of the symmetric group of degree n over a field k of characteristic zero, where q is a primitive ℓth root of unity with $n = \ell > 2$. Then letting A be the principal block of $\mathcal{H}$ and B be $\mathsf{Ext}^*_{\mathcal{H}}(k, k)$, we obtain an example with $a = n-1$, $b = n-2$, $h = n-1$ and $\ell = n$. We spell out the consequences of our main theorem in this case, in Theorem 18.3.

2 The DG Hopf algebra *R*

We begin by looking at the DG Hopf algebra R over a field k. As a graded algebra over k, R is free with odd degree generators $\xi_1, \xi_2, \xi_3, \dots$, and the differential is given by

$$d(\xi_i) = \sum_{j+k=i} \xi_j\xi_k \qquad (i \geqslant 1).$$

Thus $d(\xi_1) = 0$, $d(\xi_2) = \xi_1^2$, $d(\xi_3) = \xi_1\xi_2 + \xi_2\xi_1$, and so on.

Remark 2.1 To motivate this, we factor out the differential ideal of R generated by ξ_i for $i \geqslant h+1$ and take the DG-subalgebra R_h generated by ξ_i for $i \leqslant h-1$. The element $\mu_h = d(\xi_h)$ lies in R_h and represents the h-fold Massey power of the homology class of ξ_1 up to sign. Thus a DGA map from $\theta\colon R_h \to C$ in which $\theta(\xi_1) = c$ shows that the h-fold Massey power of $[c]$ is defined and gives an element $[\theta(\mu_h)] \in \pm\langle [c], \cdots, [c]\rangle$. Accordingly, in R itself, all Massey powers of $[\xi_1]$ contain zero.

The antipode S on R is the anti-automorphism of algebras given on generators by $S(\xi_i) = -\xi_i$. The comultiplication $\Delta\colon R \to R \otimes R$ is defined on generators by

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i.$$

Note that if ξ_1 has degree $2a-1$ then ξ_i has degree $2ia-1$. So R is either connected or coconnected, according to whether $2a-1$ is positive or negative. We shall assume that $a \neq 0$, so that $|\xi_1| \neq -1$, which implies that each graded piece is finite dimensional.

Lemma 2.2 *In* R *we have* $d^2 = 0$.

Proof To show that $d^2 = 0$, we note that $dd(\xi_i)$ has two terms for each way of writing i as a sum of three positive integers. They have opposite signs, because the elements ξ_i have odd degree, so they cancel:

$$d^2(\xi_i) = \sum_{j+k=i} (d(\xi_j)\xi_k - \xi_j d(\xi_k)) = \sum_{j+k+\ell=i} (\xi_j\xi_k\xi_\ell - \xi_j\xi_k\xi_\ell) = 0. \qquad \square$$

Lemma 2.3 *The map $\Delta\colon R \to R \otimes R$ is a map of DG algebras.*

Proof As an algebra, R is free, so specifying the map on generators gives a well defined map of algebras. We must check that it commutes with the differential. Since the ξ_j have odd degree and are primitive, ξ_j^2 and $\xi_j\xi_k+\xi_k\xi_j$ are also primitive. So $d(\xi_i) = \sum_{j+k=i} \xi_j\xi_k$ is also primitive, and hence $d\Delta(\xi_i) = \Delta d(\xi_i)$. □

Proposition 2.4 *The definitions above make R into a cocommutative DG Hopf algebra.*

Proof Lemmas 2.2 and 2.3 show that R is a DG bialgebra. It is easy to check that the antipode satisfies the identity $S(x_{(1)})x_{(2)} = x_{(1)}S(x_{(2)}) = 0$ in Sweedler notation, for elements of non-zero degree; this only needs checking on the generators, where it is clear. Cocommutativity also only needs checking on generators. □

Lemma 2.5 $H_*R = \Lambda(\xi_1)$.

Proof Define a linear map $\delta\colon R \to R$ sending a monomial of the form $\xi_1\xi_i f$ to $\xi_{i+1}f$, and sending all other monomials to zero. Then we have

$$\begin{aligned}
\delta d(\xi_1\xi_i f) &= \delta(-\xi_1(\xi_1\xi_{i-1} + \cdots + \xi_{i-1}\xi_1)f + \xi_1\xi_i df)\\
&= -(\xi_2\xi_{i-1} + \cdots + \xi_i\xi_1)f + \xi_{i+1}df\\
d\delta(\xi_1\xi_i f) &= d(\xi_{i+1}f) = (\xi_1\xi_i + \cdots + \xi_i\xi_1)f - \xi_{i+1}df\\
(\delta d + d\delta)(\xi_1\xi_i f) &= \xi_1\xi_i f,
\end{aligned}$$

while for $j > 1$ we have

$$\begin{aligned}
d\delta(\xi_j f) &= d(0) = 0\\
\delta d(\xi_j f) &= \delta((\xi_1\xi_{j-1} + \cdots + \xi_{j-1}\xi_1)f - \xi_j df) = \xi_j f\\
(d\delta + \delta d)(\xi_j f) &= \xi_j f.
\end{aligned}$$

Thus $\delta d + d\delta$ is the identity on all monomials apart from 1 and ξ_1, on which it vanishes. So δ defines a homotopy from the identity map of R to the projection onto the linear span of 1 and ξ_1. It follows that $H_*R = \Lambda(\xi_1)$. □

Definition 2.6 The *weight* of a monomial in R is the sum of the subscripts (and zero for constants). The *height* of a monomial in R is the number of generators ξ_i that have to be multiplied to give the monomial (we might call it degree, if that didn't already have a different meaning). Multiplication in R adds weights, and adds heights. If $f(\xi_1, \ldots, \xi_n)$ is an element of R, we write $f_{i,j}(\xi_1, \ldots, \xi_n)$ for the sum of the terms of f with weight i and height j. Thus $f = \sum_{i,j} f_{i,j}$.

The differential d preserves weight, and increases height by one, so that $d(f_{i,j}) = (df)_{i,j+1}$. Thus if $df = 0$ then each $d(f_{i,j}) = 0$.

3 The DG Hopf Algebra Q

In this section, we let h and ℓ be integers ≥ 2 and describe the DG Hopf algebra $Q = Q_{h,\ell}$. In terms of the previous section, the idea is that $Q_{h,\ell}$ is obtained from R_h by adjoining an ℓth root to the element $-\mu_h$. Thus a DGA map $\theta\colon Q_{h,\ell} \to C$ shows that the h-fold Massey power of $c = \theta(\xi_1)$ is defined and contains an ℓth power of the adjoined variable.

The generators for Q are $\xi_1, \dots, \xi_{h-1}$ in odd degree and τ in even degree. The relations and differential are as follows:

$$\begin{aligned} \tau\xi_i &= \xi_i\tau \qquad 1 \le i \le h-1 \\ d(\tau) &= 0 \\ \sum_{j+k=i} \xi_j\xi_k &= \begin{cases} d(\xi_i) & 1 \le i \le h-1 \\ -\tau^\ell & i = h \\ 0 & h+1 \le i \le 2h-2. \end{cases} \end{aligned}$$

The antipode is the algebra anti-automorphism given by $S(\xi_i) = -\xi_i$, $S(\tau) = -\tau$, and the comultiplication is given by

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i, \qquad \Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau.$$

We write $|\xi_1| = 2a-1$, and we assume that $a \neq 0$. The relations imply that $|\xi_i| = 2ia-1$ and $|\tau^\ell| = 2ah-2$. So writing $2b$ for $|\tau|$ we have $2b\ell = 2ah-2$, or equivalently

$$ah - b\ell = 1.$$

In particular, a and b are coprime, as are h and ℓ.

As well as this homological grading, we give Q a second, internal grading by setting

$$|\xi_i| = (2ia-1, i\ell), \qquad |\tau| = (2b, h).$$

It is easy to check that the relations, differential, and Hopf structure above respect this second grading.

Example 3.1 If $h = 2$, the algebra Q is generated by ξ_1 and τ with relations

$$\begin{aligned} d(\tau) &= 0 & \tau\xi_1 &= \xi_1\tau \\ d(\xi_1) &= 0 & \xi_1^2 &= -\tau^\ell \end{aligned}$$

In this case the differential is zero, and Q is just the graded algebra $k[\tau, \xi_1]/(\xi_1^2 + \tau^\ell)$.

Example 3.2 If $h = 3$, the algebra Q is generated by ξ_1, ξ_2 and τ with relations

$$\begin{aligned} d(\tau) &= 0 & \tau\xi_i &= \xi_i\tau \qquad 1 \leqslant i \leqslant 2 \\ d(\xi_1) &= 0 & \xi_1\xi_2 + \xi_2\xi_1 &= -\tau^\ell \\ d(\xi_2) &= \xi_1^2 & \xi_2^2 &= 0. \end{aligned}$$

Example 3.3 If $h = 4$, the algebra Q is generated by ξ_1, ξ_2, ξ_3 and τ with relations

$$\begin{aligned} d(\tau) &= 0 & \tau\xi_i &= \xi_i\tau \qquad 1 \leqslant i \leqslant 3 \\ d(\xi_1) &= 0 & \xi_1\xi_3 + \xi_2^2 + \xi_3\xi_1 &= -\tau^\ell \\ d(\xi_2) &= \xi_1^2 & \xi_2\xi_3 + \xi_3\xi_2 &= 0 \\ d(\xi_3) &= \xi_1\xi_2 + \xi_2\xi_1 & \xi_3^2 &= 0. \end{aligned}$$

Lemma 3.4 *In the algebra Q, every element has a unique expression of the form*

$$f(\xi_1, \dots, \xi_{h-2}) + \xi_{h-1}g(\xi_1, \dots, \xi_{h-2})$$

with coefficients in $k[\tau]$.

Proof The algebra relations (ignoring the differential) can be rewritten in the form

$$\xi_i \xi_{h-1} = \xi_{h-1} \phi_i(\xi_1, \dots, \xi_{h-2}),$$

with $1 \leq i \leq h-1$ (note that $\phi_{h-1} = 0$). Thus all occurrences of ξ_{h-1} may be moved to the beginning, and $\xi_{h-1}^2 = 0$. There are no relations among $\xi_1, \dots, \xi_{h-2}$. □

Definition 3.5 We shall refer to a monomial in $\xi_1, \dots, \xi_{h-2}$, or ξ_{h-1} times such a monomial, as a *standard monomial* in the variables $\xi_1, \dots, \xi_{h-1}$. The lemma shows that the standard monomials form a basis for Q as a free module over $k[\tau]$.

Lemma 3.6 *In the algebra Q, we have $d^2 = 0$.*

Proof The differential is given by

$$d(f + \xi_{h-1} g) = (df + (\xi_1 \xi_{h-2} + \dots + \xi_{h-2} \xi_1) g) - \xi_{h-1} dg.$$

On the free subalgebra generated by $\xi_1, \dots, \xi_{h-2}$, the differential is the same as in the algebra R above, and so by Lemma 2.2 we have $d^2 = 0$ on this subalgebra. Thus we have

$$\begin{aligned} d^2(f + \xi_{h-1} g) &= d(df + (\xi_1 \xi_{h-2} + \dots + \xi_{h-2} \xi_1) g - \xi_{h-1} dg) \\ &= d^2 f + (\xi_1 \xi_{h-2} + \dots + \xi_{h-2} \xi_1) dg - (\xi_1 \xi_{h-2} + \dots + \xi_{h-2} \xi_1) dg \\ &= 0. \end{aligned}$$

□

Lemma 3.7 *The map $\Delta\colon Q \to Q \otimes Q$ is a map of DG algebras.*

Proof As in Lemma 2.3, for each $i > 0$, $\sum_{j+k=i} \xi_j \xi_k$ is primitive, so both the relations and the elements $d(\xi_i)$ are primitive. Hence Δ takes relations to relations, and $d\Delta = \Delta d$. □

Proposition 3.8 *The definitions above make Q into a cocommutative DG Hopf algebra.*

Proof The proof of this is similar to the proof of Proposition 2.4, but using Lemmas 3.6 and 3.7 in place of Lemmas 2.2 and 2.3. □

4 The A_∞ Algebras A and B

In this section we introduce an A_∞ algebra A which is quasi-isomorphic to the DG algebra Q of the last section. We then describe the Koszul dual B of A. These are the A_∞ algebras discussed in our previous paper [9].

Let h and ℓ be positive integers with $h \geq 3$, and let k be a field. We are interested in the following A_∞ algebra $A = A_{h,\ell}$. The differential m_1 is zero, m_2 is the strictly associative multiplication giving A the ring structure of $k[\tau] \otimes \Lambda(\xi)$, where $|\xi| = (2a-1, \ell)$ and $|\tau| = (2b, h)$, with $ah - b\ell = 1$. We have

$$m_h(\xi, \dots, \xi) = (-1)^{h(h-1)/2} \tau^\ell, \tag{4.1}$$

which implies

$$m_h(\tau^{j_1}\xi, \dots, \tau^{j_h}\xi) = (-1)^{h(h-1)/2} \tau^{\ell + j_1 + \dots + j_h}$$

for all $j_1, \dots, j_h \geqslant 0$. All m_i for $i > 2$ on all other i-tuples of monomials give zero. We allow the elements τ and ξ to be either in positive or in negative degree, and we grade everything homologically. The relation (4.1) may be interpreted as saying that the Massey product of h copies of ξ is equal to $-\tau^\ell$, the sign being the standard one relating Massey products with A_∞ structure; see [38, Theorem 3.1], [15, Theorem 3.2].

We extend the definition to $h = 2$ by letting A be the formal A_∞ algebra $k[\tau, \xi]/(\xi^2 + \tau^\ell)$, with $|\xi| = (2a-1, \ell)$, $|\tau| = (2b, 2)$, $2a - b\ell = 1$, $a \neq 0$, and with all m_i apart from m_2 equal to zero.

We next show that the DG algebra Q (forgetting the Hopf structure) is quasi-isomorphic to A as an A_∞-algebra. In other words, $A \cong H_*(Q)$, with the A_∞ structure given by Kadeishvili's theorem [28].

Theorem 4.2 *There is a quasi-isomorphism from the DG algebra Q to the A_∞ algebra A, sending τ to τ and ξ_1 to ξ.*

Proof First, we show that H_*Q is isomorphic to A as an algebra over $k[\tau]$. The proof is similar to the proof of Lemma 2.5, but working over $k[\tau]$ instead of k. Namely, we define a linear map $\delta\colon Q \to Q$ sending a monomial of the form $\xi_1\xi_i f$ to $\xi_{i+1} f$ for $1 \leqslant i \leqslant h-2$, and all other standard monomials (see Definition 3.5) to zero. Thus $\delta(f + \xi_{h-1}g) = \delta(f)$. The same computation as in Lemma 2.5 shows that $\delta d + d\delta$ is the identity on all monomials except those in $k[\tau] + \xi_1 k[\tau]$, where it is zero. Thus δ defines a homotopy from the identity map of Q to the projection onto $k[\tau]+\xi_1 k[\tau]$. It follows that H_*Q is isomorphic to A as a ring, with τ and ξ_1 corresponding to τ and ξ. The maps m_i on H_*Q are easy to calculate using the elements ξ_i, and give zero for $i > 2$ except in the case of m_h, where it gives $m_h(\xi, \ldots, \xi) = (-1)^{h(h-1)/2}\tau^\ell$. Using Kadeishvili's theorem [28] completes the proof. $\square$

Corollary 4.3 *We have* $H_*(Q) \cong \begin{cases} k[\tau] \otimes \Lambda(\xi) & h > 2 \\ k[\tau, \xi]/(\xi^2 + \tau^\ell) & h = 2. \end{cases}$

Let B be the A_∞ algebra whose algebra structure is $k[x] \otimes \Lambda(t)$ with $|x| = (-2a, -\ell)$, $|t| = (-2b-1, -h)$,

$$m_\ell(x^{j_1}t, \ldots, x^{j_\ell}t) = (-1)^{\ell(\ell-1)/2}\, x^{h+j_1+\cdots+j_\ell},$$

and all m_i with $i > 2$ are zero on all other monomials. In the exceptional case where $\ell = 2$, we define B to be the formal A_∞ algebra $k[x, t]/(t^2 + x^h)$. It was shown in [9] that A and B are Koszul dual. Thus A is quasi-isomorphic to $\mathcal{E}nd_{\mathsf{D^b}(B)}(k)$ and B is quasi-isomorphic to $\mathcal{E}nd_{\mathsf{D^b}(A)}(k)$. Here, $\mathcal{E}nd_{\mathsf{D^b}(B)}(k)$ denotes the A_∞ endomorphism ring whose homology is

$$H_*\mathcal{E}nd_{\mathsf{D^b}(B)}(k) \cong \mathsf{End}_{\mathsf{D^b}(B)}(k),$$

and so on.

5 Hochschild Cohomology

We will recall the definition of the Hochschild cohomology of an A_∞-algebra and then calculate it for the A_∞ algebras A and B described in the previous section. The point of this is that nilpotent elements in (for example) $HH^*(A)$ control certain uniform processes of construction in the category of A-modules: we will make essential use of this in our proof of Theorem 9.7.

The bar resolution $\mathbb{B}(\mathfrak{a}) = \bigoplus_{n\geq 0} \mathfrak{a}^{\otimes(n+2)}$ of an A_∞ algebra $\mathfrak{a}$ is described in Section 3 of Getzler and Jones [20], see also Definition 12.6 of Stasheff [42]. The action of the differential on $\mathfrak{a}^{\otimes(n+2)}$ in bar notation is

$$\begin{aligned} d(x \otimes [a_1|\dots|a_n] \otimes y) &= \sum_{j=0}^{n} \pm m_{j+1}(x, a_1, \dots, a_j) \otimes [a_{j+1}|\dots|a_n] \otimes y \\ &+ \sum_{0\leqslant i+j\leqslant n} \pm x \otimes [a_1|\dots|a_i|m_j(a_{i+1}, \dots, a_{i+j})|a_{i+j+1}|\dots|a_n] \otimes y \\ &+ \sum_{j=0}^{n} \pm x \otimes [a_1|\dots|a_{n-j}] \otimes m_{j+1}(a_{n-j+1}, \dots, a_n, y), \end{aligned}$$

where the signs are determined by the usual sign conventions. It is explained in Section 3.6 of Keller [32] how to compute the signs by looking at the reduced tensor coalgebra of the suspension. Taking $\mathfrak{a}$-$\mathfrak{a}$-bimodule homomorphisms to a bimodule M, we obtain the differential on Hochschild cochains

$$\mathsf{Hom}_{\mathfrak{a},\mathfrak{a}}(\mathfrak{a}^{\otimes(n+2)}, M) \cong \mathsf{Hom}_k(\mathfrak{a}^{\otimes n}, M)$$

as follows:

$$(df)[a_1|\dots|a_n] = d(f[a_1|\dots|a_n]) + \sum_{0\leqslant i+j\leqslant n} \pm f[a_1|\dots|a_i|m_j(a_{i+1}, \dots, a_{i+j})|a_{i+j+1}|\dots|a_n],$$

see also Section 1 of Roitzheim and Whitehouse [41]. The cohomology of this complex is $HH^*(\mathfrak{a}, M)$. If $M = \mathfrak{a}$, we write $HH^*(\mathfrak{a})$ for $HH^*(\mathfrak{a}, \mathfrak{a})$.

We filter $\mathbb{B}(\mathfrak{a})$ by number of bars, $F_i\,\mathbb{B}(\mathfrak{a}) = \bigoplus_{n\leq i} \mathfrak{a}^{\otimes(n+2)}$. This gives a filtration on Hochschild cochains, for which F_i is formed by the cochains which vanish on $F_i\,\mathbb{B}(\mathfrak{a})$. With this filtration, F_0 is the whole complex and $\bigcap_i F_i = 0$. This leads to a spectral sequence in which the differentials d_n are given by the terms involving $\pm m_{n+1}$. Thus the E^1 page is the Hochschild complex of $H_*\mathfrak{a}$ with coefficients in H_*M, and the E^2 page is $HH^*(H_*\mathfrak{a}, H_*M)$. So the spectral sequence takes the form

$$HH^*(H_*\mathfrak{a}, H_*M) \Rightarrow HH^*(\mathfrak{a}, M). \tag{5.1}$$

We are numbering everything homologically, so the Hochschild degrees in $HH^*H_*\mathfrak{a}$ are negative, and the spectral sequence lives in the second and third quadrants.

Applying $\varprojlim$ with respect to i to the exact sequences $0 \to F_i \to F_0 \to F_0/F_i \to 0$, we get

$$0 = \bigcap_i F_i \to F_0 \to \varprojlim F_0/F_i \to \varprojlim{}^1 F_i \to 0.$$

So the spectral sequence is conditionally convergent, and is strongly convergent if and only if $\varprojlim{}^1 F_i = 0$, which is equivalent to $\varprojlim{}^1 E^i = 0$ in the spectral sequence, see for example Theorem 7.1 of Boardman [10]. In particular, if each graded piece of E^r is finite dimensional for some r, then the spectral sequence (5.1) is strongly convergent.

Theorem 5.2 *Given a map of exact couples $(D, E) \to (D', E')$, where both spectral sequences are conditionally convergent and live in the (homologically indexed) left half plane, if the map $E^r \to E'^r$ is an isomorphism for some r then $D \to D'$ is an isomorphism of filtered graded groups.*

Proof This follows from Theorem 7.2 of Boardman [10], since the hypotheses imply that $E^\infty \to E'^\infty$ and $\lim\limits_{\leftarrow}{}^1 E^i \to \lim\limits_{\leftarrow}{}^1 E'^i$ are isomorphisms. See also Theorem B.7 of Greenlees and May [22]. $\square$

We need to know that A, B and Q have the same Hochschild cohomology. Since our equivalences are slightly indirect we need some machinery to see the isomorphism preserves structure. Keller's article [29] recalls the definition of B_∞-algebras.

Proposition 5.3 *There are isomorphisms in the homotopy category of B_∞ algebras between the Hochschild complexes of Q, A and B.*

Proof For the algebras Q and A, we apply the main theorem in Section 3.2 of [29] to the quasi-isomorphism $Q \to A$ of Theorem 4.2 to obtain an equivalence of Hochschild complexes in the homotopy category of B_∞ algebras.

Let C be the cobar construction on Q. This is an augmented DG coalgebra, which is finite dimensional in each degree. As in Section 2 of Keller [36], this is the Koszul–Moore dual of Q. Its graded k-linear dual C^* is "the" Koszul dual of Q, and is quasi-isomorphic to B. Using the fact that C is finite dimensional in each degree, the Hochschild complex for C is isomorphic to that for C^* as a B_∞ algebra. Applying Theorem 3.3 of [36], the Hochschild complexes of Q and C are equivalent in the homotopy category of B_∞ algebras. $\square$

Certainly an isomorphism in the homotopy category of B_∞ algebras induces an isomorphism of cohomology rings.

Corollary 5.4 *We have $HH^*Q \cong HH^*A \cong HH^*B$.*

Theorem 5.5 *Suppose that $h > 2$ and $\ell > 2$. In the spectral sequence $HH^*H_*B \Rightarrow HH^*B$ the E^2 page is given by*

$$HH^*H_*B \cong H_*A \otimes H_*B \cong k[x,\tau] \otimes \Lambda(t,\xi)$$

where $|x| = (0,-2a,-\ell)$, $|t| = (0,-2b-1,-h)$, $|\xi| = (-1,2a,\ell)$, and $|\tau| = (-1,2b+1,h)$. The only non-zero differential is $d_{\ell-1}$, and this is given by $d_{\ell-1}(\xi) = \pm h x^{h-1}\tau^\ell$, $d_{\ell-1}(t) = \pm \ell x^h \tau^{\ell-1}$. There are no ungrading problems in the spectral sequence.

Proof The element t on the E^2 page corresponds to the cochain $\tilde{t}\colon [\,] \mapsto t$ in the Hochschild complex (here, $[\,]$ is the unique basis element in the bar complex of length zero). Applying the formula for the differential, we have (again using the bar notation)

$$\begin{aligned}(d\tilde{t})[\underbrace{t,\dots,t}_{\ell-1}] &= m_\ell(\tilde{t}[\,],t,\dots,t) + m_\ell(t,\tilde{t}[\,],\dots,t) + \cdots + m_\ell(t,t,\dots,\tilde{t}[\,])\\ &= \ell m_\ell(t,\dots,t) = \ell x^h.\end{aligned}$$

Using this, and the rather simple form of the A_∞ structure on B, it is not hard to see that $d\tilde{t} = \pm \ell x^h \tau^{\ell-1}$ since the two cochains take the same value on all elements of the bar resolution.

The element ξ on the E^2 page corresponds to the cochain $\tilde{\xi}\colon [x^i] \mapsto ix^{i-1}$, $[tx^i] \mapsto itx^{i-1}$. Then applying the formula for the differential, we have

$$(d\tilde{\xi})[\underbrace{t, \dots, t}_{\ell}] = \tilde{\xi}(m_\ell(t, \dots, t)) = \tilde{\xi}(x^h) = hx^{h-1}.$$

Using this, again we find that $d\tilde{\xi} = \pm hx^{h-1}\tau^\ell$ since both cochains take the same value on all elements of the bar resolution.

Examining the locations of these terms in the filtration of the bar complex giving rise to the spectral sequence, we deduce that these correspond to the differential $d_{\ell-1}$ taking t to $\pm \ell x^h \tau^{\ell-1}$ and ξ to $\pm hx^{h-1}\tau^\ell$.

Next, we show that the possible values of n for which d_n is non-zero are very restricted. The possible tridegrees (u, v, w) at the E^2-term lie in three parallel planes. Take $N = (\ell-2, \ell, -2a)$ as normal direction and consider the dot products $N \cdot (u, v, w) = (\ell-2)u + \ell v - 2aw$. We have $N \cdot |x| = 0$, $N \cdot |t| = 2 - \ell$, $N \cdot |\xi| = 2 - \ell$, and $N \cdot |\tau| = 0$. So the only possible values of $N \cdot (u, v, w)$ on the E^2 page are 0, $2 - \ell$, and $4 - 2\ell$. Furthermore, the elements with $N \cdot (u, v, w) = 4 - 2\ell$ are multiples of $t\xi$.

The differential d_n decreases u by n, increases v by $n - 1$, and leaves w unchanged. It therefore increases $N \cdot (u, v, w)$ by $2n - \ell$. Since $n \geq 2$, we first deduce that all differentials are zero on elements with $N \cdot (u, v, w) = 0$, and hence on the polynomial generators x and τ. Next, we deduce that the smallest value of n for which $d_n \neq 0$ is when $(2n-\ell)+(2-\ell) = 0$, so $n = \ell - 1$. We computed above the value of $d_{\ell-1}$ on the exterior generators t and ξ. Finally, since h and ℓ are coprime, $d_{\ell-1}$ is injective on elements with $N \cdot (u, v, w) = 4-2\ell$, and so there is no room for further differentials.

For the ungrading problem, we note that moving down one place in the filtration replaces (u, v, w) by $(u-1, v+1, w)$ and so the dot product with N increases by $N \cdot (-1, 1, 0) = 2$, while w is unchanged. The relations $hx^{h-1}\tau^\ell = 0$ and $\ell x^h \tau^{\ell-1} = 0$ therefore have no ungrading problems, and hold in HH^*B. The relations $\xi^2 = 0$ and $t^2 = 0$ have no ungrading problems, because there are no candidates with the correct value of w and with larger dot product with N. □

Theorem 5.6 *There are three cases for $HH^*Q \cong HH^*A \cong HH^*B$, according to the characteristic of the field k.*

(i) *If $p \mid h$ then $HH^*B \cong k[x, \tau]/(x^h\tau^{\ell-1}) \otimes \Lambda(\xi)$.*
(ii) *If $p \mid \ell$ then $HH^*B \cong k[x, \tau]/(x^{h-1}\tau^\ell) \otimes \Lambda(t)$.*
(iii) *If $p \nmid h$ and $p \nmid \ell$ then $HH^*B \cong (k[x, \tau] \otimes \Lambda(u))/(x^{h-1}\tau^\ell, x^h\tau^{\ell-1}, x^{h-1}\tau^{\ell-1}u)$.*

Here, we have $|x| = (-2a, -\ell)$, $|t| = (-2b-1, -h)$, $|\xi| = (2a-1, \ell)$, $|\tau| = (2b, h)$, and $|u| = (-1, 0)$.

Proof If $h > 2$ and $\ell > 2$ then this follows from Theorem 5.5, after checking that there are no ungrading problems. The element u in case (iii) represents $\pm ax\xi \pm bt\tau$ in E^∞ (recall that $ah - b\ell = 1$). If $h = 2$ then B is the formal A_∞ algebra $k[x, t]/(t^2 + x^h)$, and we can use the method of Buchweitz and Roberts [14] to compute HH^*B. If $\ell = 2$ then we can use the same method on HH^*A. □

Corollary 5.7 *We have $x^h\tau^\ell = 0$ in $HH^*Q \cong HH^*A \cong HH^*B$.*

Proof This follows from Theorem 5.6: in all three cases we have $x^h\tau^\ell = 0$. Note that if $h > 2$ and $\ell > 2$ then we see directly that the differential $d_{\ell-1}$ in Theorem 5.5 takes $\pm a\xi \pm bt$ (with appropriate signs) to $x^h\tau^\ell$. □

6 The Derived Category

Suppose first that $\mathfrak{a}$ is a DG algebra. We write $\mathsf{D}(\mathfrak{a})$ for the derived category of $\mathfrak{a}$. This is the triangulated category having as objects the left DG $\mathfrak{a}$-modules, and as arrows the homotopy classes of morphisms of DG modules, with the quasi-isomorphisms inverted. The shift functor is the suspension Σ defined by $(\Sigma M)_n = M_{n-1}$, so the triangles take the form $X \to Y \to Z \to \Sigma X$.

In the case where $H_*\mathfrak{a}$ is a Noetherian graded ring, we write $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ for the thick subcategory of $\mathsf{D}(a)$ whose objects are the $\mathfrak{a}$-modules X such that H_*X is finitely generated as an $H_*\mathfrak{a}$-module. We regard this as the analogue of the bounded derived category in this context; an extended discussion motivating this can be found in Greenlees and Stevenson [23].

Definition 6.1 An $\mathfrak{a}$-module is *homotopically projective* if the functor $\mathsf{Hom}_{\mathfrak{a}}(X, -)$ preserves quasi-isomorphisms, see for example Section 8.1 of Keller [30]. It is shown in Theorem 8.1.1 of [30] that given any module X there exists a homotopically projective module X' and a surjective quasi-isomorphism $X' \to X$. We call this a *homotopically projective resolution* of X.

Homomorphisms in $\mathsf{D}(\mathfrak{a})$ may be described as follows. Given DG $\mathfrak{a}$-modules X and Y, choose a homotopically projective module X' and a quasi-isomorphism $X' \to X$. Then

$$\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y) \cong H_*(\mathsf{Hom}_{\mathfrak{a}}(X', Y)).$$

It can be seen that $X' \to X$ is a *fibrant replacement* with respect to the projective model structure (see Hovey [27]) on $\mathfrak{a}$-modules, and $\mathsf{D}(\mathfrak{a})$ is the corresponding homotopy category.

Next, we describe the derived category of an A_∞ algebra. Suppose that $\mathfrak{a}$ is an A_∞ algebra. In this case, the modules do not form an abelian category, because of the definition of morphism of A_∞ modules. This time, the derived category $\mathsf{D}(\mathfrak{a})$ is the triangulated category having as objects the left A_∞ modules over $\mathfrak{a}$, and as arrows the homotopy classes of A_∞ morphisms. Unlike in the DG context, A_∞ quasi-isomorphisms automatically have A_∞ inverses. This is again a triangulated category, with triangles of the form $X \to Y \to Z \to \Sigma X$. For details, see Keller [31, 32]. As before, in the case where $H_*\mathfrak{a}$ is Noetherian, we write $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ for the thick subcategory whose objects are the modules with finitely generated homology.

In the case where $\mathfrak{a}$ is a DG algebra regarded as an A_∞ algebra with $m_i = 0$ for $i > 2$, the two definitions agree up to canonical equivalences of triangulated categories. If X and Y are DG $\mathfrak{a}$-modules, the homotopy classes of morphisms of A_∞ modules from X to Y are canonically isomorphic to the homotopy classes of morphisms of DG modules from X' to Y, where X' is a homotopically projective resolution of X. A suitable set of details can be found in Théorème 2.2.2.2 and Sections 2.4 and 4.1 of the thesis of Lefèvre-Hasegawa [37]. See also Theorem 4.5 of Keller [34].

In the case of the DG algebra Q of Section 3 and the A_∞ algebra A of Section 4, we have the following.

Proposition 6.2 *The bounded derived category* $\mathsf{D}^{\mathsf{b}}(Q)$ *is equivalent to* $\mathsf{D}^{\mathsf{b}}(A)$.

Proof A quasi-isomorphim of A_∞ algebras induces an equivalence of derived categories, see for example [37], Section 4.1.3. So it follows from Theorem 4.2 that $\mathsf{D}^{\mathsf{b}}(Q)$ is equivalent to $\mathsf{D}^{\mathsf{b}}(A)$. □

Remark 6.3 Although the A_∞ algebras Q, A and B carry an internal grading to make them bigraded, we do not require that the objects in the derived category carry an internal grading respected by the morphisms. Nonetheless, we shall make use of internal gradings in identifying Auslander–Reiten triangles in $\mathsf{D}_{\mathsf{sg}}(B)$ in Section 14. As we shall see, the reason this works is that the duality established in Section 10 respects grading for objects that admit one.

7 A Spectral Sequence

In this section, we give a brief reminder of the construction and convergence properties of the spectral sequence for computing Homs in the derived category $\mathsf{D}(\mathfrak{a})$ of an A_∞ algebra $\mathfrak{a}$:

$$\mathsf{Ext}^{**}_{H_*\mathfrak{a}}(H_*X, H_*Y) \Rightarrow \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y). \tag{7.1}$$

We shall make use of this in Section 12 to compute some endomorphism rings, as a preliminary to applying Auslander–Reiten theory. The construction is taken from Adams [1], and a discussion of convergence may be found in Boardman [10].

Let $\mathfrak{a}$ be an A_∞ algebra. If X is an A-module, then taking homology gives isomorphisms

$$\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(A, X) \cong \mathsf{Hom}_{H_*\mathfrak{a}}(H_*\mathfrak{a}, H_*X) \cong H_*X.$$

Choosing a set of generators of H_*X, we obtain a morphism $F_0 \to X$, where F_0 is a direct sum of shifts of $\mathfrak{a}$, with the property that $H_*F_0 \to H_*X$ is surjective. Setting $X_0 = X$, we complete to a triangle

$$F_0 \xrightarrow{k} X_0 \xrightarrow{i} X_1 \xrightarrow{j} \Sigma F_0$$

in $\mathsf{D}(\mathfrak{a})$, and the map $i\colon X_0 \to X_1$ is zero in homology. Repeating this construction, we obtain a sequence of triangles

$$\begin{tikzcd}
X \arrow[r, equal] & X_0 \arrow[rr, "i"] \arrow[dr, "j"] & & X_1 \arrow[rr, "i"] \arrow[dr, "j"] & & X_2 \arrow[rr, "i"] & & \cdots \\
& & F_0 \arrow[ul, "k"] & & F_1 \arrow[ul, "k"] & & F_2 \arrow[ul, "k"] &
\end{tikzcd}$$

where the maps marked j involve a degree shift. This has the property that the resulting sequence

$$\cdots \xrightarrow{(jk)_*} \Sigma^{-2}H_*F_2 \xrightarrow{(jk)_*} \Sigma^{-1}H_*F_1 \xrightarrow{(jk)_*} H_*F_0 \xrightarrow{k_*} H_*X \to 0$$

is a free resolution of H_*X as an $H_*\mathfrak{a}$-module.

Lemma 7.2 *We have* $\varinjlim_i X_i \simeq 0$.

Proof Any map $\Sigma^j\mathfrak{a} \to \varinjlim_i X_i$ factors through some X_i, and then the composite

$$\Sigma^j\mathfrak{a} \to X_i \to X_{i+1}$$

is zero. Thus $H_* \varinjlim_i X_i = 0$ and so $\varinjlim_i X_i \simeq 0$. ☐

If Y is another $\mathfrak{a}$-module, then taking Homs in $\mathsf{D}(\mathfrak{a})$ from the above resolution of X to Y, we obtain a diagram of long exact sequences

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r, "i^*"] & \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X_2, Y) \arrow[r, "i^*"] \arrow[dl, "k^*"] & & \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X_1, Y) \arrow[r, "i^*"] \arrow[dl, "k^*"] & & \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X_0, Y). \arrow[dl, "k^*"] \\
\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_2, Y) & & \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_1, Y) \arrow[ul, "j^*"] & & \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_0, Y) \arrow[ul, "j^*"] &
\end{tikzcd}$$

The direct sum of all these long exact sequences is an exact couple

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X_*, Y) \arrow[rr] & & \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X_*, Y). \arrow[dl] \\
& \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_*, Y) \arrow[ul] &
\end{tikzcd}$$

The spectral sequence of this exact couple has as its E^1 term

$$\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_*, Y) \cong \mathsf{Hom}_{\mathfrak{a}}(H_*F_*, H_*Y).$$

The differential is the composite

$$\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_j, Y) \xrightarrow{(jk)^*} \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(F_{j+1}, Y),$$

and so we have

$$E^2 \cong \mathsf{Ext}^{**}_{H_*\mathfrak{a}}(H_*X, H_*Y),$$

and the abutment of the spectral sequence is $\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y)$. Thus we have a filtration

$$F_j \,\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y) = \text{Image of } \left(\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X_j, Y) \to \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y)\right),$$

and

$$F_j \,\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y)_n / F_{j+1} \,\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y)_n \cong E^{\infty}_{j,n-j}.$$

In this filtration, F_0 is the whole thing, and by Lemma 7.2 we have

$$\bigcap_j F_j \,\mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(X, Y) = \varprojlim_j \mathsf{Hom}(X_j, Y) = \mathsf{Hom}_{\mathsf{D}(\mathfrak{a})}(\varinjlim_j X_j, Y) = 0.$$

As in the spectral sequence for Hochschild cohomology described in Section 5, the spectral sequence is conditionally convergent, and strongly convergent if and only if $\varinjlim_i F_i = 0$, which is equivalent to $\varinjlim_i E^i = 0$. In particular, if each graded piece of E^i is finite dimensional, the the spectral sequence is strongly convergent.

8 Inverting τ

The advantage of the explicit model Q is that the element τ is represented by a central cycle, so it is elementary to invert it.

Definition 8.1 We write K for the graded field $k[\tau, \tau^{-1}]$, and we define

$$Q[\tau^{-1}] = K \otimes_{k[\tau]} Q,$$

as a DG algebra over K. If X is a DG Q-module, we write

$$X[\tau^{-1}] = K \otimes_{k[\tau]} X$$

as a DG $Q[\tau^{-1}]$-module.

Lemma 8.2 *If X and Y are objects in $\mathsf{D}^{\mathsf{b}}(Q)$ then*

$$\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(X[\tau^{-1}], Y[\tau^{-1}]) = K \otimes_{k[\tau]} \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q)}(X, Y),$$

which we write as $\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q)}(X, Y)[\tau^{-1}]$.

Proof This follows from the fact that τ is central in Q with $d\tau = 0$, together with the fact that $H_*(X)$ is finitely generated over $H_*(Q)$. □

Similarly, we write $A[\tau^{-1}]$ for $K \otimes_{k[\tau]} A$ as an A_∞-algebra, and we have a quasi-isomorphism $Q[\tau^{-1}] \simeq A[\tau^{-1}]$ coming from Theorem 4.2. If X is an A-module, we write $X[\tau^{-1}] = K \otimes_{k[\tau^{-1}]} X$ as an $A[\tau^{-1}]$-module.

Proposition 8.3 *We have an equivalence of bounded derived categories*

$$\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}]).$$

If X and Y are objects in $\mathsf{D}^{\mathsf{b}}(A)$ then

$$\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])}(X[\tau^{-1}], Y[\tau^{-1}]) = K \otimes_{k[\tau]} \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(A)}(X, Y) = \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(A)}(X, Y)[\tau^{-1}].$$

Proof This follows from Proposition 6.2 and Lemma 8.2. □

9 Koszul Duality and Singularity Categories

We recapitulate the development of [23] in our more concrete setting.

Definition 9.1 Let $\mathfrak{a}$ be an augmented A_∞ algebra with Noetherian homology.

(i) The *singularity category* $\mathsf{D}_{\mathsf{sg}}(\mathfrak{a})$ is the quotient of $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ by $\mathsf{Thick}(\mathfrak{a})$.
(ii) The *cosingularity category* $\mathsf{D}_{\mathsf{csg}}(\mathfrak{a})$ is the quotient of $\mathsf{D}^{\mathsf{b}}(\mathfrak{a})$ by $\mathsf{Thick}(k)$.

Lemma 9.2 *Suppose that $\mathfrak{a}$ is an A_∞ algebra such that $H_*\mathfrak{a}$ is local with residue field k. If M is an $\mathfrak{a}$-module such that H_*M has finite length, then M is in $\mathsf{Thick}(k)$.*

Proof A copy of k of lowest degree in H_*M lifts to a map $k \to M$. Completing to a triangle $k \to M \to N$, the length of H_*N is one less than the length of H_*M. So by induction on length of H_*M, it follows that M is in $\mathsf{Thick}(k)$. □

For the A_∞ algebras Q and A introduced in Sections 3 and 4 and the A_∞ algebras $Q[\tau^{-1}]$ and $A[\tau^{-1}]$ described in Section 8 we have the following calculation.

Theorem 9.3 *We have equivalences of triangulated categories*

$$\mathsf{D}_{\mathsf{csg}}(Q) \simeq \mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}]) \simeq \mathsf{D}_{\mathsf{csg}}(A).$$

Proof By Propositions 6.2 and 8.3, we have compatible equivalences $\mathsf{D}^{\mathsf{b}}(Q) \simeq \mathsf{D}^{\mathsf{b}}(A)$ and $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$. Since K is flat over $k[\tau]$, we have a functor

$$K \otimes_{k[\tau]} - \colon \mathsf{D}^{\mathsf{b}}(Q) \to \mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}]).$$

Using Lemma 9.2, it kills exactly $\mathsf{Thick}(k)$, and therefore induces an equivalence $\mathsf{D}_{\mathsf{csg}}(Q) \to \mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$. Similarly,

$$K \otimes_{k[\tau]} - \colon \mathsf{D}^{\mathsf{b}}(A) \to \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$$

kills exactly $\mathsf{Thick}(k)$, and therefore induces an equivalence $\mathsf{D}_{\mathsf{csg}}(A) \to \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$. □

Theorem 9.4 *The functor $\mathcal{H}om_A(k, -)$ induces a triangulated equivalence of derived categories*

$$\mathsf{D}^{\mathsf{b}}(A) \xrightarrow{\sim} \mathsf{D}^{\mathsf{b}}(B),$$

that sends A to k and k to B. It induces triangulated equivalences

$$\mathsf{D}_{\mathsf{sg}}(A) \xrightarrow{\sim} \mathsf{D}_{\mathsf{csg}}(B), \qquad \mathsf{D}_{\mathsf{csg}}(A) \xrightarrow{\sim} \mathsf{D}_{\mathsf{sg}}(B).$$

Proof This follows from Greenlees and Stevenson [23, Theorem 9.1]. We need to show that If A and B come as part of a symmetric Gorenstein context so that the hypotheses of the theorem are satisfied. If A and B come from groups, this is explicit [23, Example 10.6]. However, even in that case the following proof is more elementary.

Since A and B are Koszul dual to each other, they are both dc-complete. It remains to show that A has a strongly Gorenstein normalisation in the sense of [23, Definition 6.3] since then by [23, Proposition 6.4] we have a symmetric Gorenstein context as required in the hypothesis of [23, Theorem 9.1]. We claim that the composite $k[\tau] \to Q \to A$ is a strongly Gorenstein normalisation. It is obvious that $k[\tau]$ is regular and Gorenstein. Since $H_*(A) = k[\tau] \otimes \Lambda(\xi)$ we see that A is small over $k[\tau]$ so that $k[\tau] \to A$ is regular. Indeed $A \simeq k[\tau] \oplus \Sigma^{2b} k[\tau]$ as $k[\tau]$-modules, and hence we have an equivalence $\mathsf{Hom}_{k[\tau]}(A, k[\tau]) \simeq \Sigma^{-2b} A$ so that $k[\tau] \to A$ is relatively Gorenstein. Thus $k[\tau] \to A$ is a strongly Gorenstein normalisation as required. Finally, we note that the normalisation of A is polynomial, as is the Koszul dual one of B. This shows the bounded derived categories in the sense of [23] agree with the concrete definition of objects with finitely generated homology as we have used here. □

Corollary 9.5 *We have an equivalence of categories $\mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$, taking k to $\mathcal{E}nd_{\mathsf{D}_{\mathsf{sg}}(B)}(k) \cong A[\tau^{-1}]$.*

Proof This follows from Theorems 9.3 and 9.4. □

Thus we can regard the central element τ of A as a periodicity operator on $\mathsf{D}_{\mathsf{sg}}(B)$ of degree $2b$, namely a natural isomorphism from the identity to Σ^{2b}. In Section 13, we shall see explicitly how to interpret this element in terms of resolutions.

Corollary 9.6 *If X and Y are objects in $\mathsf{D}^{\mathsf{b}}(B)$ then*

$$\mathsf{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(X, Y) \cong \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(B)}(X, Y)[\tau^{-1}].$$

Proof This follows from Proposition 8.3 using the equivalence of categories given in Corollary 9.5. □

The A_∞ algebras $A[\tau^{-1}]$ and $Q[\tau^{-1}]$ are regular, in the following sense.

Theorem 9.7 *Every object in* $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$ *is in* $\mathsf{Thick}(A[\tau^{-1}])$.

Proof Consider the element $x \in B$. For any B-module M with H_*M finitely generated, the fibre F of $x\colon M \to \Sigma^{-2a}M$ has the property that H_*F has finite length. So by Lemma 9.2, F is in $\mathsf{Thick}(k)$. Now under the equivalence $\mathsf{D}^{\mathsf{b}}(B) \simeq \mathsf{D}^{\mathsf{b}}(A)$, the image of k is A. So regarding x as an element of $HH^*B \cong HH^*A$, we have an action of x on $\mathsf{D}^{\mathsf{b}}(A)$, and for any A-module N in $\mathsf{D}^{\mathsf{b}}(A)$, the fibre of $x\colon N \to \Sigma^{-2a}N$ is in $\mathsf{Thick}(A)$. So we have a triangle $F \to N \xrightarrow{x} \Sigma^{-2a}N$ with F in $\mathsf{Thick}(A)$. Inverting τ, we have such a triangle in $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$ with F in $\mathsf{Thick}(A[\tau^{-1}])$. Now by Corollary 5.7, we have $x^h\tau^\ell = 0$ in HH^*A. Since τ is an isomorphism in $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$, it follows that x^h acts as zero on $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$. Therefore the fibre of $x^h\colon N \to \Sigma^{-2ah}N$ is $N \oplus \Sigma^{-2ah-1}N$. It is also in $\mathsf{Thick}(F)$, and therefore in $\mathsf{Thick}(A[\tau^{-1}])$. Hence so is N. □

Corollary 9.8 *Every object in* $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$ *is in* $\mathsf{Thick}(Q[\tau^{-1}])$.

Proof This follows from Theorem 9.7, using the fact that the equivalence $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$ of Theorem 9.3 sends $A[\tau^{-1}]$ to $Q[\tau^{-1}]$. □

Corollary 9.9 *If X and Y are objects in* $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$ *then we have natural equivalences*

(i) $\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(X, Y) \simeq \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(X, Q[\tau^{-1}]) \otimes_{Q[\tau^{-1}]} Y$,

(ii) $\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(X, Q[\tau^{-1}]), Q[\tau^{-1}]) \simeq X$.

Proof These hold for $X = Q[\tau^{-1}]$, and hence for every object in $\mathsf{Thick}(Q[\tau^{-1}])$, which by Corollary 9.8 is every object in $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$. □

10 Duality for $Q[\tau^{-1}]$-modules

Let $Q[\tau^{-1}]$ be the algebra described in Section 8. In this section, we prove a form of Tate duality for the bounded derived category of $Q[\tau^{-1}]$-modules, see Theorem 10.6. This combine the dualities $\mathsf{Hom}_K(-, K)$ (Brown–Comenetz duality) and $\mathsf{Hom}_{Q[\tau^{-1}]}(-, Q[\tau^{-1}])$ (Spanier–Whitehead duality). The proof makes essential use of Corollary 9.9.

Definition 10.1 We write $Q[\tau^{-1}]^*$ for the dual of Q with respect to K, $\mathsf{Hom}_{k[\tau]}(Q, K)$. Left and right multiplication make $Q[\tau^{-1}]^*$ into a Q-bimodule, and simultaneously a K-module with compatible actions of τ, and hence a $Q[\tau^{-1}]$-bimodule. Note that

$$\begin{aligned} Q[\tau^{-1}]^* &= \mathsf{Hom}_{k[\tau]}(Q, K) \cong \mathsf{Hom}_{k[\tau]}(Q, \mathsf{Hom}_K(K, K)) \\ &\cong \mathsf{Hom}_K(K \otimes_{k[\tau]} Q, K) = \mathsf{Hom}_K(Q[\tau^{-1}], K), \end{aligned}$$

so we can equally well regard $Q[\tau^{-1}]^*$ as $\mathsf{Hom}_K(Q[\tau^{-1}], K)$.

Similarly, if X is any K-vector space (i.e., graded K-module), we write X^* for $\mathsf{Hom}_K(X, K)$. In particular, if X is a left $Q[\tau^{-1}]$-module then X^* is a right $Q[\tau^{-1}]$-module and if X is a right $Q[\tau^{-1}]$-module then X^* is a left $Q[\tau^{-1}]$-module.

Proposition 10.2 *There is a quasi-isomorphism of $Q[\tau^{-1}]$-bimodules $Q[\tau^{-1}] \to \Sigma^{|\xi_1|}Q[\tau^{-1}]^*$*

Remark 10.3 For brevity, we have recorded the shift as $|\xi_1| = 2a - 1$, however in the presence of the τ-periodicity, this is only well defined modulo $2b$. It is more helpful to say that the shift in Tate duality is one less than the Gorenstein shift ($2a - 2b$ in this case) of A as in [24, Proposition 4.1].

We should also keep track of the internal degrees, and write

$$Q[\tau^{-1}] \xrightarrow{\sim} \Sigma^{2a-1,\ell} Q[\tau^{-1}]^*.$$

Proof The standard monomials of Definition 3.5 form a K-basis for $Q[\tau^{-1}]$. We construct a K-module homomorphism $Q[\tau^{-1}] \to \Sigma^{|\xi_1|}Q[\tau^{-1}]^*$ as follows. It takes all standard monomials to zero except 1 and ξ_1. It takes 1 to the element of $Q[\tau^{-1}]^*$ taking value 1 on ξ_1 and value zero on all other standard monomials, and it takes ξ_1 to the element of $Q[\tau^{-1}]^*$ taking value 1 on 1 and value zero on all other standard monomials. It is easy to check that this is a map of $Q[\tau^{-1}]$-bimodules, and using Corollary 4.3 that is a quasi-isomorphism. □

Proposition 10.4 *If X is a left $Q[\tau^{-1}]$-module and Y is a right $Q[\tau^{-1}]$-module, then there is a natural isomorphism of K-vector spaces*

$$\mathsf{Hom}_{Q[\tau^{-1}]}(X, \mathsf{Hom}_K(Y, K)) \cong \mathsf{Hom}_K(Y \otimes_{Q[\tau^{-1}]} X, K).$$

If Y is a $Q[\tau^{-1}]$-bimodule, this is an isomorphism of left $Q[\tau^{-1}]$-modules.

Proof This is standard. □

Corollary 10.5 *If X is homotopically projective then we have a quasi-isomorphism*

$$\mathsf{Hom}_{Q[\tau^{-1}]}(X, Q[\tau^{-1}]) \simeq \Sigma^{|\xi_1|}\mathsf{Hom}_K(X, K).$$

Proof Using Propositions 10.2 and 10.4, and the fact that X is homotopically projective, we have

$$\begin{aligned}\mathsf{Hom}_{Q[\tau^{-1}]}(X, Q[\tau^{-1}]) &\simeq \mathsf{Hom}_{Q[\tau^{-1}]}(X, \Sigma^{|\xi_1|}Q[\tau^{-1}]^*)\\ &\cong \Sigma^{|\xi_1|}\mathsf{Hom}_{Q[\tau^{-1}]}(X, \mathsf{Hom}_K(Q[\tau^{-1}], K))\\ &\cong \Sigma^{|\xi_1|}\mathsf{Hom}_K(Q[\tau^{-1}] \otimes_{Q[\tau^{-1}]} X, K)\\ &\cong \Sigma^{|\xi_1|}\mathsf{Hom}_K(X, K).\end{aligned}$$ □

Theorem 10.6 *Let X and Y be objects in* $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$*. Then there is a functorial duality of graded K-modules*

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(X, Y)^* \cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])}(Y, \Sigma^{-|\xi_1|} X).$$

Proof We may replace X and Y by homotopically projective resolutions and work with $Q[\tau^{-1}]$-module homomorphisms. Combining Corollary 9.9 (ii) with Corollary 10.5, we have

$$\mathrm{Hom}_{Q[\tau^{-1}]}(X, Q[\tau^{-1}])^* \simeq \Sigma^{-|\xi_1|} X.$$

Hence using Proposition 10.4 and Corollary 9.9 (i), we have

$$\begin{aligned}
\mathrm{Hom}_{Q[\tau^{-1}]}(X, Y)^* &= \mathrm{Hom}_K(\mathrm{Hom}_{Q[\tau^{-1}]}(X, Y), K) \\
&\simeq \mathrm{Hom}_K(\mathrm{Hom}_{Q[\tau^{-1}]}(X, Q[\tau^{-1}]) \otimes_{Q[\tau^{-1}]} Y, K) \\
&\cong \mathrm{Hom}_{Q[\tau^{-1}]}(Y, \mathrm{Hom}_K(\mathrm{Hom}_{Q[\tau^{-1}]}(X, Q[\tau^{-1}]), K)) \\
&\simeq \mathrm{Hom}_{Q[\tau^{-1}]}(Y, \Sigma^{-|\xi_1|} X).
\end{aligned}$$

□

Corollary 10.7 *Let X and Y be objects in* $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$*. Then there is a functorial duality of graded K-modules*

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])}(X, Y)^* \cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])}(Y, \Sigma^{1-2a} X).$$

If X and Y carry an internal grading, the shift is $\Sigma^{1-2a,-\ell}$.

Proof This follows from Theorem 10.6, using the equivalence of categories described in Proposition 8.3. □

Theorem 10.8 *Let X and Y be objects in* $\mathsf{D}_{\mathsf{sg}}(B)$*. Then there is a functorial duality of graded K-modules*

$$\mathrm{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(X, Y)^* \cong \mathrm{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(Y, \Sigma^{1-2a} X).$$

If X and Y carry an internal grading, the shift is $\Sigma^{1-2a,-\ell}$.

Proof This follows from Corollary 10.7 using the equivalence of categories given in Corollary 9.5. □

11 A Tour of the Two Worlds

Our aim is to understand the singularity category of B, and in particular to construct indecomposable objects. In this interlude we give a topological account of the strategy before returning to give an elementary implementation in algebra. The section can be entirely ignored by those with a strong algebraic compass.

Bearing in mind that B itself is trivial in the singularity category, it is natural to start with $X_1 = k$ and then construct objects from that. Since $\mathrm{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(k, k) = A[\tau^{-1}]$ it is natural to focus on ξ, which supplies a map $\Sigma^{2a-1}k \to k$, with mapping cone X_2 with cells in degree 0 and $2a$. We then attempt to iterate this construction by forming

$$X_s = k \cup_\xi e_k^{2a} \cup_\xi \cdots \cup_\xi e_k^{2(s-1)a}.$$

In fact X_s exists if and only if the $(s-1)$-fold Massey product $\langle \xi, \xi, \cdots, \xi \rangle$ exists and contains zero. In our case the indeterminacy is always zero. We may thus construct $X_1, \dots, X_h$, but not X_{h+1} since the h-fold Massey product is nonzero. One may also check inductively that these complexes are unique up to equivalence. Our main result will show that up to suspension this does give all indecomposables.

The counterparts $Y_s = \mathsf{Hom}_B(k, X_s)$ in A-modules will have a cell structure

$$Y_s = A \cup_\xi e_A^{2a} \cup_\xi \cdots \cup_\xi e_A^{2(s-1)a}.$$

Under the derived equivalence between finitely generated B-modules and A-modules, small B-modules (i.e., modules in $\mathsf{Thick}(B)$) correspond to A-modules with finite dimensional homology, and vice versa. Thus a B-module N is small if and only if $[k, N]_*^B$ is finite dimensional. Similarly, N corresponds to a small A-module if and only if $H_*(N)$ is finite dimensional.

Exchanging the roles of A and B, we may construct A-modules

$$V_s = k \cup_t e_k^{2b} \cup_t \cdots \cup_t e_k^{2(s-1)b}.$$

for $s = 1, 2, \dots, \ell$ but not $V_{\ell+1}$ since the ℓ-fold Massey product is nonzero. These correspond to the small B-modules

$$W_s = B \cup_t e_B^{2b} \cup_t \cdots \cup_t e_B^{2(s-1)b}.$$

Our actual method will take full advantage of the explicit and easily described models, and give concrete representatives for the objects W_s.

Remark 11.1 For the graded rings $H_*(A)$ with $h > 2$ and $H_*(B)$ with $\ell > 2$ (i.e., the formal case where the algebras are polynomial tensor exterior and all products m_i are zero for $i > 2$) the singularity and cosingularity categories are well understood, for example through the theory of maximal Cohen–Macaulay modules. See for example Proposition 2.6 of Ene and Popescu [17], which discusses modules over the ungraded completion, but the modules over the graded ring are very similar. In that case the objects X_s, Y_s, V_s, W_s exist for all $s \geqslant 0$ and the terms in each sequence are inequivalent. By contrast with our case, the singularity and cosingularity categories for $H_*(A)$ and $H_*(B)$ each contain infinitely many indecomposable objects.

12 The *B*-modules W_i in Thick(*B*)

Following the pattern set out in Section 11 we will construct explicit small B-modules (i.e., modules in $\mathsf{Thick}(B)$) $W_1, W_2, \dots, W_\ell$, starting with $W_1 = B$, and we will do so in a bigraded fashion. Background on modules over A_∞-algebras may be found in [34].

Consider the map $t\colon \Sigma^{-2b-1,-h}B \to B = W_1$. Complete this to a triangle in $\mathsf{D}^{\mathsf{b}}(B)$,

$$\Sigma^{-2b-1,-h}B \xrightarrow{t} W_1 \to W_2.$$

Then using the long exact sequence in homology, as a module over $H_*B = k[x] \otimes \Lambda(t)$ we have

$$H_*W_2 = k[x].u_2 \oplus k[x].v_2,$$

where $k[x] = H_*B/(t)$, $|u_2| = (0,0)$ and $|v_2| = (-4b-1, -2h)$. The A_∞ structure is given by

$$\begin{aligned} m_3(t, t, u_2) &= v_2 \\ m_{\ell-1}(t, \dots, t, v_2) &= (-1)^{(\ell-2)(\ell-1)/2} x^h u_2. \end{aligned}$$

The element v_2 defines a map $\Sigma^{-4b-1}B \to W_2$, which we complete to a triangle

$$\Sigma^{-4b-1,-2h}B \xrightarrow{v_2} W_2 \to W_3.$$

We have

$$H_*W_3 = k[x].u_3 \oplus k[x].v_3,$$

with $|u_3| = (0,0)$, $|v_3| = (-6b-1, -3h)$. The A_∞ structure is given by

$$\begin{aligned} m_4(t, t, t, u_3) &= v_3 \\ m_{\ell-2}(t, \dots, t, v_3) &= (-1)^{(\ell-4)(\ell-1)/2} x^h u_3. \end{aligned}$$

Continuing this way, we construct objects W_i in $\mathsf{D}^{\mathsf{b}}(B)$ ($1 \leqslant s \leqslant \ell$), finitely built from B, with

$$H_*W_i = k[x].u_i \oplus k[x].v_i,$$

with $|u_i| = (0,0)$, $|v_i| = (-2ib-1, -ih)$, and

$$\begin{aligned} m_{i+1}(\underbrace{t, \dots, t}_{i \text{ copies}}, u_i) &= v_i \\ m_{\ell-i+1}(\underbrace{t, \dots, t}_{\ell-i \text{ copies}}, v_i) &= (-1)^{(\ell-2i+2)(\ell-1)/2} x^h u_i. \end{aligned}$$

and v_i defines a triangle

$$\Sigma^{-2ib-1,-ih}B \xrightarrow{v_i} W_i \to W_{i+1}.$$

The second to last stage of this process is a module $W_{\ell-1}$ with

$$\begin{aligned} m_\ell(t, \dots, t, u_{\ell-1}) &= v_{\ell-1} \\ tv_{\ell-1} = m_2(t, v_{\ell-1}) &= (-1)^{\ell(\ell-1)/2} x^h u_{\ell-1}. \end{aligned}$$

Then something different happens. The map $\Sigma^{-2(\ell-1)b-1,-(\ell-1)h}B \to W_{\ell-1}$ still defines a triangle

$$\Sigma^{-2(\ell-1)b-1,-(\ell-1)h}B \to W_{\ell-1} \to W_\ell,$$

but the long exact sequence in homology now gives $H_*W_\ell \cong k[x]/(x^h)$, and the process cannot be continued any further. The composite $B = W_1 \to W_2 \to \cdots \to W_\ell$ shows that W_ℓ is equivalent to the quotient of B by the ideal (x^h, t). What we have seen is the following.

Theorem 12.1 *The B-module $k[x]/(x^h) = B/(x^h, t)$ is in the subcategory* $\mathsf{Thick}(B)$ *of* $\mathsf{D}^{\mathsf{b}}(B)$. *It is built from ℓ shifts of copies of B.*

13 Indecomposable objects in $\mathsf{D}_{\mathsf{sg}}(B)$

In this section, we discuss the indecomposable objects which we eventually wish to show form a complete list in $\mathsf{D}_{\mathsf{sg}}(B)$. They come with an internal grading, which we keep track of. The ideal in B generated by x^h and t is a bigraded A_∞ ideal, and the quotient

$$\bar{B} = B/(x^h, t) \cong k[x]/(x^h)$$

is a formal A_∞ algebra with $|x| = (-2a, -\ell)$. In other words, the only non-zero operation m_i on the quotient is the multiplication m_2. We begin with a discussion of $\bar{B}$-modules. The indecomposable modules are the shifts of the quotients $\bar{M}_i = k[x]/(x^i)$ for $1 \leqslant i \leqslant h$ of $\bar{B}$. For $i < j$ we have short exact sequences of $\bar{B}$-modules

$$0 \to \Sigma^{-2ia,-i\ell}\bar{M}_{j-i} \to \bar{M}_j \to \bar{M}_i \to 0. \tag{13.1}$$

We also have almost split sequences of $\bar{B}$-modules

$$\begin{gathered} 0 \to \Sigma^{-2a,-\ell}\bar{M}_1 \to \bar{M}_2 \to \bar{M}_1 \to 0 \\ 0 \to \Sigma^{-2a,-\ell}\bar{M}_2 \to \Sigma^{-2a,-\ell}\bar{M}_1 \oplus \bar{M}_3 \to \bar{M}_2 \to 0 \\ \cdots \\ 0 \to \Sigma^{-2a,-\ell}\bar{M}_{h-1} \to \Sigma^{-2a,-\ell}\bar{M}_{h-2} \oplus \bar{M}_h \to \bar{M}_{h-1} \to 0. \end{gathered}$$

This comes from the theory of almost split sequences for the Nakayama algebra $k[x]/(x^h)$, see for example Proposition 4.12 of Auslander and Reiten [6].

Pulling back the $k[x]/(x^h)$-module $\bar{M}_i$ $(1 \leqslant i \leqslant h)$ to B, we obtain a B-module M_i and a map $B \to M_i$. In particular, M_1 is the field object, with $H_*M_1 \cong k$. We shall write k for the B-module M_1. By Theorem 12.1, M_h is in the subcategory $\mathsf{Thick}(B)$ of $\mathsf{D}^{\mathsf{b}}(B)$, and therefore vanishes in $\mathsf{D}_{\mathsf{sg}}(B)$. We write X_i for the fibre of $B \to M_i$. Applying H_* to the triangle $X_i \to B \to M_i$, we see that H_*X_i is the ideal generated by t and x^i in $H_*B = k[x] \otimes \Lambda(t)$. In $\mathsf{D}_{\mathsf{sg}}(B)$, we have $\Sigma X_i \simeq M_i$ and $X_h \simeq 0$.

The exact sequence (13.1) of $\bar{B}$-modules gives rise to triangles in $\mathsf{D}^{\mathsf{b}}(B)$

$$\Sigma^{-2ia,-i\ell}X_{j-i} \to X_j \to X_i \to \Sigma^{1-2ia,-i\ell}X_{j-i}.$$

In particular, taking $j = h$, we obtain the following, critical in linking odd and even suspensions.

Proposition 13.2 *There is an equivalence $X_i \simeq \Sigma^{1-2ia,-i\ell}X_{h-i}$ in $\mathsf{D}_{\mathsf{sg}}(B)$.*

Similarly, the almost split sequences of $\bar{B}$-modules give rise to triangles in $\mathsf{D}^{\mathsf{b}}(B)$

$$\begin{gathered} \Sigma^{-2a,-\ell}X_1 \to X_2 \to X_1 \to \Sigma^{1-2a,-\ell}X_1 \\ \Sigma^{-2a,-\ell}X_2 \to \Sigma^{-2a,-\ell}X_1 \oplus X_3 \to X_2 \to \Sigma^{1-2a,-\ell}X_2 \\ \cdots \\ \Sigma^{-2a,-\ell}X_{h-1} \to \Sigma^{-2a,-\ell}X_{h-2} \oplus X_h \to X_{h-1} \to \Sigma^{1-2a,-\ell}X_{h-1}. \end{gathered} \tag{13.3}$$

Bearing in mind that X_h is zero in $\mathsf{D}_{\mathsf{sg}}(B)$, the last of these becomes

$$\Sigma^{-2a,-\ell}X_{h-1} \to \Sigma^{-2a,-\ell}X_{h-2} \to X_{h-1} \to \Sigma^{1-2a,-\ell}X_{h-1}.$$

We shall eventually see that these are almost split triangles in $\mathsf{D}_{\mathsf{sg}}(B)$.

Next, we compute the spectral sequence (7.1)

$$\mathsf{Ext}^{*,*}_{H_*B}(H_*X_i, H_*X_j) \Rightarrow \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(B)}(X_i, X_j). \tag{13.4}$$

We first assume that $\ell > 2$, and later describe the modifications necessary for the case $\ell = 2$. Resolving H_*X_i as a module over H_*B, we obtain the sequence

$$\cdots \xrightarrow{\left(\begin{smallmatrix} t & x^i \\ 0 & -t \end{smallmatrix}\right)} \Sigma^{-4b-2,-2h} H_*B \oplus \Sigma^{-2ia-2b-1,-h-i\ell} H_*B$$
$$\xrightarrow{\left(\begin{smallmatrix} t & x^i \\ 0 & -t \end{smallmatrix}\right)} \Sigma^{-2b-1,-h} H_*B \oplus \Sigma^{-2ia,-i\ell} H_*B \xrightarrow{(t,x^i)} H_*X_i \to 0.$$

Thus H_*X_i is a maximal Cohen–Macaulay H_*B-module corresponding to the following matrix factorisation of the relation $t^2 = 0$.

$$\begin{pmatrix} t & x^i \\ 0 & -t \end{pmatrix}\begin{pmatrix} t & x^i \\ 0 & -t \end{pmatrix} = t^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If X_i and X_j are B-modules of this form, we take homomorphisms from the resolution of H_*X_i to H_*X_j to obtain

$$\cdots \xleftarrow{\left(\begin{smallmatrix} t & 0 \\ x^i & -t \end{smallmatrix}\right)} \Sigma^{4b+2,2h} H_*X_j \oplus \Sigma^{2ia+2b+1,h+i\ell} H_*X_j \xleftarrow{\left(\begin{smallmatrix} t & 0 \\ x^i & -t \end{smallmatrix}\right)} \Sigma^{2b+1,h} H_*X_j \oplus \Sigma^{2ia,i\ell} H_*X_j \leftarrow 0.$$

The differential sends

$$\left(\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 \\ tx^i \end{smallmatrix}\right) \qquad \left(\begin{smallmatrix} x^j \\ 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} tx^j \\ x^{i+j} \end{smallmatrix}\right) \qquad \left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \qquad \left(\begin{smallmatrix} 0 \\ x^j \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 \\ -tx^j \end{smallmatrix}\right).$$

13.0.1 (i) The Case $j \geqslant i$:

The kernel is generated by $\left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right)$ and

$$x^{j-i}\left(\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 \\ x^j \end{smallmatrix}\right) = \left(\begin{smallmatrix} tx^{j-i} \\ x^j \end{smallmatrix}\right).$$

The image contains t times these and x^i times these.

We write α for the element of the kernel represented by the first of these elements $\left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right)$, and β for the element represented by the second $\left(\begin{smallmatrix} tx^{j-i} \\ x^j \end{smallmatrix}\right)$. Thus as a module over H_*B, the degree zero homology of the above complex is $\mathsf{Hom}_{H_*B}(H_*X_i, H_*X_j)$, and is generated by α and β with relation $x^j\alpha = t\beta$. Recalling that $H_*(A) = \mathsf{Hom}_{\mathsf{D^b}(B)}(X_1, X_1)$, the periodicity element of degree $(-1, 2b+1, h)$ in the above resolution represents the element $\tau \in H_{2b}A$, so we shall call it τ by abuse of notation. As a module over $H_*B[\tau]$ we have

$$\mathsf{Ext}^{*,*}_{H_*B}(H_*X_i, H_*X_j) = \langle\alpha, \beta\rangle/(t\alpha, x^j\alpha - t\beta, t\tau\beta, x^i\tau\beta)$$

with α in degree $(0, 2ja - 2b - 1, j\ell - h)$ and β in degree $(0, 0, 0)$.

13.0.2 (ii) The Case $i \geqslant j$:

The kernel is generated by $\left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right)$ and

$$\left(\begin{smallmatrix} t \\ 0 \end{smallmatrix}\right) + x^{i-j}\left(\begin{smallmatrix} 0 \\ x^j \end{smallmatrix}\right) = \left(\begin{smallmatrix} t \\ x^i \end{smallmatrix}\right).$$

The image contains t times these, and x^j times these.

We again write α for the element of the kernel represented by the first of these $\left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right)$ and β for the element represented by the second $\left(\begin{smallmatrix} t \\ x^i \end{smallmatrix}\right)$. This time,

$$\mathsf{Ext}^{*,*}_{H_*B}(H_*X_i, H_*X_j) = \langle\alpha, \beta\rangle/(t\alpha, x^i\alpha - t\beta, t\tau\beta, x^j\tau\beta)$$

with α in degree $(0, 2ia - 2b - 1, i\ell - h)$ and β in degree $(0, 0, 0)$.

13.0.3 (iii) The Case $i = j$

The cases $i = j$ in (i) and (ii) coincide, but there is one extra piece of structure to determine, namely the ring structure. This is determined by the statements that β is the identity endomorphism of X_i, and α is the endomorphism sending x^i to t and t to zero. So the extra relations are $\alpha^2 = 0$ and $\beta = 1$. Thus $x^i\alpha = t$, so t is a redundant generator, and the presentation becomes

$$\mathsf{Ext}^{*,*}_{H_*B}(H_*X_i, H_*X_i) \cong k[x, \tau, \alpha]/(\alpha^2, x^i\tau). \tag{13.5}$$

For the computations above, we assumed that $\ell > 2$. We now explain the modifications necessary for the case $\ell = 2$. In this case, the matrix in the resolution of H_*X_i is $\left(\begin{smallmatrix} t & x^i \\ x^{h-i} & -t \end{smallmatrix}\right)$. This corresponds to the following matrix factorisation of the relation $t^2 + x^h = 0$.

$$\begin{pmatrix} t & x^i \\ x^{h-i} & -t \end{pmatrix}\begin{pmatrix} t & x^i \\ x^{h-i} & -t \end{pmatrix} = (t^2 + x^h)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking homomorphisms into H_*X_j, the matrix for the differential is the transpose, $\left(\begin{smallmatrix} t & x^{h-i} \\ x^i & -t \end{smallmatrix}\right)$. The differential sends

$$\begin{pmatrix} t \\ 0 \end{pmatrix} \to \begin{pmatrix} -x^h \\ tx^i \end{pmatrix} \qquad \begin{pmatrix} x^j \\ 0 \end{pmatrix} \to \begin{pmatrix} tx^j \\ x^{i+j} \end{pmatrix} \qquad \begin{pmatrix} 0 \\ t \end{pmatrix} \to \begin{pmatrix} tx^{h-i} \\ x^h \end{pmatrix} \qquad \begin{pmatrix} 0 \\ x^j \end{pmatrix} \to \begin{pmatrix} x^{h+j-i} \\ -tx^j \end{pmatrix}.$$

There are more cases this time, but we content ourselves with computing $\mathsf{Ext}^{*,*}_{H_*B}(X_i, X_j)$ in the case $i = j \le h/2$, which is all we need. In this case, the kernel is generated by the element α representing $\left(\begin{smallmatrix} -x^{h-i} \\ t \end{smallmatrix}\right)$ and β representing $\left(\begin{smallmatrix} t \\ x^i \end{smallmatrix}\right)$. Again β is the identity endomorphism of X_i and α is the endomorphism sending x^i to t and t to $-x^{h-i}$, so α^2 is multiplication by $-x^{h-2i}$. This time the presentation becomes

$$\mathsf{Ext}^{*,*}_{H_*B}(H_*X_i, H_*X_i) \cong k[x, \tau, \alpha]/(\alpha^2 + x^{h-2i}, x^i\tau). \tag{13.6}$$

Theorem 13.7 *If $2i \le h$, we have*

$$\mathsf{End}_{\mathsf{D^b}(B)}(X_i) \cong k[x, \tau, \alpha]/(\alpha^2 + \lambda x^{h-2i}\tau^{\ell-2}, x^i\tau)$$

with $|x| = (-2a, -\ell)$, $|\tau| = (2b, h)$, $|\alpha| = (2ia - 2b - 1, i\ell - h)$, and λ some scalar in k.

Proof First assume that $\ell > 2$. We have to show that there are no non-zero differentials in the spectral sequence

$$\mathsf{Ext}^{*,*}_{H_*B}(H_*X_i, H_*X_i) \Rightarrow \mathsf{End}_{\mathsf{D^b}(B)}(X_i)$$

(see Section 7) whose E^2 page is given by Eq. 13.5, and we then have to address the ungrading problem for the E^∞ term. This is easier if we take into account the internal degrees, which have to be preserved by the differentials. So elements in the spectral sequence are triply graded, with $|x| = (0, -2a, -\ell)$, $|\tau| = (-1, 2b+1, h)$, $|\alpha| = (0, 2ia-2b-1, i\ell-h)$. As in the proof of Theorem 5.5, the possible tridegrees (u, v, w) at the E^2-term are very restricted. This time they lie in two parallel planes. We use the same normal vector $N = (\ell - 2, \ell, -2a)$, and consider the dot products $N \cdot (u, v, w) = (\ell - 2)u + \ell v - 2aw$. We note that $N \cdot |x| = 0$, $N \cdot |\tau| = 0$ and $N \cdot |\alpha| = 2 - \ell$. Hence $N \cdot |x^i\tau^j| = 0$ and $N \cdot |x^i\tau^j\alpha| = 2 - \ell$, and these are the only possible values of $N \cdot (u, v, w)$ for degrees of elements at E^2.

The differential d_n decreases u by n, increases v by $n-1$, and leaves w unchanged. It therefore increases $N \cdot (u, v, w)$ by $2n - \ell$. Since $n \geqslant 2$, we deduce that either $d_n = 0$ or $(2n-\ell)+(2-\ell) = 0$, so $n = \ell - 1$. In the latter case, $d_{\ell-1}$ sends x and τ to zero, and α to an element of degree $(-\ell+1, 2ia-2b+\ell-3, i\ell-h)$. There is only one such monomial in x and τ, namely $x^{h-i}\tau^{\ell-1}$. The assumption that $2i \leqslant h$ implies that this is zero, since $x^i\tau = 0$.

For the ungrading problem, since α^2 has even degree it cannot involve $\alpha x^{i_1}\tau^{i_2}$. If α^2 has the same bidegree as $x^{i_1}\tau^{i_2}$ then equating bidegree and solving, we find that $i_1 = h - 2i$, $i_2 = \ell - 2$. So we have that α^2 is a multiple of $x^{h-2i}\tau^{\ell-2}$.

The element $x^i\tau$ has degree $(-1, 2b-1-2ia, h-i\ell)$. There are no non-zero monomials in E^∞ with this internal degree, so $x^i\tau$ ungrades to zero.

Finally, in the case $\ell = 2$, B and X_i are formal, and the E^2 page (13.6) of the spectral sequence computes $\mathsf{End}_{\mathsf{D^b}(B)}(X_i)$ with no non-zero differentials or ungrading problems. □

Remark 13.8 If $\ell > 2$ and $i \leqslant h/3$ then the element $x^{h-2i}\tau^{\ell-2}$ is zero, so the value of λ only matters when $h/3 < i \leqslant h/2$. Using the models we produce in Section 16, it turns out that $\lambda = 1$ always holds, as we saw above in the case $\ell = 2$. We shall not need this information in what follows.

Theorem 13.9 *We have*

$$\mathsf{End}_{\mathsf{D_{sg}}(B)}(X_i) \cong \begin{cases} K[x,\alpha]/(\alpha^2 + \lambda x^{h-2i}\tau^{\ell-2}, x^i) & 2i \leqslant h \\ K[x,\alpha]/(\alpha^2 + \lambda x^{2i-h}\tau^{\ell-2}, x^{h-i}) & 2i \geqslant h \end{cases}$$

(recall $K = k[\tau, \tau^{-1}]$). This has finite length over K, and in particular, it is an Artinian local graded ring. The (homological) degree zero part is $k[x^{|b|}\tau^{|a|}]$; note that $x^{|b|}\tau^{|a|}$ is nilpotent, and often zero. As a module over $\mathsf{End}_{\mathsf{D_{sg}}(B)}(X_i)$, $\mathsf{Hom}_{\mathsf{D_{sg}}(B)}(X_i, X_j)$ has finite length. The Krull–Schmidt theorem holds for finite direct sums of shifts of the objects X_i.

Proof By Corollary 9.6 we have $\mathsf{End}_{\mathsf{D_{sg}}(B)}(X_i) \cong \mathsf{End}_{\mathsf{D^b}(B)}(X_i)[\tau^{-1}]$, so if $2i \leqslant h$ it follows from Theorem 13.7 that this is isomorphic to $K[x,\alpha]/(\alpha^2 + \lambda x^{2h-i}\tau^{\ell-2}, x^i)$. The structure for $2i \geqslant h$ then follows from the isomorphism $X_i \cong \Sigma^{1-2ia}X_{h-i}$ given in Proposition 13.2.

Next we compute the homological degree zero part. The degree of α is $2ia - 2b - 1$, which is odd, while the degrees of x and τ are even. If $x^j\tau^m$ has degree zero then $ja = mb$. Since a and b are coprime, this implies that j is divisible by b, m is divisible by a, and the element is a power of $x^{|b|}\tau^{|a|}$ (note that $ah - b\ell = 1$, and h and ℓ are positive, a and b have the same sign).

The E^2 page of the spectral sequence (13.4) is finitely generated over $k[\tau]$, and therefore $\mathsf{Hom}_{\mathsf{D_{sg}}(B)}(X_i, X_j) = \mathsf{Hom}_{\mathsf{D^b}(B)}(X_i, X_j)[\tau^{-1}]$ has finite length over K.

The final statement about the Krull–Schmidt theorem follows from the fact that the endomorphism rings of the X_i are local rings. □

Corollary 13.10 *The socle of the homological degree $2a-1$ part of $\mathsf{End}_{\mathsf{D_{sg}}(B)}(X_i)$, as a module over the degree zero part, is spanned by the element $\alpha x^{i-1}\tau$ in degree $(2a-1, \ell)$. None of the other monomials of homological degree $2a-1$ has internal degree ℓ.*

Proof Using Theorem 13.9, the monomials of homological degree $2a-1$ in $\mathsf{End}_{\mathsf{D}_{\mathsf{sg}}(B)}(X_i)$ are the elements $\alpha x^{i-1-j|b|}\tau^{1-j|a|}$ in degree $(2a-1, \ell + j(\ell|b| - h|a|)) = (2a-1, \ell \pm j)$. $\square$

Corollary 13.11 *The socles of the homological degree $2a-1$ parts of $\mathsf{End}_{\mathsf{D}_{\mathsf{sg}}(B)}(X_i)$ for $1 \le i \le h$ are represented by the triangles (13.3) coming from the almost split sequences of $\bar{B}$-modules.*

Proof The connecting homomorphism for the triangle (13.3) for X_i is a non-zero element of degree $(2a-1, \ell)$ in $\mathsf{End}_{\mathsf{D}_{\mathsf{sg}}(B)}(X_i)$. By Corollary 13.10, it is therefore a non-zero multiple of $\alpha x^{i-1}\tau$. $\square$

Remark 13.12 The images in $\mathsf{D}^{\mathsf{b}}(A)$ of the objects X_i of $\mathsf{D}^{\mathsf{b}}(B)$ are, up to shifts, the analogues of the modules W_i described in Section 12.

Write $Y_i = \mathcal{H}om_B(k, X_i)$ for the image of X_i in $\mathsf{D}^{\mathsf{b}}(A)$. Then Y_i is a free $k[\tau]$-module on two generators, u_i of degree $2a-1$ and v_i in degree $-2(i-1)a$. The action of ξ is given by

$$m_{i+1}(\underbrace{\xi, \dots, \xi}_{i \text{ copies}}, u_i) = v_i,$$
$$m_{h-i+1}(\underbrace{\xi, \dots, \xi}_{h-i \text{ copies}}, v_i) = (-1)^{(h-2i+2)(h-1)/2}\tau^{\ell} u_i$$

and the remaining m_j vanish.

14 Auslander–Reiten triangles

For background on Auslander–Reiten triangles in triangulated categories, see Happel [25, 26]. We shall construct them in the category $\mathsf{D}_{\mathsf{sg}}(B)$ and use them to classify the indecomposables. Although we make use of the internal grading to identify these triangles, the grading does not interfere with their existence and use for classification.

Suppose that X is an indecomposable, internally gradable object in $\mathsf{D}_{\mathsf{sg}}(B)$ with local endomorphism ring $\mathsf{End}_{\mathsf{D}_{\mathsf{sg}}(B)}(X)$. A map from another, not necessarily internally gradable object Y to X has a right inverse (i.e., induces an isomorphism from a direct summand of Y to X) if and only if the induced map $\mathsf{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(X, Y) \to \mathsf{End}_{\mathsf{D}_{\mathsf{sg}}(B)}(X)$ is surjective.

Using Theorem 10.8, the dual $\mathsf{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(X, \Sigma^{1-2a}X)$ has a simple socle as a module over $\mathsf{End}_{\mathsf{D}_{\mathsf{sg}}(B)}(X)$, and the socle is a map from X to $\Sigma^{1-2a,-\ell}X$. For the objects X_i, this socle is identified in Corollary 13.10. Choose a non-zero morphism $\alpha_X : X \to \Sigma^{1-2a,-\ell}X$ in this socle. This has the property that a map $Y \to X$ has a right inverse if and only if the composite $Y \to X \xrightarrow{\alpha_X} \Sigma^{1-2a}X$ is non-zero.

Complete to a triangle

$$\Sigma^{-2a,-\ell}X \to Z \to X \xrightarrow{\alpha_X} \Sigma^{1-2a,-\ell}X$$

in $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$. This then has the following lifting property. If a map $Y \to X$ does not have a right inverse, then it lifts to a map $Y \to Z$.

$$\begin{tikzcd}
 & & Y \arrow[d] \arrow[dl, dotted] & \\
\Sigma^{-2a}X \arrow[r] & Z \arrow[r] & X \arrow[r, "\alpha_X"] & \Sigma^{1-2a}X
\end{tikzcd}$$

Similarly, if $\Sigma^{-2a}X \to Y$ does not have a left inverse, then it extends to a map $Z \to Y$.

$$\begin{tikzcd}
\Sigma^{-1}X \arrow[r, "\Sigma^{-1}\alpha_X"] & \Sigma^{-2a}X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[dl, dotted] & X \arrow[r, "\alpha_X"] & \Sigma^{1-2a}X \\
 & Y & & &
\end{tikzcd}$$

These are the defining properties of an *Auslander–Reiten triangle*, sometimes also called an *almost split triangle*.

Theorem 14.1 *The Auslander–Reiten triangles in $\mathsf{D}_{\mathsf{sg}}(B)$ for the objects X_i are the triangles* (13.3)*, and the Auslander–Reiten translate is $\Sigma^{-2a,-\ell}$.*

Proof This follows from Corollary 13.11. □

If Z' is a direct summand of Z then the composite $Z' \to Z \to X$ is called an *irreducible morphism* if it has the property that it is not an isomorphism, but for any factorisation $Z' \to U \to X$, either $Z' \to U$ has a left inverse or $U \to X$ has a right inverse. This property is symmetric, so that if Z' is a direct summand of Z then the composite $\Sigma^{-2a}X \to Z \to Z'$ is also an irreducible morphism.

The *Auslander–Reiten quiver* is the quiver (directed graph) whose vertices correspond to the isomorphism classes of indecomposable objects and whose directed edges correspond to the irreducible morphisms. This comes with a translation Σ^{-2a} with the property that there is an arrow from Z to X if and only if there is an arrow from $\Sigma^{-2a}X$ to Z.

For the objects $\Sigma^j X_i$, the Auslander–Reiten quiver has the following form.

$$\begin{tikzcd}
 & \Sigma^{-2a}X_1 \arrow[dr] & & X_1 \arrow[dr] & & \Sigma^{2a}X_1 & \\
\cdots & & X_2 \arrow[ur] \arrow[dr] & & \Sigma^{2a}X_2 \arrow[ur] \arrow[dr] & & \cdots \\
 & X_3 \arrow[ur] \arrow[dr] & & \Sigma^{2a}X_3 \arrow[ur] \arrow[dr] & & \Sigma^{4a}X_3 & \\
 & \vdots & {} \arrow[ur] & \vdots & {} \arrow[ur] & \vdots &
\end{tikzcd}$$

This wraps round to form a cylinder, since $\Sigma^{2b}X_i$ is isomorphic to X_i via τ. Since a and b are coprime, the circumference of the cylinder is b. The height is $h-1$, and $X_{h-1} \cong \Sigma^{2a-1}X_1$ is in the bottom row. This gives a total of $b(h-1)$ isomorphism classes of objects. In the usual language of translation quivers, we therefore have the following.

Theorem 14.2 *The objects $\Sigma^j X_i$ in $\mathsf{D}_{\mathsf{sg}}(B)$ form a connected component of the Auslander–Reiten quiver, isomorphic to $\mathbb{Z}A_{h-1}/T^{|b|}$, where T is the translation functor.*

For example, if $a = 6$, $h = 6$, $b = 5$, $\ell = 7$, we get the following quiver isomorphic to $\mathbb{Z}A_5/T^5$.

$$\begin{tikzcd}
X_1 \arrow[dr] \arrow[dd, dotted, no head] & & \Sigma^2X_1 \arrow[dr] & & \Sigma^4X_1 \arrow[dr] & & \Sigma^6X_1 \arrow[dr] & & \Sigma^8X_1 \arrow[dr] & & X_1 \arrow[dd, dotted, no head] \\
& \Sigma^2X_2 \arrow[ur] \arrow[dr] & & \Sigma^4X_2 \arrow[ur] \arrow[dr] & & \Sigma^6X_2 \arrow[ur] \arrow[dr] & & \Sigma^8X_2 \arrow[ur] \arrow[dr] & & X_2 \arrow[ur] \arrow[dr] & \\
\Sigma^2X_3 \arrow[ur] \arrow[dr] \arrow[dd, dotted, no head] & & \Sigma^4X_3 \arrow[ur] \arrow[dr] & & \Sigma^6X_3 \arrow[ur] \arrow[dr] & & \Sigma^8X_3 \arrow[ur] \arrow[dr] & & X_3 \arrow[ur] \arrow[dr] & & \Sigma^2X_3 \arrow[dd, dotted, no head] \\
& \Sigma^7X_2 \arrow[ur] \arrow[dr] & & \Sigma^9X_2 \arrow[ur] \arrow[dr] & & \Sigma X_2 \arrow[ur] \arrow[dr] & & \Sigma^3X_2 \arrow[ur] \arrow[dr] & & \Sigma^5X_2 \arrow[ur] \arrow[dr] & \\
\Sigma^5X_1 \arrow[ur] & & \Sigma^7X_1 \arrow[ur] & & \Sigma^9X_1 \arrow[ur] & & \Sigma X_1 \arrow[ur] & & \Sigma^3X_1 \arrow[ur] & & \Sigma^5X_1
\end{tikzcd}$$

Here, the left and right ends are identified along the dotted lines to form a cylinder of height 5 and circumference 5, for a total of 25 isomorphism classes of objects. Note that in this example we have $X_i \simeq \Sigma^{10}X_i$ for each i, and also for the middle row $X_3 \simeq \Sigma^5 X_3$. In general, Σ^b acts as a reflection on the quiver, about the horizontal line going through the middle. There are $[h/2]$ orbits of the shift functor. If h is odd, they all have length $2|b|$, while if h is even there is just one of them with length $|b|$, namely the middle one, and the rest have length $2|b|$.

15 Classification of Indecomposables

The goal of this section is to show that every indecomposable object in $\mathsf{D}_{\mathsf{sg}}(B)$ is isomorphic to some $\Sigma^j X_i$, so that the Auslander–Reiten quiver is that given in Theorem 14.2.

The following proposition plays a role analogous to that of the Harada–Sai lemma in the representation theory of finite dimensional algebras, see for example Lemma 4.14.1 of [7].

Proposition 15.1 *The composite of any h composable irreducible morphisms between the objects $\Sigma^j X_i$ is equal to zero.*

Proof For each i and j, the sum of the composites $\Sigma^{j-2a}X_i \to \Sigma^j X_{i+1} \to \Sigma^j X_i$ and $\Sigma^{j-2a}X_i \to \Sigma^{j-2a}X_{i-1} \to \Sigma^j X_i$ is the composite of two adjacent maps in an Auslander–Reiten triangle, and therefore equal to zero. Using these relations, any composite of h composable irreducible morphisms can be rewritten as a composite involving $\Sigma^{j-2a}X_1 \to \Sigma^j X_2 \to \Sigma^j X_1$, which is zero. In other words, we can deform any path of length h so that it hits the top edge of the cylinder, without moving the ends of the path. $\square$

Proposition 15.2 *If X is any non-zero object in $\mathsf{D}_{\mathsf{sg}}(B)$ then for some $n \in \mathbb{Z}$ there is a non-zero morphism $\Sigma^n X_1 \to X$.*

Proof By Corollary 9.6, the image of $X_1 = k$ under the equivalence $\mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$ is $A[\tau^{-1}]$. So if Y is the image of X in $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$ then

$$\mathsf{Hom}_{\mathsf{D}_{\mathsf{sg}}(B)}(X_1, X) \cong \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])}(A[\tau^{-1}], Y) \cong H_*Y.$$

if there is no non-zero morphism from any $\Sigma^n X_1$ to X then $H_*Y = 0$, so Y is quasi-isomorphic to zero in $\mathsf{D}^{\mathsf{b}}(A[\tau^{-1}])$ and hence X is quasi-isomorphic to zero in $\mathsf{D}_{\mathsf{sg}}(B)$. $\square$

Theorem 15.3 *Every indecomposable object in* $\mathsf{D}_{\mathsf{sg}}(B)$ *is isomorphic to some* $\Sigma^j X_i$ *with* $1 \leqslant i < h$*,* $0 \leqslant j < |b|$*.*

Proof First we note that the set of $\Sigma^j X_i$ with $1 \leqslant i < h$ and $0 \leqslant j < b$ is the set of vertices in the Auslander–Reiten quiver described in Section 14.

Let X be an indecomposable object. Then by Proposition 15.2 there is a non-zero morphism $\Sigma^n X_1 \to X$ for some $n \in \mathbb{Z}$. Since X is indecomposable, if this has a left inverse then it is an isomorphism, and we are done. Otherwise, it factors as $\Sigma^n X_1 \to \Sigma^{n+2a} X_2 \to X$. If $\Sigma^{n+2a} X_2 \to X$ has a left inverse, again we are done. Otherwise, we obtain a factorisation

$$\Sigma^n X_1 \to \Sigma^{n+2a} X_2 \to \Sigma^{n+2a} X_1 \oplus \Sigma^{n+4a} X_3 \to X.$$

Since the composite $\Sigma^n X_1 \to \Sigma^{n+2a} X_2 \to \Sigma^{n+2a} X_1$ is zero, it follows that the composite $\Sigma^n X_1 \to \Sigma^{n+2a} X_2 \to \Sigma^{n+4a} X_3 \to X$ is non-zero. If $\Sigma^{n+4a} X_3 \to X$ is not an isomorphism, then at the next stage, we obtain a statement that a sum of two composites is non-zero, so at least one of them has to be non-zero. Continuing this way, we obtain either an isomorphism $\Sigma^j X_i \to X$ for some i, j, or a factorisation through a composite of at least h irreducible morphisms between the objects X_i. In the latter case, by Proposition 15.1, it follows that $\Sigma^n X_1 \to X$ is the zero map, contradicting the way it was chosen. □

Corollary 15.4 *The Auslander–Reiten quiver of* $\mathsf{D}_{\mathsf{sg}}(B)$ *is isomorphic to* $\mathbb{Z}A_{h-1}/T^{|b|}$*.*

Proof This follows from Theorems 14.2 and 15.3. □

Corollary 15.5 *The Krull–Schmidt theorem holds in* $\mathsf{D}_{\mathsf{sg}}(B)$*.*

Proof This follows from the last statement of Theorem 13.9 together with Theorem 15.3. □

Remark 15.6 In Remark 11.1 we noted that for the formal graded rings $H_*(B)$ the objects X_s exist for all $s \geq 1$ and are inequivalent. The singularity category retains the τ-periodicity, since $\mathsf{D}_{\mathsf{sg}}(H_*(B)) \simeq \mathsf{D}_{\mathsf{csg}}(H_*(A)) \simeq \mathsf{D}^{\mathsf{b}}(H_*(A)[1/\tau])$, but now (in the absence of Proposition 13.2) the Auslander–Reiten quiver consists of two semi-infinite cylinders of circumference $|b|$, one containing the even suspensions of the X_i and one containing the odd suspensions of the X_i.

16 Models for $\mathsf{D}_{\mathsf{sg}}(B)$

In this section we exhibit some more familiar looking categories that are equivalent as triangulated categories to $\mathsf{D}_{\mathsf{sg}}(B)$.

Theorem 15.3 gives us a presentation for the category $\mathsf{D}_{\mathsf{sg}}(B)$ in terms of the *mesh category* of the quiver. The definition of the mesh category $k(\Gamma)$ of a translation quiver Γ comes from Riedtmann [40] (see also Bongartz and Gabriel [12]), and can be found in Section I.5.6 on pages 54–55 of Happel [26].

In the case of the Auslander-Reiten quiver $\mathbb{Z}A_{h-1}/T^{|b|}$ of Corollary 15.4, the generators for the morphisms in the mesh category are the irreducible morphisms between the $\Sigma^j X_i$. The mesh relations come from the Auslander–Reiten triangles, and say that for each

i and j, the sum of the composites $\Sigma^{j-2a}X_i \to \Sigma^j X_{i+1} \to \Sigma^j X_i$ and $\Sigma^{j-2a}X_i \to \Sigma^{j-2a}X_{i-1} \to \Sigma^j X_i$ is equal to zero. At the boundaries, $i = 1$ and $i = h-1$, only one of these composites makes sense, and the corresponding relation says that this composite is equal to zero.

The classification of Krull–Schmidt triangulated categories with finitely many isomorphism classes of indecomposables is described in Section 6 of Amiot [3], see also Chapter 2 of Amiot [4] and the paper of Xiao and Zhu [43]. Let Γ be the translation quiver $\mathbb{Z}A_{h-1}/T^{|b|}$. Then it is shown in Theorem 6.5 of [3] that given a triangulated category $\mathcal{T}$ with Auslander–Reiten quiver Γ, we have an equivalence of k-linear categories between the full subcategory $\mathsf{ind}\,(\mathcal{T})$ of indecomposables in $\mathcal{T}$ and the mesh category $k(\Gamma)$. This induces an equivalence between $\mathcal{T}$ and the additive closure of $k(\Gamma)$. Applying this to $\mathsf{D}_{\mathsf{sg}}(B)$, we obtain a k-linear equivalence $k(\mathbb{Z}A_{h-1}/T^{|b|}) \to \mathsf{ind}\,\mathsf{D}_{\mathsf{sg}}(B)$, the full subcategory of indecomposables, which then extends to an equivalence from the additive closure of $k(\mathbb{Z}A_{h-1}/T^{|b|})$ to $\mathsf{D}_{\mathsf{sg}}(B)$.

There is another triangulated category with the same Auslander–Reiten quiver. Let $\mathsf{D}^{\mathsf{b}}(kA_{h-1})$ be the bounded derived category of modules for the quiver A_{h-1} over k. The Auslander–Reiten quiver of $\mathsf{D}^{\mathsf{b}}(kA_{h-1})$ is the quiver $\mathbb{Z}A_{h-1}$. The translation T of this quiver lifts to the translation T of $\mathsf{D}^{\mathsf{b}}(kA_{h-1})$. It is shown in Keller [33] that the *orbit category* $\mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}$, whose Hom sets are by definition

$$\bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(kA_{h-1})}(X, \mathsf{T}^{n|b|}(Y)),$$

is a triangulated category in such a way that the canonical functor

$$\mathsf{D}^{\mathsf{b}}(kA_{h-1}) \to \mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}$$

is a triangle functor. The Auslander–Reiten quiver of the orbit category $\mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}$ is isomorphic to $\mathbb{Z}A_{h-1}/T^{|b|}$. So Theorem 6.5 of [3] shows that there is a k-linear equivalence

$$\mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}$$

inducing the isomorphism of Auslander–Reiten quivers. We would like to know that this is an equivalence of triangulated categories. This proves to be more delicate, but another theorem of Amiot comes to our rescue.

Theorem 16.1 *There is a triangulated equivalence* $\mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}$.

Proof We would like to apply Amiot [3, Theorem 7.2]. This states that given a finite triangulated category $\mathcal{T}$ which is connected, algebraic, and standard, there exists a Dynkin diagram Γ of type A, D or E, and a self-equivalence Φ of $\mathsf{D}^{\mathsf{b}}(k\Gamma)$, such that $\mathcal{T}$ is triangle equivalent to Keller's orbit category $\mathsf{D}^{\mathsf{b}}(k\Gamma)/\Phi$. To apply the theorem, we need to check the conditions.

To say that $\mathcal{T}$ is connected means that the Auslander–Reiten quiver is connected, so we have already established that $\mathsf{D}_{\mathsf{sg}}(B)$ is connected. To say that $\mathcal{T}$ is standard means that it is equivalent to a mesh category as a k-linear category, so we have also already established that $\mathsf{D}_{\mathsf{sg}}(B)$ is standard.

To say that $\mathcal{T}$ is algebraic means that there is a Frobenius category $\mathcal{E}$ such that $\mathcal{T}$ is triangle equivalent to the stable category $\underline{\mathcal{E}}$, see Keller [35, Section 3.6]. By Proposition 8.3 and Corollary 9.5 we have $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}]) \simeq \mathsf{D}_{\mathsf{sg}}(B)$, and $\mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}])$ is algebraic by [35, Lemma 3.3 and Theorem 3.9]. It follows that $\mathsf{D}_{\mathsf{sg}}(B)$ is algebraic.

We have therefore checked the conditions for applying the theorem of Amiot. Since the Auslander–Reiten quiver of $\mathsf{D}_{\mathsf{sg}}(B)$ is $\mathbb{Z}A_{h-1}/T^{|b|}$, we have $\Gamma = A_{h-1}$ and $\Phi = \mathsf{T}^{|b|}$, and the theorem now follows. □

There is another category that looks very similar, and we apply similar techniques to make the comparison. We write $\bar{B}$ for the formal A_∞ algebra $k[x]/(x^h)$ where $|x| = -2a$. Thus there is an obvious map $B \to \bar{B}$ sending x to x and t to zero. We consider the bounded derived category $\mathsf{D}^{\mathsf{b}}(\bar{B})$ and its quotient, the singularity category $\mathsf{D}_{\mathsf{sg}}(\bar{B})$ formed by quotienting out all objects finitely built from the ring. We have objects $\bar{M}_i = k[x]/(x^i)$ in this category for $1 \le i \le h$, and $\bar{M}_h$ is zero. The analogue of Corollary 10.7 in this situation says that $\mathsf{Hom}_{\mathsf{D}_{\mathsf{sg}}(\bar{B})}(X, Y)$ is the graded vector space dual of $\mathsf{Hom}_{\mathsf{D}_{\mathsf{sg}}(\bar{B})}(Y, \Sigma^{1-2a}X)$, and hence we have Auslander–Reiten triangles

$$\Sigma^{-2a}\bar{M}_i \to \Sigma^{-2a}\bar{M}_{i-1} \oplus \bar{M}_{i+1} \to \bar{M}_i \to \Sigma^{1-2a}\bar{M}_i.$$

However, in contrast to the situation for $\mathsf{D}_{\mathsf{sg}}(B)$, the periodicity is given by

$$\bar{M}_i \cong \Sigma^{2\ell b}\bar{M}_i,$$

since $\Omega^2\bar{M}_i \cong \Sigma^{2ha}\bar{M}_i$, and $2ha - 2 = 2\ell b$. The Auslander–Reiten quiver again consists of the $\Sigma^j\bar{M}_i$. It is in the form of a cylinder, and again the height of the cylinder is $h-1$, but the circumference is $\ell|b|$ instead of $|b|$. The generators and relations for this category are given in the same way as that of $\mathsf{D}_{\mathsf{sg}}(B)$ in terms of the irreducible morphisms and the Auslander–Reiten triangles.

Theorem 16.2 *There is a triangulated equivalence* $\mathsf{D}_{\mathsf{sg}}(\bar{B}) \simeq \mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{\ell|b|}$.

Proof This is proved in the same way as Theorem 16.1, using Amiot [3, Theorem 7.2]. □

The functor $\mathsf{D}_{\mathsf{sg}}(\bar{B}) \to \mathsf{D}_{\mathsf{sg}}(B)$ sends $\bar{M}_i$ to ΣX_i and irreducible morphisms to irreducible morphisms. It wraps the Auslander–Reiten quiver of $\mathsf{D}_{\mathsf{sg}}(\bar{B})$ around that of $\mathsf{D}_{\mathsf{sg}}(B)$ exactly ℓ times. Thus it corresponds to the functor on orbit categories

$$\mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{\ell|b|} \to \mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}.$$

There is another way to achieve this wrapping around. Namely, instead of considering differential $\mathbb{Z}$-graded modules for $\bar{B}$, we consider differential $\mathbb{Z}/2|b|$-graded modules. Let us write $\mathsf{D}^{\mathsf{b}}(\bar{B}, \mathbb{Z}/2|b|)$ for this bounded derived category $\mathsf{D}_{\mathsf{sg}}(\bar{B}, \mathbb{Z}/2b)$ for the corresponding singularity category.

Theorem 16.3 *There is a triangulated equivalence* $\mathsf{D}_{\mathsf{sg}}(\bar{B}, \mathbb{Z}/2|b|) \simeq \mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|}$. *There is an equivalence of categories* $\mathsf{D}_{\mathsf{sg}}(\bar{B}, \mathbb{Z}/2|b|) \to \mathsf{D}^{\mathsf{b}}(B)$ *making the following diagram commute.*

$$\begin{tikzcd} \mathsf{D}_{\mathsf{sg}}(\bar{B}) \arrow[r] \arrow[d] & \mathsf{D}_{\mathsf{sg}}(B) \\ \mathsf{D}_{\mathsf{sg}}(\bar{B}, \mathbb{Z}/2|b|) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Proof Again, this follows by applying Amiot [3, Theorem 7.2]. □

We put all these equivalences together in the following theorem.

Theorem 16.4 *We have equivalences of triangulated categories*

$$\mathsf{D}^{\mathsf{b}}(kA_{h-1})/\mathsf{T}^{|b|} \simeq \mathsf{D}_{\mathsf{sg}}(\bar{B}, \mathbb{Z}/2|b|) \simeq \mathsf{D}_{\mathsf{sg}}(B) \simeq \mathsf{D}_{\mathsf{csg}}(A) \simeq \mathsf{D}^{\mathsf{b}}(A[\tau^{-1}]) \simeq \mathsf{D}^{\mathsf{b}}(Q[\tau^{-1}]).$$

Each of these is a finite Krull–Schmidt category with $|b|(h-1)$ isomorphism classes of indecomposable objects, in $[h/2]$ orbits of the shift functor. The Auslander–Reiten quiver is $\mathbb{Z}A_{h-1}/T^{|b|}$.

17 H^*BG and $H_*\Omega BG^{\wedge}_p$

In this section, we apply our main results to the A_∞ algebras H^*BG and $H_*\Omega BG^{\wedge}_p$ for G a finite group with cyclic Sylow p-subgroups. It is shown in [9] that these give an instance of Context 1.1.

We are interested in the following occurrences of the A_∞ algebra A of Section 4. Let p be an odd prime and k a field of characteristic p. Let G be a finite group with cyclic Sylow subgroup P of order p^n and inertial index $q = |N_G(P){:}C_G(P)| > 1$. Let C^*BG be the cochains on the classifying space BG with coefficients in k, and $C_*\Omega BG^{\wedge}_p$ be the chains on the p-completed loop space of BG, again with coefficients in k.

The A_∞ algebra structures on H^*BG and on $H_*\Omega BG^{\wedge}_p$ coming from the DG algebras C^*BG and $C_*\Omega BG^{\wedge}_p$ are described in [9]. They are Koszul dual, and we can apply the results of this paper either with $A = H^*BG$, $B = H_*\Omega BG^{\wedge}_p$, or with $A = H_*\Omega BG^{\wedge}_p$ and $B = H^*BG$.

In the case of $A = H_*\Omega BG^{\wedge}_p$, we have $a = q$, $b = q-1$, $h = p^n - (p^n-1)/q$ and $\ell = p^n$, and we must assume that $q > 1$. In the case of $A = H^*BG$ (homologically graded in negative degrees), the roles are reversed and we have $a = -(q-1)$, $b = -q$, $h = p^n$, and $\ell = p^n - (p^n-1)/q$.

Proof of Theorem 1.3 By Proposition 6.2, the bounded derived categories of the DGA algebras C^*BG and $C_*\Omega BG^{\wedge}_p$ are equivalent to those of the A_∞ algebras H^*BG and $H_*\Omega BG^{\wedge}_p$. The equivalences of categories follow by applying Theorem 9.4 to the A_∞ algebras H^*BG and $H_*\Omega BG^{\wedge}_p$. The classification of the indecomposable objects in these categories follow from Theorem 16.4. □

18 Brauer Trees and Hecke Algebras

In this section, we describe what our main theorem tells us about Brauer tree algebras. In general, a Brauer tree algebra is described by a planar embedding of a tree with e edges corresponding to the simple modules. The vertices are assigned multiplicities, which are all equal to one with the possible exception of a single vertex of multiplicity $\lambda > 1$; otherwise we set $\lambda = 1$. This parameter λ is called the *exceptional multiplicity*, even when it equals one. These data are sufficient to describe the algebra up to Morita equivalence, and an algorithm for computing projective resolutions was described by Green [21]. Brauer tree algebras were first introduced in order to describe blocks of defect one in the representation theory of finite groups by Brauer [13], and the analysis was extended to all blocks of cyclic defect by Dade [16]. A nice treatment in this context may be found in the book of

Alperin [2]. They also appear in many other contexts in representation theory, and we shall give an example of this in characteristic zero below.

We say that a simple module is a *leaf module* if it corresponds to an edge one end of which has valency one and multiplicity one. The leaf modules are all syzygies of each other, so they have isomorphic Ext algebras.

Theorem 18.1 *Let $\boldsymbol{A}$ be a Brauer tree algebra with $e > 1$ edges and exceptional multiplicity λ, and let M be a simple leaf module for $\boldsymbol{A}$. Then the A_∞ algebra $\mathsf{Ext}^*_{\boldsymbol{A}}(M, M)$ is the algebra B described in* Section 4*, with parameters $a = e$, $b = e - 1$, $\ell = \lambda e + 1$ and $h = \ell - \lambda = \lambda(e-1) + 1$, giving an instance of Context 1.1.*

The singularity category of $\mathsf{Ext}^*_{\boldsymbol{A}}(M, M)$ has finite representation type, with $\lambda(e-1)^2$ isomorphism classes of indecomposable objects, in $[(\lambda(e-1)+1)/2]$ orbits of the shift functor Σ. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{\lambda(e-1)}/T^{e-1}$, where T is the translation functor Σ^{-2e}.

The cosingularity category of $\mathsf{Ext}^*_{\boldsymbol{A}}(M, M)$ has finite representation type, with λe^2 isomorphism classes of indecomposable objects, in $[(\lambda e+1)/2]$ orbits of the shift functor Σ. The Auslander–Reiten quiver is isomorphic to $\mathbb{Z}A_{\lambda e}/T^{e}$, where T is the translation functor $\Sigma^{-2(e-1)}$.

Proof It was shown by Gabriel and Riedtmann [18] that all Brauer tree algebras with e edges and exceptional multiplicity λ are stably equivalent (indeed, even more is true: Rickard [39] showed that these are all derived equivalent, but we don't need to go that far). In particular, they are all stably equivalent to the *Brauer star algebra* which has e vertices of valency one and multiplicity one, surrounding the one remaining vertex in the middle, which has multiplicity λ. The Brauer star algebras have exactly the same presentation as the algebras in Section 2 of [9], except for the change of characteristic. In more detail, the projective indecomposables are uniserial, and the radical filtration is isomorphic to its associated graded. Some readers may find it helpful to refer to the paper Bogdanic [11] which recalls details of Brauer tree algebras and shows that the stable grading comes from a grading of the Brauer tree algebras themselves.

Under such a stable equivalence, the leaf modules correspond to simple modules or first syzygies of simple modules, for the corresponding Brauer star algebra. It follows that we may compute the A_∞ structure on these Ext algebras using exactly the same computation as in [9] and the result is as stated in the theorem.

Theorem 1.2 now describes the singularity and cosingularity categories. □

Remark 18.2 In the remaining case $e = 1$, a Brauer tree algebra is Morita equivalent to a truncated polynomial algebra, so the A_∞ structure on the Ext ring of the simple module is just like that of a cyclic p-group.

As an example, let $\mathcal{H} = \mathcal{H}(n, q)$ be the Hecke algebra of the symmetric group of degree n over a field k of characteristic zero, where q is a primitive ℓth root of unity with $\ell \geqslant 2$. This has generators $T_1, \dots, T_{n-1}$ satisfying braid relations together with the relations $(T_i + 1)(T_i - q) = 0$. The cohomology $H^*(\mathcal{H}, k) = \mathsf{Ext}^*_{\mathcal{H}}(k, k)$ was computed by Benson, Erdmann and Mikaelian [8].

In the case where $n = \ell > 2$, $H^*(\mathcal{H}, k)$ has the form $k[x] \otimes \Lambda(t)$ where $|x| = -2n + 2$ and $|t| = -2n + 3$. This corresponds to the fact that in this case the principal block of $\mathcal{H}$ is a Brauer tree algebra for a tree which is a straight line with n vertices, $n - 1$ edges, and

$\lambda = 1$, with the trivial module k at one end as a leaf module. In particular, we can apply Theorem 18.1 in this context. So the Massey product of n copies of t is equal to $-x^{n-1}$, and writing $a = n-1$, $b = n-2$, $\ell = n$, $h = n-1$, we have $\mathsf{Ext}^*_{\mathcal{H}}(k, k) = B$ as an A_∞ algebra.

Theorem 18.3 *Let $\mathcal{H} = \mathcal{H}(n, q)$ be the Hecke algebra of the symmetric group of degree n over a field k of characteristic zero, where q is a primitive ℓth root of unity. In the case $n = \ell > 2$, the singularity category of the A_∞ algebra $\mathsf{Ext}^*_{\mathcal{H}}(k, k)$ has finite representation type, with $(n-2)^2$ isomorphism classes of indecomposables, in $[(n-1)/2]$ orbits of the shift functor. The Auslander–Reiten quiver is a cylinder of height $n-2$ and circumference $n-2$.*

The cosingularity category of the A_∞ algebra $\mathsf{Ext}^*_{\mathcal{H}}(k, k)$ also has finite representation type, with $(n-1)^2$ isomorphism classes of indecomposables, in $[n/2]$ orbits of the shift functor. The Auslander–Reiten quiver is a cylinder of height $n-1$ and circumference $n-1$.

Other examples of Hecke algebras described by Brauer trees may be found in Geck [19], Ariki [5].

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Declarations

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