Determinantal Structure of Conditional Overlaps in the Complex Ginibre Ensemble

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Declaration and Acknowledgements

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree, apart from Lemma 1.2 in section 4.1, which was included in my Master's degree.

Chapters 2 and 3 are part of published work [2], done alongside Roger Tribe, Oleg Zaboronski and Gernot Akemann. My main contribution was that I first derived the formula for the reduced kernel in Theorem 1, and then I proceeded to derive Corollaries 1,2 and 3 alongside my supervisors.

Finally, most of the background theory in 5.2 has been taken directly from [13] and referenced appropriately.

Contents

1 Introduction and Motivation.

1.1 Introduction to complex Ginibre

The law of a random matrix S can be written as $P(S)\mu(dS)$, where $\mu(dS)$ is the linear measure

$$\mu(dS) = \prod_{j,k} S_{jk}^{(0)} S_{jk}^{(1)}$$

where $S_{jk}^{(0)}$ and $S_{jk}^{(1)}$ are respectively the real and imaginary part of the matrix element

$$S_{jk} = S_{jk}^{(0)} + S_{jk}^{(1)}.$$

For $P(S)$ we take [20, Ginibre (1965)]

$$P(S) = \exp[-tr(S^* S)].$$

Note that $P(S)$ is invariant under all unitary transformations.

We denote as $\mathrm{Gin}(n,\mathbb{C})$, the ensemble of $n \times n$ matrices with independent and identically distributed complex Gaussian entries (complex Ginibre ensemble). If $M \sim \mathrm{Gin}(n,\mathbb{C})$ is a complex Ginibre matrix, then it has the following properties:

$$\mathbb{E}(M_{ij}) = 0 = \mathbb{E}(\bar{M}_{ij}),\ 1 \le i,j \le n, \tag{1}$$

$$\mathbb{E}(M_{ij}M_{kl}) = 0 = \mathbb{E}(\bar{M}_{ij}\bar{M}_{kl}),\ 1 \le i,j,k,l \le n, \tag{2}$$

$$\mathbb{E}(M_{ij}\bar{M}_{kl}) = \delta_{ik}\delta_{jl}\ 1 \le i,j,k,l \le n, \tag{3}$$

where '$^-$' stands for complex conjugation. Let $\mathbf{\Lambda}^{(n)} = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ be the set of complex eigenvalues of M. This ensemble was introduced in 1965 in [17] along with its real and quaternionic counterparts. It was immediately realised in this pioneering paper that the marginal distribution of eigenvalues for $\mathrm{Gin}(n,\mathbb{C})$ can be computed and is a natural generalisation of the corresponding answer for the Gaussian Unitary Ensemble (GUE), cf. [24], to the case of a complex spectrum:

$$P_n\left(\mathbf{\Lambda}^{(n)} \in d\boldsymbol{\lambda}^{(n)}\right) = \frac{1}{Z_n}|\Delta^{(n)}(\boldsymbol{\lambda}^{(n)})|^2 e^{-\sum_{j=1}^n |\lambda_j|^2} d\boldsymbol{\lambda}^{(n)} \tag{4}$$

where $d\boldsymbol{\lambda}^{(n)} = \prod_{k=1}^n d\lambda_k d\bar{\lambda}_k$ is the Lebesgue measure on $\mathbb{C}^n$, $\Delta^{(n)}(\boldsymbol{\lambda}^{(n)}) = \prod_{i>j}^n(\lambda_i - \lambda_j)$ is the Vandermonde determinant, and $Z_n = \pi^n \prod_{j=1}^n j!$ is the normalisation constant. There is an important difference between the complex Ginibre ensemble and GUE: even though the marginal distribution of eigenvalues for $\mathrm{Gin}(n,\mathbb{C})$ can be computed analytically, the statistics of eigenvectors do not decouple from the statistics of eigenvalues. Despite this fundamental difference, it was not until the late nineties that Chalker and Mehlig initiated the study of the joint statistics of eigenvectors and eigenvalues for $\mathrm{Gin}(n,\mathbb{C})$.

Two methods can be found in [24, A.33 and A.35], for showing that the eigenvalues' joined probabilty density is given by

$$p_n(\lambda_1, \lambda_2, ..., \lambda_n) = \frac{1}{Z_n}\prod_{1\le i<j\le n}|\lambda_i - \lambda_j|^2 e^{-\sum_{i=1}^n |\lambda_i|^2},$$

but the exact value of Z_n is calculated indirectly, through the probability of all eigenvalues lying outside a circle of radius a. Let said probabilty be $P_c(a)$. This object is interesting on its own, but more importantly the methods used in the proof are also used in [11] and in this thesis as well. $P_c(a)$ is given by

$$P_c(a) = \int \cdots \int_{|\lambda_i|>a} p_n(\lambda_1, \lambda_2, ..., \lambda_n) \prod_{i=1}^n d\lambda_i d\bar{\lambda}_i.$$

Re-writing

$$\begin{aligned}
\prod_{1\le i<j\le n} |\lambda_i - \lambda_j|^2 &= \prod_{1\le i<j\le n} (\lambda_i - \lambda_j) \prod_{1\le i<j\le n} (\bar\lambda_i - \bar\lambda_j) \\
&= \det \begin{vmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} \times \det \begin{vmatrix} 1 & \dots & 1 \\ \bar\lambda_1 & \dots & \bar\lambda_n \\ \dots & \dots & \dots \\ \bar\lambda_1^{n-1} & \dots & \bar\lambda_n^{n-1} \end{vmatrix}^T \\
&= \det \begin{vmatrix} n & \sum_i \lambda_i & \dots & \sum_i \lambda_i^{n-1} \\ \sum_i \bar\lambda_i & \sum_i \bar\lambda_i \lambda_i & \dots & \sum_i \bar\lambda_i \lambda_i^{n-1} \\ \dots & \dots & \dots & \dots \\ \sum_i \bar\lambda_i^{n-1} & \sum_i \bar\lambda_i^{n-1}\lambda_i & \dots & \sum_i \bar\lambda_i^{n-1}\lambda_i^{n-1} \end{vmatrix}.
\end{aligned}$$

we get

$$\begin{aligned}
P_c(a) &= \frac{1}{Z_n} \int \cdots \int_{|\lambda_i|>a} \prod_{i=1}^n d\lambda_i d\bar\lambda_i \exp\Big(-\sum_{i=1}^n |\lambda_i|^2\Big) \\
&\times \det \begin{vmatrix} n & \sum_i \lambda_i & \dots & \sum_i \lambda_i^{n-1} \\ \sum_i \bar\lambda_i & \sum_i \bar\lambda_i \lambda_i & \dots & \sum_i \bar\lambda_i \lambda_i^{n-1} \\ \dots & \dots & \dots & \dots \\ \sum_i \bar\lambda_i^{n-1} & \sum_i \bar\lambda_i^{n-1}\lambda_i & \dots & \sum_i \bar\lambda_i^{n-1}\lambda_i^{n-1} \end{vmatrix}.
\end{aligned}$$

Because the determinant is symmetric in all λ_i, we can replace the first line with $(1, \lambda, \lambda^2, ..., \lambda^{n-1})$ and multiply the whole expression with a factor of n. With row and column computation we can then get rid of z_1 from all other lines. Continuing in a similar fashion, the probability of all eigenvalues lying outside a circle of radius a can be written as

$$\begin{aligned}
P_c(a) &= \frac{n!}{Z_n} \int \cdots \int_{|\lambda_i|>a} \prod_{i=1}^n d\lambda_i d\bar\lambda_i \exp\Big(-\sum_{i=1}^n |\lambda_i|^2\Big) \\
&\times \det \begin{vmatrix} 1 & \lambda_1 & \dots & {}^{n-1}_{1} \\ \bar\lambda_2 & \bar\lambda_2\lambda_2 & \dots & \bar\lambda_2\lambda_2^{n-1} \\ \dots & \dots & \dots & \dots \\ \bar\lambda_n^{n-1} & \bar\lambda_n^{n-1}\lambda_n & \dots & \bar\lambda_n^{n-1}\lambda_n^{n-1} \end{vmatrix}.
\end{aligned}$$

Now each row is only dependent on one of the λ_i. We can integrate them separately, alongside with the exponential factor, and by passing to polar coordinates we get

$$\int_{|\lambda|>a} e^{-|\lambda|^2} \bar\lambda^j \lambda^k d\lambda d\bar\lambda = \pi \delta_{jk} \Gamma(j+1, a^2).$$

All the non diagonal elements are zero, thus we have

$$P_c(a) = \frac{n!\pi^n}{Z_n} \prod_{j=1}^n \Gamma(j, a^2)$$

where

$$\Gamma(j,a^2) = \int_{a^2}^{\infty} e^{-x}x^{j-1}dx$$

is the incomplete gamma function. Since $P_c(0) = 1$, by taking $a = 0$, Z_n is determined to be

$$Z_n = \pi^n \prod_{j=1}^{n} j!$$

Another object of great importance to us are the correlation functions between eigenvalues. The k-point correlation function is given by (see [24])

$$\begin{aligned} \rho^{(n,k)}(\lambda_1,\lambda_2,...,\lambda_k) &= \frac{n!}{(n-k)!}\int\cdots\int p_n(\lambda_1,\lambda_2,...,\lambda_n)\prod_{i=k+1}^{n} d\lambda_i d\bar{\lambda}_i \\ &= \pi^{-k}\exp\Big(-\sum_{i=1}^{k}|\lambda_i|^2\Big)\det[K_n(\lambda_i,\lambda_j)]_{i,j=1}^{k} \end{aligned}$$

where

$$K_n(\lambda_i,\lambda_j) = \sum_{l=0}^{n-1}\frac{(\lambda_i\bar{\lambda}_j)^l}{l!}.$$

Note that the correlation functions tend to well defined limits as $n\to\infty$. Two other quantities that are worthy of mention are the 1-point correlation function, which is the density of the eigenvalues

$$\rho^{(n,1)}(\lambda) = \pi^{-1}e^{-|\lambda|^2}\sum_{l=0}^{n-1}\frac{|\lambda|^{2l}}{l!},$$

which is rotationally invariant and depends only on the distance of the eigenvalue from the origin, and the limit of the 2-point correlation as $n\to\infty$

$$\lim_{n\to\infty}\rho^{(n,2)}(\lambda_1,\lambda_2) = \pi^2\big[1-\exp(-|\lambda_1-\lambda_2|^2)\big],$$

which depends only on the distance between the two eigenvalues, λ_1 and λ_2 .

1.2 Motivation

Stability questions can be hard to justify within a static setting of the complex Ginibre ensemble, but become very natural if one considers some kind of dynamics (random or deterministic) on the space of complex matrices. Our own interest in the statistics of eigenvectors for $\mathrm{Gin}(n,\mathbb{C})$ was inspired by the study of the stochastic dynamics of eigenvectors and eigenvalues of complex matrices initiated by Z. Burda and M. Nowak, their collaborators and students, see [19, 8]. So, in order to motivate the main subject of this thesis, let us follow [6] and [18] and consider the Brownian motion M_t with values in $n\times n$ complex matrices started from zero. The fixed time $t>0$ marginal law for the process $(M_t)_{t\geq 0}$ coincides up to rescaling with the complex Ginibre ensemble.

Let $(\Lambda_{t\alpha},\boldsymbol{L}_{t\alpha},\boldsymbol{R}_{t\alpha})_{t\geq 0,1\leq\alpha\leq n}$ be the induced processes, describing the evolution of eigenvalues of M_t and the bi-orthogonal set of corresponding left and right eigenvectors,

$$\boldsymbol{L}_{t\alpha}^{*}M_t = \Lambda_{t\alpha}\boldsymbol{L}_{t\alpha}^{*},\ \ 1\leq\alpha\leq n, \tag{5}$$

$$M_t\boldsymbol{R}_{t\alpha} = \Lambda_{t\alpha}\boldsymbol{R}_{t\alpha},\ \ 1\leq\alpha\leq n, \tag{6}$$

$$\langle\boldsymbol{L}_{t\alpha},\boldsymbol{R}_{t\beta}\rangle = \delta_{\alpha,\beta},\ \ 1\leq\alpha,\beta\leq n, \tag{7}$$

where ‘ $*$ ’ denotes Hermitian conjugation and $\langle\cdot,\cdot\rangle$ stands for the Hermitian inner product on $\mathbb{C}^n$. As shown in [6] and [18], the process $(\Lambda_{t\alpha})_{t\geq 0,1\leq\alpha\leq n}$ is a complex martingale such that

$$\langle d\Lambda_{t\alpha}d\bar{\Lambda}_{t\beta}\rangle = O_{t\alpha\beta}dt, \tag{8}$$

where

$$O_{t\alpha\beta} = \langle \boldsymbol{L}_{t\alpha}, \boldsymbol{L}_{t\beta} \rangle \langle \boldsymbol{R}_{t\alpha}, \boldsymbol{R}_{t\beta} \rangle, \ 1 \le \alpha, \beta \le n \tag{9}$$

is the matrix of the overlaps between the left and the right eigenvectors of M_t. (It is worth noticing that paper [18] derives the full set of stochastic differential equations for the joint evolution of eigenvalues and eigenvectors of M_t for any matrix size n). Notice that for complex matrices, the matrix of overlaps is a non-trivial random variable as the left and the right eigenvectors are not orthogonal,

$$\langle \boldsymbol{L}_\alpha, \boldsymbol{L}_\beta \rangle \neq 0, \langle \boldsymbol{R}_\alpha, \boldsymbol{R}_\beta \rangle \neq 0, \ 1 \le \alpha < \beta \le n.$$

To study the evolution of eigenvalues corresponding to (8), it is natural to study conditional expectations

$$\mathbb{E}_n(d\Lambda_{t\alpha} d\bar{\Lambda}_{t\alpha} \mid \Lambda_{t\alpha} = \lambda_\alpha) = \mathbb{E}_n(O_{t\alpha\alpha} \mid \Lambda_{t\alpha} = \lambda_\alpha) dt, \ 1 \le \alpha \le n,$$

and

$$\mathbb{E}_n(d\Lambda_{t\alpha} d\bar{\Lambda}_{t\beta} \mid \Lambda_{t\alpha} = \lambda_\alpha, \Lambda_{t\beta} = \lambda_\beta) = \mathbb{E}_n(O_{t\alpha\beta} \mid \Lambda_{t\alpha} = \lambda_\alpha, \Lambda_{t\beta} = \lambda_\beta) dt, \ 1 \le \alpha \neq \beta \le n,$$

where $\mathbb{E}_n(\cdot)$ denotes expectation with respect to $\mathrm{Gin}(n, \mathbb{C})$. These are the conditional expectations of the diagonal and the non-diagonal overlaps originally studied in [10,11]. Furthermore, if we wish to understand the influence of a fixed set of eigenvalues on the evolution of a single eigenvalue or a pair of eigenvalues, it is reasonable to consider general conditional expectations

$$\mathbb{E}_n(O_{t\alpha_1\alpha_2} \mid \Lambda_{t\alpha_p} = \lambda_{\alpha_p}, p = 1, 2, \ldots k), \ 1 \le \alpha_p \le n, k = 1, 2, \ldots n.$$

These are the principal objects studied in the present thesis. An additional motivation for our study comes from the mathematical structure of the answers: we find that conditional expectations of overlaps are expressed in terms of determinants of matrices built out of a kernel of some integrable operator. While this structure is a well-known feature of point processes associated with the statistics of eigenvalues of random matrices, we were unaware of determinantal answers for the statistics of eigenvectors prior to starting our work.

Our work continues the mathematical study of the statistical properties of eigenvectors of non-Hermitian matrices, which has become an active research area during the past few years. In [10] and [11], Chalker and Mehlig showed that for large N, where N is the size of matrix in the complex Ginibre ensemble, the eigenvalues associated with a pair of eigenvectors are highly correlated if the two of them are close in the complex plain.

We denote as $(O_{\alpha\beta})_{\alpha,\beta=1}^n$ the overlap matrix, where $O_{\alpha\beta} = \langle L_\alpha, L_\beta \rangle \langle R_\alpha, R_\beta \rangle$ is the product of inner products of inner products associated of the left and right eigenvectors associated with the eigenvalues $\lambda_\alpha, \lambda_\beta$. In relations (10) and (11) of [11], Chalker and Mehlig define the local averages of diagonal and off-diagonal elements of the overlap matrix $O_{\alpha\beta}$ as

$$\begin{aligned} O(z) &= \langle \sigma^2 \sum_\alpha O_{\alpha\alpha} \delta(z - \lambda_\alpha) \rangle \\ O(z_1, z_2) &= \langle \sigma^2 \sum_{\alpha \neq \beta} O_{\alpha\beta} \delta(z_1 - \lambda_\alpha) \delta(z_2 - \lambda_\beta) \rangle \end{aligned}$$

which correspond respectively to (22) and (23) of this thesis, for $\sigma = 1$. In particular, for $\sigma = 1$ we use the following notation:

$$O(z) = D_{11}^{(n,1)}(z)$$

and

$$O(z_1, z_2) = D_{12}^{(n,2)}(z_1, z_2).$$

They obtain exact formulas in the form of determinants, and in the diagonal case of $O(z)$ they simplify the expression further by computing the determinant recursively. Another main focus of their work, is the derivation of expressions for $O(z)$ and $O(z_1, z_2)$, see relations (7) and (8) in [10].

There are a lot of similarities between their computations and the ones in sections 4.1 and 4.2, where we re-derive some of their result, and we also faced some of the same issues. For the diagonal overlaps, the expression for $O(z) = D_{11}^{(N,1)}(z)$ could be further simplified, because we have in our hands a 3-diagonal determinant which is easy to compute. But working with the off-diagonal overlaps, we come across a 5-diagonal determinant which is no longer manageable. Chalker and Mehlig assumed translation invariance, which is a universal argument applying to more than just the Ginibre ensembles, and by taking $z_1 = 0$ and $z_2 = z$ they were able to get an exact formula for $O(z_1, z_2)$ as N goes to infinity. Not only we wanted to avoid using the above argument, we also wanted to study more general cases, which, working the same way, would produce increasingly harder determinant. So we abandoned our original approach and focused on the determinantal structure of the overlaps. In Theorem 1, we find exact formulas of $D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$ and $D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$ as defined in (18) and (19). Having expressions for finite n and following the footsteps of Chalker and Mehlig, in Corollary 1 and 2 we proceed to compute the bulk and edge scaling limits of both diagonal and off-diagonal cases, see (37)-(40).

1.3 Other works

In [32], Walters and Starr extend the answers of [10, 11] for the conditional expectation of the diagonal overlap at $n < \infty$ to any conditioned value of the corresponding eigenvalue. This allows the authors to calculate the edge scaling limit for the conditional expectation of the diagonal overlap. It is worth stressing that our own calculations are based on the same analysis of recursion relations for the determinants of certain 3-diagonal moment matrices as in [32]. We complement it by an exact correspondence between diagonal and off-diagonal overlaps, which allows us to avoid difficulties associated with the analysis of the 5-diagonal moment matrices. Also, even though they find Chalker and Mehlig's approach to be beautiful and believe they give a very good argument which is highly plausible on the basis of mathematical reasoning, Walter and Starr do not consider the proof of (8) in [10] to be fully rigorous. As expected of course, the conditional overlaps are indeed transitionally invariant and our results in Corollary 1 do correspond with their answer.

In [15] Fyodorov studied the following object

$$P(t, z) = \langle \sum_{\alpha} \delta(O_{\alpha\alpha} - 1 - t)\delta(z - \lambda_\alpha) \rangle$$

which can be interpreted as the joint probability function of the diagonal orthogonality factor $t = O_{\alpha\alpha} - 1$ and the corresponding eigenvalue. He derived expressions for matrices of finite size N in the real Ginibre ensemble for a real eigenvalue, and in the complex Ginibre ensemble for a complex eigenvalue, and then analysed the bulk and edge scaling limits for said expressions.

In a related paper [6], Bourgade and Dubach prove that the law of the diagonal overlap conditioned on the corresponding eigenvalue is given in the bulk scaling limit by the inverse Gamma-distribution with parameter 2. This is a significant generalisation of the results of Chalker and Mehlig who managed to calculate this distribution for the matrix size $n = 2$ only. The statement follows from a representation of the diagonal overlap conditioned on all eigenvalues as a product of independent random variables. The authors also obtain new results for the variance of the off-diagonal overlaps and the two-point function of diagonal overlaps, establishing in particular the algebraic decay of the latter as a function of the distance between the corresponding eigenvalues. Other work in this field include [12], where Crawford and Rosenthal study high order moments of the overlap matrix (9), and [9], where Burda and Spisak studied the statistics of overlaps of products of Ginibre matrices.

Finally, we must mention the work of the Krakow School, which is at least partially responsible for the current renaissance of research into the joint statistics of eigenvectors and eigenvalues for random non-Hermitian matrices. Among its recent contributions most relevant to the present work is the derivation of the system of stochastic evolution equations for eigenvalues and eigenvectors, [18], which allowed its authors to express the rate of change of eigenvalue correlation functions in terms of conditional expectations of overlaps. These are the objects studied in here.

1.4 Structure of Thesis

The thesis is split in two parts. First part focuses on conditional overlaps and the contents of our paper. Section 2 presents our main results concerning conditional expectations of overlaps: the determinantal representation for $n < \infty$, the bulk and the edge scaling limits, exact algebraic asymptotic in the bulk for well separated eigenvalues. Section 3 contains the proofs as presented in our paper [2]: 3.1, 3.2 is the derivation of the determinantal representation for the conditional expectations of diagonal and off-diagonal overlaps in terms of bi-orthogonal polynomials in the complex plane; 3.3 a heuristic calculation of the correlation kernels, which shows how the result of rather complicated calculations of the following sections can be easily guessed using the assumption of the extended translational invariance; 3.4 a rigorous evaluation of correlation kernels for $n < \infty$ in terms of the exponential polynomials; 3.5 - 3.7 the calculation of various scaling limits as $n \to \infty$. The methods used in these proofs are rather classical: the determinantal structure is a consequence of Dyson's theorem reviewed in [24] and the product structure of the overlap expectations conditioned on all eigenvalues; the computation of the correlation kernel for the diagonal overlaps reduces to the inversion of the tri-diagonal moment matrix using the recursions already encountered in [10], [11] and [32]; the calculation of the kernel for the off-diagonals overlaps uses a relation between diagonal and off-diagonal overlaps established in Lemma 1 and determinantal identities, which we explore more thoroughly in Section 5. More results can be found in both [2] and our follow up paper [1].

In Section 4 we go and explore alternate approaches and proof. We show that some of the tri-diagonal can be computed directly, we present different ways of finding the kernel and the orthogonal polynomials, a different proof of Theorem 1 using contour integrals and a more detailed version of the bulk and edge limits computations of the kernel. These alterantive methods were explored at the same time as the methods that eventually were included in the paper. We hope that both sets of methods may prove useful in generalisations of these results.

The second part of the thesis revolves around a parallel research on the conditioning of determinantal and pfaffian point processes. It is general results, which I illustrate through Ginibre, but otherwise unrelated. In Section 5 we use a presumably known, but long forgotten determinantal identity, and use it to compute explicitly the kernel of conditioned determinantal and pfaffian p.p. on occupied points, empty sets or the combination of the two. We explore the discrete determinantal case, the discrete pfaffian case by finding an analogue of said identity, but for pfaffians using Tanner's identity [21] and finally the continuous determinantal point processes with the assistance of Campbell and Palm measures. Unfortunately the continuous pfaffian remains for the time being unfinished. A more detailed account of the work can be found in Section 5.

2 Statements of Results

2.1 Definitions and notation

Let $\mathrm{Gin}(n,\mathbb{C})$ be an ensemble of $N \times N$ matrices with independent complex Gaussian entries (complex Ginibre ensemble): if $M \sim \mathrm{Gin}(n,\mathbb{C})$ is a complex Ginibre matrix, then

$$\mathbb{E}(M_{ij}) = 0 = \mathbb{E}(\bar{M}_{ij}),\ 1 \le i,j \le N, \tag{10}$$

$$\mathbb{E}(M_{ij}M_{kl}) = 0 = \mathbb{E}(\bar{M}_{ij}\bar{M}_{kl}),\ 1 \le i,j,k,l \le N, \tag{11}$$

$$\mathbb{E}(M_{ij}\bar{M}_{kl}) = \delta_{ik}\delta_{jl}\ 1 \le i,j,k,l \le N, \tag{12}$$

We will be interested in the joint statistics of the overlaps and eigenvalues of $M \sim \mathrm{Gin}(n,\mathbb{C})$. Namely, we will study the following conditional expectations:

$$\mathbb{E}_N(O_{\alpha\alpha} \mid \Lambda_m = \lambda_m, m \in I),\ I \subset \{1,2,\dots,N\}, \alpha \in I \tag{13}$$

$$\mathbb{E}_N(O_{\alpha\beta} \mid \Lambda_m = \lambda_m, m \in J),\ J \subset \{1,2,\dots,N\}, \alpha,\beta \in J. \tag{14}$$

In other words , we consider the expectation of the overlaps with respect to $\mathrm{Gin}(n,\mathbb{C})$ measure conditioned on a set of eigenvalues. To be more concrete, if $M \sim \mathrm{Gin}(n,\mathbb{C})$ is parametrised using Schur coordinates (see section4.1 for more details), we compute the expected overlaps with respect to the product measure whose factors are the Haar measure for the unitary conjugation, a Gaussian measure for the upper triangular degrees of freedom, and the eigenvalue measure obtained by conditioning

$$P_n\left(\boldsymbol{\Lambda}^{(n)} \in d\boldsymbol{\lambda}^{(n)}\right) = \frac{1}{Z_n}|\Delta^{(n)}(\boldsymbol{\lambda}^{(n)})|^2 e^{-\sum_{j=1}^{n}|\lambda_j|^2} d\boldsymbol{\lambda}^{(n)}. \tag{15}$$

on a set of eigenvalues (see [24]). Due to the permutation symmetry of $\mathrm{Gin}(n,\mathbb{C})$ measure, it is sufficient to consider the following expectations:

$$\mathbb{E}_n(O_{11} \mid \Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2, \dots \Lambda_k = \lambda_k),\ k = 1,2,\dots,n, \tag{16}$$

$$\mathbb{E}_n(O_{12} \mid \Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2, \dots \Lambda_k = \lambda_k),\ k = 2,\dots,n. \tag{17}$$

Closely associated with these expectations are the following weighted multi-point intensities of the eigenvalues:

$$D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) := \mathbb{E}_n(O_{11} \mid \Lambda_1 = \lambda_1, \dots, \Lambda_k = \lambda_k)\rho^{(n,k)}(\boldsymbol{\lambda}^{(k)}), \quad k = 1,2,\dots,n, \tag{18}$$

and

$$D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) := \mathbb{E}_n(O_{12} \mid \Lambda_1 = \lambda_1, \dots, \Lambda_k = \lambda_k)\rho^{(n,k)}(\boldsymbol{\lambda}^{(k)}), \quad k = 2,\dots,n, \tag{19}$$

where $\boldsymbol{\lambda}^{(k)} = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\rho^{(n,k)}$ is the k-point correlation function (Lebesgue density for factorial moments) for $\mathrm{Gin}(n,\mathbb{C})$ eigenvalues. Recall from Mehta [24] that

$$\rho^{(n,k)}(\boldsymbol{\lambda}^{(k)}) := \frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}} \prod_{m=k+1}^{n} d\lambda_m d\bar{\lambda}_m p_n(\boldsymbol{\lambda}^{(n)}) = \det_{1\le i,j\le n}\left(K_{ev}^{(n)}(\lambda_i,\lambda_j)\right), \tag{20}$$

where

$$K_{ev}^{(n)}(x,y) = \frac{1}{\pi}e^{-|x|^2}\sum_{m=0}^{n-1}\frac{(\bar{x}y)^m}{m!} \tag{21}$$

is the kernel of the determinantal point process corresponding to the distribution of $\mathrm{Gin}(n,\mathbb{C})$ eigenvalues, see [17] and [24] for the derivation of (20) and (21). [1]
For the sake of brevity, we will refer to the expectations (18) and (19) as conditional overlaps. Notice that

$$D_{11}^{(n,1)}(\lambda) = \mathbb{E}_n\left(\sum_{\alpha=1}^{n} O_{\alpha\alpha}\delta(\Lambda_\alpha - \lambda)\right), \tag{22}$$

$$D_{12}^{(n,2)}(\lambda,\mu) = \mathbb{E}_n\left(\sum_{\alpha\neq\beta=1}^{n} O_{\alpha\beta}\delta(\Lambda_\alpha - \lambda)\delta(\Lambda_\beta - \mu)\right), \tag{23}$$

coincide with the expectations of diagonal and off-diagonal elements of the overlap matrix studied by Chalker and Mehlig: compare $D_{11}^{(n,1)}(\lambda)$ and $D_{12}^{(n,2)}(\lambda,\mu)$ with equations (10) and (11) of [11] evaluated at $\sigma = 1$.

Our starting point is the fundamental result of [10, 11] for the overlaps conditioned on *all* eigenvalues:

$$D_{11}^{(n,n)}(\boldsymbol{\lambda}^{(n)}) = n!\prod_{k=2}^{n}\left(1 + \frac{1}{|\lambda_1 - \lambda_k|^2}\right)p_n(\boldsymbol{\lambda}^{(n)}), \tag{24}$$

$$D_{12}^{(n,n)}(\boldsymbol{\lambda}^{(n)}) = -\frac{n!}{|\lambda_1 - \lambda_2|^2}\prod_{k=3}^{n}\left(1 + \frac{1}{(\lambda_1 - \lambda_k)\left(\bar{\lambda}_2 - \bar{\lambda}_k\right)}\right)p_n(\boldsymbol{\lambda}^{(n)}), \tag{25}$$

see equations (43) and (46) of [11]. How we re-derived these results will be presented in Chapter 4.1.

2.2 Results

The study of conditional expectations of overlaps is simplified due to a simple relation between D_{11} and D_{12}. In order to state this simple relation, we will treat $\{\lambda_i, \bar{\lambda}_i\}_{1\leq i\leq k}$ as independent complex variables and therefore treat conditional overlaps as functions on $\mathbb{C}^{2k}$. To obtain the final answer we will specialize to the real surface $\mathbb{C}^k \subset \mathbb{C}^{2k}$ by treating λ_i as the complex conjugate of $\bar{\lambda}_i$ for $1 \leq i \leq k$. As a slight abuse of notation, we will always refer to the value of the overlap at a point as $D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$. Let $\hat{T}$ be the following transposition acting on functions on $\mathbb{C}^{2k}$, $k \geq 2$:

$$\hat{T}f(\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2,\ldots) = f(\lambda_1,\bar{\lambda}_2,\lambda_2,\bar{\lambda}_1,\ldots), \tag{26}$$

leaving the remaining variables $\lambda_3,\bar{\lambda}_3,\ldots\lambda_k,\bar{\lambda}_k$ untouched. We have the following:

Lemma 1 (Exact relation between diagonal and off-diagonal overlaps for $n < \infty$.) For any $2 \leq k \leq n < \infty$, the functions $D_{11}^{(n,k)}$ and $D_{12}^{(n,k)}$ are entire functions on $\mathbb{C}^{2k}$. Moreover,

$$D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{e^{-|\lambda_1-\lambda_2|^2}}{1-|\lambda_1-\lambda_2|^2}\hat{T}D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}). \tag{27}$$

To state the main result we need to introduce some notations. Let

$$e_p(x) = \sum_{k=0}^{p}\frac{x^k}{k!},\ p = 0,1,2,\ldots \tag{28}$$

be the exponential polynomial of order p considered as functions on $\mathbb{C}$. Let

$$f_p(x) = (p+1)e_p(x) - xe_{p-1}(x),\ p = 0,1,\ldots, \tag{29}$$

[1] The kernel (21) can be re-written in a more symmetric form $\frac{1}{\pi}e^{-\frac{1}{2}(|x|^2+|y|^2)}\sum_{m=0}^{N-1}\frac{(\bar{x}y)^m}{m!}$ by the conjugation $K_{ev}(x,y) \to e^{\frac{1}{2}x^2}K_{ev}(x,y)e^{-\frac{1}{2}y^2}$, which does not change the correlation functions.

where we define $e_{-1}(x) \equiv 0$. The polynomials f_p are closely related to the bi-orthogonal polynomials in the complex plane associated with conditional overlaps, as we will see in a later section. Finally, let $\mathfrak{F}_n : \mathbb{C}^3 \to \mathbb{C}$ be the following polynomial in three variables:

$$\begin{aligned} \mathfrak{F}_n(x,y,z) &= e_n(xy)\cdot e_n(xz) - e_n(xyz)\cdot e_n(x)\cdot(1-x(1-y)(1-z)) \\ &\quad + \frac{(1-y)(1-z)}{n!}\cdot\frac{(xyz)^{n+1}e_n(x) - x^{n+1}e_n(xyz)}{1-yz},\ n=0,1,\ldots \end{aligned} \tag{30}$$

The following is the main result:

Theorem 1 *(Determinantal structure of conditional overlaps) For any $1 \le k \le n < \infty$,*

$$D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{f_{n-1}(|\lambda_1|^2)}{\pi}e^{-|\lambda_1|^2}\det_{2\le i,j\le k}\left(K_{11}^{(n-1)}\left(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j \mid \lambda_1,\bar\lambda_1\right)\right), \tag{31}$$

where the kernel

$$K_{11}^{(n)}(x,\bar x,y,\bar y \mid \lambda,\bar\lambda) = \omega(x,\bar x\mid\lambda,\bar\lambda)\kappa^{(n)}(\bar x,y\mid\lambda,\bar\lambda), \tag{32}$$

is a function on $\mathbb{C}^6$, which is built out of the weight

$$\omega(x,y\mid\lambda,\mu) = \frac{1}{\pi}(1+(x-\lambda)(y-\mu))e^{-xy}, \tag{33}$$

a function on $\mathbb{C}^4$, and the reduced kernel

$$\kappa^{(n)}(\bar x,y\mid\lambda,\bar\lambda) = \frac{\left((n+1)\,\mathfrak{F}_{n+1}\left(\lambda\bar\lambda,\frac{\bar x}{\bar\lambda},\frac{y}{\lambda}\right) - \lambda\bar\lambda\mathfrak{F}_n\left(\lambda\bar\lambda,\frac{\bar x}{\bar\lambda},\frac{y}{\lambda}\right)\right)}{\left(\bar x-\bar\lambda\right)^2(y-\lambda)^2 f_n\left(\lambda\bar\lambda\right)}. \tag{34}$$

Furthermore, for $k \ge 2$,

$$\begin{aligned} D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi^2}f_{n-1}(\lambda_1\bar\lambda_2)\kappa^{(n-1)}(\bar\lambda_1,\lambda_2\mid\lambda_1,\bar\lambda_2) \\ &\quad\times\det_{3\le i,j\le k}\left(K_{12}^{(n-1)}\left(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2\right)\right), \end{aligned} \tag{35}$$

where

$$\begin{aligned} K_{12}^{(n)}(x,\bar x,y,\bar y\mid u,\bar u,v,\bar v) &= \frac{\omega(x,\bar x\mid u,\bar v)}{\kappa^{(n)}(\bar u,v\mid u,\bar v)} \\ &\quad\times\det\begin{pmatrix}\kappa^{(n)}(\bar u,v\mid u,\bar v) & \kappa^{(n)}(\bar u,y\mid u,\bar v)\\ \kappa^{(n)}(\bar x,v\mid u,\bar v) & \kappa^{(n)}(\bar x,y\mid u,\bar v)\end{pmatrix}. \end{aligned} \tag{36}$$

The finite-n answer stated above enables an easy study of the large-n limits of conditional overlaps. It is well known that the global spectral density of complex eigenvalues approaches the circular law, $\lim_{n\to\infty}\rho^{(n,1)}(\sqrt{n}z) = \frac{1}{\pi}\Theta(1-|z|)$, where Θ is the Heaviside step function, [20]. Therefore, we will consider two such local, microscopic limits: the local bulk scaling limit,

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{n}D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}), \tag{37}$$

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{n}D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}), \tag{38}$$

i.e. we fix $\lambda_1,\ldots,\lambda_k$ and take the large-n limit, which places us in the vicinity of the origin[2], and the local edge scaling limit,

$$D_{11}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}D_{11}^{(n,k)}(e^{i\theta}(\sqrt{n}+\boldsymbol{\lambda}^{(k)})), \tag{39}$$

$$D_{12}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}D_{12}^{(n,k)}(e^{i\theta}(\sqrt{n}+\boldsymbol{\lambda}^{(k)})), \tag{40}$$

[2]To access the general bulk we would have to scale $z = re^{i\theta}\sqrt{n}+\lambda$, with $r<1$ and fixed λ.

i.e. we shift to the vicinity of the spectral edge at $|z| = \sqrt{n}$, fix $\lambda_1, \ldots, \lambda_k$ and then take the $N \to \infty$ limit. In the bulk, the conditional overlaps scale as n, which we already know from [11] and can be seen by averaging over all eigenvalues. It is common in random matrices for the bulk and edge to scale differently, and in this case, when the eigenvalues are on the edge the overlaps scale as $n^{1/2}$. Thus the overall rescaling of conditional overlaps used for the bulk and edge limits are factors of n^{-1} and $n^{-1/2}$ respectively. Notice also that our notations for the edge scaling limit reflect the independence of the final answer on the point at the edge of the spectrum around which we expand.

Corollary 1 *(Local bulk scaling limit of conditional overlaps)*

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty} \frac{1}{n} D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{\pi} \det_{2\le i,j\le k} \left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1) \right), \tag{41}$$

where $K_{11}^{(bulk)} : \mathbb{C}^6 \to \mathbb{C}$ *is the limiting kernel:*

$$K_{11}^{(bulk)}(u, \bar{u}, v, \bar{v} \mid \lambda, \bar{\lambda}) = \omega^{(bulk)}(u, \bar{u} \mid \lambda, \bar{\lambda}) \kappa^{(bulk)}(\bar{u}, v \mid \lambda, \bar{\lambda}), \tag{42}$$

where

$$\omega^{(bulk)}(u, \bar{u} \mid \lambda, \bar{\lambda}) = \frac{1}{\pi}(1 + (u - \lambda)(\bar{u} - \bar{\lambda}))e^{-(u-\lambda)(\bar{u}-\bar{\lambda})} \tag{43}$$

is the weight and

$$\kappa^{(bulk)}(\bar{u}, v \mid \lambda, \bar{\lambda}) = \frac{d}{dz}\left(\frac{e^z - 1}{z}\right)\Big|_{z=(\bar{u}-\bar{\lambda})(v-\lambda)}, \tag{44}$$

is the reduced kernel. Moreover,

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{1}{\pi^2} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) \det_{3\le i,j\le k} \left(K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2) \right), \tag{45}$$

where

$$\begin{aligned} K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2) &= \frac{\omega^{(bulk)}(\lambda_i, \bar{\lambda}_i \mid \lambda_1, \bar{\lambda}_2)}{\kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2)} \\ &\times \det \begin{pmatrix} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \\ \kappa^{(bulk)}(\bar{\lambda}_i, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(bulk)}(\bar{\lambda}_i, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \end{pmatrix}. \end{aligned} \tag{46}$$

As expected, conditional overlaps in the bulk are translationally invariant, meaning that $D_{11}^{(bulk,\ k)}$ and $D_{12}^{(bulk,\ k)}$ are invariant with respect to a simultaneous shift of the arguments,

$$\lambda_i \to \lambda_i + \mu, \bar{\lambda}_i \to \bar{\lambda}_i + \bar{\mu},\ 1 \le i \le k,\ \mu \in \mathbb{C}.$$

Finally, let us verify that the statement of Corollary 1 agrees with Chalker and Mehlig's answer for $D_{12}^{(bulk,2)}(\lambda_1, \lambda_2)$ obtained in [10, 11] . Specialising (45) to the particular case $k = 2$ and denoting

$$\lambda_{ij} = \lambda_i - \lambda_j,\ \bar{\lambda}_{ij} = \bar{\lambda}_i - \bar{\lambda}_j,\ 1 \le i, j \le N, \tag{47}$$

we find that

$$\begin{aligned} D_{12}^{(bulk,\ 2)}(\lambda_1, \lambda_2) &= -\frac{1}{\pi^2} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) \\ &= -\frac{1}{\pi^2} \frac{d}{dz} \left(\frac{e^z - 1}{z} \right) |_{z=-|\lambda_{12}|^2} = -\frac{1}{\pi^2} \frac{1}{|\lambda_{12}|^4} \left(1 - \left(1 + |\lambda_{12}|^2\right) e^{-|\lambda_{12}|^2} \right), \end{aligned}$$

which corresponds to Eqn. (9) of [10] for fluctuations at the origin ($z_+ = 0$ in [10]).

The finite-N results stated in Theorem 1 are also well suited for studying the statistics of overlaps at the edge. For $a \in \mathbb{C}$, let

$$F(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2} dx \equiv \frac{1}{2}\mathrm{erfc}\left(\frac{a}{\sqrt{2}}\right), \tag{48}$$

where erfc is the complementary error function, analytically continued to the complex plane. For any $a, b, c, d, f \in \mathbb{C}$, let

$$\begin{aligned} H(a,b,c,d,f) = & -\frac{\sqrt{2\pi}}{\left(1 - \sqrt{2\pi} a e^{\frac{a^2}{2}} F(a)\right)} \\ & \times \frac{d}{dx}\left[e^{\frac{(a+x)^2}{2}} \left(e^{-f} F(b+x)F(c+x) - F(d+x)F(a+x) + fF(d)F(a+x)\right)\right]\Big|_{x=0}. \end{aligned} \tag{49}$$

Corollary 2 *(Local edge scaling limit of conditional overlaps)*

$$\begin{aligned} D_{11}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} D_{11}^{(n,k)}(e^{i\theta}(\sqrt{n} + \boldsymbol{\lambda}^{(k)})) \quad = \quad & \frac{1}{\sqrt{2\pi^3}} \left(e^{-\frac{1}{2}(\lambda_1+\bar{\lambda}_1)^2} - \sqrt{2\pi}(\lambda_1 + \bar{\lambda}_1) F(\lambda_1 + \bar{\lambda}_1)\right) \\ & \times \det_{2\le i,j\le k} \left(K_{11}^{(edge)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1)\right), \end{aligned} \tag{50}$$

where $K_{11}^{(edge)} : \mathbb{C}^6 \to \mathbb{C}$ *is the limiting kernel:*

$$K_{11}^{(edge)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) = \omega^{(edge)}(x, \bar{x} \mid \lambda, \bar{\lambda}) \kappa^{(edge)}(\bar{x}, y \mid \lambda, \bar{\lambda}), \tag{51}$$

where

$$\omega^{(edge)}(x, \bar{x} \mid \lambda, \bar{\lambda}) = \frac{1}{\pi}(1 + (x-\lambda)(\bar{x} - \bar{\lambda}))e^{-x\bar{x}} \tag{52}$$

is the weight and

$$\kappa^{(edge)}(\bar{x}, y \mid \lambda, \bar{\lambda}) = e^{\bar{x}y} \frac{H\left(\lambda + \bar{\lambda}, \lambda + \bar{x}, y + \bar{\lambda}, y + \bar{x}, (\lambda - y)(\bar{\lambda} - \bar{x})\right)}{(\lambda - y)^2 (\bar{\lambda} - \bar{x})^2} \tag{53}$$

is the reduced kernel. Moreover,

$$\begin{aligned} D_{12}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) \quad = \quad & -\frac{1}{\sqrt{2\pi^5}} \left(1 - \sqrt{2\pi}(\lambda_1 + \bar{\lambda}_2) e^{\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2} F(\lambda_1 + \bar{\lambda}_2)\right) \\ & \times \frac{e^{-|\lambda_1-\lambda_2|^2 - \frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}}{\lambda_{12}^2 \bar{\lambda}_{12}^2} H(\lambda_1 + \bar{\lambda}_2, \lambda_1 + \bar{\lambda}_1, \lambda_2 + \bar{\lambda}_2, \lambda_2 + \bar{\lambda}_1, -\lambda_{12}\bar{\lambda}_{12}) \\ & \times \det_{3\le i,j\le k} \left(K_{12}^{(edge)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2)\right), \end{aligned} \tag{54}$$

where

$$\begin{aligned} K_{12}^{(edge)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2) \quad = \quad & \frac{\omega^{(edge)}(\lambda_i, \bar{\lambda}_i \mid \lambda_1, \bar{\lambda}_2)}{\kappa^{(edge)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2)} \\ & \times \ \det \begin{pmatrix} \kappa^{(edge)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(edge)}(\bar{\lambda}_1, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \\ \kappa^{(edge)}(\bar{\lambda}_i, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) & \kappa^{(edge)}(\bar{\lambda}_i, \lambda_j \mid \lambda_1, \bar{\lambda}_2) \end{pmatrix}. \end{aligned} \tag{55}$$

As expected, the translational invariance is lost at the edge. However, it is easy to check that $D_{11}^{(edge,\ k)}$ and $D_{12}^{(edge,\ k)}$ are invariant with respect to a global shift along the edge of the spectrum,

$$\lambda_m \to \lambda_m + i\mu, \bar{\lambda}_m \to \bar{\lambda}_m - i\mu,\ 1 \le m \le k,\ \mu \in \mathbb{R}.$$

It follows from the statement of Corollary 2, that for $k = 1$,

$$D_{11}^{(edge,\ 1)}(\lambda_1) = \frac{1}{\sqrt{2\pi^3}}\left(e^{-\frac{1}{2}(\lambda_1+\bar{\lambda}_1)^2} - \sqrt{2\pi}(\lambda_1+\bar{\lambda}_1)F(\lambda_1+\bar{\lambda}_1)\right), \tag{56}$$

which coincides with the answer for the edge scaling limit of the diagonal overlap obtained in [32, Corollary 4.3]. For $k = 2$, we find that

$$\begin{aligned} &D_{12}^{(edge,\ 2)}(\lambda_1,\lambda_2) = \\ &\frac{1}{\pi^2}\frac{e^{-|\lambda_1-\lambda_2|^2-\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}}{\lambda_{12}^2\bar{\lambda}_{12}^2}\frac{d}{dx}\Bigg[e^{\frac{(\lambda_1+\bar{\lambda}_2+x)^2}{2}}\Bigg(e^{\lambda_{12}\bar{\lambda}_{12}}F(\lambda_1+\bar{\lambda}_1+x)F(\lambda_2+\bar{\lambda}_2+x) \\ &-F(\lambda_2+\bar{\lambda}_1+x)F(\lambda_1+\bar{\lambda}_2+x)-\lambda_{12}\bar{\lambda}_{12}F(\lambda_2+\bar{\lambda}_1)F(\lambda_1+\bar{\lambda}_2+x)\Bigg)\Bigg]\Bigg|_{x=0}, \end{aligned} \tag{57}$$

which is apparently a new expression for the off-diagonal overlap at the edge.

There is also a different kind of relation between the scaling limits of overlaps: as we have already reviewed, the typical magnitude of the overlap in the bulk is $O(N)$, near the edge - $O(\sqrt{N})$. This is consistent with the fact that the prefactor in (50) diverges as we move back into the bulk: if $Re(\lambda) = Re(\bar{\lambda}) = R$,

$$\lim_{R\to-\infty}\left(e^{-(\lambda+\bar{\lambda})^2} - \sqrt{2\pi}(\lambda+\bar{\lambda})F(\lambda+\bar{\lambda})\right) = -\sqrt{2\pi}\lim_{R\to-\infty}(\lambda+\bar{\lambda}) = +\infty.$$

Therefore, there is no *a priori* reason for any relation between conditional overlaps in the bulk and at the edge. However, simple analysis of the answers presented in Corollaries 1 and 2 reveals the following relations:

Corollary 3

$$\lim_{R\to-\infty}\frac{D_{11}^{(edge,\ k)}(R\mathbf{1}^{(k)}+\boldsymbol{\lambda}^{(k)})}{D_{11}^{(edge,\ 1)}(R+\lambda_1)} = \frac{D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)})}{D_{11}^{(bulk,\ 1)}(\lambda_1)},\ k = 1,2,\ldots, \tag{58}$$

$$\lim_{R\to-\infty}\frac{D_{12}^{(edge,\ k)}(R\mathbf{1}^{(k)}+\boldsymbol{\lambda}^{(k)})}{D_{12}^{(edge,\ 2)}(R+\lambda_1,R+\lambda_2)} = \frac{D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)})}{D_{12}^{(bulk,\ 2)}(\lambda_1,\lambda_2)},\ k = 2,3,\ldots, \tag{59}$$

where $\mathbf{1}^{(k)} = (1,1,\ldots,1) \in \mathbb{R}^k$.

3 Proofs

3.1 Bi-orthogonal Polynomials

Recall expressions (24) and (25) for the overlaps conditioned on N eigenvalues. Averaging over all the eigenvalues but $\lambda_1,\dots,\lambda_k$, we get

$$\begin{aligned} D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= \frac{1}{Z_n}\frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}}\prod_{i=k+1}^{n} d\lambda_i d\bar{\lambda}_i |\Delta^{(n)}(\lambda_1,\lambda_2,\dots,\lambda_n)|^2 e^{-\sum_{j=1}^{n}|\lambda_j|^2} \\ &\quad\times\prod_{\ell=2}^{n}\left(1+\frac{1}{|\lambda_1-\lambda_\ell|^2}\right), \end{aligned} \tag{60}$$

$$\begin{aligned} D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= \frac{1}{Z_n}\frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}}\prod_{i=k+1}^{n} d\lambda_i d\bar{\lambda}_i |\Delta^{(n)}(\lambda_1,\lambda_2,\dots,\lambda_n)|^2 e^{-\sum_{j=1}^{n}|\lambda_j|^2} \\ &\quad\times\frac{1}{|\lambda_1-\lambda_2|^2}\prod_{\ell=3}^{n}\left(1+\frac{1}{(\lambda_1-\lambda_\ell)\left(\bar{\lambda}_2-\bar{\lambda}_\ell\right)}\right), \end{aligned} \tag{61}$$

where $Z_n = \pi^n\prod_{j=1}^{n} j!$ is the normalisation constant. Therefore,

$$\begin{aligned} D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= \frac{e^{-|\lambda_1|^2}}{Z_n}\frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}}\prod_{i=k+1}^{n} d\lambda_i d\bar{\lambda}_i |\Delta^{(n-1)}(\lambda_2,\dots,\lambda_n)|^2 \\ &\quad\times\prod_{m=2}^{n}\pi\omega(\lambda_m,\bar{\lambda}_m\mid\lambda_1,\bar{\lambda}_1), \end{aligned} \tag{62}$$

$$\begin{aligned} D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{Z_n}\frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}}\prod_{i=k+1}^{N} d\lambda_i d\bar{\lambda}_i \Delta^{(n-1)}(\lambda_2,\lambda_3,\dots,\lambda_N) \\ &\quad\times\Delta^{(n-1)}(\bar{\lambda}_1,\bar{\lambda}_3,\dots,\bar{\lambda}_n)\prod_{m=3}^{n}\pi\omega(\lambda_m,\bar{\lambda}_m\mid\lambda_1,\bar{\lambda}_2), \end{aligned} \tag{63}$$

where the integration measure is defined in both cases by the following function on $\mathbb{C}^3$:

$$\omega(z,\bar{z}|u,v) = \frac{1}{\pi}(1+(z-u)(\bar{z}-v))e^{-z\bar{z}},\ \ z,u,v\in\mathbb{C}. \tag{64}$$

How we go from (60) and (61) to (62) and (63) is presented in detail in chapter 4.2 and also [24, chapter 15].

In order to determine $D_{12}^{(n,k)}$ using Lemma 1, proved in Section 3.2 below, we need to calculate $D_{11}^{(n,k)}$ treating the complex variables $\boldsymbol{\lambda}^{(k)}$ and $\bar{\boldsymbol{\lambda}}^{(k)}$ as independent. This helps, because from a purely notational standpoint, when going from $D_{11}^{(n,k)}$ to $D_{12}^{(n,k)}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ change places. The first steps are standard, see e.g. [24], but we will go through them anyway in chapter 3. Using elementary linear algebra,

$$\begin{aligned} &|\Delta^{(n-1)}(\lambda_2,\dots,\lambda_n)|^2\prod_{m=2}^{n}\omega(\lambda_m,\bar{\lambda}_m\mid\lambda_1,\bar{\lambda}_1) \\ &= \prod_{q=0}^{n-2}\langle P_q,Q_q\rangle\det_{2\le i,j\le n}\left(K_{11}^{(n-1)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\mid\lambda_1,\bar{\lambda}_1)\right), \end{aligned} \tag{65}$$

where $K_{11}^{(n)}$ is the following kernel (of an integral operator):

$$K_{11}^{(n)}(x,\bar{x},y,\bar{y}\mid\lambda_1,\bar{\lambda}_1) = \sum_{k=0}^{n-1}\frac{\overline{P_k(x)}Q_k(y)}{\langle P_k,Q_k\rangle}\omega(x,\bar{x}\mid\lambda_1,\bar{\lambda}_1), \tag{66}$$

and $\{P_i,Q_i\}_{i=0}^{\infty}$ are holomorphic monic polynomials on $\mathbb{C}$, bi-orthogonal with respect to the weight $\omega(\cdot,\cdot\mid\lambda_1,\bar{\lambda}_1)$:

$$\langle P_i,Q_j\rangle := \int_{\mathbb{C}} dz d\bar{z}\omega(z,\bar{z}\mid\lambda_1,\bar{\lambda}_1)\overline{P_i(z)}Q_j(z) = \langle P_i,Q_i\rangle\delta_{i,j},\ \ 0\le i,j<\infty. \tag{67}$$

Note that it is not necessary to find explicit expressions for the polynomials themselves, only the kernel. Notice also that the bi-orthogonal polynomials depend on λ_1 and $\bar{\lambda}_1$ as parameters, but we will suppress this dependence in order to simplify the notation. We will establish the existence of the bi-orthogonal polynomials and the associated kernel (66) for the concrete weight ω by constructing them explicitly. For a general discussion see [3], where Akemann and Vernizzi present a way of finding orthogonal polynomials with respect to more complex weights than ω.

In what follows it will be convenient to define the reduced kernel $\kappa^{(n)}$ via

$$\begin{aligned} K_{11}^{(n)}(x,\bar{x},y,\bar{y}\,|\,\lambda_1,\bar{\lambda}_1) &= \kappa^{(n)}(\bar{x},y\,|\,\lambda_1,\bar{\lambda}_1)\omega(x,\bar{x}\,|\,\lambda_1,\bar{\lambda}_1), \\ \kappa^{(n)}(\bar{x},y\,|\,\lambda_1,\bar{\lambda}_1) &= \sum_{k=0}^{n-1}\frac{\overline{P_k(x)}Q_k(y)}{\langle P_k,Q_k\rangle}. \end{aligned} \tag{68}$$

Notice that the kernel $K_{11}^{(n)}$ is self-reproducing,

$$K_{11}^{(n)} * K_{11}^{(n)} = K_{11}^{(n)}.$$

Therefore, Dyson's theorem (see [24, chapter 5]) is applicable to the calculation of the integral in (62). [3] Substituting (65) into (62) and applying the theorem, we find that

$$\begin{aligned} D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= \frac{\pi^{n-1}n!}{Z_n}\prod_{q=0}^{n-2}\langle P_q,Q_q\rangle\cdot e^{-|\lambda_1|^2} \\ &\quad\times\det_{2\le i,j\le k}\left(K_{11}^{(n-1)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\,|\,\lambda_1,\bar{\lambda}_1)\right). \end{aligned} \tag{69}$$

Observe the emergence of the determinantal structure for the diagonal conditional overlaps. The off-diagonal overlap $D_{12}^{(n,k)}$ as a function on $\mathbb{C}^{2k}$ can now be computed using Lemma 1:

$$\begin{aligned} D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{\pi^{n-1}n!}{Z_n}\frac{e^{-\lambda_{12}\bar{\lambda}_{12}-\lambda_1\bar{\lambda}_2}}{1-\lambda_{12}\bar{\lambda}_{12}}\hat{T}\left(\prod_{q=0}^{n-2}\langle P_q,Q_q\rangle\right) \\ &\quad\times\det\left(\begin{array}{c|c} K_{11}^{(n-1)}(\lambda_2,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_1\,|\,\lambda_1,\bar{\lambda}_2) & \begin{array}{c}K_{11}^{(n-1)}(\lambda_2,\bar{\lambda}_1,\lambda_j,\bar{\lambda}_j\,|\,\lambda_1,\bar{\lambda}_2),\\ 3\le j\le k\end{array} \\ \hline \begin{array}{c}K_{11}^{(n-1)}(\lambda_i,\bar{\lambda}_i,\lambda_2,\bar{\lambda}_1\,|\,\lambda_1,\bar{\lambda}_2),\\ 3\le i\le k\end{array} & \begin{array}{c}K_{11}^{(n-1)}(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j\,|\,\lambda_1,\bar{\lambda}_2),\\ 3\le i,j\le k\end{array} \end{array}\right). \end{aligned} \tag{70}$$

It is worth stressing that $\prod_{q=0}^{n-2}\langle P_q,Q_q\rangle$ is a function of $\lambda_1,\bar{\lambda}_1$, therefore the action of $\hat{T}$ on this product is non-trivial. Recall also that $\lambda_{ij} := \lambda_i-\lambda_j$, $\bar{\lambda}_{ij} := \bar{\lambda}_i-\bar{\lambda}_j$. The determinant in the above formula can be re-written using the following determinantal identity

$$\det_{1\le i,j\le n}(a_{ij}) = a_{11}\det_{2\le i,j\le n}\left(a_{11}^{-1}\det\begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij}\end{pmatrix}\right),\ a_{11}\neq 0. \tag{71}$$

This follows from a well known identity for block determinants, see e.g. [27]:

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det(A)\det\left(D-CA^{-1}B\right), \tag{72}$$

valid for invertible matrices A. Namely, choosing for $A = a_{11}\neq 0$ in (71) we have

$$\det_{2\le i,j\le n}\begin{pmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij}\end{pmatrix} = \det(a_{11})\det_{2\le i,j\le n}\left(a_{ij}-a_{i1}a_{11}^{-1}a_{1j}\right) = a_{11}\det_{2\le i,j\le n}\left(a_{11}^{-1}(a_{11}a_{ij}-a_{i1}a_{1j})\right). \tag{73}$$

[3]The self-adjointness of a kernel is not necessary for the applicability of Dyson's theorem.

Eq. (71) can be seen as the simplest of Tanner's identities for determinants and Pfaffians, see [21] for a review. More about this identity can also be found in appendix 6.1. Applying (71) to (70) results into

$$D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{\pi^{n-2}n!}{Z_n}\hat{T}\left(\prod_{q=0}^{n-2}\langle P_q, Q_q\rangle\right)e^{-\lambda_1\bar{\lambda}_1-\lambda_2\bar{\lambda}_2}\kappa^{(n-1)}\left(\bar{\lambda}_1,\lambda_2|\lambda_1,\bar{\lambda}_2\right)$$
$$\times\det_{3\le i,j\le k}\left(\frac{\omega(\lambda_i,\bar{\lambda}_i\mid\lambda_1,\bar{\lambda}_2)}{\kappa^{(n-1)}\left(\bar{\lambda}_1,\lambda_2|\lambda_1,\bar{\lambda}_2\right)}\det\begin{pmatrix}\kappa^{(n-1)}(\bar{\lambda}_1,\lambda_2\mid\lambda_1,\bar{\lambda}_2) & \kappa^{(n-1)}(\bar{\lambda}_1,\lambda_j\mid\lambda_1,\bar{\lambda}_2)\\ \kappa^{(n-1)}(\bar{\lambda}_i,\lambda_2\mid\lambda_1,\bar{\lambda}_2) & \kappa^{(n-1)}(\bar{\lambda}_i,\lambda_j\mid\lambda_1,\bar{\lambda}_2)\end{pmatrix}\right), \tag{74}$$

which explains the structure of the claim (35), (36) of Theorem 1. The final answers for the conditional overlaps are obtained by evaluating (69) and (74) on the real surface $\mathbb{C}^k \subset \mathbb{C}^{2k}$, specified by the equations $\boldsymbol{\lambda}^{(k)} = \overline{\bar{\boldsymbol{\lambda}}^{(k)}}$.

The proof of Theorem 1 is therefore reduced to the calculation of the reduced kernel $\kappa^{(n)}$ and the inner products of the bi-orthogonal polynomials $\langle P_q, Q_q\rangle$ for $q = 0,1,2,\ldots$ The bi-orthogonal polynomials themselves are not the subject of our current investigation, therefore it is reasonable to follow the approach of [4] and derive expressions for $\kappa^{(n)}$ and $\langle P_q, Q_q\rangle$ directly in terms of the moment matrix M defined as

$$M_{ij} = \langle z^i, z^j\rangle,\ i,j \ge 0. \tag{75}$$

Let (L,D,U) be the LDU-decomposition of M. That is D is the diagonal matrix, L and U^T are the lower triangular matrices with the diagonal entries equal to 1 such that

$$M = LDU. \tag{76}$$

Therefore, $L^{-1}MU^{-1} = D$. Re-writing this identity in components we find that

$$\langle P_k, Q_l\rangle = D_{kk}\delta_{k,l},\ k,l \ge 0, \tag{77}$$

where

$$P_k(z) = \sum_{m=0}^{k}(\bar{L}^{-1})_{km}z^m,$$
$$Q_k(z) = \sum_{m=0}^{k}z^m(U^{-1})_{mk}, \tag{78}$$

for $k \ge 0$. We see that $\{P_q, Q_q\}_{k\ge 0}$ is the set of holomorphic monic polynomials bi-orthogonal with respect to the weight $\omega(\cdot,\cdot\mid\lambda_1,\bar{\lambda}_1)$. Comparing (77) with (67) we find that

$$\langle P_k, Q_k\rangle = D_{kk},\ k \ge 0. \tag{79}$$

Substituting (79) and (78) into the expression (68) for the reduced kernel we also find that

$$\kappa^{(n)}(\bar{z},z\mid\lambda_1,\bar{\lambda}_1) = \sum_{i,j=0}^{n-1}z^iC_{ij}^{(n-1)}\bar{z}^j, \tag{80}$$

where

$$C_{ij}^{(n)} = \sum_{k=0}^{n}(U^{-1})_{ik}\frac{1}{D_{kk}}(L^{-1})_{kj},\ i,j \ge 0. \tag{81}$$

At least formally, the semi-infinite matrix $C^{(n)}$ converges to M^{-1} as $N \to \infty$. Perhaps less trivially, as a consequence of the Gram-Schmidt orthogonalisation procedure, it can be also characterised as the inverse of the $(n+1)\times(n+1)$ moment matrix $(\langle z^i, z^j\rangle)_{0\le i,j} \le N$, see [4].

Now the proof of Theorem 1 has been reduced to the calculation of the LDU decomposition of the moment matrix M.

Remark. We see that the expression for the off-diagonal overlap $D_{12}^{(N,k)}$ is determinantal with the kernel expressed as the 2×2 determinant of a matrix built out of the kernel corresponding to the weight $\omega(x,\bar{x}\mid\lambda,\bar{\lambda})$. Such a structure is to be expected from the general theory of orthogonal polynomials in the complex plane developed in [3]. Really, relation (63) can be re-written as

$$\begin{aligned} D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{Z_n}\frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}}\prod_{i=k+1}^{N} d\lambda_i d\bar{\lambda}_i |\Delta^{(N-2)}(\lambda_3,\lambda_4,\ldots,\lambda_N)|^2 \\ &\quad\times\prod_{m=3}^{N}\pi(\lambda_2-\lambda_m)(\bar{\lambda}_1-\bar{\lambda}_m)\omega(\lambda_m,\bar{\lambda}_m\mid\lambda_1,\bar{\lambda}_2). \end{aligned}$$

By Dyson's theorem, the right hand side of this expression is proportional to the $(k-2)\times(k-2)$ determinant of the kernel associated with holomorphic polynomials, which are bi-orthogonal with respect to the weight

$$(u-z)(\bar{v}-\bar{z})\omega(z,\bar{z}\mid v,\bar{u}).$$

Such a kernel can be expressed in terms of a 2×2 determinant of the kernel associated with the weight $\omega(\cdot,\cdot\mid\lambda,\bar{\lambda})$, see formula (3.10) of [3]. Our present calculation can be therefore regarded as a short re-derivation of the general expression of [3] in the particular context of integration weights associated with the overlaps. The main tools used in our calculation are the analyticity and determinant identities.

3.2 Lemma 1

It follows from (62) and (63) that both $D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)})e^{\sum_{m=1}^{k}\lambda_k\bar{\lambda}_k}$ and $D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})e^{\sum_{m=1}^{k}\lambda_k\bar{\lambda}_k}$ are polynomials in $\boldsymbol{\lambda}^{(k)},\bar{\boldsymbol{\lambda}}^{(k)}$. Therefore $D_{11}^{(n,k)}$ and $D_{12}^{(n,k)}$ are entire functions on $\mathbb{C}^{2k}$.

Recall the definition of the transposition $\hat{T}$ acting on functions on $\mathbb{C}^{2k}$:

$$\hat{T}f(\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2,\ldots)=f(\lambda_1,\bar{\lambda}_2,\lambda_2,\bar{\lambda}_1,\ldots) \tag{82}$$

Comparing (62) and (63), we see that for $k\geq 2$,

$$D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})=-\frac{e^{-|\lambda_1-\lambda_2|^2}}{1-|\lambda_1-\lambda_2|^2}\hat{T}D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}), \tag{83}$$

for any $(\boldsymbol{\lambda}^{(k)},\bar{\boldsymbol{\lambda}}^{(k)})\in\mathbb{C}^{2k}$. Lemma 1 is proved. This is just an observation one can make by looking at the expressions (62) and (63). The transformation $\hat{T}$ switches $\bar{\lambda}_1$ and $\bar{\lambda}_2$ places, and then all that's left in order to go from $D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$ to $D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$, is a factor of $-\frac{e^{-|\lambda_1-\lambda_2|^2}}{1-|\lambda_1-\lambda_2|^2}$.

3.3 Heuristic derivation of $N=\infty$ results in the bulk assuming T-invariance

The calculations below are rather simple, but non-rigorous - they rest on the assumptions of the extended translational invariance of conditional overlaps in the bulk and the validity of Lemma 1 at $N=\infty$. We could try justifying these assumptions using analysis, but as it turns out, it is possible to obtain a fairly simple explicit expression for the kernel at $N<\infty$, thus enabling the study of conditional overlaps not only in the bulk of the spectrum, but also near the spectral edge.

The task of calculating bi-orthogonal polynomials (67) is considerably simpler at the special point $\lambda_1=\bar{\lambda}_1=0$. In this case the weight function reduces to

$$\omega(z,\bar{z}\mid 0,0)=\frac{1}{\pi}(1+|z|^2)e^{-|z|^2}. \tag{84}$$

The bi-orthogonal polynomials associated the weight (84) are just the monomials,

$$P_k(x)=Q_k(x)=x^k,\ \ k\geq 0. \tag{85}$$

Their inner products can also be computed explicitly,

$$\langle P_k, Q_k \rangle = k!(k+2), \;\; k \geq 0, \tag{86}$$

leading to the following kernel:

$$K_{11}^{(N)}(x, \bar{x}, y, \bar{y} \mid 0, 0) = \frac{1}{\pi}(1+|x|^2)e^{-|x|^2} \sum_{k=0}^{N-1} \frac{(\bar{x}y)^k}{(k+2)k!}. \tag{87}$$

As $N \to \infty$, the limiting kernel in the bulk is

$$K_{11}^{(bulk)}(x, \bar{x}, y, \bar{y} \mid 0, 0) \;\; = \;\; \frac{1}{\pi}(1+|x|^2)e^{-|x|^2} \kappa^{(bulk)}(\bar{x}, y \mid 0, 0), \tag{88}$$

where

$$\kappa^{(bulk)}(\bar{x}, y \mid 0, 0) = \frac{1}{(\bar{x}y)^2} + \left(\frac{1}{(\bar{x}y)} - \frac{1}{(\bar{x}y)^2} \right) e^{\bar{x}y}. \tag{89}$$

Alternatively, we can write

$$\kappa^{(bulk)}(\bar{x}, y \mid 0, 0) = \frac{d}{dz} \left(\frac{e^z - 1}{z} \right) \Big|_{z = \bar{x}y}. \tag{90}$$

The N-dependent pre-factor in the right hand side of (69) is N/π, which leads to the following answer for the conditional overlap in the bulk:

$$D_{11}^{(bulk,\ k)}(0, \lambda_2, \ldots, \lambda_k) = \frac{1}{\pi} \det_{2 \leq i,j \leq k} \left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid 0, 0) \right) \tag{91}$$

Let us *assume* the extended translational invariance for the diagonal overlaps regarded as functions on C^{2k}, which means that $D_{11}^{(bulk,\ k)}$ is invariant under the shift $\lambda_m \to \lambda_m + \epsilon$, $\bar{\lambda}_m \to \bar{\lambda}_m + \bar{\epsilon}$, $m = 1, 2, \ldots, k$, where $\epsilon, \bar{\epsilon}$ are independent complex variables. Then

$$\begin{aligned} D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) \;\; &= \;\; D_{11}^{(bulk,\ k)}(0, \lambda_2 - \lambda_1, \ldots, \lambda_k - \lambda_1) \\ &= \;\; \frac{1}{\pi} \det_{2 \leq i,j \leq k} \left(K_{11}^{(bulk)}(\lambda_i - \lambda_1, \bar{\lambda}_i - \bar{\lambda}_1, \lambda_j - \lambda_1, \bar{\lambda}_j - \bar{\lambda}_1 \mid 0, 0) \right). \end{aligned} \tag{92}$$

We conclude that

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{\pi} \det_{2 \leq i,j \leq k} \left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1) \right), \tag{93}$$

where

$$K_{11}^{(bulk)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) \;\; = \;\; \frac{1}{\pi}(1+|x-\lambda|^2)e^{-|x-\lambda|^2} \kappa^{(bulk)}(\bar{x}, y \mid \lambda, \bar{\lambda}), \tag{94}$$

and the reduced kernel is

$$\kappa^{(bulk)}(\bar{x}, y \mid \lambda, \bar{\lambda}) = \frac{d}{dz} \left(\frac{e^z - 1}{z} \right) \Big|_{z = (\bar{x} - \bar{\lambda})(y - \lambda)}, \tag{95}$$

which agrees with the statement (41) of Corollary 1.

To calculate the off-diagonal conditional overlaps, let us *assume* that the relation (27) remains valid at $N = \infty$ as well. Then

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{1}{\pi} \frac{e^{-|\lambda_1 - \lambda_2|^2}}{1 - |\lambda_1 - \lambda_2|^2} \hat{T} \det_{2 \leq i,j \leq k} \left(K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_1) \right). \tag{96}$$

Applying the determinant identity (71), we find

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{1}{\pi^2} \kappa^{(bulk)}(\bar{\lambda}_1, \lambda_2 \mid \lambda_1, \bar{\lambda}_2) \det_{3 \leq i,j \leq k} \left(K_{12}^{(bulk)}(\lambda_i, \bar{\lambda}_i, \lambda_j, \bar{\lambda}_j \mid \lambda_1, \bar{\lambda}_2) \right), \tag{97}$$

where

$$K_{12}^{(bulk)}(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j \mid \lambda_1,\bar\lambda_2) = \frac{\omega^{(bulk)}(\lambda_i,\bar\lambda_i \mid \lambda_1,\bar\lambda_2)}{\kappa^{(bulk)}(\bar\lambda_1,\lambda_2 \mid \lambda_1,\bar\lambda_2)} \times \det\begin{pmatrix} \kappa^{(bulk)}(\bar\lambda_1,\lambda_2 \mid \lambda_1,\bar\lambda_2) & \kappa^{(bulk)}(\bar\lambda_1,\lambda_j \mid \lambda_1,\bar\lambda_2) \\ \kappa^{(bulk)}(\bar\lambda_i,\lambda_2 \mid \lambda_1,\bar\lambda_2) & \kappa^{(bulk)}(\bar\lambda_i,\lambda_j \mid \lambda_1,\bar\lambda_2) \end{pmatrix}, \tag{98}$$

which agrees with the statement (45) of Corollary 1. The step of applying (71) is needed, because in (96) we do not yet have the kernel we want, we still need to get rid of the first row and column which correspond to i or j equal 2. Only in (97) do we finally have the kernel for the off-diagonal case.

3.4 The kernel for $N < \infty$

3.4.1 The LDU decomposition of the moment matrix.

We will use the relation between the kernel and the moment matrix established in Section 3.1. An explicit computation of $\langle z^i, z^j\rangle$ with the weight $\omega(\cdot,\cdot\,|,\lambda,\bar\lambda)$ defined in (64) gives

$$M_{ij} = i!\left[\delta_{ij}\left((1+\lambda\bar\lambda)+(i+1)\right) - \delta_{i+1,j}\lambda(i+1) - \delta_{i,j+1}\bar\lambda\right],\ i,j \geq 0. \tag{99}$$

Crucially, the moment matrix is tri-diagonal, which makes explicit calculations leading to the kernel possible. The recursive formulae for computing the LDU decomposition and the inverse of a tri-diagonal matrix are well-known. What makes our case special however, is that the recursions we get can be solved exactly in terms of the exponential polynomials. At some point it would be interesting to understand the algebraic reasons for the exact solvability of our problem, but in the mean time we adopt a tour de force approach.

Let μ be the following tri-dagonal matrix:

$$\mu_{ij} = \delta_{ij}\left((1+\lambda\bar\lambda)+(i+1)\right) - \delta_{i+1,j}\lambda(i+1) - \delta_{i,j+1}\bar\lambda,\ i,j \geq 0. \tag{100}$$

As M is the product of of μ and the diagonal matrix with entries $i!$, the LDU decomposition of M is easy to construct from the LDU decomposition of μ. If

$$\mu = LDU, \tag{101}$$

where $D_{pq} = d_p\delta_{pq}, L_{pq} = \delta_{pq} + l_p\delta_{p,q+1}, U_{pq} = \delta_{pq} + u_q\delta_{q,p+1},\ p,q \geq 0$, then

$$u_{p+1} = -\frac{(1+p)\lambda}{d_p},\ p \geq 0, \tag{102}$$

$$l_{p+1} = -\frac{\bar\lambda}{d_p},\ p \geq 0, \tag{103}$$

$$d_p = -d_{p-1}l_p u_p \mathbb{1}_{p\geq 1} + 2 + \lambda\bar\lambda + p,\ p \geq 0, \tag{104}$$

defining $d_{-1} \equiv 0$. Let $x = \lambda\bar\lambda$. To determine the LDU decomposition of μ we have to solve the first order non-linear recursion for d_p's:

$$d_p = 2 + x + p - \frac{px}{d_{p-1}},\ p \geq 1, \qquad d_0 = 2 + x. \tag{105}$$

This recursion can be linearised via the substitution $d_p = \frac{r_{p+1}}{r_p}$, which, upon choosing $r_0 = 1$, gives

$$r_{p+1} + pxr_{p-1} = (2 + x + p)r_p,\ p \geq 1, \qquad r_1 = 2 + x. \tag{106}$$

The unique solution of (106) is

$$r_p(x) = p! \sum_{m=0}^{p} \frac{(p+1-m)}{m!} x^m = (p+1)! e_p(x) - p! x e_{p-1}(x), \tag{107}$$

where $e_p(x) = \sum_{k=0}^{p} \frac{x^p}{p!}$ is the exponential polynomial of degree p.

Therefore,

$$r_p(x) = p! f_p(x), \; p = 0, 1, \ldots, \tag{108}$$

where f_p's are the polynomials defined in (29). Converting the LDU decomposition of μ to the LDU decomposition of M and updating notations, we find that $M = LDU$, where

$$L_{pm} = \delta_{pm} - \bar{\lambda} \frac{f_{p-1}(\lambda\bar{\lambda})}{f_p(\lambda\bar{\lambda})} \delta_{p,m+1}, \; p, m \geq 0, \tag{109}$$

$$D_{mm} = (m+1)! \frac{f_{m+1}(\lambda\bar{\lambda})}{f_m(\lambda\bar{\lambda})}, \; m \geq 0, \tag{110}$$

$$U_{mq} = \delta_{mq} - \lambda \frac{f_{m-1}(\lambda\bar{\lambda})}{f_m(\lambda\bar{\lambda})} \delta_{q,m+1}, \; m, q \geq 0. \tag{111}$$

Using the relation (77) between the LDU decomposition and the inner products of the bi-orthogonal polynomials, we conclude that

$$\langle P_p, Q_p \rangle = (p+1)! \frac{(p+2) e_{p+1}(x) - x e_p(x)}{(p+1) e_p(x) - x e_{p-1}(x)}, \tag{112}$$

which coincides with (86) at the point $x = 0$, as it should.

3.4.2 The inner products of the bi-orthogonal polynomials and the pre-factor in (69)

Now we can calculate the factor in front of the determinant in the r. h. s. of (69). Using the relation (112) we find

$$\prod_{q=0}^{n-2} \langle P_q, Q_q \rangle = \prod_{q=0}^{n-2} D_{qq} = \prod_{q=1}^{n-1} q! \cdot f_{n-1}(\lambda\bar{\lambda}). \tag{113}$$

Therefore,

$$\pi^{n-1} \frac{n!}{Z_n} \prod_{q=0}^{n-2} \langle P_q, Q_q \rangle \cdot e^{-\lambda\bar{\lambda}} = \frac{f_{n-1}(\lambda\bar{\lambda})}{\pi} e^{-\lambda\bar{\lambda}}, \tag{114}$$

which allows us to make the operation of $\hat{T}$ on the inner product explicit.

3.4.3 Inversion of the L and U factors, the bi-orthogonal polynomials and the kernel

The inverse of the lower-triangular matrix L (resp. upper triangular matrix U) is a lower (resp. upper) triangular matrix. The corresponding matrix elements can be computed directly from the relations $LL^{-1} = I$, $UU^{-1} = I$ using the explicit expressions (109) for the decomposition factors. The answer is

$$(L^{-1})_{pq} = \begin{cases} 0 & q > p, \\ 1 & q = p, \\ \bar{\lambda}^{p-q} \frac{f_q(x)}{f_p(x)} & q < p, \end{cases} \qquad (U^{-1})_{pq} = \begin{cases} \lambda^{q-p} \frac{f_p(x)}{f_q(x)} & q > p, \\ 1 & q = p, \\ 0 & q < p. \end{cases} \tag{115}$$

Substituting (115) and (110) into the formula (80) we find that

$$\kappa^{(n+1)}(\bar{\mu}, \nu \mid \lambda, \bar{\lambda}) = G^{(n)}\left(\lambda\bar{\lambda}, \frac{\bar{\mu}}{\bar{\lambda}}, \frac{\nu}{\lambda}\right), \tag{116}$$

where

$$G^{(n)}(x,y,z) = \sum_{m,n=0}^{n} f_m(x) f_n(x) y^m z^n \sum_{k=m\vee n}^{n} \frac{x^k}{(k+1)! f_k(x) f_{k+1}(x)} \tag{117}$$

is a function on $\mathbb{C}^3$ and $a \vee b := \max(a,b)$. Even though we do not use explicit expressions for the bi-orthogonal polynomials in the paper, we record them here for future use:

$$P_k(z \mid \lambda, \bar{\lambda}) = Q_k(z \mid \lambda, \bar{\lambda}) = \sum_{m=0}^{k} \lambda^{k-m} \frac{f_m(\lambda\bar{\lambda})}{f_k(\lambda\bar{\lambda})} z^m. \tag{118}$$

3.4.4 Simplification of the reduced kernel for $N < \infty$.

The above form of the reduced kernel is not well suited for studying the large-N asymptotic of the overlaps. In particular, we do not see how to calculate the large-N limit of the kernel directly from (116). Fortunately, it can be considerably simplified via a sequence of cancellations yielding formula (34). The inner sum in (116) can be simplified as follows: Let $\Phi_n : \mathbb{C} \to \mathbb{C}$ be such that

$$\Phi_n(x) := \sum_{k=0}^{n} \frac{x^k}{(k+1)! f_k(x) f_{k+1}(x)}, \tag{119}$$

where we define $\Phi_{-1} \equiv 0$. Then

$$G^{(n)}(x,y,z) = \sum_{m,n=0}^{N} f_m(x) f_n(x) y^m z^n \left(\Phi_n(x) - \Phi_{(m\vee n)-1}(x)\right). \tag{120}$$

We have the following key technical result:

Lemma 3.1:

$$\Phi_n(x) = \frac{(n+2-x)}{x^2 f_{n+1}(x)} + \frac{x-1}{x^2}, \ n = 0, 1, \dots \tag{121}$$

Proof:
For a fixed value of x, the sequence

$$\Phi_n(x) = \sum_{k=0}^{n} \frac{x^k}{(k+1)! f_k(x) f_{k+1}(x)} \tag{122}$$

satisfies the following difference equation:

$$\Phi_{n+1}(x) = \Phi_n(x) + \frac{x^{n+1}}{(n+2)! f_{n+1}(x) f_{n+2}(x)}, \tag{123}$$

$$\Phi_0(x) = \frac{1}{2+x}. \tag{124}$$

Using $f_1(x) = 2 + x$, it is easy to check that the expression (121) satisfies the initial condition (124). Assuming that Φ_n is given by (121), we find from the equation (123) that

$$\Phi_{n+1} = \frac{x-1}{x^2} + \frac{(n+2-x) f_{n+2}(x) + \frac{x^{n+3}}{(n+2)!}}{x^2 f_{n+1}(x) f_{n+2}(x)}. \tag{125}$$

A direct calculation based on the definitions (28) for the exponential polynomials e_n and (29) for the polynomials f_n confirms that

$$(n+2-x) f_{n+2}(x) + \frac{x^{n+3}}{(n+2)!} = (n+3-x) f_{n+1}(x). \tag{126}$$

Therefore,

$$\Phi_{n+1} = \frac{x-1}{x^2} + \frac{(n+3-x)}{x^2 f_{n+2}(x)}, \tag{127}$$

and Lemma 2.1 is proved by induction. □

Substituting (121) into (120), we find

$$\begin{aligned} G^{(n)}(x,y,z) &= \frac{(n+2-x)}{x^2 f_{n+1}(x)} \left(\sum_{m=0}^{n} f_m(x) y^m \right) \left(\sum_{s=0}^{n} f_s(x) z^s \right) \\ &\quad + \frac{1}{x^2} \sum_{m,s=0}^{n} (x - (m \vee s) - 1) \frac{f_m(x) f_s(x)}{f_{m\vee s}(x)} y^m z^s. \end{aligned} \tag{128}$$

Let

$$\alpha_n(x,y) := \sum_{m=0}^{n} f_m(x) y^m, m = 0, 1, \dots. \tag{129}$$

Then, the first term in the r.h.s. of (128) is equal to

$$\frac{(n+2-x)}{x^2 f_{n+1}(x)} \alpha_n(x,y) \alpha_n(x,z). \tag{130}$$

The second term can be also be expressed in terms of α_n's:

$$\begin{aligned} &\frac{1}{x^2} \sum_{m,n=0}^{n} (x - (m \vee n) - 1) \frac{f_m(x) f_n(x)}{f_{m\vee n}(x)} y^m z^n = \frac{1}{x^2} \sum_{s=0}^{n} (x-s-1) f_s(x) (yz)^s \\ &+ \frac{1}{x^2} \sum_{m>s\geq 0}^{n} (x-m-1) f_s(x) y^m z^s + \frac{1}{x^2} \sum_{0\leq m<s}^{n} (x-s-1) f_m(x) y^m z^s \\ &= \frac{1}{x^2} \left(\left(x - \omega \frac{\partial}{\partial \omega} - 1 \right) \alpha_n(x,\omega) \mid_{\omega=yz} + \psi_x(y,z) + \psi_x(z,y) \right), \end{aligned} \tag{131}$$

where

$$\psi_x(y,z) = \sum_{s>m\geq 0}^{n} (x-s-1) f_m(x) y^m z^s. \tag{132}$$

Next,

$$\begin{aligned} \psi_x(y,z) &= \left(x - z\frac{\partial}{\partial z} - 1 \right) \sum_{s>m\geq 0}^{n} f_m(x) y^m z^s \\ &= \left(x - z\frac{\partial}{\partial z} - 1 \right) \sum_{m=0}^{n} f_m(x) y^m \sum_{s=m+1}^{n} z^s \\ &= \left(x - z\frac{\partial}{\partial z} - 1 \right) \sum_{m=0}^{n} f_m(x) y^m \left(\frac{z^{n+1} - z^{m+1}}{z-1} \right) \\ &= \left(x - z\frac{\partial}{\partial z} - 1 \right) \frac{1}{(z-1)} \left(z^{n+1} \alpha_n(x,y) - z \alpha_n(x,yz) \right). \end{aligned} \tag{133}$$

Substituting (133) into (131) and then substituting the result and (130) into (128), we find that

$$\begin{aligned} G^{(n)}(x,y,z) &= \frac{(n+2-x)}{x^2 f_{n+1}(x)} \alpha_n(x,y) \alpha_n(x,z) + \frac{1}{x^2} \left(x - \omega \frac{\partial}{\partial \omega} - 1 \right) \alpha_n(x,\omega) \mid_{\omega=yz} \\ &\quad + \frac{1}{x^2} \left(x - z\frac{\partial}{\partial z} - 1 \right) \frac{z}{z-1} \left(z^n \alpha_n(x,y) - \alpha_n(x,yz) \right) \\ &\quad + \frac{1}{x^2} \left(x - y\frac{\partial}{\partial y} - 1 \right) \frac{y}{y-1} \left(y^n \alpha_n(x,z) - \alpha_n(x,yz) \right). \end{aligned} \tag{134}$$

To simplify the expression for $G^{(n)}$ further, we need an expression for α_n in terms of the exponential polynomials:

$$\begin{aligned}
\alpha_n(x,y) &= \sum_{s=0}^{n} f_n(x) y^s = \sum_{s=0}^{n} \left((s+1)e_s(x) - x e_{s-1}(x)\right) y^s \\
&= \left(\frac{\partial}{\partial y} y - xy\right) \sum_{s=0}^{n} e_s(x) y^s + x y^{n+1} e_N(x) \\
&= \left(\frac{\partial}{\partial y} y - xy\right) \frac{e_n(yx) - y^{n+1} e_n(x)}{1-y} + x y^{n+1} e_n(x).
\end{aligned} \tag{135}$$

Explicitly,

$$\alpha_n(x,y) = \frac{e_n(yx)}{(1-y)^2} - \frac{yx}{(1-y)} \frac{(yx)^n}{n!} - \left((n+2-x) - (n+1-x)y\right) \frac{y^{n+1} e_n(x)}{(1-y)^2}. \tag{136}$$

Substituting (136) into (134), computing the derivatives and grouping the terms according to the denominators we arrive at

$$x^2 G^{(n)}(x,y,z) = \frac{T_A^{(n)}(x,y,z)}{(1-y)^2(1-z)^2} + \frac{T_B^{(n)}(x,y,z)}{(1-y)(1-z)} + \frac{T_C^{(n)}(x,y,z)}{(1-y)^2(1-z)} + \frac{T_C^{(n)}(x,z,y)}{(1-z)^2(1-y)}, \tag{137}$$

where

$$\begin{aligned}
T_A^{(n)}(x,y,z) &= \frac{(n+2-x)}{f_{n+1}(x)} e_n(yx) e_n(zx) - e_n(xyz) \\
&\quad + \frac{(n+2)}{f_{n+1}(x)(n+1)!} \left((zx)^{n+1} e_n(yx) + (yx)^{n+1} e_n(zx) - (xyz)^{n+1} e_n(x)\right),
\end{aligned} \tag{138}$$

$$\begin{aligned}
T_B^{(n)}(x,y,z) &= \frac{(n+2-x)}{f_{n+1}(x)} \frac{(yx)^{n+1}}{n!} \frac{(zx)^{n+1}}{n!} + x e_n(xyz) + (n+1-x) \frac{(xyz)^{n+1}}{n!} \\
&\quad - \frac{(n+2)(n+1-x)}{f_{n+1}(x)(n+1)!} \\
&\quad \times \left(\frac{(zx)^{n+1}(yx)^{n+1}}{n!} + \frac{(yx)^{n+1}(zx)^{n+1}}{n!} + (n+1-x)(xyz)^{n+1} e_n(x) \right),
\end{aligned} \tag{139}$$

$$\begin{aligned}
T_C^{(n)}(x,y,z) = &-\frac{(n+2-x)}{f_{n+1}(x)} e_n(yx) \frac{(zx)^{n+1}}{n!} + \frac{(xyz)^{n+1}}{n!} + \frac{(n+2)}{f_{n+1}(x)(n+1)!} \\
&\times \left((n+1-x)(zx)^{n+1} e_n(yx) - \frac{(yx)^{n+1}(zx)^{n+1}}{n!} - (n+1-x) e_n(x)(xyz)^{n+1} \right).
\end{aligned} \tag{140}$$

A straightforward simplification of each of the T-terms gives:

$$\begin{aligned}
f_{n+1}(x) T_A^{(n)}(x,y,z) &= (n+2)\left(e_{n+1}(yx) e_{n+1}(zx) - e_{n+1}(xyz) e_{n+1}(x)\right) \\
&\quad - x\left(e_n(yx) e_n(zx) - e_n(xyz) e_n(x)\right),
\end{aligned} \tag{141}$$

$$\begin{aligned}
f_{n+1}(x) T_B^{(n)}(x,y,z) &= x \frac{(yx)^{n+1}(zx)^{n+1}}{n!(n+1)!} + x(n+1-x) e_n(x) \frac{(xyz)^{n+1}}{(n+1)!} \\
&\quad + x e_n(xyz) \left((n+2-x) e_n(x) + (n+2) \frac{x^{n+1}}{(n+1)!} \right),
\end{aligned} \tag{142}$$

$$f_{n+1}(x) T_C^{(n)}(x,y,z) = x \frac{(xyz)^{n+1}}{(n+1)!} e_n(x) - x e_n(yx) \frac{(zx)^{n+1}}{(n+1)!}. \tag{143}$$

Substituting (141), (142) and (143) into (137) we arrive at

$$\begin{aligned} x^2 f_{n+1}(x) G^{(n)}(x,y,z) &= \frac{(n+2)W_{n+1}(x,y,z) - xW_n(x,y,z)}{(1-y)^2(1-z)^2} \\ &+x\frac{(xyz)^{n+1}e_n(x) - (xz)^{n+1}e_n(xy)}{(n+1)!(1-y)^2(1-z)} \\ &+x\frac{(xyz)^{n+1}e_n(x) - (xy)^{n+1}e_n(xz)}{(n+1)!(1-z)^2(1-y)} \\ &-x\frac{(xyz)^{n+1}}{(n+1)!}\frac{e_{n+1}(x) + xe_n(x)}{(1-y)(1-z)}, \end{aligned} \tag{144}$$

where

$$W_n(x,y,z) := e_n(xy)e_n(xz) - (1 - x(1-y)(1-z))e_n(xyz)e_n(x),\ n \in \mathbb{N}. \tag{145}$$

This answer is already well-suited for the calculation of the kernel for the bulk and the edge scaling limits of the overlaps, but it still looks rather complicated. Fortunately, it can be re-written in a shorter form: Observing that $e_n(x) = e_{n+1}(x) - \frac{x^{n+1}}{(n+1)!}$, we find that

$$\begin{aligned} W_n(x,y,z) &= W_{n+1}(x,y,z) - e_{n+1}(xy)\frac{(xz)^{n+1}}{(n+1)!} - e_{n+1}(xz)\frac{(xy)^{n+1}}{(n+1)!} \\ &+\left(e_{n+1}(xyz)\frac{(x)^{n+1}}{(n+1)!} + e_{n+1}(x)\frac{(xyz)^{n+1}}{(n+1)!}\right)(1 - x(1-y)(1-z)) \\ &+x(1-y)(1-z)\frac{(yzx^2)^{n+1}}{((n+1)!)^2}. \end{aligned} \tag{146}$$

Expressing W_{n+1} and W_n in (144) in terms of W_{n+2} and W_{n+1}, using (146) one finds

$$\begin{aligned} x^2 f_{n+1}(x) G^{(n)}(x,y,z) &= \frac{(n+2)W_{n+2}(x,y,z) - xW_{n+1}(x,y,z)}{(1-y)^2(1-z)^2} \\ &-\frac{x}{(1-y)(1-z)}\frac{(xyz)^{n+2}}{(n+1)!}e_{n+2}(x). \end{aligned} \tag{147}$$

It is straightforward to check that

$$-x(1-y)(1-z)\frac{(xyz)^{n+2}}{(n+1)!}e_{n+2}(x) = (n+2)H_{n+2}(x,y,z) - xH_{n+1}(x,y,z), \tag{148}$$

where

$$H_n(x,y,z) := \frac{(1-y)(1-z)}{n!}\frac{x^{n+1}e_n(xyz) - (xyz)^{n+1}e_n(x)}{(yz-1)}. \tag{149}$$

It follows from (147, 148), that

$$x^2 f_{n+1}(x) G^{(n)}(x,y,z) = \frac{(n+2)\mathfrak{F}_{n+2}(x,y,z) - x\mathfrak{F}_{n+1}(x,y,z)}{(1-y)^2(1-z)^2}, \tag{150}$$

where

$$\mathfrak{F}_n(x,y,z) := W_n(x,y,z) + H_n(x,y,z), n\ \in \mathbb{N}, \tag{151}$$

is a function on $\mathbb{C}^3$ defined in (30). Finally, substituting (150) into (116) we arrive at the expression (34) for the reduced kernel. Theorem 1 is proved.

Remark. The final part of the proof following (144) is not very satisfying as it is both non-obvious and reliant on unexpected cancellations. A more direct route to (150) is to substitute the partition of unity $1 = \mathbb{1}_{m<n} + \mathbb{1}_{m\geq n}$ into the double sum in (120), represent the indicator functions as a contour integral,

$$\mathbb{1}_{m<n} = \oint \frac{dz}{2\pi i}\frac{z^{m-n}}{1-z},$$

and analyse the resulting expression for $G_N(x,y,z)$ as a sum of two contour integrals. The integrands of each of the integrals contain poles of order N as well as poles of order 1 and 2. It turns out that the contributions from the high order poles cancel, and the sum of contributions from the poles of low order gives (150). Even knowing the final answer beforehand, it took us close to 2 months to match the results. The alternative proof will be presented in detail in chapter 4.

3.5 Corollary 1

The proof is based on the following elementary remark: for any fixed $x \in \mathbb{C}$

$$\lim_{n\to\infty} e_n(x) = e^x. \tag{152}$$

Consequently,

$$\frac{f_n(x)}{n} = (1+n^{-1})e_n(x) - n^{-1}xe_{n-1}(x) \xrightarrow{n\uparrow\infty} e^x. \tag{153}$$

Therefore, the factor in front of the determinant in the r.h.s. of (31) divided by n converges for $|\lambda_1|^2 < n$ to

$$\lim_{n\to\infty} \frac{f_{n-1}(|\lambda_1|^2)}{n\pi} e^{-|\lambda_1|^2} = \frac{1}{\pi}, \tag{154}$$

as is well known from the Ginibre ensemble [24]. The large-N limit of the reduced kernel defined in (34) is most easily taken when fixing all arguments of the kernel, that is remaining in the vicinity of the origin, as the spectral edge is located at $\sqrt{N}$. The same bulk limit close to the origin was taken already in Ginibre's original paper for the complex eigenvalue correlations [17]. For our kernel we thus have

$$\begin{aligned}
\lim_{n\to\infty} \kappa^{(n)}(\bar{x}, y \mid \lambda, \bar{\lambda}) &= e^{-\lambda\bar{\lambda}} \frac{\lim_{n\to\infty} \mathfrak{F}_{n+1}\left(\lambda\bar{\lambda}, \frac{\bar{x}}{\bar{\lambda}}, \frac{y}{\lambda}\right)}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2} \\
&= \frac{e^{\bar{x}y}}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2}\left(e^{-(\bar{x}-\bar{\lambda})(y-\lambda)} - 1 + (\bar{x}-\bar{\lambda})(y-\lambda)\right) \\
&= \frac{e^{\bar{x}y}}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2} e^{(2)}(-(\bar{x}-\bar{\lambda})(y-\lambda)),
\end{aligned} \tag{155}$$

where $e^{(m)}(x) := \sum_{s=m}^{\infty} \frac{x^s}{s!}$, $m = 0, 1, \ldots$ In the second equality we used the definition (30) of the polynomials $\mathfrak{F}_n$. Therefore, the bulk scaling limit of the kernel $K_{11}^{(n)}$ defined in (32) is

$$\begin{aligned}
\lim_{n\to\infty} K_{11}^{(n)}(x,\bar{x},y,\bar{y} \mid \lambda,\bar{\lambda}) &= \frac{e^{-x\bar{x}+y\bar{x}}}{\pi}\left(1+(x-\lambda)(\bar{x}-\bar{\lambda})\right)\frac{e^{(2)}(-(\bar{x}-\bar{\lambda})(y-\lambda))}{(\bar{x}-\bar{\lambda})^2(y-\lambda)^2} \\
&= e^{(y-x)\bar{\lambda}}\frac{1}{\pi}(1+(x-\lambda)(\bar{x}-\bar{\lambda}))e^{-(x-\lambda)(\bar{x}-\bar{\lambda})} \\
&\quad \times \frac{d}{dz}\left(\frac{e^z-1}{z}\right)\Big|_{z=(\bar{x}-\bar{\lambda})(y-\lambda)}.
\end{aligned} \tag{156}$$

Notice that the factor $e^{(y-x)\bar{\lambda}}$ in the r.h.s. of (156) corresponds to the conjugation of the kernel $K(\lambda_i,\lambda_j) \to \phi(\lambda_i)K(\lambda_i,\lambda_j)\phi^{-1}(\lambda_j)$, which does not change the value of the determinant in the expression (31) for the conditional overlap. This remark allows us to write the bulk scaling limit of the kernel as

$$K_{11}^{(bulk)}(x,\bar{x},y,\bar{y} \mid \lambda,\bar{\lambda}) = \frac{1}{\pi}(1+(x-\lambda)(\bar{x}-\bar{\lambda}))e^{-(x-\lambda)(\bar{x}-\bar{\lambda})}\,\frac{d}{dz}\left(\frac{e^z-1}{z}\right)\Big|_{z=(\bar{x}-\bar{\lambda})(y-\lambda)}. \tag{157}$$

Expressions (154), (157) solve the problem of computing the large-n limit of (31), thus proving claims (41), (42), (43) and (44) of Corollary 1.

It remains to calculate the bulk scaling limit of $D_{12}^{(n,k)}$ starting with its determinantal representation (35). Using (155) and (156), one finds that the large-n limit of the pre-factor in (35) divided by n is

$$\lim_{n\to\infty} \frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{n\pi^2} f_{n-1}(\lambda_1\bar{\lambda}_2)\kappa^{(n-1)}(\bar{\lambda}_1,\lambda_2 \mid \lambda_1,\bar{\lambda}_2) = \frac{\kappa^{(bulk)}(\bar{\lambda}_1,\lambda_2 \mid \lambda_1,\bar{\lambda}_2)}{\pi^2}, \tag{158}$$

where $\kappa^{(bulk)}$ is defined in (44), and

$$\begin{aligned}\lim_{n\to\infty} K_{12}^{(n-1)}(x,\bar{x},y,\bar{y} \mid u,\bar{u},v,\bar{v}) &= e^{(y-x)\bar{v}} \frac{\omega^{(bulk)}(x,\bar{x} \mid u,\bar{v})}{\kappa^{(bulk)}(\bar{u},v \mid u,\bar{v})} \\ &\times \det\begin{pmatrix} \kappa^{(bulk)}(\bar{u},v \mid u,\bar{v}) & \kappa^{(bulk)}(\bar{u},y \mid u,\bar{v}) \\ \kappa^{(bulk)}(\bar{x},v \mid u,\bar{v}) & \kappa^{(bulk)}(\bar{x},y \mid u,\bar{v}) \end{pmatrix},\end{aligned} \tag{159}$$

where the weight $\omega^{(bulk)}$ is defined by (43). Calculating the large-n limit of (35) with the help of (158) and (159), and using the fact that conjugation of the kernel by $e^{\lambda_i\bar{\lambda}_2}$ does not change the determinant, we arrive at the characterisation (45), (46) for the bulk scaling limit of the off-diagonal conditional overlaps. Corollary 1 is proved.

3.6 Corollary 2

The calculation is based on the following two asymptotic formulae: Let us fix $a,b\in\mathbb{C}$. Then

$$\begin{aligned}\log\frac{(n+\sqrt{n}a+b)^{n+1}}{(n+1)!} &= (n+\sqrt{n}a+b) - \frac{1}{2}\log 2\pi n \\ &\quad -\frac{a^2}{2} + \frac{a-ab+\frac{1}{3}a^3}{\sqrt{n}} + O(n^{-1}),\end{aligned} \tag{160}$$

$$e_{n+k}(n+\sqrt{n}a+b) = e^{n+\sqrt{n}a+b}\left(F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi n}}\left(\frac{a^2}{3} - b + k + \frac{2}{3}\right) + O(n^{-1})\right), \tag{161}$$

where $k = 0,1,2,\ldots$. Here F is a rescaling on the complementary error function defined in (48). The derivation of the above formulae is based on Stirling's formula and the following well known integral representation of the exponential polynomials in terms of the incomplete Gamma-function:

$$e_n(x) = e^x\frac{\Gamma(n+1,x)}{\Gamma(n+1)} = \frac{e^x}{n!}\int_x^\infty t^n e^{-t}dt,\ n=0,1,2\ldots, \tag{162}$$

see [25, Chapter 8.11.10] for more details. The calculations leading to (160) and (161) are straightforward, but lengthy due to the fact that we need to know the asymptotic expansion of e_{n+k} and $\log\frac{(n+\sqrt{n}a+b)^{n+1}}{(n+1)!}$ up to and including the terms of order $n^{-1/2}$, see also [25]. My own computations of this are in 4.6 and are only included so these notes are complete.
As a consequence of (161),

$$\begin{aligned}f_{n+1}(n+\sqrt{N}a+b) &= \sqrt{\frac{n}{2\pi}}e^{n+\sqrt{n}a+b-\frac{a^2}{2}} \\ &\times\left(1-a\sqrt{2\pi}e^{\frac{a^2}{2}}F(a)\right)(1+O(n^{-\frac{1}{2}})), a,b\in\mathbb{C}.\end{aligned} \tag{163}$$

Now we are ready to calculate the edge scaling limit of conditional overlaps. Let

$$\begin{aligned}x^{(n)} &= e^{i\theta}(\sqrt{n}+x), \\ y^{(n)} &= e^{i\theta}(\sqrt{n}+y), \\ \lambda^{(n)} &= e^{i\theta}(\sqrt{n}+\lambda),\end{aligned} \tag{164}$$

where $x, y, \lambda \in \mathbb{C}$ are of order unity. Then the edge scaling limit of the factor multiplying the determinant in the r.h.s. of (31) down-scaled by $n^{-1/2}$ is

$$\lim_{n\to\infty} \frac{e^{-|\lambda^{(n)}|^2}}{\pi\sqrt{n}} f_{n-1}(|\lambda^{(n)}|^2) = \frac{1}{\sqrt{2\pi^3}} \left(e^{-\frac{1}{2}(\lambda+\bar{\lambda})^2} - \sqrt{2\pi}\left(\lambda+\bar{\lambda}\right) F(\lambda+\bar{\lambda}) \right),\ \lambda, \bar{\lambda} \in \mathbb{C}^2. \tag{165}$$

The derivation of (165) is based on (161). Note that (165) is valid for any pair of complex numbers $(\lambda, \bar{\lambda})$, not just on the real surface $\lambda = \bar{\bar{\lambda}}$, which makes it suitable for the calculation of both the diagonal and the off-diagonal overlaps.

To find the edge scaling limit of the kernel $K_{11}^{(n)}$ we substitute the expressions (164) into the formula for the kernel

$$\begin{aligned} K_{11}^{(n)}(x^{(n)}, \bar{x}^{(n)}, y^{(n)}, \bar{y}^{(n)} \mid \lambda^{(n)}, \bar{\lambda}^{(n)}) &= \frac{1 + (x^{(n)} - \lambda^{(n)})(\bar{x}^{(n)} - \bar{\lambda}^{(n)})}{\pi} e^{-x^{(n)}\bar{x}^{(n)}} \\ &\quad \times G^{(n-1)}\left(\lambda^{(n)}\bar{\lambda}^{(n)}, \frac{\bar{x}^{(n)}}{\bar{\lambda}^{(n)}}, \frac{y^{(n)}}{\lambda^{(n)}}\right), \end{aligned}$$

where $G^{(n)}$ is given by formula (147), and compute the large-n asymptotics using (160), (161) and (163). The result follows from another lengthy computation and is

$$\begin{aligned} K_{11}^{(n)}(x^{(n)}, \bar{x}^{(n)}, y^{(n)}, \bar{y}^{(n)} \mid \lambda^{(n)}, \bar{\lambda}^{(n)}) &= e^{\sqrt{n}(y-x)} \frac{1 + (x-\lambda)(\bar{x}-\bar{\lambda})}{\pi} e^{-x\bar{x}+\bar{x}y} \\ &\times \frac{H\left(\lambda+\bar{\lambda}, \lambda+\bar{x}, \bar{\lambda}+y, y+\bar{x}, (\lambda-y)(\bar{\lambda}-\bar{x})\right)}{(\lambda-y)^2(\bar{\lambda}-\bar{x})^2} \left(1 + O(n^{-1/2})\right), \end{aligned} \tag{166}$$

where the function H is defined in (49). We see that there the large-n limit of $K_{11}^{(n)}$ at the edge does not exist, but fortunately, the residual N-dependence can be eliminated by the N-dependent conjugation

$$K(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) \to e^{\sqrt{n}x} K(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) e^{-\sqrt{n}y}, \tag{167}$$

which does not change the value of the conditional overlap. Therefore we can conclude that

$$\begin{aligned} K_{11}^{(edge)}(x, \bar{x}, y, \bar{y} \mid \lambda, \bar{\lambda}) &:= \lim_{n\to\infty} e^{\sqrt{n}x} K_{11}^{(n)}(x^{(n)}, \bar{x}^{(n)}, y^{(n)}, \bar{y}^{(n)} \mid \lambda^{(n)}, \bar{\lambda}^{(n)}) e^{-\sqrt{n}y} \\ &= \frac{1 + (x-\lambda)(\bar{x}-\bar{\lambda})}{\pi} e^{-x\bar{x}+\bar{x}y} \frac{H\left(\lambda+\bar{\lambda}, \lambda+\bar{x}, \bar{\lambda}+y, y+\bar{x}, (\lambda-y)(\bar{\lambda}-\bar{x})\right)}{(\lambda-y)^2(\bar{\lambda}-\bar{x})^2}. \end{aligned} \tag{168}$$

Substituting (165) and (168) into the edge scaling limit (39) of the conditional overlap $D_{11}^{(n,k)}$ we arrive at the statement (50), (51), (52) and (53) of the Corollary 2.

To find the edge scaling limit of the off-diagonal overlap $D_{12}^{(n,k)}$, we need to substitute its expression (35) into (40) and calculate the large-n limit of the resulting sequence. As before, the calculation reduces to the evaluation of the scaling limits of the pre-factor in the r.h.s. of (35) and the kernel (36).

A straightforward computation based on (163, 168) gives

$$\begin{aligned} &\lim_{N\to\infty} (-1) \frac{e^{-|\lambda_1^{(n)}|^2 - |\lambda_2^{(n)}|^2}}{\pi^2\sqrt{n}} f_{n-1}(\lambda_1^{(n)}\bar{\lambda}_2^{(n)}) \kappa^{(n-1)}(\bar{\lambda}_1^{(n)}, \lambda_2^{(n)} \mid \lambda_1^{(n)}, \bar{\lambda}_2^{(n)}) \\ &= -\frac{e^{-|\lambda_{12}|^2 - \frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2}}{\sqrt{2\pi^5}\lambda_{12}^2\bar{\lambda}_{12}^2} \left(1 - \sqrt{2\pi}(\lambda_1+\bar{\lambda}_2) e^{\frac{1}{2}(\lambda_1+\bar{\lambda}_2)^2} F(\lambda_1+\bar{\lambda}_2)\right) \\ &\times H(\lambda_1+\bar{\lambda}_2, \lambda_1+\bar{\lambda}_1, \lambda_2+\bar{\lambda}_2, \lambda_2+\bar{\lambda}_1, -\lambda_{12}\bar{\lambda}_{12}). \end{aligned} \tag{169}$$

Similarly, introducing in addition to (164), $u^{(n)} = e^{i\theta}(\sqrt{n}+u)$, $v^{(n)} = e^{i\theta}(\sqrt{n}+v)$, we find that

$$\begin{aligned} &K_{12}^{(n)}(x^{(n)}, \bar{x}^{(n)}, y^{(n)}, \bar{y}^{(n)} \mid u^{(n)}, \bar{u}^{(n)}, v^{(n)}, \bar{v}^{(n)}) \\ &= e^{\sqrt{n}(y-x)} K_{12}^{(edge)}(x, \bar{x}, y, \bar{y} \mid u, \bar{u}, v, \bar{v}) \left(1 + O(n^{-1/2})\right), \end{aligned} \tag{170}$$

where $K_{12}^{(edge)}$ is given by (55). Substituting (169) and (170) into the right hand side of (40), performing the n-dependent conjugation (167) of the kernel inside the resulting determinant and computing the $n \to \infty$ limit, we arrive at the statements (54), (55) of the Corollary 2. Corollary 2 is proved.

3.7 Corollary 3

It follows from (41), (45), (50) and (54) that

$$D_{11}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{11}^{(bulk,\ 1)}(\lambda_1)\det_{2\le i,j\le k}(K_{11}^{(bulk)}(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_1)),\ k\ge 1, \tag{171}$$

$$D_{12}^{(bulk,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{12}^{(bulk,\ 2)}(\lambda_1,\lambda_2)\det_{3\le i,j\le k}(K_{12}^{(bulk)}(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2)),\quad k\ge 2, \tag{172}$$

$$D_{11}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{11}^{(edge,\ 1)}(\lambda_1)\det_{2\le i,j\le k}(K_{11}^{(edge)}(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_1)),\ k\ge 1, \tag{173}$$

$$D_{12}^{(edge,\ k)}(\boldsymbol{\lambda}^{(k)}) = D_{12}^{(edge,\ 2)}(\lambda_1,\lambda_2)\det_{3\le i,j\le k}(K_{12}^{(edge)}(\lambda_i,\bar\lambda_i,\lambda_j,\bar\lambda_j\mid\lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2)),\quad k\ge 2, \tag{174}$$

where the expressions for all the relevant kernels are given in Corollaries 1 and 2. We see that the ratios

$$\frac{D_{11}^{(edge,\ k)}}{D_{11}^{(edge,\ 1)}},\ \frac{D_{12}^{(edge,\ k)}}{D_{12}^{(edge,\ 2)}},\ \frac{D_{11}^{(bulk,\ k)}}{D_{11}^{(bulk,\ 1)}}\ \text{and}\ \frac{D_{12}^{(bulk,\ k)}}{D_{12}^{(bulk,\ 2)}} \tag{175}$$

are completely determined by the conjugacy classes of the corresponding kernels. Therefore, the claim of the Corollary 3 is an immediate consequence of the following relations between the kernels:

$$\lim_{R\to-\infty} e^{(R+\bar\lambda)x}K_{11}^{(edge)}(R+x,R+\bar x,R+y,R+\bar y\mid R+\lambda_1,R+\bar\lambda_1)e^{-(R+\bar\lambda)y} = K_{11}^{(bulk)}(x,\bar x,y,\bar y\mid\lambda_1,\bar\lambda_1), \tag{176}$$

$$\lim_{R\to-\infty} K_{12}^{(edge)}(R+x,R+\bar x,R+y,R+\bar y\mid R+\lambda_1,R+\bar\lambda_1,R+\lambda_2,R+\bar\lambda_2) = K_{12}^{(bulk)}(x,\bar x,y,\bar y\mid\lambda_1,\bar\lambda_1,\lambda_2,\bar\lambda_2), \tag{177}$$

Both (176) and (177) can be derived from the following asymptotic formula for the H-function (49):

$$H(-\epsilon^{-1}+a,-\epsilon^{-1}+b,-\epsilon^{-1}+c,-\epsilon^{-1}+d,f) = \left(e^{-f}-1+f\right)(1+O(\epsilon)), \tag{178}$$

where $\epsilon>0,(a,b,c,d,f)\in\mathbb{C}^5$. This formula follows directly from the standard asymptotic for the complementary error function (48): for $a<0$,

$$F(a)=1+O\left(e^{-a^2/2}\right),\ F'(a)=O\left(e^{-a^2/2}\right). \tag{179}$$

The proof of (176):

$$\begin{aligned}
&\lim_{R\to-\infty} e^{(R+\bar\lambda)x}K_{11}^{(edge)}(R+x,R+\bar x,R+y,R+\bar y\mid R+\lambda,R+\bar\lambda)e^{-(R+\bar\lambda)y}\\
&= -\lim_{R\to-\infty} e^{\bar\lambda x}\left(\frac{1+(x-\lambda)(\bar x-\bar\lambda)}{\pi z^2}e^{\bar x(y-x)}H\left(2R+a,2R+b,2R+c,2R+d,z\right)\big|_{z=(\bar x-\bar\lambda)(y-\lambda)}\right)e^{-\bar\lambda y}\\
&= \frac{1+(x-\lambda)(\bar x-\bar\lambda)}{\pi}e^{-|x-\lambda|^2}\frac{1-(1-z)e^z}{z^2}\bigg|_{z=(\bar x-\bar\lambda)(y-\lambda)} = K_{11}^{(bulk)}(x,\bar x,y,\bar y\mid\lambda,\bar\lambda).
\end{aligned}$$

To prove (177), let us first notice that (178) leads to

$$\kappa^{(edge)}(R+\bar x,R+y\mid R+\lambda,R+\bar\lambda) = e^{R^2+R(\bar x+y)-|\lambda|^2+\bar\lambda y+\lambda\bar x}\kappa^{(bulk)}(\bar x,y\mid\lambda,\bar\lambda)\left(1+O(R^{-1})\right), \tag{180}$$

where $R<0$. Also,

$$\begin{aligned}
\omega^{(edge)}(R+x,R+\bar x\mid R+\lambda,R+\bar\lambda) &= \frac{1+(x-\lambda)(y-\bar\lambda)}{\pi}e^{-|R+x|^2}\\
&= e^{-R^2-R(x+\bar x)+\lambda\bar\lambda-\bar x\lambda-x\bar\lambda}\omega^{(bulk)}(x,\bar x\mid\lambda,\bar\lambda).
\end{aligned} \tag{181}$$

Notice that the last two relations are still valid if $\kappa^{(edge)}$ and $\omega^{(edge)}$ are treated as functions on $\mathbb{C}^4$. Substituting (180), (181) into (55) we find

$$\begin{aligned} &K_{12}^{(edge)}(R+x,R+\bar{x},R+y,R+\bar{y}\mid R+\lambda_1,R+\bar{\lambda}_1,R+\lambda_2,R+\bar{\lambda}_2)\\ &\quad= e^{(R+\bar{\lambda}_2)(y-x)}K_{12}^{(bulk)}(x,\bar{x},y,\bar{y}\mid\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2)\left(1+O(R^{-1})\right). \end{aligned} \tag{182}$$

Therefore

$$\begin{aligned} \lim_{R\to-\infty} e^{(R+\bar{\lambda}_2)x}K_{12}^{(edge)}(R+x,R+\bar{x},R+y,R+\bar{y}\mid R+\lambda_1,R+\bar{\lambda}_1,R+\lambda_2,R+\bar{\lambda}_2)e^{-(R+\bar{\lambda}_2)y}&\\ = K_{12}^{(bulk)}(x,\bar{x},y,\bar{y}\mid\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2).& \end{aligned}$$

Equation (177) is established and therefore the Corollary 3 is proved.

4 Alternative Approaches and Proofs

In this section we are going to go through different approaches and alternate proofs to some of the results, and with greater detail. It will be a closer look in how we first derived said results. The main focus will be on Theorem 1. To be more precise, the focus will be on the reduced kernel (34) in particular.

We start with my computation of the weights in (24) and (25), using Eynard's method, see [14]. I will go through the computation of (16) and (17) found in 2.1, for $k = n$, which corresponds to the expectations of the overlaps conditioned on all eigenvalues.

Then I start slowly working on the orthogonal polynomials. I begin with the formula derivations for $D_{11}^{(n,1)}(\lambda)$ and $D_{12}^{(n,2)}(\lambda,\mu)$ with $\mu = 0$. This includes the computation of a few tri-diagonal determinants, some of which although not directly related, will be of future use. Having the aforementioned formulas, I will then use a lemma found in Gernot Akemann's personal notes to re-derive the bi-orthogonal polynomials $P_\kappa(z|\lambda,\bar{\lambda})$, $Q_\kappa(z|\lambda,\bar{\lambda})$ and the reduced kernel $\kappa^{(n+1)}(\bar{\mu},\nu|\lambda,\bar{\lambda})$ as presented in (116) - (118). From there I will present a different approach to the simplification of the reduced kernel in 3.4.4, in a lengthy calculation which includes contour integrals, but does not rely on knowing the final answer, nor depends on unexpected cancellations.

Final part is just a very detailed and direct computation of the limits in Corollary 1 and 2.

4.1 Overlaps conditioned on all eigenvalues

Eynard [14] found a relation between unitary and triangular integrals of invariant functions, and then proceeded to calculate such triangular ones y integrating out the elements $u_1, u_2, ..., u_{n-1}$ of the right column, reducing the size of the matrix by one and resulting in a recursion formula.
The functions F he works with have a certain 2-variable polynomial structure.
To be more precise they are of the following form.

$$\prod_k \Big(1 + tr\prod_l (x_{i_{k,l}} - A)^{-1}(y_{j_{k,l}} - B)^{-1}\Big) tr\prod_l (x_{i_{k,l}} - A)^{-1}(y_{j_{k,l}} - B)^{-1}.$$

Such examples would be:

$$F(A,B) = (1 + tr(x_1 - A)^{-1}(y_1 - B)^{-1})(1 + tr(x_2 - A)^{-1}(y_2 - B)^{-1})$$

or

$$F(A,B) = tr\big[(x_1 - A)^{-1}(y_1 - B)^{-1}(x_2 - A)^{-1}(y_2 - B)^{-1}\big].$$

Our cases are of a simpler nature, so we will only borrow the idea of integrating out a column at a time and finding a recursion formula.

We start with this small lemma:

Lemma 1.2

Let A and B be two $n \times n$ random matrices and let c be an $n \times 1$ column vector with iid complex Gaussians $N(0,t)$, independent of A and B.
Then the following equations hold:

i) $E(c^* Ac) = 2tE(trA)$

ii) $E(c^* Ac\, trB) = 2tE(trA\, trB)$

iii) $E[tr(Acc^* B)] = 2tE(trAB)$

iv) $E[(c^*Ac)(c^*Bc)] = 4t^2E(trAB) + 4t^2E(trA\,trB)$

Proof

i) For $i \neq j$ we have $E[\bar{c}_i(A)_{ij}c_j] = 0$

$$\begin{aligned}\Rightarrow E(c^*Ac) &= E\Big[\sum_{i,j}\bar{c}_i(A)_{ij}c_j\Big] = E\Big[\sum_i \bar{c}_i(A)_{ii}c_i\Big] \\ &= E\Big[\sum_i (A)_{ii}|c_i|^2\Big] = 2tE\Big[\sum_i (A)_{ii}\Big] = 2tE(trA)\end{aligned}$$

ii) Similarly with $i)$ we have

$$\begin{aligned}E(c^*Ac\,trB) &= E\Big[trB\sum_{i,j}\bar{c}_i(A)_{ij}c_j\Big] = E\Big[trB\sum_i (A)_{ii}|c_i|^2\Big] \\ &= 2tE(trA\,trB)\end{aligned}$$

iii)

$$\begin{aligned}E[tr(Acc^*B)] &= E[tr(BAcc^*)] = E\Big[\sum_{i,j}(BA)_{ij}c_j\bar{c}_i\Big] \\ &= E\Big[\sum_i (BA)_{ii}|c_i|^2\Big] = 2tE(trBA) = 2tE(trAB)\end{aligned}$$

iv)

$c_k = x_k + iy_k$ where x_k and y_k are independent real gaussians $N(0,t)$.
First note that even though $E({c_k}^2)$ is not zero, $E({\bar{c}_k}^2) = 0$. Also

$$\begin{aligned}E(|c_k|^4) &= E[(x^2+y^2)^2] \\ &= E(x^4) + E(y^4) + 2E(x^2)E(y^2) \\ &= 8t^2\end{aligned}$$

$$E[(c^*Ac)(c^*Bc)] = E\Big[\Big(\sum_{i,j}\bar{c}_i(A)_{ij}c_j\Big)\Big(\sum_{k,j}\bar{c}_k(B)_{kl}c_l\Big)\Big]$$

The only terms with non zero mean are those where
$i = j = k = l$, $i = j$ and $k = l$ or $(i,j) = (l,k)$

$$\begin{aligned}\Rightarrow E[(c^*Ac)(c^*Bc)] &= E\Big[\sum_i |c_i|^4(A)_{ii}(B)_{ii}\Big] + E\Big[\sum_{\substack{i,j\\ i\neq j}} |\bar{c}_i|^2|c_j|^2(A)_{ij}(B)_{ji}\Big] \\ &\quad + E\Big[\sum_{\substack{i,j\\ i\neq j}} |\bar{c}_i|^2|c_j|^2(A)_{ii}(B)_{jj}\Big] \\ &= 8t^2E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + 4t^2E\Big[\sum_{\substack{i,j\\ i\neq j}}(A)_{ij}(B)_{ji}\Big] \\ &\quad + 4t^2E\Big[\sum_{\substack{i,j\\ i\neq j}}(A)_{ii}(B)_{jj}\Big] \\ &= 4t^2E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + 4t^2E\Big[\sum_{\substack{i,j\\ i\neq j}}(A)_{ij}(B)_{ji}\Big] \\ &\quad + 4t^2E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + 4t^2E\Big[\sum_{\substack{i,j\\ i\neq j}}(A)_{ii}(B)_{jj}\Big] \\ &= 4t^2E[tr(AB)] + 4t^2E(trA\,trB)\end{aligned}$$

□

Remark 1: Part i) actually implies part ii), but I kept it as it commonly appears in this form in the calculations. Also note that for $\text{Gin}(n,\mathbb{C}), t = \frac{1}{2}$.

Remark 2: This is an example of the real case being potentially harder. If you look at the analogous real version of this lemma, the first 3 parts remain more or less the same, with the only difference being that the coefficients will be just t instead of $2t$. But the fourth part is quite different. First of all we have $E(|c_i|^2) = t$ and $E(|c_i|^4) = 3t^2$, but that will not actually interferer with the splitting of terms as we did in the complex case.

Note that not only the terms with $(i,j) = (l,k)$, but also the ones with $(i,j) = (k,l)$ will survive, hence

$$\begin{aligned}
E[(c^*Ac)(c^*Bc)] &= E\Big[\sum_i |c_i|^4 (A)_{ii}(B)_{ii}\Big] + E\Big[\sum_{\substack{i,j\\ i\neq j}} |c_i|^2|c_j|^2 (A)_{ij}(B)_{ji}\Big] \\
&\quad + E\Big[\sum_{\substack{i,j\\ i\neq j}} |c_i|^2|c_j|^2 (A)_{ii}(B)_{jj}\Big] + E\Big[\sum_{\substack{i,j\\ i\neq j}} |c_i|^2|c_j|^2 (A)_{ij}(B)_{ij}\Big] \\
&= 3t^2 E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + t^2 E\Big[\sum_{\substack{i,j\\ i\neq j}} (A)_{ij}(B)_{ji}\Big] \\
&\quad + t^2 E\Big[\sum_{\substack{i,j\\ i\neq j}} (A)_{ii}(B)_{jj}\Big] + t^2 E\Big[\sum_{\substack{i,j\\ i\neq j}} (A)_{ij}(B)_{ij}\Big]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow E[(c^*Ac)(c^*Bc)] &= t^2 E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + t^2 E\Big[\sum_{\substack{i,j\\ i\neq j}} (A)_{ij}(B)_{ji}\Big] \\
&\quad + t^2 E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + t^2 E\Big[\sum_{\substack{i,j\\ i\neq j}} (A)_{ii}(B)_{jj}\Big] \\
&\quad + t^2 E\Big[\sum_i (A)_{ii}(B)_{ii}\Big] + t^2 E\Big[\sum_{\substack{i,j\\ i\neq j}} (A)_{ij}(B)_{ij}\Big] \\
&= t^2 E[tr(AB)] + t^2 E[tr(A)tr(B)] + t^2 E[tr(AB^T)]
\end{aligned}$$

□

We are now in position to start working on (16) and (17) from in 2.1, for $k = n$. In order to find expressions for O_{11} and O_{12}, and show how we derived the expressions (24) and (25), we follow the footsteps of [11, p. 9-12], alongside the method used in [24, A.35] to compute the joined probability density of eigenvalues of a $\text{Gin}(n,\mathbb{C})$ matrix. These are results previously found by Chalker and Mehlig, and as I've mentioned before, the computations below alongside Lemma 1.2, are directly taken from my MsC, where they were originally included.

1) Off-diagonal Overlaps

We start with the off diagonal case, which the more 'complex' of the two. We first have to find the formula for $O_{12} =< l_1, l_2 >< r_1, r_2 >$ using Eyanrd method. For notational convenience, in this subsection we work with an $(n+2)\times(n+2)$ matrix with eigenvalues $(\lambda_1, \lambda_2, \sigma_1, ..., \sigma_n)$. Also we use $\mathbb{E}^{ev}$ to denote the expectation while conditioning over all eigenvalues. We use a two step Shur decomposition. For the right eigenvectors we have:

$$\begin{pmatrix} M & z_1 & z_2 \\ & \lambda_1 & c \\ 0 & & \lambda_2 \end{pmatrix} r_1 = \lambda_1 r_1 \qquad \text{and} \qquad \begin{pmatrix} M & z_1 & z_2 \\ & \lambda_1 & c \\ 0 & & \lambda_2 \end{pmatrix} r_2 = \lambda_2 r_2$$

where $M \sim \mathrm{Gin}(n, \mathbb{C})$ with eigenvalues $(\sigma_1, ..., \sigma_n)$ and z_1, z_2 and vectors of length n, of iid complex Gaussians.

Right away we can see that

$$r_1 = \begin{pmatrix} \underline{v_1} \\ b \\ 0 \end{pmatrix} \quad , \quad r_2 = \begin{pmatrix} \underline{v_2} \\ a \\ 1 \end{pmatrix}$$

where we can also take $b = 1$, $a \in \mathbb{C}$ and u_1 and u_2 are vectors of length n. Inserting them in the above equations we get

$$\begin{aligned} M\underline{v_2} + z_1 a + z_2 &= \lambda_2 \underline{v_2} \\ \lambda_1 a + c &= \lambda_2 a \Rightarrow a = \frac{-c}{\lambda_1 - \lambda_2} \\ M\underline{v_1} + z_1 &= \lambda_1 \underline{v_1} \end{aligned}$$

Solving for v_1 and v_2 we have

$$\begin{aligned} \underline{v_1} &= -(M - \lambda_1)^{-1} z_1 \\ \underline{v_2} &= -(M - \lambda_2)^{-1} \Big(\frac{-z_1 c}{\lambda_1 - \lambda_2} + z_2 \Big) \end{aligned}$$

As for the left eigenvectors

$$l_1 \begin{pmatrix} M & z_1 & z_2 \\ & \lambda_1 & c \\ 0 & & \lambda_2 \end{pmatrix} = \lambda_1 l_1 \qquad \text{and} \qquad l_2 \begin{pmatrix} M & z_1 & z_2 \\ & \lambda_1 & c \\ 0 & & \lambda_2 \end{pmatrix} = \lambda_2 l_2$$

We can take $l_2 = (0, 0, ..., 1)$ and $l_1 = (\underline{w}, d_1, d_2)$. For l_1 we get the equations

$$\begin{aligned} \underline{w} M &= \lambda_1 \underline{w} \quad (i) \\ \underline{w} z_1 + \lambda_1 d_1 &= \lambda_1 d_1 \quad (ii) \\ \underline{w} z_2 + c d_1 + \lambda_2 d_2 &= \lambda_1 d_2 \quad (iii) \end{aligned}$$

From (i), if $\underline{w}$ is not $\underline{0}$, then the original matrix will have a non-simple spectrum.

$$\Rightarrow P(\underline{w} = \underline{0}) = 1$$

With $\underline{w} = \underline{0}$, (ii) implies we can take $d_1 = 1$.

Finally (iii) gives $d_2 = \dfrac{c}{\lambda_1 - \lambda_2}$.

There are also some normalization constants which will go away in the final expression.
We are going to compute $< l_2, l_1 >< r_2, r_1 >$.

- $$< l_2, l_1 >= \overline{\Big(\frac{c}{\lambda_1 - \lambda_2}\Big)}$$

- $$< r_2, r_1 >= (M-\lambda_2)^{-1}\Big(\frac{-z_1 c}{\lambda_1 - \lambda_2} + z_2\Big){z_1}^*(M-\lambda_1)^{-*} - \frac{c}{\lambda_1 - \lambda_2}$$

$$\Rightarrow O_{12} = -(M-\lambda_2)^{-1}\frac{z_1 {z_1}^* |c|^2}{|\lambda_1 - \lambda_2|^2}(M-\lambda_1)^{-*} - \frac{|c|^2}{|\lambda_1 - \lambda_2|^2} + (M-\lambda_2)^{-1}\frac{-z_2 {z_1}^* \bar{c}^2}{|\lambda_1 - \lambda_2|^2}(M-\lambda_1)^{-*}$$

where $^{-*}$ denotes inversion and Hermitian conjugation together.
Taking expectations conditioned on all eigenvalues we get

$$\mathbb{E}^{ev}(O_{12}) = -\frac{\mathbb{E}^{ev}[tr(M-\lambda_1)^{-1}(M-\lambda_2)^{-*}]}{|\lambda_1 - \lambda_2|^2} - \frac{1}{|\lambda_1 - \lambda_2|^2}$$

Writing $M = ULU^*$ in Schur's form, we take ${R_n}^{-1}$ and ${U_n}^{-1}$ to be $(L-\lambda_1)$ and $(L-\lambda_2)$ respectively. Then, if $\rho_i = \lambda_1 - \sigma_i$ and $u_i = \lambda_2 - \sigma_i$, we have that

$$R_n^{-1} = \begin{pmatrix} R_{n-1}^{-1} & c \\ 0 & \rho_n \end{pmatrix} \Rightarrow R_n = \begin{pmatrix} R_{n-1} & \frac{1}{\rho_n} R_{n-1} c \\ 0 & \frac{1}{\rho_n} \end{pmatrix}$$

and

$${U_n}^{-1} = \begin{pmatrix} U_{n-1}^{-1} & c \\ 0 & u_n \end{pmatrix} \Rightarrow U_n \quad = \quad \begin{pmatrix} U_{n-1} & \frac{1}{u_n} U_{n-1} c \\ 0 & \frac{1}{u_n} \end{pmatrix}$$

$$\Rightarrow U_n^* \quad = \quad \begin{pmatrix} U_{n-1}^* & 0 \\ \frac{1}{\bar{u}_n} c^* U_{n-1}^* & \frac{1}{\bar{u}_n} \end{pmatrix}$$

We dub $W_n^{(2)} = \mathbb{E}^{ev}[tr(U_n^* R_n)]$, take on both sides conditional expectations over all eigenvalues and proceed using Eynard's way

$$\begin{aligned}
tr(U_n^* R_n) &= tr(U_{n-1}^* R_{n-1}) + \frac{c^* U_{n-1}^* R_{n-1} c}{\rho_n \bar{u}_n} + \frac{1}{\rho_n \bar{u}_n} \\
\Rightarrow W_n^{(2)} = \mathbb{E}^{ev}[tr(U_n^* R_n)] &= W_{n-1}^{(2)} + \frac{1}{\rho_n \bar{u}_n} W_{n-1}^{(2)} + \frac{1}{\rho_n \bar{u}_n} \\
\Rightarrow W_n^{(2)} + 1 &= (W_{n-1}^{(2)} + 1)\Big(1 + \frac{1}{\rho_n \bar{u}_n}\Big) \\
\Rightarrow W_n^{(2)} + 1 &= \prod_{i=1}^{n}\Big(1 + \frac{1}{\rho_i \bar{u}_i}\Big) = \prod_{i=1}^{n}\Big(1 + \frac{1}{(\lambda_1 - \sigma_i)(\bar{\lambda}_2 - \bar{\sigma}_i)}\Big)
\end{aligned}$$

$$\Rightarrow \mathbb{E}_{n+2}(O_{12} \mid \Lambda_1 = \lambda_1, \dots, \Lambda_{n+2} = \sigma_n) = \frac{1}{|\lambda_1 - \lambda_2|^2}\prod_{i=1}^{n}\Big(1 + \frac{1}{(\lambda_1 - \sigma_i)(\bar{\lambda}_2 - \bar{\sigma}_i)}\Big)$$

2) Diagonal Overlaps

Similarly to the case above (note there will be no matrix U_n), except for starting with an $(n+1)\times(n+1)$ matrix, we compute

$$\mathbb{E}^{ev}(O_{11}) = \mathbb{E}^{ev}[tr(R_n{}^* R_n)] - 1$$

where

$$R_n{}^* = \begin{pmatrix} R_{n-1}{}^* & 0^T \\ \frac{1}{\bar\rho_n} c^* R^*_{n-1} & \frac{1}{\bar\rho_n} \end{pmatrix}$$

and

$$R_n{}^* R_n = \begin{pmatrix} R_{n-1}{}^* R_{n-1} & \frac{1}{\rho_n} R_{n-1}{}^* R_{n-1} c \\ \frac{1}{\bar\rho_n} c^* R_{n-1}{}^* R_{n-1} & \frac{1}{|\rho_n|^2}(c^* R_{n-1}{}^* R_{n-1} c + 1) \end{pmatrix}$$

We dub $W_n^{(1)} = \mathbb{E}^{ev}(tr(R_n{}^* R_n))$, take expectations and once again solve the resulting recursion.

$$\begin{aligned}
W_n^{(1)} = \mathbb{E}^{ev}(tr(R_n{}^* R_n)) &= \mathbb{E}^{ev}(tr(R_{n-1}{}^* R_{n-1}) + \frac{1}{|\rho_n|^2} + \frac{1}{|\rho_n|^2} E(c^* R_{n-1}{}^* R_{n-1} c) \\
&= W_{n-1}^{(1)} + \frac{1}{|\rho_n|^2} + \frac{1}{|\rho_n|^2} E(tr(R_{n-1}{}^* R_{n-1}) \\
&= W_{n-1}^{(1)}\Big(1 + \frac{1}{|\rho_n|^2}\Big) + \frac{1}{|\rho_n|^2} \\
\Rightarrow W_n^{(1)} + 1 &= \Big(W_{n-1}^{(1)} + 1\Big)\Big(1 + \frac{1}{|\rho_n|^2}\Big)
\end{aligned}$$

$$\Rightarrow \mathbb{E}_{n+1}(O_{11} \mid \Lambda_1 = \lambda_1, \Lambda_2 = \sigma_1, \dots \Lambda_{n+1} = \sigma_n) = \prod_{i=1}^n \Big(1 + \frac{1}{|\lambda_1 - \sigma_i|^2}\Big)$$

For the sake of brevity we will denote $\prod_{i=1}^n \Big(1 + \frac{1}{|\lambda_1 - \sigma_i|^2}\Big)$ with $W_n^{(1)}$, instead of $W_n^{(1)} + 1$.

4.2 $D_{11}^{(n,1)}(\lambda)$

Here we will do nothing other than work with $D_{11}^{(n,1)}(\lambda)$. This was the first step, it's the simplest form of Theorem 1 and corresponds to what Chalker and Mehlig did in the diagonal case. As of yet, there is no reason for orthogonal polynomials, since it all comes down to the determinant of a tri-diagonal matrix. Such a calculation though becomes nigh impossible going forward, ie for $D_{11}^{(n,2)}(\lambda, \sigma)$, since even the determinant of a 5-diagonal matrix is not a reasonable calculation.

To be exact, because I find it easier to keep track of the notation this way, we will be working with $D_{11}^{(n+1,1)}(\lambda)$. So let $M \sim \text{Gin}(n, \mathbb{C})$ be an $(n+1)\times(n+1)$ matrix with eigenvalues $\lambda, \lambda_1, ..., \lambda_n$. We start with the weight

$$W_n^{(1)} = \prod_{i=1}^n \Big(1 + \frac{1}{|\lambda - \lambda_i|^2}\Big).$$

This is how we first derived ourselves the simplest case of Theorem 1.

$$D_{11}^{(n+1,1)}(\lambda) := \mathbb{E}_n(O_{11}|\lambda)\rho^{(n+1,1)}(\lambda) = \frac{n+1}{Z_{n+1}} e^{-|\lambda|^2} \int_{\mathbb{C}_n} e^{-\sum_{i=1}^n |\lambda_i|^2} |\Delta^{(n+1)}(\lambda, \boldsymbol{\lambda}^{(n)})|^2 \prod_{i=1}^n \Big(1 + \frac{1}{|\lambda - \lambda_i|^2}\Big) \prod_{i=1}^n d\lambda_i d\bar\lambda_i$$

where we average over all $\lambda_1, ...\lambda_n$ and Z_{n+1} is the normalisation factor with

$$Z_{n+1} = \prod_{j=1}^{n+1} j!\pi^{n+1}.$$

By taking separately all the terms involving λ, we rewrite the Vandermonde as follows:

$$\begin{aligned} |\Delta^{(n+1)}(\lambda, \boldsymbol{\lambda}^{(n)})|^2 &= \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_i |\lambda - \lambda_i|^2 \\ &= |\Delta^{(n)}(\boldsymbol{\lambda}^{(n)})|^2 \prod_i |\lambda - \lambda_i|^2 \end{aligned}$$

and

$$\begin{aligned} |\Delta^{(n)}(\boldsymbol{\lambda}^{(n)})|^2 &= \prod_{i<j}(\lambda_i - \lambda_j) \prod_{i<j}(\bar{\lambda}_i - \bar{\lambda}_j) \\ &= \det \begin{vmatrix} 1 & ... & 1 \\ \lambda_1 & ... & \lambda_n \\ ... & ... & ... \\ \lambda_1^{n-1} & ... & \lambda_n^{n-1} \end{vmatrix} \times \det \begin{vmatrix} 1 & ... & 1 \\ \bar{\lambda}_1 & ... & \bar{\lambda}_n \\ ... & ... & ... \\ \bar{\lambda}_1^{n-1} & ... & \bar{\lambda}_n^{n-1} \end{vmatrix}^T \\ &= \det \begin{vmatrix} n & \sum_i \lambda_i & ... & \sum_i \lambda_i^{n-1} \\ \sum_i \bar{\lambda}_i & \sum_i \bar{\lambda}_i \lambda_i & ... & \sum_i \bar{\lambda}_i \lambda_i^{n-1} \\ ... & ... & ... & ... \\ \sum_i \bar{\lambda}_i^{n-1} & \sum_i \bar{\lambda}_i^{n-1} \lambda_i & ... & \sum_i \bar{\lambda}_i^{n-1} \lambda_i^{n-1} \end{vmatrix}. \end{aligned}$$

By inserting the above expression into the integral we have

$$\begin{aligned} D_{11}^{(n+1,1)}(\lambda) &= \frac{n+1}{Z_{n+1}} e^{-|\lambda|^2} \int_{\mathbb{C}_n} \prod_{i=1}^n d\lambda_i d\bar{\lambda}_i e^{-\sum_{i=1}^n |\lambda_i|^2} \prod_{i=1}^n \Big(1 + \frac{1}{|\lambda - \lambda_i|^2}\Big) \times \det \begin{vmatrix} n & \sum_i \lambda_i & ... & \sum_i \lambda_i^{n-1} \\ \sum_i \bar{\lambda}_i & \sum_i \bar{\lambda}_i \lambda_i & ... & \sum_i \bar{\lambda}_i \lambda_i^{n-1} \\ ... & ... & ... & ... \\ \sum_i \bar{\lambda}_i^{n-1} & \sum_i \bar{\lambda}_i^{n-1} \lambda_i & ... & \sum_i \bar{\lambda}_i^{n-1} \lambda_i^{n-1} \end{vmatrix} \\ &= \frac{(n+1)!}{Z_{n+1}} e^{-|\lambda|^2} \int_{\mathbb{C}_n} \prod_{i=1}^n d\lambda_i d\bar{\lambda}_i e^{-\sum_{i=1}^n |\lambda_i|^2} \prod_{i=1}^n \Big(1 + \frac{1}{|\lambda - \lambda_i|^2}\Big) \times \det \begin{vmatrix} 1 & \lambda_1 & ... & \lambda_1^{n-1} \\ \bar{\lambda}_2 & \bar{\lambda}_2 \lambda_2 & ... & \bar{\lambda}_2 \lambda_2^{n-1} \\ ... & ... & ... & ... \\ \bar{\lambda}_n^{n-1} & \bar{\lambda}_n^{n-1} \lambda_n & ... & \bar{\lambda}_n^{n-1} \lambda_n^{n-1} \end{vmatrix}. \end{aligned}$$

since the expression is symmetric over all λ_i.

Now, since each line is expressed in terms of only one of λ_i, by merging the product and the determinant appropriately and passing the integral inside, we get to calculate integrals of the form

$$\int e^{-|\lambda_i|^2} \bar{\lambda}_i^{j-1} \lambda_i^{k-1} (|\lambda - \lambda_i|^2 + 1) d\lambda_i$$

where j and k range from 1 to n.

We write

$$|\lambda - \lambda_i|^2 + 1 = (|\lambda|^2 + 1 + |\lambda_i|^2) - (\lambda\bar{\lambda}_i + \bar{\lambda}\lambda_i)$$

where the first parenthesis will generate the diagonal terms and the second the off-diagonal ones of the resulting 3-diagonal determinant. The rest of the terms to be 0.

With $a_{j,k}$ I will denote the (j,k) position on the matrix and not a particular element. Those will be a_k, b_k and c_k. We expect some factors of π which we will take separately as they will cancel with the normalisation constant.

Remark: Note that this matrix is none other than M from 3.4.1, as shown in (99).

- diagonal elements (j=k)

$$\int e^{-|\lambda_i|^2} |\lambda_i|^{2(k-1)} (|\lambda|^2 + 1 + |\lambda_i|^2) d\lambda_i = (|\lambda|^2 + 1) \int e^{-|\lambda_i|^2} |\lambda_i|^{2(k-1)} d\lambda_i + \int e^{-|\lambda_i|^2} |\lambda_i|^{2k} d\lambda_i$$

By taking polar coordinates and setting $r^2 = x$ we get

$$\begin{aligned}
&= \pi(|\lambda|^2 + 1) \int e^{-r^2} r^{2(k-1)} 2r dr + \pi \int e^{-r^2} r^{2k} 2r dr \\
&= \pi(|\lambda|^2 + 1) \int e^{-x} x^{(k-1)} dx + \pi \int e^{-x} x^k dx \\
&= \pi(|\lambda|^2 + 1)\Gamma(k) + \pi\Gamma(k+1) \\
&= \pi[(|\lambda|^2 + 1)(k-1)! + k!]
\end{aligned}$$

$$\Rightarrow a_{k,k} = (|\lambda|^2 + 1)(k-1)! + k! = (k-1)!(|\lambda|^2 + k + 1).$$

- upper diagonal elements (j+1=k)

$$\begin{aligned}
\int e^{-|\lambda_i|^2} \bar{\lambda}_i^{k-1} \lambda_i^{k} (\lambda\bar{\lambda}_i) d\lambda_i &= \lambda \int e^{-|\lambda_i|^2} |\lambda_i|^{2k} d\lambda_i \\
&= \pi\lambda\Gamma(k) \\
&= \pi\lambda k!
\end{aligned}$$

$$\Rightarrow -a_{k,k+1} = \lambda k!$$

times a factor of π.

- lower diagonal elements (j=k+1)

Similarly we get $-a_{k+1,k} = \bar{\lambda} k!$ times a factor of π.

By setting $a_k = |\lambda|^2 + k + 1$ and $b_k = \lambda$, the initial expression becomes:

$$D_{11}^{(n+1,1)}(\lambda) = \frac{(n+1)!\pi^n}{Z_{n+1}} e^{-|\lambda|^2} \det \begin{vmatrix} a_1 & -\bar{b}_1 & 0 & \dots & \dots & 0 \\ -b_1 & a_2 & -2!\bar{b}_2 & 0 & \dots & . \\ 0 & -2!b_2 & 2!a_3 & -3!\bar{b}_3 & \dots & . \\ . & 0 & -3!b_3 & . & . & . \\ . & . & . & . & . & -(n-1)!\bar{b}_{n-1} \\ 0 & \dots & \dots & \dots & -(n-1)!b_{n-1} & (n-1)!a_n \end{vmatrix}$$

$$= \frac{(n+1)\pi^n}{Z_{n+1}} \prod_{j=1}^{n} j! e^{-|\lambda|^2} \det \begin{vmatrix} a_1 & -\bar{b}_1 & 0 & \dots & \dots & 0 \\ -b_1 & a_2 & -2\bar{b}_2 & 0 & \dots & 0 \\ 0 & -b_2 & a_3 & -3\bar{b}_3 & \dots & . \\ . & 0 & -b_3 & . & . & . \\ . & . & . & . & . & -(n-1)\bar{b}_{n-1} \\ 0 & \dots & \dots & \dots & -b_{n-1} & a_n \end{vmatrix}.$$

Note that the minus' do not matter, because if one the products in the developed determinant contain the $(i, i+1)$ element, it will also contain the $(i+1, i)$, and thus they will cancel.

Finally, since

$$Z_{n+1} = \prod_{j=1}^{n+1} j! \pi^{n+1}$$

we have

$$\frac{(n+1)\pi^n}{Z_{n+1}} \prod_{j=1}^{n} j! = \frac{1}{\pi} \frac{1}{n!}.$$

And now to calculate the $n \times n$ determinant

$$g_n = \det \begin{vmatrix} |\lambda|^2+2 & \bar{\lambda} & 0 & \dots & \dots & 0 \\ \lambda & |\lambda|^2+3 & 2\bar{\lambda} & \dots & \dots & . \\ 0 & \lambda & |\lambda|^2+4 & 3\bar{\lambda} & \dots & . \\ . & . & \lambda & . & . & . \\ . & . & . & . & . & (n-1)\bar{\lambda} \\ 0 & \dots & \dots & \dots & \lambda & |\lambda|^2+n+1 \end{vmatrix}$$

We proceed by eliminating the lower diagonal elements by using row and columns computations.
Step by step:

- after eliminating a_{21} we have

$$\begin{aligned} a_{22} &= a_2 - \frac{|b_1|^2}{a_{11}} \\ &= |\lambda|^2 + 3 - \frac{|\lambda|^2|}{|\lambda|^2 + 2} \\ &= \frac{(|\lambda|^2 + 3)(|\lambda|^2 + 2) - |\lambda|^2|}{|\lambda|^2 + 2} \\ &= \frac{|\lambda|^4 + 4|\lambda|^2 + 3!}{|\lambda|^2 + 2!} \end{aligned}$$

- after eliminating a_{32}

$$\begin{aligned} a_{33} &= a_3 - \frac{2|b_2|^2}{a_{22}} \\ &= |\lambda|^2 + 4 - \frac{2|\lambda|^2(|\lambda|^2 + 2)}{|\lambda|^4 + 4|\lambda|^2 + 3!} \\ &= \frac{|\lambda|^6 + 6|\lambda|^4 + 18|\lambda|^2 + 4!}{|\lambda|^4 + 4|\lambda|^2 + 3!} \end{aligned}$$

- after eliminating a_{43}

$$a_{44} = \frac{|\lambda|^8 + 8|\lambda|^6 + 36|\lambda|^4 + 96|\lambda|^2 + 5!}{|\lambda|^6 + 6|\lambda|^4 + 18|\lambda|^2 + 4!}$$

So far we have

$$\begin{aligned} g_1 &= |\lambda|^2 + 2! \\ g_2 &= |\lambda|^4 + 4|\lambda|^2 + 3! \\ g_3 &= |\lambda|^6 + 6|\lambda|^4 + 18|\lambda|^2 + 4! \\ g_4 &= |\lambda|^8 + 8|\lambda|^6 + 36|\lambda|^4 + 96|\lambda|^2 + 5! \end{aligned}$$

Now taking into that its integral of the form

$$\int e^{-|\lambda|^2}|\lambda|^{2(n-l)}d\lambda$$

will produce an $(n-l)!$ term, it's not so hard to spot the following pattern.

$$\begin{aligned} g_1 &= |\lambda|^2 + 2(1!) \\ g_2 &= |\lambda|^4 + 2(2)|\lambda|^2 + 3(2!) \\ g_3 &= |\lambda|^6 + 2(3)|\lambda|^4 + 3(2.3)|\lambda|^2 + 4(3!) \\ g_4 &= |\lambda|^8 + 2(4)|\lambda|^6 + 3(3.4)|\lambda|^4 + 4(2.3.4)|\lambda|^2 + 5(4!) \\ &\ \vdots \\ g_n &= \sum_{l=0}^{n} \frac{n!}{(n-l)!}(l+1)|\lambda|^{2(n-l)} \end{aligned}$$

or

$$g_n \quad = \quad \sum_{l=0}^{n} \frac{n!}{l!}(n-l+1)|\lambda|^{2l} = \sum_{l=0}^{n} \binom{n}{l}(l+1)!|\lambda|^{2l}$$

We have to check that this formula satisfies the recursion

$$g_n = (|\lambda|^2 + n + 1)g_{n-1} - (n-1)|\lambda|^2 g_{n-2}$$

We can take $|\lambda|^2 = x$ so I'll have to write less.

$$\begin{aligned}
RHS &= (x+n+1)\sum_{l=0}^{n-1}\frac{(n-1)!}{l!}(n-l)x^l - (n-1)x\sum_{l=0}^{n-2}\frac{(n-2)!}{l!}(n-l-1) \\
&= (n+1)\sum_{l=0}^{n-1}\frac{(n-1)!}{l!}(n-l)x^l + \sum_{l=1}^{n}\frac{(n-1)!}{(l-1)!}(n-l+1)x^l - (n-1)\sum_{l=1}^{n-1}\frac{(n-2)!}{(l-1)!}(n-l) \\
&= (n+1)\sum_{l=0}^{n-1}\frac{(n-1)!}{l!}(n-l)x^l + \sum_{l=1}^{n}\frac{(n-1)!}{l!}l(n-l+1)x^l - \sum_{l=1}^{n-1}\frac{(n-1)!}{l!}l(n-l).
\end{aligned}$$

We will independently check that the terms for $l = 0$, $l = n$ and $0 < l < n$ on each side match.

- $l = 0$

$$(n+1)\frac{(n-1)!}{0!}n + 0 + 0 = (n+1)!$$

- $l = n$

$$0 + \frac{(n-1)!}{(n-1)!}1 + 0 = 1$$

- $0 < l < n$

$$\begin{aligned}
\sum_{l=1}^{n-1}\frac{(n-1)!}{l!}[(n+1)(n-l) + l(n-l+1) - l(n-l)] &= \sum_{l=1}^{n-1}\frac{(n-1)!}{l!}[(n^2 + -nl + n - l) + l] \\
&= \sum_{l=1}^{n-1}\frac{(n-1)!}{l!}n(n-l+1) = \sum_{l=1}^{n-1}\frac{n!}{l!}(n-l+1)
\end{aligned}$$

Remark: Note that $g_n(x)$ is actually just $n!f_n(x)$.

So altogether we get

$$D_{11}^{(n+1,1)}(\lambda) = \frac{1}{\pi}\frac{1}{n!}e^{-|\lambda|^2}g_n(|\lambda|^2) = \frac{e^{-|\lambda|^2}}{\pi}f_n(|\lambda|^2)$$

which corresponds with Theorem 1. This way has already almost reached it's limits. Even for the simplest case of the off-diagonal overlaps $D_{12}^{(n,2)}(\lambda,\sigma)$, or the diagonal overlap $D_{11}^{(n,2)}(\lambda,\sigma)$, we have to resolve in heuristic calculations by assuming translation invariance at $n \to \infty$ and take $\sigma = 0$. Otherwise we end up with the determinant of a 5-diagonal matrix, which is no longer a realistic calculation. I'll omit this part, as taking $\sigma = 0$ yields very similar tri-diagonal determinants and it's more or less the same calculation. Having said that...

4.3 Slightly different $W_n^{(1)}$ weights

Before we move to bi-orthogonal polynomials, I'll go on a small tangent. Nothing in this section is used for any of the alternative proofs, although they could be, had we used a different approach. We are going to do the same calculations, but this time using different weights, in particular

$$W_{n,k}^{(1)} = \prod \Big(1 + \frac{k}{|\lambda - \lambda_i|^2}\Big).$$

We do this mostly for future reference and because it is possible, but actually these weights do start to appear when we use Eynard's way to find a recursion, not for the overlaps, but for example quantities such as the square of an overlap or the product of overlaps. Also I will briefly explain further down how it can be used for the reduced kernel, instead of the LDU decomposition in 3.4.1.

We start with the case of $k = 2$, where

$$W_{n,k}^{(1)} = \prod \Big(1 + \frac{k}{|\lambda - \lambda_i|^2}\Big).$$

The differences only start to appear after we pass the integral inside the determinant and working the same way as in 4.2 we end up with integrals of the form:

$$\int e^{-|\lambda_i|^2} \bar{\lambda}_i^j {\lambda_i}^k (|\lambda - \lambda_i|^2 + 2) d\lambda$$

Notice that the only difference will be a '+1' to all the diagonal elements and thus we have to calculate the determinant:

$$g_n^{(2)} = det \begin{vmatrix} |\lambda|^2+3 & \bar{\lambda} & 0 & ... & ... & 0 \\ \lambda & |\lambda|^2+4 & 2\bar{\lambda} & ... & ... & . \\ 0 & \lambda & |\lambda|^2+5 & 3\bar{\lambda} & ... & . \\ . & . & \lambda & . & . & . \\ . & . & . & . & . & (n-1)\bar{\lambda} \\ 0 & ... & ... & ... & \lambda & |\lambda|^2+n+2 \end{vmatrix}$$

Again by eliminating the λ one by one we get:

- $$a_{22} = |\lambda|^2 + 4 - \frac{|\lambda|^2}{|\lambda|^2+3} = \frac{|\lambda|^4 + 6|\lambda|^2 + 3.4}{|\lambda|^2+3}$$

- $$a_{33} = |\lambda|^2 + 5 - \frac{2|\lambda|^2(|\lambda|^2+3)}{|\lambda|^4 + 6|\lambda|^2 + 3.4} = \frac{|\lambda|^6 + 9|\lambda|^4 + 36|\lambda|^2 + 3.4.5}{|\lambda|^4 + 6|\lambda|^2 + 3.4}$$

- $$a_{44} = |\lambda|^2 + 6 - \frac{3|\lambda|^2(|\lambda|^4 + 6|\lambda|^2 + 3.4)}{|\lambda|^6 + 9|\lambda|^4 + 36|\lambda|^2 + 3.4.5} = \frac{|\lambda|^8 + 12|\lambda|^6 + 72|\lambda|^4 + 240|\lambda|^2 + 3.4.5.6}{|\lambda|^6 + 9|\lambda|^4 + 36|\lambda|^2 + 3.4.5}$$

and thus

$$
\begin{aligned}
g_1^{(2)} &= |\lambda|^2 + 3 \\
g_2^{(2)} &= |\lambda|^4 + 6|\lambda|^2 + 3.4 \\
g_3^{(2)} &= |\lambda|^6 + 9|\lambda|^4 + 36|\lambda|^2 + 3.4.5 \\
g_4^{(2)} &= |\lambda|^8 + 12|\lambda|^6 + 72|\lambda|^4 + 240|\lambda|^2 + 3.4.5.6
\end{aligned}
$$

A bit harder to spot a pattern, but if we mutiply g_2, g_3 and g_4 with $e^{-|\lambda|^2}$ and integrate λ over $\mathbb{C}$, we get respectively

- $2! + 3.2! + 6.2!$
- $3! + 3.3! + 6.3! + 10.3!$
- $4! + 3.4! + 6.4! + 10.4! + 15.4!$

which suggests

$$
\begin{aligned}
g_n^{(2)} &= \sum_{l=0}^{n} \frac{n!}{(n-l)!} \frac{(l+1)(l+2)}{2} |\lambda|^{2(n-l)} \\
&= \sum_{l=0}^{n} \frac{n!}{l!} \frac{(n-l+1)(n-l+2)}{2} |\lambda|^{2l}.
\end{aligned}
$$

Once again we set $x = |\lambda|^2$ and we want the above expression to satisfy the following recursion:

$$
\begin{aligned}
g_n^{(2)} &= (x+n+2)g_{n-1}^{(2)} - x(n-1)g_{n-2}^{(2)} \\
&= (n+2)\sum_{l=0}^{n-1} \frac{(n-1)!}{2l!}(n-l)(n-l+1)x^l + \sum_{l=1}^{n} \frac{(n-1)!}{2l!} l(n-l+1)(n-l+2)x^l - \sum_{l=1}^{n-1} \frac{(n-1)!}{2l!} l(n-l)(n-l+1)
\end{aligned}
$$

Again, we check independently that the terms for $l = 0$, $l = n$ and $0 < l < n$ on each side match.

- $l = 0$

$$
(n+2)\frac{(n-1)!}{2}n(n+1) + 0 + 0 = \frac{(n+2)!}{2}
$$

- $l = n$

$$
0 + \frac{(n-1)!}{2n!} n1.2 + 0 = 1
$$

- $0 < l < n$

$$
\begin{aligned}
RHS &= \sum_{l=1}^{n-1} \frac{(n-1)!}{2l!}[(n+2)(n-l)(n-l+1) + l(n-l+1)(n-l+2) - l(n-l)(n-l+1)] \\
&= \sum_{l=1}^{n-1} \frac{(n-1)!}{2l!}\{[n(n-l+2) - 2l](n-l+1) + 2l(n-l+1)\} \\
&= \sum_{l=1}^{n-1} \frac{n!}{l!} \frac{(n-l+1)(n-l+2)}{2}.
\end{aligned}
$$

We move on to

$$W^{(1)}_{n,k} = \prod\Big(1 + \frac{k}{|\lambda - \lambda_i|^2}\Big).$$

Similarly to before, for a general k, the resulting determinant we have to compute is

$$g_n^{(k)} = det\begin{vmatrix} |\lambda|^2+k+1 & \bar\lambda & 0 & \dots & \dots & 0 \\ \lambda & |\lambda|^2+k+2 & 2\bar\lambda & \dots & \dots & . \\ 0 & \lambda & |\lambda|^2+k+3 & 3\bar\lambda & \dots & . \\ . & . & \lambda & . & . & . \\ . & . & . & . & . & (n-1)\bar\lambda \\ 0 & \dots & \dots & \dots & \lambda & |\lambda|^2+k+n \end{vmatrix}$$

Working the same way we guess that

$$\begin{aligned} g_n^{(k)} &= \sum_{l=0}^{n} \frac{n!}{(n-l)!}\binom{l+k}{k}|\lambda|^{2(n-l)} \\ &= \sum_{l=0}^{n} \frac{n!}{l!}\binom{n-l+k}{k}|\lambda|^{2l}. \end{aligned}$$

Checking that it satisfies the recursion

$$g_n^{(k)} = (x+n+k)g_{n-1}^{(k)} - x(n-1)g_{n-2}^{(k)}$$

$$RHS = (n+k)\sum_{l=0}^{n-1}\frac{(n-1)!}{l!}\binom{n-l+k-1}{k}x^l + \sum_{l=1}^{n}\frac{(n-1)!}{l!}l\binom{n-l+k}{k}x^l - \sum_{l=1}^{n-1}\frac{(n-1)!}{l!}l\binom{n-l+k-1}{k}x^l$$

- $l = 0$

$$(n+k)(n-1)!\binom{n+k-1}{k} + 0 + 0 = (n+k)!(n-1)!\frac{(n+k-1)!}{k!(n-1)!} = \frac{(n+k)!}{k!}$$

- $l = n$

$$0 + \frac{(n-1)!}{n!}n\binom{k}{k} + 0 = 1$$

- $0 < l < n$

$$\begin{aligned} RHS &= \sum_{l=1}^{n-1}\frac{(n-1)!}{l!}\frac{(n-l+k-1)!}{(n-l)!k!}\Big[(n+k)(n-l) + l(n-l+k) - l(n-l)\Big] \\ &= \sum_{l=1}^{n-1}\frac{(n-1)!}{l!}\frac{(n-l+k-1)!}{(n-l)!k!}n(n-l+k) \\ &= \sum_{l=1}^{n-1}\frac{n!}{l!}\binom{n-l+k}{k}. \end{aligned}$$

□

As I said before, we can use the formulas for $g_n^{(k)}$ to find the reduced kernel. They enable us to find the adjoint of M and consequently, the inverse. I am only going to give a sketch of this calculation. We have the general form of $\kappa^{(n)}$ written in terms of the bi-orthogonal polynomials

$$\kappa^{(n)}(x,y) = \sum_k \frac{P_k(x)\overline{Q}_k(y)}{c_k}$$

which we can rewrite as

$$\begin{aligned}\kappa^{(n)}(x,y) &= \sum_k \sum_{l\leq k} \sum_{s\leq k} \frac{p_{kl}x^l\bar{y}^s q_{ks}}{c_k} \\ &= tr(C^{-1}PWQ^T) \\ &= tr(Q^TC^{-1}PW) \\ &= tr(M^{-1}W)\end{aligned}$$

where $W_{ij} = x^i\bar{y}^j$, C is diagonal and P,Q are upper trigonal. We essentially have the LDU decomposition of M^{-1} as seen in 3.4.1, where $M_{ij} =< z_i, z_j >$, with respect to $d\omega_\lambda(z) = \omega_\lambda(z)dzd\bar{z} = (|\lambda - z|^2+1)e^{-|z|^2}dzd\bar{z}$.. It comes down to calculating the inverse of M. We are also going to need $\prod_{k=1}^n c_k$, but because of the structure of P and Q, we can see from $Q^TC^{-1}P = M^{-1}$ that $\prod_{k=1}^n c_k$=detM, which we have calculated previously.

Let $\{A\}_{ij}$ be the cofactor matrix of M. Then

$$M^{-1} = \frac{1}{detM}A^T.$$

Because M is tri-diagonal all of the elements of A end up being the product of 2 determinants, both of the form $g_n^{(k)}$, times a factor of λ. Then all that needs to be done is multiply with W and take the trace. Still, this is not my method of choice.

4.4 Bi-Orthogonal Polynomials

In this section we are going to use an alternate method to the LDU decomposition, of calculating the reduced kernel $\kappa^{(n)}$ as shown in 3.4.3, in (116) and (117). This approach is more in line with the determinantal calculations so far in this chapter.

The aim is to show that the reduced kernel is given by

$$\kappa^{(n+1)}(x,\bar{y}) = \sum_s \frac{P_s(x)Q_s(\bar{y})}{c_s} = \frac{1}{\pi}\sum_{s=1}^n \frac{|\lambda|^{2s}}{f_s(|\lambda|^2)f_{s+1}(|\lambda|^2)(s+1)!}\sum_{i=1}^s \left(\frac{x}{\lambda}\right)^i f_i(|\lambda|^2)\sum_{j=1}^s \left(\frac{\bar{y}}{\bar{\lambda}}\right)^j f_j(|\lambda|^2).$$

This corresponds to (116) if we change the order of summation.

I'm going to use the following lemma, which I found in some of Gernot Akemann's personal notes. This way yield not only a formula for the reduced kernel $\kappa_n(x,y)$, but also formulas for the polynomials themselves, which although we don't need, is nice to have.

Lemma 4.1: For some $\omega(z):\mathbb{C}\to\mathbb{C}$, define for ease $< k,l >$ to be

$$< k,l > := \int z^k\bar{z}^j\omega(z)dzd\bar{z}.$$

for some weight $\omega(z)$.

Let Δ_n be the following determinant:

$$\Delta_n = \begin{vmatrix} <0,0> & <1,0> & \cdots & <n-1,0> \\ <0,1> & <1,1> & \cdots & <n-1,1> \\ \vdots & \vdots & \ddots & \vdots \\ <0,n-1> & <1,n-1> & \cdots & <n-1,n-1> \end{vmatrix}$$

If we define the polynomials $P_n(z)$ and $Q_n(\bar{z})$ to b

$$P_n(z) = \begin{vmatrix} <0,0> & <1,0> & \cdots & <n,0> \\ <0,1> & <1,1> & \cdots & <n,1> \\ \vdots & \vdots & \ddots & \vdots \\ <0,n-1> & <1,n-1> & \cdots & <n,n-1> \\ 1 & z & \cdots & z^n \end{vmatrix} \Delta_n^{-1}$$

and

$$Q_n(\bar{z}) = \begin{vmatrix} <0,0> & <0,1> & \cdots & <0,n> \\ <1,0> & <1,1> & \cdots & <1,n> \\ \vdots & \vdots & \ddots & \vdots \\ <n-1,0> & <n-1,1> & \cdots & <n-1,n> \\ 1 & \bar{z} & \cdots & \bar{z}^n \end{vmatrix} \Delta_n^{-T}$$

then $P_n(z)$ and $Q_n(\bar{z})$ form a family of monic bi-orthogonal polynomials with respect to the weighted measure $\omega(z)dzd\bar{z}$.

Proof: It's easy to see that they are monic.

For bi-orthogonality one just has to notice that

$$<P_n(z), \bar{z}^k> = \begin{cases} 0 & k < n \\ \frac{\Delta_{n+1}}{\Delta_n} & k = n \end{cases}$$

and

$$<z^k, Q_n(\bar{z})> = \begin{cases} 0 & k < n \\ \frac{\Delta_{n+1}^{-T}}{\Delta_n^{T}} & k = n \end{cases}$$

$\square$

Now we are finally ready to find to the bi-orthogonal polynomials and all we have to do is apply the lemma for the measure:

$$d\omega_\lambda(z) = \omega_\lambda(z)dzd\bar{z} = (|\lambda - z|^2 + 1)e^{-|z|^2}dzd\bar{z}.$$

For this measure the polynomials take the following form:

$$P_n(z) = \begin{vmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & c_{n-1} & a_n & b_n \\ 1 & z & \cdots & z^{n-2} & z^{n-1} & z^n \end{vmatrix} \Delta_n(z)^{-1}.$$

First we already know what Δ_n is from 4.2, where we also computed all the a_i, b_i, c_i. It's the first tri-diagonal determinant we computed.

$$\begin{aligned} \Delta_n &= \pi^n \prod_{i=1}^{n-1} i! \sum_{l=0}^{n} \frac{n!}{l!}(n-l+1)|\lambda|^{2l} \\ &= \pi^n \prod_{i=1}^{n} i! \sum_{l=0}^{n} \frac{n-l+1}{l!}|\lambda|^{2l} \\ &= \pi^n \prod_{i=1}^{n} i! f_n(|\lambda|^{2l}) \end{aligned}$$

where

$$f_n(x) = \sum_{l=0}^{n} \frac{n-l+1}{l!} x^l.$$

Let d_k be the coefficient of z_k in the polynomial $P_n(z)$. Then

$$\begin{aligned} d_k &= \Delta_n^{-1} k! f_k(|\lambda|^2)(-1)^{n-k} b_{k+1} b_{k+2} \cdots b_n \\ &= (-1)^{n-k} \Delta_n^{-1} \pi^{n-k} \lambda^{n-k} k! \prod_{i=k+1}^{n} f_k(|\lambda|^2) \\ &= \frac{(-\lambda)^n}{f_n(|\lambda|^2)} \frac{f_k}{\lambda^k} \\ \Rightarrow P_n(z) &= \frac{(-\lambda)^n}{f_n(|\lambda|^2)} \sum_{k=0}^{n} \left(\frac{z}{\lambda}\right)^k f_k(|\lambda|^2) \end{aligned}$$

where again $f_n(x) = \sum_{l=0}^{n} \frac{n-l+1}{l!} x^l$.

Similarly

$$Q_n(\bar{z}) = \frac{(-\bar{\lambda})^n}{f_n(|\lambda|^2)} \sum_{k=0}^{n} \left(\frac{\bar{z}}{\bar{\lambda}}\right)^k f_k(|\lambda|^2).$$

Now all that's left for the kernel is the $c_k =< P_k(z), Q_k(\bar{z}) >$. Remember that $P_k(z)$ and $Q_k(\bar{z})$ are monic and $< P_k(z), \bar{z}^l >= 0$ for $l < k$. This means that

$$
\begin{aligned}
c_k &= < P_k(z), \bar{z}^k >= \frac{\Delta_{k+1}}{\Delta_k} \\
&= \frac{\pi^{k+1}\prod_{i=0}^{k+1} i! f_{k+1}(|\lambda|^2)}{\pi^k \prod_{i=0}^{k} i! f_k(|\lambda|^2)} \\
&= \pi(k+1)!\frac{f_{k+1}(|\lambda|^2)}{f_k(|\lambda|^2)}.
\end{aligned}
$$

Now we finally have

$$
\kappa^{(n)}(x,\bar{y}|\lambda,\bar{\lambda}) = \sum_s \frac{P_s(x)Q_s(\bar{y})}{c_s} = \frac{1}{\pi}\sum_{s=1}^{n} \frac{|\lambda|^{2s}}{f_s(|\lambda|^2)f_{s+1}(|\lambda|^2)(s+1)!}\sum_{i=1}^{s}\left(\frac{x}{\lambda}\right)^i f_i(|\lambda|^2)\sum_{j=1}^{s}\left(\frac{\bar{y}}{\bar{\lambda}}\right)^j f_j(|\lambda|^2).
$$

If we substitute λ with ν and $\bar{\lambda}$ with $\bar{\mu}$, we get the more general form:

$$
\kappa^{(n)}(x,\bar{y}|\nu,\bar{\mu}) = \frac{1}{\pi}\sum_{s=1}^{n} \frac{(\nu\bar{\mu})^s}{f_s(\nu\bar{\mu})f_{s+1}(\nu\bar{\mu})(s+1)!}\sum_{i=1}^{s}\left(\frac{x}{\nu}\right)^i f_i(\nu\bar{\mu})\sum_{j=1}^{s}\left(\frac{\bar{y}}{\bar{\mu}}\right)^j f_j(\nu\bar{\mu})
$$

which, after changing the order of summation, can be written as in (117)

$$
\kappa^{(n)}(x,\bar{y},\nu,\bar{\mu}) = G^{(n)}(\frac{x}{\nu},\frac{\bar{y}}{\bar{\mu}},\nu\bar{\mu})
$$

where

$$
G^{(n)}(x,y,z) = \frac{1}{\pi}\sum_{i,j=1}^{n} x^i y^j f_i(z) f_j(z) \sum_{s=i\vee j}^{n} \frac{z^s}{f_s(z)f_{s+1}(z)(s+1)!}.
$$

where

$$
f_i(x) = \sum_{s=0}^{i} \frac{i-s+1}{s!}x^s.
$$

Remark: Thanks to Lemma 1, we have a relation between diagonal and off diagonal overlaps, which is the reason that knowing the reduce kernel is sufficient for both cases. Had this not been the case, then we would need the kernel corresponding to the off-diagonal weight of the form

$$
\omega^{(2)}_{\nu\mu}(z) = (\bar{z}-\bar{\nu})(z-\mu)[1-(z-\nu)(\bar{z}-\bar{\mu})]e^{-|z|^2}
$$

If we call

$$
\omega^{(1)}_{\nu\mu}(z) = [1-(z-\nu)(\bar{z}-\bar{\mu})]e^{-|z|^2}
$$

then $\omega^{(2)}_{\nu\mu}(z)$ can be written as

$$
\omega^{(2)}_{\nu\mu}(z) = (\bar{z}-\bar{\nu})(z-\mu)\omega^{(1)}_{\nu\mu}(z).
$$

Note again that we know the orthogonal polynomials and kernel with respect to $\omega^{(1)}_{\nu\mu}(z)$, since they are the same as for $\omega_\lambda(z) = (1+|z-\lambda|^2)e^{-|z|^2}$ if we just substitute λ with ν and $\bar{\lambda}$ with $\bar{\mu}$.

As it has been already mentioned, if we know the orthogonal polynomials and kernel for $\omega^{(1)}_{\nu\mu}(z)$, there are formulas in [3], which give both the orthogonal polynomials and kernel for a measure of the form $\omega^{(2)}_{\nu\mu}(z) = (\bar{z}-\bar{\nu})(z-\mu)\omega^{(1)}_{\nu\mu}(z)$. But this creates a huge mess we would like to avoid, or more like had to avoid. Lemma 1 is of huge importance to the study off-diagonal overlaps.

4.5 Alternate way to simplify the reduced kernel

Now we are almost ready for the final part of the simplification of the reduced kernel, where with the help of contour integrals we are going to derive relation (150) from 3.4.4. First we need to massage $G^{(n)}(x,y,z)$ a bit further. We require a lemma similar to 3.3.

Let $k = i \vee j$

Lemma 4.2 Call

$$M_{n,k}(z) = \sum_{s=k}^{n} \frac{z^s}{f_s(z) f_{s+1}(z)(s+1)!}.$$

Then

$$M_{n,k}(z) = \frac{z^k}{f_k(z) f_{n+1}(z)(n+1)!} \sum_{s=0}^{n-k} \frac{(n+1)!}{(s+k+2)!} [(n-k-s+1)(s+1)+(k+1)] z^s.$$

Proof: Not much too this proof really. We notice that

$$M_{k+1,k}(z) = \frac{z+(k+3)}{f_k(z) f_{k+2}(z)(k+2)!} z^k$$

$$M_{k+2,k}(z) = \frac{z^2+(k+5)z+(k+3)(k+4)}{f_k(z) f_{k+3}(z)(k+3)!} z^k$$

$$M_{k+3,k}(z) = \frac{z^3+(k+7)z^2+(k+4)(k+7)z+(k+3)(k+4)(k+5)}{f_k(z) f_{k+4}(z)(k+4)!} z^k$$

and so on. Now this may seem very out of context, and it probably is, but the numerators are given by the following determinants:

$$h_n = det \begin{vmatrix} |\lambda|^2+k+3 & (k+2)\bar{\lambda} & 0 & \dots & \dots & 0 \\ \lambda & |\lambda|^2+k+4 & (k+3)\bar{\lambda} & \dots & \dots & . \\ 0 & \lambda & |\lambda|^2+k+5 & (k+4)\bar{\lambda} & \dots & . \\ . & . & \lambda & . & . & . \\ . & . & . & . & . & (k+n)\bar{\lambda} \\ 0 & \dots & \dots & \dots & \lambda & |\lambda|^2+k+n+2 \end{vmatrix}$$

which I happen to have already computed to be

$$h_n(z) = \sum_{s=0}^{n-k} \frac{(n+1)!}{(s+k+2)!} [(n-k-s+1)(s+1)+k+1] z^s.$$

Nothing unusual to it, just one more tri-diagonal determinant in a line of many. The only relation I can see, is that for $k=0$, h_n is the tri-diagonal determinant I had to compute for $D_{11}^{(n,2)}(\lambda,\sigma)$ with $\sigma=0$. So I'm guessing the numerators of $M_{n,k}$ correspond to the main sub-determinants of h_n.

Anyway, having already seen the pattern, the answer is:

$$M_{n,k}(z) = \frac{z^k}{f_k(z) f_{n+1}(z)(n+1)!} \sum_{s=0}^{n-k} \frac{(n+1)!}{(s+k+2)!} [(n-k-s+1)(s+1)+(k+1)] z^s.$$

□

Working a bit on $M_{n,k}(z)$ and plugging it back in, we derive the following expression

$$\begin{aligned}\pi G^{(n)}(x,y,z) &= \frac{n+3}{f_{n+1}(z)z^2}\sum_{i,j=0}^{n} x^i y^j \frac{f_i(z)f_j(z)}{f_{i\vee j}(z)} A_{n,i\vee j}(z) && (i)\\ &-\frac{1}{f_{n+1}(z)z^2}\sum_{i,j=0}^{n} x_i y_j \frac{f_i(z)f_j(z)}{f_{i\vee j}(z)} B_{n,i\vee j}(z) && (ii)\end{aligned}$$

where

$$\begin{aligned}A_{n,j}(z) &= z e_{j+1}^{n+1}(z) - (j+1)e_{j+2}^{n+2}(z)\\ B_{n,j}(z) &= z^2 e_j^n(z) - zj e_{j+1}^{n+1}(z) - (j+1)e_{j+2}^{n+2}(z)\end{aligned}$$

and

$$e_j^n(z) = \sum_{s=j}^{n}\frac{z^s}{s!}.$$

Now we are ready.

4.5.1 $n = \infty$

Although $n < \infty$ is what we are after, since this is a very long calculation and we are interested in both cases, with $n = \infty$ corresponding to the bulk limit in Corollary 1, I find it wise to start with $n = \infty$ to understand how the process works. Note that $f_n(z) = (n+1)e^n(z) - ze^{n-1}(z)$, therefore (ii) goes to 0 as n goes to ∞. Thus taking $n \to \infty$ is significantly simpler to work with.

As for (i), we break it in two parts, C_1 and C_2

$$\begin{aligned}\sum_{i,j=0}^{\infty} x^i y^j \frac{f_i(z)f_j(z)}{f_{i\vee j}(z)} A_{\infty,i\vee j}(z) &= \sum_{i,j=0}^{\infty} x^i y^j f_i(z) A_{\infty,j}(z) 1_{j\geq i} + \sum_{i,j=0}^{\infty} x^i y^j f_j(z) A_{\infty,i}(z) 1_{j<i}\\ &= C_1 + C_2.\end{aligned}$$

Going to use the following identity to rewrite the 2 sums. Might as well call it a lemma.

Lemma 4.3: Using contour integrals, the following indicator functions can be re-written as

$$\begin{aligned}1_{j\geq i} &= \sum_{t=i}^{\infty} 1_{j=t} = \sum_{t=i}^{\infty}\oint dw \frac{w^{t-j}}{2\pi i w} = \oint dw \frac{w^{i-j}}{2\pi i w}\sum_{t=0}^{\infty} w^t \qquad (|w|<1)\\ &= \oint dw \frac{w^{i-j}}{2\pi i w(1-w)}\end{aligned}$$

and similarly

$$1_{j<i} = \oint dw \frac{w^{j-i}}{2\pi i(1-w)}$$

where $w \in \mathbb{C}$ with $|w| < 1$. □

Thus we get the following expressions for C_1 and C_2

$$C_1 = \oint \frac{dw}{2\pi i w(1-w)}\sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j A_{\infty,j}(z)\sum_{i=0}^{\infty}(xw)^i f_i(z)$$

and

$$C_2 = \oint \frac{dw}{2\pi i(1-w)}\sum_{j=0}^{\infty}(yw)^j f_j(z)\sum_{i=0}^{\infty}\left(\frac{x}{w}\right)^i A_{\infty,i}(z).$$

Here is the main idea.
Both C_1 and C_2 are integrals over circles $\Gamma = \{re^{i\theta} : 0 \le \theta \le 2\pi\}$, for which all we need so far is $r < 1$. This gives us a lot of leaway. Also notice how they seem to be one transformation away from eachother. Now w is nothing but a dummy variable we eventually need to get rid off. What we aim to do is find Γ_1 and Γ_2 for C_1 and C_2 respectively, such that no poles are included inside the circles other than 0. Then we will apply the tranformation $w \to \frac{1}{w}$ to Γ_2 and shrink the new $\tilde{\Gamma}_2$ down to Γ_1. This will yield $-C_1$ plus all the residues, and those residues is the answer we are looking for.

First things first, we need to find Γ_1 and Γ_2. We start with C_1. The expression for C_1 includes two sums on which we will work on separately.

- $$\begin{aligned}\sum_{i=0}^{\infty}(xw)^i f_i(z) &= \sum_{i=0}^{\infty}\sum_{s=0}^{i}(xw)^i\frac{i-s+1}{s!}z^s = \sum_{s=0}^{\infty}\frac{z^s}{s!}\sum_{i=s}^{\infty}(xw)^i(i-s+1)\\ &= \sum_{s=0}^{\infty}\frac{(xwz)^s}{s!}\sum_{i=0}^{\infty}(xw)^i(i+1) = \frac{e^{xwz}}{(1-xw)^2}.\end{aligned}$$

  From here we can see that we want $|w| < |x|^{-1}$.

  Now for the second sum,

- $$\begin{aligned}\sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j A_{\infty,j}(z) &= \sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j[ze_{j+1}(z)-(j+1)e_{j+2}(z)]\\ &= \sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j ze_{j+1}(z) \qquad (1)\\ &\quad -\sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j(j+1)e_{j+2}(z). \qquad (2)\end{aligned}$$

  This is also too long to be done altogether, so we once again split it in two parts, (1) and (2).

  $$\begin{aligned}(1) &= \sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j ze_{j+1}(z) = z\sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j\sum_{s=j+1}^{\infty}\frac{z^s}{s!} = z\sum_{s=1}^{\infty}\frac{z^s}{s!}\sum_{j=0}^{s-1}\left(\frac{y}{w}\right)^j\\ &= \frac{z}{1-\frac{y}{w}}\sum_{s=1}^{\infty}\frac{z^s}{s!}\left[1-\left(\frac{y}{w}\right)^s\right] = \frac{z}{1-\frac{y}{w}}(e^z - e^{\frac{zy}{w}}).\end{aligned}$$

$$\begin{aligned}(2) &= -\sum_{j=0}^{\infty}\left(\frac{y}{w}\right)^j(j+1)\sum_{s=j+2}^{\infty}\frac{z^s}{s!} = \sum_{s=2}^{\infty}\frac{z^s}{s!}\sum_{j=0}^{s-2}\left(\frac{y}{w}\right)^j(j+1) = -\sum_{s=2}^{\infty}\frac{z^s}{s!}\frac{(s-1)\left(\frac{y}{w}\right)^s - s\left(\frac{y}{w}\right)^{s-1}+1}{\left(1-\frac{y}{w}\right)^2}\\ &= \frac{-1}{\left(1-\frac{y}{w}\right)^2}\left(\sum_{s=2}^{\infty}\frac{z^s}{s!}\left(\frac{y}{w}\right)^s(s-1) - \sum_{s=2}^{\infty}\frac{z^s}{s!}\left(\frac{y}{w}\right)^{s-1}s + \sum_{s=2}^{\infty}\frac{z^s}{s!}\right)\\ &= \frac{-1}{\left(1-\frac{y}{w}\right)^2}\left[\frac{zy}{w}\left(e^{\frac{zy}{w}}-1\right) - \left(e^{\frac{zy}{w}} - 1 - \frac{zy}{w} + ze^{\frac{y}{w}} - z\right) + (e^z - 1 - z)\right]\\ &= \frac{-1}{\left(1-\frac{y}{w}\right)^2}\left[\left(\frac{zy}{w} - 1 - z\right)e^{\frac{zy}{w}} + e^z\right].\end{aligned}$$

from where we can see that we also need $|w| < |y|$.

Putting back everything together in the initial expression for C_1, we have

$$\begin{aligned}
\Rightarrow C_1 &= \int_{\Gamma_1} \frac{dw}{2\pi i w(1-w)} \left[\frac{e^{xwz}}{(1-xw)^2} \left(\frac{z}{1-\frac{y}{w}} (e^z - e^{\frac{zy}{w}}) - \frac{1}{\left(1-\frac{y}{w}\right)^2} \left[\left(\frac{zy}{w} - 1 - z \right) e^{\frac{zy}{w}} + e^z \right] \right) \right] \\
&= \int_{\Gamma_1} \frac{dw}{2\pi i w(1-w)} \left[\frac{e^{xwz}}{(1-xw)^2} \left(\frac{ze^z}{1-\frac{y}{w}} - \frac{e^z}{\left(1-\frac{y}{w}\right)^2} \right) + \frac{e^{xwz}}{(1-xw)^2} \frac{e^{\frac{zy}{w}}}{\left(1-\frac{y}{w}\right)^2} \right]
\end{aligned}$$

where now we know we want $\Gamma_1 = \{re^{i\theta} : 0 \le \theta \le 2\pi\}$ with $r < 1, \frac{1}{|x|}, |y|$.

Note that the first part in the last expression for C_1, the one which does not include any $e^{\frac{zy}{w}}$ temrs, gives 0 since there are no poles. So, by setting

$$g(x) = \frac{e^{xz}}{(1-x)^2}$$

we have that

$$C_1 = \int_{\Gamma_1} \frac{dw}{2\pi i w(1-w)} g(xw) g\left(\frac{y}{w}\right)$$

and similarly, for it is the same calculation,

$$C_2 = \int_{\Gamma_2} \frac{dw}{2\pi i(1-w)} g(yw) g\left(\frac{x}{w}\right)$$

where $\Gamma_2 = \{re^{i\theta} : 0 \le \theta \le 2\pi\}$ with $r < 1, \frac{1}{|y|}, |x|$.

By applying the change of variables $w \longleftrightarrow \frac{1}{w}$ in C_2 it becomes

$$C_2 = -\int_{\tilde{\Gamma}_2} \frac{dw}{2\pi i w(1-w)} g\left(\frac{y}{w}\right) g(xw)$$

with $\tilde{\Gamma}_2 = \{re^{i\theta} : 0 \le \theta \le 2\pi\}$, $r > 1, \frac{1}{|x|}, |y|$.

Now, by shrinking $\tilde{\Gamma}_2$ down to Γ_1, we get the residue contribution from the 3 poles and C_2 can be written as

$$\begin{aligned}
C_2 &= -\int_{\Gamma_1} \frac{dw}{2\pi i w(1-w)} g(xw) g\left(\frac{y}{w}\right) - Res\left[\frac{g(xw) g\left(\frac{y}{w}\right)}{w(1-w)} \Big| 1, \frac{1}{x}, y \right] \\
\Rightarrow C_1 + C_2 &= -Res\left[\frac{g(xw) g\left(\frac{y}{w}\right)}{w(1-w)} \Big| 1, \frac{1}{x}, y \right].
\end{aligned}$$

- $$\begin{aligned}
Res\left[\frac{g(xw) g\left(\frac{y}{w}\right)}{w(1-w)} \Big| 1 \right] &= \lim_{w \to 1} (w-1) \frac{g(xw) g\left(\frac{y}{w}\right)}{w(1-w)} \\
&= -g(x) g(y) = -\frac{e^{xz}}{(1-x)^2} \frac{e^{yz}}{(1-y)^2}.
\end{aligned}$$

- $$\begin{aligned} Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big| y\right] &= \lim_{w\to y}\frac{d}{dw}\left[\frac{(w-y)^2}{w(1-w)}\frac{e^{xwz}}{(1-xw)^2}\frac{e^{\frac{yz}{w}}}{\left(1-\frac{y}{w}\right)^2}\right] \\ &= \lim_{w\to y}\frac{d}{dw}\left[\frac{e^{xwz+\frac{yz}{w}}}{(1-xw)^2}\frac{w}{1-w}\right]. \end{aligned}$$

Going to compute the derivative separately

$$\frac{d}{dw}\left[\frac{e^{xwz+\frac{yz}{w}}}{(1-xw)^2}\frac{w}{1-w}\right] = \frac{d}{dw}\left[\frac{e^{xwz+\frac{yz}{w}}}{(1-xw)^2}\right]\frac{w}{1-w} + \frac{e^{xwz+\frac{yz}{w}}}{(1-xw)^2}\frac{d}{dw}\left[\frac{w}{1-w}\right]$$

$$= \quad e^{xwz+\frac{yz}{w}}\left[\frac{\left(xw-\frac{yz}{w^2}\right)(1-xw)^2-2(1-xw)(-x)}{(1-xw)^4}\frac{w}{1-w}+\frac{1}{(1-xw)^2}\frac{1(1-w)-w(-1)}{(1-w)^2}\right]$$

$$= \quad e^{xwz+\frac{yz}{w}}\left[\frac{\left(xw-\frac{yz}{w^2}\right)(1-xw)+2x}{(1-xw)^3}\frac{w}{1-w}+\frac{1}{(1-xw)^2(1-w)^2}\right]$$

$$= \quad e^{xwz+\frac{yz}{w}}\left\{\frac{[\left(xw-\frac{yz}{w^2}\right)(1-xw)+2x]w(1-w)+1-xw}{(1-xw)^3(1-w)^2}\right\}$$

$$\begin{aligned} \Rightarrow Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big| y\right] &= \lim_{w\to y}\frac{e^{xwz+\frac{yz}{w}}}{(1-xw)^3(1-w)^2}\left\{[\left(xw-\frac{yz}{w^2}\right)(1-xw)+2x]w(1-w)+1-xw\right\} \\ &= \frac{e^{xyz+z}}{(1-y)^2(1-xy)^3}\left[z(1-xy)^2(y-1)+xy-2xy^2+1\right]. \end{aligned}$$

- $$\begin{aligned} Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big|\frac{1}{x}\right] &= \lim_{w\to\frac{1}{x}}\frac{d}{dw}\left[\frac{\left(w-\frac{1}{x}\right)^2}{w(1-w)}\frac{e^{xwz}}{(1-xw)^2}\frac{e^{\frac{yz}{w}}}{\left(1-\frac{y}{w}\right)^2}\right] \\ &= \lim_{w\to\frac{1}{x}}\frac{d}{dw}\left[\frac{e^{xwz+\frac{yz}{w}}}{(xw-xy)^2}\frac{w}{1-w}\right] \end{aligned}$$

Once again the derivative is

$$\frac{d}{dw}\left[\frac{e^{xwz+\frac{yz}{w}}}{(xw-xy)^2}\frac{w}{1-w}\right] = \frac{d}{dw}\left[\frac{e^{xwz+\frac{yz}{w}}}{(xw-xy)^2}\right]\frac{w}{1-w} + \frac{e^{xwz+\frac{yz}{w}}}{(xw-xy)^2}\frac{d}{dw}\left[\frac{w}{1-w}\right]$$

$$= \quad e^{xwz+\frac{yz}{w}}\left[\frac{\left(xw-\frac{yz}{w^2}\right)(xw-xy)-2x}{(xw-xy)^3}\frac{w}{1-w}+\frac{1}{(xw-xy)^2(1-w)^2}\right]$$

$$\Rightarrow Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big|\frac{1}{x}\right] = \lim_{w\to\frac{1}{x}}\frac{e^{xwz+\frac{yz}{w}}}{(xw-xy)^3(1-w)^2}\Big\{\big[\big(xw-\frac{yz}{w^2}\big)(xw-xy)-2x\big]w(1-w)+xw-xy\Big\}$$

$$= \frac{e^{xyz+z}}{(1-x)^2(1-xy)^3}\Big[xz(1-xy)^2(x-1)-x^2+2x-x^3y\Big].$$

Now we just need to have faith.

$$Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big|y\right]+Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big|\frac{1}{x}\right] = \frac{e^{xyz+z}}{(1-y)^2(1-xy)^3}\Big[z(1-xy)^2(y-1)+xy-2xy^2+1\Big]$$
$$+\frac{e^{xyz+z}}{(1-x)^2(1-xy)^3}\Big[xz(1-xy)^2(x-1)-x^2+2x-x^3y\Big]$$

$$= e^{xyz+z}\frac{[z(y-1)(1-xy)^2(1-x)^2+xz(1-xy)^2(x-1)(y-1)^2]+[(xy-2xy^2+1)(1-x)^2+(-x^2+2x-x^3y)(1-y)^2]}{(1-y)^2(1-x)^2(1-xy)^3}$$

$$= e^{xyz+z}\frac{[-z(x-1)(y-1)(1-xy)^3]+[(1-xy)^3]}{(1-y)^2(1-x)^2(1-xy)^3} = \frac{e^{xyz+z}}{(x-1)^2(y-1)^2}\Big[1-z(x-1)(y-1)\Big].$$

Altogether we have

$$\pi G_\infty(x,y,z) = -\frac{e^{-z}}{z^2}Res\left[\frac{g(xw)g\left(\frac{y}{w}\right)}{w(1-w)}\Big|1,\frac{1}{x},y\right]$$

$$= -\frac{e^{-z}}{z^2}\Big\{-\frac{e^{xz+yz}}{(x-1)^2(y-1)^2}+\frac{e^{xyz+z}}{(x-1)^2(y-1)^2}\Big[1-z(x-1)(y-1)\Big]\Big\}.$$

Substituting back to the original variables $z=\nu\mu$, $x=\frac{x}{\nu}$ and $y=\frac{y}{\mu}$ we get the final answer

$$\kappa^{(bulk)}(x,y|\nu,\mu) = \frac{1}{\pi}\frac{e^{xy}}{(x-\nu)^2(y-\mu)^2}\Big[e^{(x-\nu)(y-\mu)}-1+(x-\nu)(y-\mu)\Big].$$

□

Here you may notice some discrepancies between this answer and the one given by (44) in Corollary 1, but that's because (43) and (44) have been written in a more symmetric way, while this result uses ω as in (33) and without the $\frac{1}{\pi}$.

Now is time for the longer computation, which is thankfully is no longer that much longer for we have done some of the work.

4.5.2 Finite n

Reminder that

$$\pi G^{(n)}(x,y,z) = \frac{n+3}{f_{n+1}(z)z^2}\sum_{i,j=0}^{n}x^iy^j\frac{f_i(z)f_j(z)}{f_{i\vee j}(z)}A_{n,i\vee j}(z) \qquad (i)$$
$$-\frac{1}{f_{n+1}(z)z^2}\sum_{i,j=0}^{n}x_iy_j\frac{f_i(z)f_j(z)}{f_{i\vee j}(z)}B_{n,i\vee j}(z) \qquad (ii)$$

where

$$A_{n,j}(z) = ze_{j+1}^{n+1}(z) - (j+1)e_{j+2}^{n+2}(z)$$

$$B_{n,j}(z) = z^2 e_j^n(z) - zje_{j+1}^{n+1}(z) - (j+1)e_{j+2}^{n+2}(z)$$

and

$$e_j^n(z) = \sum_{s=j}^{n} \frac{z^s}{s!}.$$

The main difference for finite n, is that (ii) is no longer 0. Still, there is also more work that needs to be done for (i). But now we know the poles and those do not change.

For part (i) we have

$$C_1 = \oint \frac{dw}{2\pi i w(1-w)} \sum_{j=0}^{n} \left(\frac{y}{w}\right)^j A_{n,j}(z) \sum_{i=0}^{n} (xw)^i f_i(z)$$

and

$$C_2 = \oint \frac{dw}{2\pi i(1-w)} \sum_{j=0}^{n} (yw)^j f_j(z) \sum_{i=0}^{n} \left(\frac{x}{w}\right)^i A_{n,i}(z).$$

Like before, C_1 and C_2 are very similar, thus we need to do the work for only one of them. We again take each of the sums in C_1 separately. Since we know the poles, now the aim is to simple split the parts with singularity at 0 from the rest.

- $$\begin{aligned}
\sum_{i=0}^{n} (xw)^i f_i(z) &= \sum_{i=0}^{n}\sum_{s=0}^{i} (xw)^i \frac{i-s+1}{s!} z^s = \sum_{s=0}^{n} \frac{z^s}{s!} \sum_{i=s}^{n} (xw)^i (i-s+1) = \sum_{s=0}^{n} \frac{(xwz)^s}{s!} \sum_{i=0}^{n} (xw)^i (i+1) \\
&= \sum_{s=0}^{n} \frac{(xwz)^s}{s!} \frac{(n-s+1)(xw)^{n-s+2} - (n-s+2)(xw)^{n-s+1} + 1}{(1-xw)^2} \\
&= \frac{1}{(1-xw)^2} \sum_{s=0}^{n} \frac{(xwz)^s}{s!} + \sum_{s=0}^{n} \frac{(xwz)^s}{s!} \frac{(n-s+1)(xw)^{n-s+2} - (n-s+2)(xw)^{n-s+1}}{(1-xw)^2} \\
&= \frac{1}{(1-xw)^2} \sum_{s=0}^{n+2} \frac{(xwz)^s}{s!} + w^{n+1} D(x,z,w) = g_{n+2}(xw) + w^{n+1} D(x,z,w)
\end{aligned}$$

  where $D(x,z,w)$ has no singularity at $w=0$ and that's all we care about.

- This computation is very similar to before and yields

$$\sum_{j=0}^{n}\left(\frac{y}{w}\right)^{j}A_{n,j}(z) = \sum_{j=0}^{n}\left(\frac{y}{w}\right)^{j}[ze_{j+1}^{n+1}(z)-(j+1)e_{j+2}^{n+2}(z)]$$

$$= \frac{e_{n+2}(\frac{yz}{w})}{\left(1-\frac{y}{w}\right)^2}+\frac{z\left(1-\frac{y}{w}\right)e_{n+1}(z)-e_{n+2}(z)}{\left(1-\frac{y}{w}\right)^2}$$

$$= g_{n+2}\left(\frac{y}{w}\right)+\frac{z\left(1-\frac{y}{w}\right)e_{n+1}(z)-e_{n+2}(z)}{\left(1-\frac{y}{w}\right)^2}$$

where

$$g_n(x)=\frac{e^n(xz)}{(1-x)^2}.$$

To avoid confusion we call

$$e_n(x)=e_0^n(x)=\sum_{s=0}^{n}\frac{z^s}{s!}$$

as in

$$e_j^n(z)=\sum_{s=j}^{n}\frac{z^s}{s!}$$

for $j=0$.

$$\Rightarrow C_1=\oint_{\Gamma_1}\frac{dw}{2\pi iw(1-w)}\Big[g_{n+2}(xw)+w^{n+1}D(x,z,w)\Big]\left[g_{n+2}\left(\frac{y}{w}\right)+\frac{z\left(1-\frac{y}{w}\right)e_{n+1}(z)-e_{n+2}(z)}{\left(1-\frac{y}{w}\right)^2}\right]$$

with Γ_1 same as before.

Note that

$$\oint_{\Gamma_1}\frac{dw}{2\pi iw(1-w)}w^{n+1}D(x,z,w)\left[g_{n+2}\left(\frac{y}{w}\right)+\frac{z\left(1-\frac{y}{w}\right)e_{n+1}(z)-e_{n+2}(z)}{\left(1-\frac{y}{w}\right)^2}\right]=0$$

since $g_{n+2}\left(\frac{y}{w}\right)$ has a pole at 0 of degree n, thus the whole expression has no singularities inside of Γ_1.

Therefore

$$C_1=\oint_{\Gamma_1}\frac{dw}{2\pi iw(1-w)}g_{n+2}(xw)g_{n+2}\left(\frac{y}{w}\right)$$

and

$$C_2=\oint_{\Gamma_2}\frac{dw}{2\pi i(1-w)}g_{n+2}(yw)g_{n+2}\left(\frac{x}{w}\right).$$

Working the same way as before, we apply the change of variables $w=\dfrac{1}{w}$ on C_2 and get

$$C_2=-\int_{\tilde{\Gamma}_2}\frac{dw}{2\pi iw(1-w)}g_{n+2}\left(\frac{y}{w}\right)g_{n+2}(xw)$$

$$\Rightarrow C_1+C_2=-Res\left[\frac{g_{n+2}(xw)g_{n+2}\left(\frac{y}{w}\right)}{w(1-w)}\Big|1,\frac{1}{x},y\right].$$

This is just more residue computations I'm going to omit. So after a bit of suffering and substituting back to the original variables $z = \nu\mu$, $x = \frac{x}{\nu}$ and $y = \frac{y}{\mu}$ we have

$$(i) = \frac{(n+3)e_{n+2}(xy)e_{n+2}(\nu\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}\left[\frac{e_{n+2}(x\mu)e_{n+2}(y\nu)}{e_{n+2}(xy)e_{n+2}(\nu\mu)} - 1 + (x-\nu)(y-\mu) + H_{n+2}(x,y,\nu,\mu)\right]$$

where

$$H_n(x,y,\nu,\mu) \quad = \quad \frac{(x-\nu)(y-\mu)}{n!e_n(xy)e_n(\nu\mu)}\left(\frac{(\nu\mu)^{n+1}e_n(xy) - (xy)^{n+1}e_n(\nu\mu)}{xy-\nu\mu}\right)$$

or without substituting back to the original values.

$$(i) = \frac{(n+3)e_{n+2}(xyz)e_{n+2}(z)}{f_{n+1}(z)z^2(x-1)^2(y-1)^2}\left[\frac{e_{n+2}(xz)e_{n+2}(yz)}{e_{n+2}(xy)e_{n+2}(z)} - 1 + z(x-1)(y-1) + H_{n+2}(x,y,z)\right]$$

where

$$H_n(x,y,z) = \frac{z(x-1)(y-1)}{n!e_n(xyz)e_n(z)}\left(\frac{z^n e_n(xyz) - xy(xyz)^{n+1}e_n(z)}{xy-1}\right).$$

If we call

$$F_n(x,y,z) = -Res\left[\frac{g_n(xw)g_n\left(\frac{y}{w}\right)}{w(1-w)}\Big|1,\frac{1}{x},y\right]$$

then

$$(i) = \frac{n+3}{f_{n+1}(z)z^2}F_{n+2}(x,y,z).$$

We also have part (ii) to deal with
We once again split it into two parts C_3 and C_4.

$$\begin{aligned}\sum_{i,j=0}^n x^i y^j \frac{f_i(z)f_j(z)}{f_{i\vee j(z)}}B_{n,i\vee j}(z) &= \sum_{i,j=0}^n x^i y^j f_i(z)B_{n,j}(z)1_{j\geq i} + \sum_{i,j=0}^n x^i y^j f_j(z)B_{n,i}(z)1_{j<i} \\ &= C_3 + C_4\end{aligned}$$

where

$$B_{n,j}(z) = z^2 e_j^n(z) - zje_{j+1}^{n+1}(z) - (j+1)e_{j+2}^{n+2}(z).$$

The idea is still the same and exactly like before we get

$$C_3 = \oint \frac{dw}{2\pi i w(1-w)}\sum_{j=0}^n \left(\frac{y}{w}\right)^j B_{n,j}(z)\sum_{i=0}^n (xw)^i f_i(z)$$

and

$$C_4 = \oint \frac{dw}{2\pi i(1-w)}\sum_{j=0}^n (yw)^j f_j(z)\sum_{i=0}^n \left(\frac{x}{w}\right)^i B_{n,i}(z).$$

Again, dealing with only C_3 suffices. We have already done the work for $\sum_{i=0}^n (xw)^i f_i(z)$, so we focus on

$$\sum_{j=0}^n \Big(\frac{y}{w}\Big)^j B_{n,j}(z) = \sum_{j=0}^n \Big(\frac{y}{w}\Big)^j \Big(z^2 e_j^n(z) - zje_{j+1}^{n+1}(z) - (j+1)e_{j+2}^{n+2}(z)\Big).$$

We will do this in three steps so we don't get lost.

- $$\begin{aligned}\sum_{j=0}^n \Big(\frac{y}{w}\Big)^j z^2 e_j^n(z) &= z^2 \sum_{j=0}^n \Big(\frac{y}{w}\Big)^j \sum_{s=j}^n \frac{z^s}{s!} = z^2 \sum_{s=0}^n \frac{z^s}{s!} \sum_{j=0}^s \Big(\frac{y}{w}\Big)^j \\ &= z^2 \sum_{s=0}^n \frac{z^s}{s!} \frac{1-\big(\frac{y}{w}\big)^{s+1}}{1-\frac{y}{w}} \\ &= \frac{z^2}{1-\frac{y}{w}}\Big[e_n(z) - \frac{y}{w} e_n\Big(\frac{zy}{w}\Big)\Big]\end{aligned}$$

- $$\begin{aligned}-\sum_{j=0}^n \Big(\frac{y}{w}\Big)^j zje_{j+1}^{n+1}(z) &= -z\sum_{j=0}^n \Big(\frac{y}{w}\Big)^j j \sum_{j+1}^{n+1} \frac{z^s}{s!} = -z \sum_{s=1}^{n+1} \frac{z^s}{s!} \sum_{j=0}^{s-1} \Big(\frac{y}{w}\Big)^j [(j+1)-1] \\ &= -z \sum_{s=1}^{n+1} \frac{z^s}{s!} \frac{s\big(\frac{y}{w}\big)^{s+1} - (s+1)\big(\frac{y}{w}\big)^s + 1}{\big(1-\frac{y}{w}\big)^2} + z\sum_{s=1}^{n+1} \frac{z^s}{s!} \frac{\big(1-\frac{y}{w}\big)^s}{1-\frac{y}{w}} \\ &= \frac{z^2\frac{y}{w}}{1-\frac{y}{w}} e^n\Big(\frac{zy}{w}\Big) + \frac{z\frac{y}{w}}{\big(1-\frac{y}{w}\big)^2} e^{n+1}\Big(\frac{y}{w}\Big) + \frac{z\frac{y}{w}}{\big(1-\frac{y}{w}\big)^2}\end{aligned}$$

- and we have already computed

$$-\sum_{j=0}^n \Big(\frac{y}{w}\Big)^j (j+1)e_{j+2}^{n+2} = \frac{1}{\big(1-\frac{y}{w}\big)^2}\Big[z\Big(1-\frac{y}{w}\Big)e^{n+1}\Big(\frac{zy}{w}\Big) + e^{n+2}\Big(\frac{zy}{w}\Big) - e^{n+2}(z)\Big]$$

Altogether we have

$$\begin{aligned}\Rightarrow \sum_{j=0}^n \Big(\frac{y}{w}\Big)^j B_{n,j}(z) &= z\frac{e^{n+1}\big(\frac{yz}{w}\big)}{\big(1-\frac{y}{w}\big)^2} + \frac{e^{n+2}\big(\frac{zy}{w}\big)}{\big(1-\frac{y}{w}\big)^2} + \frac{z^2\big(1-\frac{y}{w}\big)e^n(z) + z\frac{y}{w}e^{n+1}(z) - e^{n+2}(z)}{\big(1-\frac{y}{w}\big)^2} \\ &= zg_{n+1}\Big(\frac{y}{w}\Big) + g_{n+2}\Big(\frac{y}{w}\Big) + w\tilde{D}_n(y,z,w)\end{aligned}$$

where $\tilde{D}_n(y,z,w)$ has no singularity at $w = 0$.

Therefore the expression for C_3 becomes

$$C_3 = \int_{\Gamma_1} \frac{dw}{2\pi i w(1-w)} \left[g_{n+2}(xw) + w^{n+1} D_n(x,z,w) \right] \left[z g_{n+1}\left(\frac{y}{w}\right) + g_{n+2}\left(\frac{y}{w}\right) + w\tilde{D}_n(y,z,w) \right]$$

After getting rid of all the terms with no singularities at 0, all that remains is

$$C_3 = \int_{\Gamma_1} \frac{dw}{2\pi i w(1-w)} \left[z g_{n+1}(xw) g_{n+1}\left(\frac{y}{w}\right) + g_{n+2}(xw) g_{n+2}\left(\frac{y}{w}\right) \right]$$

and similarly for C_4

$$C_4 = \int_{\Gamma_2} \frac{dw}{2\pi i(1-w)} \left[z g_{n+1}(yw) g_{n+1}\left(\frac{x}{w}\right) + g_{n+2}(yw) g_{n+2}\left(\frac{x}{w}\right) \right].$$

Following the same argument as before, we change variables on C_4, shrink $\tilde{\Gamma}_2$ to Γ_1 and get

$$\begin{aligned} C_3 + C_4 &= -zRes\left[\frac{g_{n+1}(xw) g_{n+1}\left(\frac{y}{w}\right)}{w(1-w)} \Big| 1, \frac{1}{x}, y \right] - Res\left[\frac{g_{n+2}(xw) g_{n+2}\left(\frac{y}{w}\right)}{w(1-w)} \Big| 1, \frac{1}{x}, y \right] \\ &= zF_{n+1}(x,y,z) + F_{n+2}(x,y,z) \\ \Rightarrow (ii) &= \frac{1}{f_{n+1}(z) z^2} F_{n+2}(x,y,z) + \frac{z}{f_{n+1}(z) z^2} F_{n+1}(x,y,z). \end{aligned}$$

So, summing (i) and $-(ii)$, the final expression for G_n is

$$\begin{aligned} \pi G_n(x,y,z) &= \frac{n+2}{f_{n+1}(z)(z)^2} F_{n+2}(x,y,z) - \frac{z}{f_{n+1}(z) z^2} F_{n+1}(x,y,z) \\ &= \frac{(n+2) e_{n+2}(xyz) e_{n+2}(z)}{f_{n+1}(z) z^2 (x-1)^2 (y-1)^2} \left[\frac{e_{n+2}(xz) e_{n+2}(yz)}{e_{n+2}(xyz) e_{n+2}(z)} - 1 + z(x-1)(y-1) + H_{n+2}(x,y,z) \right] \\ &\quad - \frac{z e_{n+1}(xyz) e_{n+1}(z)}{f_{n+1}(z) z^2 (x-1)^2 (y-1)^2} \left[\frac{e_{n+1}(xz) e_{n+1}(yz)}{e_{n+1}(xyz) e_{n+1}(z)} - 1 + z(x-1)(y-1) + H_{n+1}(x,y,z) \right] \end{aligned}$$

where

$$H_n(x,y,z) = \frac{z(x-1)(y-1)}{n! e_n(xyz) e_n(z)} \left(\frac{z^n e_n(xyz) - xy(xyz)^n e_n(z)}{xy-1} \right).$$

After changing back to the original variable, the final expression for the reduced kernel for finite n is

$$\begin{aligned}
\pi\kappa^{(n+1)}(x,y|\nu,\mu) &= \frac{n+2}{f_{n+1}(\nu\mu)(\nu\mu)^2}F_{n+2}(x,y,\nu\mu) - \frac{\nu\mu}{f_{n+1}(\nu\mu)(\nu\mu)^2}F_{n+1}(x,y,\nu\mu) \\
&= \frac{(n+2)e_{n+2}(xy)e_{n+2}(\nu\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}\left[\frac{e_{n+2}(x\mu)e_{n+2}(y\nu)}{e_{n+2}(xy)e_{n+2}(\nu\mu)} - 1 + (x-\nu)(y-\mu) + H_{n+2}(x,y,\nu,\mu)\right] \\
&\quad - \frac{\nu\mu e_{n+1}(xy)e_{n+1}(\nu\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}\left[\frac{e_{n+1}(x\mu)e_{n+1}(y\nu)}{e_{n+1}(xy)e_{n+1}(\nu\mu)} - 1 + (x-\nu)(y-\mu) + H_{n+1}(x,y,\nu,\mu)\right]
\end{aligned}$$

where

$$H_n(x,y,\nu,\mu) = \frac{(x-\nu)(y-\mu)}{n!e_n(xy)e_n(\nu\mu)}\left(\frac{(\nu\mu)^{n+1}e_n(xy) - (xy)^{n+1}e_n(\nu\mu)}{xy-\nu\mu}\right).$$

So

$$\kappa^{(n)}(x,y|\nu,\mu) = \frac{1}{\pi}\frac{(n+1)F_{n+1}(x,y,\nu,\mu) - \nu\mu F_n(x,y,\nu,\mu)}{f_n(\nu\mu)(\nu\mu)^2}$$

which although may not look like it at first glance, does correspond to (34) from Theorem 1, if you take $\frac{1}{\pi}$ to be part of the weight and take into account the definition of $\mathfrak{F}_n$. Through $\mathfrak{F}_n$, (34) is a much slicker way to write down the reduced kernel, but I also do like F_n for you can write the reduce kernel in terms of it, even without computing the residues. Not that this $H_n(x,y,z)$ and the one used in (149) are not the same, so it is a bit of an abuse of notation. But they are almost the same, since we where actively trying to match the two answers. I've kept them as they are to include some context as to why we are handling the sum in 3.4.4 the way that we do.

Even having the above answer, matching it with (134) from chapter 3.4.4, took us more than 2 months. Needless to say, getting to the final expression from (134) is not at all obvious and would probably be nigh impossible.

4.6 Bulk and edge scaling limits

In this section I'll present in much greater detail the computation regarding the bulk and edge scaling limits, from Corollaries 1 and 2. Nothing interesting happens here, just an endless computation as $n \to \infty$. For those calculations I'll use the reduced kernel $\kappa^{(n)}$ as shown in chapter 4.5.

4.6.1 Bulk Scaling Limit (Corollary 1)

Diagonal Overlaps:

In the bulk, the conditional overlaps scale as n, thus we use the following rescaling:

$$O_{11} = \frac{1}{n} < r_1, r_1 >< l_1, l_1 >$$

A quick reminder of where we are and what we are doing. We have

$$D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) := \mathbb{E}(O_{11}|\lambda_1, \lambda_2, ..., \lambda_k)\rho^{(n,k)}(\boldsymbol{\lambda}^{(k)})$$

for $k = 1, ..., n$, where $\boldsymbol{\lambda}^{(k)} = (\lambda_1, \lambda_k ..., \lambda_k)$ and $\rho^{(n,k)}(\boldsymbol{\lambda}^{(k)})$ is the k-point correlation function.

Then

$$D_{11}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty} \frac{1}{n} D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}).$$

We have from Theorem 1 that

$$D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{f_{n-1}(|\lambda_1|^2)}{\pi} e^{-|\lambda_1|^2} \det_{2\le i,j\le k} \left[K_{11}^{(n-1)}(\lambda_i, \bar{\lambda}_j)\right]$$

where

$$K_{11}^{(n)}(x, \bar{y}) = \omega(x, \bar{x}, \lambda_1, \bar{\lambda}_1)\kappa^{(n)}(x, \bar{y}|\lambda_1, \bar{\lambda}_1).$$

We want to compute the bulk scaling limit as $n \to \infty$. Only two things we have to do. First we take the limit of the term outside the determinant as $n \to \infty$, alongside the rescaling factor of n.

$$\frac{f_{n-1}(|\lambda_1|^2)}{n\pi} e^{-|\lambda_1|^2} = \frac{n e_{n-1}(|\lambda_1|^2) - |\lambda_1|^2 e_{n-2}(|\lambda_1|^2)}{n\pi} e^{-|\lambda_1|^2} \overset{\text{n}\to\infty}{=} \frac{1}{\pi}$$

Secondly we take the limit of the reduced kernel. For $\kappa^{(n)}$ we can, and have seen in 4.5, that in the bulk, not only the terms H_n disappear, but there is also no contribution from the second half. But here we shall use the reduced kernel for finite n and then take n to infinity in order to be consistent.

$$\begin{aligned}
\kappa^{(n)}(x, \bar{y}|\lambda_1, \bar{\lambda}_1) &= \frac{1}{\pi} \frac{(n+2)e_{n+2}(x\bar{y})e_{n+2}(\lambda_1\bar{\lambda}_1)}{f_{n+1}(\lambda_1\bar{\lambda}_1)(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_1)^2} \left[\frac{e_{n+2}(x\bar{\lambda}_1)e_{n+2}(\bar{y}\lambda_1)}{e_{n+2}(x\bar{y})e_{n+2}(\lambda_1\bar{\lambda}_1)} - 1 + (x-\lambda_1)(\bar{y}-\bar{\lambda}_1) + H_{n+2}(x,\bar{y},\lambda_1,\bar{\lambda}_1)\right] \\
&\quad - \frac{\lambda_1\bar{\lambda}_1 e_{n+1}(x\bar{y})e_{n+1}(\lambda_1\bar{\lambda}_1)}{f_{n+1}(\lambda_1\bar{\lambda}_1)(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_1)^2} \left[\frac{e_{n+1}(x\bar{\lambda}_1)e_{n+1}(\bar{y}\nu)}{e_{n+1}(x\bar{y})e_{n+1}(\lambda_1\bar{\lambda}_1)} - 1 + (x-\nu)(\bar{y}-\bar{\lambda}_1) + H_{n+1}(x,\bar{y},\lambda_1,\bar{\lambda}_1)\right] \\
&\overset{\text{n}\to\infty}{=} \frac{1}{\pi} \frac{e^{x\bar{y}}}{(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_1)^2} \left[e^{-(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)} - [1-(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)]\right] = \kappa^{(bulk)}
\end{aligned}$$

So for the bulk limit of the diagonal overlaps we have

$$D_{11}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{1}{\pi} \det_{2\le i,j\le k} \left[K_{11}^{(bulk)}(\lambda_i, \bar{\lambda}_j)\right]$$

where

$$\begin{aligned}
K_{11}^{(bulk)}(x, \bar{y}) &= \frac{1}{\pi} \frac{(1+|x-\lambda_1|^2)e^{-x\bar{x}+x\bar{y}}}{(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_1)^2} \left[e^{-(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)} - [1-(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)]\right] \\
&= \frac{1}{\pi} \frac{(1+|x-\lambda_1|^2)e^{-(x-\lambda_1)(\bar{x}-\bar{\lambda}_1)}}{(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_1)^2} \left[e^{(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)}(x-\lambda_1)(\bar{y}-\bar{\lambda}_1) - \left[e^{(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)} - 1\right]\right] \\
&= \frac{(1+|x-\lambda_1|^2)e^{-|x-\lambda_1|^2}}{\pi} \frac{d}{dz}\left(\frac{e^z-1}{z}\right)\Bigg|_{z=(x-\lambda_1)(\bar{y}-\bar{\lambda}_1)}
\end{aligned}$$

$\square$

This corresponds to (41)-(44) from Corollary 1.

Off-diagonal Ovelaps:

From Theorem 1 we have that the weighted multi-point intensities of the eigenvalues are given by

$$D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi} f_{n-1}(\lambda_1\bar{\lambda}_2)\kappa^{(n-1)}(\lambda_2,\bar{\lambda}_1|\lambda_1,\bar{\lambda}_2) \det_{3\le i,j\le k}\Big[K_{12}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big].$$

We want to compute the bulk scale limit

$$D_{12}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{n}D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$$

We already have $K_{12}^{(bulk)}$, $\kappa^{(bulk)}$ and the weight ω does not change. Then a straightforward computation yields

$$\begin{aligned}
D_{12}^{(bulk,k)}(\boldsymbol{\lambda}^{(k)}) &= \lim_{n\to\infty}\frac{1}{n}D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) \\
&= \lim_{n\to\infty} -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi}\frac{ne_{n-1}(\lambda_1\bar{\lambda}_2)-\lambda_1\bar{\lambda}_2 e_{n-2}(\lambda_1\bar{\lambda}_2)}{n}\kappa^{(n-1)}(\lambda_2,\bar{\lambda}_1|\lambda_1,\bar{\lambda}_2)\det_{3\le i,j\le k}\Big[K_{12}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big] \\
&= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi}e^{\lambda_1\bar{\lambda}_2}\kappa^{(bulk)}(\lambda_2,\bar{\lambda}_1|\lambda_1,\bar{\lambda}_2)\det_{3\le i,j\le k}\Big[K_{12}^{(bulk)}(\lambda_i,\bar{\lambda}_j)\Big] \\
&= -\frac{e^{-|\lambda_1-\lambda_2|^2}}{\pi^2(\lambda_2-\lambda_1)^2(\bar{\lambda}_1-\bar{\lambda}_2)^2}\Big[e^{-(\lambda_2-\lambda_1)(\bar{\lambda}_1-\bar{\lambda}_2)}-[1-(\lambda_2-\lambda_1)(\bar{\lambda}_1-\bar{\lambda}_2)]\Big]\det_{3\le i,j\le k}\Big[K_{12}^{(bulk)}(\lambda_i,\bar{\lambda}_j)\Big] \\
&= -\frac{1}{\pi^2}\frac{\Big[-|\lambda_1-\lambda_2|^2e^{-|\lambda_1-\lambda_2|^2}-(e^{-|\lambda_1-\lambda_2|^2}-1)\Big]}{-|\lambda_1-\lambda_2|^4}\det_{3\le i,j\le k}\Big[K_{12}^{(bulk)}(\lambda_i,\bar{\lambda}_j)\Big] \\
&= -\frac{1}{\pi^2}\det_{3\le i,j\le k}\Big[K_{12}^{(bulk)}(\lambda_i,\bar{\lambda}_j)\Big]\frac{d}{dz}\Big(\frac{e^z-1}{z}\Big)\Big|_{z=-|\lambda_1-\lambda_2|^2}
\end{aligned}$$

where

$$K_{12}^{(bulk)}(x,\bar{y}) = \frac{\omega(x,\bar{x}|\lambda_1,\bar{\lambda}_2)}{\kappa^{(bulk)}(\lambda_2,\bar{\lambda}_1|\lambda_1,\bar{\lambda}_2)}\det\begin{pmatrix}\kappa^{(bulk)}(x,\bar{y}|\lambda_1,\bar{\lambda}_2) & \kappa^{(bulk)}(x,\bar{\lambda}_1|\lambda_1,\bar{\lambda}_2) \\ \kappa^{(bulk)}(\lambda_2,\bar{y}|\lambda_1,\bar{\lambda}_2) & \kappa^{(bulk)}(\lambda_2,\bar{\lambda}_1|\lambda_1,\bar{\lambda}_2)\end{pmatrix}$$

$$\omega(x,\bar{y}|\lambda_1,\bar{\lambda}_2) = [1+(x-\lambda_1)(\bar{y}-\bar{\lambda}_2)]e^{-x\bar{y}}$$

and

$$\kappa^{(bulk)}(x,\bar{y}|\lambda_1,\bar{\lambda}_2) = \frac{1}{\pi}\frac{e^{x\bar{y}}}{(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_2)^2}\Bigg[e^{-(x-\lambda_1)(\bar{y}-\bar{\lambda}_2)}-[1-(x-\lambda_1)(\bar{y}-\bar{\lambda}_2)]\Bigg].$$

4.6.2 Edge Scaling Limit (Corollary 2)

Here we prove Corollary 2, but this time in great detail. It's not even an alternate proof, I'm just going to go through all the miniscule computations that are required. We need to include terms of order $n^{-\frac{1}{2}}$, which makes this a very lengthy computation. It's as straightforward and long as it gets, so even though I'll go in great detail, it will be light on explanations as it feels as more of an appendix entry.

We set $x = \sqrt{n}+x'$, $y = \sqrt{n}+y'$, $\nu = \sqrt{n}+\nu'$, $\mu = \sqrt{n}+\mu'$. Also $z = \nu\mu = n+\sqrt{n}(\nu'+\mu')+\nu'\mu' = n+\sqrt{n}a+b$, where $a = \nu' + \mu'$ and $b = \nu'\mu'$.

Note that it makes no difference as $n \to \infty$ whether we have $\kappa^n(\sqrt{n}+x', \sqrt{n}+\overline{y}')$ or $\kappa^{n-2}(\sqrt{n}+x', \sqrt{n}+\overline{y}')$.

Before we even go near the reduced kernel, we need to go through a long list terms which appear down the road.

Step by step.

-
$$\begin{aligned}
\frac{z^{n+1}}{n!\sqrt{n}} &= \frac{(n+\sqrt{n}a+b)^{n+1}}{n!\sqrt{n}} \frac{n^{n+1}}{n!\sqrt{n}}\Big(1+\frac{a}{\sqrt{n}}+\frac{b}{n}\Big)^{n+1} = \frac{n^{n+1}}{n!\sqrt{n}} e^{(n+1)log\left(1+\frac{a}{\sqrt{n}}+\frac{b}{n}\right)} \\
&= \frac{n^{n+\frac{1}{2}}}{n!} \exp\left\{(n+1)\Big[\Big(\frac{a}{\sqrt{n}}+\frac{b}{n}\Big)-\frac{1}{2}\Big(\frac{a}{\sqrt{n}}+\frac{b}{n}\Big)^2+\Big(\frac{a}{\sqrt{n}}+\frac{1}{3}\frac{b}{n}\Big)^3+O(n^{-4})\Big]\right\} \\
&= \frac{n^{n+\frac{1}{2}}}{n!} \exp\Big[\sqrt{n}a+b-\frac{a^2}{2}+\frac{a}{\sqrt{n}}-\frac{ab}{\sqrt{n}}+\frac{a^3}{3\sqrt{n}}+O(n^{-1})\Big] \\
&= \frac{n^{n+\frac{1}{2}}}{n!} e^{\sqrt{n}a+b-\frac{a^2}{2}}\Big[1+\frac{1}{\sqrt{n}}\Big(a-ab+\frac{a^3}{3}\Big)+O(n^{-1})\Big] \\
&= \frac{e^n}{\sqrt{2\pi}} e^{\sqrt{n}a+b-\frac{a^2}{2}}\Big[1+\frac{1}{\sqrt{n}}\Big(a-ab+\frac{a^3}{3}\Big)+O(n^{-1})\Big] \\
&= \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi}}\Big[1+\frac{1}{\sqrt{n}}\Big(a-ab+\frac{a^3}{3}\Big)+O(n^{-1})\Big]
\end{aligned}$$

  where we used the Stirling formula for $w = n$.

$$\Gamma(w+1) = \sqrt{2\pi}w^{w+\frac{1}{2}}e^{-w}\Big[1+\frac{1}{12w}+\frac{1}{128w^2}+O(w^{-3})\Big].$$

  Two other similar relations that will be useful are

$$\frac{z^{n+1}}{(n+1)!} = \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big[1+\frac{1}{\sqrt{n}}\Big(a-ab+\frac{a^3}{3}\Big)+O(n^{-1})\Big]$$

  and

$$\frac{z^{n+2}}{(n+2)!} = \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big[1+\frac{1}{\sqrt{n}}\Big(2a-ab+\frac{a^3}{3}\Big)+O(n^{-1})\Big].$$

- $$\begin{aligned}
e_n(z) &= \frac{e^z}{n!}\int_z^\infty t^n e^{-t}dt \overset{t=\tau z}{=} \frac{e^z}{n!}z^{n+1}\int_1^\infty \tau^n e^{-\tau z}d\tau = \frac{e^z}{n!}z^{n+1}\int_1^\infty e^{nlog\tau-\tau z}d\tau \\
&\overset{\tau=t+1}{=} \frac{e^z}{n!}z^{n+1}\int_0^\infty e^{n\log(t+1)-(t+1)z}dt = \frac{z^{n+1}}{n!}\int_0^\infty \exp\Big[n\Big(t-\frac{t^2}{2}+\frac{t^3}{3}+O(t^4)\Big)-(n+\sqrt{n}a+b)t\Big]dt \\
&\overset{t=\frac{t}{\sqrt{n}}}{=} \frac{z^{n+1}}{n!\sqrt{n}}\int_0^\infty \exp\Big[-\frac{t^2}{2}-at-\frac{bt}{\sqrt{n}}+\frac{t^3}{3\sqrt{n}}+O(n^{-1})\Big]dt \\
&= \frac{z^{n+1}}{n!\sqrt{n}}\int_0^\infty e^{-\frac{t^2}{2}-at}\Big[1+\frac{1}{\sqrt{n}}\Big(-bt+\frac{t^3}{3}\Big)+O(n^{-1})\Big]dt \\
&= \frac{z^{n+1}}{n!\sqrt{n}}\Big[\int_0^\infty e^{-\frac{t^2}{2}-at}dt+\frac{1}{\sqrt{n}}\int_0^\infty e^{-\frac{t^2}{2}-at}\Big(\frac{t^3}{3}-bt\Big)dt+O(n^{-1})\Big].
\end{aligned}$$

Reminder that

$$F(a) = \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-\frac{t^2}{2}}dt.$$

Before continuing with $e_n(z)$, we need to step aside and compute

$$\int_0^\infty e^{-\frac{t^2}{2}-at}\Big(\frac{t^3}{3}-bt\Big)dt$$

separately, by splitting it into two parts:

1)

$$\begin{aligned}
b\int_0^\infty(-t)e^{-\frac{t^2}{2}-at}dt &= b\frac{\partial}{\partial a}\int_0^\infty e^{-\frac{t^2}{2}-at}dt = b\frac{\partial}{\partial a}e^{\frac{a^2}{2}}\int_a^\infty e^{-\frac{t^2}{2}}dt = bae^{\frac{a^2}{2}}\int_a^\infty e^{-\frac{t^2}{2}}dt - be^{\frac{a^2}{2}}e^{-\frac{a^2}{2}} \\
&= ab\sqrt{2\pi}e^{\frac{a^2}{2}}F(a) - b
\end{aligned}$$

and

2)

$$\begin{aligned}
\int_0^\infty \frac{t^3}{3} e^{-\frac{t^2}{2}-at} dt &= -\frac{1}{3}\frac{\partial^3}{\partial a^3}\int_0^\infty e^{-\frac{t^2}{2}-at} dt = -\frac{1}{3}\frac{\partial^3}{\partial a^3} e^{\frac{a^2}{2}} \int_a^\infty e^{-\frac{t^2}{2}} dt = -\frac{1}{3}\frac{\partial^2}{\partial a^2}\Big[ae^{\frac{a^2}{2}} \int_a^\infty e^{-\frac{t^2}{2}} dt - 1\Big] \\
&= -\frac{1}{3}\frac{\partial}{\partial a}\Big[\Big(e^{\frac{a^2}{2}} + a^2 e^{\frac{a^2}{2}}\Big)\int_a^\infty e^{-\frac{t^2}{2}} dt - a\Big] \\
&= -\frac{1}{3}\Big[\Big(3ae^{\frac{a^2}{2}} + a^3 e^{\frac{a^2}{2}}\Big)\int_a^\infty e^{-\frac{t^2}{2}} dt - (1+a^2) - 1\Big] \\
&= \Big(-\frac{a^3}{3} - a\Big)\sqrt{2\pi}e^{\frac{a^2}{2}} F(a) + \frac{a^2+2}{3}.
\end{aligned}$$

Altogether we have

$$\int_0^\infty e^{-\frac{t^2}{2}-at}\Big(\frac{t^3}{3} - bt\Big)dt = \Big(-\frac{a^3}{3} + ab - a\Big)\sqrt{2\pi}e^{\frac{a^2}{2}} F(a) + \frac{a^2+2}{3} - b.$$

Back to $e_n(z)$.

$$\begin{aligned}
\Rightarrow e_n(z) &= \frac{\sqrt{2\pi}e^{\frac{a^2}{2}} z^{n+1}}{n!\sqrt{n}}\Big\{F(a) + \frac{1}{\sqrt{n}}\Big[\Big(-\frac{a^3}{3} + ab - a\Big)F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\frac{a^2+2-3b}{3}\Big] + O(n^{-1})\Big\} \\
&= e^z\Big[1 + \frac{1}{\sqrt{n}}\Big(\frac{a^3}{3} - ab + a\Big) + O(n^{-1})\Big]\Big\{F(a) + \frac{1}{\sqrt{n}}\Big[\Big(-\frac{a^3}{3} + ab - a\Big)F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\frac{a^2+2-3b}{3}\Big] + O(n^{-1})\Big\} \\
&= e^z\Big[F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi n}}\frac{a^2+2-3b}{3} + O(n^{-1})\Big].
\end{aligned}$$

Note that

$$e_{n+k}(z) = e^z\Big[F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big(\frac{a^2+2-3b}{3} + k\Big) + O(n^{-1})\Big].$$

This can be seen from the term

$$\int_0^\infty e^{(n+k)\log(t+1)-(t+1)z} dt = \int_0^\infty e^{n\log(t+1)-(t+1)z}(1+t)^k dt = \int_0^\infty e^{n\log(t+1)-(t+1)z}[1+kt+O(t^2)]dt.$$

- $$\begin{aligned} e_{n+2}(z) &= e_{n+1}(z) + \frac{z^{n+2}}{(n+2)!} = e_{n+1}(z) + \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big[1 + \frac{2a - ab + \frac{a^3}{3}}{\sqrt{n}} + O(n^{-1})\Big] \\ &= e_{n+1}(z) + \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi n}} + e^z O(n^{-1}). \end{aligned}$$

- $$\begin{aligned} f_{n+1}(z) &= (n+2)e_{n+1}(z) - ze_n(z) = (n+2)\Big[e_n(z) + \frac{z^{n+1}}{(n+1)!}\Big] - ze_n(z) \\ &= (n+2)\Big\{e_n(z) + \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big[1 + \frac{1}{\sqrt{n}}\Big(a - ab + \frac{a^3}{3}\Big) + O(n^{-1})\Big]\Big\} - ze_n(z) \\ &= (-a\sqrt{n} - b + 2)e_n(z) + \sqrt{\frac{n}{2\pi}}e^{z-\frac{a^2}{2}} + \frac{e^{z-\frac{a^2}{2}}}{\sqrt{2\pi}}\Big(\frac{a^3}{3} - ab + a\Big) + e^z O(n^{-\frac{1}{2}}) \\ &= \sqrt{\frac{n}{2\pi}}e^{z-\frac{a^2}{2}}\Bigg[e^{-z+\frac{a^2}{2}}\Big(-a + \frac{2-b}{\sqrt{n}}\Big)\sqrt{2\pi}e_n(z) + 1 + \frac{1}{\sqrt{n}}\Big(\frac{a^3}{3} - ab + a\Big) + O(n^{-1})\Bigg] \\ &= \sqrt{\frac{n}{2\pi}}e^{z-\frac{a^2}{2}}\Bigg\{ -a\Big[\sqrt{2\pi}e^{\frac{a^2}{2}}F(a) + \frac{1}{\sqrt{n}}\frac{a^2+2-3b}{3} + O(n^{-1})\Big] + 1 + \frac{1}{\sqrt{n}}\Big(\frac{a^3}{3} - ab + a\Big) + O(n^{-\frac{1}{2}})\Bigg\} \\ &= \sqrt{\frac{n}{2\pi}}e^{z-\frac{a^2}{2}}\Big[1 - a\sqrt{2\pi}e^{\frac{a^2}{2}}F(a) + O(n^{-\frac{1}{2}})\Big]. \end{aligned}$$

- $$\begin{aligned} e_{n+2}(x\mu)e_{n+2}(y\nu) &= \Bigg(e_{n+1}(x\mu) + \frac{e^{x\mu-\frac{(x'+\mu')^2}{2}}}{\sqrt{2\pi n}} + e^{x\mu}O(n^{-1})\Bigg)\Bigg(e_{n+1}(y\nu) + \frac{e^{y\nu-\frac{(y'+\nu')^2}{2}}}{\sqrt{2\pi n}} + e^{y\nu}O(n^{-1})\Bigg) \\ &= e_{n+1}(x\mu)e_{n+1}(y\nu) + \frac{1}{\sqrt{2\pi n}}\Big[e^{x\mu-\frac{(x'+\mu')^2}{2}}e_{n+1}(y\mu) + e^{y\nu-\frac{(y'+\nu')^2}{2}}e_{n+1}(x\mu)\Big] + e^{x\mu+y\nu}O(n^{-1}). \end{aligned}$$

Once again, we are going to work on them separately. Also a, c and b, d are going to be used in the place of sums and products respectively.

$$
\begin{aligned}
\frac{e^{x\mu-\frac{a^2}{2}}}{\sqrt{2\pi n}}e_{n+1}(y\mu) &= \frac{e^{x\mu-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big[e_n(y\nu)+\frac{(y\nu)^{n+1}}{(n+1)!}\Big] = \frac{e^{x\mu-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Bigg[e_n(y\nu)+\frac{e^{y\nu-\frac{c^2}{2}}}{\sqrt{2\pi n}}\Big(1+\frac{c-cd+\frac{d^3}{3}}{\sqrt{n}}\Big)\Bigg] \\
&= \frac{e^{x\mu-\frac{a^2}{2}}}{\sqrt{2\pi n}}e_n(y\nu)+e^{x\mu+y\nu}O(n^{-1}) \\
&= \frac{e^{x\mu-\frac{a^2}{2}}}{\sqrt{2\pi n}}e^{y\nu}\Big[F(c)+\frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi n}}\frac{c^2+2-3d}{3}+O(n^{-1})\Big]+e^{x\mu+y\nu}O(n^{-1}) \\
&= \frac{e^{x\mu+y\nu}}{\sqrt{2\pi n}}e^{-\frac{a^2}{2}}F(c)+e^{x\mu+y\nu}O(n^{-1}) = \frac{e^{x\mu+y\nu}}{\sqrt{2\pi n}}e^{-\frac{(x'+\mu')^2}{2}}F(y'+\nu')+e^{x\mu+y\nu}O(n^{-1})
\end{aligned}
$$

$$
\Rightarrow e_{n+2}(x\mu)e_{n+2}(y\nu) = e_{n+1}(x\mu)e_{n+1}(y\nu)+\frac{e^{x\mu+y\nu}}{\sqrt{2\pi n}}\Big[e^{-\frac{(x'+\mu')^2}{2}}F(y'+\nu')+e^{-\frac{(y'+\nu')^2}{2}}F(x'+\mu')+O(n^{-\frac{1}{2}})\Big]
$$

and similarly

$$
e_{n+2}(xy)e_{n+2}(\mu\nu) = e_{n+1}(xy)e_{n+1}(\mu\nu)+\frac{e^{xy+\mu\nu}}{\sqrt{2\pi n}}\Big[e^{-\frac{(x'+y')^2}{2}}F(\mu'+\nu')+e^{-\frac{(\mu'+\nu')^2}{2}}F(x'+y')+O(n^{-\frac{1}{2}})\Big]
$$

We are going to come back to $e_{n+1}(xy)e_{n+1}(\nu\mu)$ later.

- Let

$$
h_n(x,y,\nu,\mu) = \frac{(\nu\mu)^{n+1}e_n(xy)-(xy)^{n+1}e_n(\nu\mu)}{n!}
$$

Then

$$
\begin{aligned}
\tilde{H}_n(x,y,\nu\mu) &= \frac{(n+2)e^{n+2}(xy)e^{n+2}(\nu\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}H_{n+2}(x,y,\nu,\mu)-\frac{\nu\mu e^{n+1}(xy)e^{n+1}(\nu\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}H_{n+1}(x,y,\nu,\mu) \\
&= \frac{(x-\nu)(y-\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2(xy-\nu\mu)}\Big[(n+2)h_{n+2}(x,y,\nu,\mu)-(\nu\mu)h_{n+1}(x,y,\nu,\mu)\Big].
\end{aligned}
$$

We dub $J=(n+2)h_{n+2}(x,y,\nu,\mu)-(\nu\mu)h_{n+1}(x,y,\nu,\mu)$ and compute it separately.

$$
\begin{aligned}
J &= (n+2)\frac{(\nu\mu)^{n+3}e_{n+2}(xy)-(xy)^{n+3}e_{n+2}(\nu\mu)}{(n+2)!}-(\nu\mu)\frac{(\nu\mu)^{n+2}e_{n+1}(xy)-(xy)^{n+2}e_{n+1}((\nu\mu))}{(n+1)!}\\
&= \frac{(\nu\mu)^{n+3}e_{n+1}(xy)}{(n+1)!}+\frac{(xy)^{n+2}(\nu\mu)^{n+3}}{(n+1)!(n+2)!}-\frac{(\nu\mu)^{n+3}e_{n+1}(xy)}{(n+1)!}-\frac{(xy)^{n+3}e_{n+1}(\nu\mu)}{(n+1)!}\\
&\quad -\frac{(xy)^{n+3}(\nu\mu)^{n+2}}{(n+1)!(n+2)!}+\frac{\nu\mu(xy)^{n+2}e_{n+1}(\nu\mu)}{(n+1)!}\\
&= \frac{(\nu\mu-xy)(xy)^{n+2}}{(n+1)!}\frac{(\nu\mu)^{n+2}}{(n+2)!}+\frac{(xy)^{n+2}(\nu\mu-xy)}{(n+1)!}e_{n+1}(\nu\mu)\\
&= -\frac{(xy-\nu\mu)(xy)^{n+2}}{(n+1)!}e_{n+2}(\nu\mu)\\
\Rightarrow \tilde{H}_n(x,y,\nu\mu) &= -\frac{(x-\nu)(y-\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}\frac{(xy)^{n+2}}{(n+1)!}e_{n+2}(\nu\mu).
\end{aligned}
$$

- $$\frac{(xy)^{n+2}}{(n+1)!}e_{n+2}(\nu\mu)=(n+2)\frac{(xy)^{n+2}}{(n+2)!}e_{n+2}(\nu\mu)=$$

  $$=(n+2)\frac{e^{xy-\frac{c^2}{2}}}{\sqrt{2\pi n}}\Big[1+\frac{1}{\sqrt{n}}\Big(2c-cd+\frac{c^3}{3}\Big)+O(n^{-1})\Big]e^{\nu\mu}\Big[F(a)+\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big(\frac{a^2+2-3b}{3}+2\Big)+O(n^{-1})\Big]$$

  $$=\sqrt{\frac{n}{2\pi}}e^{xy+\nu\mu}e^{-\frac{(x'+y')^2}{2}}F(\nu'+\mu')+e^{xy+\nu\mu}O(1)=\sqrt{\frac{n}{2\pi}}e^{xy+\nu\mu}\Big[e^{-\frac{(x'+y')^2}{2}}F(\nu'+\mu')+O(n^{-\frac{1}{2}})\Big].$$

- $$1-(x-\nu)(y-\mu)=1-[(\sqrt{n}+x')-(\sqrt{n}+\nu')][(\sqrt{n}+y')-(\sqrt{n}+\mu')]=1-(x'-\nu')(y'-\mu').$$

-

$$\Rightarrow \quad e_{n+2}(x\mu)e_{n+2}(y\nu) - [1-(x-\nu)(y-\mu)]e_{n+2}(xy)e_{n+2}(\nu\mu) =$$

$$\begin{aligned} = \quad & e_{n+1}(x\mu)e_{n+1}(y\nu) - [1-(x'-\nu')(y'-\mu')]e_{n+1}(xy)e_{n+1}(\nu\mu) \\ & + \frac{e^{x\mu+y\nu}}{\sqrt{2\pi n}}\Big[e^{-\frac{(x'+\mu')^2}{2}}F(y'+\nu') + e^{-\frac{(y'+\nu')^2}{2}}F(x'+\mu') + O(n^{-\frac{1}{2}})\Big] \\ & -[1-(x'-\nu')(y'-\mu')]\frac{e^{xy+\mu\nu}}{\sqrt{2\pi n}}\Big[e^{-\frac{(x'+y')^2}{2}}F(\mu'+\nu') + e^{-\frac{(\mu'+\nu')^2}{2}}F(x'+y') + O(n^{-\frac{1}{2}})\Big]. \end{aligned}$$

-

$$\begin{aligned} e_{n+1}(xy)e_{n+1}(\nu\mu) &= e^{xy+\nu\mu}\Big[F(a) + \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi n}}\Big(\frac{a^2+2-3b}{3}+1\Big) + O(n^{-1})\Big] \times \\ & \quad \times\Big[F(c) + \frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi n}}\Big(\frac{c^2+2-3d}{3}+1\Big) + O(n^{-1})\Big] \\ &= e^{xy+\nu\mu}\Big[F(x'+y')F(\nu'+\mu') + O(n^{-\frac{1}{2}})\Big] \\ &= e^{2n+\sqrt{n}(x'+y'+\nu'+\mu')}e^{x'y'+\nu'\mu'}\Big[F(x'+y')F(\nu'+\mu') + O(n^{-\frac{1}{2}})\Big]. \end{aligned}$$

As boring and incoherent as that might have been, we now have all the terms for the reduce kernel and we are ready to substitute them in the following expression.

$$\begin{aligned} \kappa^{(n)}(x,y|\nu,\mu) &= \frac{\pi^{-1}(n+2)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}\Bigg[e_{n+2}(x\mu)e_{n+2} - [1-(x-\nu)(y-\mu)]e_{n+2}(xy)e_{n+2}(\nu\mu) + H_{n+2}(x,y,\nu,\mu)\Bigg] \\ &- \frac{\pi^{-1}(\nu\mu)}{f_{n+1}(\nu\mu)(x-\nu)^2(y-\mu)^2}\Bigg[e_{n+1}(x\mu)e_{n+1} - [1-(x-\nu)(y-\mu)]e_{n+1}(xy)e_{n+1}(\nu\mu) + H_{n+1}(x,y,\nu,\mu)\Bigg] \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \pi K_n(x,y|\nu,\mu) \quad = \quad & \frac{n+2-\nu\mu}{f_{n+1}(\nu\mu)(x'-\nu')^2(y'-\mu')^2}\Bigg[e_{n+1}(x\mu)e_{n+1}(y\nu)-[1-(x'-\nu')(y'-\mu')]e_{n+1}(xy)e_{n+1}(\nu\mu)\Bigg] \quad (i)\\
& +\frac{n+2}{f_{n+1}(\nu\mu)(x'-\nu')^2(y'-\mu')^2}\Bigg\{\frac{e^{x\mu+y\nu}}{\sqrt{2\pi n}}\Big[e^{-\frac{(x'+\mu')^2}{2}}F(y'+\nu')+e^{-\frac{(y'+\nu')^2}{2}}F(x'+\mu')\Big]\\
& -[1-(x'-\nu')(y'-\mu')]\frac{e^{xy+\mu\nu}}{\sqrt{2\pi n}}\Big[e^{-\frac{(x'+y')^2}{2}}F(\mu'+\nu')+e^{-\frac{(\mu'+\nu')^2}{2}}F(x'+y')\Big]\Bigg\} \quad (ii)\\
& -\frac{(x'-\nu')(y'-\mu')}{f_{n+1}(\nu\mu)(x'-\nu')^2(y'-\mu')^2}\sqrt{\frac{n}{2\pi}}e^{xy+\nu\mu}e^{-\frac{(x'+y')^2}{2}}F(\nu'+\mu') \quad (iii)\\
& +\frac{(n+2)O(n^{-1})}{f_{n+1}(\nu\mu)(x'-\nu')^2(y'-\mu')^2}\Bigg[\frac{e^{x\mu+y\nu}-e^{xy+\mu\nu}}{\sqrt{2\pi}}-[1-(x'-\nu')(y'-\mu')]\frac{e^{xy+\mu\nu}}{\sqrt{2\pi}}\Bigg]. \quad (iv)
\end{aligned}
$$

Note that the terms $e^{xy+\nu\mu}$ and $e^{x\mu+y\nu}$ are dependent on n, since x,y,ν,μ scale as $\sqrt{n}$, which means the reduced kernel on each own blows up as $n\to\infty$, but we are not yet ready for the whole expression. So let $\tilde{\kappa}^{(n)}$ be the reduced reduced kernel such that:

$$
\kappa^{(n)}(x,y)=e^{n+\sqrt{n}(x'+y')}\tilde{\kappa}^{(n)}(x,y).
$$

This definition is restricted only to the calculation of the edge limit, since it makes no sense in any other context. This was done so we can take $n\to\infty$ independently for $\tilde{K}_n(x,\bar{y})$.

Now we are going to compute $(i)-(iv)$, but for $\tilde{K}_n(x,y)=e^{-n-\sqrt{n}(x'+y')}K_n(x,y)$.

(i)
- $$
\begin{aligned}
\frac{n+2-\nu\mu}{f_{n+1}(\nu\mu)}e^{-n-\sqrt{n}(x'+y')} \quad &= \quad \frac{-\sqrt{n}(\nu'+\mu')+2-\nu'\mu'}{\sqrt{\frac{n}{2\pi}}e^{\nu\mu-\frac{a^2}{2}}\Big[1-a\sqrt{2\pi}e^{\frac{a^2}{2}}F(a)+O(n^{-\frac{1}{2}})\Big]}e^{-n-\sqrt{n}(x'+y')}\\
&= \quad \frac{-\sqrt{2\pi}(\nu'+\mu')e^{\frac{(\nu'+\mu')^2}{2}-\nu'\mu'}+O(n^{-\frac{1}{2}})}{1-\sqrt{2\pi}(\nu'+\mu')e^{\frac{(\nu'+\mu')^2}{2}}F(\nu'+\mu')+O(n^{-\frac{1}{2}})}e^{-2n-\sqrt{n}(x'+y'+\nu'+\mu')}.
\end{aligned}
$$
- $$
\begin{aligned}
&e_{n+1}(x\mu)e_{n+1}(y\nu)-[1-(x'-\nu')(y'-\mu')]e_{n+1}(xy)e_{n+1}(\nu\mu)=e^{2n+\sqrt{n}(x'+y'+\nu'+\mu')}\times\\
&\times\Bigg[e^{x'\mu'+y'\nu'}F(x'+\mu')F(y'+\nu')-[1-(x'-\nu')(y'-\mu')]e^{x'y'+\nu'\mu'}F(x'+y')F(\nu'+\mu')+O(n^{-\frac{1}{2}})\Bigg].
\end{aligned}
$$

Since the primes are not needed anymore, we are dropping them.
By this I mean we change (x',y',μ',ν') back to (x,y,μ,ν).

$$\Rightarrow (i) \quad = \quad \frac{-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}-\nu\mu}+O(n^{-\frac{1}{2}})}{\Big[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)+O(n^{-\frac{1}{2}})\Big](x-\nu)^2(y-\mu)^2}\times$$

$$\times\Bigg[e^{x\mu+y\nu}F(x+\mu)F(y+\nu)-[1-(x-\nu)(y-\mu)]e^{xy+\nu\mu}F(x+y)F(\nu+\mu)+O(n^{-\frac{1}{2}})\Bigg]$$

$$\overset{n\to\infty}{=} \quad \frac{-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}-\nu\mu}}{\Big[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)\Big](x-\nu)^2(y-\mu)^2}\times$$

$$\times\Bigg[e^{x\mu+y\nu}F(x+y)F(\nu+\mu)-[1-(x-\nu)(y-\mu)]e^{xy+\nu\mu}F(x+y)F(\nu+\mu)\Bigg]$$

$$= \quad \frac{-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}e^{xy}}{\Big[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)\Big](x-\nu)^2(y-\mu)^2}\times$$

$$\times\Bigg[e^{-(x-\nu)(y-\mu)}F(x+\mu)F(y+\nu)-[1-(x-\nu)(y-\mu)]F(x+y)F(\nu+\mu)\Bigg].$$

(ii)

$$\frac{(n+2)e^{-n-\sqrt{n}(x'+y')}}{f_{n+1}(\nu\mu)\sqrt{2\pi n}} = \frac{(n+2)e^{-n-\sqrt{n}(x'+y')}}{\sqrt{\frac{n}{2\pi}}e^{\nu\mu-\frac{a^2}{2}}\Big[1-a\sqrt{2\pi}e^{\frac{a^2}{2}}F(a)+O(n^{-\frac{1}{2}})\Big]\sqrt{2\pi n}}$$

$$= \frac{e^{\frac{(\nu'+\mu')^2}{2}}e^{-\nu'\mu'}+O(n^{-1})}{1-\sqrt{2\pi}(\nu'+\mu')e^{\frac{(\nu'+\mu')^2}{2}}F(\nu'+\mu')+O(n^{-\frac{1}{2}})}e^{-2n-\sqrt{n}(x'+y'+\nu'+\mu')}.$$

Thus, once again reverting back to (x, y, ν, μ), we have:

$$\Rightarrow (ii) \quad = \quad \frac{e^{\frac{(\nu+\mu)^2}{2}}e^{xy}+O(n^{-1})}{\Big[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)+O(n^{-\frac{1}{2}})\Big](x-\nu)^2(y-\mu)^2}\times$$

$$\times\Bigg\{e^{-(x-\nu)(y-\mu)}\Big[e^{-\frac{(x+\mu)^2}{2}}F(y+\nu)+e^{-\frac{(y+\nu)^2}{2}}F(x+\mu)\Big]-$$

$$-[1-(x-\nu)(y-\mu)]\Big[e^{-\frac{(x+y)^2}{2}}F(\mu+\nu)+e^{-\frac{(\mu+\nu)^2}{2}}F(x+y)\Big]\Bigg\}$$

$$\overset{n\to\infty}{=} \quad \frac{e^{\frac{(\nu+\mu)^2}{2}} e^{xy}}{\left[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)\right](x-\nu)^2(y-\mu)^2} \times$$

$$\times\left\{e^{-(x-\nu)(y-\mu)}\left[e^{-\frac{(x+\mu)^2}{2}}F(y+\nu)+e^{-\frac{(y+\nu)^2}{2}}F(x+\mu)\right]-\right.$$

$$\left.-[1-(x-\nu)(y-\mu)]\left[e^{-\frac{(x+y)^2}{2}}F(\mu+\nu)+e^{-\frac{(\mu+\nu)^2}{2}}F(x+y)\right]\right\}.$$

(iii) Similarly, this gives:

$$-\frac{e^{\frac{(\nu+\mu)^2}{2}} e^{xy}}{\left[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)\right](x-\nu)^2(y-\mu)^2}(x-\nu)(y-\mu)e^{-\frac{(x+y)^2}{2}}F(\mu+\nu)$$

and cancels out with a term from (ii).

(iv)

- $$\begin{aligned}\frac{(n+2)O(n^{-1})}{f_{n+1}(\nu\mu)}e^{-n-\sqrt{n}(x'+y')} &= \frac{O(1)e^{-n-\sqrt{n}(x'+y')}}{\sqrt{\frac{n}{2\pi}}e^{\nu\mu-\frac{a^2}{2}}\left[1-a\sqrt{2\pi}e^{\frac{a^2}{2}}F(a)+O(n^{-\frac{1}{2}})\right]} \\ &= O(n^{-\frac{1}{2}})\frac{O(1)e^{-n-\sqrt{n}(x'+y'+\mu'+\nu')}}{O(1)+O(n^{-\frac{1}{2}})} \\ &= O(n^{-\frac{1}{2}})e^{-n-\sqrt{n}(x'+y'+\mu'+\nu')}.\end{aligned}$$

- $$\frac{e^{x\mu+y\nu}-e^{xy+\mu\nu}}{\sqrt{2\pi}}-[1-(x'-\nu')(y'-\mu')]\frac{e^{xy+\mu\nu}}{\sqrt{2\pi}} = e^{-n-\sqrt{n}(x'+y'+\mu'+\nu')}O(1)$$

$$\Rightarrow (iv) = O(n^{-\frac{1}{2}}) \overset{n\to\infty}{=} 0.$$

Altogether we have that

$$\begin{aligned}\tilde{\kappa}^{(edge)}\infty(x,y) &= \frac{1}{\pi}\frac{e^{\frac{(\nu+\mu)^2}{2}} e^{xy}}{\left[1-\sqrt{2\pi}(\nu+\mu)e^{\frac{(\nu+\mu)^2}{2}}F(\nu+\mu)\right](x-\nu)^2(y-\mu)^2} \times \\ &\times\left\{-\sqrt{2\pi}(\nu+\mu)\left[e^{-(x-\nu)(y-\mu)}F(x+\mu)F(y+\nu)-[1-(x-\nu)(y-\mu)]F(x+y)F(\nu+\mu)\right]\right. \\ &+e^{-(x-\nu)(y-\mu)}\left[e^{-\frac{(x+\mu)^2}{2}}F(y+\nu)+e^{-\frac{(y+\nu)^2}{2}}F(x+\mu)\right]-[1-(x-\nu)(y-\mu)]e^{-\frac{(\mu+\nu)^2}{2}}F(x+y) \\ &\left.-e^{-\frac{(x+y)^2}{2}}F(\mu+\nu)\right\}.\end{aligned}$$

And now we can finally start computing the edge limit of the reduced kernel for D_{11} and D_{12}.

Off-diagonal Overlaps

Remember that for the edge scaling limit we rescale the conditional overlaps by $\sqrt{n}$ instead of n.

As K_n is the reduced kernel, the expression for the full kernel is:

$$K_{12}^{(n)}(x,\bar{y}) = \frac{\omega(x,\bar{x}|\nu,\bar{\mu})}{\kappa^{(n)}(\mu,\bar{\nu}|\nu,\bar{\mu})} \det\begin{pmatrix} \kappa^{(n)}(x,\bar{y}|\nu,\bar{\mu}) & \kappa^{(n)}(x,\bar{\nu}|\nu,\bar{\mu}) \\ \kappa^{(n)}(\mu,\bar{y}|\nu,\bar{\mu}) & \kappa^{(n)}(\mu,\bar{\nu}|\nu,\bar{\mu}) \end{pmatrix}$$

where

$$\omega(x,\bar{y}|\nu,\bar{\mu}) = [1 + (x-\nu)(\bar{y}-\bar{\mu})]e^{-x\bar{y}}$$

Then

$$\begin{aligned} K_{12}^{(n)}(x,\bar{y}) &= \frac{\omega(x,\bar{x})}{e^{n+\sqrt{n}(\mu'+\bar{\nu}')}\tilde{\kappa}^{(n)}(\mu,\bar{\nu})} \det\begin{pmatrix} e^{n+\sqrt{n}(x'+\bar{y}')}\tilde{\kappa}^{(n)}(x,\bar{y}) & e^{n+\sqrt{n}(x'+\bar{\nu}')}\tilde{\kappa}^{(n)}(x,\bar{\nu}) \\ e^{n+\sqrt{n}(\mu'+\bar{y}')}\tilde{\kappa}^{(n)}(\mu,\bar{y}) & e^{n+\sqrt{n}(\mu'+\bar{\nu}')}\tilde{\kappa}^{(n)}(\mu,\bar{\nu}) \end{pmatrix} \\ &= \frac{[1-(x'-\nu')(\bar{x}'-\bar{\mu}')]e^{-|x|^2}e^{n+\sqrt{n}(x+\bar{y})}}{\tilde{\kappa}^{(n)}(\mu,\bar{\nu})} \det\begin{pmatrix} \tilde{\kappa}^{(n)}(x,\bar{y}) & \tilde{\kappa}^{(n)}(x,\bar{\nu}) \\ \tilde{\kappa}^{(n)}(\mu,\bar{y}) & \tilde{\kappa}^{(n)}(\mu,\bar{\nu}) \end{pmatrix}. \end{aligned}$$

Again from Theorem one, the expression for $D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$ is given by

$$D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi} f_{n-1}(\lambda_1\bar{\lambda}_2)\kappa^{(n-1)}(\lambda_2,\bar{\lambda}_1) \det_{3\le i,j\le k}\Big[K_{12}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big].$$

Define $\tilde{L}_{12}^{(n)}(x,\bar{y})$ to be

$$\tilde{L}_{12}^{(n)}(x,\bar{y}) = \frac{1+(x'-\lambda_1')(\bar{x}'-\bar{\lambda}_2')}{\tilde{\kappa}^{(n)}(\lambda_2,\bar{\lambda}_1)} \det\begin{pmatrix} \tilde{\kappa}^{(n)}(x,\bar{y}) & \tilde{\kappa}^{(n)}(x,\bar{\lambda}_1) \\ \tilde{\kappa}^{(n)}(\lambda_2,\bar{y}) & \tilde{\kappa}^{(n)}(\lambda_2,\bar{\lambda}_1) \end{pmatrix}.$$

Then

$$\begin{aligned} \det_{3\le i,j\le k}\Big[K_{12}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big] &= \prod_{s=3}^{k} e^{-|\lambda_s|^2}e^{n+\sqrt{n}(\lambda_s'+\bar{\lambda}_s')} \det_{3\le i,j\le k}\Big[\tilde{L}_{12}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big] \\ &= \prod_{s=3}^{k} e^{-\lambda_s'\bar{\lambda}_s'} \det_{3\le i,j\le k}\Big[\tilde{L}_{12}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big] \end{aligned}$$

and last but not least

$$
\begin{aligned}
\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi}\kappa^{(n-1)}(\lambda_2,\bar\lambda_1)f_{n-1}(\lambda_1\bar\lambda_2) &= \frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi}e^{n+\sqrt{n}(\lambda_2'+\bar\lambda_1')}\tilde\kappa^{(n-1)}(\lambda_2,\bar\lambda_1)\sqrt{\frac{n}{2\pi}}e^{\lambda_1\bar\lambda_2-\frac{(\lambda_1'+\bar\lambda_2')^2}{2}}\times \\
&\times\Big[1-\sqrt{2\pi}(\lambda_1'+\bar\lambda_2')e^{\frac{(\lambda_1'+\bar\lambda_2')^2}{2}}F(\lambda_1'+\bar\lambda_2')+O(n^{-\frac{1}{2}})\Big] \\
&= \sqrt{\frac{n}{2\pi}}\frac{e^{-(\lambda_1'-\lambda_2')(\bar\lambda_1'-\bar\lambda_2')}}{\pi}e^{-\lambda_2'\bar\lambda_1'}e^{-\frac{(\lambda_1'+\bar\lambda_2')^2}{2}}\tilde\kappa^{(n-1)}(\lambda_2,\bar\lambda_1)\times \\
&\times\Big[1-\sqrt{2\pi}(\lambda_1'+\bar\lambda_2')e^{\frac{(\lambda_1'+\bar\lambda_2')^2}{2}}F(\lambda_1'+\bar\lambda_2')+O(n^{-\frac{1}{2}})\Big].
\end{aligned}
$$

So by dropping the primes this time for good, rescaling by $\sqrt{n}$ and taking $n\to\infty$ we finally have

$$
D_{12}^{(edge,k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}D_{12}^{(n,k)}(e^{i\theta}(\sqrt{n}+\boldsymbol{\lambda}^{(k)})) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}D_{12}^{(n,k)}(\sqrt{n}+\boldsymbol{\lambda}^{(k)})
$$

$$
\begin{aligned}
&= -\frac{e^{-(\lambda_1-\lambda_2)(\bar\lambda_1-\bar\lambda_2)}}{\pi\sqrt{2\pi}}e^{-\lambda_2\bar\lambda_1}e^{-\frac{(\lambda_1+\bar\lambda_2)^2}{2}}\tilde\kappa^{(edge)}(\lambda_2,\bar\lambda_1)\prod_{s=3}^{k}e^{-|\lambda_s|^2}\det_{3\le i,j\le k}\Big[\tilde L_{12}^{(egde)}(\lambda_i,\bar\lambda_j)\Big]\times \\
&\times\Big[1-\sqrt{2\pi}(\lambda_1+\bar\lambda_2)e^{\frac{(\lambda_1+\bar\lambda_2)^2}{2}}F(\lambda_1+\bar\lambda_2)\Big]
\end{aligned}
$$

$$
= -\frac{e^{-\frac{\lambda_1^2}{2}-\frac{\bar\lambda_2^2}{2}}}{\pi\sqrt{2\pi}}\tilde\kappa^{(edge)}(\lambda_2,\bar\lambda_1)\prod_{s=1}^{k}e^{-|\lambda_s|^2}\Big[1-\sqrt{2\pi}(\lambda_1+\bar\lambda_2)e^{\frac{(\lambda_1+\bar\lambda_2)^2}{2}}F(\lambda_1+\bar\lambda_2)\Big]\det_{3\le i,j\le k}\Big[L_{12}^{(egde)}(\lambda_i,\bar\lambda_j)\Big]
$$

where

$$
\tilde L_{12}^{(egde)}(\lambda_i,\bar\lambda_j) = \frac{1+(x-\lambda_1)(\bar x-\bar\lambda_2)}{\tilde\kappa^{(edge)}(\lambda_2,\bar\lambda_1)}\det\begin{pmatrix}\tilde\kappa^{(edge)}(x,\bar y) & \tilde\kappa^{(edge)}(x,\bar\lambda_1)\\ \tilde\kappa^{(edge)}(\lambda_2,\bar y) & \tilde\kappa^{(edge)}(\lambda_2,\bar\lambda_1)\end{pmatrix}
$$

and

$$
\begin{aligned}
\tilde\kappa^{(edge)}(x,\bar y) &= \frac{1}{\pi}\frac{e^{\frac{(\lambda_1+\bar\lambda_2)^2}{2}}e^{x\bar y}}{\Big[1-\sqrt{2\pi}(\lambda_1+\bar\lambda_2)e^{\frac{(\lambda_1+\bar\lambda_2)^2}{2}}F(\lambda_1+\bar\lambda_2)\Big](x-\lambda_1)^2(\bar y-\bar\lambda_2)^2}\times \\
&\times\Big\{-\sqrt{2\pi}(\lambda_1+\bar\lambda_2)\Big[e^{-(x-\lambda_1)(\bar y-\bar\lambda_2)}F(x+\bar\lambda_2)F(\bar y+\lambda_1)-[1-(x-\lambda_1)(\bar y-\bar\lambda_2)]F(x+\bar y)F(\lambda_1+\bar\lambda_2)\Big] \\
&+e^{-(x-\lambda_1)(\bar y-\bar\lambda_2)}\Big[e^{-\frac{(x+\bar\lambda_2)^2}{2}}F(\bar y+\lambda_1)+e^{-\frac{(\bar y+\lambda_1)^2}{2}}F(x+\bar\lambda_2)\Big] \\
&-[1-(x-\lambda_1)(\bar y-\bar\lambda_2)]e^{-\frac{(\lambda_1+\bar\lambda_2)^2}{2}}F(x+\bar y) \\
&-e^{-\frac{(x+\bar y)^2}{2}}F(\lambda_1+\bar\lambda_2)\Big\}.
\end{aligned}
$$

In particular for $k=2$ we have that

$$D_{12}^{(edge,2)}(\boldsymbol{\lambda}^{(2)}) = -\frac{e^{-\frac{\lambda_1^2}{2}-\frac{\bar{\lambda}_2^2}{2}}}{\pi\sqrt{2\pi}}\tilde{K}_\infty(\lambda_2,\bar{\lambda}_1)\prod_{s=1}^{2}e^{-|\lambda_s|^2}\Big[1-\sqrt{2\pi}(\lambda_1+\bar{\lambda}_2)e^{\frac{(\lambda_1+\bar{\lambda}_2)^2}{2}}F(\lambda_1+\bar{\lambda}_2)\Big]$$

$$\begin{aligned}
&= -\frac{1}{\pi^2}\frac{e^{-|\lambda_1-\lambda_2|^2}}{\sqrt{2\pi}(x-\lambda_1)^2(\bar{y}-\bar{\lambda}_2)^2}\times \\
&\times\Bigg\{-\sqrt{2\pi}(\lambda_1+\bar{\lambda}_2)\Bigg[e^{|\lambda_1-\lambda_2|^2}F(\lambda_2+\bar{\lambda}_2)F(\bar{\lambda}_1+\lambda_1)-[1+|\lambda_1-\lambda_2|^2)]F(\lambda_2+\bar{\lambda}_1)F(\lambda_1+\bar{\lambda}_2)\Bigg] \\
&\quad +e^{|\lambda_1-\lambda_2|^2}\Big[e^{-\frac{(\lambda_2+\bar{\lambda}_2)^2}{2}}F(\bar{\lambda}_1+\lambda_1)+e^{-\frac{(\bar{\lambda}_1+\lambda_1)^2}{2}}F(\lambda_2+\bar{\lambda}_2)\Big] \\
&\quad -[1+|\lambda_1-\lambda_2|^2]e^{-\frac{(\lambda_1+\bar{\lambda}_2)^2}{2}}F(\lambda_2+\bar{\lambda}_1)-e^{-\frac{(\lambda_2+\bar{\lambda}_1)^2}{2}}F(\lambda_1+\bar{\lambda}_2)\Bigg\}.
\end{aligned}$$

Diagonal Overlaps:

For the diagonal overlaps we have from Theorem 1

$$D_{11}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) = \frac{f_{n-1}(|\lambda_1|^2)}{\pi}e^{-|\lambda_1|^2}\det_{2\le i,j\le k}\Big[K_{11}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big].$$

Then

$$D_{11}^{(edge,k)}(\boldsymbol{\lambda}^{(k)}) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}D_{11}^{(n,k)}(e^{i\theta}(\sqrt{n}+\boldsymbol{\lambda}^{(k)})) = \lim_{n\to\infty}\frac{1}{\sqrt{n}}D_{11}^{(n,k)}(\sqrt{n}+\boldsymbol{\lambda}^{(k)})$$

$$\begin{aligned}
&= \lim_{n\to\infty}\sqrt{\frac{n}{2\pi}}\frac{e^{\lambda_1\bar{\lambda}_1-\frac{(\lambda_1'+\bar{\lambda}_1')^2}{2}}\Big[1-\sqrt{2\pi}(\lambda_1'+\bar{\lambda}_1')e^{\frac{(\lambda_1'+\bar{\lambda}_1')^2}{2}}F(\lambda_1'+\bar{\lambda}_1')+O(n^{-\frac{1}{2}})\Big]}{\sqrt{n}\pi}e^{-|\lambda_1|^2}\times \\
&\quad\times\prod_{s=2}^{k}e^{-\lambda_s'\bar{\lambda}_s'}\det_{2\le i,j\le k}\Big[\tilde{L}_{11}^{(n-1)}(\lambda_i,\bar{\lambda}_j)\Big]
\end{aligned}$$

$$\text{(dropping primes)} = \frac{e^{-\frac{(\lambda_1+\bar{\lambda}_1)^2}{2}}}{\pi\sqrt{2\pi}}\Big[1-\sqrt{2\pi}(\lambda_1+\bar{\lambda}_1)e^{\frac{(\lambda_1+\bar{\lambda}_1)^2}{2}}F(\lambda_1+\bar{\lambda}_1)\Big]\prod_{s=2}^{k}e^{-|\lambda_s|^2}\det_{2\le i,j\le k}\Big[\tilde{L}_{11}^{(edge)}(\lambda_i,\bar{\lambda}_j)\Big]$$

where

$$\tilde{L}_{11}^{(n)}(x,\bar{y}) = (1+|x-\lambda_1|^2)\tilde{\kappa}^{(n-1)}(x,\bar{y}|\lambda_1,\bar{\lambda}_1)$$

and

$$\tilde{L}_{11}^{(edge)}(x,\bar{y}) = (1+|x-\lambda_1|^2)\tilde{\kappa}^{(edge)}(x,\bar{y}|\lambda_1,\bar{\lambda}_1).$$

5 Conditioning on Determinantal and Pfaffian Point Processes

In this chapter we first consider determinantal point process on a countable set (to be concrete, on $\mathbb{Z}$). We show that the point process, conditioned on whether a finite subset of points are either occupied or empty, remains a determinantal point process, and give a formula for the new kernel (see Theorem 2 below). We extend this to Pfaffian point process in Theorem 3, which I will not present yet, for it will make little sense out of context. The proofs are elementary, and reduce to certain finite determinantal and Pfaffian identities which we recall in an appendix. The derivation of these expressions comes from standard conditioning through Bayes formula. Using this method it is not immediately clear that the resulting point process is a determinantal one, so we use the determinantal identity first found in [21] to rearrange the elements and produce an explicit expression for the new kernel. In section 5.1.1 we work on discrete determinantal point processes and prove Theorem 2. In section 5.1.2 we do the same work, but for discrete Pfaffian point processes, and prove Theorem 3, which can be seen as the natural expansion of Theorem 2 for Pfaffians. In Section 5.2 we extend these results to the natural continuum analogues for point processes on $\mathbb{R}$, though for the time we have yet to explore the Pfaffian case. We explore alternative proofs using of the modern point process techniques of Campbell and Palm measures to state the results for conditioned processes and characterise processes via Laplace functionals, which mesh well with the algebraic structure. This is on-going work. As a concrete example of the conditioning, we show in the final section that the complex Ginibre ensemble, conditioned to have k points at the origin, remains determinantal and we give an explicit formula for the new kernel.

One of our hopes is that we will be able use this study to get some results on rigidity (see [7,16,22] for reference). Say we have a point process on $\mathbb{R}$ and let $(a,b) \in \mathbb{R}$ be an interval. That point process is called rigid, if by conditioning on everything outside the interval (a,b), we can determine the exact number of particles inside (a,b). We hope we will be able to use our expressions for the kernels of the conditioned determinantal point process to find under which circumstance a determinantal p.p. has the rigidity property.finally we give an example of conditioning using the point process formed by the eigenvalues on the complex Ginibre ensemble.

This chapter has little to nothing to do with overlaps and the previous chapters. It's a tangent I went on after coming across a determinantal identity while we were computing $D_{12}^{(n,3)}(\lambda_1,\lambda_2,\lambda_3)$ in two different ways. Remember that from Theorem 1 we have

$$\begin{aligned} D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)}) &= -\frac{e^{-|\lambda_1|^2-|\lambda_2|^2}}{\pi^2} f_{n-1}(\lambda_1\bar{\lambda}_2)\kappa^{(n-1)}(\bar{\lambda}_1,\lambda_2 \mid \lambda_1,\bar{\lambda}_2) \\ &\quad \times \det_{3\le i,j\le k}\left(K_{12}^{(n-1)}\left(\lambda_i,\bar{\lambda}_i,\lambda_j,\bar{\lambda}_j \mid \lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2\right)\right) \end{aligned}$$

where from (36) in Theorem 1 we know that $K_{12}^{(n-1)}$ itself is given by a 2x2 determinant

$$\begin{aligned} K_{12}^{(n)}(x,\bar{x},y,\bar{y} \mid u,\bar{u},v,\bar{v}) &= \frac{\omega(x,\bar{x} \mid u,\bar{v})}{\kappa^{(n)}(\bar{u},v \mid u,\bar{v})} \\ &\quad \times \det\begin{pmatrix} \kappa^{(n)}(\bar{u},v \mid u,\bar{v}) & \kappa^{(n)}(\bar{u},y \mid u,\bar{v}) \\ \kappa^{(n)}(\bar{x},v \mid u,\bar{v}) & \kappa^{(n)}(\bar{x},y \mid u,\bar{v}) \end{pmatrix}. \end{aligned}$$

Without the use of transposition

$$\hat{T}f(\lambda_1,\bar{\lambda}_1,\lambda_2,\bar{\lambda}_2,\ldots) = f(\lambda_1,\bar{\lambda}_2,\lambda_2,\bar{\lambda}_1,\ldots),$$

at first in was not at all obvious that $K_{12}^{(n)}$ could be written as a 2x2 determinant, and heuristic computations of $D_{12}^{(n,k)}(\boldsymbol{\lambda}^{(k)})$ yielded a determinant one size larger that in Theorem 1 and not in determinantal form. Turns out, that was because of this identity

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.$$

which can be generalised as follows:

Lemma: Suppose we have an $n \times n$ matrix with elements a_{ij}. For some $1 \leq k \leq n-2$, let K be

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & & . \\ \vdots & & \ddots & . \\ a_{k1} & . & . & a_{kk} \end{pmatrix} = K.$$

Then the following identity is true for every $1 \leq k \leq n-2$

$$\left| \begin{array}{cccc|cc} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & & . & & \\ \vdots & & \ddots & . & & \vdots \\ a_{k1} & . & . & a_{kk} & & \\ \hline \vdots & & & & \ddots & \\ a_{n1} & & \cdots & & & a_{nn} \end{array} \right| = \frac{1}{|K|^{n-k-1}} \begin{vmatrix} \begin{vmatrix} K & & a_{1,k+1} \\ & & . \\ & & . \\ a_{k+1,1} & \cdots & a_{k+1,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} K & & a_{1,n} \\ & & . \\ & & . \\ a_{k+1,1} & \cdots & a_{k+1,n} \end{vmatrix} \\ . & & . \\ . & & . \\ \begin{vmatrix} K & & a_{1,k+1} \\ & & . \\ & & . \\ a_{n,1} & \cdots & a_{n,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} K & & a_{1,n} \\ & & . \\ & & . \\ a_{n,1} & \cdots & a_{nn} \end{vmatrix} \end{vmatrix}.$$

where K is a $k \times k$ matrix, the large determinat is of size $(n-k) \times (n-k)$, and its elements are $(k+1) \times (k+1)$ determinants of matrices comprised by the matrix K by a the row $(\alpha_{h,j})_{j=1}^{k+1}$ and a column $(\alpha_{i,g})_{i=1}^{k+1}$, for $k+1 \leq h, g \leq n$. We have re-written an $n \times n$ determinant as a ratio of an $(n-k) \times (n-k)$ determinant with $(k+1) \times (k+1)$ determinants as elements, over a $k \times k$ determinant to the power o$n-k-1$.

Proof of this Lemma is in the appendix.

What we did was proceed to used this identity to find explicit formulas for the kernels of determinant and Pfaffian point process, conditioned on having particles at certain points.

Random Point Processes: A small introduction on the random p.p. we will be working with. Nothing fancy, just some basic information.

Let $\mathbb{X}$ be either $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ or $\mathbb{C}$. A locally finite collection of points X in $\mathbb{X}$ is called a point configuration. The set of all point configurations in $\mathbb{X}$ is denoted by $Conf(\mathbb{X})$. A random point process in $\mathbb{X}$ is a probability measure on $Conf(\mathbb{X})$. Under mild conditions on the process (reference Borodin), a sequence of symmetric measures $(\rho_n)_{n=1}^{\infty}$ exists on $\mathbb{X}^n$, where ρ_n is called the nth correlation measure.

In the discrete cases of $\mathbb{Z}$ or $\mathbb{N}$, the probabilistic meaning of the nth correlation function is:

$$\rho_n(x_1, x_2, ..., x_n) = Pr\{\text{there exists a particle in each point } x_i\}.$$

If $\mathbb{X} = \mathbb{R}$, then ρ_n is the density function of the probability of finding particles in each of the infinitesimal intervals around x_i:

$$\rho_n(x_1, x_2, ..., x_n) dx_1 dx_2 ... dx_n = Pr\{\text{there exists a particle in each of the intervals } (x_i, x_i + dx_i)\}.$$

Suppose we have a point process P on $\mathbb{X}$, such that all the correlation functions exist. The point process P is called determinantal if there exist a function $K : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$\rho_n(x_1, x_2 ... x_n) = \det_{1 \le i,j \le n} [K(x_i, x_j)].$$

Similarly, a point process P is called a Pfaffian if the correlation functions are given by

$$\rho_n(x_1, x_2 ... x_n) = \underset{1 \le i,j \le n}{pf} [K(x_i, x_j)]$$

where this time

$$K(x,y) = \begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix}$$

and $K(x,y)$ is antisymmetric.

Theorem 2: Suppose we have a discrete point process like above. Then if we condition on any finite number k of occupied and unoccupied points, say at $x_1, x_2, \ldots, x_k$, we still get a determinantal p.p. with kernel $\tilde{K}(x_i, x_j)$ given by

$$\frac{\left| \begin{array}{c|c} \Large D_k & \begin{matrix} K(x_1, x_j) \\ \vdots \\ K(x_k, x_1) \end{matrix} \\ \hline K(x_i, x_j) \quad \ldots K(x_i, x_k) & K(x_i, x_j) \end{array} \right|}{|D_k|}$$

where D_k is a $k \times k$ matrix, whose structure depends on the points conditioned. Specifically, D_k will be the matrix $(D_k)_{ij} = K(x_i, x_j)$ for $i \neq j$, $(D_k)_{ii} = K(x_i, x_i)$ if x_i is conditioned to be occupied and $(D_k)_{ii} = K(x_i, x_i) - 1$ if x_i is conditioned to be unoccupied. □

5.1 Discrete Point Processes

5.1.1 Determinantal Point Processes

Suppose we have a simple point processes on a discrete state space X, such as $\mathbb{N}$ or $\{1, 2, ..., M\}$, of which all the correlation functions exist. Let there be a kernel $K : X \times X \to \mathbb{R}$ or $\mathbb{C}$ such that the correlation functions are given by

$$\rho_n(x_1, x_2, ..., x_n) = \det_{1 \le i,j \le n} (K(x_i, x_j)).$$

In this section we are going to be using the determinantal identity to prove the following

Theorem 2: Suppose we have a discrete point process like above. Then if we condition on any finite number k of occupied and unoccupied points, say at $x_1, x_2, \ldots, x_k$, we still get a determinantal p.p. with kernel $\tilde{K}(x_i, x_j)$ given by

$$
\frac{\left|\begin{array}{c|c} \multirow{3}{*}{D_k} & K(x_1,x_j) \\ & \vdots \\ & K(x_k,x_1) \\ \hline K(x_i,x_j)\ \ldots K(x_i,x_k) & K(x_i,x_j) \end{array}\right|}{|D_k|}
$$

where D_k is a $k \times k$ matrix, whose structure depends on the points conditioned. Specifically, D_k will be the matrix $(D_k)_{ij} = K(x_i,x_j)$ for $i \neq j$, $(D_k)_{ii} = K(x_i,x_i)$ if x_i is conditioned to be occupied and $(D_k)_{ii} = K(x_i,x_i) - 1$ if x_i is conditioned to be unoccupied.

Proof:

We write $N_{x_i} = 1$ if there exist a particle at point $x_i \in X$, and $N_{x_i} = 0$ if not.
We also denote $J_n = \{1,2,...,n\}$. For ease, when working with determinants, we will substitute $K(x_i,x_j)$ with a_{ij}.

1) We start by conditioning on 1 point.

$$
P(N_{x_i} = 1, J_n \backslash \{1\} | N_{x_1} = 1) = \frac{P(N_{x_i} = 1, J_n)}{P(N_{x_1} = 1)} = \frac{\det\limits_{1\le i,j\le n}(K(x_i,x_j))}{K(x_1,x_1)}.
$$

Using the determinantal identities from 4.1, we have

$$
\det_{1\le i,j\le n}(K(x_i,x_j)) = \begin{vmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix} = \frac{1}{(a_{11})^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \ldots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \ldots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-1)\times(n-1)}
$$

$$
\Rightarrow \frac{\det(K(x_i,x_j))}{K(x_1,x_1)} = \begin{vmatrix} \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11}} & \ldots & \frac{\begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix}}{a_{11}} \\ \vdots & \ddots & \vdots \\ \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix}}{a_{11}} & \ldots & \frac{\begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix}}{a_{11}} \end{vmatrix}_{(n-1)\times(n-1)}.
$$

So by denoting

$$
\tilde{K}(x_i,x_j) = \frac{\begin{vmatrix} K(x_1,x_1) & K(x_1,x_j) \\ K(x_i,x_1) & K(x_i,x_j) \end{vmatrix}}{K(x_1,x_1)}
$$

we have

$$
P(N_{x_i} = 1, J_n \backslash \{1\} | N_{x_1} = 1) = \det_{2\le i,j\le n}(\tilde{K}(x_i,x_j))
$$

and thus a new determinantal point process.

2) For $k < n$, we now condition on k points

$$P(N_{x_i} = 1, J_n \backslash J_k | N_{x_i} = 1, J_k) = \frac{P(N_{x_i} = 1, J_n)}{P(N_{x_i} = 1, J_k)} = \frac{\det\limits_{1 \le i,j \le n}(K(x_i, x_j))}{\det\limits_{1 \le i,j \le k}(K(x_i, x_j))}.$$

Calling

$$\det_{1 \le i,j \le k}(K(x_i, x_j)) = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} = |D_k|$$

we write

$$\det_{1 \le i,j \le n}(K(x_i, x_j)) = \frac{1}{|D_k|^{n-k-1}} \begin{vmatrix} \begin{vmatrix} D_k & & a_{1,k+1} \\ & & \cdot \\ & & \cdot \\ a_{k+1,1} & \dots & a_{k+1,k+1} \end{vmatrix} & \dots & \begin{vmatrix} D_k & & a_{1n} \\ & & \cdot \\ & & \cdot \\ a_{k+1,1} & \dots & a_{k+1,n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} D_k & & a_{1,k+1} \\ & & \cdot \\ & & \cdot \\ a_{n1} & \dots & a_{n,k+1} \end{vmatrix} & \dots & \begin{vmatrix} D_k & & a_{1n} \\ & & \cdot \\ & & \cdot \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \end{vmatrix}.$$

So once again by defining a new kernel $\tilde{K}(x_i, x_j)$ to be

$$\frac{\begin{vmatrix} D_k & & K(x_1, x_j) \\ & & \cdot \\ & & \cdot \\ K(x_i, x_1) & \dots & K(x_i, x_j) \end{vmatrix}}{|D_k|}$$

we get

$$P(N_{x_i} = 1, J_n \backslash J_k | N_{x_i} = 1, J_k) = \det_{k+1 \le i,j \le n}(\tilde{K}(x_i, x_j)).$$

3) Conditioning on empty points

$$P(N_{x_i}=1,J_n\backslash\{1\}|N_{x_1}=0) = \frac{P(N_{x_i}=1,J_n\backslash\{1\})-P(N_{x_i}=1,J_n)}{1-P(N_{x_1}=1)}$$

$$= \frac{\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}{1-a_{11}}$$

$$= \frac{\begin{vmatrix} a_{11}-1 & a_{12} & . & . & a_{1n} \\ a_{21} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}}{a_{11}-1}$$

$$= \frac{1}{(a_{11})^{n-1}} \begin{vmatrix} \begin{vmatrix} a_{11}-1 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}-1 & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11}-1 & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}-1 & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\begin{vmatrix} a_{11}-1 & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{a_{11}-1} & \cdots & \frac{\begin{vmatrix} a_{11}-1 & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix}}{a_{11}-1} \\ \vdots & \ddots & \vdots \\ \frac{\begin{vmatrix} a_{11}-1 & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix}}{a_{11}-1} & \cdots & \frac{\begin{vmatrix} a_{11}-1 & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix}}{a_{11}-1} \end{vmatrix}$$

$$= \det_{2\le i,j\le n}(\tilde{K}(x_i,x_j))$$

where

$$\tilde{K}(x_i,x_j) = \frac{\begin{vmatrix} a_{11}-1 & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix}}{a_{11}-1}.$$

4) Let's see how this works for 2 unoccupied points

$$P(N_{x_i}=1,J_n\backslash J_2|N_{x_i}=0,J_2) = \frac{P(N_{x_i}=1,J_n\backslash J_2 \quad \& \quad N_{x_i}=0,J_2)}{P(N_{x_i}=0,J_2)}$$

$$= \frac{P(N_{x_i}=1,J_n\backslash J_2)-P(N_{x_i}=1,J_n\backslash\{1\})-P(N_{x_i}=1,J_n\backslash\{2\})+P(N_{x_i}=1,J_n)}{1-P(N_{x_1}=1)-P(N_{x_2}=1)+P(N_{x_i}=1,J_2)}.$$

In terms of determinants

$$numerator = \begin{vmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} & . & . & a_{1n} \\ a_{31} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}$$

$$- \begin{vmatrix} a_{22} & a_{23} & . & . & a_{1n} \\ a_{32} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} & . & . & a_{1n} \\ a_{43} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22}-1 & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix} - \begin{vmatrix} a_{22}-1 & a_{23} & . & . & a_{1n} \\ a_{32} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}-1 & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22}-1 & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}$$

$$= \frac{1}{\begin{vmatrix} a_{11}-1 & a_{12} \\ a_{21} & a_{22}-1 \end{vmatrix}^{n-3}} \begin{vmatrix} \begin{vmatrix} a_{11}-1 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}-1 & a_{12} & a_{1n} \\ a_{21} & a_{22}-1 & a_{2n} \\ a_{31} & a_{32} & a_{3n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11}-1 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{n1} & a_{n2} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}-1 & a_{12} & a_{1n} \\ a_{21} & a_{22}-1 & a_{2n} \\ a_{n1} & a_{n2} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-2)\times(n-2)}.$$

$$denominator = 1 - a_{11} - a_{22} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}-1 & a_{12} \\ a_{21} & a_{22}-1 \end{vmatrix}.$$

<u>Introducing new notation</u>: $b_{ij} = a_{ij} - 1$

By setting

$$\tilde{K}(x_i, x_j) = \frac{\begin{vmatrix} b_{11} & a_{12} & a_{1j} \\ a_{21} & b_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix}}{\begin{vmatrix} b_{11} & a_{12} \\ a_{21} & b_{22} \end{vmatrix}}$$

we have

$$P(N_{x_i} = 1, J_n \backslash J_2 | N_{x_i} = 0, J_2) = \det_{3 \le i,j \le n}(\tilde{K}(x_i, x_j)).$$

5) **Lemma 5.1.1:** The following identity holds true:

$$P(N_{x_i} = 1, J_n \backslash J_k \quad \& \quad N_{x_i} = 0, J_k) = (-1)^k \left|\begin{array}{cccc|ccc} b_{11} & a_{12} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & b_{22} & & . & & & \\ \vdots & & \ddots & . & & & \vdots \\ a_{k1} & . & . & b_{kk} & & & \\ \hline a_{k+1,1} & & & & a_{k+1,k+1} & & \\ \vdots & & & & & \ddots & \\ a_{n1} & & \cdots & & & & a_{nn} \end{array}\right|.$$

(Proof in the appendix)

Remark: For $k = n$ we have

$$P(N_{x_i} = 0, J_n) = (-1)^n \det(K - I) = \det(I - K)$$

where $K_{ij} = K(x_i, x_j) = a_{ij}$.

6) Call $\quad P(N_{x_i} = 0, J_k) = \left|D_k\right|$.

Then

$$P(N_{x_i} = 1, J_n \backslash J_k | N_{x_i} = 0, J_k) \quad = \quad \frac{P(N_{x_i} = 1, J_n \backslash J_k \quad \& \quad N_{x_i} = 0, J_k)}{P(N_{x_i} = 0, J_k)}$$

$$= \frac{(-1)^k \left|\begin{array}{ccc|cc} & D_k & a_{1,k+1} & \cdots & a_{1n} \\ & & . & . & . \\ & & . & . & . \\ a_{k+1,1} & \cdots & a_{k+1,k+1} & & \\ \hline \vdots & & & \ddots & \\ a_{n1} & \cdots & & & a_{nn} \end{array}\right|}{(-1)^k \left|D_k\right|}$$

$$= \frac{1}{|D_k|^{n-k}} \begin{vmatrix} \begin{vmatrix} D_k & \begin{matrix} a_{1,k+1} \\ \vdots \end{matrix} \\ a_{k+1,1} \ \dots & a_{k+1,k+1} \end{vmatrix} & \dots & \begin{vmatrix} D_k & \begin{matrix} a_{1n} \\ \vdots \end{matrix} \\ a_{k+1,1} \ \dots & a_{k+1,n} \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} D_k & \begin{matrix} a_{1,k+1} \\ \vdots \end{matrix} \\ a_{n1} \ \dots & a_{n,k+1} \end{vmatrix} & \dots & \begin{vmatrix} D_k & \begin{matrix} a_{1n} \\ \vdots \end{matrix} \\ a_{n1} \ \dots & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-k)\times(n-k)}$$

$$= \det_{k+1 \le i,j \le n} (\tilde{K}(x_i, x_j))$$

where, similarly with before, $\tilde{K}(x_i, x_j)$ is given by

$$\frac{\begin{vmatrix} D_k & \begin{matrix} K(x_1, x_j) \\ \vdots \end{matrix} \\ K(x_i, x_1) \ \dots & K(x_i, x_j) \end{vmatrix}}{|D_k|}.$$

7) Conditioning on any mixture of occupied and unoccupied points still produces a determinantal point process with a similar kernel, where D_k will be the matrix $(D_k)_{ij} = K(x_i, x_j)$ for $i \neq j$, $(D_k)_{ii} = K(x_i, x_i)$ if x_i is conditioned to be occupied and $(D_k)_{ii} = K(x_i, x_i) - 1$ if x_i is conditioned to be unoccupied.

□

5.1.2 Pfaffian Point Processes

Pfaffian Identity

We start by finding an analogue of the determinantal identity, but for Pfaffians. This is pretty much a bit more explained and denser version of Knuth's first pages [21].

Following Knuth's notation, for a set of indexes $\mathbb{X}$ and function f satisfying the skew symmetry

$$f(xy) = -f(yx) \quad x, y \in \mathbb{X}$$

we define the Pfaffian

$$f(x_1 x_2 ... x_{2n}) = \sum_{\mu \in M(a)} s(\mu) f(y_1 y_2) ... f(y_{2n-1} y_{2n})$$

where we sum over all the $(2n-1)!!$ perfect matching $\{y_1, y_2\} \cup ... \cup \{y_{2n-1} y_{2n}\}$ of $\{1, 2, ..., 2n\}$, and $s(\mu)$ is the sign of the permutation that takes $(x_1, x_2, ..., x_{2n})$ to $(y_1, ..., y_{2n})$.
i.e.

$$f(xyzw) = f(xy)f(zw) - f(xz)f(yw) + f(xw)f(yz) = pf \begin{pmatrix} 0 & f(xy) & f(xz) & f(xw) \\ -f(xy) & 0 & f(yz) & f(yw) \\ -f(xz) & -f(yz) & 0 & f(zw) \\ -f(xw) & -f(yw) & -f(zw) & 0 \end{pmatrix}.$$

Another way of defining a Pfaffian is the following recursion

$$\begin{aligned} f(\epsilon) &= 1 \\ f(x_1x_2...x_{2n}) &= \sum_{j=2}^{n} f(x_1x_j)f(x_{j+1}...x_{2n}x_2...x_j) \end{aligned}$$

where ϵ is the null word.

Also, for two arbitrary words a, b made up of indexes in $\mathbb{X}$, i.e. $a = x_1x_2...x_{2n}$, we define $s(a,b) = 0$ if either a or b has a repeated letter or if b contains a letter not in a. Otherwise we define $s(a,b)$ to be the sign of the permutation that takes a into the word $b(a/b)$. For example, if $a = x_1x_2x_3x_4x_5x_6x_7x_8$ and $b = x_4x_5x_6$, then $b(a/b)$ would be $x_4x_5x_6(x_1x_2x_3x_7x_8)$ and the sign $s(a,b)$ of the permutation would be -1, since we had 3 letters move to the left 3 times each.

Under this notation we have the following identity by Tanner

$$f(a)f(ab) = \sum_y s(b,xy)f(axy)f(ab/xy), \qquad \text{for all } x \in b.$$

The smaller non trivial case is for $|b| = 4$ and gives

$$f(a)f(axyzw) = f(axy)f(azw) - f(axz)f(ayw) + f(axw)f(ayz).$$

Now let a be a fixed word with $|a|$ even, and define

$$g(b) = \frac{f(ab)}{f(a)}.$$

Tanner's identity tells us that

$$g(b) = \sum_y s(b,xy)g(xy)g(b/xy), \qquad \text{for all } x \in b.$$

This is a generalization of the definition by recurrence. We already knew that $g(b)$ is a Pfaffian, but now we also know that all the identities for f also hold for g. In particular we have that

$$\begin{aligned} g(b) &= \sum_{\mu \in M(a)} s(\mu)g(y_1y_2)...g(y_{2n-1}y_{2n}) \\ \Rightarrow \frac{f(ab)}{f(a)} &= \sum_{\mu \in M(a)} s(\mu)\frac{f(ay_1y_2)}{f(a)}...\frac{f(ay_{2n-1}y_{2n})}{f(a)} \end{aligned}$$

where we know for which matrix this is the Pfaffian of.

In other words we now have a similar way of writing a ratio of two Pfaffians as we do for determinants.

Here we are going to take $\mathbb{X}$ to be $\mathbb{N}$. Similarly to before, we dub $f(i,j) = a_{ij}$. i.e.

$$\begin{aligned} f(1234) &= f(12)f(34) - f(13)f(24) + f(14)f(23) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \\ &= pf\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}. \end{aligned}$$

In the examples we won't reach double digits for it to be confusing.

The simplest non trivial example of how to use this identity is for $|a|$=2 and $|b|$=4.
Let's take $a = 12$ and $b = 3456$.
Then

$$\frac{pf\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ & & 0 & a_{34} & a_{35} & a_{36} \\ & & & 0 & a_{45} & a_{46} \\ & & & & 0 & a_{56} \\ & & & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & a_{12} \\ & 0 \end{pmatrix}} = \frac{f(123456)}{f(12)} = g(3456) = pf\begin{pmatrix} 0 & g_{34} & g_{35} & g_{36} \\ & 0 & g_{45} & g_{46} \\ & & 0 & g_{56} \\ & & & 0 \end{pmatrix}$$

where

$$g_{ij} = \frac{f(12ij)}{f(ij)} = \frac{pf\begin{pmatrix} 0 & a_{12} & a_{1i} & a_{1j} \\ & 0 & a_{2i} & a_{2j} \\ & & 0 & a_{ij} \\ & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & a_{12} \\ & 0 \end{pmatrix}}.$$

Now we can move on to conditioning. We aim to prove a theorem on conditioning for the Pfaffian point processes, analogue to Theorem 2 for the determinantal case.

Theorem 3: Suppose now we have a Pfaffian point processes on a discrete space, of which all the correlation functions exist and are given by

$$\rho_n(x_1, ..., x_n) = \underset{1 \le i,j \le n}{pf} (K(x_i, x_j)).$$

Then the point process derived by conditioning on a finite number k of occupied and unoccupied points, say $x_1, x_2, \dots, x_k$ is also Pfaffian with kernel given by

$$\begin{aligned} \tilde{K}(x_i, x_j) &= \begin{pmatrix} \tilde{K}_{11}(x_i, x_j) & \tilde{K}_{12}(x_i, x_j) \\ \tilde{K}_{21}(x_i, x_j) & \tilde{K}_{22}(x_i, x_j) \end{pmatrix} \\ &= \frac{1}{pf(J_K)} \begin{pmatrix} pf\begin{pmatrix} J_K & u_1^{(k)}(x_i) & u_1^{(k)}(x_j) \\ & 0 & K_{11}(x_i, x_j) \\ & & 0 \end{pmatrix} & pf\begin{pmatrix} J_K & u_1^{(k)}(x_i) & u_2^{(k)}(x_j) \\ & 0 & K_{12}(x_i, x_j) \\ & & 0 \end{pmatrix} \\ pf\begin{pmatrix} J_K & u_2^{(k)}(x_i) & u_1^{(k)}(x_j) \\ & 0 & K_{21}(x_i, x_j) \\ & & 0 \end{pmatrix} & pf\begin{pmatrix} J_K & u_2^{(k)}(x_i) & u_2^{(k)}(x_j) \\ & 0 & K_2(x_i, x_j) \\ & & 0 \end{pmatrix} \end{pmatrix} \end{aligned}$$

where

$$J_K = \underset{1 \le i,j \le k}{\det} J_{ij}$$

is a $2k \times 2k$ antisymmetric matrix with each element given by a 2×2 matrix. $J_{ij} = K(x_i, x_j)$ for $i \neq j$, $J_{ii} = K(x_i, x_i)$ if we condition on x_i being occupied and

$$J_{ii} = C(x_i, x_i) = \begin{pmatrix} 0 & K_{12}(x_i, x_i) - 1 \\ \\ -K_{12}(x_i, x_i) + 1 & 0 \end{pmatrix}$$

if x_i is conditioned to be unoccupied.

Finally $u_1^{(k)}$ and $u_2^{(k)}$ are defined to be

$$u_1^{(k)}(y) = \begin{pmatrix} K_{11}(x_1, y) \\ K_{21}(x_1, y) \\ K_{11}(x_2, y) \\ K_{21}(x_2, y) \\ \vdots \\ K_{11}(x_k, y) \\ K_{21}(x_k, y) \end{pmatrix} \qquad \text{and} \qquad u_2^{(k)}(y) = \begin{pmatrix} K_{12}(x_1, y) \\ K_{22}(x_1, y) \\ K_{12}(x_2, y) \\ K_{22}(x_2, y) \\ \vdots \\ K_{12}(x_k, y) \\ K_{22}(x_k, y) \end{pmatrix}$$

Proof: We start by introducing the following lemma.

Lemma 5.1.2: $f(1, 2..., 2n) - f(3, 4, ..., 2n) := \tilde{f}(2, 2n)$ is the Pfaffian of the matrix

$$\begin{pmatrix} 0 & a_{12} - 1 & a_{13} & \cdots & a_{1,2n} \\ & 0 & a_{23} & & \\ & & 0 & & \vdots \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

i.e.

$$pf \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix} - pf \begin{pmatrix} 0 & a_{34} \\ & 0 \end{pmatrix} = pf \begin{pmatrix} 0 & a_{12} - 1 & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix}.$$

(Proof in the appendix).

Remark: Similar identity holds for

$$f(1, 2, ..., 2n) - f(1, 2, ..., 2i, 2i + 3, ..., 2n) = pf(L)$$

where $L_{km} = \begin{cases} f(km), & (k, m) \neq (2i + 1, 2i + 2) \\ \\ f(2i + 1, 2i + 2) - 1, & (k, m) = (2i + 1, 2i + 2). \end{cases}$

Suppose now we have a Pfaffian point processes on a discrete space, of which all the correlation functions exist and are given by

$$\rho_n(x_1, ..., x_n) = \underset{1 \leq i,j \leq n}{pf} (K(x_i, x_j))$$

where $K(x, y)$ is a 2×2 block given by

$$K(x, y) = \begin{pmatrix} K_{11}(x, y) & K_{12}(x, y) \\ \\ K_{21}(x, y) & K_{22}(x, y) \end{pmatrix}.$$

Now if

$$K(x_i, x_i) = \begin{pmatrix} 0 & a_{2i-1,2i} \\ -a_{2i-1,2i} & 0 \end{pmatrix}$$

define

$$C(x_i,x_i) = \begin{pmatrix} 0 & a_{2i-1,2i}-1 \\ -a_{2i-1,2i}+1 & 0 \end{pmatrix}.$$

Note that for easier notation, we are going to be focusing on conditioning on empty points. If instead we want to have particles at certain points, we just replace $C(x_i,x_i)$ with $K(x_i,x_i)$ as needed.

Lemma 5.1.3: $P(N_{x_i}=1, J_n\backslash J_k \quad \& \quad N_{x_i}=0, J_k)$ is equal to

$$(-1)^k pf\left(\begin{array}{cccc|ccc} C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_k) & & \cdots & K(x_1,x_n) \\ & C(x_2,x_2) & & & & & \\ & & \ddots & \vdots & & & \vdots \\ & & & C(x_k,x_k) & & & \\ \hline & & & & K(x_{k+1},x_{k+1}) & & \\ & & & & & \ddots & \\ & & & & & & K(x_n,x_n) \end{array}\right).$$

(Proof in the appendix).

Now we are going to start computing the kernel for the conditioned point processes.
We will need to use the a_{ij}, b_{ij} notation and revert back to the elements of $K(x,y)$ afterwards.

i)

$$P(N_{x_i}=1, J_n\backslash\{1\}|N_{x_1}=0) = \frac{P(N_{x_i}=1, J_n\backslash\{1\} \quad \& \quad N_{x_1}=0)}{P(N_{x_1}=0)}$$

$$= \frac{pf\begin{pmatrix} C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_n) \\ & K(x_2,x_2) & & \\ & & \ddots & \vdots \\ & & & K(x_n,x_n) \end{pmatrix}}{pf(C(x_1,x_1))} = \frac{\begin{pmatrix} 0 & b_{12} & a_{13} & a_{14} & \cdots & a_{1,2n} \\ & 0 & a_{23} & a_{24} & & \\ & & 0 & & & \\ & & & \ddots & & \vdots \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}}$$

$$= \frac{f(1,2,3,...,2n)}{f(1,2)} = g(3,...,2n)$$

$$= pf\begin{pmatrix} 0 & g_{34} & g_{35} & \cdots & g_{3,2n} \\ & 0 & g_{45} & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

where, as seen before,

$$g_{ij} = \frac{f(12ij)}{f(ij)} = \frac{pf\begin{pmatrix} 0 & b_{12} & a_{1i} & a_{1j} \\ & 0 & a_{2i} & a_{2j} \\ & & 0 & a_{ij} \\ & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}}.$$

Passing from the (a_{ij}, b_{ij}) to the $K_{lm}(x_i, x_j)$ notation.

Let's take $a_{23} \sim (2,3)$.
To get (x_i, x_j), for an even index you divide by 2 and for an odd you add 1 and then divide by 2.

$$(2,3) \rightsquigarrow (x_1, x_2)$$

For K_{lm} it's

$$(2,3) \rightsquigarrow (\text{even}, \text{odd}) \rightsquigarrow (2,1)$$

$$\Rightarrow a_{23} = K_{21}(x_1, x_2)$$

other examples

$$\begin{aligned} a_{14} &= K_{12}(x_1, x_2) \\ a_{56} &= K_{12}(x_3, x_3) \end{aligned}$$

Now let's take the following block

$$\begin{pmatrix} g_{35} & g_{36} \\ g_{45} & g_{46} \end{pmatrix} = \frac{1}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}} \begin{pmatrix} pf\begin{pmatrix} 0 & b_{12} & a_{13} & a_{15} \\ & 0 & a_{23} & a_{25} \\ & & 0 & a_{35} \\ & & & 0 \end{pmatrix} & pf\begin{pmatrix} 0 & b_{12} & a_{13} & a_{16} \\ & 0 & a_{23} & a_{26} \\ & & 0 & a_{36} \\ & & & 0 \end{pmatrix} \\ pf\begin{pmatrix} 0 & b_{12} & a_{14} & a_{15} \\ & 0 & a_{24} & a_{25} \\ & & 0 & a_{45} \\ & & & 0 \end{pmatrix} & pf\begin{pmatrix} 0 & b_{12} & a_{14} & a_{16} \\ & 0 & a_{24} & a_{26} \\ & & 0 & a_{46} \\ & & & 0 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}} \begin{pmatrix} pf\begin{pmatrix} 0 & b_{12} & K_{11}(x_1,x_2) & K_{11}(x_1,x_3) \\ & 0 & K_{21}(x_1,x_2) & K_{21}(x_1,x_3) \\ & & 0 & K_{11}(x_2,x_3) \\ & & & 0 \end{pmatrix} & pf\begin{pmatrix} 0 & b_{12} & K_{11}(x_1,x_2) & K_{12}(x_1,x_3) \\ & 0 & K_{21}(x_1,x_2) & K_{22}(x_1,x_3) \\ & & 0 & K_{12}(x_2,x_3) \\ & & & 0 \end{pmatrix} \\ pf\begin{pmatrix} 0 & b_{12} & K_{12}(x_1,x_2) & K_{11}(x_1,x_3) \\ & 0 & K_{22}(x_1,x_2) & K_{21}(x_1,x_3) \\ & & 0 & K_{21}(x_2,x_3) \\ & & & 0 \end{pmatrix} & pf\begin{pmatrix} 0 & b_{12} & K_{12}(x_1,x_2) & K_{12}(x_1,x_3) \\ & 0 & K_{22}(x_1,x_2) & K_{22}(x_1,x_3) \\ & & 0 & K_{22}(x_2,x_3) \\ & & & 0 \end{pmatrix} \end{pmatrix}$$

where $b_{12} = K_{12}(x_i, x_j)$.

So by denoting

$$\tilde{K}(x_i,x_j) = \begin{pmatrix} \tilde{K}_{11}(x_i,x_j) & \tilde{K}_{12}(x_i,x_j) \\ \\ \tilde{K}_{21}(x_i,x_j) & \tilde{K}_{22}(x_i,x_j) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{pf\begin{pmatrix} 0 & b_{12} & K_{11}(x_1,x_i) & K_{11}(x_1,x_j) \\ & 0 & K_{21}(x_1,x_i) & K_{21}(x_1,x_j) \\ & & 0 & K_{11}(x_i,x_j) \\ & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}} & \dfrac{pf\begin{pmatrix} 0 & b_{12} & K_{11}(x_1,x_i) & K_{12}(x_1,x_j) \\ & 0 & K_{21}(x_1,x_i) & K_{22}(x_1,x_j) \\ & & 0 & K_{12}(x_i,x_j) \\ & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}} \\ \\ \dfrac{pf\begin{pmatrix} 0 & b_{12} & K_{12}(x_1,x_i) & K_{11}(x_1,x_j) \\ & 0 & K_{22}(x_1,x_i) & K_{21}(x_1,x_j) \\ & & 0 & K_{21}(x_i,x_j) \\ & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}} & \dfrac{pf\begin{pmatrix} 0 & b_{12} & K_{12}(x_1,x_i) & K_{12}(x_1,x_j) \\ & 0 & K_{22}(x_1,x_i) & K_{22}(x_1,x_j) \\ & & 0 & K_{22}(x_i,x_j) \\ & & & 0 \end{pmatrix}}{pf\begin{pmatrix} 0 & b_{12} \\ & 0 \end{pmatrix}} \end{pmatrix}$$

we have that $P(N_{x_i} = 1, J_n\backslash\{1\}|N_{x_1} = 0) = \underset{2\leq i,j\leq n}{pf}(\tilde{K}(x_i,x_j))$.

ii)

$$P(N_{x_i} = 1, J_n\backslash J_k|N_{x_i} = 0, J_k) = \frac{P(N_{x_i} = 1, J_n\backslash J_k \quad \& \quad N_{x_i} = 0, J_k)}{P(N_{x_i} = 0, J_k)}$$

Here

$$\begin{aligned} P(N_{x_i} = 0, J_k) &= pf\begin{pmatrix} C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_k) \\ \\ & C(x_2,x_2) & & \\ \\ & & \ddots & \vdots \\ \\ & & & C(x_k,x_k) \end{pmatrix} \\ &= f(1,2,...,2k) = pf(J_K) \end{aligned}$$

It's worthwhile to note that the f we are using here is such that

$$\begin{cases} f(2i-1,2i) = b_{2i-1,2i}, & \text{for } i = 1,...,k \\ \\ f(i,j) = a_{ij}, & \text{otherwise.} \end{cases}$$

Also the following results hold for the general case where, similarly to the determinantal point processes, we condition on any mixture of occupied and unoccupied points. It will simply produce an accordingly different matrix J_K.

Passing to Knuth's notation, we have that

$$
\begin{aligned}
P(N_{x_i}=1, J_n\backslash J_k | N_{x_i}=0, J_k) &= \frac{f(1,2,...,2n)}{f(1,2,..,.2k)} = g(2k+1,...,2n) \\
&= pf\begin{pmatrix} 0 & g_{2k+1,2k+2} & g_{2k+1,2k+3} & \cdots & g_{2k+1,2n} \\ & 0 & g_{2k+2,2k+3} & & \vdots \\ & & 0 & & \\ & & & \ddots & g_{2n-1,2n} \\ & & & & 0 \end{pmatrix}
\end{aligned}
$$

Let's look at the block $\begin{pmatrix} g_{2k+1,2k+3} & g_{2k+1,2k+4} \\ g_{2k+2,2k+3} & g_{2k+2,2k+4} \end{pmatrix}$ which might as well be a random one.

We have $\frac{(2k+1)+1}{2} = k+1$, $\frac{(2k+3)+1}{2} = k+2$, so this block will be $\tilde{K}(x_{k+1}, x_{k+2})$.

Now, before this becomes an uncontrollable mess, we define

$$
u_1^{(k)}(y) = \begin{pmatrix} K_{11}(x_1,y) \\ K_{21}(x_1,y) \\ K_{11}(x_2,y) \\ K_{21}(x_2,y) \\ \vdots \\ K_{11}(x_k,y) \\ K_{21}(x_k,y) \end{pmatrix} \quad \text{and} \quad u_2^{(k)}(y) = \begin{pmatrix} K_{12}(x_1,y) \\ K_{22}(x_1,y) \\ K_{12}(x_2,y) \\ K_{22}(x_2,y) \\ \vdots \\ K_{12}(x_k,y) \\ K_{22}(x_k,y) \end{pmatrix}
$$

Then for example

$$
g_{2k+1,2k+3} = \frac{f(1,...,2k,2k+1,2k+3)}{f(1,...,2k)} = \frac{1}{pf(J_K)} pf\begin{pmatrix} J_K & u_1^{(k)}(x_{k+1}) & u_1^{(k)}(x_{k+2}) \\ & 0 & K_{11}(x_{k+1},x_{k+2}) \\ & & 0 \end{pmatrix}
$$

So by defining

$$
\begin{aligned}
\tilde{K}(x_i,x_j) &= \begin{pmatrix} \tilde{K}_{11}(x_i,x_j) & \tilde{K}_{12}(x_i,x_j) \\ \tilde{K}_{21}(x_i,x_j) & \tilde{K}_{22}(x_i,x_j) \end{pmatrix} \\
&= \frac{1}{pf(J_K)} \begin{pmatrix} pf\begin{pmatrix} J_K & u_1^{(k)}(x_i) & u_1^{(k)}(x_j) \\ & 0 & K_{11}(x_i,x_j) \\ & & 0 \end{pmatrix} & pf\begin{pmatrix} J_K & u_1^{(k)}(x_i) & u_2^{(k)}(x_j) \\ & 0 & K_{12}(x_i,x_j) \\ & & 0 \end{pmatrix} \\ pf\begin{pmatrix} J_K & u_2^{(k)}(x_i) & u_1^{(k)}(x_j) \\ & 0 & K_{21}(x_i,x_j) \\ & & 0 \end{pmatrix} & pf\begin{pmatrix} J_K & u_2^{(k)}(x_i) & u_2^{(k)}(x_j) \\ & 0 & K_2(x_i,x_j) \\ & & 0 \end{pmatrix} \end{pmatrix}
\end{aligned}
$$

we have that

$$P(N_{x_i} = 1, J_n \backslash J_k | N_{x_i} = 0, J_k) = \underset{k+1 \le i,j \le n}{pf} (\tilde{K}(x_i, x_j)).$$

$\square$

Observation: Suppose we have a discrete Pfaffian point process with correlation functions $\rho_n(x_1, x_2, ..., x_n) = \underset{1 \le i,j \le n}{pf} (K(x_i, x_j))$. The kernels of all the known Pfaffian point processes have the following derived form:

$$K(x,y) = \begin{pmatrix} sgn(y-x)F|y-x| & F'(y-x) \\ -F'(y-x) & -F''(y-x) \end{pmatrix} = \begin{pmatrix} sgn(y-x)F|y-x| & \frac{d}{dy}F(y-x) \\ \frac{d}{dx}F(y-x) & \frac{d^2}{dxdy}F(y-x) \end{pmatrix}.$$

The question is, do Pfaffian point processes derived by conditioning retain the above form? Well, yes they do.

First note that $K_{12}(x,y) = -K_{21}(y,x) \Rightarrow F'(y-x) = F'(x-y)$, thus F' is even and F'' is odd. By condition on one point, say b, having a particle, we get a new Pfaffian p.p. with

$$\tilde{\rho}_n(x_1, x_2, \cdots, x_n) = P(N_{x_i} = 1, i = 1, \cdots, n | N_b = 1) = pf(\tilde{K}(x_i, x_j)$$

where the new kernel $\tilde{K}$ is give by

$$\tilde{K}(x,y) = \frac{1}{pf\begin{pmatrix} 0 & c \\ & 0 \end{pmatrix}} \begin{pmatrix} pf\begin{pmatrix} 0 & c & K_{11}(b,x) & K_{11}b(y,) \\ & 0 & K_{21}(b,x) & K_{21}(b,y) \\ & & 0 & K_{11}(x,y) \\ & & & 0 \end{pmatrix} & pf\begin{pmatrix} 0 & c & K_{11}(b,x) & K_{12}(b,y) \\ & 0 & K_{21}(b,x) & K_{22}(b,y) \\ & & 0 & K_{12}(x,y) \\ & & & 0 \end{pmatrix} \\ pf\begin{pmatrix} 0 & c & K_{12}(b,x) & K_{11}(b,y) \\ & 0 & K_{22}(b,x) & K_{21}(b,y) \\ & & 0 & K_{21}(b,y) \\ & & & 0 \end{pmatrix} & pf\begin{pmatrix} 0 & c & K_{12}(b,x) & K_{12}(b,y) \\ & 0 & K_{22}(b,x) & K_{22}(b,y) \\ & & 0 & K_{22}(x,y) \\ & & & 0 \end{pmatrix} \end{pmatrix}$$

where c is a constant.

Without loss of generality, I'm going to assume $x < b < y$. Then, forgetting about the denominator, we have

- $$\begin{aligned} \tilde{K}_{11}(x,y) &= cK_{11}(x,y) - K_{11}(b,x)K_{21}(b,y) + K_{21}(b,x)K_{11}(b,y) \\ &= sgn(y-x)cF|y-x| + sgn(x-b)F|x-b|F'(y-b) - F'(x-b)sgn(y-b)F|y-b| \\ &= cF(y-x) - F(b-x)F'(y-b) - F'(b-x)F(y-b). \end{aligned}$$

- $$\frac{d}{dy}\tilde{K}_{11}(x,y) = cF' - F(b-x)F''(y-b) - F'(b-xF'(y-b)$$

and

$$\begin{aligned}
\tilde{K}_{12}(x,y) &= cF'(y-x) + F''(y-b)sgn(x-b)F|x-b| - F'(y-b)F'(x-b) \\
&= cF'(y-x) - F(b-x)F''(y-b) - F'(y-b)F'(b-x) \\
&= \frac{d}{dy}\tilde{K}_{11}(x,y).
\end{aligned}$$

- $$\frac{d}{dx}\tilde{K}_{11}(x,y) = -cF'(y-x) + F'(b-x)F'(y-b) + F''(b-x)F(y-b)$$

$$\begin{aligned}
\tilde{K}_{21}(x,y) &= -cF'(y-x) + F'(x-b)F'(y-b) - sgn(y-b)F|y-b|F''(x-b) \\
&= -cF'(y-x) + F'(b-x)F'(y-b) + F''(b-x)F(y-b) \\
&= \frac{d}{dx}\tilde{K}_{11}(x,y).
\end{aligned}$$

- $$\frac{d^2}{dxdy}\tilde{K}_{11}(x,y) = -cF''(y-x) + F'(b-x)F''(y-b) + F''(b-x)F'(y-b)$$

and

$$\begin{aligned}
\tilde{K}_{22}(x,y) &= -cF''(y-x) + F'(x-b)F''(y-b) - F'(y-b)F''(x-b) \\
&= -cF''(y-x) + F'(b-x)F''(y-b) + F''(b-x)F'(y-b) \\
&= \frac{d^2}{dxdy}\tilde{K}_{11}(x,y).
\end{aligned}$$

As we can see, the kernel of the new Pfaffian process indeed retain the same form.

5.2 Continuous Determinantal Point Processes

Conditioning of point processes plays a key role in the papers on rigidity of random matrix ensembles, see [7,16,22], where conditioning on an infinite region is done. Janossi densities, which lead immediately to the kernels for processes conditioned on empty regions, are also well explored (see [5] and [28]). There are also some papers on the use of determinantal point processes in statistical modelling, where condtioning is discussed [26]. These papers seem to be for certain special classes of kernels, or special matrix ensembles, and we do not know of results indicating the complete generality of conditioning, and there is also almost nothing for Pfaffian point processes. We hope our results will lead to new kernels for conditioned particle systems, where coalescing and annihilating systems naturally lead to Pfaffian point processes [29], [30].

First we will be needing a bit of theory which is taken directly from [13, Chapter 12]. The aim of this section is to see how Theorem 2 works for continuous determinantal point processes. and use the determinantal identity to get similar expressions for the kernels of the conditioned p.p. Apart from the fact that instead of conditioning on empty points, now we have to start conditioning on empty sets which means we'll come across Fredholm determinants, we also have to be very careful about conditioning on occupied points, since these are null sets. Before anything else, we have to make sure what we are doing even makes sense. We felt that the best way to go about it, is through Palm measures [13, 31].

We start by building up the machinery we'll be using and showing the connection between Palm measures and conditioning. First step is to show that taking a random measure ξ on $\mathbb{R}$ with distribution P and conditioning on having a particle on a point $x \in \mathbb{R}$, yields a Palm measure P_x (Lemma 5.2.1). In this section we will only work with determinantal point processes, but in the future we want to extend those results to Pfaffian p.p. too. Next, in section 5.2.2., we take the Laplace functional $L[f]$ of a random measure with distribution P and define $L_x[f]$, which is the Laplace functional with respect to P_x. By using Lemma 5.2.2., which offers a relation between $L[f]$ and $L_x[f]$, and Proposition 5.2.5., which expresses the Laplace functional of a determinantal point process as a Fredholm determinant, we where able to show that conditioning on a determinantal p.p. having a particle at a point x, produces a new determinantal p.p. with kernel

$$K_x(y,z) = K(y,z) - \frac{K(x,z)K(y,x)}{K(x,x)},$$

which correspond with our previous results. Conditioning on having particles on more that one point can be done inductively the same way it was done in 5.1. Working with Palm measures is convenient, but we also want to verify the results by conditioning on events of positive probability and then taking the limit, as we do so in Lemmas 5.2.8, 5.2.9 and 5.2.10. This unifies the discrete and continuous cases and give a common tool for both of them. Lemma 5.2.8 corresponds to the main result of this section, which we have rigorously proven. As for Lemma 5.2.9, I believe the expression for a kernel of a determinantal point process conditioned on not having any particles in a set A, is already an established result. Lemma 5.2.10 yields what we believe to be the kernel of a determinantal p.p. conditioned on both an empty set A and a particle at point x. Once again, this can be inductively expanded to n particles at points $x_1, ..., x_n$. While there are strong indications that is the correct kernel, it is just a naive Bayes computation and we still do not have a rigorous proof. It is one of our future goals, alongside conditioning on having more than one particles on a point x, which the Palm measures allow, and getting analogous results for the continuous Pfaffian point processes.

5.2.1 Campbell and Palm Measures

Let ξ be a random measure with distribution P on a complete separable metric space X (in our case $X = \mathbb{R}, \mathbb{C}$) and $\hat{M}(X)$ be the set of boundedly finite measures on X.

For $A \in B(X)$ and $U \in (\hat{M}(X))$, the Campbell measure can be introduced on the product space $W = X \times \hat{M}(X)$ to be the set function

$$C_P(A \times U) = E(\xi(A)\mathbb{1}_U(\xi)) = \int_U \int_A \xi(dx)P(d\xi)$$

(Note: A more exact notation would be $E_P[\xi_{(\omega)}(A)1_U(\xi_{(\omega)})]$)

Let $\mathscr{B}_{\mathscr{W}}$ be the Borel σ-field $\mathscr{B}_{\mathscr{X}} \times \mathscr{B}_{\hat{\mathscr{M}}(\mathscr{X})}$. To be exact $\mathscr{B}_{\mathscr{W}}$ is generated by all $A \times U \in W$ with $A \in \mathscr{B}_{\mathscr{X}}$ and $U \in \mathscr{B}_{\hat{\mathscr{M}}(\mathscr{X})}$.

Proposition 5.2.1:([13]) $C_P(A \times U)$ extends uniquely to a measure on $\mathscr{B}_{\mathscr{W}}$.

Proposition 5.2.2: For $\mathscr{B}_{\mathscr{W}}$ measurable functions $g(x,\xi)$ that are either non-negative or C_P integrable

$$\begin{aligned}\int_W g(x,\xi)C_P(dx\times d\xi) &= E\Big(\int_X g(x,\xi)\xi(dx)\Big)\\ &= \int_{\hat{M}(X)}\int_X g(x,\xi)\xi(dx)P(d\xi).\end{aligned}$$

If the first moment measure exists, by taking U to be the whole of $\hat{M}(X)$ we get

$$C_P(A\times\hat{M}(X)) = E(\xi(A)) = M(A).$$

In this case the previous lemma yields

$$E\Big(\int_X g(x)\xi(dx)\Big) = \int_X g(x)\int_{\hat{M}(X)}\xi(dx)P(d\xi) = \int_X g(x)M(dx).$$

For fixed U, $C_P(\cdot\times U)$ is absolutely continuous wrt $M(\cdot)$ $\Big(C_P(\cdot\times U)\ll M(\cdot)\Big)$, thus we can introduce the Radon-Nikodym derivative as a $\mathscr{B}_{\mathscr{W}}$ measurable function $P_x(U)$ satisfying $\forall A\in\mathscr{B}_{\mathscr{W}}$

$$\int_A P_x(U)M(dx) = C_P(A\times U).$$

$P_x(U)$ is defined uniquely up to a set of M-measure zero. Furthermore, the family of $\{P_x(U)\}$ can be chosen so that

1) for each fixed $x\in\mathscr{B}_{\mathscr{X}}$, $P_x(U)$ is a probability measure on $U\in\mathscr{B}_{\hat{\mathscr{M}}(\mathscr{X})}$.

2) for each fixed $U\in\mathscr{B}_{\hat{\mathscr{M}}(\mathscr{X})}$, $P_x(U)$ is a measurable function of x, M-integrable on bounded subsets of X.

Each such measure $P_x(\cdot)$ is called a local Palm distribution and a family of such measures satisfying 1) and 2), a Palm kernel.

Proposition 5.2.3: Let ξ be a random measure for which the first moment measure exists. Then ξ admits a family of local Palm distributions $P_{\cdot}(\cdot)$, which is defined uniquely up to a set of M-measure zero and satisfy for each $\mathscr{B}_{\mathscr{X}}$ measurable functions g that is either non-negative or C_P integrable

$$E\Big(\int_X g(x,\xi)\xi(dx)\Big) = \int_W g(x,\xi)C_p(dx\times d\xi) = \int_X\int_{\hat{M}(X)} g(x,\xi)P_x(d\xi)M(dx).$$

Proof: For a function $g(x,\xi) = \mathbb{1}_A(x)\mathbb{1}_U(\xi)$ we have

$$\begin{aligned}E\Big(\int_X \mathbb{1}_A(x)\mathbb{1}_U(\xi)\xi(dx)\Big) &= \int_X\int_{\hat{M}(X)}\mathbb{1}_A(x)\mathbb{1}_U(\xi)P_x(d\xi)M(dx)\\ \Leftrightarrow C_P(A\times U) &= \int_A P_x(U)M(dx).\end{aligned}$$

The rest of the proof is same as in the previous proposition.

□

Now this is one of the most important results we will need, hidden as an exercise in [13], in a version I don't own. This is where we first connect Palm measures and conditioning.

Lemma 5.2.1: Let ξ be a random measure on $\mathbb{R}$. If ρ_1 is continuous, ρ_2 is locally bounded and $P_x(U)$ is continuous in x for some fixed $U \in \mathscr{B}_{\hat{\mathscr{M}}(\mathbb{R})}$

$$\lim_{\epsilon \to 0} P\big(\xi \in U | \xi(x - \epsilon, x + \epsilon) > 0\big) = P_x(U).$$

Proof: We can assume wlog that $x = 0$ and call $\xi(-\epsilon, \epsilon) = Z_\epsilon$. Then

$$P\big(\xi \in U | Z_\epsilon > 0\big) = \frac{P\big(\xi \in U, Z_\epsilon > 0\big)}{P\big(Z_\epsilon > 0\big)}.$$

1) **Denominator**

We have $P\big(Z_\epsilon > 0\big) = E(\mathbb{1}_{Z_\epsilon > 0})$. We can rewrite $\mathbb{1}_{Z_\epsilon > 0}$ the following way

$$\begin{aligned} Z_\epsilon &= \mathbb{1}_{Z_\epsilon > 0} + (Z_\epsilon - 1)\mathbb{1}_{Z_\epsilon > 1} \\ \Rightarrow \mathbb{1}_{Z_\epsilon > 0} &= Z_\epsilon - (Z_\epsilon - 1)\mathbb{1}_{Z_\epsilon > 1} \\ \Rightarrow E(\mathbb{1}_{Z_\epsilon > 0}) &= E(Z_\epsilon) - E\big((Z_\epsilon - 1)\mathbb{1}_{Z_\epsilon > 1}\big). \end{aligned}$$

Also

$$(Z_\epsilon - 1)\mathbb{1}_{Z_\epsilon > 1} \leq Z_\epsilon(Z_\epsilon - 1) \Rightarrow E\big((Z_\epsilon - 1)\mathbb{1}_{Z_\epsilon > 1}\big) \leq E\big(Z_\epsilon(Z_\epsilon - 1)\big).$$

We have that

$$E\big(Z_\epsilon(Z_\epsilon - 1)\big) = \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \rho_2(z_1, z_2) dz_1 dz_2 \leq 4\epsilon^2 M$$

for $M > 0$, since ρ_2 is locally bounded.

$$\Rightarrow E\big((Z_\epsilon - 1)\mathbb{1}_{Z_\epsilon > 1}\big) = o(\epsilon).$$

As for $E(Z_\epsilon)$

$$\begin{aligned} E(Z_\epsilon) &= \int_{-\epsilon}^{\epsilon} \rho_1(z) dz = \int_{-\epsilon}^{\epsilon} \Big(\rho_1(z) - \rho_1(0)\Big) dz + 2\epsilon\rho_1(0) \\ &\leq 2\epsilon \sup_{|z| < \epsilon} |\rho_1(z) - \rho_1(0)| + 2\epsilon\rho_1(0) = 2\epsilon\rho_1(0) + o(\epsilon). \end{aligned}$$

Put together we get

$$P\big(Z_\epsilon > 0\big) = E(\mathbb{1}_{Z_\epsilon > 0}) = 2\epsilon\rho_1(0) + o(\epsilon).$$

2) **Numerator**

$$P\big(\xi\in U, Z_\epsilon>0\big)=E(\mathbb{1}_U(\xi)\mathbb{1}_{Z_\epsilon>0})=E(\mathbb{1}_U(\xi)Z_\epsilon)-E\big(\mathbb{1}_U(\xi)(Z_\epsilon-1)\mathbb{1}_{Z_\epsilon>1}\big).$$

We already know that

$$E\big(\mathbb{1}_U(\xi)(Z_\epsilon-1)\mathbb{1}_{Z_\epsilon>1}\big)=o(\epsilon).$$

Let $g(y,\xi)=\mathbb{1}_U(\xi)\mathbb{1}_{(-\epsilon,\epsilon)}(y)$. Then

$$\begin{aligned}
E(\mathbb{1}_U(\xi)Z_\epsilon) &= E\big(\mathbb{1}_U(\xi)\xi(-\epsilon,\epsilon)\big)=E\Big(\int_{\mathbb{R}}\mathbb{1}_U(\xi)\mathbb{1}_{(-\epsilon,\epsilon)}(y)\xi(dy)\Big)=E\Big(\int_{\mathbb{R}}g(y,\xi)\xi(dy)\Big)\\
&= \int_{\mathbb{R}}g(y,\xi)P_y(\xi)M(dy)=\int_{-\epsilon}^{\epsilon}\int_U P_y(\xi)\rho_1(y)dy=\int_{-\epsilon}^{\epsilon}P_y(U)\rho_1(y)dy\\
&= 2\epsilon\rho_1(0)P_0(U)+o(\epsilon)
\end{aligned}$$

$$\Rightarrow \lim_{\epsilon\to0}P\big(\xi\in U|\xi(-\epsilon,\epsilon)>0\big) = \lim_{\epsilon\to0}\frac{P\big(\xi\in U,Z_\epsilon>0\big)}{P\big(Z_\epsilon>0\big)}=\lim_{\epsilon\to0}\frac{2\epsilon\rho_1(0)P_0(U)+o(\epsilon)}{2\epsilon\rho_1(0)+o(\epsilon)}=P_0(U).$$

□

Note 1) This shows that if ξ is a simple point process in $\mathbb{R}$ then P_x can be interpreted as the law of ξ conditioned on x being a point in $\mathbb{R}$.

Note 2) If $x\to P_x$ is continuous then the Palm measure is defined uniquely.

5.2.2 Laplace Functionals

Let $L[f]$ be the Laplace functional of a random measure with distribution P and, for $f\in BM_+(X)$ (non negative functions on X with bounded support), let $L[f;x]=L_x[f]$ be the the Laplace functional derived from the Palm kernel

$$L_x[f]=\int_{\hat{M}(X)}\exp\Big(-\int_X f(y)\xi(dy)\Big)P_x(d\xi).$$

Lemma 5.2.2:([13]) Let ξ be a random measure with finite first moment measure M, and $L[f]$, $L_x[f]$ the Laplace functionals associated with the original measure and it's Palm measure respectively. Then for $f,g\in BM_+(X)$, which is the set of non-negative measurable functions of bounded support in X, the functionals $L[f]$ and $L_x[f]$ satisfy the relation

$$D_gL[f]=\lim_{\epsilon\to0}\frac{L[f+\epsilon g]-L[f]}{\epsilon}=-\int_X g(x)L_x[f]M(dx).$$

Example: Poisson Point Process.

Let μ be a Poisson p.p. on the real line with intensity λ. First we need the Laplace functional. We take a simple function $f=\sum_{i=1}^n\mathbb{1}_{A_i}$. Then

$$
\begin{aligned}
L[f] &= E\Big(e^{-\int f(x)\mu(dx)}\Big) = E\Big(e^{-\sum_{i=1}^n c_i\mu(A_i)}\Big) = \prod_{i=1}^n E\Big(e^{-c_i\mu(A_i)}\Big) \\
&= \prod_{i=1}^n e^{-\lambda(A_i)(1-e^{-c_i})} = \exp\Big(-\sum_{i=1}^n -\lambda(A_i)(1-e^{-c_i})\Big) = \exp\Big(-\sum_{i=1}^n \int_{A_i}(1-e^{-c_i})\lambda(dx)\Big) \\
&= \exp\Big(-\int(1-e^{-f(x)})\lambda(dx)\Big)
\end{aligned}
$$

because $1-e^{-f(x)} = 0$ for $x \notin \bigcup_{i=1}^n A_i$.

For a general $f \in BM_+(R)$ we find $f_n \nearrow f$ pointwise as $n \to \infty$, by monotone convergence theorem $\mu(f_n) \to \mu(f)$ and by dominated convergence $L[f_n] \to L[f]$

$$\Rightarrow L[f] = \exp\Big(-\int(1-e^{-f(x)})\lambda(dx)\Big).$$

The directional derivative is also given by

$$
\begin{aligned}
\left.\frac{dL(f+\epsilon g)}{d\epsilon}\right|_{\epsilon=0} &= \left.-L(f+\epsilon g)\frac{d}{d\epsilon}\int_X(1-e^{-f(x)-\epsilon g(x)})\lambda(dx)\right|_{\epsilon=0} \\
&= \left.-L(f+\epsilon g)\int_X g(x)(1-e^{-f(x)-\epsilon g(x)})\lambda(dx)\right|_{\epsilon=0} \\
&= -L[f]\int_X g(x)(1-e^{-f(x)})\lambda(dx)
\end{aligned}
$$

$$\Rightarrow L_x[f] = e^{-f(x)}L[f] = L_{\delta_x}[f]L[f]$$

which was expected. We get back the original process plus a deterministic point mass at x. By $L_{\delta_x}[f]$ I mean the Laplace functional of a delta function. So we get back the Poisson process, plus an extra deterministic point at x.

5.2.3 Higher Order Campbell and Palm measures

We define the second order Campbell measure on $W^{(2)} = X^2 \times \hat{M}(X)$ by setting

$$C_P^{(2)}(A \times B \times U) = E\big(\xi(A)\xi(B)\mathbb{1}_U(\xi)\big).$$

Countable additivity works the same way as in the first order case. We are going to show that is σ-finite. Let $\{A_m : m \in \mathbb{N}\}$ be a cover of X, same as before. We define

$$U_{m_1m_2n} = \{\xi : \xi(A_{m_1})\xi(A_{m_2}) \le n\}.$$

Then

$$C_P^{(2)}(A_{m_1} \times A_{m_2} \times U_{m_1m_2n}) = \int_{A_{m_1}}\int_{A_{m_2}}\int_{U_{m_1m_2n}} \xi(dx)\xi(dy)P(d\xi) \le nP(U_{m_1m_2n}).$$

Now let $(x, y, \xi) \in W^{(2)}$. Since $\{A_m : m \in \mathbb{N}\}$ is a cover of X, $\exists m_1, m_2 \in \mathbb{N}$ such that $x \in A_{m_1}$ and $y \in A_{m_1}$. ξ is boundedly finite, thus $\exists n$ such that
$xi(A_{m_1})\xi(A_{m_2}) \le n \Rightarrow (x, y, \xi) \in A_{m_1} A_{m_2} U_{m_1 m_2 n}$, which means those set s form a cover of $W^{(2)}$. By Carathéodory's Theorem the set function $C_P^{(2)}$ extends uniquely to a measure on $\mathscr{B}_{\mathscr{W}}{}^{(2)} = \mathscr{B}_{\mathscr{X}} \times \mathscr{B}_{\mathscr{X}} \times \mathscr{B}_{\hat{\mathscr{M}}(\mathscr{X})}$.

Of course the same way we can define

$$C_P^{(n)}(A_1 \times A_2 \cdots \times A_n \times U) = E\big(\xi(A_1)\xi(A_2)\cdots\xi(A_n)\mathbb{1}_U(\xi)\big)$$

which extends uniquely to a σ-finite measure on $\mathscr{B}_{\mathscr{W}}{}^{(n)}$.

Lemma 5.2.3: ([13]) For $\mathscr{B}_{\mathscr{W}}{}^{(n)}$ measurable functions $g(x_1, x_2, \cdots, x_n, \xi)$ that are either non-negative or $C_P^{(n)}$ integrable

$$\begin{aligned}\int_{W^{(n)}} g(x_1, x_2, \cdots, \xi) C_P^{(n)}(dx_1, dx_2, \cdots, d\xi) &= E\Big(\int_{X^n} g(x_1, x_2, \cdots, \xi) dx_1 dx_2 \cdots d\xi\Big) \\ &= \int_{\hat{M}(X)} \int_{X^n} g(x_1, x_2, \cdots, \xi)\xi(dx_1)\xi(dx_2)\cdots P(d\xi).\end{aligned}$$

When the nth moment measure M_n exists, for $U = \hat{M}(X)$ we have

$$C_P^{(n)}\big(A_1 \times A_2 \cdots \times A_n \times \hat{M}(X)\big) = E\big(\xi(A_1)\xi(A_2)\cdots\xi(A_n)\mathbb{1}\big) = M_n(A_1 \times A_2 \cdots \times A_n).$$

Again, for a fixed U, $C_P^{(n)}(\cdot \times U)$ is absolutely continuous to $M_n(\cdot)$, so we can introduce the Palm measure as the Radon-Nikodym derivative and the $\mathscr{B}_{\mathscr{X}}{}^n$ measurable function $P(x_1, \cdots, x_n, U) = P_{\bar{x}}(U)$

$$\int_{A_1} \cdots \int_{A_n} P(x_1, \cdots, x_n, U) M_n(dx_1, \cdots, dx_n) = C_P^{(n)}(A_1 \times A_2 \cdots \times A_n \times U).$$

We also have the following:

Lemma 5.2.4: ([13]) Let ξ be a random measure for which the nth moment measure M_n exists. Then same as before, but for a general n

$$\begin{aligned}E\Big(\int_{X^n} g(\bar{x}, \xi)\xi(d\bar{x})\Big) &= \int_{W^{(n)}} g(\bar{x}, \xi) C_P^{(n)}(d\bar{x}, d\xi) \\ &= \int_{X^n} \int_{W^{(n)}} g(\bar{x}, \xi) P_{\bar{x}}(d\xi) M_n(d\bar{x}).\end{aligned}$$

We have already defined $D_g L[f]$ to be the directional derivative in direction g and have proved that

$$D_g L[f] = \lim_{\epsilon \to 0} \frac{L[f + \epsilon g] - L[f]}{\epsilon} = -\int_X g(x) L_x[f] M(dx).$$

Taking this one step further I claim

Proposition 5.2.4: Let $f, g \in BM_+(R)$. With the same conditions as before

$$D^{(2)}_{g,h}L[f] = \lim_{\epsilon\to\infty}\frac{D_gL[f+\epsilon h]-D_gL[f]}{\epsilon} = \int_X\int_X g(x)h(y)L_{xy}[f]M_2(dx_1,dx_2)$$

where $L_{xy}[f]$ is the Laplace functional of f with respect to the second order Palm measure $P_{xy}(d\xi)$.

Proof: We first take a step back

$$\begin{aligned}
D_gL[f] &= -\int_X g(x)L_x[f]M(dx) = -\int_X g(x)\int_{\hat{M}(X)} e^{-\xi(f)}P_x(d\xi)M(dx)\\
&= -\int_{\hat{M}(X)}\int_X g(x)e^{-\xi(f)}P(d\xi)\xi(dx)\\
\Rightarrow D_gL[f+\epsilon h]-D_gL[f] &= -\int_{\hat{M}(X)}\int_X g(x)\Big(e^{-\xi(f+\epsilon h)}-e^{-\xi(f)}\Big)P(d\xi)\xi(dx).
\end{aligned}$$

We compute separately

$$\begin{aligned}
e^{-\xi(f+\epsilon g)}-e^{-\xi(f)} &= e^{-\xi(f)}\Big(e^{-\epsilon\int_X h(y)\xi(dy)}-1\Big)\\
&= e^{-\xi(f)}\Big(1-\epsilon\int_X h(y)\xi(dy)+o(\epsilon)-1\Big)\\
&= -\epsilon e^{-\xi(f)}\xi(h)+o(\epsilon)\\
\Rightarrow \lim_{\epsilon\to\infty}\frac{D_gL[f+\epsilon h]-D_gL[f]}{\epsilon} &= \int_{\hat{M}(X)}\int_X g(x)\xi(y)e^{-\xi(f)}P(d\xi)\xi(dx)\\
&= \int_{\hat{M}(X)}\int_X\int_X g(x)h(y)e^{-\xi(f)}P(d\xi)\xi(dx)\xi(dy)\\
&= \int_{\hat{M}(X)}\int_X\int_X g(x)h(y)e^{-\xi(f)}P_{xy}(d\xi)M_2(dx,dy)\\
&= \int_X\int_X g(x)h(y)L_{xy}[f]M_2(dx,dy).
\end{aligned}$$

□

Before continuing, we prove the following lemma

Lemma 5.2.5: Let $\{X_i\}_{i=1}^{\infty}$, be a random p.p. on $\mathbb{R}$ with correlation functions $\rho_n(x_1,x_2,\cdots,x_n)$. Then for disjoint $A,B\in\mathbb{R}$

$$\int_A\int_B \rho_2(x,y)dxdy = E(N(A)N(B))$$

and

$$\int_{A^2}\int_B \rho_3(x,y,z)dxdydz = E[N(A)(N(A)-1)N(B)]$$

where $N(A)$ and $N(B)$ are the number of particles in A and B respectively.

Proof: Let $m(x)$ be the empirical density

$$m(x) = \sum_i \delta(x-x_i).$$

Then by definition

$$\rho_1(x) := E(m(x))$$

and for $x \neq y$

$$\rho_2 := E(m(x)m(y)) = \sum_{i\neq j} E(\delta(x-x_i)\delta(y-y_i)).$$

- $$\int_A \rho_1(x)dx = E\Big(\sum_{i=1}^{\infty} 1_A(x_i)\Big) = E(N(A)).$$

- $$\begin{aligned}\int_{A^2} \rho_2(x,y)dxdy &= \Big(\sum_{i\neq k} 1_A(x_i)1_A(x_j)\Big) = E\Big(\sum_{i,j} 1_A(x_i)1_A(x_j) - \sum_i 1_A(x_i)\Big). \\ &= E[N(A)(N(A)-1)]\end{aligned}$$

- Similarly, $\int_{A^k} \rho_k(x_1, x_2, \cdots, x_k)dx_1dx_2\cdots dx_k = E(P_k(N(A))$, where $P_k(x)$ is a monic polynomial of degree k. Since for $N(A) = 0, 1, \cdots, k-1$ we have $P_k(N(A)) = 0$, we can deduce that

$$P_k(N(A)) = N(A)(N(A)-1)(N(A)-2)\cdots(N(A)-k+1)$$

i)

$$\begin{aligned}\int_A \int_B \rho_2(x,y)dxdy &= \int_A \int_B E\Big(\sum_{i\neq j} \delta(x-x_i)\delta(y-x_j)\Big)dxdy \\ &= E\Big(\sum_{i\neq j} 1_A(x_i)1_B(x_j)\Big) = E\Big(\sum_{i,j} 1_A(x_i)1_B(x_j)\Big) \\ &= E(N(A)N(B))\end{aligned}$$

ii)

$$\begin{aligned}\int_{A^2} \int_B \rho_3(x,y,z)dxdydz &= E\Big(\sum_{i\neq j\neq k\neq i} 1_A(x_i)1_A(x_j)1_B(x_k)\Big) \\ &= E\Big(\sum_k 1_B(x_k)\sum_{i\neq j} 1_A(x_i)1_A(x_j)\Big) \\ &= E\Big(N(B)\big[\sum_{i,j} 1_A(x_i)1_A(x_j) - \sum_i 1_A(x_i)\big]\Big) \\ &= E\Big[N(A)(N(A)-1)N(B)\Big]\end{aligned}$$

where we also used that for either $i = k$ or $j = k$, $1_A(x_i)1_A(x_j)1_B(x_k) = 0$.

□

Lemma 5.2.6: Assume we have a point process on $\mathbb{R}$ for which ρ_1, ρ_2, ρ_3 are continuous and $P_{xy}(U)$ is continuous in both x and y for some fixed U. Then

$$\lim_{\epsilon \to 0} P(\xi \in U | \xi(x-\epsilon, x+\epsilon) > 0, \xi(y-\epsilon, y+\epsilon) > 0) = P_{x,y}(U).$$

Proof: The proof is similar to that of lemma 4.4.1.

$$P(\xi \in U | \xi(x-\epsilon, x+\epsilon) > 0, \xi(y-\epsilon, y+\epsilon) > 0) = \frac{P(\xi \in U, Z_\epsilon^x > 0, Z_\epsilon^y > 0)}{P(Z_\epsilon^x > 0, Z_\epsilon^y > 0)}$$

- **Denominator**

  $$P(Z_\epsilon^x > 0, Z_\epsilon^y > 0) \quad = \quad E(\mathbb{1}_{Z_\epsilon^x > 0} \mathbb{1}_{Z_\epsilon^y > 0})$$

  We once again use that $1_{Z_\epsilon^x > 0} = Z_\epsilon^x - (Z_\epsilon^x - 1)\mathbb{1}_{Z_\epsilon^x > 1}$.

  $$\Rightarrow P(Z_\epsilon^x > 0, Z_\epsilon^y > 0) = E(Z_\epsilon^x Z_\epsilon^y - Z_\epsilon^x (Z_\epsilon^y - 1)\mathbb{1}_{Z_\epsilon^y > 1} - Z_\epsilon^y (Z_\epsilon^x - 1)\mathbb{1}_{Z_\epsilon^x > 1} + (Z_\epsilon^x - 1)(Z_\epsilon^y - 1)\mathbb{1}_{Z_\epsilon^x > 1}\mathbb{1}_{Z_\epsilon^y > 1}).$$

  I claim that $P(Z_\epsilon^x > 0, Z_\epsilon^y > 0) = 4\epsilon^2 \rho_2(x,y) + o(\epsilon^2)$. Term by term, we have:

  i)

  $$Z_\epsilon^x (Z_\epsilon^y - 1)\mathbb{1}_{Z_\epsilon^y > 1} \quad \leq \quad Z_\epsilon^x Z_\epsilon^y (Z_\epsilon^y - 1)$$

  and

  $$\begin{aligned} E(Z_\epsilon^x Z_\epsilon^y (Z_\epsilon^y - 1)) &= \int_{A_\epsilon^2} \int_{B_\epsilon} \rho_3(z_1, z_2, z_3) dz_1 dz_2 dz_3 \\ &\leq 8\epsilon^3 \sup_{A_\epsilon^2 \times B_\epsilon} |\rho_3| \end{aligned}$$

  ii) $Z_\epsilon^y (Z_\epsilon^x - 1) 1_{Z_\epsilon^x > 1}$ and $(Z_\epsilon^x - 1)(Z_\epsilon^y - 1) 1_{Z_\epsilon^x > 1} 1_{Z_\epsilon^y > 1}$ are similar to i).

  iii)

  $$\begin{aligned} E(Z_\epsilon^x Z_\epsilon^y) &= \int_{x-\epsilon}^{x+\epsilon} \int_{y-\epsilon}^{y+\epsilon} \rho_2(z_1, z_2) dz_1 dz_2 \\ &= \int_{x-\epsilon}^{x+\epsilon} \int_{y-\epsilon}^{y+\epsilon} \rho_2(z_1, z_2) - \rho_(x,y) dz_1 dz_2 + 4\epsilon^2 \rho_2(x,y) \\ &= \leq 4\epsilon^2 \sup_{(z_1, z_2) \in A_\epsilon \times B_\epsilon} |\rho_2(z_1, z_2) - \rho_2(x,y)| + 4\epsilon^2 \rho_2(x,y) \\ &= 4\epsilon^2 \rho_2(x,y) + o(\epsilon^2) \end{aligned}$$

- **Numerator**

  Reminder that Z_ϵ^x is just short for $\xi(x-\epsilon, x+\epsilon)$, which just counts the number of points in $(x-\epsilon, x+\epsilon)$.
  The numerator can now be written as follows:

  $$E(\mathbb{1}_U(\xi), \mathbb{1}_{Z_\epsilon^x > 0}, \mathbb{1}_{Z_\epsilon^x > 0}) = E(\mathbb{1}_U(\xi) \xi(A_\epsilon) \xi(B_\epsilon)) + o(\epsilon^2).$$

We define $g(z_1, z_2, \xi) = \mathbb{1}_{A_\epsilon}(z_1)\mathbb{1}_{B_\epsilon}(z_2)\mathbb{1}_U(\xi)$. Then

$$E(\mathbb{1}_U(\xi)\xi(A_\epsilon)\xi(B_\epsilon)) = \int_{\mathbb{R}^2}\int_{\hat{M}(\mathbb{R})} g(z_1, z_2, \xi)P(d\xi)\xi(dz_1)\xi(dz_2)$$

$$= \int_{\mathbb{R}^2}\int_{\hat{M}(\mathbb{R})} g(z_1, z_2, \xi)P_{z_1,z_2}(d\xi)M(dz_1 \times dz_2) = \int_{A_\epsilon}\int_{B_\epsilon} P_{z_1,z_2}(U)M(dz_1 \times dz_2)$$

$$= \int_{x-\epsilon}^{x+\epsilon}\int_{y-\epsilon}^{y+\epsilon} \rho_2(z_1, z_2)P_{z_1,z_2}(U)dz_1dz_2 = 4\epsilon^2\rho_2(x, y)P_{x,y}(U) + o(\epsilon^2)$$

Thus, by dividing both numerator and denominator by $4\epsilon^2$ and sending $\epsilon \to 0$, we get

$$\lim_{\epsilon\to 0} P(\xi \in U|\xi(x - \epsilon, x + \epsilon) > 0, \xi(y - \epsilon, y + \epsilon) > 0) = P_{x,y}(U).$$

□

Working the exact same way, but with P_x instead of P, we can show

$$\lim_{\epsilon\to 0} P_x(\xi \in U|\xi(y - \epsilon, y + \epsilon) > 0) = (P_x)_y(U).$$

A question I have yet to answer is the following: What happens if we condition on having two particles at the same point, ie

$$\lim_{\epsilon\to 0} P(\xi \in U|\xi(x - \epsilon, x + \epsilon) > 1) =?$$

While I have yet to solve this, I present a rigid example in 5.2.6 as what happens if we condition on having many particles at the origin, on the determinantal p.p. form by the eigenvalues of the Ginibre ensemble.

5.2.4 Determinantal Point Processes

This is what everything has been leading to. For a determinantal point process on $\mathbb{R}$ with kernel K, we will show that the Palm measure P_x corresponds to a determinantal p.p. away from x, with kernel $K_x(y, z) = K(y, z) - \frac{K(x,z)K(y,x)}{K(x,x)}$. Not only this is the result we got from Theorem 2, but we also use the knowledge of what we expect the kernel to be, in order to prove it.

Proposition 5.2.5: Let $f \in BM_+(R)$ and K be a kernel such that $TrK = \int_A K(x, x)dx$ is finite for all bounded sets A. The Laplace functionals for a determinantal point process ξ with kernel K are given by the Fredholm determinants

$$L[f] = E\big(e^{-\xi(f)}\big) = \det(1 - M_{1-e^{-f}}K).$$

Proof: Let $\xi = \sum_{i=1}^{\infty} \delta_{x_i}$ be a d.p.p. on $\mathbb{R}$ with correlation functions ρ_n, and ϕ be a function on $\mathbb{R}$ s.t. the kernel $\phi(x)K(x, y)$ defines an operator $M_\phi K$ in L_2 and $Tr(M_\phi K)$ is also finite. Then

$$E\Big(\prod_{i=1}^{\infty}(1 - \phi(x_i)\Big) = \det(1 - M_\phi K).$$

To show this we start with the identity

$$\begin{aligned}
\prod_{i=1}^{\infty}\big(1-\phi(x_i)\big) &= 1-\sum_{i=1}^{\infty}\phi(x_i)+\sum_{i_1<i_2}\phi(x_{i_1})\phi(x_{i_2})-\dots+\sum_{i_1<i_2<\dots<i_k}(-1)^k\phi(x_{i_1})\phi(x_{i_2})\dots\phi(x_{i_k})+\dots \\
\Rightarrow E\big(\prod_{i=1}^{\infty}(1-\phi(x_i)\big) &= \sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\int\cdots\int\phi(x_1)\phi(x_2)\dots\phi(x_k)\rho_k(x_1,x_2,\dots,x_k)\prod_{i=1}^{k}dx_i \\
&= \det(I-M_\phi K)
\end{aligned}$$

In order to get the Laplace funtional of a d.p.p. we set $\phi = 1-e^f$ and work as following

$$\begin{aligned}
E\big(\prod_{i=1}^{\infty}(1-\phi(x_i)\big) &= E\Big(e^{\sum_i \log(1-\phi(x_i)}\Big)=E\Big(e^{\int\log(1-\phi(x)\xi(dx)}\Big) \\
&= E\Big(e^{-\int f(x)\xi(dx)}\Big)=E\big(e^{-\xi(f)}\big)=L[f]
\end{aligned}$$

□

Lemma 5.2.7: Suppose the L_2 operator K_t, where $K_t(x,y)$ is a C_1 function on t. Then

$$\frac{d}{dt}\det(I+K_t)=\det(I+K_t)Tr\Big(\dot{K}_t(1+K_t)^{-1}\Big).$$

(Proof in the appendix)

Using the previous lemma and

$$\frac{d}{d\epsilon}M_{1-e^{-f-\epsilon g}}\Big|_{\epsilon=0}=M_{ge^{-f}}$$

we get

$$-D_gL[f]=\det(1-M_{1-e^{-f}}K)Tr\Big(M_{ge^{-f}}K(1-M_{1-e^{-f}}K)^{-1}\Big).$$

Expanding the trace as an integral we have

$$\begin{aligned}
-D_gL[f] &= \det(1-M_{1-e^{-f}}K)\int_{\mathbb{R}}g(x)e^{-f(x)}K(1-M_{1-e^{-f}}K)^{-1}(x,x)dx \\
&= \int_{\mathbb{R}}g(x)\det(1-M_{1-e^{-f}}K)\frac{e^{-f(x)}}{K(x,x)}K(1-M_{1-e^{-f}}K)^{-1}(x,x)K(x,x)dx \\
\Rightarrow L_x[f] &= \det(1-M_{1-e^{-f}}K)\frac{e^{-f(x)}}{K(x,x)}K(1-M_{1-e^{-f}}K)^{-1}(x,x).
\end{aligned}$$

We predict that the new kernel will be

$$K_x(y,z)=K(y,z)-\frac{K(x,z)K(y,x)}{K(x,x)}$$

which can be written as $K_x = K-F_x$, where $F_x = K(x,x)^{-1}K(\cdot,x)K(x,\cdot)$, showing it is a rank one perturbation.

But as seen in the Poisson p.p. example, the Palm measure also accounts for an extra deterministic point mass on x, on which we conditioned. So we expect

$$\begin{aligned}
L_x[f] &= L_{\delta_x}[f]\det(1 - M_{1-e^{-f}}(K - F_x)) \\
&= e^{-f(x)}\det(1 - M_{1-e^{-f}}K)\det\left(1 + M_{1-e^{-f}}F_x(1 - M_{1-e^{-f}}K)^{-1}\right).
\end{aligned}$$

The operator $M_{1-e^{-f}}F_x(1 - M_{1-e^{-f}}K)^{-1}$ is still rank one, which means it has only one eigenvalue and thus the characteristic polynomial is of degree one.

The characteristic polynomial is given by

$$\det(1 + zK) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\int\cdots\int\rho_n(x_1,\cdots,x_n)\prod_{i=1}^{n}dxi.$$

Then for kernels of the form $K = u\circ v$, for $z = 1$ we get the Fredholm determinant $det(1+u\circ v) = 1+\int uv$

Call $1 - e^f = \phi$. We want to show that

$$\frac{1}{K(x,x)}K(1 - M_\phi K)^{-1}(x,x) = \det(1 + M_\phi F_x(1 - M_\phi K)^{-1}).$$

First we rewrite the RHS, starting with writing down the kernel

$$\begin{aligned}
M_\phi F_x(1 - M_\phi K)^{-1}(z,w) &= \frac{\phi(z)}{K(x,x)}K(z,x)\int K(x,y)(1 - M_{\phi K}K)^{-1}(y,w)dy \\
\Rightarrow \det(1 + M_\phi F_x(1 - M_\phi K)^{-1}) &= 1 + \int\frac{\phi(z)}{K(x,x)}K(z,x)\int K(x,y)(1 - M_{\phi K}K)^{-1}(y,z)dydz.
\end{aligned}$$

It suffices to show that

$$K(1 - M_\phi K)^{-1}(x,x) = K(x,x) + \int\phi(z)K(z,x)\int K(x,y)(1 - M_{\phi K}K)^{-1}(y,z)dydz$$

which in operator language translates to

$$K(1 - M_\phi K)^{-1} = K + K(1 - M_\phi K)^{-1}M_\phi K$$

on the diagonal. But these operators are equal in general:

$$\begin{aligned}
K + K(1 - M_\phi K)^{-1}M_\phi K &= K + K(1 - M_\phi K)^{-1}M_\phi K - K(1 - M_\phi K)^{-1} + K(1 - M_\phi K)^{-1} \\
&= K + K(1 - M_\phi K)^{-1}(M_\phi K - 1) + K(1 - M_\phi K)^{-1} = K(1 - M_\phi K)^{-1} \\
\Rightarrow L_x[f] &= e^{-f(x)}\det(1 - M_{1-e^{-f}}K_x) = L_{\delta_x}[f]\det(1 - M_{1-e^{-f}}K_x)
\end{aligned}$$

which means we have our kernel. We can get rid of the point mass at x by using test function f that disappear on x, i.e. $f(x) = 0 \Rightarrow \phi(x) = 0$.

5.2.5 Conditioning

Now back to the old tricks.

Just like in the discrete case, we start by conditioning on having particles on certain points. The difference is that in the continuous case that has probability zero, so we need to condition on a point process having a particle in a small increment and then sent the length of said increment to 0. The goal is to condition on any finite amount of points plus an empty integral and find a formula for the kernel, much like we did in the discrete case.

We will prove the following 3 Lemmas, all of which are counterparts to the results we got for the discrete case.

Lemma 5.2.8:

$$\lim_{\epsilon\to 0} P(Z_\delta^x > 0 | Z_\epsilon^y > 0) = \frac{\begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{vmatrix}}{K(y,y)} \delta + o(\delta)$$

Lemma 5.2.9:

$$P(Z_\delta^x > 0 | N(A) = 0) = K|_A (I - K|_A)^{-1}_{(x,x)} \delta + o(\delta)$$

and

Lemma 5.2.10:

$$\lim_{\epsilon\to 0} P(Z_\delta^x > 0 | N(A) = 0, Z_\epsilon^y > 0) \quad = \quad \tilde{K}|_A (I - \tilde{K}|_A)^{-1}_{(x,x)} \delta + o(\delta)$$

where

$$\tilde{K}(x,z) = \frac{\begin{vmatrix} K(x,z) & K(x,y) \\ K(y,z) & K(y,y) \end{vmatrix}}{K(y,y)}.$$

i) We will use the following notation. Z_ϵ^x is the number of particles in $(x, x+\epsilon)$ and $N(A)$ is the number of points in a set A. We start by conditioning on a point process having a particle in $(y, y+\epsilon)$.

   We will show that

   Lemma 5.2.8:

$$\lim_{\epsilon\to 0} P(Z_\delta^x > 0 | Z_\epsilon^y > 0) = \frac{\begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{vmatrix}}{K(y,y)} \delta + o(\delta)$$

   Proof:

$$\begin{aligned} P(Z_\delta^x > 0 | Z_\epsilon^y > 0) &= \frac{P(Z_\delta^x > 0, Z_\epsilon^y > 0)}{P(Z_\epsilon^y > 0)} \\ &= \frac{E(\mathbb{1}_{Z_\delta^x > 0} \mathbb{1}_{Z_\epsilon^y > 0)})}{E(\mathbb{1}_{Z_\epsilon^y > 0)}}. \end{aligned}$$

We have already come across these quantities before and we have:

$$\begin{aligned} P(Z^y_\epsilon > 0) &= E(\mathbb{1}_{Z^y_\epsilon > 0}) = E(Z^y_\epsilon) + o(\epsilon) \\ &= \int_y^{y+\epsilon} K(w,w)dw + o(\epsilon) = \epsilon K(y,y) + o(\epsilon). \end{aligned}$$

Similarly

$$\begin{aligned} P(Z^x_\delta > 0, Z^y_\epsilon > 0) &= E(\mathbb{1}_{Z^x_\delta > 0}\mathbb{1}_{Z^y_\epsilon > 0}) = E(Z^x_\delta Z^y_\epsilon) + \epsilon o(\delta) + o(\epsilon) \\ &= \int_x^{x+\delta}\int_y^{y+\epsilon} \begin{vmatrix} K(w_1,w_1) & K(w_1,w_2) \\ K(w_2,w_1) & K(w_2,w_2) \end{vmatrix} dw_1 dw_2 + \epsilon o(\delta) + o(\epsilon) \\ &= \epsilon\delta \begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{vmatrix} + \epsilon o(\delta) + o(\epsilon). \end{aligned}$$

Let's see in more detail on the side why this last bit holds true.

$$\begin{aligned} &\int_x^{x+\delta}\int_y^{y+\epsilon} \begin{vmatrix} K(w_1,w_1) & K(w_1,w_2) \\ K(w_2,w_1) & K(w_2,w_2) \end{vmatrix} dw_1 dw_2 \\ = &\int_x^{x+\delta}\int_y^{y+\epsilon} \Big[K(w_1,w_1)K(w_2,w_2) - K(w_1,w_2)K(w_2,w_1)\Big] dw_1 dw_2 \\ = &\int_x^{x+\delta} \Big[\epsilon K(w_1,w_1) \sup_{w_2\in(y,y+\epsilon)} |K(w_2,w_2) - K(y,y)| + \epsilon K(w_1,w_1)K(y,y) \\ &-\epsilon \sup_{w_2\in(y,y+\epsilon)} |K(w_1,w_2)K(w_2,w_1) - K(w_1,y)K(y,w_1)| - \epsilon K(w_1,y)K(y,w_1)\Big] dw_1 \end{aligned}$$

$$\begin{aligned} &= \epsilon \int_x^{x+\delta} \Big[K(w_1,w_1)K(y,y) - K(w_1,y)K(y,w_1)\Big] dw_1 + \delta o(\epsilon) \\ &= \epsilon\delta K(y,y) \sup_{w_1\in(x,x+\delta)} |K(w_1,w_1) - K(x,x)| + \epsilon\delta K(x,x)K(y,y) \\ &\quad - \sup_{w_1\in(x,x+\delta)} |K(w_1,y)K(y,w_1) - K(x,y)K(y,x)| - \epsilon\delta K(x,y)K(y,x) + \delta o(\epsilon) \\ &= \epsilon\delta \begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{vmatrix} + \epsilon o(\delta) + \delta o(\epsilon). \end{aligned}$$

Dividing numerator and denominator by ϵ and sending it to 0, we get

$$\lim_{\epsilon\to 0} P(Z^x_\delta > 0 | Z^y_\epsilon > 0) = \frac{\begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{vmatrix}}{K(y,y)}\delta + o(\delta)$$

$\square$

From there we can see that for the new point process derived from conditioning we have

$$\rho_1(x) = \tilde{K}(x,x) = \frac{\begin{vmatrix} K(x,x) & K(x,y) \\ K(y,x) & K(y,y) \end{vmatrix}}{K(y,y)}.$$

So all the identities from the discrete case so far carry over to the continuous one, since we've seen how we can inductively condition on more particles.

Before we continue, we will include this well know proposition.

Proposition 5.2.6: For a determinantal point process with correlation functions given by $\rho_n(x_1, x_2, ..., x_n) = \det_{1 \le i,j \le n}(K(x_i, x_j))$, where the kernel K is continuous, we have that

$$P(N(A) = 0) = det(I - K|_A) = D_A$$

where $det(I - K|_A)$ is the Fredholm determinant

$$D_A = 1 - \int\limits_A K(z_1, z_1) dz_1 + \frac{1}{2!} \int\limits_{A^2} \begin{vmatrix} K(z_1, z_1) & K(z_1, z_2) \\ K(z_2, z_1) & K(z_2, z_2) \end{vmatrix} dz_1 dz_2 - \frac{1}{3!} \int\limits_{A^3} \cdots$$

$\square$

ii) We continue with conditioning on an empty set A. We will show that

Lemma 5.2.9:

$$P(Z^x_\delta > 0 | N(A) = 0) = K|_A (I - K|_A)^{-1}_{(x,x)} \delta + o(\delta)$$

Proof:

$$P(Z^x_\delta > 0 | N(A) = 0) = \frac{P(Z^x_\delta > 0, N(A) = 0)}{P(N(A) = 0)} = \frac{E(\mathbb{1}_{Z^x_\delta > 0} \mathbb{1}_{N(A)=0})}{D_A}.$$

Without dwelling again too much on the indicators and subsequent continuity arguments (I'll give a more concrete one in the next, more general case), we have

$$E(\mathbb{1}_{Z^x_\delta > 0} \mathbb{1}_{N(A)=0}) \quad = \quad E(Z^x_\delta (1-1)^{N(A)}) + o(\delta)$$

and

$$E(Z^x_\delta (1-1)^{N(A)}) = E(Z^x_\delta) - E(Z^x_\delta N(A)) + E\Big(Z^x_\delta \frac{N(A)(N(A)-1)}{2!}\Big) - \cdots$$

$$= \int\limits_x^{x+\delta} K(w_1, w_1) dw_1 - \int\limits_x^{x+\delta} \int\limits_A \begin{vmatrix} K(w_1, w_1) & K(w_1, z_1) \\ K(z_1, w_1) & K(z_1, z_1) \end{vmatrix} dw_1 dz_1 + \frac{1}{2!} \int\limits_x^{x+\delta} \int\limits_{A^2} \cdots$$

$$= \delta K(x,x) + o(\delta) - \delta \int\limits_A \begin{vmatrix} K(x,x) & K(x,z_1) \\ K(z_1,x) & K(z_1,z_1) \end{vmatrix} dz_1 + \cdots$$

By defining as $R(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the following function

$$R(x,y) = K(x,y) - \int\limits_A \begin{vmatrix} K(x,y) & K(x,z_1) \\ K(z_1,y) & K(z_1,z_1) \end{vmatrix} dz_1 + \frac{1}{2!}\int\limits_{A^2} \begin{vmatrix} K(x,y) & K(x,z_1) & K(x,z_2) \\ K(z_1,y) & K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,y) & K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1 dz_2$$
$$-\frac{1}{3!}\int\limits_{A^3} |4\times 4| dz_1 dz_2 dz_3 + \frac{1}{4!}\int\limits_{A^4} \cdots$$

we can write

$$P(Z_\delta^x > 0 | N(A) = 0) = \frac{R(x,x)}{D_A}\delta + o(\delta)$$

Now we have from Lax [23, Chapter 24] the following 2 equations linking $R(x,y)$ and $K(x,y)$

$$D_A K(x,y) - R(x,y) + \int_{\mathbb{R}} K(x,z)R(z,y)dz = 0$$
$$D_A K(x,y) - R(x,y) + \int_{\mathbb{R}} R(x,z)K(z,y)dz = 0$$

If we write them in operator language and also the operator $I-K$ is invertible, then from the second equation we get

$$R - RK|_A = D_A K|_A \Rightarrow R(I - K|_A) = D_A K|_A \Rightarrow R = D_A K|_A (I-K|_A)^{-1}$$

$$\Rightarrow P(Z_\delta^x > 0 | N(A) = 0) = K|_A (I - K|_A)^{-1}_{(x,x)}\delta + o(\delta).$$

□

iii) Now we move one on the more general case where we condition both on an empty set A and a particle at y. This may be redundant, since we can just condition first on an set of points being occupied, get our new kernel $\tilde{K}$, and then condition on an empty set. But given that I have yet to condition the other way around, meaning first condition on an empty set and then on a set of occupied points, and then match the two answers, I decided to include it.

Lemma 5.2.10:

$$\lim_{\epsilon\to 0} P(Z_\delta^x > 0 | N(A) = 0, Z_\epsilon^y > 0) = \tilde{K}|_A (I - \tilde{K}|_A)^{-1}_{(x,x)}\delta + o(\delta)$$

where

$$\tilde{K}(x,z) = \frac{\begin{vmatrix} K(x,z) & K(x,y) \\ K(y,z) & K(y,y) \end{vmatrix}}{K(y,y)}.$$

Proof:

$$P(Z_\delta^x > 0 | N(A) = 0, Z_\epsilon^y > 0) = \frac{P(Z_\delta^x > 0, N(A) = 0, Z_\epsilon^y > 0)}{P(N(A) = 0, Z_\epsilon^y > 0)}$$

Denominator

$$P(N(A)=0, Z_\epsilon^y > 0) = E(\mathbb{1}_{N(A)=0}\mathbb{1}_{Z_\epsilon^y>0}) = E(\mathbb{1}_{N(A)=0}Z_\epsilon^y) + o(\epsilon)$$

$$= \epsilon\Bigg(K(y,y) - \int\limits_A \begin{vmatrix} K(y,y) & K(y,z_1) \\ K(z_1,y) & K(z_1,z_1) \end{vmatrix} dz_1 + \\ + \frac{1}{2!}\int\limits_{A^2} \begin{vmatrix} K(y,y) & K(y,z_1) & K(y,z_2) \\ K(z_1,y) & K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,y) & K(z_2,z_1) & K(z_2,z_2) \end{vmatrix} dz_1 dz_2 \cdots \Bigg) + o(\epsilon)$$

$$= \epsilon K(y,y)\Bigg(1 - \int\limits_A \frac{\begin{vmatrix} K(y,y) & K(y,z_1) \\ K(z_1,y) & K(z_1,z_1) \end{vmatrix}}{K(y,y)} dz_1 \\ + \frac{1}{2!}\int\limits_{A^2} \frac{\begin{vmatrix} K(y,y) & K(y,z_1) & K(y,z_2) \\ K(z_1,y) & K(z_1,z_1) & K(z_1,z_2) \\ K(z_2,y) & K(z_2,z_1) & K(z_2,z_2) \end{vmatrix}}{K(y,y)} dz_1 dz_2 \cdots \Bigg) + o(\epsilon)$$

$$= \epsilon K(y,y)\Bigg(1 - \int\limits_A \tilde{K}(z_1,z_1) dz_1 + \\ + \frac{1}{2!}\int\limits_{A^2} \frac{\begin{vmatrix} \begin{vmatrix} K(y,y) & K(y,z_1) \\ K(z_1,y) & K(z_1,z_1) \end{vmatrix} & \begin{vmatrix} K(y,y) & K(y,z_2) \\ K(z_1,y) & K(z_1,z_2) \end{vmatrix} \\ \begin{vmatrix} K(y,y) & K(y,z_1) \\ K(z_2,y) & K(z_2,z_1) \end{vmatrix} & \begin{vmatrix} K(y,y) & K(y,z_2) \\ K(z_2,y) & K(z_2,z_2) \end{vmatrix} \end{vmatrix}}{K^2(y,y)} dz_1 dz_2 \cdots \Bigg) + o(\epsilon)$$

$$= \epsilon K(y,y)\Bigg(1 - \int\limits_A \tilde{K}(z_1,z_1) dz_1 + \frac{1}{2!}\int\limits_{A^2} \begin{vmatrix} \tilde{K}(z_1,z_1) & \tilde{K}(z_1,z_2) \\ \tilde{K}(z_2,z_1) & \tilde{K}(z_2,z_2) \end{vmatrix} dz_1 dz_2 \cdots \Bigg) + o(\epsilon)$$

$$= \epsilon K(y,y)\tilde{D}_A + o(\epsilon)$$

where

$$\tilde{K}(x_1,x_2) = \frac{\begin{vmatrix} K(x_1,x_2) & K(x_1,y) \\ K(y,x_2) & K(y,y) \end{vmatrix}}{K(y,y)}$$

Numerator

$$\begin{aligned} P(Z_\delta^x > 0, N(A)=0, Z_\epsilon^y > 0) &= E(\mathbb{1}_{Z_\delta^x>0}\mathbb{1}_{Z_\epsilon^y>0}\mathbb{1}_{N(A)=0}) \\ &= E(Z_\delta^x Z_\epsilon^y \mathbb{1}_{N(A)=0}) + \epsilon o(\delta) + o(\epsilon) \end{aligned}$$

where

$$E(Z_\delta^x Z_\epsilon^y \mathbb{1}_{N(A)=0}) = E(Z_\delta^x Z_\epsilon^y) - E(Z_\delta^x Z_\epsilon^y N(A)) - \frac{1}{2}E((Z_\delta^x Z_\epsilon^y N(A)(N(A)-1)) + \cdots$$

$$
\begin{aligned}
&= \int_x^{x+\delta}\int_y^{y+\epsilon} \begin{vmatrix} K(w_1,w_1) & K(w_1,w_2) \\ K(w_2,w_1) & K(w_2,w_2) \end{vmatrix} dw_1 dw_2 \\
&\quad - \int_x^{x+\delta}\int_y^{y+\epsilon}\int_A \begin{vmatrix} K(w_1,w_1) & K(w_1,w_2) & K(w_1,z_1) \\ K(w_2,w_1) & K(w_2,w_2) & K(w_1,z_1) \\ K(z_1,w_1) & K(z_1,w_2) & K(z_1,z_1) \end{vmatrix} dw_1 dw_2 dz_1 + o(\epsilon) \\
&= \epsilon\delta K(y,y)\tilde{R}(x,x) + \epsilon o(\delta) + o(\epsilon).
\end{aligned}
$$

This probably requires an explanation. First of all

$$
\begin{aligned}
\tilde{R}(x_1,x_2) &= \tilde{K}(x_1,x_2) - \int_A \begin{vmatrix} \tilde{K}(x_1,x_2) & \tilde{K}(x_1,z_1) \\ \tilde{K}(z_1,x_2) & \tilde{K}(z_1,z_1) \end{vmatrix} dz_1 \\
&\quad + \frac{1}{2!}\int_{A^2} \begin{vmatrix} \tilde{K}(x_1,x_2) & \tilde{K}(x_1,z_1) & \tilde{K}(x_1,z_2) \\ \tilde{K}(z_1,x_2) & \tilde{K}(z_1,z_1) & \tilde{K}(z_1,z_2) \\ \tilde{K}(z_2,x_2) & \tilde{K}(z_2,z_1) & \tilde{K}(z_2,z_2) \end{vmatrix} dz_1 dz_2 - \frac{1}{3!}\int_{A^3} |4\times 4| dz_1 dz_2 dz_3 + \dots
\end{aligned}
$$

where again

$$
\tilde{K}(x_1,x_2) = \frac{\begin{vmatrix} K(x_1,x_2) & K(x_1,y) \\ K(y,x_2) & K(y,y) \end{vmatrix}}{K(y,y)}.
$$

What we did was take outside a factor of $K(y,y)$, consequently dividing each term by it, and then used the determinantal identity to write each ratio as one determinant with elements $\tilde{K}_{i,j}$.

Also here is a more detailed explanation as to why the rest of the term are $\epsilon o(\delta) + o(\epsilon)$. Let's take the term with the $n\times n$ determinant. Using

$$
detA = \sum_{\sigma\in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma i}
$$

said term can be written as

$$
\begin{aligned}
&\frac{1}{(n-2)!}\int_x^{x+\delta}\int_y^{y+\epsilon}\int_{A^{n-2}} \sum_{\sigma\in S_n} sgn(\sigma) K(w_1,\sigma(w_1))K(\sigma^{-1}(w_2),w_2)\prod_{i=1}^{n-2} K(z_i,\sigma(z_1)) dw_1 dw_2 dz_1 \cdots dz_{n-2} \\
&= \frac{1}{(n-2)!}\int_{A^{n-2}} dz_1\cdots dz_{n-2} \sum_{\sigma\in S_n} sgn(\sigma)\prod_{i=1}^{n-2} K(z_i,\sigma(z_1)) \int_x^{x+\delta}\int_y^{y+\epsilon} dw_1 dw_2 K(w_1,\sigma(w_1))K(\sigma^{-1}(w_2),w_2)
\end{aligned}
$$

where the double integral inside has been already computed.

Note also that all the $o(\epsilon)$ terms may include δ, we just don't care about them, since we expect them to disappear as $\epsilon \to 0$.

Currently we have managed to write the conditional probability as follows:

$$
P(Z_\delta^x > 0 | N(A) = 0, Z_\epsilon^y > 0) = \frac{\delta\epsilon K(y,y)\tilde{R}(x,x) + \epsilon o(\delta) + o(\epsilon)}{\epsilon K(y,y)\tilde{D}_A + o(\epsilon)}.
$$

Again, by taking $\epsilon \to 0$, we get

$$\begin{aligned}\lim_{\epsilon\to 0} P(Z^x_\delta > 0 | N(A) = 0, Z^y_\epsilon > 0) &= \frac{K(y,y)\tilde{R}(x,x)}{K(y,y)\tilde{D}_A}\delta + o(\delta) \\ &= \tilde{K}|_A(I - \tilde{K}|_A)^{-1}_{(x,x)}\delta + o(\delta).\end{aligned}$$

$\square$

Remark: While we have assumed y is not in A, the same method would work if we take $y \in A$ and instead of $N(A) = 0$ we condition on $N(A/(y, y+\epsilon) = 0$.

Conditioning on an empty set A and having particles in n points will produce a determinantal point process with kernel $\tilde{K}|_A(I - \tilde{K}|_A)_{(x,y)}$ where $\tilde{K}$ is the ratio of an $(n+1)\times(n+1)$ determinant by an $n \times n$ determinant, same as in the discrete case. I don't believe it's worth going through those huge calculations just to show something there no doubt it holds true.

5.2.6 Complex Ginibre Example

We have the correlation function of the eigenvalues which for a determinantal point process. We will use this example to see what happens if we condition on having to particles at the same point, i.e. the origin.

$$\rho_n(z_1, z_2, ..., z_n) = \pi^{-n} \exp\left(-\sum_{i=1}^{n} |z_i|^2\right) \det_{1\le i,j\le n} (K(z_i, z_j))$$

where

$$K(x,y) = e^{x\bar{y}}.$$

All the calculations will be made with respect to the measure

$$\mu(dz) = \frac{e^{-|z|^2}}{\pi} dz$$

i) Conditioning on having one particle at the origin, the new kernel will be

$$\tilde{K}(x,y) = \frac{\begin{vmatrix} K(0,0) & K(0,y) \\ K(x,0) & K(x,y) \end{vmatrix}}{K(0,0)} = \frac{\begin{vmatrix} 1 & 1 \\ 1 & K(x,y) \end{vmatrix}}{1} = e^{x\bar{y}} - 1.$$

ii) Conditioning on two particles at the origin results in $0/0$, so we conditioning on a particle at $a \to 0$. Also we are going to use as $K(x,y)$, the new kernel above. Thus once again we have

$$\begin{aligned}\tilde{K}(x,y) &= \lim_{a\to 0} \frac{\begin{vmatrix} K(a,a) & K(a,y) \\ K(x,a) & K(x,y) \end{vmatrix}}{K(a,a)} = \lim_{a\to 0} \frac{\begin{vmatrix} e^{|a|^2} - 1 & e^{a\bar{y}} - 1 \\ e^{x\bar{a}} - 1 & e^{x\bar{y}} - 1 \end{vmatrix}}{e^{|a|^2} - 1} \\ &= \lim_{a\to 0}\left[e^{x\bar{y}} - 1 - \frac{(e^{x\bar{a}} - 1)(e^{a\bar{y}} - 1)}{e^{|a|^2} - 1}\right]\end{aligned}$$

Calculating separately

$$\lim_{a\to 0}\frac{(e^{x\bar a}-1)(e^{a\bar y}-1)}{e^{|a|^2}-1} = \lim_{a\to 0}\frac{\left(x\bar a+\frac{(x\bar a)^2}{2!}+O(\bar a^3)\right)\left(a\bar y+\frac{(a\bar y)^2}{2!}+O(a^3)\right)}{|a|^2+O(|a|^4)}$$

$$= \lim_{a\to 0}\frac{x\bar y|a|^2+\frac{x\bar y(a+\bar a)|a|^2}{2!}+O(|a|^2)}{|a|^2+O(|a|^4)} = x\bar y$$

$$\Rightarrow \tilde K(x,y) = e^{x\bar y}-1-x\bar y.$$

The pattern is obvious by now, so assuming the kernel for the complex Ginibre conditioned on having $k-1$ particles at the origin is

$$K(x,y) = e^{x\bar y}-\sum_{i=0}^{k-1}\frac{(x\bar y)^i}{i!}$$

then for k particles at $(0,0)$ we have

$$\tilde K(x,y) = K(x,y)-\lim_{a\to 0}\frac{K(x,a)K(a,y)}{K(a,a)}$$

$$\lim_{a\to 0}\frac{K(x,a)K(a,y)}{K(a,a)} = \lim_{a\to 0}\frac{\left(\frac{(x\bar a)^k}{k!}+O(\bar a^{k+1})\right)\left(\frac{(a\bar y)^k}{k!}+O(a^{k+1})\right)}{\frac{|a|^{2k}}{k!}+O(|a|^{2k+2})}$$

$$= \lim_{a\to 0}\frac{\frac{(x\bar y)^k}{(k!)^2}|a|^{2k}+O(|a|^{\frac{4k+1}{2}})}{\frac{|a|^{2k}}{k!}+O(|a|^{2k+1})} = \frac{(x\bar y)^k}{k!}$$

$$\Rightarrow \tilde K(x,y) = e^{x\bar y}-\sum_{i=0}^{k}\frac{(x\bar y)^i}{i!}.$$

6 Appendix

6.1 Proof of the Determinantal Identity

Here we prove this presumably known, but long forgotten determinantal identity.

Lemma Suppose we have an $n \times n$ matrix with elements a_{ij}. For some $1 \le k \le n-2$, let K be

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & & \cdot \\ \vdots & & \ddots & \cdot \\ a_{k1} & \cdot & \cdot & a_{kk} \end{pmatrix} = K.$$

Then the following identity is true for every $1 \le k \le n-2$

$$\left|\begin{array}{cccc|cc} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \cdot & & \\ \vdots & & \ddots & \cdot & & \vdots \\ a_{k1} & \cdot & \cdot & a_{kk} & & \\ \hline \vdots & & & & \ddots & \\ a_{n1} & & \cdots & & & a_{nn} \end{array}\right| = \frac{1}{|K|^{n-k-1}} \left| \begin{array}{ccc} \left|\begin{array}{ccc} K & & a_{1,k+1} \\ & & \cdot \\ & & \cdot \\ a_{k+1,1} & \cdots & a_{k+1,k+1} \end{array}\right| & \cdots & \left|\begin{array}{ccc} K & & a_{1,n} \\ & & \cdot \\ & & \cdot \\ a_{k+1,1} & \cdots & a_{k+1,n} \end{array}\right| \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \left|\begin{array}{ccc} K & & a_{1,k+1} \\ & & \cdot \\ & & \cdot \\ a_{n,1} & \cdots & a_{n,k+1} \end{array}\right| & \cdots & \left|\begin{array}{ccc} K & & a_{1,n} \\ & & \cdot \\ & & \cdot \\ a_{n,1} & \cdots & a_{nn} \end{array}\right| \end{array} \right|.$$

Proof:

It's going to be down in multiple steps to showcase how it works and the forms we will be using most often.

i) We start with the simplest case we showed before, to understand where the cancellations occur

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{11}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix}.$$

On the left we have 6 terms and on the right 8. Note that the only terms we don't want cancel out

$$\frac{(-a_{12}a_{21})(-a_{13}a_{31})}{a_{11}} - \frac{(-a_{21}a_{13})(-a_{12}a_{31})}{a_{11}} = 0.$$

ii) Let's begin generalizing

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \frac{1}{(a_{11})^2} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{12} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} \end{vmatrix}$$

$$
\begin{aligned}
RHS &= \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{(a_{11})^2}
\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \\[2ex]
\begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}
\end{vmatrix}
- \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}}{(a_{11})^2}
\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\[2ex]
\begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}
\end{vmatrix} \\
&+ \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}}{(a_{11})^2}
\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\[2ex]
\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix}
\end{vmatrix}.
\end{aligned}
$$

Using $i)$ we get

$$
\begin{aligned}
RHS &= \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)
\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
- \left(a_{32} - \frac{a_{12}a_{31}}{a_{11}}\right)
\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \\
&+ \left(a_{42} - \frac{a_{12}a_{41}}{a_{11}}\right)
\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix}.
\end{aligned}
$$

By adding and subtracting

$$
a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
$$

and breaking the above expression into two sums we have that RHS is equal to

$$
\begin{aligned}
&-a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
+ a_{22} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
- a_{32} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
+ a_{42} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix} \\
&+ \frac{a_{12}}{a_{11}} \left(a_{11} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
- a_{21} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
+ a_{31} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}
- a_{41} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{vmatrix} \right).
\end{aligned}
$$

The first part is the expansion of the initial 4×4 determinant with respect to the second column and the second part is

$$
\frac{a_{12}}{a_{11}} \begin{vmatrix} a_{11} & a_{11} & a_{13} & a_{14} \\ a_{21} & a_{21} & a_{23} & a_{24} \\ a_{31} & a_{31} & a_{33} & a_{34} \\ a_{41} & a_{41} & a_{43} & a_{44} \end{vmatrix} = 0.
$$

iii)

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\
a_{21} & \cdot & & & \cdot \\
\cdot & & \cdot & & \cdot \\
\cdot & & & \cdot & \cdot \\
a_{n1} & \cdot & \cdot & \cdot & a_{nn}
\end{vmatrix}
= \frac{1}{(a_{11})^{n-2}}
\begin{vmatrix}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdot\ \cdot & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\[2ex]
\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \cdot & & \cdot \\
\cdot & & \cdot & \cdot \\
\cdot & & \cdot & \cdot \\
\begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \cdot & \cdot\ \cdot & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix}
\end{vmatrix}.
$$

The proof will be by induction following the same steps as in ii)

$$RHS = \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}{(a_{11})^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ . & & . \\ . & & . \\ \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{vmatrix} + \dots +$$

$$+ \frac{(-1)^{n-1} \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix}}{(a_{11})^{n-2}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ . & & . \\ . & & . \\ \begin{vmatrix} a_{11} & a_{13} \\ a_{n-1,1} & a_{n-1,3} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n-1,1} & a_{n-1,n} \end{vmatrix} \end{vmatrix}.$$

Using the induction step for $k = n - 1$ we get

$$RHS = \left(a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right) \begin{vmatrix} a_{11} & a_{13} & . & . & a_{1n} \\ a_{31} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix} + \dots + (-1)^{n-1}\left(a_{n2} - \frac{a_{n1}a_{12}}{a_{11}}\right) \begin{vmatrix} a_{11} & a_{13} & . & . & a_{1n} \\ a_{21} & . & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n-1,1} & . & . & . & a_{n-1,n} \end{vmatrix}.$$

Again, by adding and subtracting

$$a_{12} \begin{vmatrix} a_{21} & a_{23} & . & . & a_{1n} \\ a_{31} & a_{33} & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & . & . & . & a_{nn} \end{vmatrix}$$

and splitting into two sums, the first will be the expansion of the initial $n \times n$ determinant and the second will give

$$\frac{a_{12}}{a_{11}} \begin{vmatrix} a_{11} & a_{11} & . & . & a_{1n} \\ a_{21} & a_{21} & & & . \\ . & & . & & . \\ . & & & . & . \\ a_{n1} & a_{n1} & . & . & a_{nn} \end{vmatrix} = 0.$$

iv) In the 4×4 case, the matrix can also be written as follows

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \end{vmatrix}.$$

Proof:

$$RHS = \frac{1}{(a_{11})^2}\frac{1}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}$$

$$= \frac{1}{(a_{11})^2} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} \end{vmatrix} = LHS.$$

First we used $i)$ for each of the 3×3 determinant. On the second step we used again $i)$, this time the other way around for a 3×3 determinant with elements 2×2 determinants. Finally we used $ii)$, all while having so much fun typing this.

v) The above relations can be generalized as follows:

An $n{\times}n$ determinant can be written as an $(n{-}k){\times}(n{-}k)$ determinant determinant with $(k{+}1){\times}(k{+}1)$ determinants as elements, divided by a $k{\times}k$ determinant to the power of $n{-}k{-}1$, for every $1 \leq k \leq n{-}2$.

We have already denoted K to be

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & & \cdot \\ \vdots & & \ddots & \cdot \\ a_{k1} & \cdot & \cdot & a_{kk} \end{pmatrix} = K.$$

We will know prove the more general form of the identity

$$\left|\begin{array}{cccc|cc} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \cdot & & \\ \vdots & & \ddots & \cdot & & \vdots \\ a_{k1} & \cdot & \cdot & a_{kk} & & \\ \hline \vdots & & & & \ddots & \\ a_{n1} & & \cdots & & & a_{nn} \end{array}\right| = \frac{1}{|K|^{n-k-1}} \begin{vmatrix} \begin{vmatrix} K & \begin{matrix} a_{1,k+1} \\ \cdot \\ \cdot \end{matrix} \\ a_{k+1,1} \ \cdots & a_{k+1,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} K & \begin{matrix} a_{1,n} \\ \cdot \\ \cdot \end{matrix} \\ a_{k+1,1} \ \cdots & a_{k+1,n} \end{vmatrix} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \begin{vmatrix} K & \begin{matrix} a_{1,k+1} \\ \cdot \\ \cdot \end{matrix} \\ a_{n,1} \ \cdots & a_{n,k+1} \end{vmatrix} & \cdots & \begin{vmatrix} K & \begin{matrix} a_{1,n} \\ \cdot \\ \cdot \end{matrix} \\ a_{n,1} \ \cdots & a_{nn} \end{vmatrix} \end{vmatrix}.$$

We want to prove the above relation for every n and every $1 \leq k \leq n - 2$.
We have proved this for $n = 4$.
Assuming this true for every $s < n$, we now want to prove it for $s = n$.

We know it's true for $s = n$ and $k = 1$. So it suffices to prove that if the relation hold for $s = n$ and $1, 2, .., k-1$, then it also holds for $s = n$ and k.

First we write each element of the right determinant as a 2×2 determinant.

i.e. the $(1,1)$ element

$$\left|\begin{array}{cccc|cc} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1k} & a_{1,k+1} \\ a_{21} & a_{22} & & \cdot & & \\ \vdots & & \ddots & \cdot & & \vdots \\ a_{k-1,1} & \cdot & \cdot & a_{k-1,k-1} & & \\ \hline a_{k1} & & & & a_{kk} & \\ a_{k+1,1} & & \cdots & & & a_{k+1,k+1} \end{array}\right| = \frac{1}{|L|} \left| \begin{array}{cc} \begin{vmatrix} L & a_{1k} \\ & \cdot \\ & \cdot \\ a_{k1} \ \dots & a_{kk} \end{vmatrix} & \begin{vmatrix} L & a_{1,k+1} \\ & \cdot \\ & \cdot \\ a_{k1} \ \dots & a_{k,k+1} \end{vmatrix} \\ \begin{vmatrix} L & a_{1k} \\ & \cdot \\ & \cdot \\ a_{k+1,1} \ \dots & a_{k+1,k} \end{vmatrix} & \begin{vmatrix} L & a_{1,k+1} \\ & \cdot \\ & \cdot \\ a_{k+1,1} \ \dots & a_{k+1,k+1} \end{vmatrix} \end{array} \right|$$

where L is th top left block matrix.
Also note that the top left element of each determinant will be $|K|$. Thus the initial relation takes the following form:

$$RHS = \frac{1}{|L|^{n-k}} \frac{1}{|K|^{n-k-1}} \left| \begin{array}{ccc} \begin{vmatrix} |K| & | & | \\ | \ | & | & | \end{vmatrix} & \dots & \begin{vmatrix} |K| & | & | \\ | \ | & | & | \end{vmatrix} \\ \vdots & \ddots & \vdots \\ \begin{vmatrix} |K| & | & | \\ | \ | & | & | \end{vmatrix} & \dots & \begin{vmatrix} |K| & | & | \\ | \ | & | & | \end{vmatrix} \end{array} \right|_{(n-k)\times(n-k)} .$$

We now have an $(n-k) \times (n-k)$ determinant with 2×2 determinants as elements in the correct form to use $iii)$ an write it as an $(n-k+1) \times (n-k+1)$ determinant. The exponent of $|K|$ will be $(n-k+1) - 2 = n-k-1$.

$$RHS = \frac{1}{|L|^{n-k}} \left| \begin{array}{cccc} \begin{vmatrix} L & a_{1k} \\ & \cdot \\ & \cdot \\ a_{k1} \ \dots & a_{kk} \end{vmatrix} & \begin{vmatrix} L & a_{1,k+1} \\ & \cdot \\ & \cdot \\ a_{k1} \ \dots & a_{k,k+1} \end{vmatrix} & \dots & \begin{vmatrix} L & a_{1n} \\ & \cdot \\ & \cdot \\ a_{k1} \ \dots & a_{kn} \end{vmatrix} \\ \begin{vmatrix} L & a_{1k} \\ & \cdot \\ & \cdot \\ a_{k+1,1} \ \dots & a_{k+1,k} \end{vmatrix} & \begin{vmatrix} L & a_{1,k+1} \\ & \cdot \\ & \cdot \\ a_{k+1,1} \ \dots & a_{k+1,k+1} \end{vmatrix} & & \vdots \\ \vdots & & \ddots & \\ \begin{vmatrix} L & a_{1k} \\ & \cdot \\ & \cdot \\ a_{n1} \ \dots & a_{nk} \end{vmatrix} & & \dots & \begin{vmatrix} L & a_{1n} \\ & \cdot \\ & \cdot \\ a_{n1} \ \dots & a_{n} \end{vmatrix} \end{array} \right|$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ a_{n1} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix} .$$

In the last step we used the induction step inside the induction.

$\square$

Remark: I have always started from the top left corner because of convenience, but each of the determinants can be re-written in a number of different ways.

i.e.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \frac{1}{\begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \\ \\ \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix}.$$

6.2 Proof of Lemma 5.1.1

Lemma 5.1.1: The following identity holds true:

$$P(N_{x_i}=1, J_n\backslash J_k \quad \& \quad N_{x_i}=0, J_k) = (-1)^k \left|\begin{array}{cccc|ccc} b_{11} & a_{12} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & b_{22} & & . & & & \\ \vdots & & \ddots & . & & & \vdots \\ a_{k1} & . & . & b_{kk} & & & \\ \hline a_{k+1,1} & & & & a_{k+1,k+1} & & \\ \vdots & & & & & \ddots & \\ a_{n1} & & \cdots & & & & a_{nn} \end{array}\right|.$$

Proof: By induction, presented in the appendix.

We have already proved it for $k=1,2$.
Assuming it holds for $s \leq k-1 < n$, we are going to prove it for $s=k \leq n$.

$$P(N_{x_i}=1, J_n\backslash J_k \quad \& \quad N_{x_i}=0, J_k) = P(N_{x_i}=1, J_n\backslash J_k \quad \& \quad N_{x_i}=0, J_{k-1}) - P(N_{x_i}=1, J_n\backslash J_{k-1} \quad \& \quad N_{x_i}=0, J_{k-1})$$

$$= (-1)^{k-1} \left|\begin{array}{cccc|ccc} b_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & b_{22} & & . & & & \\ \vdots & & \ddots & . & & & \vdots \\ a_{k-1,1} & . & . & b_{k-1,k-1} & & & \\ \hline a_{k+1,1} & & & & a_{k+1,k+1} & & \\ \vdots & & & & & \ddots & \\ a_{n1} & & \cdots & & & & a_{nn} \end{array}\right| - (-1)^{k-1} \left|\begin{array}{cccc|ccc} b_{11} & a_{12} & \cdots & a_{1,k} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & b_{22} & & . & & & \\ \vdots & & \ddots & . & & & \vdots \\ a_{k,1} & . & . & a_{k,k} & & & \\ \hline a_{k+1,1} & & & & a_{k+1,k+1} & & \\ \vdots & & & & & \ddots & \\ a_{n1} & & \cdots & & & & a_{nn} \end{array}\right|$$

$$= (-1)^k \left|\begin{array}{cccc|ccc} b_{11} & a_{12} & \cdots & a_{1k} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & b_{22} & & . & & & \\ \vdots & & \ddots & . & & & \vdots \\ a_{k1} & . & . & b_{kk} & & & \\ \hline a_{k+1,1} & & & & a_{k+1,k+1} & & \\ \vdots & & & & & \ddots & \\ a_{n1} & & \cdots & & & & a_{nn} \end{array}\right|.$$

□

6.3 Proof of Lemma 5.1.2

Lemma 5.1.2: $f(1,2...,2n) - f(3,4,...,2n) := \tilde{f}(2,2n)$ is the Pfaffian of the matrix

$$\begin{pmatrix} 0 & a_{12}-1 & a_{13} & \cdots & a_{1,2n} \\ & 0 & a_{23} & & \\ & & 0 & & \vdots \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$$

i.e.

$$pf\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix} - pf\begin{pmatrix} 0 & a_{34} \\ & 0 \end{pmatrix} = pf\begin{pmatrix} 0 & a_{12}-1 & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix}.$$

Proof:

$$\tilde{f}(2,2n) \quad = \quad \sum_{\mu_n} s(\mu_n) f(y_1y_2)...f(y_{2n-1}y_{2n}) - \sum_{\mu_{n-1}} s(\mu_{n-1}) f(x_1x_2)...f(x_{2n-3}x_{2n-2})$$

$$\left[\begin{array}{c} (y_1,...,y_{2n}) \text{ is a permutation of } (1,2,...,2n) \text{ and} \\ (x_1,...,x_{2n-2}) \text{ is a permutation of } (3,4,...,2n) \end{array}\right]$$

$$\begin{aligned} = & \sum_{\{y_{2i-1}y_{2i}\}\neq\{1,2\}} s(\mu_n) f(y_1y_2)...f(y_{2n-1}y_{2n}) + f(12)\sum_{\mu_{n-1}} s(\mu_{n-1}) f(x_1x_2)...f(x_{2n-3}x_{2n-2}) \\ & - \sum_{\mu_{n-1}} s(\mu_{n-1}) f(x_1x_2)...f(x_{2n-3}x_{2n-2}) \end{aligned}$$

$$= \sum_{\{y_{2i-1}y_{2i}\}\neq\{1,2\}} s(\mu_n) f(y_1y_2)...f(y_{2n-1}y_{2n}) + (f(12)-1)\sum_{\mu_{n-1}} s(\mu_{n-1}) f(x_1x_2)...f(x_{2n-3}x_{2n-2})$$

$$= \sum_{\mu_n} s(\mu_n) h(y_1y_2)...h(y_{2n-1}y_{2n})$$

where $\begin{cases} h(ij) = f(ij) \text{ for } (i,j) \neq (1,2) \text{ and} \\ h(12) = f(12) - 1. \end{cases}$ □

Remark: Similar identity holds for

$$f(1,2,...,2n) - f(1,2,...,2i,2i+3,...,2n) = pf(L)$$

where $L_{km} = \begin{cases} f(km), & (k,m) \neq (2i+1,2i+2) \\ f(2i+1,2i+2) - 1, & (k,m) = (2i+1,2i+2). \end{cases}$

6.4 Proof of Lemma 5.1.3

Lemma 5.1.3: $P(N_{x_i}=1, J_n\backslash J_k \quad \& \quad N_{x_i}=0, J_k)$ is equal to

$$(-1)^k pf\left(\begin{array}{cccc|ccc} C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_k) & & \cdots & K(x_1,x_n) \\ & C(x_2,x_2) & & & & & \\ & & \ddots & \vdots & & & \vdots \\ & & & C(x_k,x_k) & & & \\ \hline & & & & K(x_{k+1},x_{k+1}) & & \\ & & & & & \ddots & \\ & & & & & & K(x_n,x_n) \end{array}\right).$$

Proof: This proof is quite similar to the equivalent determinantal one. We will show by induction that it's true $\forall n$ and $\forall k \le n$.

- $n=2,\ k=1$ is Lemma 5.1.2

- $n=2, k=2$

$$\begin{aligned} P(N_{x_i}=0,J_2) &= 1-P(N_{x_1}=1)-P(N_{x_2}=1)+P(N_{x_i}=1,J_2) \\ &= 1-pf\begin{pmatrix} 0 & a_{12} \\ & 0 \end{pmatrix} - pf\begin{pmatrix} 0 & a_{34} \\ & 0 \end{pmatrix} + pf\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix} \\ &= -pf\begin{pmatrix} 0 & a_{12}-1 \\ & 0 \end{pmatrix} + pf\begin{pmatrix} 0 & a_{12}-1 & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix} \\ &= pf\begin{pmatrix} 0 & a_{12}-1 & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34}-1 \\ & & & 0 \end{pmatrix} \\ &= pf\begin{pmatrix} C(x_1,x_1) & K(x_1,x_2) \\ & C(x_2,x_2) \end{pmatrix}. \end{aligned}$$

Assuming it holds $\forall s<n$ and $\forall k \le s$ we will prove it for $s=n$. We know it's true for $k=1$, so it's sufficient to assume that it holds for $k-1$ and prove it for k.

$$\begin{aligned} P(N_{x_i}=1,J_n\backslash J_k \quad \& \quad N_{x_i}=0,J_k) &= P(N_{x_i}=1,J_n\backslash J_k \quad \& \quad N_{x_i}=0,J_{k-1}) \\ &\quad -P(N_{x_i}=1,J_n\backslash J_{k-1} \quad \& \quad N_{x_i}=0,J_{k-1}) \end{aligned}$$

$$
= (-1)^{k-1} pf \left(\begin{array}{cccc|cc}
C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_{k-1}) & \cdots & K(x_1,x_n) \\
 & C(x_2,x_2) & & & & \\
 & & \ddots & \vdots & & \vdots \\
 & & & C(x_{k-1},x_{k-1}) & & \\
\hline
 & & & & K(x_{k+1},x_{k+1}) & \\
 & & & & \ddots & \\
 & & & & & K(x_n,x_n)
\end{array}\right)
$$

$$
-(-1)^{k-1} pf \left(\begin{array}{cccc|cc}
C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_{k-1}) & \cdots & K(x_1,x_n) \\
 & C(x_2,x_2) & & & & \\
 & & \ddots & \vdots & & \vdots \\
 & & & C(x_{k-1},x_{k-1}) & & \\
\hline
 & & & & K(x_k,x_k) & \\
 & & & & \ddots & \\
 & & & & & K(x_n,x_n)
\end{array}\right)
$$

$$
= (-1)^{k} pf \left(\begin{array}{cccc|cc}
C(x_1,x_1) & K(x_1,x_2) & \cdots & K(x_1,x_k) & \cdots & K(x_1,x_n) \\
 & C(x_2,x_2) & & & & \\
 & & \ddots & \vdots & & \vdots \\
 & & & C(x_k,x_k) & & \\
\hline
 & & & & K(x_{k+1},x_{k+1}) & \\
 & & & & \ddots & \\
 & & & & & K(x_n,x_n)
\end{array}\right)
$$

$\square$

6.5 Proof of Lemma 5.2.7

Lemma 5.2.7: Suppose the L_2 operator K_t, where $K_t(x,y)$ is a C_1 function on t. Then

$$\frac{d}{dt}\det(I+K_t)=\det(I+K_t)Tr\Big(\dot{K}_t(1+K_t)^{-1}\Big)$$

Proof:

$$\begin{aligned}
LHS &= \lim_{\epsilon\to 0}\frac{\det(1+K_{t+\epsilon})-\det(1+K_t)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\det(1+K_t)\frac{\det\big[(1+K_t)^{-1}(1+K_{t+\epsilon})\big]-1}{\epsilon}\\
&= \det(1+K_t)\lim_{\epsilon\to 0}\frac{\det\big[(1+K_t)^{-1}+(1+K_t)^{-1}K_{t+\epsilon}+(1+K_t)^{-1}K_t-(1+K_t)^{-1}K_t\big]-1}{\epsilon}\\
&= \det(1+K_t)\lim_{\epsilon\to 0}\frac{\det\big[(1+K_t)^{-1}(1+K_t)+(1+K_t)^{-1}(K_{t+\epsilon}-K_t)\big]-1}{\epsilon}\\
&= \det(1+K_t)\lim_{\epsilon\to 0}\frac{\det\big[1+(1+K_t)^{-1}(K_{t+\epsilon}-K_t)\big]-1}{\epsilon}\\
&= \det(1+K_t)\lim_{\epsilon\to 0}\frac{1+Tr\Big((1+K_t)^{-1}(K_{t+\epsilon}-K_t)\Big)+o(\epsilon)-1}{\epsilon}\\
&= \det(1+K_t)Tr\Big(\dot{K}_t(1+K_t)^{-1}\Big).
\end{aligned}$$

$\square$

On the second to last line, we expanded the Fredholm determinant, kept the first 2 terms and the rest are of $o(\epsilon)$.

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