Mathematische Zeitschrift
https://doi.org/10.1007/s00209-022-03031-6

Mathematische Zeitschrift

# Two-dimensional cycle classes on $\overline{\mathcal{M}}_{0,n}$

**Rohini Ramadas[1] · Rob Silversmith[1]**

Received: 8 May 2020 / Accepted: 17 March 2022


**Abstract**

For each $n \geq 5$, we give an $S_n$-equivariant basis for $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, as well as for $H_{2(n-5)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Such a basis exists for $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ and for $H_{2(n-4)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, but it is not known whether one exists for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ when $3 \leq k \leq n-6$.

## 1 Introduction

The moduli space $\overline{\mathcal{M}}_{0,n}$ is an $(n-3)$-dimensional smooth projective variety that parametrizes stable $n$-marked genus-zero curves. $\overline{\mathcal{M}}_{0,n}$ admits a stratification whose intricate combinatorics are reflected in the structure of its homology groups. These groups have been described, in terms of generators and relations, first by Keel [10] and then by Kontsevich and Manin [12]. The groups were described further by Kapranov [9], Fulton-MacPherson [7], Manin [13], Getzler [8], Yuzvinsky [18] and others.

$\overline{\mathcal{M}}_{0,n}$ carries a natural $S_n$-action; this induces $S_n$-actions on the homology groups of $\overline{\mathcal{M}}_{0,n}$. Indeed, for $n \geq 5$, $S_n$ is the full automorphism group of $\overline{\mathcal{M}}_{0,n}$ [3, 5]. Getzler [8] gave an algorithm for computing the character of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ as an $S_n$-representation, and Bergström and Minabe [2] later gave a recursive formula for the character using spaces of weighted stable curves.

Recall that a finite-dimensional representation $V$ of a finite group $G$ is called a *permutation representation* if $V$ has a *permutation basis*; that is, a basis $B$ of $V$ such that the action of $G$ on $V$ restricts to an action on $B$. We ask:

**Question 1** *Is $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ a permutation representation of $S_n$ for all $n \geq 3$ and $k \geq 0$?*

Farkas and Gibney [6] gave an affirmative answer if $k = n - 4$ by producing a permutation basis of divisors. Since Poincaré duality induces $S_n$-equivariant isomorphisms $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \rightarrow H_{2(n-3-k)}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, this also implies an affirmative answer if $k = 1$. The first author found [15] a permutation basis for $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ that is not Poincaré dual to the basis of Farkas and Gibney—interestingly, the two bases are not isomorphic as $S_n$-sets for even $n$.

✉ Rob Silversmith
Rob.Silversmith@warwick.ac.uk

Rohini Ramadas
Rohini.Ramadas@warwick.ac.uk

[1] Warwick Mathematics Institute, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, UK

We give an affirmative answer in the case $k = 2$ (and therefore also $k = n - 5$):

**Theorem** 2.8. *Let $\mathbf{P}^1_{2,n}$ denote the $S_n$-set of all subsets $A \subseteq \{1, \ldots, n\}$ such that $5 \leq |A| \leq n$ and $|A|$ is odd. Let $\mathbf{P}^2_{2,n}$ denote the $S_n$-set of unordered pairs $\{P_1, P_2\}$, where $P_1$ and $P_2$ are disjoint subsets of $\{1, \ldots, n\}$ of cardinality at least 3. Then $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is a permutation representation of $S_n$, with a permutation basis in equivariant bijection with $\mathbf{P}^1_{2,n} \sqcup \mathbf{P}^2_{2,n}$, for all $n \geq 3$.*

For general $k$, Question 1 appears to be open. The known bases of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ are constructed via recursion on $n$, which involves treating the $n$-th marked point as special. For example, Kapranov [9] and Yuzvinsky [18] described bases (recursively and in closed form, respectively) for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ that are permutation bases with respect to the $S_{n-1}$ action induced by permuting all but the $n$th point. In contrast, a permutation basis for $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, if it exists, would treat the $n$ marked points symmetrically.

Very recent work of Castravet and Tevelev [4] on the derived category of $\overline{\mathcal{M}}_{0,n}$ implies that $H_*(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = \bigoplus_{k=0}^{n-3} H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is a permutation representation; however, this does not answer Question 1, as the permutation basis given for $H_*(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ consists of classes not of pure degree. (They are Chern characters of certain vector bundles.)

### 1.1 An $S_n$-equivariant filtration of $H_{2k}(\overline{\mathcal{M}}_{0,n})$ and the proof of Theorem 2.8

Theorem 2.8 follows from the stronger Theorem 2.7, which we now explain. In [15], the first author defined (see Definition 4) an $S_n$-equivariant decomposition

$$H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong \bigoplus_{r=1}^{\min\{k,n-2-k\}} Q^r_{k,n}. \tag{1}$$

The first author also showed (Theorem 2.3) that for all $k \geq 0$ and $n \geq 3$, $Q^1_{k,n}$ has a permutation basis, in equivariant bijection with the $S_n$-set $\mathbf{P}^1_{k,n}$ defined in Theorem 2.3. We prove:

**Theorem** 2.7. *For all $k \geq 0$ and $n \geq 3$, $Q^2_{k,n}$ has a permutation basis, in equivariant bijection with the $S_n$-set $\mathbf{P}^2_{k,n}$ defined in Definition 2.5.*

**Corollary** 2.9. *The $S_n$-permutation representation $\mathbb{Q}(\mathbf{P}^1_{k,n} \sqcup \mathbf{P}^2_{k,n})$ is a subrepresentation of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ for all $k$ and $n$.*

By (1), $H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = Q^1_{1,n}$ and $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong Q^1_{2,n} \oplus Q^2_{2,n}$. Thus Theorems 2.3 and 2.7 imply Theorem 2.8. Note also that by Poincaré duality, for $n \leq 8$, these express $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ as a permutation representation for all $k$.

### 1.2 Experimental evidence towards Question 1

Let $V$ be a finite-dimensional permutation representation of $G$ with permutation basis $B$. Then $B$ is a disjoint union of $G$-orbits, i.e. transitive $G$-sets. It is important to note that if $B_1$ and $B_2$ are two permutation bases for $V$, they need not be isomorphic as $G$-sets. However, the number of orbits of $B_1$ is equal to the dimension of the $G$-fixed subspace of $V$, thus equal to the number of orbits of $B_2$. Recall that every transitive $G$-set is equivariantly isomorphic to the $G$-set of left cosets of some subgroup of $G$. This gives a computational strategy for checking whether a given $G$-representation $W$ is a permutation representation — namely, one enumerates the subgroups of $G$ and the associated characters, then does an exhaustive search for the character of $W$ as a non-negative integer combination of those characters. This is doable in the GAP computer algebra system for $n \leq 10$.

J. Bergström and S. Minabe shared with us the characters of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ for $n \leq 20$. We now give some observations inferred from their data, from Theorem 2.7, and from a GAP analysis as above. The smallest example where Theorems 2.3 and 2.7 do not answer Question 1 is $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$.

- $Q^3_{3,9}$ is not a permutation representation. We computed the character of $Q^3_{3,9}$ by subtracting those of $Q^1_{3,9}$ and $Q^2_{3,9}$ (from Theorems 2.3 and 2.7) from that of $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$ (from Bergström and Minabe's data). We then used the strategy outlined in the above paragraph to show that $Q^3_{3,9}$ is not a permutation representation.
- Despite the previous point, $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$ is a permutation representation. We showed this by again applying the strategy of the previous paragraph; in fact, GAP produced two inequivalent decompositions of $H_6(\overline{\mathcal{M}}_{0,9}, \mathbb{Q})$ into transitive permutation representations. It follows that in these two decompositions, the two permutation subrepresentations $\mathbb{Q}\mathbf{P}^1_{3,9}$ and $\mathbb{Q}\mathbf{P}^2_{3,9}$ *cannot* both appear.
- $H_{2k}(\overline{\mathcal{M}}_{0,10}, \mathbb{Q})$ is a permutation representation for all $k$, by the same exhaustive computer search.

### 1.3 Outline of the paper

In Sect. 2, we recall fundamental results about the homology groups of $\overline{\mathcal{M}}_{0,n}$, and define $Q^r_{k,n}$, $\mathbf{P}^1_{k,n}$, and $\mathbf{P}^2_{k,n}$. In Sect. 3.1 we outline the proof of Theorem 2.7, and the complete proof is in Sect. 3.2. In Sect. 4, we state a conjectural formula for the dimension of $Q^r_{k,n}$ in general.

We would like to thank J. Bergström and S. Minabe for generously sharing data they had generated (based on their recursive formula) of the characters of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ for $n \leq 20$ — this was crucial in allowing us to formulate and check conjectures. We are grateful to Bernd Sturmfels, Renzo Cavalieri, Maria Monks Gillespie and Melody Chan for useful conversations, and to the two anonymous referees for their thoughtful comments.

The first author was supported by NSF grants DMS-0943832 and DMS-1703308, and by a postdoctoral position at Brown University. The second author was supported by NSF grants DMS-0602191 and DMS-1645877, and by a postdoctoral position at Northeastern University.

## 2 Background and notation

**Definition 2.1** Let $\mathbf{S}_{k,n}$ denote the set of stable trees with $n-2-k$ vertices, $n-3-k$ edges, and $n$ marked half-edges (labeled by the set $\{1, \ldots, n\}$). (A tree is *stable* if each vertex has valence at least 3.) $\mathbf{S}_{k,n}$ is in canonical bijection with the set of (closed) $k$-dimensional boundary strata in $\overline{\mathcal{M}}_{0,n}$.

We will abuse notation, and refer to an element of $\mathbf{S}_{k,n}$ interchangeably as a marked tree, as a closed $k$-dimensional subvariety of $\overline{\mathcal{M}}_{0,n}$, and as a cycle class in $H_{2k}(\overline{\mathcal{M}}_{0,n})$ or various quotients thereof. For example, it is a well-known result of Keel [10] that $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is spanned by $\mathbf{S}_{k,n}$. Kontsevich and Manin [12] gave a spanning set for the kernel of the map $\mathbb{Q}\mathbf{S}_{k,n} \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Let $\sigma \in \mathbf{S}_{k+1,n}$, let $v$ be a vertex of $\sigma$ with valence at least 4, and let $A, B, C, D$ be four edges or half-edges incident to $v$. Let $\mathbf{T}$ denote the set of edges and half-edges incident to $v$, other than $A$, $B$, $C$, $D$. We define a relation

$$\mathcal{R} = \mathcal{R}(\sigma, v, A, B, C, D) = \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AB\mathbf{U}_1|CD\mathbf{U}_2) - \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AC\mathbf{U}_1|BD\mathbf{U}_2) \in \mathbb{Q}\mathbf{S}_{k,n}. \tag{2}$$

We now explain this notation, which we will continue to use. Here $(AB\mathbf{U}_1|CD\mathbf{U}_2) \in \mathbf{S}_{k,n}$ is the tree obtained from $\sigma$ by replacing $v$ with two vertices $v'$, $v''$ connected by an edge $e'$, with the edges or half-edges $\{A, B\} \cup \mathbf{U}_1$ incident to $v'$ and the edges or half-edges $\{C, D\} \cup \mathbf{U}_2$ incident to $v''$. Note that contracting $e'$ yields $\sigma$ again. Kontsevich and Manin showed that the relations obtained this way (varying over all choices of $\sigma, v, A, B, C, D$) span the kernel of the map $\mathbb{Q}\mathbf{S}_{k,n} \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.

Note that for any $\sigma \in \mathbf{S}_{k,n}$, we have

$$\sum_v (\mathrm{val}(v) - 3) = k, \tag{3}$$

where the sum is over vertices of $\sigma$. Thus to any $\sigma \in \mathbf{S}_{k,n}$ we may associate a partition $\lambda_\sigma$ of $k$, namely the set of nonzero summands on the right-hand side of (3). Since $|V(\sigma)| = n-2-k$, $\lambda_\sigma$ has at most $n - 2 - k$ parts. (As a partition of $k$, $\lambda_\sigma$ also has at most $k$ parts.)

**Definition 2.2** [16] For $r \in \{1, \ldots, \min\{k, n-2-k\}\}$, let $\mathbf{S}^{\geq r}_{k,n} \subseteq \mathbf{S}_{k,n}$ be the subset consisting of those $\sigma \in \mathbf{S}_{k,n}$ such that $\lambda_\sigma$ has at least $r$ parts. This defines an $S_n$-invariant filtration

$$\mathbf{S}_{k,n} = \mathbf{S}^{\geq 1}_{k,n} \supseteq \mathbf{S}^{\geq 2}_{k,n} \supseteq \cdots \supseteq \mathbf{S}^{\geq k}_{k,n}.$$

Since $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is spanned by $\mathbf{S}_{k,n}$, this also defines an $S_n$-invariant filtration

$$H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = Q^{\geq 1}_{k,n} \supseteq Q^{\geq 2}_{k,n} \supseteq \cdots \supseteq Q^{\geq k}_{k,n}, \tag{4}$$

where

$$Q^{\geq r}_{k,n} = \mathrm{Span}(\mathbf{S}^{\geq r}_{k,n}) \subseteq H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}).$$

Finally, let $\mathbf{S}^r_{k,n} = \mathbf{S}^{\geq r}_{k,n} \setminus \mathbf{S}^{\geq r+1}_{k,n}$ and $Q^r_{k,n} = Q^{\geq r}_{k,n}/Q^{\geq r+1}_{k,n}$. Note that $\mathbf{S}^r_{k,n}$ spans $Q^r_{k,n}$. Note also that the $S_n$-invariant filtration (4), together with Maschke's Theorem, induces an isomorphism of $S_n$-representations

$$H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong \bigoplus_{r=1}^{\min\{k,n-2-k\}} Q^r_{k,n}.$$

In [16], it is shown that the filtration (4) is preserved by pushforwards along forgetful morphisms of moduli spaces as well as along "Hurwitz correspondences", which are algebraic correspondences that generalize forgetful maps. In [17], $Q^1_{k,n}$ is described explicitly as a permutation representation. We briefly recall the relevant results here. Recall that Mumford [14] and Arbarello-Cornalba [1] defined, for every $k$, a codimension-$k$ class $\kappa_k \in H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Given a subset $T \subset \{1, \ldots, n\}$ with $|T| \geq 3$, let $\pi_T : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,|T|}$ be the forgetful map remembering only the marked points in $T$, and set $\kappa^T_k := \pi^*_T(\kappa_k) \in H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. Let

$$\mathcal{K}^k = \mathrm{Span}\{\kappa^T_k \mid T \subset \{1, \ldots, n\}\} \subset H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}).$$

**Theorem 2.3** ([17])

1. *The subspace* $\mathcal{K}^k \subset H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ *is the annihilator, under the intersection pairing, of the subspace* $Q^{\geq 2}_{k,n} \subset H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$*, and there is therefore a natural* $S_n$*-equivariant isomorphism* $\mathcal{K}^k \cong (Q^1_{k,n})^\vee$.
2. *The set* $\{\kappa_k^T \mid T \subset \{1, \ldots, n\}, |T| \geq 3, |T| \equiv (k+3) \mod 2\}$ *is a permutation basis for* $\mathcal{K}^k$.

Let $\mathbf{P}^1_{k,n} := \{T \subseteq \{1, \ldots, n\} : k+3 \leq |T| \leq n, \ |T| \equiv k+3 \mod 2\}$. Since every finite-dimensional $S_n$ representation is isomorphic to its dual, we obtain that $Q^1_{k,n} \cong \mathbb{Q}\mathbf{P}^1_{k,n}$ as $S_n$-representations. Since $Q^{\geq 2}_{1,n} \subseteq H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ is zero by definition, this implies:

**Corollary 2.4** ([17]) *For* $n \geq 3$*, there are isomorphisms of* $S_n$*-representations*

$$H_2(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong Q^1_{1,n} \cong \mathbb{Q}\mathbf{P}^1_{1,n} \cong \mathbb{Q}\{T \subseteq \{1, \ldots, n\} : 4 \leq |T| \leq n, \ |T| \ \textit{even}\}.$$

**Definition 2.5** Let $\mathbf{P}^2_{k,n}$ denote the set of unordered pairs $\{(P_1, \alpha_1), (P_2, \alpha_2)\}$, where:

- $P_1$ and $P_2$ are disjoint subsets of $\{1, \ldots, n\}$, and
- $\alpha_1$ and $\alpha_2$ are positive integers whose sum is $k$, and
- $P_i \geq \alpha_i + 2$ for $i \in \{1, 2\}$.

Observe that Definition 2.5 is consistent with the notation $\mathbf{P}^2_{2,n}$ from Theorem 2.8.

**Definition 2.6** We define an $S_n$-equivariant map $W_n : \mathbf{S}^2_{k,n} \to \mathbf{P}^2_{k,n}$ as follows. For $\sigma \in \mathbf{S}^2_{k,n}$, let $v_1$ and $v_2$ be the two vertices of $\sigma$ with valence at least 4. There is a unique edge $e_1$ incident to $v_1$ that is contained in any path from $v_1$ to $v_2$, and similarly an edge $e_2$ incident to $v_2$. Cutting the two edges $e_1$ and $e_2$ (into unmarked half-edges) yields three connected components $\sigma_1$, $\sigma'$, and $\sigma_2$. (Note: the graphs $\sigma_1$ and $\sigma_2$ have a single "central" vertex, with trivalent subtrees incident to it. The graph $\sigma'$ is itself a trivalent subtree.)

Let $\alpha_1 = \mathrm{val}(v_1) - 3$, and let $P_1 \subseteq \{1, \ldots, n\}$ be the set of marked half-edges in $\sigma_1$. Similarly let $\alpha_2 = \mathrm{val}(v_2) - 3$, and let $P_2 \subseteq \{1, \ldots, n\}$ be the set of marked half-edges in $\sigma_2$.

Then we define $W_n(\sigma) := \{(P_1, \alpha_1), (P_2, \alpha_2)\}$. Observe that $W_n(\sigma) \in \mathbf{P}^2_{k,n}$, since:

- $P_1$ and $P_2$ are disjoint (by definition),
- $\alpha_1 + \alpha_2 = k$ (this follows from the fact that $\sigma$ is $k$-dimensional), and
- $|P_i| \geq \alpha_i + 2$ for $i \in \{1, 2\}$. (This is equivalent to $|P_i| \geq \mathrm{val}(v_i) - 1$, which follows from the fact that by stability, every edge incident to $v_i$, except $e_1$, is on a nonrepeating path from $v_i$ to at least one marked half-edge in $\sigma_1$.)

**Theorem 2.7** *For any* $n > 3$ *and* $2 \leq k \leq n-4$*,* $Q^2_{k,n}$ *is isomorphic (as an* $S_n$*-module) to* $\mathbb{Q}\mathbf{P}^2_{k,n}$.

Together with Theorem 2.3 and the fact that $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ consists of the two graded pieces $Q^1_{2,n}$ and $Q^2_{2,n}$, Theorem 2.7 implies:

**Theorem 2.8** *For* $n \geq 3$*,* $H_4(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \cong \mathbb{Q}\mathbf{P}^1_{2,n} \oplus \mathbb{Q}\mathbf{P}^2_{2,n}$ *as* $S_n$*-representations.*

**Corollary 2.9** *For* $n \geq 3$ *and* $k \geq 0$*,* $\mathbb{Q}\mathbf{P}^1_{k,n} \oplus \mathbb{Q}\mathbf{P}^2_{k,n}$ *is a subrepresentation of* $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.

# 3 Proof of Theorem 2.7

## 3.1 Outline of Proof

We now introduce further filtrations

$$\begin{aligned} \mathbf{S}^2_{k,n} &= (\mathbf{S}^2_{k,n})^{\geq 0} \supseteq (\mathbf{S}^2_{k,n})^{\geq 1} \supseteq \cdots \supseteq (\mathbf{S}^2_{k,n})^{\geq n-k-4} \\ \mathbf{P}^2_{k,n} &= (\mathbf{P}^2_{k,n})^{\geq 0} \supseteq (\mathbf{P}^2_{k,n})^{\geq 1} \supseteq \cdots \supseteq (\mathbf{P}^2_{k,n})^{\geq n-k-4} \\ Q^2_{k,n} &= (Q^2_{k,n})^{\geq 0} \supseteq (Q^2_{k,n})^{\geq 1} \supseteq \cdots \supseteq (Q^2_{k,n})^{\geq n-k-4}, \end{aligned}$$

defined as follows.

**Definition 3.1** For $\sigma \in \mathbf{S}^2_{k,n}$, let $\sigma_1$, $\sigma_2$, and $\sigma'$ be as in Definition 2.6. Then for $0 \leq b \leq n-k-4$, we let $(\mathbf{S}^2_{k,n})^{\geq b}$ denote the subset consisting of those trees $\sigma$ such that $\sigma'$ has at least $b$ marked half-edges. Let

$$(Q^2_{k,n})^{\geq b} = \operatorname{Span}((\mathbf{S}^2_{k,n})^{\geq b}) \subseteq Q^2_{k,n} \qquad \text{and} \qquad (\mathbf{P}^2_{k,n})^{\geq b} = W_n((\mathbf{S}^2_{k,n})^{\geq b}) \subseteq \mathbf{P}^2_{k,n}.$$

We also define the graded pieces

$$\begin{aligned} (\mathbf{S}^2_{k,n})^{b} &= (\mathbf{S}^2_{k,n})^{\geq b} \setminus (\mathbf{S}^2_{k,n})^{\geq b+1} \\ (Q^2_{k,n})^{b} &= (Q^2_{k,n})^{\geq b} / (Q^2_{k,n})^{\geq b+1} \\ (\mathbf{P}^2_{k,n})^{b} &= (\mathbf{P}^2_{k,n})^{\geq b} \setminus (\mathbf{P}^2_{k,n})^{\geq b+1}. \end{aligned}$$

We write $\rho_{n,b}$ for the natural surjective linear map $\mathbb{Q}(\mathbf{S}^2_{k,n})^b \to (Q^2_{k,n})^b$. Observe that $W_n$ restricts to a surjection $W_{n,b} : (\mathbf{S}^2_{k,n})^b \to (\mathbf{P}^2_{k,n})^b$. In other words, we have

$$\begin{array}{ccc} & \mathbb{Q}(\mathbf{S}^2_{k,n})^b & \\ {\scriptstyle \rho_{n,b}} \swarrow & & \searrow {\scriptstyle W_{n,b}} \\ (Q^2_{k,n})^b & & \mathbb{Q}(\mathbf{P}^2_{k,n})^b \end{array}$$

where we abuse notation and use $W_{n,b}$ to refer to the map induced on free $\mathbb{Q}$-vector spaces by $W_{n,b}$. We will show (Lemma 3.2) that $W_{n,b}$ factors equivariantly through $\rho_{n,b}$, and (Lemma 3.7) that $\rho_{n,b}$ factors equivariantly through $W_{n,b}$; the two resulting maps $(Q^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ and $\mathbb{Q}(\mathbf{P}^2_{k,n})^b \to (Q^2_{k,n})^b$ are therefore inverses, by a straightforward diagram chase.

## 3.2 Proofs of Lemmas 3.2 and 3.7

We now prove:

**Lemma 3.2** *Fix nonnegative integers $n \geq 4$, $k \leq n-3$, and $b \leq n-k-4$. The map $W_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ descends equivariantly to an $S_n$-equivariant map $\overline{W_{n,b}} : (Q^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$.*

The proof is by induction on $b$. The base case $b = 0$ (for all $n$) is Proposition 3.3. The inductive step is Proposition 3.5.

**Proposition 3.3** *$W_{n,0} : \mathbb{Q}(\mathbf{S}^2_{k,n})^0 \to \mathbb{Q}(\mathbf{P}^2_{k,n})^0$ descends equivariantly to an $S_n$-equivariant map $\overline{W_{n,0}} : (Q^2_{k,n})^0 \to \mathbb{Q}(\mathbf{P}^2_{k,n})^0$.*

The proof is complicated, so we first give a discussion. Note that

$$(Q^2_{k,n})^0 = \frac{\mathbb{Q}\mathbf{S}^{\geq 2}_{k,n}}{R^{\geq 2} + \mathbb{Q}\mathbf{S}^{\geq 3}_{k,n} + \mathbb{Q}(\mathbf{S}^2_{k,n})^{\geq 1}}.$$

Here $R^{\geq 2} := R \cap \mathbb{Q}\mathbf{S}^{\geq 2}_{k,n} \subseteq \mathbb{Q}\mathbf{S}^{\geq 2}_{k,n}$, where $R \subseteq \mathbb{Q}\mathbf{S}_{k,n}$ is the kernel of the map $\mathbb{Q}\mathbf{S}_{k,n} \to H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. It is difficult to work directly with $R^{\geq 2}$, due to the fact that the filtration $(\mathbf{S}^{\geq 2}_{k,n})^\bullet$ does not interact in a predictable way with the Kontsevich-Manin relations (2). Our strategy is as follows. We will find a map $\tilde{W}_n : \mathbb{Q}\mathbf{S}_{k,n} \to \mathbb{Q}\mathbf{P}^2_{k,n}$ such that:

(i) The restriction of $\tilde{W}_n$ to $\mathbb{Q}(\mathbf{S}^2_{k,n})^0$ is $W_{n,0}$,
(ii) The image of the restriction of $\tilde{W}_n$ to $\mathbb{Q}(\mathbf{S}^2_{k,n})^{\geq 1}$ lies in $\mathbb{Q}(\mathbf{P}^2_{k,n})^{\geq 1}$,
(iii) The restriction of $\tilde{W}_n$ to $\mathbb{Q}\mathbf{S}^{\geq 3}_{k,n}$ is zero, and
(iv) For any Kontsevich-Manin relation $\mathcal{R} \in \mathbb{Q}\mathbf{S}_{k,n}$, $\tilde{W}_n(\mathcal{R}) \in \mathbb{Q}(\mathbf{P}^2_{k,n})^{\geq 1}$.

Conditions 3.2, 3.2, and 3.2 imply that $\tilde{W}_n$ descends to a map

$$\frac{\mathbb{Q}\mathbf{S}_{k,n}}{R + \mathbb{Q}\mathbf{S}^{\geq 3}_{k,n} + \mathbb{Q}(\mathbf{S}^2_{k,n})^{\geq 1}} \to \mathbb{Q}(\mathbf{P}^2_{k,n})^0.$$

Restricting to $\mathbb{Q}\mathbf{S}^{\geq 2}_{k,n} \subseteq \mathbb{Q}\mathbf{S}_{k,n}$, Condition 3.2 gives a map $\overline{W_{n,0}} : (Q^2_{k,n})^0 \to \mathbb{Q}(\mathbf{P}^2_{k,n})^0$ descended from $W_{n,0}$.

***Remark 3.4*** The existence of a map satisfying conditions 3.2–3.2 is quite mysterious to us; we found $\tilde{W}_n$ by computer experimentation. The fact that the definition of $\tilde{W}_n$ (see (5) below) is quite complicated makes us wonder if there is some simpler, more conceptual description that might simplfy the following proof.

***Proof of Proposition 3.3*** Let $\sigma \in \mathbf{S}^1_{k,n}$, and let $v$ be the unique vertex of $\sigma$ with valence $\geq 4$. Then $\mathrm{val}(v) = k + 3$. We define a set partition $\Pi \vdash \{1, \ldots, n\}$ induced by $\sigma$, where $i_1$ and $i_2$ are in the same part if and only if the corresponding marked half-edges are in the same connected component of the complement of $v$ in $\sigma$. Note that $\Pi$ has $k + 3$ parts.

Suppose $\Gamma = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n})^0$ is such that $P_1$ is a union of $s_1$ parts of $\Pi$ (so necessarily $P_2$ is a union of the remaining $s_2 := k + 3 - s_1$ parts of $\Pi$). Let $e_\Pi(\Gamma) = \min\{s_1 - \alpha_1, s_2 - \alpha_2\}$. We define:

$$\tilde{W}_n(\sigma) := \sum_{\substack{\Gamma \in (\mathbf{P}^2_{k,n})^0 \\ \Gamma = \{(P_1,\alpha_1),(P_2,\alpha_2)\} \\ P_1 \text{ is a union of parts of } \Pi.}} \frac{(-1)^{e_\Pi(\Gamma)}}{2} \Gamma. \tag{5}$$

This defines $\tilde{W}_n$ on $\mathbf{S}^1_{k,n}$. We also define

$$\tilde{W}_n(\sigma) = \begin{cases} W_n(\sigma) & \sigma \in \mathbf{S}^2_{k,n} \\ 0 & \sigma \in \mathbf{S}^{\geq 3}_{k,n}. \end{cases} \tag{6}$$

Extending by linearity, these collectively define a map $\tilde{W}_n : \mathbb{Q}\mathbf{S}_{k,n} \to \mathbb{Q}\mathbf{P}^2_{k,n}$, which clearly satisfies conditions 3.2, 3.2, and 3.2 above.

Let $\mathcal{R} = \mathcal{R}(\sigma, v, A, B, C, D) \in \mathbb{Q}\mathbf{S}_{k,n}$ be a Kontsevich-Manin relation as in (2). We must show $\tilde{W}_n(\mathcal{R}) \in \mathbb{Q}(\mathbf{P}^2_{k,n})^{\geq 1}$. There are several cases, which we treat separately – Case VI is the hardest by far:

**I** $\sigma$ has $\geq 4$ vertices with valence $\geq 4$.
**II** $\sigma$ has 3 vertices $v$, $v'$, $v''$ with valence $\geq 4$, and $\mathrm{val}(v) \geq 5$.
**III** $\sigma$ has 3 vertices $v$, $v'$, $v''$ with valence $\geq 4$, and $\mathrm{val}(v) = 4$.
**IV** $\sigma$ has exactly 2 vertices $v$, $v'$ with valence $\geq 4$, and $\mathrm{val}(v) > 4$.
**V** $\sigma$ has exactly 2 vertices $v$, $v'$ with valence $\geq 4$, and $\mathrm{val}(v) = 4$.
**VI** $v$ is the unique vertex of $\sigma$ with $\mathrm{val}(v) \geq 4$.

**Case I.** In this case, every term of $\mathcal{R}$ is in $\mathbf{S}_{k,n}^{\geq 3}$, so $\tilde{W}_n(\mathcal{R}) = 0$.

**Case II.** Again, every term of $\mathcal{R}$ is in $\mathbf{S}_{k,n}^{\geq 3}$, so $\tilde{W}_n(\mathcal{R}) = 0$.

**Case III.**

In this case, $\mathcal{R}$ has exactly two terms, with opposite signs. These two trees (which both have exactly two vertices with valence $\geq 4$) differ only in the rearrangement of trivalent subtrees. Thus $\tilde{W}_n$ does not distinguish between them, i.e. $\tilde{W}_n(\mathcal{R}) = 0$.

**Case IV.** All but four terms of $\mathcal{R}$ are in $\mathbf{S}_{k,n}^{\geq 3}$, so we may ignore them. If $v$ and $v'$ are not adjacent, then these four terms are all in $(\mathbf{S}_{k,n}^{2})^{\geq 1}$, hence $\tilde{W}_n(\mathcal{R}) \in (\mathbf{P}_{k,n}^{2})^{\geq 1}$. If $v$ and $v'$ are adjacent, connected by an edge $e$, there are two subcases:

(1) One of $A$, $B$, $C$, $D$ is $e$ (without loss of generality, $D = e$), or
(2) None of $A$, $B$, $C$, $D$ is $e$.

In Sect. 3.2, let $\mathbf{T}$ be the set of edges or half-edges incident to $v$ other than $A$, $B$, $C$, $D$. In the notation of Sect. 2, the four remaining terms of $\mathcal{R}$ are:

$$(AB\mathbf{T}|Ce) + (AB|Ce\mathbf{T}) - (AC\mathbf{T}|Be) - (AC|Be\mathbf{T}).$$

Note that by definition $\tilde{W}_n(AB|Ce\mathbf{T}) = \tilde{W}_n(AC|Be\mathbf{T})$. The first and third terms are sent to $(\mathbf{P}_{k,n}^{2})^{\geq 1}$. Thus $\tilde{W}_n(\mathcal{R}) \in (\mathbf{P}_{k,n}^{2})^{\geq 1}$.

In Sect. 3.2, let $\mathbf{T}$ be the set of edges or half-edges incident to $v$ other than $A$, $B$, $C$, $D$, $e$. The four remaining terms of $\mathcal{R}$ are:

$$(ABe\mathbf{T}|CD) + (AB|CDe\mathbf{T}) - (ACe\mathbf{T}|BD) - (AC|BDe\mathbf{T}).$$

Observe that

$$\tilde{W}_n(ABe\mathbf{T}|CD) = \tilde{W}_n(AB|CDe\mathbf{T}) = \tilde{W}_n(ACe\mathbf{T}|BD) = \tilde{W}_n(AC|BDe\mathbf{T}).$$

Thus $\tilde{W}_n(\mathcal{R}) = 0$.

**Case V.** $\mathcal{R}$ has two terms with opposite sign, each of which is a tree with exactly one vertex with valence $\geq 4$. They induce the same partition $\Pi \vdash \{1, \ldots, n\}$ (as in (5)), hence they cancel after applying $\tilde{W}_n$.

**Case VI.** We have $\mathrm{val}(v) = k + 4$. We write $\mathbf{T}$ for the set of edges and half-edges incident to $v$ other than $A$, $B$, $C$, $D$. Note that $|\mathbf{T}| = k$.

Recall that

$$\mathcal{R} = \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AB\mathbf{U}_1|CD\mathbf{U}_2) - \sum_{\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}} (AC\mathbf{U}_1|BD\mathbf{U}_2) \in \mathbb{Q}\mathbf{S}_{k,n}.$$

Note that terms where $\mathbf{U}_1 = \emptyset$ or $\mathbf{U} = \emptyset$ are in $\mathbf{S}_{k,n}^{1}$, and terms where $\mathbf{U}_1, \mathbf{U}_2 \neq \emptyset$ are in $\mathbf{S}_{k,n}^{2}$. We will apply (5) or (6) accordingly.

We also introduce an abuse of notation that will now be convenient. By definition, $A$, $B$, $C$, $D$, and the elements of $\mathbf{T}$ denote edges or half-edges of $\sigma$ incident to $v$. We use the same symbols to refer to subsets of $\{1, \ldots, n\}$, where e.g. $A$ is the set of all $i \in \{1, \ldots, n\}$

such that the path from $v$ to the $i$th marked half-edge contains the edge $A$. (If $A$ is itself the $i$th marked half-edge, then we write $A = \{i\}$.)

Let $\Gamma = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n})^0$. Note that by definition of $(\mathbf{P}^2_{k,n})^0$, we have $P_1 \cup P_2 = \{1, \ldots, n\}$. We need to show that the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero. We do this argument in cases again; $\Gamma$ must be of one of the following types:

*Type 0.* At least one of the sets $A, B, C, D \subseteq \{1, \ldots, n\}$, or a part of $\mathbf{T}$, has nonempty intersection with both $P_1$ and $P_2$.

*Type 1.* Two of $A, B, C, D$ are subsets of $P_1$, and the other two are subsets of $P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 1.1* $A, B \subseteq P_1$ and $C, D \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 1.2.* $A, C \subseteq P_1$ and $B, D \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 1.3.* $A, D \subseteq P_1$ and $B, C \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 2.* Three of $A, B, C, D$ are subsets of $P_1$, and the remaining one is a subset of $P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 2.1* $A, B, C \subseteq P_1$ and $D \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 2.2.* $A, C, D \subseteq P_1$ and $B \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 2.3.* $A, B, D \subseteq P_1$ and $C \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 2.4.* $B, C, D \subseteq P_1$ and $A \subseteq P_2$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

*Type 3.* $A, B, C, D \subseteq P_1$, and no part of $\mathbf{T}$ has nonempty intersection with both $P_1$ and $P_2$.

**Caution.** It is tempting to use the $S_4$-action that permutes $A, B, C, D$, but one must be very careful in doing so. The sets $A, B, C, D$ have been fixed, and may have different cardinalities; we may only invoke symmetry if our arguments do not refer to any specific properties of $A, B, C, D$.

**If $\Gamma$ is of type 0, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** By (5) and (6), no terms of $\mathcal{R}$ contribute a term of type 0 to $\tilde{W}_n(\mathcal{R})$.

**If $\Gamma$ is of type 1.1, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** Let $\Gamma$ be of type 1.1, so we may write

$$\Gamma = \left\{\left(A \cup B \cup \bigcup_{S \in \mathbf{T}_1} S, \alpha_1\right), \left(C \cup D \cup \bigcup_{S \in \mathbf{T}_2} S, \alpha_2\right)\right\}$$

for some partition $\mathbf{T}_1 \sqcup \mathbf{T}_2 = \mathbf{T}$, and for some $\alpha_1, \alpha_2$. Note that $\alpha_1 + \alpha_2 = k = |\mathbf{T}| = |\mathbf{T}_1| + |\mathbf{T}_2|$ . Let

$$\ell = \alpha_1 - |\mathbf{T}_1| = |\mathbf{T}_2| - \alpha_2.$$

**Claim: If $\ell = 0$, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.**

***Proof of claim*** First, note that in this case $|\mathbf{T}_1| = \alpha_1 \geq 1$, i.e. $\mathbf{T}_1 \neq \emptyset$. Similarly $\mathbf{T}_2 \neq \emptyset$. By definition, every term of $\mathcal{R}$ is of the form $(AB\mathbf{U}_1|CD\mathbf{U}_2)$ or $-(AC\mathbf{U}_1|BD\mathbf{U}_2)$, for some partition $\mathbf{U}_1 \sqcup \mathbf{U}_2 = \mathbf{T}$. In terms of the latter form, $\Gamma$ appears with coefficient zero by (5) and (6).

In a term $(AB\mathbf{U}_1|CD\mathbf{U}_2)$ with $\mathbf{U}_1, \mathbf{U}_2 \neq \emptyset$ and $\mathbf{U}_1 \neq \mathbf{T}_1$, $\Gamma$ also appears with coefficient zero by (6). Thus the only terms of $\mathcal{R}$ that contribute to the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ are:

$$(AB\mathbf{T}_1|CD\mathbf{T}_2), \qquad (AB|CD\mathbf{T}), \qquad \text{and} \qquad (AB\mathbf{T}|CD). \tag{7}$$

By definition,

$$\tilde{W}_n((AB\mathbf{T}_1|CD\mathbf{T}_2)) = 1 \cdot \Gamma.$$

The boundary stratum $(AB|CD\mathbf{T})$ has a single vertex with valence $\geq 4$, with corresponding set partition $\Pi = \{A \cup B, C, D\} \cup \mathbf{T} \vdash \{1, \ldots, n\}$. Using the notation in the paragraph preceding (5) (with respect to our fixed $\Gamma \in (\mathbf{P}^2_{k,n})^0$), we have $s_1 = 1 + |\mathbf{T_1}| = 1 + \alpha_1$, so $s_1 - \alpha_1 = 1$. Thus $s_2 - \alpha_2 = 2$, so $e_\Pi(\Gamma) = s_1 - \alpha_1 = 1$. The coefficient of $\Gamma$ in $\tilde{W}_n((AB|CD\mathbf{T}))$ is thus $-1/2$.

The set partition associated to $(AB\mathbf{T}|CD)$ is $\Pi = \{A, B, C \cup D\} \cup \mathbf{T}$. In this case we have $s_1 = 2 + |\mathbf{T_1}| = 2 + \alpha_1$, so $s_1 - \alpha_1 = 2$. Thus $s_1 - \alpha_1 = 1$, so $e_\Pi(\Gamma) = s_2 - \alpha_2 = 1$. The coefficient of $\Gamma$ in $\tilde{W}_n((AB\mathbf{T}|CD))$ is thus $-1/2$.

In total, the coefficient is $1 - 1/2 - 1/2 = 0$, proving the claim.

**Claim: If $\ell \neq 0$, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.**

***Proof of claim*** Note that $\alpha_1, \alpha_2 \geq 1$ imply $-|\mathbf{T}_1| + 1 \leq \ell \leq k - |\mathbf{T}_2| - 1$.

By the same argument as the one preceding (7), the only terms of $\mathcal{R}$ that contribute to the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ are

$$(AB\mathbf{T}|CD) \qquad \text{and} \qquad (AB|CD\mathbf{T}).$$

(Note that $(AB\mathbf{T}_1|CD\mathbf{T}_2)$ does not contribute because the values of $\alpha_1, \alpha_2$ in $\tilde{W}_n((AB\mathbf{T}_1|CD\mathbf{T}_2))$ do not match those in $\Gamma$.

As above, $(AB|CD\mathbf{T})$ induces set partition $\Pi_1 = \{A \cup B, C, D\} \cup \mathbf{T}$. Using the notation in the paragraph preceding (5) (with respect to our fixed $\Gamma \in (\mathbf{P}^2_{k,n})^0$), we have $s_1 = 1 + |\mathbf{T_1}| = 1 + \alpha_1 - \ell$, so $s_1 - \alpha_1 = 1 - \ell$ (and $s_2 - \alpha_2 = 2 + \ell$). Thus $e_{\Pi_1}(\Gamma) = \min\{1 - \ell, 2 + \ell\}$.

On the other hand, $(AB\mathbf{T}|CD)$ induces set partition $\Pi_2 = \{A, B, C \cup D\} \cup \mathbf{T}$. In this case $s_1 = 2 + |\mathbf{T_1}| = 2 + \alpha_1 - \ell$, so $s_1 - \alpha_1 = 2 - \ell \leq 1$ (and $s_2 - \alpha_2 = 1 + \ell \geq 2$). Thus $e_{\Pi_2}(\Gamma) = \min\{1 + \ell, 2 - \ell\}$.

If $\ell > 0$, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is $(-1)^{1-\ell}/2 + (-1)^{2-\ell}/2 = 0$. If $\ell < 0$, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is $(-1)^{2+\ell}/2 + (-1)^{1+\ell}/2 = 0$. This proves the claim.

We conclude that the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.

**If $\Gamma$ is of type 1.2, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** The argument for the type 1.1 case did not refer to any properties of the sets $A, B, C, D$, so this case follows by symmetry.

**If $\Gamma$ is of type 1.3, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** No terms of $\mathcal{R}$ contribute a term of type 1.3 to $\tilde{W}_n(\mathcal{R})$.

**If $\Gamma$ is of type 2.1, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** The only terms of $\mathcal{R}$ that contribute to the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ are:

$$(AB|CD\mathbf{T}) \qquad \text{and} \qquad -(AC|BD\mathbf{T}).$$

The corresponding set partitions are $\Pi_1 = \{A \cup B, C, D\} \cup \mathbf{T}$ and $\Pi_1 = \{A \cup C, B, D\} \cup \mathbf{T}$. By definition, $e_{\Pi_1}(\Gamma) = e_{\Pi_2}(\Gamma)$, so the coefficients of $\Gamma$ in $\tilde{W}_n((AB|CD\mathbf{T}))$ and $\tilde{W}_n(-(AC|BD\mathbf{T}))$ cancel.

**If $\Gamma$ is of type 2.2, 2.3, or 2.4, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** The argument for type 2.1 did not refer to any properties of the sets $A, B, C, D$, so these cases follow by symmetry.

**If $\Gamma$ is of type 3, the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ is zero.** The only terms of $\mathcal{R}$ that contribute to the coefficient of $\Gamma$ in $\tilde{W}_n(\mathcal{R})$ are:

$$(AB|CD\mathbf{T}), \qquad (AB\mathbf{T}|CD), \qquad -(AC|BD\mathbf{T}), \qquad \text{and} \qquad -(AC\mathbf{T}|BD).$$

The corresponding set partitions are $\Pi_1 = \{A \cup B, C, D\} \cup \mathbf{T}$, $\Pi_2 = \{A, B, C \cup D\} \cup \mathbf{T}$, $\Pi_3 = \{A \cup C, B, D\} \cup \mathbf{T}$, and $\Pi_1 = \{A, C, B \cup D\} \cup \mathbf{T}$. By definition, $e_{\Pi_1}(\Gamma) = e_{\Pi_2}(\Gamma) = e_{\Pi_3}(\Gamma) = e_{\Pi_4}(\Gamma)$, so the coefficients of $\Gamma$ in $\tilde{W}_n((AB|CD\mathbf{T}))$, $\tilde{W}_n((AB\mathbf{T}|CD))$, $\tilde{W}_n(-(AC|BD\mathbf{T}))$, and $\tilde{W}_n(-(AC\mathbf{T}|BD))$ cancel.

Altogether, we conclude that $\tilde{W}_n(\mathcal{R}) = 0$. This completes Case VI and the proof of the base case.

Next we prove the inductive step.

**Proposition 3.5** *Suppose $W_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ descends $S_n$-equivariantly to $(Q^2_{k,n})^b$ for a fixed $b$ and $n$. Then $W_{n,b+1} : \mathbb{Q}(\mathbf{S}^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{P}^2_{k,n+1})^{b+1}$ descends $S_{n+1}$-equivariantly to $(Q^2_{k,n+1})^{b+1}$. In particular, if Lemma 3.2 holds for $(b, n)$ for all $n$ (and fixed $b$), then it holds for $(b+1, n)$ for all $n$.*

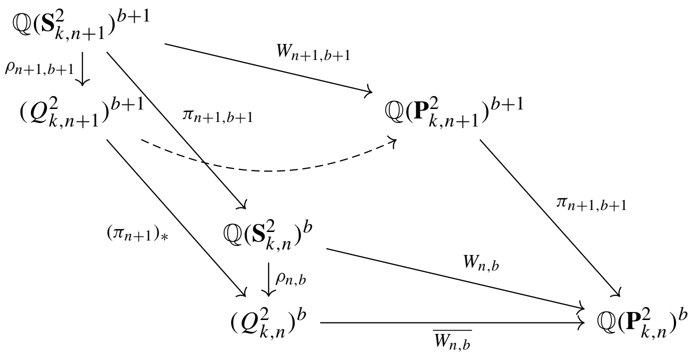
***Proof*** Suppose $W_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ descends equivariantly to $(Q^2_{k,n})^b$. Let $\pi_{n+1}$ be the forgetful map that forgets the last point. Note that $(\pi_{n+1})_*((Q^2_{k,n+1})^{\geq b+1}) \subseteq (Q^2_{k,n})^{\geq b}$. (This holds because for any tree in $(\mathbf{S}^2_{k,n+1})^{\geq b+1}$, the corresponding boundary stratum is either contracted in dimension by $\pi_{n+1}$, or maps isomorphically to a boundary stratum in $(\mathbf{S}^2_{k,n})^{\geq b}$.) Thus there is an induced map on quotients

$$(\pi_{n+1})_* : (Q^2_{k,n+1})^{b+1} \to (Q^2_{k,n})^b.$$

By the inductive hypothesis, there is a map

$$\overline{W_{n,b}} : (Q^2_{k,n})^b \to (\mathbf{P}^2_{k,n})^b,$$

such that the lower-right triangle commutes in the following diagram:




Here the map $\pi_{n+1,b+1} : \mathbb{Q}(\mathbf{S}^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{S}^2_{k,n})^b$ sends a tree $\sigma$ to the tree obtained by contracting the $(n+1)$-st marked half-edge and stabilizing (see e.g. [11]) if the $(n+1)$-st marked half-edge is on $\sigma'$ (using notation from Definition 2.6), and to zero otherwise. The map $\pi_{n+1,b+1} : \mathbb{Q}(\mathbf{P}^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{P}^2_{k,n})^b$ sends a pair $\{(P_1, \alpha_1), (P_2, \alpha_2)\}$ to itself if $n+1 \in (P_1 \cup P_2)^C$, and to zero otherwise. The entire diagram is easily checked to commute, essentially using the fact that if $\sigma \in (\mathbf{S}^2_{k,n+1})^{b+1}$ is such that $n+1 \in \sigma_1$, then

$(\pi_{n+1})_*(\rho_{n+1,b+1}(\sigma)) \in (Q^2_{k,n})^{\geq b+1}$. We are trying to prove the existence of the dashed arrow.

Let $r \in \mathbb{Q}(\mathbf{S}^2_{k,n+1})^{b+1}$ be such that $\rho_{n+1,b+1}(r) = 0$. Then certainly

$$\overline{W_{n,b}}((\pi_{n+1})_*(\rho_{n+1,b+1}(r))) = 0,$$

so by commutativity, $W_{n+1,b+1}(r)$ lies in

$$\ker(\pi_{n+1,b+1}) = \mathrm{Span}\{\{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n+1})^{b+1} : n+1 \in P_1 \cup P_2\}.$$

Then by $S_{n+1}$-symmetry, $W_{n+1,b+1}(r)$ actually lies in the smaller subspace

$$\mathrm{Span}\{\{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n+1})^{b+1} : P_1 \cup P_2 = \{1, \ldots, n+1\}\}.$$

Since $b+1 > 0$, this subspace is trivial, so $W_{n+1,b+1}(r) = 0$. Thus $W_{n+1,b+1}$ descends to a (clearly $S_{n+1}$-equivariant) map $\overline{W_{n+1,b+1}} : (Q^2_{k,n+1})^{b+1} \to \mathbb{Q}(\mathbf{P}^2_{k,n+1})^{b+1}$. This completes the proof, and we conclude Lemma 3.2.

We have just shown that $W_{n,b}$ descends to $(Q^2_{k,n})^b$. Next we must prove that $\rho_{n,b}$ descends to $(\mathbf{P}^2_{k,n})^b$. We will use the following well-known fact.

**Proposition 3.6** *Let $\sigma \in \mathbf{S}_{k,n}$, and suppose $\sigma'$ can be obtained from $\sigma$ by rearranging a trivalent subtree. (That is, suppose there exist trivalent subtrees of $\sigma$ and $\sigma'$ whose complements are isomorphic as marked trees, and such that the two subtrees have the same marking set.) Then $\sigma$ and $\sigma'$ have the same class in $H_{2k}(\overline{\mathcal{M}}_{0,n})$.*

***Sketch of proof*** Contracting the relevant trivalent subtrees, $\sigma$ and $\sigma'$ can both be written as the pushforward from the same product $\overline{\mathcal{M}}_{0,n_0} \times \overline{\mathcal{M}}_{0,n_1} \times \cdots \times \overline{\mathcal{M}}_{0,n_m}$, of the class $[\mathrm{pt}] \times 1 \times \cdots \times 1$, for different choices of pt $\in \overline{\mathcal{M}}_{0,n_0}$, the factor corresponding to the contracted subtrees.

**Lemma 3.7** *Fix nonnegative integers $n \geq 4$, $k \leq n-3$, and $b \leq n-k-4$. The map $\rho_{n,b} : \mathbb{Q}(\mathbf{S}^2_{k,n})^b \to (Q^2_{k,n})^b$ descends equivariantly to a map $\overline{\rho_{n,b}} : \mathbb{Q}(\mathbf{P}^2_{k,n})^b \to (Q^2_{k,n})^b$.*

***Proof*** We must check that $\rho_{n,b}(\ker(W_{n,b})) = 0$. Note that as an induced map on free vector spaces, $W_{n,b}$ has kernel spanned by

$$\{\sigma - \sigma' : \sigma, \sigma' \in (S^2_{k,n})^b, W_{n,b}(\sigma) = W_{n,b}(\sigma')\}.$$

Thus it suffices to show that if $\sigma, \sigma' \in (S^2_{k,n})^b$ and $W_{n,b}(\sigma) = W_{n,b}(\sigma')$, then $\rho_{n,b}(\sigma) = \rho_{n,b}(\sigma')$. We consider two cases: $b = n-k-4$ and $b < n-k-4$.

First, suppose $b = n-k-4$. Then for any $\sigma \in (\mathbf{S}^2_{k,n})^{\geq b}$, the two vertices $v_1$ and $v_2$ are leaves, and $W_{n,b}(\sigma)$ determines $\sigma$ up to rearrangement of the trivalent subtree obtained by deleting $v_1$ and $v_2$. Thus $\rho_{n,b}$ descends to $(\mathbf{P}^2_{k,n})^b$ by Proposition 3.6. Next, suppose $b < n-k-4$. Let $\tau \in (\mathbf{S}^2_{k,n})^b$. We will next define an algorithm ((8) below) for producing trees $\tau' \in (\mathbf{S}^2_{k,n})^b$ such that $\rho_{n,b}(\tau') = \rho_{n,b}(\tau)$. Note that Proposition 3.6 is another such algorithm. We will then prove that if $\sigma, \sigma' \in (S^2_{k,n})^b$ and $W_{n,b}(\sigma) = W_{n,b}(\sigma')$, then $\sigma'$ can be obtained from $\sigma$ via these two algorithms.

Let $\tau \in (\mathbf{S}^2_{k,n})^{\geq b}$. Since $b < n-k-4$, at least one of $v_1$ and $v_2$ is not a leaf. Suppose without loss of generality that $v_1$ is not a leaf. Let $e_1$ be the edge incident to $v_1$ that is on the path from $v_1$ to $v_2$. Fix an edge $e \neq e_1$, and write $e = \{v_1, v'\}$. Let $\overline{\tau}$ be the tree obtained by contracting $e$ to a vertex $v_1 = v'$. We consider Kontsevich-Manin relations

$\mathcal{R} = \mathcal{R}(\overline{\tau}, v_1 = v', A, B, C, e_1)$, where $A, B \neq e$ are incident to $v'$ in $\tau$, and $C \neq e$ is incident to $v_1$ in $\tau$. All but four terms of $\mathcal{R}$ are in $\mathbf{S}^{\geq 3}_{k,n}$. The remaining terms are:

$$(AB\mathbf{T}|Ce_1) + (AB|Ce_1\mathbf{T}) - (AC\mathbf{T}|Be_1) + (AC|Be_1\mathbf{T})$$

where $\mathbf{T}$ is the set of edges and half-edges incident to $v_1 = v'$ other than $A, B, C, e_1$. Note that the first and third terms are in $(\mathbf{S}^2_{k,n})^{\geq b+1}$, hence are zero in $(Q^2_{k,n})^b$. We are left with the relation

$$(AB|Ce_1\mathbf{T}) = (AC|Be_1\mathbf{T}). \tag{8}$$

The left side is $\tau$, and the right side is another element of $(\mathbf{S}^2_{k,n})^b$.

Fix $P = \{(P_1, \alpha_1), (P_2, \alpha_2)\} \in (\mathbf{P}^2_{k,n})^b$. We must now show that if $\sigma, \sigma' \in (W_{n,b})^{-1}(P)$, then $\sigma'$ can be obtained from $\sigma$ via the two algorithms (8) and Proposition 3.6. We will instead show that any $\sigma \in (W_{n,b})^{-1}(P)$ can be manipulated via (8) into a standard form. That is, we will find $\sigma_0 \in (W_{n,b})^{-1}(P)$ such that for any $\sigma \in (W_{n,b})^{-1}(P)$, $\sigma_0$ can be obtained from $\sigma$ via the two algorithms. Let $\sigma_0$ be as follows: On $v_1$, the first $\alpha_1 + 1$ elements of $P_1$ (in increasing order) are half-edges incident to $v_1$, and the remaining elements of $P_1$ form a trivalent subtree incident to $v_1$. Similarly for $v_2$, and the remaining points $(P_1 \cup P_2)^C$ of course form a trivalent subtree connecting $v_1$ and $v_2$. This defines $\sigma_0$ up to rearrangement of trivalent subtrees; this is sufficient in light of Proposition 3.6.

Fix $\sigma \in (W_{n,b})^{-1}(P)$. If the first $\alpha_1 + 1$ marked half-edges of $P_1$ are incident to $v$, then $v_1$ is already in the desired form. If not, let $e, \neq e_1$ be incident to $v_1$ such that subtree corresponding to $e = \{v_1, v'\}$ contains (at least) one of the first $\alpha_1 + 1$ marked half-edges. Using Proposition 3.6, rearrange the subtree so that this half-edge is incident to $v'$, then apply (8), where $B$ is the chosen half-edge, and $C$ is any edge or half-edge incident to $v_1$ that is not equal to $e$, nor to any of the first $\alpha_1 + 1$ marked half-edges. (Such a flag must exist, since there are $\alpha_1 + 1$ edges and half-edges other than $e$ and $e_1$, and we have assumed that one of the first $\alpha_1 + 1$ marked half-edges is on the subtree corresponding to $e$.) In the resulting tree, the number of the first $\alpha_1 + 1$ marked half-edges incident to $v_1$ has increased by 1. We repeat until this number is $\alpha_1 + 1$, then apply the same argument to $v_2$. The result is $\sigma_0$; this completes the proof.

## 4 A conjectural dimension formula

We end with a conjectural formula for the dimension of $Q^r_{k,n}$.

**Conjecture 4.1** *The dimension of $Q^r_{k,n}$ is*

$$\sum_{\substack{\alpha_1,\ldots,\alpha_r \geq 1 \\ \alpha_1+\cdots+\alpha_r = k}} \sum_{\substack{p_1,\ldots,p_r,b \geq 0 \\ p_1+\cdots+p_r+b = n+r-2 \\ p_i \geq \alpha_i + 2}} \frac{1}{r!} \binom{n+r-2}{p_1, \ldots, p_r, b}.$$

The case $r = 1$ appears in [15]. The case $r = 2$ follows from Theorem 2.7, since the formula gives the cardinality of $\mathbf{P}^2_{k,n}$. Summing over $r \in \{1, \ldots, \min\{k, n-2-k\}\}$ gives a conjectural closed formula for the dimension of $H_{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$. This formula agrees with the actual dimension for $n \leq 20$ and all $k$, but we have not been able to find it in the literature. (It is not obvious to us whether or not the formula is recoverable from e.g. [18].)

**Acknowledgements** Most of this work was conducted while both authors were visiting the Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig. We thank Bernd Sturmfels for the invitations, as well as the entire institute for providing a stimulating work environment.



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