Optimising an output of a slip line field using a Chaotic Swamp Optimisation algorithm

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Abstract

In solving earth pressure problems, many input parameters are needed to account for more realistic soil properties and complex boundary conditions. Finding an optimal output while considering many inputs can be costly and time consuming. The input parameters themselves may be significantly uncertain due to lack of laboratory test data and variations in design and construction stages. Adopting an optimisation technique can be useful in such a problem. In this study, a chaotic particle swamp optimisation (CPSO) algorithm is applied to three different earth pressure problems, i. e. an excavation problem, a plane strain tunnel problem and a suffusion sinkhole problem, to determine their global optimal outputs and corresponding input parameters. The problems are solved using a slip line theory satisfying static equilibrium and the Mohr-Coulomb failure criterion. This improved PSO algorithm is found to show better performances, in accuracy and computational effort, compared to conventional Monte Carlo simulations (MCS).

Keywords chosen from ICE Publishing list

earth pressure, limit state analysis, uncertainty, reliability & risk

List of notations

τ	is the shear stress
c'	is the effective cohesion
σ'	is the effective mean stress
φ'	is the effective friction angle
f_0	is a parameter to be optimised
M	is a set of interval parameters z_i
$\underline{z}_k, \overline{z}_k$	are the lower value and upper value of interval parameter z^I_k
α	is an interval change ratio
n_p	is the number of particles x_i in swamp X
$x_i(t)$, $v_i(t)$	are the position and velocity of particle x_i at iteration t
H	is the height of a rigid retaining wall, the depth of suffosion sinkhole
β	is the inclination angle of a rigid retaining to the vertical axis
γ_t	is the total unit weight of a soil
q_s	is a uniform pressure over the soil surface
δ'	is an interface friction angle
L	is a plastic horizontal distance on the soil surface
T	is the deepest plastic point coinciding with the toe of a retaining wall

P_n	is the normal force on a retaining wall
$w_i()$	is the coefficient of time varying inertial weight
r_1^{CM}, r_2^{CM}	are the random numbers using chaotic Zaslavskii map
c_1, c_2	are the cognitive and social parameters
$x_i^P()$	is the local best ever position of particles
$x_i^G()$	is the global best ever position of particles
n_ξ, n_η	are the number of ξ, η curves
n_i	is the number of iterations
n_{MCS}	is the number of Monte Carlo simulations
D	is the diameter of a circular tunnel
C	is the plastic distance around a circular tunnel
r_0	is the radius of a suffosion sinkhole
p_0	is the internal pressure on the tunnel wall

1. Introduction

Earth pressure problems in geotechnical engineering can be solved using analytical methods such as limit equilibrium methods and the slip line theory. The limit equilibrium methods (Coulomb, 1776; Terzaghi, 1943; Chen, 1975) treat soil as rigid blocks and use force and moment equilibriums to solve for failure surfaces corresponding the lowest factor of safety. The slip line theory (Sokolovski, 1954; Salencon, 1974) considers plastic zones (in addition to rigid blocks) where equations of static equilibrium are satisfied continuously.

Earth pressure problems when formulated using the limit equilibrium methods and the slip line theory may contain few to many parameters. Considering more realistic soil properties and complex boundary conditions (geometry, load, displacement, etc.) generally increases the number of parameters. For example, bearing capacity problems may contain only 2 parameters (Prandtl, 1920) up to more than 10 parameters (Pakdel et al, 2019). Slope stability problems may involve only 2 parameters (e. g. infinite slope in homogenous soil) up to more than 10 parameters when considering 3-dimensional, unsaturated condition and seismic effects (Yang and Wei, 2021). In many problems formulated by the limit equilibrium methods, accounting accurately for geometries could easily require more than 10 parameters (Chen, 1975). Finding an optimal value of output in such problems becomes time consuming.

Optimization is used extensively in geotechnical engineering, especially in computational limit analysis (Sloan, 2013). Developed from the principle of maximum work and applied to a perfectly plastic materials obeying the associated flow rule (Michalowski, 2005), the method of limit analysis has been extended to elasto-plastic non-associative materials accounting for steady state flow types. Solution procedures for realistic boundary value problems (BVPs) have been automated in computer programmes. Much of this advancement has been enabled by increasingly efficient and user-friendly optimisation algorithms.

Another potential advantage of optimisation is that it can deal with geotechnical designs containing uncertain features of soil parameters and variations introduced by practical design and construction constraints. Due to the inherent heterogeneity of geomaterial and the costs associated with field investigation, it has never been the case that soil parameters are deterministic values. There is always a certain degree of subjectivity in assessing the characteristic parameters of soil in the early stage of design. Although the probabilistic-based approach is preferred to feature the uncertainty of design parameters, this approach, normally, requires to collect a great amount of information from site investigation and laboratory experimentation to estimate the random distribution of parameters, which is not feasible for all applications. In some circumstances, other mathematical models should be considered as alternative options. A distribution model where the design parameter is known to be bracketed within a range is considered here. This type of design parameter is also referred to as interval parameters.

Although there are many algorithms to tackle optimization problems, PSO proposed by (Kennedy and Eberhart, 1995) is a promising technique compared to other optimization algorithms (Elbeltagi et al., 2005) as fewer number of iteration is implemented in attain the same or better results. The proficiency of PSO has been validated and demonstrated in many engineering problems (Venter and Sobieszczanski-Sobieski, 2004; Do et al., 2020; Plevris and Papadrakakis, 2011).

PSO have been applied to earth pressure analyses, pile and foundation design, tunnelling and underground space engineering technology (Hajihassani et al., 2018). In particular, PSO and their variants have been applied successfully to limit equilibrium analyses of retaining wall (Gandomi et al., 2015), homogenous slope impacted by seismic loadings (Gordan et al., 2016) and shallow earth footing (Kashani et al., 2021). Random field theory has been applied to a stochastic slip line method to predict the quasistatic bearing capacity of strip footing (Johari et al., 2017). To the authors' knowledge, there has been no application of PSO in the slip line theory. This paper presents one way an improved PSO algorithm could be used to find a globally optimal value of a slip line field. Applications of the improved PSO algorithm are demonstrated in three examples to show the simplicity of the method and its better performance compared to conventional Monte Carlo simulations (MCS). Utilising this improved PSO algorithm could save significant time when solving earth pressure problems with many parameters.

2. Statement of the boundary value problem

A body of cohesive frictional soil is at static equilibrium, and due to boundary actions, parts of the soil are yielding in accordance with the Mohr-Coulomb failure criterion $\tau = c' + \sigma' \tan(\varphi')$, where τ is the shear stress, c' is the effective cohesion, σ' is the effective mean stress and φ' is the effective friction angle (in other parts, the soil remains elastic/undisturbed). The governing equations for such a soil body were derived and applied to many stability problems in soil mechanics previously (Sokolovski, 1954), with solutions typically found using the finite difference method. Depending on what are known and unknown at the boundaries, at least four distinct BVPs have been identified (Booker, 1970; Sokolovski, 1954; Salencon, 1974), and the method of solution for each is well established.

In solving practical problems, however, these BVPs are combined so that quantities can be integrated (along slip line directions) from known to unknown boundaries (where desired quantities can be extracted). This is a trial-and-error procedure which could be costly time-wise. When a desired quantity lying on an unknown boundary is a global minimum or maximum, PSO could be used to estimate the corresponding combination of parameters. As mentioned previously in the introduction, utilisation of an optimization algorithm could be reasonable also because soil and design and construction-related parameters are generally not deterministic. In view of optimisation problem, the procedure in present study can be defined as:

$$Minimise\ \ f_0\ \ with\ \ M = \left[z_i^I, i \in [1,n]\right]$$

$$Subject\ to \begin{cases} \underline{z}_1 \le z_1^I \le \overline{z}_1 \\ \vdots \\ \underline{z}_k \le z_k^I \le \overline{z}_k \\ \vdots \\ \underline{z}_n \le z_n^I \le \overline{z}_n \end{cases}$$

1.

where f_0 is the parameter under consideration in current study and M is a set of interval parameters z_i whose values are uncertain-but-bounded, rather than deterministic. $\underline{z}_k$ and $\bar{z}_k$ are, respectively, the lower and upper values of interval parameter z_k^I. The uncertainty of all parameters is described by the interval change ratio α. Denote mean value of parameter z_k^I as z_m, then $\underline{z}_k$ and $\bar{z}_k$ are the defined as

$$\underline{z}_k = z_m - z_m\alpha$$

2.

$$\overline{z}_k = z_m + z_m\alpha$$

3.

In order to take advantages of PSO as well as improve the performance of global multidimensional optimisation for current study, PSO adopted in this research is enhanced by a merger between the backbone of RLHNPSO (Do et al., 2014) and integration of chaotic Zaslavsky map (Zaslavsky, 1978) into global search. This approach can be denoted as Quasi chaotic based particle swarm optimisation (CPSO).

Consider a swarm X consisting n_p particles x_i, each particle x_i is given numeric values for input problem parameters (for example, effective cohesion c' and effective friction angle φ') to be within the possible range of known variation. The present algorithm retains the nature-inspired features of the tradition PSO in which x_i flies over time in a multidimensional design space to search for its optimal position by updating its position $x_i(t)$ and velocity $v_i(t)$ simultaneously based on the predefined fitness function. Noted, t is signified as the iteration for interest. All related parameters adopted for the current study can be found in the background of RLHNPSO (Do et al., 2014). In present algorithm, the feature of chaotic map is integrated into the equation of velocity to improve the stochastic search technique for an improved PSO. The advantages of chaos embedded PSO algorithms were first highlighted in Alatas et al. (2009). This chaotic feature would dramatically accelerate the process of updating particles via velocity in searching for their optimum positions to reach the global optimisation. This is demonstrated in a comparison between the current PSO technique and the traditional PSO method in section 3. The chaotic map-based velocity $v_i(t)$ is, then, formulated as:

$$v_i(t+1) = w_i(t+1)v_i(t) + c_1 r_1^{CM}\left(x_i^P(t) - x_i(t)\right) + c_2 r_2^{CM}\left(x_i^G(t) - x_i(t)\right)$$

4.

where r_1^{CM} and r_2^{CM} are evaluated by chaotic Zaslavskii map while w_i is coefficient of time varying inertial weight.

The position of particle x_i is, then, updated over time as:

$$x_i(t+1) = x_i(t) + v_i(t+1)$$

5.

The updates on $x_i(t)$ and $v_i(t)$ aim to satisfy the fitness function whereby the parameter f_0 is minimised. This parameter will be detailed in the section of numerical examples.

3. Numerical examples in earth pressure problems

3.1 The retaining force needed to support an excavation

A cohesive frictional soil is retained by a rigid wall of height H inclined at an angle β to the vertical axis. The soil has a total unit weight of γ_t. A uniform pressure q_s acts over the soil surface. At limiting equilibrium, an interface friction angle $0° \leq \delta' \leq \varphi'$ is mobilised between the soil and the wall, the plastic zone extends a horizontal distance L over the ground surface, and the deepest plastic point (T) coincides with the toe of the wall (Figure 1(a)). It is required to calculate a precise combination of parameters (c', φ', H, β, γ_t, q_s, δ', L), whose values vary within known ranges, that minimises the retaining force P_n. In this demonstrative example, the mean values of all components of particles are given as (c'_m, φ'_m, β_m, γ_{t_m}, q_{s_m}, δ'_m, L_m) = (10 kPa, 35°, 5°, 19.62 kN/m^3, 10 kPa, 5°, 2 m). The subscript m denotes design parameter's mean values. While a degree of uncertainty in the material parameters (c', φ', γ_t, δ') is inevitable (e. g. due to limited experimental data available, non-uniform soil fabric, inadequate idealisation introduced by the Mohr-Coulomb failure model), the source of uncertainty in the geometric defining variables (H, β) and ground surcharge loading (q_s, L) is more related to practical design and construction constraints (e. g. availability of suitable construction equipments/materials, variations introduced/specified by engineers/designers, spatiotemporal restrictions by authorities). Nevertheless, the interval change ratio α can be modified to account for different levels of uncertainty in applying PSO to real-world applications (i. e. α→0 as uncertainty is reduced to certainty). Different α can be adopted for different parameters. For illustrational purpose, α=0.2 is adopted in this example.

The PSO algorithm was applied to solve the above problem by adopting the following parameters: P_{n_s}=0 kN/m run, c_1=c_2=2, coefficient of w is adopted from RLHNPSO, r_1^{CM} and r_2^{CM} are assigned via the chaotic mapping procedure discussed in section 2. For this problem, a mesh density with n_ξ=n_η=20 where n_ξ, n_η denote the number of ξ, η curves used in each slip line BVP, respectively, as shown in Figure 1(b) is used because it gives sufficiently accurate P_{n_c}. It is noted that the letter *s*, *c* in a subscript (e. g. P_{n_s}, P_{n_c}) indicates a specified and a computed value, respectively.

The results of the PSO algorithm are compared with the results of MCS. A parametric study is conducted using the number of particles (n_p) = 5, 10, 20, the number of iterations (n_i)=20, 50, 100, the number of MCS (n_{MCS})=10^3, 10^4, 10^5, 10^6. The results are listed in Table 1, which shows that $|P_{n\_c}$-$P_{n\_s}|$= 32.1769 kN/m run (hence, the minimised P_n is 32.1769 kN/m run) for all PSO cases but the case (n_p, n_i) = (5, 20). This is smaller than the minimised P_{n_c} = 33.149 kN/m run obtained even for n_{MCS}=10^6. The time required to execute the PSO algorithm is much shorter than the corresponding MCS algorithm ran in the same computing environment (Intel® Core ™ i7-6700 CPU @ 3.40GHz, RAM of 32.0GB on 64-bit Operating System & x64-based processor).

To examine how the iteration number n_i and the number of used particles n_p affect the convergence of the output, a parametric study on these parameters was conducted with the numerical example in subsection 3.1. The results are shown in Figure 1(c) for n_p =10 particles with the different numbers of iterations, and in Figure 1(d) for n_i =20 iterations with different numbers of used particles.

The increase in value of n_i dramatically enhances the chance of targeting the optimal value as shown in Figure 1(c) (an adequate number n_i is needed to avoid the premature convergence of PSO). Similar observation can be made for n_p in Figure 1(d) which shows that a sufficient number of n_p is required to obtain the convergence. The adopted values of n_p and n_i, in other words, directly affect the optimization of the output.

3.2 The critical depth at the heading of a plane strain tunnel

A purely frictional soil (c'=0 kPa, φ'>0°) at the heading of a circular tunnel of diameter D is prevented from collapse by applying an internal pressure p_0 to the wall. The soil has a total unit weight of γ_t. At limiting equilibrium, the soil around the tunnel becomes plastic while the soil further away is elastic. It is required to calculate a precise combination of parameters (φ', p_0, γ_t, D) that minimises the plastic distance C (Figure 2(a)). (φ'_m, p_{0_m}, γ_{t_m}, $D_{\_m}$) = (25°, 200 kPa, 19.62 kN/m^3, 2.5 m) is used to demonstrate this problem. The source of uncertainty over the tunnel diameter D and the internal pressure on the tunnel wall p_0 are contributed by practical constraints. Different uncertainty levels can be applied in PSO by adjusting α. In this example, the interval change ratio α=0.1 is adopted for simplicity (the effect of a variation in α is considered in subsection 3.3).

The PSO algorithm was applied to solve the above problem by adopting C_s = 0 m while α, c_1, c_2, w, r_1^{CM} and r_2^{CM} in section 3.1 are also adopted in this section. A mesh density with n_ξ=n_η=20 (Figure 2(b)) is used because it gives sufficiently accurate C_c.

Comparison between PSO and MCS results is shown in Table 2. It shows that $|C_c$-$C_s|$= 4.4022 m (hence, the minimised C_c is 4.4022 m) for all PSO cases. This is smaller than the minimised C_c = 4.4237 m obtained by MCS, even for n_{MCS}=10^6.

In each iteration of PSO, particles x_i is updated by assigning any value within the given ranges (boundaries and local values) to search for its optimal positions. For the current problems, the optimal value of the output is found based on the combination of end points of interval variables, not based on the collection of the local values within the ranges of interval variables. This can result in PSO giving premature convergence with a small number of iterations (Figures 2(c).i-v) where design parameters reach local values of interval variables before reaching their global values to attain the global optimisation. Figures 2(c).i-v show how $|C_c\text{-}C_s|$ and design parameters φ', γ_t, p_0, D vary over iterations to target the optimal value.

As mentioned in XXX, the global values of the output for current study are achieved by the combination of end points of interval variables whose values vary within closed sets. While the traditional Monte Carlo Simulation is often adopted in the literature for verifying the uncertainty analysis involving interval variables (termed as the interval analysis), another approach that is also sufficient for the current analysis is the combinatorial approach by which all possible scenarios of combining the endpoints of interval parameters are assessed. However, this approach is only available for the small number of interval variables due to the rapid escalation in the number of possible solutions, 2^k where k is the number of interval variables as identified in (Do et al., 2020). To remove this limitation for the interval analysis, the intelligent optimisation technique, CPSO is, therefore, adopted for the current analysis and is recommended for the analysis of uncertain-but-bounded problems.

3.3 Minimum depth of a suffosion sinkhole

A suffosion sinkhole is an axisymmetric subsidence formed in non-cohesive soil, often initiated by the withdrawal of the ground water table (Arakaki Rengifo et al., 2021; Ford and Williams, 2007; White, 1988). Here, a suffosion sinkhole is idealised to have radius r_0 and reaches a depth H at limiting equilibrium (Figure 3(a)). The soil surrounding the sinkhole has a total unit weight of γ_t. A tension crack layer imposes a surcharge $q_s=2c'\cot\varphi'/[\sin\varphi'/(1-\sin\varphi')]$ on the ground surface. A setback horizontal distance L can be estimated by observing the ground surface for signs of deformation. It is required to calculate a precise combination of parameters (c', φ', r_0, γ_t, q_s) that minimises H. A reasonable combination of parameters set (c'_m, φ'_m, r_{0_m}, L_m, γ_{t_m}) =

(10 kPa, 15°, 20 m, 1 m, 19.62 kN/m^3) is adopted to demonstrate how the PSO algorithm can be applied. The source of uncertainty over the sinkhole radius r_0 and the setback horizontal distance L is contributed by observational constraints. Different uncertainty levels can be applied in PSO by adjusting α.

The PSO algorithm was applied with the following parameters: $H_s = 0$ m. The parameters c_1, c_2, w, r_1, r_2 in this section are assigned as in section 3.1. α = 0.05, 0.1 are considered in this section in addition to only α = 0.2 in section 3.1. A mesh density with $n_\xi=n_\eta=50$ (Figure 3(b)) is used to approximate H_c with sufficient accuracy.

Comparison between PSO results and MCS results is shown in Table 3(a), 3(b), 3(c) for α =0.05, 0.1, 0.2, respectively. It shows that $|H_c-H_s|$= 1.3888, 1.3557, 1.2774 m (hence, the minimised H_c is 1.3888, 1.3557, 1.2774 m) in Table 3(a), 3(b), 3(c), respectively. They are smaller than the corresponding minimised H_c obtained by MCS, even for $n_{MCS}=10^6$. The time required to execute the PSO algorithm is again much shorter than the corresponding MCS algorithm.

To demonstrate advantages of improved PSO (CPSO), a comparison between the current PSO technique and the traditional PSO is made via a parametric study with different particles and depicted in Figure 3(c). It is shown that the performance of CPSO surpasses the performance of PSO, especially when a small number of particles is used in an analysis. In other words, the tradition PSO can give the premature convergence for the output with small number of particles while CPSO can reach the global values under the same condition. This advantage would benefit the optimisation procedure for large scale problems in which the small number of particles will result in less computational effort to achieve the global optimisation. Another advantage of CPSO is that it can accelerate the convergence compared to the traditional PSO which may require more iterations for the same number of particles. This acceleration is due to the improvement of the stochastic search technique using chaotic map mentioned in section 2. Considerable advantages of chaos embedded particle swarm optimisation can also be found in Alatas et al. (2009).

4. Conclusions

An optimisation technique has been shown to effectively aid preliminary design calculations in geotechnical engineering. In particular, an improved PSO algorithm was applied to three earth pressure problems to optimise specific outputs while accounting for uncertain inputs i. e. soil properties (e. g. due to lack of data from laboratory experiments and in situ tests), geometry and boundary conditions (e. g. due to practical design and construction constraints). The earth pressure problems were simplified to demonstrate the optimisation procedure more clearly. Each problem contains multiple input parameters to show that determining the combination of input parameters corresponding to an optimal output is not trivial using conventional methods e. g. MCS. In the three problems considered, advantages of using the improved PSO method far outweighed using MCS, in that the PSO method requires a smaller number of iterations and achieves a better accuracy. The present study shows that it is uncomplicated to incorporate a PSO algorithm into an established solution method such as the slip line theory, thus global optimisation can be considered as a promising approach for handling more intricate hybrid uncertainty in geotechnical designs.

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Figure captions

Figure 1(a). The retaining force needed to support an excavation.

Figure 1(b). Slip line mesh adopted for the retaining wall problem.

Figure 1(c). Convergence history over n_i.

Figure 1(d). Convergence history over n_p.

Figure 2(a). Critical depth of a plane strain tunnel.

Figure 2(b). Slip line mesh adopted for the plane strain tunnel problem.

Figure 2(c). Convergence history of $|C_c\text{-}C_s|$ in terms of design variables over iterations.

Figure 3(a). Minimum depth of a suffosion sinkhole.

Figure 3(b). Slip line mesh adopted for the suffosion sinkhole problem.

Figure 3(c). Convergence history of $|H_c\text{-}H_s|$ over different values of n_p.

Tables

Table 1. Results of PSO compared to MCS for a retaining wall problem (wall height *H* is not a design parameter; it is a calculated value from the optimisation procedure and is included here for completeness)

Table 2. Results of PSO compared to MCS for the plane strain tunnel problem

Table 3(a). Results of PSO compared to MCS for the suffosion sinkhole problem (α =0.05)

Table 3(b). Results of PSO compared to MCS for the suffosion sinkhole problem (α =0.1)

Table 3(c). Results of PSO compared to MCS for the suffosion sinkhole problem (α =0.2)