Quasi-Local Mass in 3D Riemannian Manifolds with a Scalar Curvature Lower Bound

by

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Contents

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Thank you Mum, for making me who I am.

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Declarations

Section 1 uses various known results to help explain and motivate the thesis, all of which are cited in the text. The main sources are [Bar89; HI01; Bra01; Mon13].

Section 2.1 introduces the notation, which is inspired by a combination of [Lee18; PX09; Mon13; Mag12]. Section 2.2 contains some basic results about conformal metrics, which can be found, for example, in [Lee18]. Section 2.3 uses some well-known formulas for the variation of some geometric quantities in Lemma 2.12, which can be found, for example, in [HP96; CM11; Bra97]. The proof of Lemma 2.14 is taken from [LMS11].

Sections 3 and 4 both use methods from [Mon13; Mon10; PX09].

Section 5 uses some well-known classical results of Killing-Hopf and Schur, and corollaries thereof. Also, some classical results of Wolf are used in the proof of Theorem 1.33 [Wol11, Theorems 3.3.3 and 3.5.1].

Section 6 uses some well-known work of Nash [Nas56], De Giorgi [De 61], Simon [Sim83, Theorem 3.18] and Tamanini [Tam82, Theorem 1], as well as some modern results, including [Nar14, Theorem 1], [Nar09, Theorem 1], [MN19, Theorem 2], [MS17, (6-9)] and [Cho+21, Theorem C.2].

Section 7 uses another classical result of Wolf [Wol11, Theorem 2.7.1], and the well-known Bishop-Gromov volume comparison theorem [BC64; Gro99].

The main results of this thesis have been submitted as a paper to a journal. Currently, it exists in preprint form [MT21].

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work, except where otherwise indicated or cited in the text. This thesis has not been submitted for a degree at any other university.

Abstract

This thesis proves rigidity theorems for three-dimensional Riemannian manifolds with scalar curvature bounded below by $6K$, where $K \in \{-1,0,1\}$, by placing restrictions on the Hawking mass of surfaces. The primary tool will be the geometry of *perturbed geodesic spheres*.

The majority of the work focusses on the case $K=0$, for which we prove both a local and global rigidity result. The first states that if every point in an open subset Ω has a neighbourhood $U \subset \Omega$ such that the supremum of the Hawking mass of surfaces contained in U is non-positive, then Ω is locally isometric to Euclidean $\mathbb{R}^3$. Taking Ω to be the ambient manifold and further assuming it is *asymptotically locally simply connected* (which encompasses as a special case, the standard *asymptotically flat* property), we will prove that it must be globally isometric to Euclidean $\mathbb{R}^3$. The method involves computing the Taylor expansion of the Hawking mass of *optimal* perturbed geodesic spheres, which will be positive when the space is non-flat. This will allow us to prove a positive lower bound (comprised of curvature tensors) on the Bartnik mass of (non-flat) open sets, once we prove that perturbed geodesic spheres are *outer-minimising*.

The proof of the outer-minimising property requires the framework of *sets of finite perimeter*. Specifically, for each element of a sequence of shrinking perturbed geodesic spheres, we consider the corresponding set with least perimeter that contains it. We prove convergence and regularity properties for this new sequence and determine that, eventually, the boundaries of its elements are the original spheres.

Later, we will extend some of our results to the case $K \neq 0$, where the model space is the complete, simply connected Riemannian manifold of constant sectional curvature K. In order to extend the global rigidity theorem to the case $K=-1$, we consider an alternative asymptotic condition, namely the *global asymptotic volume property*, which compares the volume of large balls to those in the model space.

1 Introduction

The relationship between mass and the geometry of a manifold has been studied extensively in recent times. Two important results, the Riemannian Positive Mass Theorem and the Riemannian Penrose Inequality, both give conclusions about the total mass of a three-dimensional Riemannian manifold, given some geometric assumptions. The former, proved first by Schoen and Yau [SY79] using minimal surfaces methods, and then by Witten [Wit81] using spinorial techniques, asserts that if (M, g) is an *asymptotically flat* (Definition 1.2) Riemannian manifold with non-negative scalar curvature, then the total (ADM) mass [ADM62] is non-negative, with the zero case only achieved by flat $\mathbb{R}^3$. The latter, proved independently by Huisken and Ilmanen [HI01], and Bray [Bra01], gives a positive lower bound on the total mass in terms of the area of the horizons of black holes contained in a Riemannian manifold satisfying the same geometric conditions. These Riemannian manifolds fit into the framework of General Relativity as space-like hypersurfaces with zero second fundamental form, in a four dimensional Lorentzian space-time that satisfies Einstein's field equation. Non-negative scalar curvature corresponds to assuming the *dominant energy condition* and a zero *cosmological constant*. In fact the first theorem generalises to include hypersurfaces with non-zero second fundamental form and was proved by Schoen and Yau [SY81]. The same generalisation of the second theorem is the full Penrose Conjecture, and is still open.

This thesis is primarily concerned with the interaction between the shape of an ambient manifold, and two types of *quasi-local mass* inside bounded regions; those of Hawking [Haw68] and Bartnik [Bar89]. For the Hawking mass we will prove a rigidity-type result, contained across Theorems 1.28 and 1.33, which could be described as the local and global versions respectively. Informally, it says; if there is no mass locally anywhere in a given 3D Riemannian manifold with non-negative scalar curvature, then the only possibilities for the manifold are flat, i.e. its curvature tensor is zero everywhere. In the global version, adding the further assumption of asymptotically flat, or more generally *asymptotically locally simply connected* (Definition 1.29), reduces the options to just one; flat $\mathbb{R}^3$. We will then use the Hawking mass result to prove the Bartnik mass theorems (Theorems 1.34 and 1.35), the first of which is also of rigidity-type and is in fact already contained in [HI01]; we provide a different proof although we will use some ideas therein. In the second we give a lower bound for the Bartnik mass in terms of the Hawking mass of *perturbed geodesic spheres* (Definition 1.25).

Defining a local mass in General Relativity is more difficult than in the Newtonian setting where we can integrate the mass density function over the required region. This is because, due to the Equivalence Principle, there is no well defined notion of gravitational energy density [MTW73, Section 20.4] [Pen82]. Thus, if we want to compute the total mass/energy inside a finite region contributed by both matter and gravity, a new method is required. The numerous versions of quasi-local mass that have been defined over the last few decades (see [Lee19, Chapter 6] or [Sza09]), are attempts to solve this problem. There are various natural properties one would like quasi-local mass to satisfy, such as positivity and monotonicity. The two versions appearing in this thesis have some, but not all of them, which we will detail below.

There is a third potential contribution towards the quasi-local mass if one considers a space-time with non-zero cosmological constant Λ. This means that there is a non-zero *vacuum energy* everywhere in the manifold, which could then affect the amount of mass/energy measured in a finite region. For the majority of this thesis we will set $\Lambda = 0$, but in Section 7 we will briefly discuss some extensions of our results to the $\Lambda \neq 0$ case.

Remark 1.1. Throughout this work, unless otherwise specified, a manifold M is allowed to have non-empty, disconnected boundary ∂M.

We use the following notion of asymptotically flat.

Definition 1.2. *An n-dimensional Riemannian manifold (M,g) is called asymptotically flat (AF) if there is a compact $\mathcal{K} \subset M$ and a diffeomorphism $\phi : M \setminus \mathcal{K} \to \mathbb{R}^n \setminus \overline{B_1^{\bar{g}}(0)}$ which induces coordinates at infinity such that*

$$g_{\mu\nu} = \delta_{\mu\nu} + \sigma_{\mu\nu} \quad \textit{where} \quad |x|^{|\alpha|}|(\partial^\alpha \sigma_{\mu\nu})(x)| = \mathcal{O}(|x|^{-\tau}) \quad \textit{as } |x| \to \infty$$

for some $\tau > \frac{n-2}{2}$ and all multi-indices α, with $|\alpha| = 0,1,2,3$. Also, we require the scalar curvature Sc *to be integrable over M.*

Remark 1.3. Here, and throughout, we only consider one-ended AF manifolds.

Remark 1.4. Requiring that Sc be integrable is not redundant. Take $n = 3$ and the decay on the metric to be $\tau = \frac{3}{5}$. Since the scalar curvature depends on two derivatives of the metric, this allows for $\mathrm{Sc} = |x|^{-\frac{13}{5}}$ in the AF chart, which is not integrable over $\mathbb{R}^3 \setminus \overline{B_1^{\bar{g}}(0)}$.

Definition 1.5. *The total mass of an AF, n-dimensional Riemannian manifold (M,g) is*

$$m_{ADM}(M) := \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r^{n-1}} (\partial_\nu g_{\mu\mu} - \partial_\mu g_{\mu\nu})\hat{N}^\nu \, dV_{g_{S_r^{n-1}}}$$

where ω_{n-1} is the volume of the unit sphere S^{n-1} and $\hat{N}$ and $dV_{S_r^{n-1}}$ are the unit normal and volume form for the sphere of radius r respectively.

Remark 1.6. In [Bar86], Bartnik shows that this definition is independent of the chart ϕ, and finite.

Remark 1.7. Recall that the spatial Schwartzschild manifold of mass m is the prototypical example of an asymptotically flat manifold and in this case $m_{ADM} = m$ (see Section 2.4).

1.1 Hawking Mass

Definition 1.8. *Let (M,g) be a 3D Riemannian manifold and $\Sigma \subset (M,g)$ an isometrically immersed sphere. The Hawking mass of Σ is defined to be*

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|_{g_\Sigma}}{(16\pi)^3}}(16\pi - W(\Sigma)) \tag{1.1}$$

where $W(\Sigma) := \int_\Sigma H^2 dV_{g_\Sigma}$ is the Willmore functional.

Remark 1.9. Note that in some texts, including [Wil97], the definition of the mean curvature H differs from ours by a factor of $\frac{1}{2}$ and the 16π becomes 4π in the definition of m_H.

Evidently, if Σ is a minimal surface ($H \equiv 0$), then its Hawking mass is positive.

Perhaps the simplest example to compute is the standard sphere of radius r in $\mathbb{R}^3$. Using the standard parametrisation

$$\int_{\mathbb{S}^2_r} H^2 dV_{g_{\mathbb{S}^2_r}} = \int_0^{2\pi}\int_0^{\pi} \frac{4}{r^2} r^2 \sin\phi \, d\phi \, d\theta = 16\pi$$

which gives $m_H(\mathbb{S}^2_r) = 0$. However, one can show that in $\mathbb{R}^3$ the round sphere is the unique minimiser of the Willmore functional (ignoring scaling) [Wil97]. Thus, the Hawking mass can be negative and is therefore not a perfect solution to the general problem of quantifying the mass in a finite region. However, it does have some useful properties when certain conditions are put on the ambient manifold, as its use in the proof of the Riemannian Penrose Inequality shows.

Theorem 1.10 (Riemannian Penrose Inequality)**.** *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Then*

$$m_{ADM}(M) \geq \sqrt{\frac{|\partial M|}{16\pi}}$$

with equality if and only if (M,g) is isometric to the spatial Schwarzschild manifold of mass $m_{ADM}(M)$.

Remark 1.11. Conjectured by Penrose [Pen73], the first proof of Theorem 1.10 was by Huisken and Ilmanen in [HI01], where the area term on the right hand side of the inequality above is in fact only the area of (any) one of the components of ∂M. Later, in [Bra01], Bray's proof includes the area of all the components of ∂M and allows it to be *outer-minimising*. A compact surface Σ is outer-minimising if the area of any other compact surface enclosing it is at least that of Σ. This is true for an *outermost* compact minimal surface, i.e. one not enclosed by any other, like the boundary in Theorem 1.10 [Lee19, Theorem 4.7]. Furthermore, an outermost compact minimal surface is a (union of) sphere(s). See [Bra01, Section 8] or [HI01, Lemma 4.1].

Since ∂M is a minimal surface, the inequality in Theorem 1.10 can be written as $m_{ADM}(M) \geq m_H(\partial M)$. This highlights the fact that the Hawking mass only depends on the geometry of the surface $\Sigma \subset M$, and not on the region it encloses.

The Hawking mass was instrumental in Huisken and Ilmanen's proof of Theorem 1.10. They used its monotonicity along certain families of surfaces inside (M,g). More precisely, if Σ_t is the family generated by the weak inverse mean curvature flow, starting from a boundary component, then $m_H(\Sigma_{t_1}) \leq m_H(\Sigma_{t_2})$ for $t_1 < t_2$ [HI01; Ger73]. Furthermore, as noted by the authors in [HI01], this monotonicity is valid for any initial surface that is outer-minimising, which means their proof of the Riemannian Penrose Inequality also proves a similar inequality for any outer-minimising sphere.

Lemma 1.12. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. If $\Sigma \subset M$ is an outer-minimising sphere, then*

$$m_{ADM}(M) \geq m_H(\Sigma)$$

Remark 1.13. If Σ happens to be minimal then the conditions in the lemma force it to be part of ∂M and so the Riemannian Penrose Inequality applies directly.

The Hawking mass also satisfies an exhaustion property for AF (M,g), whereby $\lim_{r\to\infty} m_H(\Sigma_r) = m_{ADM}(M)$ for a family Σ_r of so-called *nearly round* spheres [SWW09]. This includes, for example, constant mean curvature spheres, foliations of which were proved to exist by Ye [Ye96] and Huisken and Yau [HY96]. If Σ is a *stable* constant mean curvature sphere, then Christodoulou and Yau [CY88] proved that $m_H(\Sigma) \geq 0$ (also assuming non-negative scalar curvature). See also the more recent work [MWX20] by Miao, Wang and Xie, where the authors consider an alternative to the stability condition.

Next, we state modified versions of the first two main theorems proved in this thesis. See Sections 1.3, 2.1 and 1.4 for the relevant definitions and more accurate statements.

Theorem (Local rigidity Theorem 1.28)**.** *Let (M,g) be a 3D Riemannian manifold and let $\Omega \subset M$ be an open subset with non-negative scalar curvature. If every $p \in \Omega \setminus \partial M$ admits a neighbourhood $U \subset M \setminus \partial M$ such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0$$

then $\Omega \setminus \partial M$ is locally isometric to $(\mathbb{R}^3, \bar{g})$.

Theorem (Global rigidity Theorem 1.33)**.** *Let (M,g) be a connected, complete, 3D Riemannian manifold without boundary and with non-negative scalar curvature. If every $p \in M$ admits a neighbourhood U such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0$$

then (M^3, g) is isometric to a space-form of zero sectional curvature. Furthermore, if (M^3, g) is AF, then it is isometric to $(\mathbb{R}^3, \bar{g})$.

1.2 Bartnik Mass

The Bartnik mass is a localised version of the total mass of an AF manifold.

Definition 1.14. *Let (M, g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. For a bounded, open set $\Omega \subset M$ with smooth topological boundary $\partial\Omega$, the Bartnik mass of Ω is*

$$\tilde{m}_B(\Omega) := \inf\{m_{ADM}(N) : N \in \mathcal{A}\}$$

where $\mathcal{A}$ is the set of AF, complete, 3D Riemannian manifolds with non-negative scalar curvature, into which Ω isometrically embeds, where ∂N is the only compact, minimal surface in N.

Remark 1.15. The purpose of ruling out compact minimal surfaces in the extensions is to ensure that Ω is not enclosed by one. If this were allowed then so called "thin neck" extensions could be found with arbitrariy small ADM mass (see [Bar89; Bra97]) and thus $\tilde{m}_B$ would always be zero.

Remark 1.16. We could define the Bartnik mass by allowing Ω to be an arbitrary compact, 3D Riemannian manifold with non-negative scalar curvature, without reference to an ambient space. But then it would not always be clear whether $\tilde{m}_B(\Omega)$ is finite (i.e. that Ω has at least one extension). Instead, we have assumed there exists at least one extension of Ω, namely (M, g), its ambient space.

In his original definition Bartnik [Bar89] only allowed manifolds without boundary and excluded all compact minimal surfaces, whereas this version was defined by Huisken and Ilmanen [HI01] and includes manifolds with boundary as long as the boundary is minimal. The latter authors conjectured that the two versions are in fact equal. Since their version allows more extensions (the AF manifolds in the definition), it is clear that it cannot be bigger than the Bartnik version. But, as the Penrose inequality indicates, the presence of a compact minimal boundary (which models the existence of a black hole), will only increase the total mass. Therefore it seems that these extra extensions allowed by Huisken and Ilmanen may not affect the infimum over all extensions and thus their version would not be smaller than the Bartnik version either.

This definition is relatively complicated (compared to the Hawking mass), although it does benefit from being non-negative (thanks to the Riemannian Positive Mass theorem) and monotonic in the sense that $\Omega_1 \subset \Omega_2$ implies $\tilde{m}_B(\Omega_1) \leq \tilde{m}_B(\Omega_2)$. Also, the mass of an expanding sequence of spheres inside an AF ambient space, approaches its total mass [HI01, Exhaustion Property 9.2]. The requirements that the extension manifolds N be AF with non-negative scalar curvature are necessary for m_{ADM} to make sense and be non-negative. However, other aspects of the definition can be changed. For example, the embedding condition may be relaxed so that we include extensions with a boundary to which we identify $\partial\Omega$ in such a way to make the resulting metric only Lipschitz across the join. As mentioned above, we could also change the minimal surface condition in various ways (as we will do below), in part so that the definition only depends on the boundary geometry, rather than the whole of Ω (as noted earlier, the Hawking mass already satisifes this) and in part to make it less restrictive. Some changes have side-effects though, including loss of monotonicity. For a detailed discussion of these possibilities, see [Jau19; McC20; Bar02; And19; Sza09].

A big drawback of the Bartnik mass is its computational difficulty. However, by constructing *collar extensions*, Schoen and Mantoulidis [MS15] managed to prove that $\tilde{m}_B(\Omega) = m_H(\partial\Omega)$ when the first eigenvalue of the operator $-\Delta + K$ on $\partial\Omega$ is positive, where K is the Gauss curvature (in particular, when $\partial\Omega$ is a stable, minimal sphere). Using the same methods, the authors in [Ced+17, Corollary 1.1] prove an upper bound on the Bartnik mass in terms of the Hawking mass when $\partial\Omega$ has positive constant mean curvature, positive Gauss curvature and satisfies a certain integral condition. See also [MX19] for more work on upper bounds for the Bartnik mass. While it is clear from the definition that one should expect upper bounds by providing a specific extension, the issue of finding explicit lower bounds is more subtle. The latter is one of the goals of the present work.

In this thesis we will use the Hawking mass as a tool to prove the results about the Bartnik mass. In particular, we will be interested in the Hawking mass of certain outer-minimising surfaces, so it will be useful to employ a modified version of the Bartnik mass that applies the outer-minimising condition. To this end, we first consider the variant due to Bray [Bra01].

Definition 1.17. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature. For a bounded, open set $\Omega \subset M$ with smooth, outer-minimising topological boundary $\partial\Omega$, the Bartnik mass of Ω is*

$$\mathring{m}_B(\Omega) := \inf\{m_{ADM}(N) : N \in \mathcal{A}\}$$

where $\mathcal{A}$ is the set of AF, complete, 3D Riemannian manifolds with non-negative scalar curvature, into which Ω isometrically embeds, where $\partial\Omega \subset N$ is outer-minimising, i.e. $|\partial\Omega| \leq |\Sigma|$ for any surface Σ enclosing $\partial\Omega$.

This version satisfies its own monotonicity property [BC04].

Lemma 1.18. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature. If $\Omega_2 \subset \Omega_1$ are bounded, open sets and both $\partial\Omega_2$ and $\partial\Omega_1$ are outer-minimising, then*

$$\mathring{m}_B(\Omega_2) \leq \mathring{m}_B(\Omega_1)$$

In order to make use of both Lemma 1.12 and Lemma 1.18 in the proof of our Bartnik mass theorems (via Proposition 6.2), we need to combine the two previous versions into one.

Definition 1.19. *Let (M, g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. For a bounded, open set $\Omega \subset M$ with smooth, outer-minimising topological boundary $\partial\Omega$, the Bartnik mass of Ω is*

$$m_B(\Omega) := \inf\{m_{ADM}(N) : N \in \mathcal{A}\}$$

where $\mathcal{A}$ is the set of AF, complete, 3D Riemannian manifolds with non-negative scalar curvature, into which Ω isometrically embeds, where ∂N is the only compact, minimal surface in N and $\partial\Omega \subset N$ is outer-minimising.

Thanks to Lemma 1.12, and since every extension induces the same mean curvature on $\partial\Omega$, we have the following result.

Lemma 1.20. *Let (M, g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. For a bounded, open set $\Omega \subset M$ with smooth, outer-minimising topological boundary $\partial\Omega \cong \mathbb{S}^2$*

$$m_B(\Omega) \geq m_H(\partial\Omega)$$

We now state the third and fourth main theorems proved in this thesis. Again, see Sections 1.3, 2.1 and 1.4 for the relevant definitions.

Theorem (Local rigidity Theorem 1.34)**.** *Let (M, g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Let $\Omega \subset M$ be a bounded, open set with smooth, outer-minimising topological boundary $\partial\Omega$. If $m_B(\Omega) = 0$ then $\Omega \setminus \partial M$ is locally isometric to $(\mathbb{R}^3, \bar{g})$.*

As mentioned previously, this theorem was proven by Huisken and Ilmanen for the $\tilde{m}_B$ version of the Bartnik mass (and, therefore, for the m_B version too, since $\tilde{m}_B \leq m_B$) using their work on the weak inverse mean curvature flow [HI01, Positivity Property 9.1]. We prove the result for m_B, using different methods.

Theorem (Lower bound Theorem 1.35)**.** *Let (M, g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Let $\Omega \subset M$ be a bounded, open set with smooth, outer-minimising topological boundary $\partial\Omega$. Let $p \in \Omega \setminus \partial M$. For small enough ρ, the following lower bound holds:*

$$m_B(\Omega) \geq \frac{1}{12}\mathrm{Sc}_p\rho^3 + \left(\frac{1}{120}\Delta\mathrm{Sc}(p) + \frac{1}{90}\|S_p\|^2 - \frac{1}{144}\mathrm{Sc}_p^2\right)\rho^5 + \mathcal{O}(\rho^6)$$

where S is the traceless Ricci tensor.

Remark 1.21. When Ω is non-flat, there is a point p for which the lower bound above is positive.

Remark 1.22. Note that this theorem is only interesting when Ω does not contain some connected component of ∂M. Indeed, if Ω contains some connected component Σ of ∂M, then in every extension N the Riemannian Penrose Inequality [Bra01; HI01] yields $m_{ADM}(N) \geq \sqrt{\frac{|\Sigma|}{16\pi}}$ and thus $m_B(\Omega) \geq \sqrt{\frac{|\Sigma|}{16\pi}}$, which is a *definite* lower bound on $m_B(\Omega)$. Therefore, the *infinitesimal* lower bound above is more useful when $\partial M \cap \Omega = \emptyset$.

Remark 1.23. Let us also mention the recent work [Wiy18] by Wiygul, where the first order Taylor expansion of the Bartnik mass is computed for closed geodesic balls of small radius $\rho > 0$ and center $p \in M$, giving $\frac{1}{12}\mathrm{Sc}_p\rho^3$. Under the additional condition that the Riemann curvature tensor vanishes at p, the first order Taylor expansion of the Bartnik mass for such geodesic balls is given by $\frac{1}{120}\Delta\mathrm{Sc}(p)\rho^5$. Note that these results are in accordance with the lower bound in the theorem above, which holds without the assumption that the Riemann curvature tensor vanishes at p. See Section 6.4.

1.3 Perturbed Geodesic Spheres - Notation and Motivation

1.3.1 Notation

A more detailed notation section will be given in Section 2.1 but, for continuity, we will introduce the perturbed geodesic spheres now. Here, (M,g) is an arbitrary 3D Riemannian manifold and $\exp_p$ is the exponential map at $p \in M$. We write $\mathcal{R}$, Ric, Sc and $S := \mathrm{Ric} - \frac{1}{3}\mathrm{Sc} \cdot g$ for the Riemann curvature endomorphism, Ricci curvature tensor, scalar curvature and traceless Ricci tensor respectively. Let $\Theta \in \mathbb{S}^2 \subset T_pM$.

Definition 1.24. *The geodesic sphere centered at $p \in M \setminus \partial M$ of radius $\rho > 0$ is the surface*

$$S_{p,\rho} := \exp_p(\rho\Theta)$$

for ρ small enough to ensure $\exp_p$ is a diffeomorphism.

To prove the main theorems, we will compute the mass inside *perturbed* geodesic spheres, i.e. normal graphs over geodesic spheres.

Definition 1.25. *A perturbed geodesic sphere centered at $p \in M \setminus \partial M$ of radius $\rho > 0$ is a surface*

$$S_{p,\rho}(w) := \exp_p(\rho(1-w)\Theta)$$

for ρ small enough to ensure $\exp_p$ is a diffeomorphism, and

$$w \in C^{k,\alpha}(S^2)$$

for some $k \geq 2$ and $0 < \alpha < 1$.

Later we will compute expansions of various geometric quantities for perturbed spheres where the following notation (taken from [PX09; Mon13]) will be useful. For $a \in \mathbb{N}$, we denote by $\mathcal{L}_p^{(a)}(w)$ an arbitrary linear combination of the function w together with its partial derivatives, up to order a. The coefficients of $\mathcal{L}_p^{(a)}$ may depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C = C_p > 0$ independent on $\rho \in (0,1)$ such that

$$\|\mathcal{L}_p^{(a)}(w)\|_{C^{k,\alpha}(\mathbb{S}^2)} \leq C\|w\|_{C^{k+a,\alpha}(\mathbb{S}^2)} \tag{1.2}$$

Similarly, given $a, b \in \mathbb{N}$, we denote with $\mathcal{Q}_p^{(b)(a)}(w)$ an arbitrary nonlinear combination, of order at least b, of the function w together with its partial derivatives, up to order a, such that $\mathcal{Q}_p^{(b)(a)}(0) = 0$ for every $p \in M$. The coefficients of the Taylor expansion of $\mathcal{Q}_p^{(b)(a)}(w)$ in powers of w and its partial derivatives may depend on ρ and p but, for all $k \in \mathbb{N}$, there exists a constant $C = C_p > 0$ independent on $\rho \in (0,1)$ such that

$$\|\mathcal{Q}_p^{(b)(a)}(w) - \mathcal{Q}_p^{(b)(a)}(\bar{w})\|_{C^{k,\alpha}(\mathbb{S}^2)} \leq C\big(\|w\|_{C^{k+a,\alpha}(\mathbb{S}^2)} + \|\bar{w}\|_{C^{k+a,\alpha}(\mathbb{S}^2)}\big)^{b-1} \times \|w - \bar{w}\|_{C^{k+a,\alpha}(\mathbb{S}^2)}, \tag{1.3}$$

provided $\|w\|_{C^a(\mathbb{S}^2)}, \|\bar{w}\|_{C^a(\mathbb{S}^2)} \leq 1$. We write $\mathcal{O}(\rho^d)$ to denote an arbitrary smooth function on $\mathbb{S}^2$ (or an open set in $\mathbb{R}^3$, depending on the context), dependent on p, whose norm is bounded by a constant (independent of p) times ρ^d. The norm will depend on context and will be either the absolute value $|.|$, the Holder norm $\|.\|_{C^{k,\alpha}}$ or the C^k norm $\|.\|_{C^k}$ for all $k \in \mathbb{N}$.

We note the formulas for the volume of a perturbed geodesic sphere and the perturbed geodesic ball $B_{p,\rho}(w)$ enclosed by it, proved in [PX09, Appendix], which will be useful later:

$$\begin{aligned}
|S_{p,\rho}(w)|_{\mathring{g}} = |\mathbb{S}^2|_{g_{\mathbb{S}^2}} &\left[1 - \frac{1}{18}\mathrm{Sc}_p\rho^2 + \frac{1}{5400}\left(5\mathrm{Sc}_p^2 + 8\|\mathrm{Ric}\|^2 - 3\|\mathcal{R}\|^2 - 18\Delta_g\mathrm{Sc}\right)\rho^4\right]\rho^2 \\
&+ \left(\int_{\mathbb{S}^2} w^2 dV_{g_{\mathbb{S}^2}} + \frac{1}{2}\int_{\mathbb{S}^2} |\nabla w|^2 dV_{g_{\mathbb{S}^2}} - 2\int_{\mathbb{S}^2} w dV_{g_{\mathbb{S}^2}}\right)\rho^2 + \frac{2}{3}\int_{\mathbb{S}^2} \mathrm{Ric}(\Theta,\Theta) w dV_{g_{\mathbb{S}^2}}\rho^4 \\
&+ \int_{\mathbb{S}^2} \mathcal{O}(\rho^7) + \rho^5\mathcal{L}_p^{(2)}(w) + \rho^4\mathcal{Q}_p^{(2)(2)}(w) + \rho^2\mathcal{Q}_p^{(3)(2)}(w) dV_{g_{\mathbb{S}^2}}
\end{aligned} \tag{1.4}$$

$$\begin{aligned}
|B_{p,\rho}(w)|_g = \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3} &\left[1 - \frac{1}{30}\mathrm{Sc}_p\rho^2 + \frac{1}{12600}\left(5\mathrm{Sc}_p^2 + 8\|\mathrm{Ric}\|^2 - 3\|\mathcal{R}\|^2 - 18\Delta_g\mathrm{Sc}\right)\rho^4\right]\rho^3 \\
&+ \left(\int_{\mathbb{S}^2} w^2 dV_{g_{\mathbb{S}^2}} - \int_{\mathbb{S}^2} w dV_{g_{\mathbb{S}^2}}\right)\rho^3 + \frac{1}{6}\int_{\mathbb{S}^2} \mathrm{Ric}(\Theta,\Theta) w dV_{g_{\mathbb{S}^2}}\rho^5 \\
&+ \int_{\mathbb{S}^2} \mathcal{O}(\rho^8) + \rho^6\mathcal{L}_p^{(2)}(w) + \rho^5\mathcal{Q}_p^{(2)(2)}(w) + \rho^3\mathcal{Q}_p^{(3)(2)}(w) dV_{g_{\mathbb{S}^2}}
\end{aligned} \tag{1.5}$$

1.3.2 Motivation

In Theorems 1.28 and 1.33, we will consider the supremum of the Hawking mass for surfaces contained in a given neighbourhood. In particular, its proof will focus on certain perturbed geodesic spheres $S_{p,\rho}(w)$ (Definition 1.25). The next proposition indicates that they are a good choice of surfaces to analyse. Indeed, we show that they are the natural competitors for the supremum of the Hawking mass among area constrained surfaces contained in a small ball. This principle has already been shown in related results by Lamm and Metzger [LM13], who proved $W^{2,2}$-closeness to a geodesic sphere under a small energy assumption, and Laurain and Mondino [LM14], who proved smooth convergence to a geodesic sphere under a milder energy assumption.

Proposition 1.26. *Let (M,g) be a 3D Riemannian manifold and let $\Sigma_j \subset M$ be a sequence of maximisers of m_H under area constraint and Hausdorff converging to a point $\bar{p} \in M$. Then $\nabla \mathrm{Sc}(\bar{p}) = 0$ and eventually, up to a subsequence, Σ_j are perturbed geodesic spheres $S_{p_j,\rho_j}(w_j)$.*

Proof. First of all, recall that non-orientable closed two-dimensional surfaces cannot be embedded in $\mathbb{R}^3$, but only immersed (i.e. with self-intersections). By considering their *n-conformal area*, Li and Yau [LY82] showed that for such a surface Σ in any $\mathbb{R}^n$

$$\int_\Sigma |H|^2 dV_{g_\Sigma} \geq 32\pi$$

In particular, see [LY82, Theorem 6] but note the different convention used in the definition of the mean curvature. For us, this means

$$\inf\{W(\Sigma) : \Sigma \subset \mathbb{R}^3 \text{ closed, non-orientable surface}\} \geq 32\pi > 16\pi \tag{1.6}$$

Next, from the proof of the Willmore conjecture by Marques and Neves [MN14] we know that

$$\inf\{W(\Sigma) : \Sigma \subset \mathbb{R}^3 \text{ closed surface with genus}(\Sigma) \geq 1\} \geq 8\pi^2 > 16\pi \tag{1.7}$$

Therefore, any surface in $\mathbb{R}^3$ described by (1.6) or (1.7) would have a negative Hawking mass (technically the m_H from Definition 1.8 is only for spheres, but this is just the motivation). Using normal coordinates, ϕ, centred at $\bar{p}$, for large j the surfaces $\Sigma_j \subset M$ are isometric to $\phi(\Sigma_j) \subset (\mathbb{R}^3, \bar{g} + h)$ with the induced metric. Here $h = \mathcal{O}(|x - \bar{p}|^2)$ is the perturbation arising from the Taylor expansion of the metric g in normal coordinates (see Lemma 3.1). In [MS14] the authors estimate the difference between the geometric quantities when computed using the different metrics $\bar{g}$ and $\bar{g} + h$. In particular they find that (see [MS14, Lemma 2.4])

$$W_{\bar{g}}(\Sigma) \le \frac{3}{2} W_{\bar{g}+h}(\Sigma) + 1 \tag{1.8}$$

Combining (1.6) and (1.7) with (1.8) yields

$$\inf\{W(\Sigma) : \Sigma \subset M \text{ closed non-orientable surface}\} \ge \frac{2(32\pi - 1)}{3} > 16\pi \tag{1.9}$$

and

$$\inf\{W(\Sigma) : \Sigma \subset M \text{ closed surface with genus}(\Sigma) \ge 1\} \ge \frac{2(8\pi^2 - 1)}{3} > 16\pi \tag{1.10}$$

This means any of the surfaces $\Sigma_j \subset (M, g)$ fitting the descriptions in (1.9) or (1.10) would have a negative Hawking mass too (for large j). In other words, for any sequence $\Sigma_j \subset (M, g)$ Hausdorff converging to a point $\bar{p}$, with Σ_j either non-orientable or of genus at least one, there is a constant $C > 0$ such that

$$\limsup_{j\to\infty} \frac{m_H(\Sigma_j)}{\sqrt{|\Sigma_j|}} \le -C \tag{1.11}$$

On the other hand, considering a sequence of geodesic spheres $S_{p,\rho}$ yields

$$\lim_{\rho\to 0} \frac{m_H(S_{p,\rho})}{\sqrt{|S_{p,\rho}|}} = 0 \tag{1.12}$$

Thus, the area constrained maximising condition could not be satisfied by such $\Sigma_j \subset M$. This is because the geodesic sphere of area $|\Sigma_j|$ is a competitor for the surface Σ_j, and (1.11) and (1.12) imply that eventually the Hawking mass of this sphere will be greater than the corresponding Σ_j. Hence the Σ_j must eventually be topological spheres. Similarly, since the Σ_j are maximisers of m_H, (1.12) implies

$$\liminf_{j\to\infty} \frac{m_H(\Sigma_j)}{\sqrt{|\Sigma_j|}} \ge 0$$

Or, equivalently

$$\limsup_{j\to\infty} W(\Sigma_j) \le 16\pi < 32\pi$$

Therefore we can apply [LM14, Theorem 1.2] (see also [LM10]) to infer that $\nabla \mathrm{Sc}(\bar{p}) = 0$ and that, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled surfaces $\tilde{\Sigma}_j$ have fixed area 1, $\tilde{\Sigma}_j$ converge smoothly (up to a subsequence) to a round sphere in 3D Euclidean space (note, again, the different convention for the mean curvature which means the bound in the cited paper is actually 8π). This means that the $\tilde{\Sigma}_j$ must eventually be graphs over the sphere with the graph functions smoothly converging to zero, i.e. Σ_j are eventually perturbed spheres.

$\square$

With the motivation for considering perturbed geodesic spheres now clear, we introduce the following function space:

$$C^{4,\alpha}(S^2)^{\perp} := C^{4,\alpha}(\mathbb{S}^2) \cap \mathrm{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)]^{\perp} \subset L^2(\mathbb{S}^2)$$

Later, in Lemma 4.2, we will restrict the perturbation w to this function space, which is natural for a number of reasons. First, since the Euler-Lagrange equation for the Willmore functional ((1.17) with $\lambda = 0$) is a fourth-order PDE, it makes sense to require at least four derivatives for w. Second, the Hölder condition will allow us to apply an argument from [Mon10, Lemma 4.4] which relies on using Schauder estimates. Finally, restricting the Willmore equation to perturbed spheres yields a PDE where the main operator is $\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)$ (see Lemma 4.2). In order to prove the uniqueness of a solution via the Contraction Mapping Theorem, as done in [Mon10], we further restrict to $C^{4,\alpha}(S^2)^{\perp}$ so that we can invert the operator $\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)$.

We note that the constraints on these critical points of the Willmore functional are more than just area, as applied in Proposition 1.26. Solving the area constrained Euler-Lagrange equation within $C^{4,\alpha}(S^2)^{\perp}$ corresponds not to general area constrained critical points, but to *area-position* constrained critical points. This is because $\mathrm{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)]$ contains both constant functions (corresponding to scaling) and the coordinate functions (corresponding to translations).

The analysis of constrained critical points in $C^{4,\alpha}(S^2)^{\perp}$ will be used purely as motivation for using the expansion (4.24), proved in Lemma 4.2, which we will then apply at any $p \in M$, regardless of whether $S_{p,\rho}(w)$ actually describe constrained critical points of the Willmore functional/Hawking mass. They will be called *optimal* perturbed spheres and will be the key geometric objects in this thesis.

Remark 1.27. In fact, using [Mon10, Lemma 5.3], the perturbations w_j in Proposition 1.26 can be seen as elements of $C^{4,\alpha}(S^2)^{\perp}$; although the statement of [Mon10, Lemma 5.3] is for critical points of W, the same proof holds generally for area constrained critical points using that the Lagrange multipliers are bounded, thanks to [LM14, Lemma 2.2].

1.4 Main Theorems

With the notation introduced in the previous section, we can now give precise statements of the main results in this thesis.

Theorem 1.28. *Let (M,g) be a 3D Riemannian manifold and let $\Omega \subset M$ be an open subset with non-negative scalar curvature. If every $p \in \Omega \setminus \partial M$ admits a neighbourhood $U \subset M \setminus \partial M$ such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0 \tag{1.13}$$

or, more generally, if

$$\limsup_{\rho \downarrow 0} \rho^{-5} m_H(S_{p,\rho}(w)) \leq 0, \quad \forall p \in \Omega \setminus \partial M \tag{1.14}$$

where $S_{p,\rho}(w)$ is the optimally perturbed geodesic sphere with w as in (4.24), then $\Omega \setminus \partial M$ is locally isometric to $(\mathbb{R}^3, \bar{g})$.

Definition 1.29. *A Riemannian manifold (M, g) is Asymptotically Locally Simply Connected (ALSC) if it is non-compact and for all $R > 0$, and any diverging sequence $\{p_n\} \subset M$, there exists $N(R) \geq 1$ such that for all $n > N(R)$ the balls $B_R^g(p_n)$ are simply connected.*

Remark 1.30. $\{p_n\}$ is a diverging sequence when, for any fixed $\bar{p} \in M$, we have $\mathrm{d}(\bar{p}, p_n) \to \infty$ *as* $n \to \infty$, where d is the Riemannian distance function induced by a Riemannian metric g.

Remark 1.31. Note that our ALSC condition is satisfied by AF manifolds since, in this case, the balls $B_R^g(p_n)$ are eventually diffeomorphic to 3D Euclidean balls, which are simply connected. Not all ALSC manifolds are AF though; for example *asymptotically conical* and C^0*-asymptotically locally Euclidean* manifolds [MN16]. For some specific examples, see Section 2.4. Also note that a simply connected manifold need not be ALSC. An illuminating example is a half infinite cylinder with a spherical cap on one end (or instead Hamilton's Cigar soliton; see Section 2.4). Any ball with radius bigger than the radius of the cylinder will fail to be simply connected for n large enough.

Definition 1.32. *A Riemannian manifold (M, g) is called a space-form if it is complete, connected and has constant sectional curvature.*

Theorem 1.33. *Let (M, g) be a connected, complete, 3D Riemannian manifold without boundary and with non-negative scalar curvature. If every $p \in M$ admits a neighbourhood U such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0 \tag{1.15}$$

or, more generally, if

$$\limsup_{\rho \downarrow 0} \rho^{-5} m_H(S_{p,\rho}(w)) \leq 0, \quad \forall p \in M \tag{1.16}$$

where $S_{p,\rho}(w)$ is the optimally perturbed geodesic sphere with w as in (4.24), then (M^3, g) is isometric to a space-form of zero sectional curvature. Furthermore, if (M^3, g) is ALSC, then it is isometric to $(\mathbb{R}^3, \bar{g})$.

The main step in proving the above theorems will be computing the Taylor expansion (4.40) for the Hawking mass of the optimally perturbed spheres. Combining this with Proposition 6.2, which confirms that perturbed geodesic spheres have the outer-minimising property, we will prove the following two theorems.

Theorem 1.34. *Let (M, g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Let $\Omega \subset M$ be a bounded, open set with smooth, outer-minimising topological boundary $\partial\Omega$. If $m_B(\Omega) = 0$ then $\Omega \setminus \partial M$ is locally isometric to $(\mathbb{R}^3, \bar{g})$.*

Theorem 1.35. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Let $\Omega \subset M$ be a bounded, open set with smooth, outer-minimising topological boundary $\partial\Omega$. Let $p \in \Omega \setminus \partial M$. For small enough ρ, the following lower bound holds:*

$$m_B(\Omega) \geq \frac{1}{12}\mathrm{Sc}_p\rho^3 + \left(\frac{1}{120}\Delta\mathrm{Sc}(p) + \frac{1}{90}\|S_p\|^2 - \frac{1}{144}\mathrm{Sc}_p^2\right)\rho^5 + \mathcal{O}(\rho^6)$$

1.5 Summary of Proofs of Main Theorems

As the two theorems concerning the Bartnik mass depend on the work done in proving the Hawking mass theorems, we will prove the latter first.

The main idea in the proof of Theorems 1.28 and 1.33 is similar to the one in [Mon13, Section 3], and also builds on [Mon10; PX09]. We will utilise Taylor expansions of the geometric quantities involved in the Hawking mass, which will inevitably contain curvature terms. Combined with the assumptions in the theorems, we will get information about the curvature of the manifold.

For simplicity we will first compute the Taylor expansion of the Willmore energy in the *unperturbed* case, i.e. $W(S_{p,\rho}) = W(S_{p,\rho}(0))$, using normal coordinate expansions of the geometric quantities we will find in Lemma 3.1 in Section 3.1. In Section 3.2 we will compute the corresponding Hakwing mass to be

$$\begin{aligned} m_H(S_{p,\rho}) &= \sqrt{\frac{|S_{p,\rho}|_{\mathring{g}}}{(16\pi)^3}}\left(\frac{8\pi}{3}\mathrm{Sc}_p\rho^2 - \left[\frac{4\pi}{27}\mathrm{Sc}_p^2 - \frac{4\pi}{15}\Delta\mathrm{Sc}(p)\right]\rho^4 + \mathcal{O}(\rho^5)\right) \\ &= \frac{1}{12}\mathrm{Sc}_p\rho^3 - \left(\frac{1}{144}\mathrm{Sc}_p^2 - \frac{1}{120}\Delta\mathrm{Sc}(p)\right)\rho^5 + \mathcal{O}(\rho^6) \end{aligned}$$

where we have simplified using (1.4) with $w = 0$. Analysing this equation shows an explicit connection between the scalar curvature of M at p and the Hawking mass $m_H(S_{p,\rho})$. Specifically, for small ρ, if the scalar curvature is positive at p then so is $m_H(S_{p,\rho})$. Alternatively, $m_H(S_{p,\rho}) = 0$ implies the scalar curvature is zero at p too. If it's true for every $p \in \Omega$ then we have the scalar curvature is identically zero in Ω. However, no extra interesting information is yielded once we set $\mathrm{Sc} \equiv 0$.

Combined with the motivation in Section 1.3.2, this suggests exploiting the Taylor expansion in the perturbed case. Firstly, we will consider the expansion for spheres $S_{p,\rho}(w)$ which are critical points of m_H (equivalently, critical points of W) under area constraint and $w \in C^{4,\alpha}(S^2)^\perp$. Note that the area constraint is necessary due to the area factor in the definition of the Hawking mass. Thus, in Lemma 4.1 we compute the necessary expansions of geometric quantities in a similar way to Lemma 3.1, which can then be substituted into the area constrained Euler-Lagrange PDE for the Willmore functional. This PDE can be found by combining (2.17), which will be zero for a critical point, with (2.16), which must be zero to satisfy the area constraint, as in [LMS11]. This yields

$$2\Delta_{\mathring{g}} H + H(H^2 - 4D + 2\mathrm{Ric}(\hat{N},\hat{N})) = \lambda H \tag{1.17}$$

where $\lambda \in \mathbb{R}$. Restricting (1.17) to perturbed spheres in $C^{4,\alpha}(S^2)^{\perp}$, which we do in Lemma 4.2, allows an application of the Contraction Mapping Theorem, yielding a unique solution satisfying the following Taylor expansion:

$$w = \Big(-\frac{1}{6}\mathrm{Ric}(\Theta,\Theta) + \frac{1}{18}\mathrm{Sc}_p\Big)\rho^2 + \mathcal{O}(\rho^3) \tag{1.18}$$

where $\limsup_{\rho\to 0}\rho^{-3}\left\|\mathcal{O}(\rho^3)\right\|_{C^{4,\alpha}(S^2)} < \infty$.

Using this discussion of constrained critical points as motivation, for any $p \in M$ we simply refer to perturbed spheres $S_{p,\rho}(w)$ whose perturbation satisfies (1.18) as *optimal*. These optimal spheres are the central geometric objects in our main theorems. As noted in Section 1.3.2, we stress that the optimal spheres are in general *not* area constrained critical points of the Willmore functional, but rather area-position constrained critical points. However, this is not relevant to our proofs of the main theorems, where we simply use these optimal spheres as the "test" surfaces for testing the positivity of the Hawking mass.

Thus, we will compute the Hawking mass of optimally perturbed spheres in Section 4.3 to be

$$\begin{aligned} m_H(S_{p,\rho}(w)) &= \sqrt{\frac{|S_{p,\rho}(w)|_{\mathring{g}}}{(16\pi)^3}}\left(\frac{8\pi}{3}\mathrm{Sc}_p\rho^2 + \left[\frac{4\pi}{15}\Delta\mathrm{Sc}(p) + \frac{16\pi}{45}\|S_p\|^2 - \frac{4\pi}{27}\mathrm{Sc}_p^2\right]\rho^4 + \mathcal{O}(\rho^5)\right) \\ &= \frac{1}{12}\mathrm{Sc}_p\rho^3 + \left(\frac{1}{120}\Delta\mathrm{Sc}(p) + \frac{1}{90}\|S_p\|^2 - \frac{1}{144}\mathrm{Sc}_p^2\right)\rho^5 + \mathcal{O}(\rho^6) \end{aligned}$$

where we have simplified using (1.4). Note the difference with the unperturbed case is the presence of the traceless Ricci tensor S. The scalar curvature and supremum assumptions in the theorem will then imply that $\mathrm{Sc} \equiv 0$ and so the leading term becomes $C\|S_p\|^2$, where $C > 0$. Since this is non-negative and the expansion itself is assumed non-positive, we get $S \equiv 0$ too. To conclude the proof of Theorem 1.28 we will apply a well-known result of Schur (Corollary 5.4), which implies that the sectional curvature is also zero. To finish proving Theorem 1.33 we will also use the well-known result of Killing-Hopf (Corollary 5.2), which forces the ambient manifold to be a space-form. A case by case analysis will show that the only ALSC space-form is Euclidean $\mathbb{R}^3$ (Theorems 3.3.3 and 3.5.1 in [Wol11] will facilitate this).

We will now outline the proofs of Theorems 1.34 and 1.35. Recall that $B_{p,\rho}(w)$ is the bounded region enclosed by $S_{p,\rho}(w)$. The initial idea is that for any bounded, open subset U of M we can find a small perturbed sphere $S_{p,\rho}(w)$ contained in U. Thus, by the monotonicity of the Bartnik mass (Lemma 1.18), it makes sense to bound the Bartnik mass of U from below by the Bartnik mass of $B_{p,\rho}(w)$. Then, if the

Bartnik mass of U is zero, as assumed in Theorem 1.34, the Bartnik mass of $B_{p,\rho}(w)$ is also zero. Finally, we can bound the Bartnik mass of $B_{p,\rho}(w)$ from below by the Hawking mass of $S_{p,\rho}(w)$ (Lemma 1.20), which means we can appeal to Theorem 1.28 in order to conclude Theorem 1.34. Similarly for Theorem 1.35, we bound the Bartnik mass of U by the Hawking mass of $S_{p,\rho}(w)$, then use the Taylor expansion of $m_H(S_{p,\rho}(w))$ computed above to achieve an asymptotic bound for $m_B(U)$.

A key requirement for the above argument to work is the outer-minimising property. Indeed, to apply Lemmas 1.18 and 1.20 requires that both $S_{p,\rho}(w)$ and ∂U are outer-minimising. Therefore the bulk of the proof of Theorems 1.34 and 1.35 deals with proving the next proposition (Proposition 6.2).

Proposition (Proposition 6.2)**.** *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Fix $p \in M \setminus \partial M$ and consider an arbitrary sequence of perturbed geodesic spheres $S_{p,\rho_n}(w_n)$ satisfying $\rho_n \to 0$ and $\|w_n\|_{C^1(S^2)} \to 0$ as $n \to \infty$. Then there exist $N(p) > 0$ such that $S_{p,\rho_n}(w_n)$ is outer-minimising for every $n \geq N$.*

We will now summarize the four main steps of its proof, given in Section 6.3. The proof uses the framework of finite perimeter sets. Specifically, for any $\rho, \|w\|_{C^1}$, we will consider the finite perimeter set $\Omega_{p,\rho,w}$ which minimises perimeter amoung sets containing $B_{p,\rho}(w)$. In other words, $\Omega_{p,\rho,w}$ is the *minimising hull* of $B_{p,\rho}(w)$ [HI01]. We will show that $\partial^*\Omega_{p,\rho,w} = \partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$ when $\rho, \|w\|_{C^1}$ are small enough, and thus $S_{p,\rho}(w)$ is outer-minimising.

Step 1 - Volume control.

In Section 6.1 we will prove some helpful lemmas concerning the geometry of the manifold (M,g) from the proposition. The most important result uses the AF chart to prove that (M,g) has bounded geometry, i.e. bounded sectional curvature and injectivity radius. Then, in Section 6.2 we will use some results about the isoperimetric profile in manifolds with bounded geometry (namely [MN19, Theorem 2] and [Nar14, Theorem 1]) to obtain an isoperimetric inequality for AF manifolds, contained in the lemma below.

Lemma (Lemma 6.10)**.** *If (M,g) is an AF, complete, 3D Riemannian manifold, then for every $v_0 > 0$ there exists $C = C(v_0) > 0$, such that*

$$\mathrm{P}_g(E) \geq C|E|_g^{\frac{2}{3}} \quad \textit{for every subset } E \subset M \textit{ of finite perimeter, with } |E|_g \in (0, v_0].$$

This inequality will be important in proving the following result, which gives us some control over the volume of the sets $\Omega_{p,\rho,w}$ as $\rho \to 0$.

Lemma (Lemma 6.19)**.** *Let (M,g) and $S_{p,\rho_n}(w_n)$ be as in Proposition 6.2. For the corresponding sequence of finite perimeter sets Ω_{p,ρ_n,w_n}, there exists a constant $\hat{C}$ such that*

$$0 < \hat{C}^{-1} \leq \liminf_{n\to\infty} \frac{|\Omega_{p,\rho_n,w_n}|_g}{\rho_n^3} \leq \limsup_{n\to\infty} \frac{|\Omega_{p,\rho_n,w_n}|_g}{\rho_n^3} \leq \hat{C} < \infty$$

Notice that control of the volume with repesct to the g metric proved in the previous lemma translates to a unifom bound on $|\Omega_{p,\rho_n,w_n}|_{g_{\rho_n}}$, where $g_{\rho_n} = \rho_n^{-2} g$ are new, scaled metrics. This will be useful in the next step of the proof.

Step 2 - Blow-up and local convergence to a Euclidean ball.

In Section 6.2 we will prove various results relating the perimeters of sets with respect to the metrics $\bar{g}$, g and g_ρ. In particular, using the formula for the metric in normal coordinates (see (3.1) and (6.3)), we will show the following relationship.

Lemma (Lemma 6.14)**.** *Let $F \subset (M, g_\rho)$ be a set of finite perimeter and $\phi^p_{g_\rho}$ be the normal coordinate chart centered at p. Then*

$$\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(F), B^{\bar{g}}_r(\phi^p_{g_\rho}(q))) = (1 + \mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(F, B^{g_\rho}_r(q)) + \mathcal{O}(\rho^2 r^4)$$

This lemma will also be useful in step 3 when we consider the regularity of the sequence $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$. But we will use it here to help prove the local convergence too. Note that the relationship shows that

$$\limsup_{n\to\infty} \mathrm{P}_{\bar{g}}(\phi^p_{g_{\rho_n}}(F_n)) = \limsup_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(F_n)$$

as long as $\mathrm{P}_{g_{\rho_n}}(F_n)$ is uniformly bounded. We will apply this to the sequence $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ to help prove

$$\mathrm{P}_{\bar{g}}(\bar{\Omega}) \leq \mathrm{P}_{\bar{g}}(B^{\bar{g}}_1(0)) \qquad \text{and} \qquad B^{\bar{g}}_1(0) \subset \bar{\Omega}$$

where $\bar{\Omega}$ is the limit of the sequence (obtained via a diagonal argument). Together with the uniform bound on $|\Omega_{p,\rho_n,w_n}|_{g_{\rho_n}}$ from step 1, this will allow us to apply the Euclidean isoperimetric inequality to get $|\bar{\Omega} \Delta B^{\bar{g}}_1(0)|_{\bar{g}} = 0$. Hence we will get the desired convergence

$$\phi^p_{g_\rho}(\Omega_{p,\rho,w}) \xrightarrow{L^1_{loc}} B^{\bar{g}}_1(0)$$

Step 3 - Improving the convergence via regularity theory.

In this step we will use a definition and regularity result from Tamanini [Tam82], which built on the celebrated work of De Giorgi [De 61].

Definition. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and $V \subset \mathbb{R}^n$ open and bounded. Then*

$$\Psi(E,V) := \mathrm{P}_{\bar{g}}(E,V) - \inf\{\mathrm{P}_{\bar{g}}(F,V) | F\Delta E \subset\subset V\}$$

Theorem (Theorem 6.24)**.** *Let U be an open subset of $\mathbb{R}^n$, and E a set of finite perimeter satisfying*

$$\Psi(E,B_r(q)) \leq Cr^{n-1+2\alpha} \tag{1.19}$$

for some $\alpha \in (0,1)$ and for all $q \in U$ and $r \in (0,R)$, where C and R are positive constants. Then the reduced boundary $\partial^* E$ is a $C^{1,\alpha}$-hypersurface in U and

$$\mathcal{H}^k((\partial E \setminus \partial^* E) \cap U) = 0 \qquad \forall k > n-8$$

Moreover, assuming that (1.19) holds uniformly for a sequence E_h, L^1-locally convergent to E_∞, then for any sequence of points $q_h \in \partial E_h$ converging to $q_\infty \in \partial^* E_\infty$, there is an h' such that $q_h \in \partial^* E_h$ for $h > h'$ and the unit outer normal to ∂E_h at q_h converges to the unit outer normal to ∂E_∞ at q_∞.

In order to apply this to the sequence $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$, we will need a number of estimates. The first we will prove in Section 6.2.

Lemma (Lemma 6.15)**.** *Let $F \subset (M,g_\rho)$ be a set of finite perimeter which is stationary for perimeter in a bounded open set $U \subset M$ (i.e. zero first variation and, in particular, zero mean curvature). Then there exists constants $C = C(U, \mathrm{P}_{g_\rho}(F,U))$ and $r_0 = r_0(U, \mathrm{P}_{g_\rho}(F,U)) > 0$ such that, for $r < r_0$, we have*

$$\mathrm{P}_{g_\rho}(F, B^{g_\rho}_r(q)) \leq Cr^2$$

where $B^{g_\rho}_r(q) \subset U$.

The proof uses the theory of rectifiable varifolds, as described in Simon [Sim83], which in particular applies to sets of finite perimeter. The main tool will be the monotonicity formula for varifolds, which we will translate to our framework. This inequality will be used to get a better estimate in the relationship between g_ρ-perimeter and $\bar{g}$-perimeter, found in step 2, in the case $F = \Omega_{p,\rho,w}$ (note that away from the intersection points with $S_{p,\rho}(w)$, by construction $\Omega_{p,\rho,w}$ is locally perimeter minimising).

Lemma (Lemma 6.22)**.**

$$\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(\Omega_{p,\rho,w}), B^{\bar{g}}_r(\phi^p_{g_\rho}(q))) = \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(\rho^2 r^4)$$

The next useful estimate, for proving the conditions in the Tamanini theorem, is the following.

Lemma (Lemma 6.17)**.** *Let $B \subset (M,g_\rho)$ be a bounded open set with C^2 boundary. Then, for small enough* *r, there exists a constant $C = C(B) > 0$ such that, for every $q \in \overline{B}$*

$$\mathrm{P}_{g_\rho}(B) \leq \mathrm{P}_{g_\rho}(G) + Cr^3 \quad \forall\, G\Delta B \subset\subset B_r^{g_\rho}(q)$$

We prove this in Section 6.2, along the lines of the proof of $(6-9)$ in [MS17].

To apply the regularity theorem, we will switch from $\bar{g}$-perimeters to g_ρ-perimeters using the relationsips described above and then use the proven estimates to conclude that (1.19) holds for $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$.

Step 4 - $\partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$

This final step will be done in detail at the end of Section 6.3. The previous steps amount to the fact that the elements of the sequence $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ are eventually graphs of $C^{1,\frac{1}{2}}$-functions over the sphere $\partial B^{\bar{g}}_1(0) = \mathbb{S}^2$, as $\rho \to 0$. For a given ρ, $\phi^p_{g_\rho}(S_{p,\rho}(w))$ is also a graph over $\mathbb{S}^2$, and so we can in fact consider each $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ as a graph over $\phi^p_{g_\rho}(S_{p,\rho}(w))$ instead. Furthermore, since $S_{p,\rho}(w) \subset \Omega_{p,\rho,w}$, we know that the graph functions are non-negative. Thus, differentiating the area functional reveals

$$\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w}) \geq \mathrm{A}_{g_\rho}(S_{p,\rho}(w))$$

with equality if and only if the graph function is identically zero. But the definition of $\Omega_{p,\rho,w}$ implies

$$\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w}) \leq \mathrm{A}_{g_\rho}(S_{p,\rho}(w))$$

Therefore, it must be that, for small ρ, $\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w}) = \mathrm{A}_{g_\rho}(S_{p,\rho}(w))$ and hence the graph function is identically zero. Therefore $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w}) = \phi^p_{g_\rho}(S_{p,\rho}(w))$ and $\partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$. Hence $S_{p,\rho}(w)$ is outer-minimising for small ρ and $\|w\|_{C^1}$. This ends the proof of the proposition and therefore also of Theorems 1.34 and 1.35.

2 Preliminaries

2.1 Notations

We adopt the Einstein summation convention for repeated indices and agree that latin index letters (e.g. $i, j, k, l, ...$) run from 1 to 2, while greek index letters (e.g. $\mu, \nu, \eta, \lambda, ...$) run from 1 to $n > 2$.

For an n-dimensional Riemannian manifold (M, g) with Levi-Civita connection ∇, we define the Riemann curvature endomorphism to be

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector fields X, Y, Z. By lowering an index we define the Riemann curvature tensor by

$$\mathrm{Rm}(X,Y,Z,W) = g(\mathcal{R}(Z,W)Y,X)$$

and therefore the Ricci curvature tensor as the trace over the first and third indices of Rm, which we write as $\mathrm{Ric} = \mathrm{tr}_g^{13}\mathrm{Rm}$. In particular, for $n = 3$ this means at $p \in M$ with an orthonormal basis E_μ of T_pM, we have

$$\begin{aligned}
\mathrm{Ric}(X_p,Y_p) &= \mathrm{Rm}(E_1,X_p,E_1,Y_p) + \mathrm{Rm}(E_2,X_p,E_2,Y_p) + \mathrm{Rm}(E_3,X_p,E_3,Y_p) \\
&= g(\mathcal{R}(E_1,Y_p)X_p,E_1) + g(\mathcal{R}(E_2,Y_p)X_p,E_2) + g(\mathcal{R}(E_3,Y_p)X_p,E_3) \\
&= -g(\mathcal{R}(Y_p,E_1)X_p,E_1) - g(\mathcal{R}(Y_p,E_2)X_p,E_2) - g(\mathcal{R}(Y_p,E_3)X_p,E_3)
\end{aligned}$$

using the anti-symmetry of the Riemann curvature in the last two entries. Note that we will usually suppress the subscript p to keep the formulas as clean as possible. We then have the scalar curvature, denoted by Sc, as the trace of the Ricci tensor.

$$\mathrm{Sc}_p = \mathrm{Ric}(E_1,E_1) + \mathrm{Ric}(E_2,E_2) + \mathrm{Ric}(E_3,E_3)$$

A key tensor used later will be the traceless Ricci tensor (for $n = 3$)

$$S := \mathrm{Ric} - \frac{1}{3}\mathrm{Sc} \cdot g$$

We will also use the following useful fact (for $n = 3$):

$$
\begin{aligned}
\|S\|^2 &= g(\mathrm{Ric} - \frac{1}{3}\mathrm{Sc}\cdot g, \mathrm{Ric} - \frac{1}{3}\mathrm{Sc}\cdot g) \\
&= g(\mathrm{Ric},\mathrm{Ric}) - \frac{2}{3}\mathrm{Sc}\cdot g(\mathrm{Ric}, g) + \frac{1}{9}\mathrm{Sc}^2\cdot g(g,g) \\
&= g(\mathrm{Ric},\mathrm{Ric}) - \frac{2}{3}\mathrm{Sc}\cdot g_{\eta\mu}R_{\lambda\nu}g^{\eta\lambda}g^{\mu\nu} + \frac{1}{9}\mathrm{Sc}^2\cdot g_{\eta\mu}g_{\lambda\nu}g^{\eta\lambda}g^{\mu\nu} \\
&= g(\mathrm{Ric},\mathrm{Ric}) - \frac{2}{3}\mathrm{Sc}\cdot \delta^{\lambda}_{\mu}R^{\mu}_{\lambda} + \frac{1}{9}\mathrm{Sc}^2\cdot \delta^{\lambda}_{\lambda} \\
&= g(\mathrm{Ric},\mathrm{Ric}) - \frac{2}{3}\mathrm{Sc}^2 + \frac{1}{3}\mathrm{Sc}^2 \\
&= \|\mathrm{Ric}\|^2 - \frac{1}{3}\mathrm{Sc}^2
\end{aligned}
$$

We denote the Hessian of a smooth function $f \in C^\infty(M)$ by $\nabla^2 f$, its Laplacian by $\Delta f = \mathrm{tr}_g(\nabla^2 f)$ and its gradient by $\mathrm{grad} f$. More generally, for a tensor A we have $\nabla_X\nabla_Y A = \nabla^2 A(\ldots,Y,X) + \nabla_{\nabla_X Y}A$. If $X = Y$ then we write $\nabla^2_X := \nabla_X\nabla_X A$ and similarly for higher derivatives. We denote the divergence of a vector field by $div_g(X) = \mathrm{tr}(\nabla X)$.

Occasionally we will use the *Kulkarni-Nomizu product*, defined as

$$
A \owedge B(X,Y,Z,W) := A(X,Z)B(Y,W) + A(Y,W)B(X,Z) - A(X,W)B(Y,Z) - A(Y,Z)B(X,W)
$$

for symmetric 2-tensors A and B. It satisfies the property contained in the next lemma.

Lemma 2.1 (Lemma 7.22 in [Lee18]). *For a symmetric 2-tensor A on an n-dimensional Riemannian manifold (M,g)*

$$
tr^{13}_g\big(A \owedge g\big) = \big(tr_g A\big)g + (n-2)A \tag{2.1}
$$

Proof. Evaluating the left hand side of (2.1) on basis vector fields yields

$$
\begin{aligned}
\Big(\mathrm{tr}^{13}_g\big(A \owedge g\big)\Big)_{\mu\nu} &= g^{\eta\lambda}\big(A \owedge g\big)_{\eta\mu\lambda\nu} \\
&= g^{\eta\lambda}\Big(A_{\eta\lambda}g_{\mu\nu} + A_{\mu\nu}g_{\eta\lambda} - A_{\eta\nu}g_{\mu\lambda} - A_{\mu\lambda}g_{\eta\nu}\Big) \\
&= \mathrm{tr}_g(A)g_{\mu\nu} + nA_{\mu\nu} - A_{\mu\nu} - A_{\mu\nu} \\
&= \mathrm{tr}_g(A)g_{\mu\nu} + (n-2)A_{\mu\nu}
\end{aligned}
$$

□

Euclidean space as a Riemannian manifold will be written $(\mathbb{R}^n, \bar{g})$, where $\bar{g}$ is the standard Euclidean metric, i.e. at each point $x \in \mathbb{R}^n$, $\bar{g}$ is the dot product on $T_x\mathbb{R}^n$.

For the rest of this subsection, set $n = 3$. Let $\Sigma \subset (M, g)$ be a closed surface and take X, Y to be vector fields on Σ. Since the normal bundle $N\Sigma$ is 1-dimensional, we have the scalar second fundamental form h, defined as

$$h(X,Y) = g(\nabla_X Y, \hat{N}) = -g(\nabla_X \hat{N}, Y) \tag{2.2}$$

where $\hat{N}$ is the inward unit normal and the second equality follows from the Weingarten equation. The two eigenvalues k_1 and k_2 of h at a point $p \in \Sigma$ are the principal curvatures at p and we use the convention that the mean curvature $H = k_1 + k_2$, and denote their product by $D = k_1 k_2$. The formulas we will use later to compute them are

$$H = tr_{\mathring{g}}(h) = \mathring{g}^{ij} h_{ij} \qquad D = \det(\mathring{g}^{ik} h_{kj}) = \frac{\det h_{ij}}{\det \mathring{g}_{kl}}$$

where $\mathring{g}$ is the induced metric on Σ (sometimes we will write g_Σ instead). Note that when $(M,g) = (\mathbb{R}^3, \bar{g})$, D is the classical (intrinsic) Gaussian curvature. The area of Σ is

$$\mathrm{A}(\Sigma) := \int_\Sigma dV_{\mathring{g}} = \int_{\phi(\Sigma)} (\phi^{-1})^*(dV_{\mathring{g}}) = \int_{\phi(\Sigma)} \sqrt{\det \mathring{g}_{ij}} dx^1 dx^2$$

for some local coordinates x^i and chart ϕ.

We will make extensive use of *normal coordinates*, so we introduce them now. A curve γ is a geodesic if $\nabla_{\gamma'} \gamma' = 0$. At any point $p \in M$, there is a unique (maximal) geodesic starting with any given direction $V \in T_pM$ ([Lee18],Corollary 4.28). This allows us to define the exponential map

$$\exp_p : T_pM \to M$$

where the image of a tangent vector is the image of the associated geodesic after time 1. One can show that the exponential map is smooth and in fact a diffeomorphism when we restrict to a small enough neighbourhood of the origin in T_pM. Thus, given any orthonormal basis E_μ of T_pM we have a *normal coordinate chart*

$$\phi = \exp_p^{-1} : U \to T_pM \cong \mathbb{R}^3$$

where U is a neighbourhood of p, $\phi(p) = 0$, and we identify T_pM with $\mathbb{R}^3$ by $x^\mu E_\mu = x^1 E_1 + x^2 E_2 + x^3 E_3 \cong$

(x^1, x^2, x^3). The normal coordinate vector fields induced by ϕ are

$$(\phi^{-1})_* \frac{\partial}{\partial x^\mu} = (\exp_p)_* \frac{\partial}{\partial x^\mu} := \partial_\mu \tag{2.3}$$

At p they satisfy

$$\partial_\mu(p) = d_0\phi^{-1}(\frac{\partial}{\partial x^\mu}(0)) = d_0(\exp_p)(E_\mu) = Id(E_\mu) = E_\mu$$

where we used that the differential of the exponential map is just the identity on $T_0(T_pM) = T_pM$. Therefore the metric coefficients satisfy

$$(g_p)_{\mu\nu} = g_p(\partial_\mu(p), \partial_\nu(p)) = g_p(E_\mu, E_\nu) = \delta_{\mu\nu}$$

by assumption that E_μ are orthonormal. Now we introduce polar coordinates on T_pM using the standard parametrisation of the unit sphere. For $\Theta \in \mathbb{S}^2 \subset T_pM$

$$\Theta = \Theta^\mu E_\mu = (\sin\theta^1 \cos\theta^2)E_1 + (\sin\theta^1 \sin\theta^2)E_2 + (\cos\theta^1)E_3, \quad \text{for} \quad 0 < \theta^1 < \pi,\ 0 < \theta^2 < 2\pi$$

Then, for $\rho \in (0, \infty)$ we have the parametrisation of $T_pM \setminus 0$.

$$\psi^{-1}(\rho, \theta^1, \theta^2) = \rho\Theta$$

We have the induced polar coordinate vector fields obtained via the push-forward of the vector fields ∂_{θ^i}.

$$(\psi^{-1})_*\partial_{\theta^1} = \rho\partial_{\theta^1}\Theta^\mu E_\mu = \rho\Big((\cos\theta^1\cos\theta^2)E_1 + (\cos\theta^1\sin\theta^2)E_2 + (-\sin\theta^1)E_3\Big) =: \rho\Theta_1 \tag{2.4}$$

$$(\psi^{-1})_*\partial_{\theta^2} = \rho\partial_{\theta^2}\Theta^\mu E_\mu = \rho\Big((-\sin\theta^1\sin\theta^2)E_1 + (\sin\theta^1\cos\theta^2)E_2\Big) =: \rho\Theta_2 \tag{2.5}$$

Remark 2.2. Since $(g_p)_{\mu\nu} = \delta_{\mu\nu}$, using (2.4) and (2.5), we have $g_p(\Theta,\Theta) = g_p(\Theta_1,\Theta_1) = 1$, $g_p(\Theta,\Theta_i) = 0$, $g_p(\Theta_1,\Theta_2) = 0$ and $g_p(\Theta_2,\Theta_2) = \sin^2\theta^1$. Defining $\bar{\Theta}_2 := \frac{1}{\sin\theta^1}\Theta_2$ produces another orthonormal basis of T_pM: $\{\Theta, \Theta_1, \bar{\Theta}_2\}$.

Composing the polar coordinate chart with the normal coordinate chart at p yields the *polar normal coordinate chart* $\phi_P := \psi \circ \phi$ on $M \setminus p$.

Definition. *The geodesic sphere centered at* $p \in M \setminus \partial M$ *of radius* $\rho > 0$ *is the surface*

$$S_{p,\rho} := \exp_p(\rho\Theta)$$

for ρ small enough to ensure $\exp_p$ is a diffeomorphism.

The induced polar normal coordinate vector fields on $S_{p,\rho}$, denoted by Z_i, are

$$Z_i := (\exp_p(\rho\Theta))_* \partial_{\theta^i} = \partial_{\theta^i}\left[\phi \circ \exp_p\left(\rho\Theta(\theta^1,\theta^2)\right)\right]^\mu \partial_\mu = \rho\,\partial_{\theta^i}\Theta^\mu \partial_\mu \tag{2.6}$$

where we used (2.4), (2.5) and (2.3). To prove the main theorems, we will compute the mass inside *perturbed* geodesic spheres, i.e. normal graphs over geodesic spheres.

Definition. *A perturbed geodesic sphere centered at $p \in M \setminus \partial M$ of radius $\rho > 0$ is a surface*

$$S_{p,\rho}(w) := \exp_p(\rho(1-w)\Theta)$$

for ρ small enough to ensure $\exp_p$ is a diffeomorphism, and

$$w \in C^{k,\alpha}(S^2)$$

for some $k \geq 2$ and $0 < \alpha < 1$.

The induced polar normal coordinate vector fields on $S_{p,\rho}(w)$, denoted by Z_i^w, are

$$\begin{aligned} Z_i^w := (\exp_p(\rho(1-w)\Theta))_* \partial_{\theta^i} &= \partial_{\theta^i}\left[\phi \circ \exp_p\left(\rho\left(1-w(\theta^1,\theta^2)\right)\Theta(\theta^1,\theta^2)\right)\right]^\mu \partial_\mu \\ &= \rho((1-w)\partial_{\theta^i}\Theta^\mu - w_i\Theta^\mu)\partial_\mu \end{aligned} \tag{2.7}$$

For further notation regarding perturbed geodesic spheres, see Section 1.3.1.

The Riemannian measure μ_g induced by the Riemannian metric g is defined on Borel sets by

$$\mu_g(E) := \int_E dV_g$$

where dV_g is the volume form. We will henceforth write $|E|_g := \mu_g(E)$.

The relative perimeter of a Borel set E inside an open set $V \subset M$ is defined to be

$$\mathrm{P}_g(E,V) := \sup_X \left\{ \int_{E\cap V} div_g(X) dV_g \mid X \in C_c^1(V \setminus \partial M, TM), ||X||_{\infty,g} \leq 1 \right\} \tag{2.8}$$

where $||X||_{\infty,g}$ is the sup norm with respect to the metric g. When $V = M$ we get the full perimeter $\mathrm{P}_g(E)$. If $\mathrm{P}_g(E,V) < \infty$ for all bounded V then we say E is a *set of locally finite perimeter* and if $\mathrm{P}_g(E) < \infty$ then we say E is a *set of finite perimeter*. Note that ∂M does not contribute to the perimeter.

A sequence E_n locally converges to E if, for every $V \subset\subset M$, $\lim_{n\to\infty} |(E_n \Delta E) \cap V|_g = 0$, also written $E_n \xrightarrow{\text{loc}} E$. A sequence E_n converges to E if $\lim_{n\to\infty} |E_n \Delta E|_g = 0$, also written $E_n \to E$. Note that, in this context, Δ denotes the symmetric difference of two sets.

$$E\Delta F := (E \cup F) \setminus (E \cap F)$$

By Riesz's theorem [Mag12, Theorem 4.7 and Proposition 12.1], there is a vector-valued Radon measure μ_E associated to a set of finite perimeter $E \subset M$, representing the distributional gradient of the characteristic function of E, such that the (generalised) Gauss-Green formula

$$\int_E div_g(X) dV_g = \int_M g(X,u)\ d|\mu_E| \quad \forall X \in C_c^1(M \setminus \partial M, TM), \text{where } u \in TM, |u|_g = 1 \text{ and } \mu_E = u|\mu_E|$$

still holds, and its total variation measure $|\mu_E|$ satisfies $|\mu_E|(V) = \mathrm{P}_g(E,V)$. When E is open and ∂E is C^1, we have $\mu_E = \hat{N}\mathcal{H}_g^2|_{\partial E}$, where $\mathcal{H}_g^2|_{\partial E}$ is the two dimensional Hausdorff measure, restricted to ∂E, $\hat{N}$ is the outer unit normal to ∂E, and we recover the usual Gauss-Green formula.

Let $\mathrm{spt}(\mu_E) \subset M$ be the support of μ_E. We define the *reduced boundary* $\partial^* E$ to be the following set:

$$\partial^* E = \left\{ x \in \mathrm{spt}(\mu_E) : \lim_{r\to 0} \frac{\mu_E(B_r^g(x))}{|\mu_E|(B_r^g(x))} \text{ exists and has unit length} \right\}$$

Although rarely used explicitly in this thesis, we mention that the reduced boundary is contained in the topological boundary, that is $\partial^* E \subset \partial E$, and if E is open with C^1 boundary, then $\partial^* E = \partial E$. The only result we will use about the reduced boundary is De Giorgi's Theorem [De 61] which states that $\mathrm{P}_g(E,V) = \mathcal{H}_g^2(\partial^* E \cap V)$. This will be applied in the proof of Lemma 6.15.

We now state some more known results about sets of finite perimeter, used later. Their proof can be found in [Mag12] where the author proves them for Euclidean space, but analogous arguments apply to a general Riemannian manifold. Alternatively, see [Vol10] where the author proves them for functions of *bounded variation* in a Riemannian manifold, which applies to sets of finite perimeter because their characteristic functions have bounded variation. Finally, one can first Nash-embed the ambient manifold into a high dimensional Euclidean space and then use the theory of currents or varifolds. We take this approach later when proving Lemma 6.15, following the work of Simon [Sim83].

Lemma 2.3. *Let E and F be sets of finite perimeter in (M,g). Then both $E\cap F$ and $E\cup F$ are sets of finite perimeter and the following inequality holds for all open $V\subset M$:*

$$\mathrm{P}_g(E\cup F,V)+\mathrm{P}_g(E\cap F,V)\leq \mathrm{P}_g(E,V)+\mathrm{P}_g(F,V)$$

Theorem 2.4 (Lower semi-continuity of perimeter)**.** *Let $E_n\subset(M,g)$ be a sequence of locally finite perimeter sets. If, for every $V\subset\subset M$*

$$|(E_n\Delta E)\cap V|_g\to 0 \qquad \textit{and} \qquad \limsup_{n\to\infty}\mathrm{P}_g(E_n,V)<\infty$$

then E is a set of locally finite perimeter such that, for every open $V'\subset M$

$$\mathrm{P}_g(E,V')\leq \liminf_{n\to\infty}\mathrm{P}_g(E_n,V')$$

Theorem 2.5 (Compactness for sets of finite perimeter)**.** *Let $E_n\subset(M,g)$ be a sequence of finite perimeter sets. If there exists $R>0$ and $p\in M$ such that*

$$\sup \mathrm{P}_g(E_n)<\infty \qquad \textit{and} \qquad E_n\subset B_R^g(p) \quad \forall n$$

then there exists a set E of finite perimeter such that

$$|E_n\Delta E|_g\to 0 \qquad \textit{and} \qquad E\subset B_R^g(p)$$

Finally, positive constants are denoted by $C,D,...$ and their value is allowed to vary from line to line. When we want to stress the dependence of the constants on parameters, we add subscripts or parentheses e.g. $C_r, C(r)$.

2.2 Conformal Metrics

A metric $\tilde{g}$ on a smooth, n-dimensional manifold M is conformal to another metric g if $\tilde{g}=e^{2f}g$ for a smooth function f on M. In this subsection we recall some facts about such metrics, namely how the curvatures of $(M,\tilde{g})$ relate to (M,g). Some of the proofs are from [Lee18, Chapter 7].

Lemma 2.6.

$$\tilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - g(X,Y)\mathrm{grad} f \tag{2.9}$$

Proof. By definition we have

$$\begin{aligned}
\tilde{\Gamma}^{\lambda}_{\eta\mu} &= \frac{1}{2}\tilde{g}^{\lambda\nu}(\partial_\eta \tilde{g}_{\mu\nu} + \partial_\mu \tilde{g}_{\eta\nu} - \partial_\nu \tilde{g}_{\eta\mu}) \\
&= \frac{1}{2}e^{-2f}g^{\lambda\nu}(\partial_\eta(e^{2f}g_{\mu\nu}) + \partial_\mu(e^{2f}g_{\eta\nu}) - \partial_\nu(e^{2f}g_{\eta\mu})) \\
&= \frac{1}{2}e^{-2f}g^{\lambda\nu}\left[e^{2f}(\partial_\eta g_{\mu\nu} + \partial_\mu g_{\eta\nu} - \partial_\nu g_{\eta\mu}) + 2e^{2f}(\partial_\eta f g_{\mu\nu} + \partial_\mu f g_{\eta\nu} - \partial_\nu f g_{\eta\mu})\right] \\
&= \Gamma^{\lambda}_{\eta\mu} + \partial_\eta f \delta^{\lambda}_{\mu} + \partial_\mu f \delta^{\lambda}_{\eta} - g^{\lambda\nu}\partial_\nu f g_{\eta\mu}
\end{aligned}$$

Thus, for vector fields X, Y

$$\begin{aligned}
(\tilde{\nabla}_X Y)^{\lambda} &= X(Y^{\lambda}) + X^{\eta}Y^{\mu}\tilde{\Gamma}^{\lambda}_{\eta\mu} \\
&= X(Y^{\lambda}) + X^{\eta}Y^{\mu}(\Gamma^{\lambda}_{\eta\mu} + \partial_\eta f \delta^{\lambda}_{\mu} + \partial_\mu f \delta^{\lambda}_{\eta} - g^{\lambda\nu}\partial_\nu f g_{\eta\mu}) \\
&= (\nabla_X Y)^{\lambda} + X^{\eta}Y^{\lambda}\partial_\eta f + X^{\lambda}Y^{\mu}\partial_\mu f - g_{\eta\mu}X^{\eta}Y^{\mu}g^{\lambda\nu}\partial_\nu f
\end{aligned}$$

which is the coordinate version of (2.9). $\square$

Lemma 2.7.

$$\tilde{\mathrm{Rm}} = e^{2f}\big(\mathrm{Rm} - (\nabla^2 f) \owedge g + (df \otimes df) \owedge g - \frac{1}{2}|df|^2_g (g \owedge g)\big) \tag{2.10}$$

Proof. First, since $[\partial_\eta, \partial_\mu] = 0$, we have

$$\begin{aligned}
\tilde{\mathcal{R}}(\partial_\eta, \partial_\mu)\partial_\lambda &= \tilde{R}_{\eta\mu\lambda}{}^{\nu}\partial_\nu \\
&= \tilde{\nabla}_{\partial_\eta}\tilde{\nabla}_{\partial_\mu}\partial_\lambda - \tilde{\nabla}_{\partial_\mu}\tilde{\nabla}_{\partial_\eta}\partial_\lambda \\
&= \tilde{\nabla}_{\partial_\eta}(\tilde{\Gamma}^{\nu}_{\mu\lambda}\partial_\nu) - \tilde{\nabla}_{\partial_\mu}(\tilde{\Gamma}^{\nu}_{\eta\lambda}\partial_\nu) \\
&= \partial_\eta \tilde{\Gamma}^{\nu}_{\mu\lambda}\partial_\nu + \tilde{\Gamma}^{\nu}_{\mu\lambda}\tilde{\nabla}_{\partial_\eta}\partial_\nu - \partial_\mu \tilde{\Gamma}^{\nu}_{\eta\lambda}\partial_\nu - \tilde{\Gamma}^{\nu}_{\eta\lambda}\tilde{\nabla}_{\partial_\mu}\partial_\nu \\
&= \big(\partial_\eta \tilde{\Gamma}^{\nu}_{\mu\lambda} - \partial_\mu \tilde{\Gamma}^{\nu}_{\eta\lambda}\big)\partial_\nu + \tilde{\Gamma}^{\nu}_{\mu\lambda}\tilde{\Gamma}^{\alpha}_{\eta\nu}\partial_\alpha - \tilde{\Gamma}^{\nu}_{\eta\lambda}\tilde{\Gamma}^{\alpha}_{\mu\nu}\partial_\alpha \\
&= \big(\partial_\eta \tilde{\Gamma}^{\nu}_{\mu\lambda} - \partial_\mu \tilde{\Gamma}^{\nu}_{\eta\lambda} + \tilde{\Gamma}^{\alpha}_{\mu\lambda}\tilde{\Gamma}^{\nu}_{\eta\alpha} - \tilde{\Gamma}^{\alpha}_{\eta\lambda}\tilde{\Gamma}^{\nu}_{\mu\alpha}\big)\partial_\nu
\end{aligned}$$

and therefore

$$\tilde{R}_{\eta\mu\lambda\nu} = \tilde{g}_{\nu\alpha}\tilde{R}_{\eta\mu\lambda}{}^{\alpha} = \tilde{g}_{\nu\alpha}\left(\partial_\eta\tilde{\Gamma}^{\alpha}_{\mu\lambda} - \partial_\mu\tilde{\Gamma}^{\alpha}_{\eta\lambda} + \tilde{\Gamma}^{\gamma}_{\mu\lambda}\tilde{\Gamma}^{\alpha}_{\eta\gamma} - \tilde{\Gamma}^{\gamma}_{\eta\lambda}\tilde{\Gamma}^{\alpha}_{\mu\gamma}\right) \tag{2.11}$$

Working in g-normal coordinates at a point p, we have the simplifications

$$\begin{aligned} g_{\eta\mu} &= \delta_{\eta\mu} \\ \partial_\lambda g_{\eta\mu} &= 0 \\ \Gamma^{\lambda}_{\eta\mu} &= 0 \end{aligned}$$

at p. Therefore, using the above calculation, at p we have

$$\tilde{\Gamma}^{\lambda}_{\eta\mu} = \partial_\eta f\delta^{\lambda}_{\mu} + \partial_\mu f\delta^{\lambda}_{\eta} - g^{\lambda\nu}\partial_\nu f g_{\eta\mu}$$

Also we have

$$\begin{aligned} (\nabla^2 f)_{\eta\mu} &= \nabla^2 f(\partial_\eta, \partial_\mu) \\ &= \nabla_{\partial_\mu}(\nabla_{\partial_\eta} f) - \nabla_{\nabla_{\partial_\eta}\partial_\mu} f \\ &= \partial_\mu\partial_\eta f - \Gamma^{\lambda}_{\eta\mu}\partial_\lambda f \\ &= \partial_\mu\partial_\eta f \end{aligned}$$

and

$$R_{\eta\mu\lambda}{}^{\nu} = \partial_\eta\Gamma^{\nu}_{\mu\lambda} - \partial_\mu\Gamma^{\nu}_{\eta\lambda} + \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\nu}_{\eta\alpha} - \Gamma^{\alpha}_{\eta\lambda}\Gamma^{\nu}_{\mu\alpha} = \partial_\eta\Gamma^{\nu}_{\mu\lambda} - \partial_\mu\Gamma^{\nu}_{\eta\lambda}$$

at p. Finally, we have

$$\partial_\alpha\tilde{\Gamma}^{\lambda}_{\eta\mu} = \partial_\alpha\Gamma^{\lambda}_{\eta\mu} + \partial_\alpha\partial_\eta f\delta^{\lambda}_{\mu} + \partial_\alpha\partial_\mu f\delta^{\lambda}_{\eta} - g^{\lambda\nu}\partial_\alpha\partial_\nu f g_{\eta\mu}$$

where we used $\partial_\alpha g_{\eta\mu} = 0 = \partial_\alpha g^{\eta\mu}$ at p. Now we can expand (2.11) in g-normal coordinates at p.

$$
\begin{aligned}
\tilde{R}_{\eta\mu\lambda\nu} &= e^{2f} g_{\nu\alpha}\Big[\big(\partial_\eta\Gamma^\alpha_{\mu\lambda} + \partial_\eta\partial_\mu f\delta^\alpha_\lambda + \partial_\eta\partial_\lambda f\delta^\alpha_\mu - g^{\alpha\beta}\partial_\eta\partial_\beta f g_{\mu\lambda}\big) \\
&\quad - \big(\partial_\mu\Gamma^\alpha_{\eta\lambda} + \partial_\mu\partial_\eta f\delta^\alpha_\lambda + \partial_\mu\partial_\lambda f\delta^\alpha_\eta - g^{\alpha\beta}\partial_\mu\partial_\beta f g_{\eta\lambda}\big) \\
&\quad + \big(\partial_\mu f\delta^\gamma_\lambda + \partial_\lambda f\delta^\gamma_\mu - g^{\gamma\beta}\partial_\beta f g_{\mu\lambda}\big)\big(\partial_\eta f\delta^\alpha_\gamma + \partial_\gamma f\delta^\alpha_\eta - g^{\alpha\varepsilon}\partial_\varepsilon f g_{\eta\gamma}\big) \\
&\quad - \big(\partial_\eta f\delta^\gamma_\lambda + \partial_\lambda f\delta^\gamma_\eta - g^{\gamma\beta}\partial_\beta f g_{\eta\lambda}\big)\big(\partial_\mu f\delta^\alpha_\gamma + \partial_\gamma f\delta^\alpha_\mu - g^{\alpha\varepsilon}\partial_\varepsilon f g_{\mu\gamma}\big)\Big] \\
&= e^{2f}\Big[g_{\nu\alpha}R_{\eta\mu\lambda}{}^{\alpha} + g_{\nu\lambda}\partial_\eta\partial_\mu f + g_{\nu\mu}\partial_\eta\partial_\lambda f - g_{\mu\lambda}\partial_\eta\partial_\nu f - g_{\nu\lambda}\partial_\mu\partial_\eta f - g_{\nu\eta}\partial_\mu\partial_\lambda f + g_{\eta\lambda}\partial_\mu\partial_\nu f \\
&\quad + g_{\nu\lambda}\partial_\mu f\partial_\eta f + g_{\nu\eta}\partial_\mu f\partial_\lambda f - g_{\eta\lambda}\partial_\mu f\partial_\nu f + g_{\nu\mu}\partial_\lambda f\partial_\eta f + g_{\nu\eta}\partial_\lambda f\partial_\mu f - g_{\eta\mu}\partial_\lambda f\partial_\nu f \\
&\quad - g_{\mu\lambda}\partial_\nu f\partial_\eta f - g^{\gamma\beta}g_{\mu\lambda}g_{\nu\eta}\partial_\beta f\partial_\gamma f + g_{\mu\lambda}\partial_\eta f\partial_\nu f - g_{\nu\lambda}\partial_\eta f\partial_\mu f - g_{\nu\mu}\partial_\eta f\partial_\lambda f + g_{\mu\lambda}\partial_\eta f\partial_\nu f \\
&\quad - g_{\nu\eta}\partial_\lambda f\partial_\mu f - g_{\nu\mu}\partial_\lambda f\partial_\eta f + g_{\mu\eta}\partial_\lambda f\partial_\nu f + g_{\eta\lambda}\partial_\nu f\partial_\mu f + g^{\gamma\beta}g_{\eta\lambda}g_{\nu\mu}\partial_\beta f\partial_\gamma f - g_{\eta\lambda}\partial_\mu f\partial_\nu f\Big] \\
&= e^{2f}\Big[R_{\eta\mu\lambda\nu} - \partial_\eta\partial_\nu f g_{\mu\lambda} - \partial_\mu\partial_\lambda f g_{\nu\eta} + \partial_\eta\partial_\lambda f g_{\nu\mu} + \partial_\mu\partial_\nu f g_{\eta\lambda} + g_{\nu\eta}\partial_\lambda f\partial_\mu f + g_{\mu\lambda}\partial_\eta f\partial_\nu f \\
&\quad - g_{\nu\mu}\partial_\lambda f\partial_\eta f - g_{\eta\lambda}\partial_\mu f\partial_\nu f + g^{\gamma\beta}g_{\eta\lambda}g_{\nu\mu}\partial_\beta f\partial_\gamma f - g^{\gamma\beta}g_{\mu\lambda}g_{\nu\eta}\partial_\beta f\partial_\gamma f\Big] \\
&= e^{2f}\big(R_{\eta\mu\lambda\nu} - ((\nabla^2 f)\owedge g)_{\eta\mu\lambda\nu} + ((df\otimes df)\owedge g)_{\eta\mu\lambda\nu} - \frac{1}{2}|df|^2_g(g\owedge g)_{\eta\mu\lambda\nu}\big)
\end{aligned}
$$

This proves (2.10) at an arbitrary point p in normal coordinates. Hence it is true everywhere. $\square$

Lemma 2.8.

$$\tilde{\mathrm{Ric}} = \mathrm{Ric} - (n-2)(\nabla^2 f) + (n-2)df\otimes df - \Big(\Delta f + (n-2)|df|^2_g\Big)g \tag{2.12}$$

Proof.

$$
\begin{aligned}
\tilde{\mathrm{Ric}} &= \mathrm{tr}^{13}_{\tilde{g}}\tilde{\mathrm{Rm}} \\
&= e^{-2f}\mathrm{tr}^{13}_g\Big[e^{2f}\big(\mathrm{Rm} - (\nabla^2 f)\owedge g + (df\otimes df)\owedge g - \frac{1}{2}|df|^2_g(g\owedge g)\big)\Big] \\
&= \mathrm{Ric} - \mathrm{tr}^{13}_g\Big[(\nabla^2 f)\owedge g\Big] + \mathrm{tr}^{13}_g\Big[(df\otimes df)\owedge g\Big] - \frac{1}{2}|df|^2_g\mathrm{tr}^{13}_g\Big[g\owedge g\Big] \\
&= \mathrm{Ric} - (n-2)(\nabla^2 f) - \mathrm{tr}_g(\nabla^2 f)g + (n-2)df\otimes df + \mathrm{tr}_g(df\otimes df)g \\
&\quad - \frac{1}{2}|df|^2_g\Big[(n-2)g + (\mathrm{tr}_g g)g\Big] \\
&= \mathrm{Ric} - (n-2)(\nabla^2 f) - (\Delta f)g + (n-2)df\otimes df + |df|^2_g g - (n-1)|df|^2_g g \\
&= \mathrm{Ric} - (n-2)(\nabla^2 f) + (n-2)df\otimes df - \Big(\Delta f + (n-2)|df|^2_g\Big)g
\end{aligned}
$$

where in the fourth line we used (2.1) and in the fifth line we used the fact that $\mathrm{tr}_g g = g^{\eta\mu}g_{\eta\mu} = \delta^\mu_\mu = n$ and $\mathrm{tr}_g(df\otimes df) = g^{\eta\mu}(df\otimes df)_{\eta\mu} = g^{\eta\mu}(df)_\eta(df)_\mu = g(df,df) = |df|^2_g$. $\square$

Lemma 2.9.

$$\tilde{\mathrm{Sc}} = e^{-2f}\Big(\mathrm{Sc} - 2(n-1)\Delta f - (n-1)(n-2)|df|_g^2\Big) \tag{2.13}$$

Proof.

$$\begin{aligned}
\tilde{\mathrm{Sc}} &= \mathrm{tr}_{\tilde{g}}\tilde{\mathrm{Ric}} \\
&= e^{-2f}\mathrm{tr}_g\tilde{\mathrm{Ric}} \\
&= e^{-2f}\Big[\mathrm{Sc} - (n-2)\Delta f + (n-2)|df|_g^2 - n\Big(\Delta f + (n-2)|df|_g^2\Big)\Big]
\end{aligned}$$

where we traced (2.12) term by term. Collecting like terms yields (2.13). □

Now let $s = e^{2f}$. Then $\tilde{g} = sg$, $\tilde{g}^{-1} = s^{-1}g^{-1}$ and $\partial_\eta \tilde{g}_{\mu\lambda} = \partial_\eta s g_{\mu\lambda} + s\partial_\eta g_{\mu\lambda}$. Thus, the Christoffel symbols become

$$\begin{aligned}
\tilde{\Gamma}^\lambda_{\eta\mu} &= \frac{s^{-1}g^{\lambda\nu}}{2}\Big[s\Big(\partial_\eta \tilde{g}_{\mu\nu} + \partial_\mu \tilde{g}_{\eta\nu} - \partial_\nu \tilde{g}_{\eta\mu}\Big) + \partial_\eta s g_{\mu\nu} + \partial_\mu s g_{\eta\nu} - \partial_\nu s g_{\eta\mu}\Big] \\
&= \Gamma^\lambda_{\eta\mu} + \frac{s^{-1}}{2}\Big[\partial_\eta s \delta^\lambda_\mu + \partial_\mu s \delta^\lambda_\eta - g^{\lambda\nu}\partial_\nu s g_{\eta\mu}\Big]
\end{aligned}$$

and then (2.9) changes to

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{s^{-1}}{2}\Big[X(f)Y + Y(f)X - g(X,Y)\mathrm{grad} f\Big]$$

We can rewrite (2.13) in terms of s as follows:

$$\begin{aligned}
\Delta f &= \Delta\Big(\frac{1}{2}\ln s\Big) \\
&= g^{\eta\mu}\Big[\partial_\mu\partial_\eta\Big(\frac{1}{2}\ln s\Big) - \partial_\lambda\Big(\frac{1}{2}\ln s\Big)\Gamma^\lambda_{\eta\mu}\Big] \\
&= \frac{1}{2}g^{\eta\mu}\Big[\partial_\mu\Big(\frac{\partial_\eta s}{s}\Big) - \frac{\partial_\lambda s}{s}\Gamma^\lambda_{\eta\mu}\Big] \\
&= \frac{1}{2s}g^{\eta\mu}\Big[\partial_\mu\partial_\eta s - \frac{\partial_\eta s\partial_\mu s}{s} - \partial_\lambda s\Gamma^\lambda_{\eta\mu}\Big] \\
&= \frac{1}{2s}\Big[\Delta s - \frac{|ds|_g^2}{s}\Big]
\end{aligned}$$

$$|df|_g^2 = g(d\Big(\frac{1}{2}\ln s\Big), d\Big(\frac{1}{2}\ln s\Big)) = \frac{1}{4s^2} g(ds,ds) = \frac{1}{4s^2}|ds|_g^2$$

so that

$$\begin{aligned}
\tilde{\mathrm{Sc}} &= s^{-1}\Big(\mathrm{Sc} - 2(n-1)\left[\frac{1}{2s}\Big[\Delta s - \frac{|ds|_g^2}{s}\Big]\right] - (n-1)(n-2)\Big[\frac{1}{4s^2}|ds|_g^2\Big]\Big) \\
&= s^{-1}\mathrm{Sc} - s^{-2}(n-1)\Delta s - \frac{s^{-3}}{4}(n-1)(n-6)|ds|_g^2 \qquad (2.14)
\end{aligned}$$

Now, if $s = u^{\frac{4}{n-2}}$ (for $n \neq 2$) then we have

$$\begin{aligned}
\Delta\Big(u^{\frac{4}{n-2}}\Big) &= g^{\eta\mu}\Big[\partial_\mu\partial_\eta\Big(u^{\frac{4}{n-2}}\Big) - \partial_\lambda\Big(u^{\frac{4}{n-2}}\Big)\Gamma^\lambda_{\eta\mu}\Big] \\
&= \frac{4g^{\eta\mu}}{n-2}\Big[\partial_\mu\Big(u^{\frac{6-n}{n-2}}\partial_\eta u\Big) - u^{\frac{6-n}{n-2}}\partial_\lambda u\Gamma^\lambda_{\eta\mu}\Big] \\
&= \frac{4g^{\eta\mu}}{n-2}\Big[\frac{6-n}{n-2}u^{\frac{8-2n}{n-2}}\partial_\mu u\partial_\eta u + u^{\frac{6-n}{n-2}}\partial_\mu\partial_\eta u - u^{\frac{6-n}{n-2}}\partial_\lambda u\Gamma^\lambda_{\eta\mu}\Big] \\
&= \frac{4(6-n)}{(n-2)^2}u^{\frac{8-2n}{n-2}}|du|_g^2 + \frac{4}{n-2}u^{\frac{6-n}{n-2}}\Delta u
\end{aligned}$$

$$|d\Big(u^{\frac{4}{n-2}}\Big)|_g^2 = g(d\Big(u^{\frac{4}{n-2}}\Big), d\Big(u^{\frac{4}{n-2}}\Big)) = (\frac{4}{n-2})^2(u^{\frac{6-n}{n-2}})^2 g(du,du) = \frac{16}{(n-2)^2}u^{\frac{12-2n}{n-2}}|du|_g^2$$

which means (2.14) becomes

$$\begin{aligned}
\tilde{\mathrm{Sc}} &= u^{-\frac{4}{n-2}}\mathrm{Sc} - u^{-\frac{8}{n-2}}(n-1)\Delta\Big(u^{\frac{4}{n-2}}\Big) - \frac{u^{-\frac{12}{n-2}}}{4}(n-1)(n-6)|d\Big(u^{\frac{4}{n-2}}\Big)|_g^2 \\
&= u^{-\frac{4}{n-2}}\mathrm{Sc} - u^{-\frac{8}{n-2}}(n-1)\Big[\frac{4(6-n)}{(n-2)^2}u^{\frac{8-2n}{n-2}}|du|_g^2 + \frac{4}{n-2}u^{\frac{6-n}{n-2}}\Delta u\Big] \\
&\quad - \frac{u^{-\frac{12}{n-2}}}{4}(n-1)(n-6)\Big[\frac{16}{(n-2)^2}u^{\frac{12-2n}{n-2}}|du|_g^2\Big] \\
&= u^{-\frac{4}{n-2}}\mathrm{Sc} - \frac{4(n-1)}{n-2}u^{-\frac{(n+2)}{n-2}}\Delta u \qquad (2.15)
\end{aligned}$$

Remark 2.10. The main difference between (2.14) and (2.15) is that the latter doesn't have a "du" term.

Remark 2.11. If $\mathrm{Sc} = 0$, for example when $g = \bar{g}$, then (2.15) shows that $\tilde{\mathrm{Sc}} = 0$ is equivalent to u being harmonic.

2.3 Variation of Geometric Quantities

For a closed surface $\Sigma \subset (M, g)$, define a normal variation of Σ by a smooth map $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ such that $\partial_t F = f_t N_t$, for a smooth function f_t and outward unit normal N_t on the surface $\Sigma_t := F(\Sigma, t)$. Induce the pullback metric $g_t := F_t^* g$ on each Σ_t. Thus $(\Sigma_0, g_0) = (\Sigma, \mathring{g})$. With these notations, we recall the variation equations for some geometric quantities, whose proof can be found in [HP96; CM11; Bra97], or the Appendix to this work (Section 9.2).

Lemma 2.12 (Variation of Geometric Quantities)**.**

i) $\partial_t (g_t)_{ij} = 2 f_t (h_t)_{ij}$

ii) $\partial_t dV_{g_t} = f_t H_t dV_{g_t}$

iii) $\partial_t N_t = -\mathrm{grad}_{\Sigma_t} f_t$

iv) $\partial_t (h_t)_{ij} = -f_t \mathrm{Rm}(N_t, \partial_j, N_t, \partial_i) - g(\nabla_{\partial_i} \mathrm{grad}_{\Sigma_t} f_t, \partial_j) + f_t g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} N_t)$

v) $\partial_t H_t = -\Delta_{g_t} f_t - f_t (|h_t|^2 + \mathrm{Ric}(N_t, N_t))$

Lemma 2.13 (First Variation of the Area Functional)**.**

$$\partial_t|_0 (\mathrm{A}(\Sigma_t)) = \int_\Sigma f_0 H dV_{\mathring{g}} \tag{2.16}$$

Proof. By Lemma 2.12 we have

$$\partial_t (\mathrm{A}(\Sigma_t)) = \int_\Sigma \partial_t dV_{g_t} = \int_\Sigma f_t H_t dV_{g_t}$$

and evaluating at $t = 0$ completes the proof. □

Lemma 2.14 (First Variation of the Willmore Functional)**.**

$$\partial_t|_0 (W(\Sigma_t)) = \int_\Sigma \Big(-2\Delta_{\mathring{g}} H - H(H^2 - 4D + 2\mathrm{Ric}(\hat{N}, \hat{N})) \Big) f_0 \, dV_{\mathring{g}} \tag{2.17}$$

and therefore Σ is a Willmore surface if and only if it satisfies

$$-2\Delta_{\mathring{g}} H - H(H^2 - 4D + 2\mathrm{Ric}(\hat{N}, \hat{N})) = 0$$

Proof. We follow the proof given in [LMS11].

$$
\begin{aligned}
\partial_t(W(\Sigma_t)) &= \partial_t \int_\Sigma H_t^2 \, dV_{g_t} \\
&= \int_\Sigma 2H_t \partial_t(H_t) dV_{g_t} + H_t^2 \partial_t(dV_{g_t}) \\
&= \int_\Sigma 2H_t(-\Delta_{g_t} f_t - f_t(|h_t|^2 + \mathrm{Ric}(N_t, N_t))) dV_{g_t} + f_t H_t^3 dV_{g_t} \\
&= \int_\Sigma -2H_t \Delta_{g_t} f_t - 2 f_t H_t (|h_t|^2 - \frac{1}{2} H_t^2 + \mathrm{Ric}(N_t, N_t)) \, dV_{g_t} \\
&= \int_\Sigma -2 f_t \Delta_{g_t} H_t - 2 f_t H_t (|h_t|^2 - \frac{1}{2} H_t^2 + \mathrm{Ric}(N_t, N_t)) \, dV_{g_t} \\
&= \int_\Sigma \Big(-2\Delta_{g_t} H_t - 2H_t(|h_t|^2 - \frac{1}{2} H_t^2 + \mathrm{Ric}(N_t, N_t)) \Big) f_t \, dV_{g_t}
\end{aligned}
$$

where we have used Lemma 2.12, and integrated by parts (recall $\partial\Sigma = \emptyset$) so that f factored out. Evaluating at $t = 0$, we can expand $|h|^2$ in an orthonormal frame which diagonalizes h, so that

$$
\begin{aligned}
|h|^2 = k_1^2 + k_2^2 &= (k_1 + k_2)^2 - 2k_1 k_2 \\
&= H^2 - 2D
\end{aligned}
$$

where k_1, k_2 are the principle curvatures. This gives the desired integral. If Σ minimises the functional for any normal variation, then the integral in (2.17) will be equal to zero for any f, hence

$$
-2\Delta_{\mathring{g}} H - H(H^2 - 4D + 2\mathrm{Ric}(\hat{N}, \hat{N})) = 0
$$

□

2.4 Examples

These examples are 3D Riemannian manifolds relevant to the conditions in the main theorems.

Example 2.15. For any given (M, g) with zero scalar curvature, Remark 2.11 shows that we get another by taking $(M, u^4 g)$ for harmonic u. For example, the spatial Schwarzschild manifold of mass m is

$$
\left(\mathbb{R}^3 \setminus 0, \left(1 + \frac{m}{2|x|} \right)^4 \bar{g} \right)
$$

and we have

$$\Delta_{\bar{g}}\Big(1+\frac{m}{2|x|}\Big)=\sum_{\mu}\partial_\mu\partial_\mu\Big(1+\frac{m}{2|x|}\Big)=\sum_{\mu}\partial_\mu\Big(-\frac{mx^\mu}{2|x|^3}\Big)=\sum_{\mu}-\frac{m}{2}\Big(|x|^{-3}-3(x^\mu)^2|x|^{-5}\Big)=0$$

Therefore the spatial Schwarzschild manifold has zero scalar curvature. In general, for $(\mathbb{R}^3\setminus 0, u^4\bar{g})$ we can use harmonic function theory ([ABR01], Chapter 10), which gives the following Laurent expansion:

$$u(x)=\sum_{m=0}^{\infty}p_m(x)+\sum_{m=0}^{\infty}\frac{q_m(x)}{|x|^{2m+1}}$$

Restricting to functions u which satisfy $u\to a$ at infinity means we have

$$u(x)=a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})$$

Thus we can attempt to compute the ADM mass of $(\mathbb{R}^3\setminus 0, u^4\bar{g})$. Firstly

$$(u^4\delta_{\mu\nu})_\mu=4\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(-\frac{bx^\mu}{|x|^3}+\mathcal{O}(|x|^{-3})\Big)\delta_{\mu\nu}$$

and

$$(u^4\delta_{\mu\mu})_\nu=4\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(-\frac{bx^\nu}{|x|^3}+\mathcal{O}(|x|^{-3})\Big)\delta_{\mu\mu}$$

Therefore the integrand of $m_{ADM}((\mathbb{R}^3\setminus 0, u^4\bar{g}))$ is

$$
\begin{aligned}
\Big((u^4\delta_{\mu\nu})_\mu - (u^4\delta_{\mu\mu})_\nu\Big)\frac{x^\nu}{|x|} &= 4\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(-\frac{bx^\mu}{|x|^3}+\mathcal{O}(|x|^{-3})\Big)\delta_{\mu\nu}\frac{x^\nu}{|x|} \\
&\quad -4\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(-\frac{bx^\nu}{|x|^3}+\mathcal{O}(|x|^{-3})\Big)\delta_{\mu\mu}\frac{x^\nu}{|x|} \\
&= \sum_\nu -4\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(\frac{b(x^\nu)^2}{|x|^4}+\mathcal{O}(|x|^{-4})\Big) \\
&\quad +\sum_\nu 12\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(\frac{b(x^\nu)^2}{|x|^4}+\mathcal{O}(|x|^{-4})\Big) \\
&= 8b\Big(a+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^3\Big(|x|^{-2}+\mathcal{O}(|x|^{-4})\Big)
\end{aligned}
$$

Thus, (using $r = |x|$)

$$
\begin{aligned}
m_{ADM}((\mathbb{R}^3\setminus 0, u^4\bar{g})) &= \lim_{r\to\infty}\frac{2b}{\omega_2}\Big(a+\frac{b}{r}+\mathcal{O}(r^{-2})\Big)^3\Big(r^{-2}+\mathcal{O}(r^{-4})\Big)\int_{S_r^2} dV_{S_r^2} \\
&= \lim_{r\to\infty} 2b\Big(a+\frac{b}{r}+\mathcal{O}(r^{-2})\Big)^3\Big(1+\mathcal{O}(r^{-2})\Big) \\
&= 2a^3 b
\end{aligned}
$$

Unfortunately, this quantity is not well defined because it is multiplied by C when computed in a coordinate system obtained by scaling with a factor of C. The invariant quantity is in fact $2ab$ and so when $a = 1$, the above calculation is valid. This makes sense because, when $a = 1$ we can show that $(\mathbb{R}^3\setminus 0, u^4\bar{g})$ is AF (with $\tau = 1$) as follows. Rewrite the metric as

$$
\Big(1+\frac{b}{|x|}+\mathcal{O}(|x|^{-2})\Big)^4\delta_{\mu\nu} = \Big(1+\mathcal{O}(|x|^{-1})\Big)\delta_{\mu\nu} = \delta_{\mu\nu}+\mathcal{O}(|x|^{-1})\delta_{\mu\nu}
$$

and then, for any μ, ν and multi-index α, we have

$$
|x|^{|\alpha|}\partial^\alpha(\mathcal{O}(|x|^{-1})\delta_{\mu\nu}) = |x|^{|\alpha|}\mathcal{O}(|x|^{-1-|\alpha|}) = \mathcal{O}(|x|^{-1})
$$

The case with $a = 1$, $b = \frac{m}{2}$ and no lower order terms yields the spatial Schwarzschild manifold of mass m which, by the above calculation, has an ADM mass of m too.

Example 2.16. Another class of relevant examples is *asymptotically flat graphs in* $\mathbb{R}^4$. These arise by considering functions $f \in C^\infty(\mathbb{R}^3)$ such that either $f(x) \to C$ or $f(x) \to \infty$, as $r = |x| \to \infty$, where $x \in \mathbb{R}^3$

and $C \in \mathbb{R}$. Letting (M, g_f) be the image of f as a subset of $\mathbb{R}^4$ with the induced metric yields

$$(M, g_f) = (\mathbb{R}^3, \bar{g} + df \otimes df) \quad \text{where} \quad (g_f)_{\mu\nu} = \delta_{\mu\nu} + \partial_\mu f \partial_\nu f$$

Requiring

$$\sum_{|\alpha|=1}^{4} r^{|\alpha|-1} \partial^\alpha f = \mathcal{O}(r^{-\frac{\tau}{2}})$$

for some $\tau > \frac{1}{2}$ implies that (M, g_f) is asymptotically flat, as per Definition 1.2. More generally, for a bounded, open set $\Omega \subset \mathbb{R}^3$ we can consider $f \in C^\infty(\mathbb{R}^3 \setminus \Omega)$. The spatial Schwarzschild manifold fits into this class as the graph of the following function:

$$\mathbb{R}^3 \setminus B_{2m}(0) \to \mathbb{R} \qquad r \mapsto \sqrt{8m(r-2m)}$$

The scalar curvature of any graph over Euclidean space can be computed in terms of the function f using

$$\mathrm{Sc}_{g_f} = \frac{1}{1+|\mathrm{grad}_{\bar{g}} f|_{\bar{g}}^2} \Big[f_{\mu\mu} f_{\nu\nu} - f_{\mu\nu} f_{\mu\nu} - \frac{2 f_\nu f_\eta}{1+|\mathrm{grad}_{\bar{g}} f|_{\bar{g}}^2} \big(f_{\mu\mu} f_{\nu\eta} - f_{\mu\nu} f_{\mu\eta} \big) \Big]$$

See [Geo10] for the proofs and much more detail about asymptotically flat graphs. An explicit example of a graph with non-negative scalar curvature is given by the 3D elliptic paraboloid, whose graph function is $f(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2$, so that $f_\mu = 2x^\mu$, $f_{\mu\nu} = 0$ for $\mu \neq \nu$, and $f_{\mu\mu} = 2$. Plugging these in to the above formula and summing over μ, ν and η yields

$$\mathrm{Sc}_{g_f} = \frac{8|\mathrm{grad}_{\bar{g}} f|_{\bar{g}}^2 + 24}{\big(1+|\mathrm{grad}_{\bar{g}} f|_{\bar{g}}^2\big)^2} > 0$$

The metric in this case is not asymptotically flat, since in coordinates we have $(g_f)_{\mu\nu} = \delta_{\mu\nu} + 4x^\mu x^\nu$, but it is ALSC because the radius of its cross-section grows without bound.

Example 2.17. We can look for further examples by considering warped products. For (M^1, g_1), (N^2, g_2) and a positive function $f \in C^\infty(M)$ we form the warped product $(M \times N, g_f)$, where $g_f = g_1 + f^2 g_2$. The next result is a special case of Theorem 2.1 in [DD87].

Lemma 2.18. *The scalar curvature* Sc_{g_f} *of* $(M \times N, g_f)$ *satisfies*

$$v\mathrm{Sc}_{g_f} = v^{-\frac{1}{3}} \mathrm{Sc}_{g_2} - \frac{8}{3} v'' \tag{2.18}$$

where Sc_{g_2} *is the scalar curvature of* (N^2, g_2) *and* $v = f^{\frac{3}{2}}$.

Proof. We mimic the method in [DD87] and write $g_f = f^2(f^{-2}g_1 + g_2) =: f^2 g_3$, where g_3 is another metric on $M \times N$. Since M is one dimensional, it is locally isometric to an interval with the standard metric and so $\mathrm{Sc}_{g_1} = 0$ and its Laplacian is just the second derivative. Thus, by (2.14) we have $\mathrm{Sc}_{f^{-2}g_1} = f^2 \mathrm{Sc}_{g_1} = 0$. This implies that $\mathrm{Sc}_{g_3} = \mathrm{Sc}_{f^{-2}g_1} + \mathrm{Sc}_{g_2} = \mathrm{Sc}_{g_2}$, where we have used that g_3 is just the product metric on $(M, f^{-2}g_1) \times (N, g_2)$. Next, applying (2.15) to $(M \times N, g_3)$ where the conformal factor is f^2, we have

$$\begin{aligned}\mathrm{Sc}_{g_f} = \mathrm{Sc}_{f^2 g_3} &= u^{-4}\mathrm{Sc}_{g_3} - 8u^{-5}\Delta_{g_3} u \\ &= u^{-4}\mathrm{Sc}_{g_2} - 8u^{-5}\Delta_{g_3} u \end{aligned} \tag{2.19}$$

where $f^2 = u^4$ for some $u \in C^\infty(M)$. In order to change the Laplacian term so that it is with respect to the metric g_1 we use that $u \in C^\infty(M)$ and the standard coordinate formula to compute

$$\begin{aligned}\Delta_{g_3} u = \Delta_{f^{-2}g_1} u &:= (\det f^{-2}g_1)^{-\frac{1}{2}} \partial_i \Big((f^{-2}g_1)^{ij} (\det f^{-2}g_1)^{\frac{1}{2}} \partial_j u \Big) \\ &= f(g_1)^{ij} \partial_i f \partial_j u + f^2 \Delta_{g_1} u \\ &= u^3(2u'u' + uu'') \end{aligned} \tag{2.20}$$

Now we can get rid of the term containing the first derivative by letting $u = v^s$ and using the product rule

$$(v^s)'' = sv^{s-1}v'' + s(s-1)v^{s-2}v'v'$$

Substituting this into (2.20) shows that both $v'v'$ terms would have a v^{2s-2} factor, so they could be combined. To make the resulting coefficient zero, we would need s to satisfy $2s^2 = -s(s-1)$, which implies $s = \frac{1}{3}$. Thus, letting $u = v^{\frac{1}{3}}$ and inserting (2.20) into (2.19), we obtain

$$\mathrm{Sc}_{g_f} = v^{-\frac{4}{3}} \mathrm{Sc}_{g_2} - \frac{8}{3} v^{-1} v''$$

which is equivalent to (2.18). $\square$

By inspection, we can see that for $\mathrm{Sc}_{g_f} = 0$ and Sc_{g_2} equal to a constant, choosing

$$v(r) = \left(\left(\frac{\mathrm{Sc}_{g_2}}{2} \right)^{\frac{1}{2}} r + A \right)^{\frac{3}{2}}$$

satisfies the equation, for any constant A. In particular, for $\mathrm{Sc}_{g_2} = 2$ and $A = 0$ (so that $v(r) = r^{\frac{3}{2}}$ and $f(r) = r$), we get $(\mathbb{R}^3, \bar{g})$ and the spatial Schwarzschild manifold of mass m (in different coordinates to Example 2.15) as

$$(M \times N, g_f) = \left((0,\infty) \times \mathbb{S}^2, dr^2 + r^2 g_{\mathbb{S}^2}\right)$$
$$(M \times N, g_f) = \left((2m,\infty) \times \mathbb{S}^2, \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2}\right)$$

Alternatively, if $\mathrm{Sc}_{g_2} = 0$ and $A \neq 0$ (so that $v(r) \equiv A^{\frac{3}{2}}$ and $f(r) \equiv A$), then we get, for example

$$(M \times N, g_f) = \left(\mathbb{R} \times T^2, \bar{g} + A^2 g_{T^2}\right)$$

where (T^2, g_{T^2}) is the flat torus. While the first two warped products are both ALSC (in fact AF), the latter isn't, because any ball with large enough radius can't avoid the topology of the torus.

Another way a warped product can fail to be ALSC is when it shrinks to a cusp at infinity. For example

$$(M \times N, g_f) = \left([1,\infty) \times \mathbb{S}^2, dr^2 + e^{-2r} g_{\mathbb{S}^2}\right)$$

Substituting $v = e^{-\frac{3r}{2}}$ and $\mathrm{Sc}_{g_2} = 2$ into (2.18) yields $\mathrm{Sc}_{g_f} = 2e^{2r} - 6 > 0$.

Finally, we mention that in Lemma 3.3 of [Eji81], it is shown that the equivalent form of (2.18) obtained by substituting in f

$$\mathrm{Sc}_{g_f} = \frac{\mathrm{Sc}_{g_2}}{f^2} - \frac{2(f')^2}{f^2} - \frac{4f''}{f}$$

has a positive, periodic solution when Sc_{g_f} and Sc_{g_2} are positive constants.

Example 2.19. Hamilton's Cigar soliton (self-similar solution to the Ricci flow) is the complete, rotationally symmetric Riemannian manifold

$$\left(\mathbb{R}^2, \frac{dx^2 + dy^2}{1 + x^2 + y^2}\right)$$

Using polar coordinates, the metric becomes

$$\frac{dr^2 + r^2 d\theta^2}{1 + r^2}$$

and therefore, as $r \to \infty$, the cross-section at r tends towards a unit circle. Thus it is asymptotically cylindrical and therefore not ALSC, because a ball of radius bigger than 1 will eventually contain a non-trivial loop if we take it far enough towards infinity. The scalar curvature of the Cigar is

$$\frac{4}{1+r^2} > 0$$

Consider now a 3D analogue of the Cigar. That is, a complete, 3D, rotationally symmetric (gradient) Ricci soliton with positive sectional curvature (thus positive scalar curvature), proved to exist by Bryant in [Bry05, Theorem 1]. The cited author shows that its metric, written as the warped product $\left(\mathbb{R}^3, dr^2 + f^2 g_{\mathbb{S}^2}\right)$, satisfies $f(r) = \mathcal{O}(r^{\frac{1}{2}})$ as $r \to \infty$ and that its sectional curvatures decay at least inverse linearly (see also [CLN06, Section 4.6]). Thus, unlike the Cigar, the 3D Bryant soliton is ALSC since the radius of its cross-section grows without bound. However, it is not AF because $f(r)$ is only $\mathcal{O}(r^{\frac{1}{2}})$, not $\mathcal{O}(r)$.

3 Geodesic Spheres - Calculations

3.1 Normal Coordinate Expansions of Geometric Quantities

In this subsection we compute the expansions of the geometric quantities for a geodesic sphere, inside an arbitrary 3D Riemannian manifold, that are needed in order to find $W(S_{p,\rho})$ and therefore $m_H(S_{p,\rho})$. The methods used in the proofs are mostly the same as in [Mon13; PX09], and the perturbed case in Section 4 will be similar. We want expansions for $\mathring{g}$ and H, since then we can compute

$$W(S_{p,\rho}) := \int_{S_{p,\rho}} H^2 \, dV_{\mathring{g}} = \int_{\mathbb{S}^2} H^2 \sqrt{\det \mathring{g}} \, d\theta^1 d\theta^2$$

Lemma 3.1. *Let (M,g) be a 3D Riemannian manifold and $p \in M$. The following expansions hold in normal coordinates at p:*

$$\begin{aligned} i)\ g_{\mu\nu} = \delta_{\mu\nu} &+ \frac{1}{3} g(\mathcal{R}(\Theta,E_\mu)\Theta,E_\nu)\rho^2 + \frac{1}{6} g(\nabla_\Theta \mathcal{R}(\Theta,E_\mu)\Theta,E_\nu)\rho^3 \\ &+ \frac{1}{20} g(\nabla^2_\Theta \mathcal{R}(\Theta,E_\mu)\Theta,E_\nu)\rho^4 + \frac{2}{45} g(\mathcal{R}(\Theta,E_\mu)\Theta,E_\tau) g(\mathcal{R}(\Theta,E_\nu)\Theta,E_\tau)\rho^4 \\ &+ \mathcal{O}(\rho^5) \end{aligned} \tag{3.1}$$

$$\begin{aligned} ii)\ \mathring{g}_{ij} = g^{\mathbb{S}^2}_{ij}\rho^2 &+ \frac{1}{3} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^4 + \frac{1}{6} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^5 \\ &+ \frac{1}{20} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^6 + \frac{2}{45} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\rho^6 \\ &+ \mathcal{O}(\rho^7) \end{aligned}$$

$$\begin{aligned} iii)\ H^2 = \frac{4}{\rho^2} &- \frac{4}{3}\mathrm{Ric}(\Theta,\Theta) + g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho \\ &+ \Bigg[\frac{2}{5} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) + \frac{16}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau) \\ &- \frac{4}{9} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) + \frac{1}{9}\mathrm{Ric}(\Theta,\Theta)^2 \Bigg] \rho^2 + \mathcal{O}(\rho^3) \end{aligned}$$

$$
\begin{aligned}
iv)\ \sqrt{\det \mathring{g}} = \sin\theta^1 \rho^2 \Bigg[& 1 - \frac{1}{6}\mathrm{Ric}(\Theta,\Theta)\rho^2 + \frac{1}{12} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3 \\
& + \frac{1}{40} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^4 \\
& + \frac{1}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\rho^4 \\
& + \frac{1}{18} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\rho^4 \\
& - \frac{1}{18} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2\rho^4 - \frac{1}{72}\mathrm{Ric}(\Theta,\Theta)^2\rho^4 \Bigg] + \mathcal{O}(\rho^7)
\end{aligned}
$$

where all the inner products and curvatures on the right hand sides are computed at p.

Remark 3.2. It is useful to remark that (3.1), combined with the symmetries of the curvature tensor and Remark 2.2, shows

$$
g(\Theta^\mu\partial_\mu, \partial_{\theta^1}\Theta^\nu\partial_\nu) = g(\Theta^\mu\partial_\mu, \partial_{\theta^2}\Theta^\nu\partial_\nu) = 0 \qquad \text{and} \qquad g(\Theta^\mu\partial_\mu, \Theta^\nu\partial_\nu) = 1
$$

Proof. i) This is achieved by using a variation through geodesics; see [LP87], [Sak96, Chapter 2] or [SY10, Chapter 5] for example (or, using a more general method, shown in [Gra04], Chapter 9).

Consider the vectors $x, W \in T_pM \cong \mathbb{R}^3$. For every $s \in \mathbb{R}$ small enough the curve $\gamma_s(t) := \exp_p(t(x+sW))$ is a geodesic in M starting at p. In normal coordinates it is represented by $\gamma_s(t) = t(x+sW)$, with initial velocity $x+sW$. Varying s, we get a variation through geodesics with variation field $X(t) := \frac{\partial}{\partial s}\gamma_s(t) = tW$. Denoting the tangent vector field to each geodesic by $T := \gamma_s'(t)$, we prove two properties in the lemmas below.

Lemma 3.3 (Jacobi equation)**.** $\nabla_T^2 X := \nabla_T\nabla_T X = \mathcal{R}(T,X)T$

Proof. Since every curve in the variation is a geodesic, we have $\nabla_T T = 0$ by definition. Also, both X and T are the push-forward of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ which means their commutator is zero. Therefore, by the torsion-free property, we have

$$
0 = [T,X] = \nabla_T X - \nabla_X T
$$

Thus

$$
\begin{aligned}
0 = \nabla_X\nabla_T T &= \nabla_T\nabla_X T - \nabla_{[T,X]}T - \mathcal{R}(T,X)T \\
&= \nabla_T\nabla_T X - \mathcal{R}(T,X)T
\end{aligned}
$$

where we have used the definition of the Riemann curvature endomorphism. $\square$

Lemma 3.4.

$$g(\nabla_T^n \mathcal{R}(T,Y)T, Z) = g(Y, \nabla_T^n \mathcal{R}(T,Z)T) \tag{3.2}$$

for any vector fields Y, Z and non-negative integer n.

Proof. We will prove by induction. First, (3.2) is true for $n = 0$ because

$$\begin{aligned} g(\mathcal{R}(T,Y)T, Z) &= \mathrm{Rm}(T,Z,Y,T) \\ &= \mathrm{Rm}(Y,T,T,Z) \\ &= \mathrm{Rm}(T,Y,Z,T) \\ &= g(\mathcal{R}(T,Z)T, Y) \\ &= g(Y, \mathcal{R}(T,Z)T) \end{aligned}$$

where we used the symmetries of the curvature tensor. Now assume that (3.2) is true for $n = k$ and differentiate both sides, yielding

$$\begin{aligned} \nabla_T g(\nabla_T^k \mathcal{R}(T,Y)T, Z) &= g(\nabla_T \left(\nabla_T^k \mathcal{R}(T,Y)T\right), Z) + g(\nabla_T^k \mathcal{R}(T,Y)T, \nabla_T Z) \\ &= g(\nabla_T^{k+1} \mathcal{R}(T,Y)T, Z) + g(\nabla_T^k \mathcal{R}(T, \nabla_T Y)T, Z) \\ &\quad + g(\nabla_T^k \mathcal{R}(T,Y)T, \nabla_T Z) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \nabla_T g(Y, \nabla_T^k \mathcal{R}(T,Z)T) &= g(\nabla_T Y, \nabla_T^k \mathcal{R}(T,Z)T) + g(Y, \nabla_T (\nabla_T^k \mathcal{R}(T,Z)T)) \\ &= g(\nabla_T Y, \nabla_T^k \mathcal{R}(T,Z)T) + g(Y, \nabla_T^{k+1} \mathcal{R}(T,Z)T) \\ &\quad + g(Y, \nabla_T^k \mathcal{R}(T, \nabla_T Z)T) \end{aligned} \tag{3.4}$$

where we used compatibility of ∇ with g, the product rule, and $\nabla_T T = 0$. Equating (3.3) and (3.4), and using the induction assumption to cancel like terms, gives (3.2) for $n = k+1$. Thus, by induction, it is true for all non-negative n. $\square$

Now, consider the function $f(t) := g_{\exp_p(tx)}(X(t), X(t))$. To compute the Taylor expansion for f around 0 we need to compute the derivatives at 0. Using Lemmas 3.3 and 3.4, we compute

$$\nabla_T g(X,X) = 2g(\nabla_T X, X)$$

$$\begin{aligned}
\nabla_T^2 g(X,X) &= 2\nabla_T g(\nabla_T X, X) \\
&= 2g(\mathcal{R}(T,X)T, X) + 2g(\nabla_T X, \nabla_T X)
\end{aligned}$$

$$\begin{aligned}
\nabla_T^3 g(X,X) &= 2\nabla_T g(\mathcal{R}(T,X)T, X) + 2\nabla_T g(\nabla_T X, \nabla_T X) \\
&= 2g(\nabla_T(\mathcal{R}(T,X)T), X) + 2g(\mathcal{R}(T,X)T, \nabla_T X) + 4g(\mathcal{R}(T,X)T, \nabla_T X) \\
&= 2g(\nabla_T \mathcal{R}(T,X)T, X) + 2g(\mathcal{R}(T,\nabla_T X)T, X) + 6g(\mathcal{R}(T,X)T, \nabla_T X) \\
&= 2g(\nabla_T \mathcal{R}(T,X)T, X) + 8g(X, \mathcal{R}(T,\nabla_T X)T)
\end{aligned}$$

$$\begin{aligned}
\nabla_T^4 g(X,X) &= 2\nabla_T g(\nabla_T \mathcal{R}(T,X)T, X) + 8\nabla_T g(\mathcal{R}(T,X)T, \nabla_T X) \\
&= 2g(\nabla_T^2 \mathcal{R}(T,X)T, X) + 2g(\nabla_T \mathcal{R}(T,\nabla_T X)T, X) + 2g(\nabla_T \mathcal{R}(T,X)T, \nabla_T X) \\
&\quad + 8g(\nabla_T \mathcal{R}(T,X)T, \nabla_T X) + 8g(\mathcal{R}(T,\nabla_T X)T, \nabla_T X) \\
&\quad + 8g(\mathcal{R}(T,X)T, \mathcal{R}(T,X)T) \\
&= 12g(X, \nabla_T \mathcal{R}(T,\nabla_T X)T) + 8g(\mathcal{R}(T,\nabla_T X)T, \nabla_T X) \\
&\quad + 8g(\mathcal{R}(T,X)T, \mathcal{R}(T,X)T) + 2g(\nabla_T^2 \mathcal{R}(T,X)T, X)
\end{aligned}$$

$$\begin{aligned}
\nabla_T^5 g(X,X) &= 12\nabla_T g(X, \nabla_T \mathcal{R}(T,\nabla_T X)T) + 8\nabla_T g(\mathcal{R}(T,\nabla_T X)T, \nabla_T X) \\
&\quad + 8\nabla_T g(\mathcal{R}(T,X)T, \mathcal{R}(T,X)T) + 2\nabla_T g(\nabla_T^2 \mathcal{R}(T,X)T, X) \\
&= 12g(\nabla_T X, \nabla_T \mathcal{R}(T,\nabla_T X)T) + 12g(X, \nabla_T^2 \mathcal{R}(T,\nabla_T X)T) \\
&\quad + 12g(X, \nabla_T \mathcal{R}(T,\nabla_T^2 X)T) + 8g(\nabla_T \mathcal{R}(T,\nabla_T X)T, \nabla_T X) \\
&\quad + 8g(\mathcal{R}(T,\nabla_T^2 X)T, \nabla_T X) + 8g(\mathcal{R}(T,\nabla_T X)T, \nabla_T^2 X) \\
&\quad + 16g(\nabla_T \mathcal{R}(T,X)T, \mathcal{R}(T,X)T) + 16g(\mathcal{R}(T,\nabla_T X)T, \mathcal{R}(T,X)T) \\
&\quad + 2g(\nabla_T^3 \mathcal{R}(T,X)T, X) + 2g(\nabla_T^2 \mathcal{R}(T,\nabla_T X)T, X) + 2g(\nabla_T^2 \mathcal{R}(T,X)T, \nabla_T X) \\
&= 20g(\nabla_T \mathcal{R}(T,\nabla_T X)T, \nabla_T X) + 16g(\nabla_T^2 \mathcal{R}(T,\nabla_T X)T, X) \\
&\quad + 28g(\nabla_T \mathcal{R}(T,X)T, \mathcal{R}(T,X)T) + 32g(\mathcal{R}(T,\nabla_T X)T, \mathcal{R}(T,X)T) \\
&\quad + 2g(\nabla_T^3 \mathcal{R}(T,X)T, X)
\end{aligned}$$

$$
\begin{aligned}
\nabla_T^6 g(X,X) &= 20\nabla_T g(\nabla_T \mathcal{R}(T,\nabla_T X)T,\nabla_T X) + 16\nabla_T g(\nabla_T^2 \mathcal{R}(T,\nabla_T X)T,X) \\
&\quad + 28\nabla_T g(\nabla_T \mathcal{R}(T,X)T,\mathcal{R}(T,X)T) + 32\nabla_T g(\mathcal{R}(T,\nabla_T X)T,\mathcal{R}(T,X)T) \\
&\quad + 2\nabla_T g(\nabla_T^3 \mathcal{R}(T,X)T,X) \\
&= 20g(\nabla_T^2 \mathcal{R}(T,\nabla_T X)T,\nabla_T X) + 20g(\nabla_T \mathcal{R}(T,\nabla_T^2 X)T,\nabla_T X) \\
&\quad + 20g(\nabla_T \mathcal{R}(T,\nabla_T X)T,\nabla_T^2 X) + 16g(\nabla_T^3 \mathcal{R}(T,\nabla_T X)T,X) \\
&\quad + 16g(\nabla_T^2 \mathcal{R}(T,\nabla_T^2 X)T,X) + 16g(\nabla_T^2 \mathcal{R}(T,\nabla_T X)T,\nabla_T X) \\
&\quad + 28g(\nabla_T^2 \mathcal{R}(T,X)T,\mathcal{R}(T,X)T) + 28g(\nabla_T \mathcal{R}(T,\nabla_T X)T,\mathcal{R}(T,X)T) \\
&\quad + 28g(\nabla_T \mathcal{R}(T,X)T,\nabla_T \mathcal{R}(T,X)T) + 28g(\nabla_T \mathcal{R}(T,X)T,\mathcal{R}(T,\nabla_T X)T) \\
&\quad + 32g(\nabla_T \mathcal{R}(T,\nabla_T X)T,\mathcal{R}(T,X)T) + 32g(\mathcal{R}(T,\nabla_T^2 X)T,\mathcal{R}(T,X)T) \\
&\quad + 32g(\mathcal{R}(T,\nabla_T X)T,\nabla_T \mathcal{R}(T,X)T) + 32g(\mathcal{R}(T,\nabla_T X)T,\mathcal{R}(T,\nabla_T X)T) \\
&\quad + 2g(\nabla_T^4 \mathcal{R}(T,X)T,X) + 2g(\nabla_T^3 \mathcal{R}(T,\nabla_T X)T,X) + 2g(\nabla_T^3 \mathcal{R}(T,X)T,\nabla_T X) \\
&= 36g(\nabla_T^2 \mathcal{R}(T,\nabla_T X)T,\nabla_T X) + 100g(\nabla_T \mathcal{R}(T,\nabla_T X)T,\mathcal{R}(T,X)T) \\
&\quad + 20g(\nabla_T^3 \mathcal{R}(T,\nabla_T X)T,X) + 44g(\nabla_T^2 \mathcal{R}(T,X)T,\mathcal{R}(T,X)T) \\
&\quad + 28g(\nabla_T \mathcal{R}(T,X)T,\nabla_T \mathcal{R}(T,X)T) + 60g(\nabla_T \mathcal{R}(T,X)T,\mathcal{R}(T,\nabla_T X)T) \\
&\quad + 32g(\mathcal{R}(T,\nabla_T^2 X)T,\mathcal{R}(T,X)T) + 32g(\mathcal{R}(T,\nabla_T X)T,\mathcal{R}(T,\nabla_T X)T) \\
&\quad + 2g(\nabla_T^4 \mathcal{R}(T,X)T,X)
\end{aligned}
$$

where we have written most terms in a form which makes the next simplification as easy as possible. Thus, evaluating at $t=0$, we have $X(0)=0$ and $\nabla_T X(0)=W$, which means any term $g_p(\cdot,\cdot)$ with one of the entries being X or $\nabla_T^n \mathcal{R}(T,X)T$, will be zero (the latter case is true after applying Lemma 3.4). This yields

$$
\begin{aligned}
&\nabla_T g_p(X,X)|_0 = 0 \\
&\nabla_T^2 g_p(X,X)|_0 = 2g_p(W,W) \\
&\nabla_T^3 g_p(X,X)|_0 = 0 \\
&\nabla_T^4 g_p(X,X)|_0 = 8g_p(\mathcal{R}(T(0),W)T(0),W) \\
&\nabla_T^5 g_p(X,X)|_0 = 20g_p(\nabla_{T(0)} \mathcal{R}(T(0),W)T(0),W) \\
&\nabla_T^6 g_p(X,X)|_0 = 36g_p(\nabla_{T(0)}^2 \mathcal{R}(T(0),W)T(0),W) + 32g_p(\mathcal{R}(T(0),W)T(0),\mathcal{R}(T(0),W)T(0))
\end{aligned}
\tag{3.5}
$$

We could continue this process to get more terms. On the one hand we have, for a point near 0

$$f(t) = g_{\exp_p(tx)}(X(t), X(t)) = g_{\exp_p(tx)}(W,W)t^2$$

and on the other, using (3.5), we have the Taylor expansion

$$\begin{aligned} f(t) = {} & g_p(W,W)t^2 + \frac{1}{3} g_p(\mathcal{R}(T(0),W)T(0),W)t^4 + \frac{1}{6} g_p(\nabla_{T(0)}\mathcal{R}(T(0),W)T(0),W)t^5 \\ & + \frac{1}{20} g_p(\nabla^2_{T(0)}\mathcal{R}(T(0),W)T(0),W)t^6 + \frac{2}{45} g_p(\mathcal{R}(T(0),W)T(0),\mathcal{R}(T(0),W)T(0))t^6 \\ & + \mathcal{O}(t^7) \end{aligned}$$

Putting these together and letting $T(0) = x = \Theta$ and $t = \rho$ we get

$$\begin{aligned} g_{\exp_p(\rho\Theta)}(W,W) = {} & g_p(W,W) + \frac{1}{3} g_p(\mathcal{R}(\Theta,W)\Theta,W)\rho^2 + \frac{1}{6} g_p(\nabla_\Theta\mathcal{R}(\Theta,W)\Theta,W)\rho^3 \\ & + \frac{1}{20} g_p(\nabla^2_\Theta\mathcal{R}(\Theta,W)\Theta,W)\rho^4 + \frac{2}{45} g_p(\mathcal{R}(\Theta,W)\Theta,\mathcal{R}(\Theta,W)\Theta)\rho^4 \\ & + \mathcal{O}(\rho^5) \end{aligned} \tag{3.6}$$

Now, take W to be the normal coordinate vector fields ∂_μ. Applying the polarization identity for vector spaces

$$g(\partial_\mu, \partial_\nu) = \frac{1}{4}(g(\partial_\mu + \partial_\nu, \partial_\mu + \partial_\nu) - g(\partial_\mu - \partial_\nu, \partial_\mu - \partial_\nu))$$

to (3.6), simplifying using linearity, the symmetry of the curvature tensor and recalling that $\partial_\mu(p) = E_\mu$, yields (3.1).

ii) We can use (3.1) to get an expansion for $\mathring{g}$ by recalling that the induced metric is just the ambient metric restricted to the coordinate vector fields Z_i of $S_{p,\rho}$. Plugging (2.6) into g, using (3.1) and $g_p(\Theta_i, \Theta_j) = g^{\mathbb{S}^2}_{ij}$ gives the result.

iii) In order to find an expansion for the mean curvature, first we need one for the second fundamental form h of $S_{p,\rho}$, since then we have $H := h_{ij}\mathring{g}^{ij}$. To do this, we allow ρ to vary in T_pM. Then we have the three coordinate vector fields

$$
\begin{aligned}
Z_0 &= \exp_*(\partial_\rho \rho\Theta) = \Theta^\mu \partial_\mu \\
Z_1 &= \exp_*(\rho\Theta_1) = \rho\partial_{\theta^1}\Theta^\mu \partial_\mu \\
Z_2 &= \exp_*(\rho\Theta_2) = \rho\partial_{\theta^2}\Theta^\mu \partial_\mu
\end{aligned}
$$

Note that we have the following symmetry:

$$
\begin{aligned}
g(\nabla_{Z_i} Z_0, Z_j) &= \partial_i g(Z_0, Z_j) - g(Z_0, \nabla_{Z_i} Z_j) \\
&= -g(Z_0, \nabla_{Z_i} Z_j) \\
&= -g(Z_0, \nabla_{Z_j} Z_i) \\
&= g(\nabla_{Z_j} Z_0, Z_i)
\end{aligned}
$$

where we have used compatibility and symmetry of the connection, and that $g(Z_0, Z_j) = 0$. Using this and the Weingarten equation with inward unit normal $\hat{N} = -Z_0$, we get

$$
\begin{aligned}
h_{ij} = -g(\nabla_{Z_i}\hat{N}, Z_j) = g(\nabla_{Z_i} Z_0, Z_j) &= \frac{1}{2}(g(\nabla_{Z_i} Z_0, Z_j) + g(Z_i, \nabla_{Z_j} Z_0)) \\
&= \frac{1}{2}(g(\nabla_{Z_0} Z_i, Z_j) + g(Z_i, \nabla_{Z_0} Z_j)) \\
&= \frac{1}{2} Z_0(g(Z_i, Z_j)) \\
&= \frac{1}{2}\partial_\rho \mathring{g}_{ij}
\end{aligned}
$$

Differentiating the expansion for $\mathring{g}_{ij}$ above gives

$$
\begin{aligned}
h_{ij} = g_{ij}^{\mathbb{S}^2}\rho &+ \frac{2}{3} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3 + \frac{5}{12} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^4 \\
&+ \frac{3}{20} g(\nabla_\Theta^2 \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^5 + \frac{2}{15} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\rho^5 \\
&+ \mathcal{O}(\rho^6)
\end{aligned}
$$

Next, to get an expansion for $\mathring{g}^{ij}$ we use the following formula for the inverse of a sum of matrices, where A is invertible:

$$(A+B\rho^2+C\rho^3+D\rho^4)^{-1} = A^{-1} - A^{-1}BA^{-1}\rho^2 - A^{-1}CA^{-1}\rho^3 - A^{-1}DA^{-1}\rho^4 \\ + A^{-1}BA^{-1}BA^{-1}\rho^4 + \mathcal{O}(\rho^5)$$

which is applicable here because A is just the metric of the round sphere

$$A = g_{ij}^{\mathbb{S}^2} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta^1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta^1} \end{bmatrix} := g_{\mathbb{S}^2}^{ij}$$

Ignoring higher order terms, we get

$$\begin{aligned}
\mathring{g}^{ij} &= \mathring{g}_{ij}^{-1} \\
&= \rho^{-2}\Bigg[g_{ij}^{\mathbb{S}^2} + \frac{1}{3} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^2 + \frac{1}{6} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3 \\
&\quad + \frac{1}{20} g(\nabla_\Theta^2 \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^4 + \frac{2}{45} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\rho^4 \Bigg]^{-1} \\
&= g_{\mathbb{S}^2}^{ij}\rho^{-2} - \frac{1}{3} g_{\mathbb{S}^2}^{il} g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) g_{\mathbb{S}^2}^{kj} - \frac{1}{6} g_{\mathbb{S}^2}^{il} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) g_{\mathbb{S}^2}^{kj}\rho \\
&\quad - \frac{1}{20} g_{\mathbb{S}^2}^{il} g(\nabla_\Theta^2 \mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) g_{\mathbb{S}^2}^{kj}\rho^2 \\
&\quad - \frac{2}{45} g_{\mathbb{S}^2}^{il} g(\mathcal{R}(\Theta,\Theta_l)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_k)\Theta,E_\tau) g_{\mathbb{S}^2}^{kj}\rho^2 \\
&\quad + \frac{1}{9} g_{\mathbb{S}^2}^{il} g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) g_{\mathbb{S}^2}^{kn} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m) g_{\mathbb{S}^2}^{mj}\rho^2 + \mathcal{O}(\rho^3)
\end{aligned}$$

Now we can finally obtain an expansion for H by multiplying the expansions of h_{ij} and $\mathring{g}^{ij}$ and carefully collecting terms, remembering that $g_{ij}^{\mathbb{S}^2}$ and $g_{\mathbb{S}^2}^{ij}$ are inverses and so in particular $g_{ij}^{\mathbb{S}^2} g_{\mathbb{S}^2}^{ij} = \delta_i^i = 2$.

$$\begin{aligned}
H &= h_{ij}\mathring{g}^{ij} \\
&= 2\rho^{-1} - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)\rho + \frac{1}{4} g_{\mathbb{S}^2}^{ij} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^2 \\
&\quad + \Bigg[\frac{1}{10} g_{\mathbb{S}^2}^{ij} g(\nabla_\Theta^2 \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) + \frac{4}{45} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau) \\
&\quad - \frac{1}{9} g_{\mathbb{S}^2}^{il} g_{\mathbb{S}^2}^{kj} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) \Bigg]\rho^3 + \mathcal{O}(\rho^4)
\end{aligned}$$

where the second term arises because $\Theta, \Theta_1, \bar{\Theta}_2$ are in fact an orthonormal basis of T_pM (where the bar indicates that Θ_2 has been normalized by dividing by $\sin\theta^1$), and so we have

$$\begin{aligned}
\frac{1}{3} g_{\mathbb{S}^2}^{ij} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) &= \frac{1}{3}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + \frac{1}{\sin^2\theta^1} g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\Big] \\
&= \frac{1}{3}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\Big] \\
&= -\frac{1}{3}\mathrm{Ric}(\Theta,\Theta)
\end{aligned}$$

by definition of the Ricci curvature at p and the fact that $g(\mathcal{R}(\Theta,\Theta)\Theta,\Theta) = \mathrm{Rm}(\Theta,\Theta,\Theta,\Theta) = 0$ by anti-symmetry. Note that the minus sign is just a result of the convention used when defining the curvature tensor. The result is now obtained by squaring the expansion for H.

iv) Since $\mathring{g}$ can be viewed as a 2×2 matrix, its determinant can be computed by writing out the components and using the standard formula $\mathring{g}_{11}\,\mathring{g}_{22} - \mathring{g}_{12}\,\mathring{g}_{21}$.

$$\begin{aligned}
\det\mathring{g} = \sin^2\theta^1\rho^4 &+ \frac{1}{3}g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\rho^6 + \frac{\sin^2\theta^1}{3}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)\rho^6 \\
&+ \frac{1}{6}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\rho^7 + \frac{\sin^2\theta^1}{6}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)\rho^7 \\
&+ \frac{1}{20}g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\rho^8 + \frac{\sin^2\theta^1}{20}g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)\rho^8 \\
&+ \frac{2}{45}g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau)\rho^8 \\
&+ \frac{2\sin^2\theta^1}{45}g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau)\rho^8 \\
&+ \frac{1}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\rho^8 \\
&- \frac{1}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_1)\rho^8 + \mathcal{O}(\rho^9)
\end{aligned}$$

where we have used the fact that $g^{\mathbb{S}^2}_{12} = g^{\mathbb{S}^2}_{21} = 0$ which means most of the terms in the $\mathring{g}_{12}\,\mathring{g}_{21}$ part are zero. We can rewrite this in a nicer way by using some of the notations introduced above.

$$\det \mathring{g} = \sin^2\theta^1\rho^4\Bigg[1 - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)\rho^2 + \frac{1}{6}g^{ij}_{\mathbb{S}^2}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3$$
$$+\frac{1}{20}g^{ij}_{\mathbb{S}^2}g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^4$$
$$+\frac{2}{45}g^{ij}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\rho^4$$
$$+\frac{1}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\rho^4$$
$$-\frac{1}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2\rho^4\Bigg] + \mathcal{O}(\rho^9) \tag{3.7}$$

This also helps the final step because we can easily use the following Taylor expansion around zero:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$$

with $1+x$ replaced by the term inside the brackets in (3.7). This gives the result.

□

3.2 The Hawking Mass in a Geodesic Sphere

In this subsection we again use the methods in [Mon13].

Proposition 3.5. *Let (M,g) be a 3D Riemannian manifold and $p \in M$. Then*

$$m_H(S_{p,\rho}) = \sqrt{\frac{|S_{p,\rho}|_{\mathring{g}}}{(16\pi)^3}}\left(\frac{8\pi}{3}\mathrm{Sc}_p\rho^2 - \left[\frac{4\pi}{27}\mathrm{Sc}_p^2 - \frac{4\pi}{15}\Delta\mathrm{Sc}(p)\right]\rho^4 + \mathcal{O}(\rho^5)\right) \tag{3.8}$$

Proof. Plugging in the expansions from the previous section, we compute the Willmore energy of $S_{p,\rho}$ up to fourth order.

$$
\begin{aligned}
W(S_{p,\rho}) &= \int_{\mathbb{S}^2} H^2 \sqrt{\det \mathring{g}}\, d\theta^1 d\theta^2 \\
&= \int_{\mathbb{S}^2} \Big(4 - 2\mathrm{Ric}(\Theta,\Theta)\rho^2 + \frac{4}{3} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3 \\
&+ \Big[\frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) \\
&+ \frac{4}{9} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau) \\
&- \frac{4}{9} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) \\
&+ \frac{2}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) \\
&- \frac{2}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2 + \frac{5}{18}\mathrm{Ric}(\Theta,\Theta)^2 \Big] \rho^4 \Big) dV_{g_{\mathbb{S}^2}}
\end{aligned}
$$

Firstly we notice that the second and third ρ^4 terms cancel as we can take E_τ to be the orthonormal basis $\Theta, \Theta_1, \bar{\Theta}_2$. Similarly, the fourth and fifth ρ^4 can also be simplified by using the following identities:

$$
\begin{aligned}
g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) &= -(\mathrm{Ric}(\Theta_1,\Theta_1) + g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)) \\
g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) &= -(\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) + g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)) \\
g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2) &= -\mathrm{Ric}(\Theta_1,\bar{\Theta}_2) \\
\mathrm{Ric}(\Theta_1,\Theta_1) + \mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) &= \mathrm{Sc}_p - \mathrm{Ric}(\Theta,\Theta) \\
g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2) &= -\frac{1}{2}\mathrm{Sc}_p + \mathrm{Ric}(\Theta,\Theta)
\end{aligned}
\tag{3.9}
$$

where Sc_p is the scalar curvature at p. Applying these, we get

$$
\begin{aligned}
W(S_{p,\rho}) = \int_{\mathbb{S}^2} \Big(& 4 - 2\mathrm{Ric}(\Theta,\Theta)\rho^2 + \frac{4}{3} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3 \\
&+ \Big[\frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) \\
&+ \frac{1}{9}\mathrm{Sc}_p \mathrm{Ric}(\Theta,\Theta) - \frac{1}{18}\mathrm{Sc}_p^2 + \frac{5}{18}\mathrm{Ric}(\Theta,\Theta)^2 \\
&- \frac{2}{9}(\mathrm{Ric}(\Theta_1,\bar{\Theta}_2)^2 - \mathrm{Ric}(\Theta_1,\Theta_1)\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)) \Big] \rho^4 \Big) dV_{g_{\mathbb{S}^2}}
\end{aligned}
\tag{3.10}
$$

This is simpler because we have a strategy for integrating all the terms in the integrand, which we will now describe, starting with the most complicated, ρ^4 term. Recall that, at p, we have Θ, Θ_1, $\bar{\Theta}_2 \in T_pM$, where

$$
\begin{aligned}
\Theta &= (\sin\theta^1\cos\theta^2, \sin\theta^1\sin\theta^2, \cos\theta^1)\\
\Theta_1 &= (\cos\theta^1\cos\theta^2, \cos\theta^1\sin\theta^2, -\sin\theta^1)\\
\bar{\Theta}_2 &= (-\sin\theta^2, \cos\theta^2, 0)
\end{aligned}
$$

We can instead write them using the normal coordinates x, y, z as

$$
\begin{aligned}
\Theta &= (x, y, z)\\
\Theta_1 &= \left(\frac{xz}{\sqrt{x^2+y^2}}, \frac{yz}{\sqrt{x^2+y^2}}, -\sqrt{x^2+y^2}\right)\\
\bar{\Theta}_2 &= \left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\right)
\end{aligned}
$$

We can use these to integrate the fifth ρ^4 term because all the components of the Ricci tensor are evaluated at p, so they can be pulled out of the integral, leaving polynomials in x, y and z which we can integrate. Expanding the Ricci terms in these coordinates gives

$$
\begin{aligned}
\mathrm{Ric}(\Theta_1, \bar{\Theta}_2)^2 = \Big(&-R_{11}\frac{xyz}{x^2+y^2} + R_{12}\frac{x^2z}{x^2+y^2} - R_{21}\frac{y^2z}{x^2+y^2}\\
&+ R_{22}\frac{xyz}{x^2+y^2} + R_{31}y - R_{32}x\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Ric}(\Theta_1, \Theta_1) = \Big(&R_{11}\frac{x^2z^2}{x^2+y^2} + 2R_{12}\frac{xyz^2}{x^2+y^2} - 2R_{13}xz + R_{22}\frac{y^2z^2}{x^2+y^2}\\
&- 2R_{23}yz + R_{33}(x^2+y^2)\Big)
\end{aligned}
$$

$$
\mathrm{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) = \Big(R_{11}\frac{y^2}{x^2+y^2} - 2R_{12}\frac{xy}{x^2+y^2} + R_{22}\frac{x^2}{x^2+y^2}\Big)
$$

When we combine these terms to make the one we have in the integrand, we can ignore any term which contains an odd power of x, y or z. This is because any such term will have zero integral over $\mathbb{S}^2$. For example, consider $x^3z^2y^4$: the integral over the half of the sphere where $x > 0$ will cancel out the integral over the half where $x < 0$. It is also useful to note the integrals below, using index notation for x, y and z:

$$\begin{aligned}\int_{\mathbb{S}^2} (x^\mu)^2 \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{3} \\ \int_{\mathbb{S}^2} (x^\mu)^2 (x^\nu)^2 \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{15} \qquad \mu \neq \nu \\ \int_{\mathbb{S}^2} (x^\mu)^4 \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{5}\end{aligned} \tag{3.11}$$

Applying these simplifications yields

$$\begin{aligned}\mathrm{Ric}(\Theta_1, \bar{\Theta}_2)^2 - \mathrm{Ric}(\Theta_1, \Theta_1)\mathrm{Ric}(\bar{\Theta}_2, \bar{\Theta}_2) &= R_{12}^2 z^2 + R_{13}^2 y^2 + R_{23}^2 x^2 - R_{11} R_{22} z^2 \\ &\quad - R_{11} R_{33} y^2 - R_{22} R_{33} x^2\end{aligned}$$

and the integral of the fifth ρ^4 term becomes

$$\begin{aligned}&\int_{\mathbb{S}^2} -\frac{2}{9} \left(R_{12}^2 z^2 + R_{13}^2 y^2 + R_{23}^2 x^2 - R_{11} R_{22} z^2 - R_{11} R_{33} y^2 - R_{22} R_{33} x^2 \right) dV_{g_{\mathbb{S}^2}} \\ &= -\frac{8\pi}{27} \left(R_{12}^2 + R_{13}^2 + R_{23}^2 - R_{11} R_{22} - R_{11} R_{33} - R_{22} R_{33} \right) \\ &= -\frac{4\pi}{27} \left(\|\mathrm{Ric}\|^2 - \mathrm{Sc}_p^2 \right)\end{aligned}$$

Similar methods allow us to compute the second, third and fourth ρ^4 terms as

$$\begin{aligned}\int_{\mathbb{S}^2} \frac{1}{9} \mathrm{Sc}_p \mathrm{Ric}(\Theta, \Theta) \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{27} \mathrm{Sc}_p^2 \\ \int_{\mathbb{S}^2} -\frac{1}{18} \mathrm{Sc}_p^2 \, dV_{g_{\mathbb{S}^2}} &= -\frac{2\pi}{9} \mathrm{Sc}_p^2 \\ \int_{\mathbb{S}^2} \frac{5}{18} \mathrm{Ric}(\Theta, \Theta)^2 \, dV_{g_{\mathbb{S}^2}} &= \frac{2\pi}{27} (2\|\mathrm{Ric}\|^2 + \mathrm{Sc}_p^2)\end{aligned}$$

To rewrite the final ρ^4 term, first note that since we are using normal coordinates, we have $\Gamma_{ij}^k(p) = 0$ and therefore $\nabla_\Theta \Theta_l = (\Theta(\Theta_l^k) + \Theta^i \Theta_l^j \Gamma_{ij}^k)\partial_k = 0$ at p. Here we have used the radially constant extensions of Θ_l in order to compute the derivative and the fact that Θ is the velocity of a radial geodesic as in the proof of (3.1). Thus, by compatibility of the metric

$$
\begin{aligned}
\frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) &= \frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta\big(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta\big),\Theta_j) \\
&= \frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta\big(\nabla_\Theta\big(\mathcal{R}(\Theta,\Theta_i)\Theta\big)\big),\Theta_j) \\
&= \frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta\big(\mathcal{R}(\Theta,\Theta_i)\Theta\big),\Theta_j) \\
&= \frac{1}{2} g^{ij}_{\mathbb{S}^2} \nabla^2_\Theta g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) \\
&= \frac{1}{2}\Big[\nabla^2_\Theta g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + \sin^{-2}\theta^1 \nabla^2_\Theta g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\Big] \\
&= \frac{1}{2}\Big[\nabla^2_\Theta g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + \nabla^2_\Theta\big(\sin^{-2}\theta^1 g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\big) \\
&\quad - \nabla^2_\Theta\big(\sin^{-2}\theta^1\big) g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2) \\
&\quad - 2\nabla_\Theta\big(\sin^{-2}\theta^1\big)\nabla_\Theta\big(g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\big)\Big] \\
&= \frac{1}{2}\Big[\nabla^2_\Theta g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + \nabla^2_\Theta\big(\sin^{-2}\theta^1 g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\big)\Big] \\
&= \frac{1}{2}\nabla^2_\Theta\big(g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\big) \\
&= -\frac{1}{2}\nabla^2_\Theta\big(\mathrm{Ric}(\Theta,\Theta)\big) \\
&= -\frac{1}{2}\nabla_\Theta\big(\nabla_\Theta\big(\mathrm{Ric}(\Theta,\Theta)\big)\big) \\
&= -\frac{1}{2}\nabla_\Theta\big(\nabla_\Theta \mathrm{Ric}(\Theta,\Theta)\big) \\
&= -\frac{1}{2}\nabla^2_\Theta \mathrm{Ric}(\Theta,\Theta) \\
&= -\frac{1}{2}\big[\nabla^2 \mathrm{Ric}(\Theta,\Theta,\Theta,\Theta) + \nabla_{\nabla_\Theta\Theta}\mathrm{Ric}(\Theta,\Theta)\big] \\
&= -\frac{1}{2}\nabla^2 \mathrm{Ric}(\Theta,\Theta,\Theta,\Theta)
\end{aligned}
$$

where we have repeatedly used the product rule and, in the seventh line, the fact that

$$\nabla_\Theta\big(\sin^{-2}\theta^1\big) = -2\sin^{-3}\theta^1 \nabla_\Theta\big(\sin\theta^1\big) = 0$$

which is true because

$$\nabla_\Theta \sin\theta^1 = -\nabla_\Theta g_p(\Theta_1,E_3) = -g_p(\nabla_\Theta\Theta_1,E_3) - g_p(\Theta_1,\nabla_\Theta E_3) = 0$$

which follows since E_3 is a parallel vector field and using $\nabla_\Theta\Theta_i = 0$ at p. Now we can integrate, using index notation with $\Theta = x^\mu E_\mu$ and recalling that the components of the tensor $\nabla^2\mathrm{Ric}$ are evaluated at p.

$$
\begin{aligned}
\int_{\mathbb{S}^2} -\frac{1}{2}\nabla^2 \mathrm{Ric}(\Theta,\Theta,\Theta,\Theta)\, dV_{g_{\mathbb{S}^2}} &= -\frac{1}{2}\int_{\mathbb{S}^2} \nabla_\mu \nabla_\nu R_{\sigma\tau} x^\mu x^\nu x^\sigma x^\tau \, dV_{g_{\mathbb{S}^2}} \\
&= -\nabla_\mu \nabla_\nu R_{\sigma\tau} \frac{1}{2}\int_{\mathbb{S}^2} x^\mu x^\nu x^\sigma x^\tau \, dV_{g_{\mathbb{S}^2}} \\
&= -\frac{4\pi}{30}\Big(3\sum_\mu \nabla_\mu \nabla_\mu R_{\mu\mu} + \sum_{\mu\neq\nu} \nabla_\mu \nabla_\mu R_{\nu\nu} \\
&\quad + \sum_{\mu\neq\nu} \nabla_\mu \nabla_\nu R_{\mu\nu} + \sum_{\mu\neq\nu} \nabla_\mu \nabla_\nu R_{\nu\mu}\Big) \\
&= -\frac{2\pi}{15}\Big(\nabla_\mu \nabla_\mu R_{\nu\nu} + \nabla_\mu \nabla_\nu R_{\mu\nu} + \nabla_\mu \nabla_\nu R_{\nu\mu}\Big) \\
&= -\frac{2\pi}{15}\Big(\nabla_\mu \nabla_\mu R_{\nu\nu} + 2\nabla_\mu \nabla_\nu R_{\mu\nu}\Big) \\
&= -\frac{4\pi}{15}\Delta \mathrm{Sc}(p)
\end{aligned}
$$

where in the last line we have used the definition of the Laplacian, scalar curvature Sc, and the Contracted Bianchi Identity. Similarly, the ρ^3 term can be written as

$$
\frac{4}{3} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) = -\frac{4}{3}\nabla_\Theta \mathrm{Ric}(\Theta,\Theta) = -\frac{4}{3}\nabla \mathrm{Ric}(\Theta,\Theta,\Theta)
$$

which, by linearity of $\nabla\mathrm{Ric}$, changes sign if we replace Θ with $-\Theta$ and so it integrates to 0 over $\mathbb{S}^2$. The remaining terms are computed using the same method as above. Collecting everything together, we finally get

$$
W(S_{p,\rho}) = 16\pi - \frac{8\pi}{3}\mathrm{Sc}_p \rho^2 + \left(\frac{4\pi}{27}\mathrm{Sc}_p^2 - \frac{4\pi}{15}\Delta \mathrm{Sc}(p)\right)\rho^4 + \mathcal{O}(\rho^5) \tag{3.12}
$$

which in turn gives the desired expansion for the Hawking mass. □

4 Perturbed Geodesic Spheres - Calculations

In this section we use many of the methods found in [Mon13; PX09].

4.1 Normal Coordinate Expansions of Perturbed Geometric Quantities

Recall that the Taylor expansions we found in Lemma 3.1 were for the *unperturbed* geodesic spheres (i.e. $w = 0$). Now we turn to the perturbed case. Once w is introduced to the parametrisation, the expressions naturally become a little more complicated, but the same methods work. Note that throughout this section we work in an arbitrary 3D Riemannian manifold.

Lemma 4.1. *Let (M^3, g) be a 3D Riemannian manifold and $p \in M$. The following expansions hold in normal coordinates at p:*

$$
\begin{aligned}
i)\ g_{\mu\nu} = {} & \delta_{\mu\nu} + \frac{1}{3}g(\mathcal{R}(\Theta, E_\mu)\Theta, E_\nu)(1-w)^2\rho^2 + \frac{1}{6}g(\nabla_\Theta\mathcal{R}(\Theta, E_\mu)\Theta, E_\nu)(1-w)^3\rho^3 \\
& + \frac{1}{20}g(\nabla^2_\Theta\mathcal{R}(\Theta, E_\mu)\Theta, E_\nu)(1-w)^4\rho^4 \\
& + \frac{2}{45}g(\mathcal{R}(\Theta, E_\mu)\Theta, E_\tau)g(\mathcal{R}(\Theta, E_\nu)\Theta, E_\tau)(1-w)^4\rho^4 + \mathcal{O}(\rho^5) + \rho^5\mathcal{L}_p^{(0)}(w) \\
& + \rho^5\mathcal{Q}_p^{(2)(0)}(w)
\end{aligned}
$$

$$
\begin{aligned}
ii)\ \mathring{g}_{ij} = {} & g_{ij}^{\mathbb{S}^2}(1-w)^2\rho^2 + w_i w_j \rho^2 + \frac{1}{3}g(\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^4\rho^4 \\
& + \frac{1}{6}g(\nabla_\Theta\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^5\rho^5 + \frac{1}{20}g(\nabla^2_\Theta\mathcal{R}(\Theta, \Theta_i)\Theta, \Theta_j)(1-w)^6\rho^6 \\
& + \frac{2}{45}g(\mathcal{R}(\Theta, \Theta_i)\Theta, E_\tau)g(\mathcal{R}(\Theta, \Theta_j)\Theta, E_\tau)(1-w)^6\rho^6 + \mathcal{O}(\rho^7) + \rho^7\mathcal{L}_p^{(0)}(w) \\
& + \rho^7\mathcal{Q}_p^{(2)(0)}(w)
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
iii)\ \mathring{g}^{ij} = {} & g_{\mathbb{S}^2}^{ij}(1-w)^{-2}\rho^{-2} - g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}w_l w_k(1-w)^{-4}\rho^{-2} - \frac{1}{3}g_{\mathbb{S}^2}^{il}g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kj} \\
& - \frac{1}{6}g_{\mathbb{S}^2}^{il}g(\nabla_\Theta\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kj}(1-w)\rho - \frac{1}{20}g_{\mathbb{S}^2}^{il}g(\nabla^2_\Theta\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kj}(1-w)^2\rho^2 \\
& - \frac{2}{45}g_{\mathbb{S}^2}^{il}g(\mathcal{R}(\Theta, \Theta_l)\Theta, E_\tau)g(\mathcal{R}(\Theta, \Theta_k)\Theta, E_\tau)g_{\mathbb{S}^2}^{kj}(1-w)^2\rho^2 \\
& + \frac{1}{9}g_{\mathbb{S}^2}^{il}g(\mathcal{R}(\Theta, \Theta_l)\Theta, \Theta_k)g_{\mathbb{S}^2}^{kn}g(\mathcal{R}(\Theta, \Theta_n)\Theta, \Theta_m)g_{\mathbb{S}^2}^{mj}(1-w)^2\rho^2 + \mathcal{O}(\rho^3) \\
& + \rho^3\mathcal{L}_p^{(0)}(w) + \rho^2\mathcal{Q}_p^{(2)(0)}(w) + \rho^{-2}\mathcal{Q}_p^{(4)(1)}(w)
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
iv)\ h_{ij} &= g^{\mathbb{S}^2}_{ij}(1-w)\rho + (\nabla^2_{\mathbb{S}^2} w)_{ij}\rho + w_k g^{kl}_{\mathbb{S}^2}\Big(g^{\mathbb{S}^2}_{jl} w_i + g^{\mathbb{S}^2}_{il} w_j - g^{\mathbb{S}^2}_{ij} w_l\Big)\rho + \frac{1}{2} g^{\mathbb{S}^2}_{ij} g^{kl}_{\mathbb{S}^2} w_k w_l \rho \\
&+ \frac{2}{3} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^3\rho^3 \\
&+ \frac{1}{6} w_k g^{kn}_{\mathbb{S}^2} g^{ml}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\Big(\partial_i g^{\mathbb{S}^2}_{jl} + \partial_j g^{\mathbb{S}^2}_{il} - \partial_l g^{\mathbb{S}^2}_{ij}\Big)\rho^3 \\
&- \frac{1}{6} w_k g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l) + \partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l) - \partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big)\rho^3 \\
&+ \frac{5}{12} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^4\rho^4 + \frac{3}{20} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^5\rho^5 \\
&+ \frac{2}{15} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^5\rho^5 + \mathcal{O}(\rho^6) + \rho^4 \mathcal{L}^{(1)}_p(w) \\
&+ \rho\, \mathcal{Q}^{(3)(2)}_p(w) + \rho^3 \mathcal{Q}^{(2)(1)}_p(w)
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
v)\ H &= 2\rho^{-1} + (2+\Delta_{\mathbb{S}^2}) w\rho^{-1} + 2w(w+\Delta_{\mathbb{S}^2} w)\rho^{-1} \\
&+ \frac{1}{6} w_k g^{ij}_{\mathbb{S}^2} g^{kn}_{\mathbb{S}^2} g^{ml}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\Big(\partial_i g^{\mathbb{S}^2}_{jl} + \partial_j g^{\mathbb{S}^2}_{il} - \partial_l g^{\mathbb{S}^2}_{ij}\Big)\rho \\
&- \frac{1}{6} w_k g^{ij}_{\mathbb{S}^2} g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l) + \partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l) - \partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big)\rho \\
&- \frac{1}{3} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla^2_{\mathbb{S}^2} w)_{ij}\rho - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)(1-w)\rho \\
&+ \frac{1}{4} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^2\rho^2 \\
&+ \Bigg[\frac{1}{10} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) + \frac{4}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau) \\
&- \frac{1}{9} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)\Bigg](1-w)^3\rho^3 + \mathcal{O}(\rho^4) \\
&+ \rho^2 \mathcal{L}^{(1)}_p(w) + \rho\, \mathcal{Q}^{(2)(1)}_p(w) + \rho^{-1} \mathcal{Q}^{(3)(2)}_p(w)
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
vi)\ \det \mathring{g} &= \sin^2\theta^1 \rho^4 \Bigg[(1-w)^4 + g^{ij}_{\mathbb{S}^2} w_i w_j - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)(1-w)^6\rho^2 \\
&+ \frac{1}{6} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^7\rho^3 \\
&+ \frac{1}{20} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^8\rho^4 \\
&+ \frac{2}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^8\rho^4 \\
&+ \frac{1}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(1-w)^8\rho^4 \\
&- \frac{1}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2(1-w)^8\rho^4\Bigg] + \mathcal{O}(\rho^9) \\
&+ \rho^9 \mathcal{L}^{(0)}_p(w) + \rho^6 \mathcal{Q}^{(2)(1)}_p(w) + \rho^4 \mathcal{Q}^{(4)(1)}_p(w)
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
vii)\ \det h = {} & \sin^2\theta^1(1-w)^2\rho^2 + \sin^2\theta^1\Delta_{\mathbb{S}^2}w(1-w)\rho^2 + \sin^2\theta^1 g^{ij}_{\mathbb{S}^2}w_iw_j\rho^2 \\
& + \Big[(\nabla^2_{\mathbb{S}^2}w)_{11}(\nabla^2_{\mathbb{S}^2}w)_{22} - (\nabla^2_{\mathbb{S}^2}w)^2_{12}\Big]\rho^2 \\
& + \frac{2}{3}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2}w)_{22} + g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{11} \\
& - 2g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{12} - \sin^2\theta^1\mathrm{Ric}(\Theta,\Theta)(1-w)^4\Big]\rho^4 \\
& + \frac{\sin^2\theta^1}{6}\Big[w_k g^{ij}_{\mathbb{S}^2}g^{kn}_{\mathbb{S}^2}g^{ml}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\Big(\partial_i g^{\mathbb{S}^2}_{jl} + \partial_j g^{\mathbb{S}^2}_{il} - \partial_l g^{\mathbb{S}^2}_{ij}\Big) \\
& - w_k g^{ij}_{\mathbb{S}^2}g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l) + \partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l) - \partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big)\Big]\rho^4 \\
& + \sin^2\theta^1\Big[\frac{5}{12}g^{ij}_{\mathbb{S}^2}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^5\rho^5 \\
& + \frac{3}{20}g^{ij}_{\mathbb{S}^2}g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^6\rho^6 \\
& + \frac{2}{15}g^{ij}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_\tau)(1-w)^6\rho^6\Big] \\
& + \frac{4\sin^2\theta^1}{9}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar\Theta_2)\Theta,\bar\Theta_2) - g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar\Theta_2)^2\Big](1-w)^6\rho^6 \\
& + \mathcal{O}(\rho^7) + \rho^5\mathcal{L}^{(2)}_p(w) + \rho^2\mathcal{Q}^{(3)(2)}_p(w) + \rho^4\mathcal{Q}^{(2)(1)}_p(w)
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
viii)\ H^2 - 4D = {} & (\Delta_{\mathbb{S}^2}w)^2\rho^{-2} - \frac{4}{\sin^2\theta^1}(\nabla^2_{\mathbb{S}^2}w)_{11}(\nabla^2_{\mathbb{S}^2}w)_{22}\rho^{-2} \\
& + \frac{4}{\sin^2\theta^1}(\nabla^2_{\mathbb{S}^2}w)^2_{12}\rho^{-2} \\
& + \frac{8}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{12} \\
& - \frac{4}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2}w)_{22} \\
& - \frac{4}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{11} \\
& - \frac{2}{3}\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2}w - \frac{4}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar\Theta_2)\Theta,\bar\Theta_2)\rho^2 \\
& + \frac{4}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar\Theta_2)^2\rho^2 + \frac{1}{9}\mathrm{Ric}(\Theta,\Theta)^2\rho^2 + \mathcal{O}(\rho^3) + \rho\mathcal{L}^{(2)}_p(w) + \mathcal{Q}^{(2)(2)}_p(w) \\
& + \rho^{-2}\mathcal{Q}^{(3)(2)}_p(w)
\end{aligned}
\tag{4.7}
$$

$$
ix)\ \mathrm{Ric}(\hat N,\hat N) = \mathrm{Ric}(\Theta,\Theta) + \mathcal{L}^{(1)}_p(w) + \mathcal{Q}^{(2)(1)}_p(w) \tag{4.8}
$$

where all the inner products and curvatures on the right hand sides are computed at p and $\mathcal{L}$ and $\mathcal{Q}$ are linear and non-linear combinations of w and its derivatives respectively, as defined in Section 2.1.

Proof. i) The expansion is achieved by replacing Θ with $(1-w)\Theta$ in (3.1).

ii) The expression for the induced metric follows by applying g to the coordinate vector fields Z^w_i on $S_{p,\rho}(w)$, found in (2.7), and applying Remark 3.2.

iii) Since the induced metric is still a perturbation of $g_{\mathbb{S}^2}$, we can use the same formula as in Lemma 3.1 to get the required expansion.

iv) First we need an expression for the unit normal vector to $S_{p,\rho}(w)$. To find it, consider

$$\tilde{N} := -\Theta + a^j Z_j^w$$

where we would like to find a^j such that $\tilde{N}$ is orthogonal to both Z_1^w and Z_2^w. Computing

$$\begin{aligned} g(\tilde{N}, Z_i^w) &= g(-\Theta + a^j Z_j^w, Z_i^w) \\ &= -g(\Theta, Z_i^w) + a^j g(Z_j^w, Z_i^w) \\ &= -g(\Theta, \rho((1-w)\Theta_i - w_i\Theta)) + a^j \mathring{g}_{ij} \\ &= \rho w_i + a^j \mathring{g}_{ij} \end{aligned}$$

where we again use that $g(\Theta,\Theta) = 1$ and $g(\Theta,\Theta_i) = 0$. Therefore, to satisfy orthogonality, we need to take a^j such that $a^j \mathring{g}_{ij} = -w_i\rho$, or $a^j = -\mathring{g}^{ij} w_i \rho$. To normalize, we need

$$\begin{aligned} g(\tilde{N}, \tilde{N}) &= g(-\Theta + a^j Z_j^w, -\Theta + a^i Z_i^w) \\ &= g(\Theta,\Theta) - g(\Theta, a^i Z_i^w) - g(\Theta, a^j Z_j^w) + g(a^j Z_j^w, a^i Z_i^w) \\ &= 1 - a^i g(\Theta, \rho((1-w)\Theta_i - w_i\Theta)) - a^j g(\Theta, \rho((1-w)\Theta_j - w_j\Theta)) \\ &\quad + a^i a^j \mathring{g}_{ij} \\ &= 1 + a^i w_i \rho + a^j w_j \rho + a^i a^j \mathring{g}_{ij} \\ &= 1 + a^j w_j \rho \\ &= 1 - \mathring{g}^{ij} w_i w_j \rho^2 \end{aligned}$$

which means we have found the (inward) unit normal vector

$$\hat{N} = (1 - \mathring{g}^{kl} w_k w_l \rho^2)^{-\frac{1}{2}} \Big(-\Theta - \mathring{g}^{ij} w_i ((1-w)\Theta_j - w_j\Theta)\rho^2 \Big)$$

Now we can use the Taylor expansion around 0 of $(1-x)^{-\frac{1}{2}}$ with $x = \mathring{g}^{ij} w_i w_j \rho^2$ to get

$$g(\tilde{N},\tilde{N})^{-\frac{1}{2}} = 1 + \frac{1}{2}\mathring{g}^{ij} w_i w_j \rho^2 + \mathcal{Q}_p^{(4)(1)}(w) \tag{4.9}$$

Together with (4.2), we obtain

$$\begin{aligned}\hat{N} = &-\Theta - g_{\mathbb{S}^2}^{ij} w_i \Theta_j + \rho^2 \mathcal{L}_p^{(1)}(w)\Theta_1 + \rho^2 \mathcal{L}_p^{(1)}(w)\Theta_2 + \mathcal{Q}_p^{(2)(1)}(w)\Theta_1 \\ &+ \mathcal{Q}_p^{(2)(1)}(w)\Theta_2 + \mathcal{Q}_p^{(2)(1)}(w)\Theta\end{aligned} \tag{4.10}$$

We now compute $\tilde{h}_{ij} := -g(\nabla_{Z_i^w}\tilde{N}, Z_j^w)$ as a first step in computing h_{ij}. We have

$$\begin{aligned}\tilde{h}_{ij} &= -g(\nabla_{Z_i^w}(-\Theta + a^k Z_k), Z_j^w) \\ &= g(\nabla_{Z_i^w}\Theta, Z_j^w) - g(\nabla_{Z_i^w} a^k Z_k^w, Z_j^w) \\ &= \frac{w_i}{1-w} g(\Theta, Z_j^w) - \frac{w_i}{1-w} g(\Theta, Z_j^w) + g(\nabla_{Z_i^w}\Theta, Z_j^w) - g(\nabla_{Z_i^w} a^k Z_k^w, Z_j^w) \\ &= \frac{w_i}{1-w} g(\Theta, Z_j^w) + \frac{1}{1-w}[(1-w) g(\nabla_{Z_i^w}\Theta, Z_j^w) - w_i g(\Theta, Z_j^w)] - g(\nabla_{Z_i^w} a^k Z_k^w, Z_j^w) \\ &= \frac{w_i}{1-w} g(\Theta, Z_j^w) + \frac{1}{1-w} g((1-w)\nabla_{Z_i^w}\Theta + Z_i^w(1-w)\Theta, Z_j^w) - g(\nabla_{Z_i^w} a^k Z_k^w, Z_j^w) \\ &= \frac{w_i}{1-w} g(\Theta, Z_j^w) + \frac{1}{1-w} g(\nabla_{Z_i^w}((1-w)\Theta), Z_j^w) - g(\nabla_{Z_i^w} a^k Z_k^w, Z_j^w)\end{aligned} \tag{4.11}$$

Let's compute the three terms in (4.11) seperately. For the first one, we use the definition of Z_i^w and the fact that $g(\Theta,\Theta) = 1$ and $g(\Theta,\Theta_i) = 0$, to yield

$$\frac{w_i}{1-w} g(\Theta, Z_j^w) = \frac{w_i}{1-w} g(\Theta, \rho((1-w)\Theta_j - w_j\Theta)) = -\frac{w_i w_j \rho}{1-w} \tag{4.12}$$

Now we proceed in a similar way to the unperturbed case and consider ρ as a variable, giving

$$Z_0^w = \exp_*(\partial_\rho(\rho(1-w)\Theta)) = (1-w)\Theta$$

Since

$$
\begin{aligned}
g(\nabla_{Z_i^w}((1-w)\Theta), Z_j^w) &= Z_i^w(g((1-w)\Theta, Z_j^w)) - g((1-w)\Theta, \nabla_{Z_i^w} Z_j^w) \\
&= Z_i^w(g((1-w)\Theta, \rho((1-w)\Theta_j - w_j\Theta))) - g((1-w)\Theta, \nabla_{Z_i^w} Z_j^w) \\
&= Z_i^w(\rho(w-1)w_j) - g((1-w)\Theta, \nabla_{Z_i^w} Z_j^w) \\
&= \rho(w_i w_j + w w_{ji} - w_{ji}) - g((1-w)\Theta, \nabla_{Z_i^w} Z_j^w)
\end{aligned}
$$

is symmetric in i and j, we have

$$
g(\nabla_{Z_i^w}((1-w)\Theta), Z_j^w) = g(\nabla_{Z_j^w}((1-w)\Theta), Z_i^w)
$$

Thus, we compute

$$
\begin{aligned}
g(\nabla_{Z_i^w}((1-w)\Theta), Z_j^w) &= \frac{1}{2}(g(\nabla_{Z_i^w}((1-w)\Theta), Z_j^w) + g(\nabla_{Z_j^w}((1-w)\Theta), Z_i^w)) \\
&= \frac{1}{2}(g(\nabla_{Z_i^w} Z_0^w, Z_j^w) + g(\nabla_{Z_j^w} Z_0^w, Z_i^w)) \\
&= \frac{1}{2}(g(\nabla_{Z_0^w} Z_i^w, Z_j^w) + g(\nabla_{Z_0^w} Z_j^w, Z_i^w)) \\
&= \frac{1}{2} Z_0^w(g(Z_i^w, Z_j^w)) \\
&= \frac{1}{2}\partial_\rho \mathring{g}_{ij}
\end{aligned}
\tag{4.13}
$$

which sorts out the second term in (4.11). The final term becomes

$$
\begin{aligned}
g(\nabla_{Z_i^w} a^k Z_k^w, Z_j^w) &= Z_i^w(a^k g(Z_k^w, Z_j^w)) - a^k g(Z_k^w, \nabla_{Z_i^w} Z_j^w) \\
&= Z_i^w(-\mathring{g}^{lk} w_l \rho \mathring{g}_{kj}) + \mathring{g}^{lk} w_l \rho \mathring{\Gamma}^m_{ij} \mathring{g}_{km} \\
&= -w_{ji}\rho + w_l \mathring{\Gamma}^l_{ij}\rho \\
&= -(\nabla^2_{\mathring{g}} w)_{ij}\rho
\end{aligned}
\tag{4.14}
$$

Substituting (4.12), (4.13) and (4.14) into (4.11) gives

$$\tilde{h}_{ij} = -\frac{w_i w_j \rho}{1-w} + \frac{1}{2(1-w)}\partial_\rho \mathring{g}_{ij} + (\nabla^2_{\mathring{g}} w)_{ij}\rho \tag{4.15}$$

To further expand $\tilde{h}_{ij}$, we will first combine the first two terms of (4.15). Differentiating (4.1) with respect to ρ gives

$$\begin{aligned}
\partial_\rho \mathring{g}_{ij} &= 2g^{\mathbb{S}^2}_{ij}(1-w)^2\rho + 2w_i w_j \rho + \frac{4}{3}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^4\rho^3 \\
&+ \frac{5}{6}g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^5\rho^4 + \frac{3}{10}g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^6\rho^5 \\
&+ \frac{4}{15}g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^6\rho^5 + \mathcal{O}(\rho^6) + \rho^6\mathcal{L}^{(0)}_p(w) \\
&+ \rho^6\mathcal{Q}^{(2)(0)}_p(w)
\end{aligned}$$

and so the first term in (4.15) cancels the second term of $\partial_\rho \mathring{g}_{ij}$ after it is multiplied by $\frac{1}{2(1-w)}$. This leaves

$$\begin{aligned}
\tilde{h}_{ij} &= g^{\mathbb{S}^2}_{ij}(1-w)\rho + (\nabla^2_{\mathring{g}} w)_{ij}\rho + \frac{2}{3}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^3\rho^3 \\
&+ \frac{5}{12}g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^4\rho^4 + \frac{3}{20}g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^5\rho^5 \\
&+ \frac{2}{15}g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^5\rho^5 + \mathcal{O}(\rho^6) + \rho^6\mathcal{L}^{(0)}_p(w) \\
&+ \rho^6\mathcal{Q}^{(2)(0)}_p(w)
\end{aligned} \tag{4.16}$$

and therefore we are left to compute the term $(\nabla^2_{\mathring{g}} w)_{ij}$. By definition we have

$$(\nabla^2_{\mathring{g}} w)_{ij} = w_{ij} - \mathring{\Gamma}^k_{ij} w_k = w_{ij} - [\frac{1}{2}\mathring{g}^{kl}(\partial_i \mathring{g}_{jl} + \partial_j \mathring{g}_{il} - \partial_l \mathring{g}_{ij})]w_k \tag{4.17}$$

Differentiating (4.1) term by term shows

$$\begin{aligned}
\partial_i \mathring{g}_{jl} &= \partial_i(g^{\mathbb{S}^2}_{jl})(1-w)^2\rho^2 - 2g^{\mathbb{S}^2}_{jl} w_i \rho^2 + \frac{1}{3}\partial_i(g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l))\rho^4 \\
&+ \mathcal{O}(\rho^5) + \rho^4\mathcal{L}^{(1)}_p(w) + \rho^2\mathcal{Q}^{(2)(2)}_p(w)
\end{aligned} \tag{4.18}$$

Combining (4.2) and (4.18), we get

$$
\begin{aligned}
\mathring{\Gamma}_{ij}^{k} &= \Gamma_{ij}^{k} + g_{\mathbb{S}^2}^{kl}(g_{ij}^{\mathbb{S}^2} w_l - g_{jl}^{\mathbb{S}^2} w_i - g_{il}^{\mathbb{S}^2} w_j) \\
&\quad + \frac{1}{6} g_{\mathbb{S}^2}^{kl}(\partial_i(g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l)) + \partial_j(g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l)) - \partial_l(g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)))\rho^2 \\
&\quad - \frac{1}{6} g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)(\partial_i(g_{jl}^{\mathbb{S}^2}) + \partial_j(g_{il}^{\mathbb{S}^2}) - \partial_l(g_{ij}^{\mathbb{S}^2}))\rho^2 \\
&\quad + \mathcal{O}(\rho^3) + \rho^2 \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(2)}(w)
\end{aligned} \tag{4.19}
$$

where Γ_{ij}^{k} are the Christoffel symbols of $g_{\mathbb{S}^2}$. Substituting (4.17) and (4.19) into (4.16) gives

$$
\begin{aligned}
\tilde{h}_{ij} &= g_{ij}^{\mathbb{S}^2}(1-w)\rho + (\nabla_{\mathbb{S}^2}^2 w)_{ij}\rho + w_k g_{\mathbb{S}^2}^{kl}\left(g_{jl}^{\mathbb{S}^2} w_i + g_{il}^{\mathbb{S}^2} w_j - g_{ij}^{\mathbb{S}^2} w_l\right)\rho \\
&\quad + \frac{2}{3} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^3\rho^3 \\
&\quad + \frac{1}{6} w_k g_{\mathbb{S}^2}^{kn} g_{\mathbb{S}^2}^{ml} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\left(\partial_i g_{jl}^{\mathbb{S}^2} + \partial_j g_{il}^{\mathbb{S}^2} - \partial_l g_{ij}^{\mathbb{S}^2}\right)\rho^3 \\
&\quad - \frac{1}{6} w_k g_{\mathbb{S}^2}^{kl}\left(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l) + \partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l) - \partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\right)\rho^3 \\
&\quad + \frac{5}{12} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^4\rho^4 + \frac{3}{20} g(\nabla_\Theta^2 \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^5\rho^5 \\
&\quad + \frac{2}{15} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^5\rho^5 + \mathcal{O}(\rho^6) + \rho^4 \mathcal{L}_p^{(1)}(w) \\
&\quad + \rho \mathcal{Q}_p^{(3)(2)}(w) + \rho^3 \mathcal{Q}_p^{(2)(1)}(w)
\end{aligned}
$$

Now, to complete the proof, we note that

$$
\begin{aligned}
h_{ij} &:= -g(\nabla_{Z_i^w}\hat{N}, Z_j^w) \\
&= -g(\nabla_{Z_i^w} g(\tilde{N},\tilde{N})^{-\frac{1}{2}}\tilde{N}, Z_j) \\
&= -Z_i^w(g(\tilde{N},\tilde{N})^{-\frac{1}{2}}) g(\tilde{N}, Z_j^w) - g(\tilde{N},\tilde{N})^{-\frac{1}{2}} g(\nabla_{Z_i^w}\tilde{N}, Z_j^w) \\
&= g(\tilde{N},\tilde{N})^{-\frac{1}{2}}\tilde{h}_{ij}
\end{aligned}
$$

where we have used that $\tilde{N}$ is orthogonal to Z_j^w. Using (4.9), we pick up an extra term, obtaining

$$
h_{ij} = \tilde{h}_{ij} + \frac{1}{2} g_{ij}^{\mathbb{S}^2} g_{\mathbb{S}^2}^{kl} w_k w_l \rho
$$

and therefore (4.3) too.

v) Since $H = h_{ij}\mathring{g}^{ij}$, we multiply (4.3) and (4.2) to obtain (4.4), where we have used some simplifications. First, as in the unperturbed case, we have

$$\begin{aligned}\frac{1}{3}g^{ij}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) &= \frac{1}{3}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + \frac{1}{\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\Big]\\ &= \frac{1}{3}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\Big]\\ &= -\frac{1}{3}\mathrm{Ric}(\Theta,\Theta)\end{aligned}$$

because $\Theta, \Theta_1, \bar{\Theta}_2$ are in fact an orthonormal basis of T_pM (where the bar indicates that Θ_2 has been normalized by dividing by $\sin\theta^1$). This simplifies the multiplication of the third $\mathring{g}^{ij}$ term with the first h_{ij} term. Next, we use the Taylor expansions

$$\begin{aligned}(1-w)^{-1} &= 1 + w + w^2 + \mathcal{O}(w^3)\\ (1-w)^{-2} &= 1 + 2w + \mathcal{O}(w^2)\\ (1-w)^{-3} &= 1 + \mathcal{O}(w)\end{aligned}$$

to simplify the multiplication of h_{ij} by the first and second terms of $\mathring{g}^{ij}$ respectively. Here, we also use the definition $\Delta_{\mathbb{S}^2} := g^{ij}_{\mathbb{S}^2}(\nabla^2_{\mathbb{S}^2}w)_{ij}$. Furthermore, the terms containing $w_l w_k$ cancel each other out. Finally, we use the following calculation:

$$g^{ij}_{\mathbb{S}^2}w_k g^{kl}_{\mathbb{S}^2}g^{\mathbb{S}^2}_{ij}w_l - g^{ij}_{\mathbb{S}^2}w_k g^{kl}_{\mathbb{S}^2}g^{\mathbb{S}^2}_{jl}w_i - g^{ij}_{\mathbb{S}^2}w_k g^{kl}_{\mathbb{S}^2}g^{\mathbb{S}^2}_{il}w_j = 2w_k g^{kl}_{\mathbb{S}^2}w_l - w_k g^{ki}_{\mathbb{S}^2}w_i - w_k g^{kj}_{\mathbb{S}^2}w_j = 0$$

which shows that the first $\mathring{g}^{ij}$ term multiplied by the third h_{ij} term is zero.

vi) As in the unperturbed case, we compute $\det\mathring{g} = \mathring{g}_{11}\mathring{g}_{22} - \mathring{g}_{12}\mathring{g}_{21}$ to achieve (4.5). Note that $g_{\mathbb{S}^2}$ and $g^{-1}_{\mathbb{S}^2}$ are diagonal which means many of the $\mathring{g}_{12}\mathring{g}_{21}$ terms are zero. We also use the fact that $g^{\mathbb{S}^2}_{11} = \sin^2\theta^1 g^{22}_{\mathbb{S}^2}$ and $g^{\mathbb{S}^2}_{22} = \sin^2\theta^1 g^{11}_{\mathbb{S}^2}$, so that, for example, the sixth term in (4.5) arises as follows:

$$
\begin{aligned}
&\frac{2\sin^2\theta^1}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^8\rho^8 \\
&= \frac{2\sin^2\theta^1}{45}\Big(g^{11}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau) \\
&\quad + g^{22}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau)\Big)(1-w)^8\rho^8 \\
&= \frac{2}{45}\Big(g^{\mathbb{S}^2}_{22} g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau) \\
&\quad + g^{\mathbb{S}^2}_{11} g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau)\Big)(1-w)^8\rho^8 \\
&= g^{\mathbb{S}^2}_{11}(1-w)^2\rho^2 \times \frac{2}{45} g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_2)\Theta,E_\tau)(1-w)^6\rho^6 \\
&\quad + g^{\mathbb{S}^2}_{22}(1-w)^2\rho^2 \times \frac{2}{45} g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_1)\Theta,E_\tau)(1-w)^6\rho^6
\end{aligned}
$$

vii) Similarly we compute $\det h = h_{11}h_{22} - h_{12}h_{21}$ to prove (4.6), where, for example, the second term results from

$$
\begin{aligned}
\sin^2\theta^1 \Delta_{\mathbb{S}^2} w(1-w)\rho^2 &= \sin^2\theta^1 g^{ij}_{\mathbb{S}^2}(\nabla^2_{\mathbb{S}^2} w)_{ij}(1-w)\rho^2 \\
&= \sin^2\theta^1\Big(g^{11}_{\mathbb{S}^2}(\nabla^2_{\mathbb{S}^2} w)_{11} + g^{22}_{\mathbb{S}^2}(\nabla^2_{\mathbb{S}^2} w)_{22}\Big)(1-w)\rho^2 \\
&= \Big(g^{\mathbb{S}^2}_{22}(\nabla^2_{\mathbb{S}^2} w)_{11} + g^{\mathbb{S}^2}_{11}(\nabla^2_{\mathbb{S}^2} w)_{22}\Big)(1-w)\rho^2 \\
&= g^{\mathbb{S}^2}_{11}(1-w)\rho \times (\nabla^2_{\mathbb{S}^2} w)_{22}\rho \\
&\quad + g^{\mathbb{S}^2}_{22}(1-w)\rho \times (\nabla^2_{\mathbb{S}^2} w)_{11}\rho
\end{aligned}
$$

viii) First we evaluate H^2 by squaring (4.4).

$$
\begin{aligned}
H^2 = &\left[4+4(2+\Delta_{\mathbb{S}^2})w+8w(w+\Delta_{\mathbb{S}^2}w)+((2+\Delta_{\mathbb{S}^2})w)^2\right]\rho^{-2}\\
&+\left[\frac{2}{3}w_k g^{ij}_{\mathbb{S}^2} g^{kn}_{\mathbb{S}^2} g^{ml}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\left(\partial_i g^{\mathbb{S}^2}_{jl}+\partial_j g^{\mathbb{S}^2}_{il}-\partial_l g^{\mathbb{S}^2}_{ij}\right)\right.\\
&-\frac{2}{3}w_k g^{ij}_{\mathbb{S}^2} g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l)+\partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l)-\partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big)\\
&\left.-\frac{2}{3}\mathrm{Ric}(\Theta,\Theta)(2+\Delta_{\mathbb{S}^2}w)-\frac{4}{3}g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla^2_{\mathbb{S}^2}w)_{ij}\right] \qquad (4.20)\\
&+\left[g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\right]\rho\\
&+\left[\frac{2}{5}g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)+\frac{16}{45}g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\right.\\
&\left.-\frac{4}{9}g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)+\frac{1}{9}\mathrm{Ric}(\Theta,\Theta)^2\right]\rho^2\\
&+\mathcal{O}(\rho^3)+\rho\mathcal{L}^{(1)}_p(w)+\mathcal{Q}^{(2)(1)}_p(w)+\rho^{-2}\mathcal{Q}^{(3)(2)}_p(w)
\end{aligned}
$$

Now, to compute $D=\frac{\det h}{\det \mathring{g}}$ we first use the Taylor expansion $(1+x)^{-1}=1-x+x^2+\mathcal{O}(x^3)$ with (4.5), to get

$$
\begin{aligned}
\frac{1}{\det\mathring{g}} = &\frac{1}{\sin^2\theta^1(1-w)^4\rho^4}\left[1-g^{ij}_{\mathbb{S}^2}w_iw_j+\frac{1}{3}\mathrm{Ric}(\Theta,\Theta)(1-w)^2\rho^2\right.\\
&-\frac{1}{6}g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^3\rho^3\\
&-\frac{1}{20}g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^4\rho^4\\
&-\frac{2}{45}g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^4\rho^4 \qquad (4.21)\\
&-\frac{1}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar\Theta_2)\Theta,\bar\Theta_2)(1-w)^4\rho^4\\
&+\frac{1}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar\Theta_2)^2(1-w)^4\rho^4\\
&\left.+\frac{1}{9}\mathrm{Ric}(\Theta,\Theta)^2(1-w)^4\rho^4+\mathcal{O}(\rho^5)+\rho^5\mathcal{L}^{(0)}_p(w)+\rho^2\mathcal{Q}^{(2)(1)}_p(w)+\mathcal{Q}^{(4)(1)}_p(w)\right]
\end{aligned}
$$

Multiplying (4.21) and (4.6), we have

$$
\begin{aligned}
D = \Big[& 1 + 2w + \Delta_{\mathbb{S}^2} w + 3w\Delta_{\mathbb{S}^2} w + 3w^2 \\
&+ \frac{1}{\sin^2\theta^1}((\nabla^2_{\mathbb{S}^2} w)_{11}(\nabla^2_{\mathbb{S}^2} w)_{22} - (\nabla^2_{\mathbb{S}^2} w)^2_{12})\Big]\rho^{-2} \\
&+ \frac{2}{3\sin^2\theta^1}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2} w)_{22} + g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2} w)_{11} \\
&- 2g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2} w)_{12}\Big] + \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)(\Delta_{\mathbb{S}^2} w - 1) \\
&+ \frac{1}{6}\Big[w_k g^{ij}_{\mathbb{S}^2} g^{kn}_{\mathbb{S}^2} g^{ml}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\Big(\partial_i g^{\mathbb{S}^2}_{jl} + \partial_j g^{\mathbb{S}^2}_{il} - \partial_l g^{\mathbb{S}^2}_{ij}\Big) \\
&- w_k g^{ij}_{\mathbb{S}^2} g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l) + \partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l) - \partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big)\Big] \\
&+ \frac{1}{4} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)\rho \\
&+ \Big[\frac{1}{10} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) + \frac{4}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_\tau) \\
&+ \frac{1}{3} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) - \frac{1}{3} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2 \\
&- \frac{1}{9}\mathrm{Ric}(\Theta,\Theta)^2\Big](1-w)^2\rho^2 + \mathcal{O}(\rho^3) + \rho\mathcal{L}^{(2)}_p(w) + \mathcal{Q}^{(2)(2)}_p(w) + \rho^{-2}\mathcal{Q}^{(3)(2)}_p(w)
\end{aligned} \tag{4.22}
$$

Thus, to achieve (4.7), we combine (4.20) and (4.22), noting the following. Firstly, four times the first line of (4.22) subtracted from the first line of (4.20) leaves $(\Delta_{\mathbb{S}^2} w)^2\rho^{-2}$. Also, the second and third lines of (4.20) are cancelled exactly by the fifth and sixth lines of (4.22) (after they are multiplied by 4). Similarly, the ρ terms are cancelled exactly. For the remaining ρ^0 terms, we compute

$$
\begin{aligned}
&-\frac{4}{3} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla^2_{\mathbb{S}^2} w)_{ij} \\
&= -\frac{4}{3}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2} w)_{11} + \frac{2}{\sin^2\theta^1} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2} w)_{12} \\
&\quad + \frac{1}{\sin^2\theta^1} g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(\nabla^2_{\mathbb{S}^2} w)_{22}\Big]
\end{aligned}
$$

Subtracting 4 times the ρ^0 terms from D which contain Hessian components, yields

$$
\begin{aligned}
&-\frac{4}{3}g^{il}_{\mathbb{S}^2}g^{kj}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla^2_{\mathbb{S}^2}w)_{ij}\\
&-\frac{8}{3\sin^2\theta^1}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2}w)_{22}+g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{11}\\
&-2g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{12}\Big]\\
&=\frac{8}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{12}-\frac{8}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2}w)_{22}\\
&\quad-\frac{8}{3}g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(\nabla^2_{\mathbb{S}^2}w)_{11}-\frac{4}{3}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2}w)_{11}\\
&\quad-\frac{4}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(\nabla^2_{\mathbb{S}^2}w)_{22}\\
&=\frac{8}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)(\nabla^2_{\mathbb{S}^2}w)_{12}-\frac{4}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)(\nabla^2_{\mathbb{S}^2}w)_{22}\\
&\quad-\frac{4}{3}g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(\nabla^2_{\mathbb{S}^2}w)_{11}+\frac{4}{3}\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2}w
\end{aligned}
$$

where, to get the final equality, we have used half of the second and third terms, together with the fourth and fifth terms to make $\frac{4}{3}\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2}w$. This is true because

$$
\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2}w=-\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)+g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\Big]\Big[(\nabla^2_{\mathbb{S}^2}w)_{11}+\frac{1}{\sin^2\theta^1}(\nabla^2_{\mathbb{S}^2}w)_{22}\Big]
$$

Now the ρ^0 term is completed by collecting like terms. To get the ρ^2 terms, we compute

$$
\begin{aligned}
&-\frac{4}{9}g^{il}_{\mathbb{S}^2}g^{kj}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)\\
&=-\frac{4}{9}\Big[g^{11}_{\mathbb{S}^2}g^{11}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)^2+g^{11}_{\mathbb{S}^2}g^{22}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)^2+g^{22}_{\mathbb{S}^2}g^{11}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_1)^2\\
&+g^{22}_{\mathbb{S}^2}g^{22}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)^2\Big]\\
&=-\frac{4}{9}\Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)^2+g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2+g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\Theta_1)^2\\
&+g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)^2\Big]\\
&=-\frac{4}{9}\Big[\mathrm{Ric}(\Theta,\Theta)^2-2g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)+2g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2\Big]
\end{aligned}
\tag{4.23}
$$

where we have used that

$$
\begin{aligned}
\operatorname{Ric}(\Theta,\Theta)^2 &= \Big[g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)+g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\Big]^2 \\
&= g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)^2+2g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) \\
&\quad + g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)^2
\end{aligned}
$$

and the symmetries of the curvature tensor, which imply

$$
g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2) = g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\Theta_1)
$$

Using (4.23) we can now easily calculate the ρ^2 terms of (4.7).

ix) Plugging in the expression in (4.10) for the unit normal, we get (4.8).

□

4.2 An Optimal Perturbation

As outlined in Section 1.3.2 and motivated by Proposition 1.26, it is natural to choose perturbed geodesic spheres $S_{p,\rho}(w)$ as the surfaces to "test" the positivity of the Hawking mass. In this section we consider such surfaces at a given point $p \in M$, and find an expansion of the perturbation w, in terms of the radius ρ, satisfied by small $S_{p,\rho}(w)$ which have $w \in C^{4,\alpha}(S^2)^{\perp}$ and solve the area constrained Euler-Lagrange equation for the Willmore functional, assuming they exist. Inspired by this, at any $p \in M$, the spheres $S_{p,\rho}(w)$ satisfying the same expansion will be the key geometric objects in the proof of our main theorems; the *optimal* spheres.

Lemma 4.2. *Let $p \in M$. There exists $\rho_0 > 0$ and $r > 0$ such that if $S_{p,\rho}(w)$ with $w \in C^{4,\alpha}(S^2)^{\perp}$ and $(\rho, w) \in (0,\rho_0] \times B(0,r)$ is a critical point of the Willmore functional under area constraint, then w satisfies the following expansion:*

$$
w = \Big(-\frac{1}{6}\operatorname{Ric}(\Theta,\Theta)+\frac{1}{18}\mathrm{Sc}_p\Big)\rho^2+\mathcal{O}(\rho^3) \tag{4.24}
$$

where $\limsup_{\rho\to 0}\rho^{-3}\left\|\mathcal{O}(\rho^3)\right\|_{C^{4,\alpha}(S^2)} < \infty$.

Proof. **Step 1** - PDE setup.

Fix $p \in M$. Recall that if $S_{p,\rho}(w)$ is a critical point of the Willmore functional under area constraint, then it satisfies the associated PDE (1.17)

$$2\Delta_{\mathring{g}}H + H(H^2 - 4D + 2\mathrm{Ric}(\hat{N},\hat{N})) = \lambda H \tag{4.25}$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier and N is the inward pointing unit normal vector [LMS11]. The geometric expansions of Section 4.1 give us

$$\begin{aligned}
\mathring{g}^{ij} &= g_{\mathbb{S}^2}^{ij}\rho^{-2} + \mathcal{O}(\rho^0) + \rho^{-2}\mathcal{L}_p^{(0)}(w) + \rho^{-2}\mathcal{Q}_p^{(2)(1)}(w) \\
\mathring{\Gamma}_{ij}^k &= \Gamma_{ij}^k + \mathcal{O}(\rho^2) + \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(2)}(w) \\
H &= 2\rho^{-1} + (2+\Delta_{\mathbb{S}^2})w\rho^{-1} - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)\rho + \mathcal{O}(\rho^2) + \rho\mathcal{L}_p^{(2)}(w) \\
&\quad + \rho^{-1}\mathcal{Q}_p^{(2)(2)}(w) \\
H^2 - 4D &= \mathcal{O}(\rho^2) + \mathcal{L}_p^{(2)}(w) + \rho^{-2}\mathcal{Q}_p^{(2)(2)}(w) \\
\mathrm{Ric}(\hat{N},\hat{N}) &= \mathrm{Ric}(\Theta,\Theta) + \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(1)}(w)
\end{aligned} \tag{4.26}$$

Using equations (4.26) and the fact that $\Delta_{\mathring{g}}v = \mathring{g}^{ij}(v_{ij} - \mathring{\Gamma}_{ij}^k v_k)$ for a function v, we get

$$\begin{aligned}
\Delta_{\mathring{g}}v &= v_{ij}\left[g_{\mathbb{S}^2}^{ij}\rho^{-2} + \mathcal{O}(\rho^0) + \rho^{-2}\mathcal{L}_p^{(0)}(w) + \rho^{-2}\mathcal{Q}_p^{(2)(1)}(w)\right] \\
&\quad - v_k\left[g_{\mathbb{S}^2}^{ij}\rho^{-2} + \mathcal{O}(\rho^0) + \rho^{-2}\mathcal{L}_p^{(0)}(w) + \rho^{-2}\mathcal{Q}_p^{(2)(1)}(w)\right]\left[\Gamma_{ij}^k + \mathcal{O}(\rho^2) + \mathcal{L}_p^{(1)}(w) + \mathcal{Q}_p^{(2)(2)}(w)\right] \\
&= \Delta_{\mathbb{S}^2}v\rho^{-2} + \mathcal{O}(\rho^0)(v_{ij}+v_k) + \rho^{-2}\mathcal{L}_p^{(1)}(w)(v_{ij}+v_k) + \rho^{-2}\mathcal{Q}_p^{(2)(2)}(v_{ij}+v_k)
\end{aligned} \tag{4.27}$$

By linearity, to compute $\Delta_{\mathring{g}}H$ we can split H and successively compute $\Delta_{\mathring{g}}v$ for the terms below:

$$v \in \left\{2\rho^{-1},\ (2+\Delta_{\mathbb{S}^2})w\rho^{-1},\ -\frac{1}{3}\mathrm{Ric}(\Theta,\Theta)\rho,\ \mathcal{O}(\rho^2) + \rho\mathcal{L}_p^{(2)}(w) + \rho^{-1}\mathcal{Q}_p^{(2)(2)}(w)\right\}$$

using (4.27).

• $v = 2\rho^{-1}$

In this case v is constant, so $\Delta_{\mathring{g}}v = 0$.

• $v = (2+\Delta_{\mathbb{S}^2})w\rho^{-1}$

In this case we have $(v_{ij}+v_k) = \rho^{-1}\mathcal{L}_p^{(4)}(w)$ and therefore

$$\Delta_{\mathring{g}} v = \Delta_{\mathbb{S}^2}(2+\Delta_{\mathbb{S}^2}) w \rho^{-3} + \rho^{-1}\mathcal{L}_p^{(4)}(w) + \rho^{-3}\mathcal{Q}_p^{(2)(4)}(w) \tag{4.28}$$

• $v = -\frac{1}{3}\mathrm{Ric}(\Theta,\Theta)\rho$

Now $(v_{ij} + v_k) = \mathcal{O}(\rho)$, giving

$$\Delta_{\mathring{g}} v = -\frac{1}{3}\Delta_{\mathbb{S}^2}\mathrm{Ric}(\Theta,\Theta)\rho^{-1} + \mathcal{O}(\rho) + \rho^{-1}\mathcal{L}_p^{(1)}(w) + \rho^{-1}\mathcal{Q}_p^{(2)(2)}(w) \tag{4.29}$$

• $v = \mathcal{O}(\rho^2) + \rho\mathcal{L}_p^{(2)}(w) + \rho^{-1}\mathcal{Q}_p^{(2)(2)}(w)$

Finally, $(v_{ij} + v_k) = \mathcal{O}(\rho^2) + \rho\mathcal{L}_p^{(4)}(w) + \rho^{-1}\mathcal{Q}_p^{(2)(4)}(w)$ which yields

$$\Delta_{\mathring{g}} v = \mathcal{O}(\rho^0) + \rho^{-1}\mathcal{L}_p^{(4)}(w) + \rho^{-3}\mathcal{Q}_p^{(2)(4)}(w) \tag{4.30}$$

Combining (4.28) - (4.30) gives

$$\Delta_{\mathring{g}} H = \Delta_{\mathbb{S}^2}(2+\Delta_{\mathbb{S}^2}) w \rho^{-3} - \frac{1}{3}\Delta_{\mathbb{S}^2}\mathrm{Ric}(\Theta,\Theta)\rho^{-1} + \mathcal{O}(\rho^0) + \rho^{-1}\mathcal{L}_p^{(4)}(w) + \rho^{-3}\mathcal{Q}_p^{(2)(4)}(w) \tag{4.31}$$

To compute the other term on the left hand side of (4.25), we use equations (4.26) again. This reveals

$$H(H^2 - 4D + 2\mathrm{Ric}(\hat{N},\hat{N})) = 4\mathrm{Ric}(\Theta,\Theta)\rho^{-1} + \mathcal{O}(\rho) + \rho^{-1}\mathcal{L}_p^{(2)}(w) + \rho^{-3}\mathcal{Q}_p^{(2)(2)}(w) \tag{4.32}$$

Finally, using that the λs are bounded as $\rho \to 0$ ([LM14, Lemma 2.2]), the term on the right hand side of (4.25) is

$$\lambda H = 2\lambda\rho^{-1} + \mathcal{O}(\rho) + \rho^{-1}\mathcal{L}_p^{(2)}(w) + \rho^{-1}\mathcal{Q}_p^{(2)(2)}(w) \tag{4.33}$$

Substituting (4.31) - (4.33) into (4.25) yields the PDE for perturbed spheres.

$$\begin{aligned}\Delta_{\mathbb{S}^2}(2+\Delta_{\mathbb{S}^2}) w \rho^{-3} - \frac{1}{3}\Delta_{\mathbb{S}^2}\mathrm{Ric}(\Theta,\Theta)\rho^{-1} + 2\mathrm{Ric}(\Theta,\Theta)\rho^{-1} = \lambda\rho^{-1} &+ \mathcal{O}(\rho^0) + \rho^{-1}\mathcal{L}_p^{(4)}(w) \\ &+ \rho^{-3}\mathcal{Q}_p^{(2)(4)}(w)\end{aligned}$$

Rearranging gives

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)w = \Big(\frac{1}{3}\Delta_{\mathbb{S}^2}\mathrm{Ric}(\Theta,\Theta) - 2\mathrm{Ric}(\Theta,\Theta) + \lambda\Big)\rho^2 + \mathcal{O}(\rho^3) + \rho^2\mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w) \tag{4.34}$$

Step 2 - Uniqueness.

Next we use the argument from [Mon10, Lemma 4.4]. First, we have the orthogonal projection

$$P : L^2(S^2) \to \mathrm{Ker}[\Delta_{S^2}(\Delta_{S^2}+2)]^{\perp}$$

Then, (4.34) implies

$$P\Big[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)w + \mathcal{O}(\rho^2) + \rho^2\mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w)\Big] = 0. \tag{4.35}$$

Recall that $w \in C^{4,\alpha}(\mathbb{S}^2)^{\perp} = C^{4,\alpha}(\mathbb{S}^2) \cap \mathrm{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)]^{\perp}$, and the operator $\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)$ is invertible on the space orthogonal to its kernel. Thus, denoting its inverse by K, applying K to 4.35 and rearranging, yields

$$w = K \circ P\Big[\mathcal{O}(\rho^2) + \rho^2\mathcal{L}_p^{(4)}(w) + \mathcal{Q}_p^{(2)(4)}(w)\Big] =: F_{p,\rho}(w) \tag{4.36}$$

In [Mon10, Lemma 4.4] the author uses Schauder estimates to show that $F_{p,\rho}(w)$ is a bounded operator. Combining this with the estimates (1.2) and (1.3), they show that (for all p in a compact set) $F_{p,\rho}(w)$ is in fact a contraction on a small enough ball $B(0,r) \subset C^{4,\alpha}(\mathbb{S}^2)^{\perp}$, for small enough ρ. Thus, by the Contraction Mapping Theorem, there exists $\rho_0 > 0$ and $r > 0$ such that, for $\rho \in (0,\rho_0]$, there exists a unique solution of (4.36) in $B(0,r) \subset C^{4,\alpha}(\mathbb{S}^2)^{\perp}$ (i.e. a unique solution of (4.34) in $B(0,r) \subset C^{4,\alpha}(\mathbb{S}^2)^{\perp}$).

Step 3 - Asymptotic expansion.

We will show that the expansion (4.24) holds. Firstly, set the ansatz

$$w = w_2\rho^2 + \mathcal{O}(\rho^3) \tag{4.37}$$

where $w_2 \in C^{4,\alpha}(\mathbb{S}^2)^{\perp}$ depends on p, but not ρ, and $\limsup_{\rho\to 0}\rho^{-3}\left\|\mathcal{O}(\rho^3)\right\|_{C^{4,\alpha}(\mathbb{S}^2)} < \infty$. Substituting (4.37) into (4.34) gives

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)w_2\rho^2 = \Big(\frac{1}{3}\Delta_{\mathbb{S}^2}\mathrm{Ric}(\Theta,\Theta) - 2\mathrm{Ric}(\Theta,\Theta) + \lambda\Big)\rho^2 + \mathcal{O}(\rho^3) \tag{4.38}$$

Therefore, since the ρ^2 terms are dominant as $\rho \to 0$, we deduce

$$\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)w_2 = \frac{1}{3}\Delta_{\mathbb{S}^2}\mathrm{Ric}(\Theta,\Theta) - 2\mathrm{Ric}(\Theta,\Theta) + \lambda. \tag{4.39}$$

To solve this PDE we can use knowledge of the eigenfunctions of $\Delta_{\mathbb{S}^2}$. As before, we have $\Theta = x^\mu E_\mu \in T_pM$ and so (at p) we can write the Ricci curvature in the following way:

$$\begin{aligned}
\mathrm{Ric}(\Theta,\Theta) &= \mathrm{Ric}(x^\mu E_\mu, x^\nu E_\nu) \\
&= R_{\mu\nu}x^\mu x^\nu \\
&= \sum_{\mu\neq\nu} R_{\mu\nu}x^\mu x^\nu + \sum_{\mu} R_{\mu\mu}x^\mu x^\mu \\
&= \sum_{\mu\neq\nu} R_{\mu\nu}x^\mu x^\nu + \sum_{\mu} R_{\mu\mu}\left((x^\mu)^2 - \frac{1}{3}\right) + \frac{1}{3}\sum_{\mu} R_{\mu\mu}
\end{aligned}$$

where $\sum_\mu R_{\mu\mu}$ is just the scalar curvature at p. Now, since Θ is on the unit sphere, we have

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$$

which, after rearranging, gives

$$(x^1)^2 - \frac{1}{3} = \frac{1}{3}\Big(((x^1)^2 - (x^2)^2) + ((x^1)^2 - (x^3)^2)\Big)$$

and similarly for $(x^2)^2 - \frac{1}{3}$ and $(x^3)^2 - \frac{1}{3}$. But since $(x^\mu)^2 - (x^\nu)^2$ and $x^\mu x^\nu$ are eigenfunctions of $\Delta_{\mathbb{S}^2}$ with eigenvalue -6, we have shown that

$$\mathrm{Ric}(\Theta,\Theta) - \frac{1}{3}\mathrm{Sc}_p = \sum_{\mu\neq\nu} R_{\mu\nu}x^\mu x^\nu + \sum_{\mu} R_{\mu\mu}\left((x^\mu)^2 - \frac{1}{3}\right)$$

is just a sum of constants times eigenfunctions, and is therefore also an eigenfunction with eigenvalue -6. Now, (4.39) becomes

$$\begin{aligned}
\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)w_2 &= \frac{1}{3}\Delta_{\mathbb{S}^2}\left(\mathrm{Ric}(\Theta,\Theta) - \frac{1}{3}\mathrm{Sc}_p\right) - 2\mathrm{Ric}(\Theta,\Theta) + \lambda \\
&= -4\left(\mathrm{Ric}(\Theta,\Theta) - \frac{1}{3}\mathrm{Sc}_p\right) - \frac{2}{3}\mathrm{Sc}_p + \lambda
\end{aligned}$$

We can solve this by taking $\lambda = \frac{2}{3}\mathrm{Sc}_p$ and noting that, for any constant A

$$\begin{aligned}\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)\left[A\left(\mathrm{Ric}(\Theta,\Theta)-\frac{1}{3}\mathrm{Sc}_p\right)\right] &= \Delta_{\mathbb{S}^2}\left[-4A\left(\mathrm{Ric}(\Theta,\Theta)-\frac{1}{3}\mathrm{Sc}_p\right)\right] \\ &= 24A\left(\mathrm{Ric}(\Theta,\Theta)-\frac{1}{3}\mathrm{Sc}_p\right)\end{aligned}$$

Therefore, we take $A = -\frac{1}{6}$, so that

$$w_2 = -\frac{1}{6}\mathrm{Ric}(\Theta,\Theta)+\frac{1}{18}\mathrm{Sc}_p$$

We have shown that the ansatz (4.37) yields a solution of (4.25). Therefore, by the uniqueness shown in step 2, this proves (4.24) provided $\|w\|_{C^{4,\alpha}(S^2)} < r$. □

4.3 The Hawking Mass in an Optimally Perturbed Geodesic Sphere

Proposition 4.3. *Let $S_{p,\rho}(w) \subset M$ be an optimally perturbed geodesic sphere, i.e. w satisfies (4.24). Then the Hawking mass of $S_{p,\rho}(w)$ has the following expansion:*

$$\begin{aligned}m_H(S_{p,\rho}(w)) &:= \sqrt{\frac{|S_{p,\rho}(w)|_{\mathring{g}}}{(16\pi)^3}}\Big(16\pi - W(S_{p,\rho}(w))\Big) \\ &= \sqrt{\frac{|S_{p,\rho}(w)|_{\mathring{g}}}{(16\pi)^3}}\left(\frac{8\pi}{3}\mathrm{Sc}_p\rho^2+\left[\frac{4\pi}{15}\Delta\mathrm{Sc}(p)+\frac{16\pi}{45}\|S_p\|^2-\frac{4\pi}{27}\mathrm{Sc}_p^2\right]\rho^4+\mathcal{O}(\rho^5)\right)\end{aligned} \tag{4.40}$$

where Sc *is the scalar curvature and $S := \mathrm{Ric} - \frac{1}{3}\mathrm{Sc}\cdot g$ is the traceless Ricci tensor.*

Proof. **Step 1** - Computing the Willmore functional integrand.

To compute the Hawking mass we need expansions for the terms in the Willmore functional $W(S_{p,\rho}(w)) := \int_{S_{p,\rho}(w)} H^2\, dV_{\mathring{g}} = \int_{\mathbb{S}^2} H^2\sqrt{\det\mathring{g}}\; d\theta^1 d\theta^2$. Using (4.5), the Taylor expansion $\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\mathcal{O}(x^3)$ and the fact that $w = \mathcal{O}(\rho^2)$, we have

$$
\begin{aligned}
\sqrt{\det \mathring{g}} = \sin\theta^1 \rho^2 \Bigg[1 + \frac{1}{2}\Big[& -4w + 6w^2 - 4w^3 + w^4 + g^{ij}_{\mathbb{S}^2} w_i w_j - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)(1-w)^6\rho^2 \\
& + \frac{1}{6} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^7\rho^3 + \frac{1}{20} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^8\rho^4 \\
& + \frac{2}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^8\rho^4 \\
& - \frac{1}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2(1-w)^8\rho^4 \\
& + \frac{1}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(1-w)^8\rho^4\Big] \\
& - \frac{1}{8}\Big[-4w + 6w^2 - 4w^3 + w^4 + g^{ij}_{\mathbb{S}^2} w_i w_j - \frac{1}{3}\mathrm{Ric}(\Theta,\Theta)(1-w)^6\rho^2 \\
& + \frac{1}{6} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^7\rho^3 + \frac{1}{20} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)(1-w)^8\rho^4 \\
& + \frac{2}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)(1-w)^8\rho^4 \\
& - \frac{1}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2(1-w)^8\rho^4 \\
& + \frac{1}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)(1-w)^8\rho^4\Big]^2 \Bigg] + \mathcal{O}(\rho^7)
\end{aligned}
$$

Ignoring higher order terms, we reach

$$
\begin{aligned}
\sqrt{\det \mathring{g}} = \sin\theta^1 \rho^2 \Bigg[& (1-w)^2 + \frac{1}{2} g^{ij}_{\mathbb{S}^2} w_i w_j - \frac{1}{6}\mathrm{Ric}(\Theta,\Theta)\rho^2 + \frac{2}{3} w \mathrm{Ric}(\Theta,\Theta)\rho^2 \\
& + \frac{1}{12} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^3 + \frac{1}{40} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\rho^4 \\
& + \frac{1}{45} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\rho^4 - \frac{1}{18} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2\rho^4 \\
& + \frac{1}{18} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\rho^4 - \frac{1}{72}\mathrm{Ric}(\Theta,\Theta)^2\rho^4 \Bigg] + \mathcal{O}(\rho^7)
\end{aligned} \tag{4.41}
$$

Doing the same with (4.20) yields

$$
\begin{aligned}
H^2 = &\left[4+4(2+\Delta_{\mathbb{S}^2})w+8w(w+\Delta_{\mathbb{S}^2}w)+((2+\Delta_{\mathbb{S}^2})w)^2\right]\rho^{-2} \\
&+\left[\frac{2}{3}w_k g_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{kn}g_{\mathbb{S}^2}^{ml}g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\left(\partial_i g_{jl}^{\mathbb{S}^2}+\partial_j g_{il}^{\mathbb{S}^2}-\partial_l g_{ij}^{\mathbb{S}^2}\right)\right. \\
&-\frac{2}{3}w_k g_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{kl}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l)+\partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l)-\partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big) \\
&\left.-\frac{2}{3}\mathrm{Ric}(\Theta,\Theta)(2+\Delta_{\mathbb{S}^2}w)-\frac{4}{3}g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla_{\mathbb{S}^2}^2 w)_{ij}\right] \\
&+\left[g_{\mathbb{S}^2}^{ij}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\right]\rho \\
&+\left[\frac{2}{5}g_{\mathbb{S}^2}^{ij}g(\nabla_\Theta^2\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)+\frac{16}{45}g_{\mathbb{S}^2}^{ij}g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\right. \\
&\left.-\frac{4}{9}g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)+\frac{1}{9}\mathrm{Ric}(\Theta,\Theta)^2\right]\rho^2+\mathcal{O}(\rho^3)
\end{aligned}
\tag{4.42}
$$

Multiplying (4.42) by (4.41), we obtain the integrand of the Willmore functional evaluated on a perturbed geodesic sphere.

$$
\begin{aligned}
H^2\sqrt{\det\mathring{g}} = \sin\theta^1 &\left[\left[4+4\Delta_{\mathbb{S}^2}w+4w\Delta_{\mathbb{S}^2}w+(\Delta_{\mathbb{S}^2}w)^2+2g_{\mathbb{S}^2}^{ij}w_iw_j\right]\right. \\
&+\left[\frac{2}{3}w_k g_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{kn}g_{\mathbb{S}^2}^{ml}g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\left(\partial_i g_{jl}^{\mathbb{S}^2}+\partial_j g_{il}^{\mathbb{S}^2}-\partial_l g_{ij}^{\mathbb{S}^2}\right)\right. \\
&-\frac{2}{3}w_k g_{\mathbb{S}^2}^{ij}g_{\mathbb{S}^2}^{kl}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l)+\partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l)-\partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big) \\
&-\frac{4}{3}g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla_{\mathbb{S}^2}^2 w)_{ij}-2\mathrm{Ric}(\Theta,\Theta)+4w\mathrm{Ric}(\Theta,\Theta) \\
&\left.-\frac{4}{3}\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2}w\right]\rho^2+\left[\frac{4}{3}g_{\mathbb{S}^2}^{ij}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\right]\rho^3 \\
&+\left[\frac{1}{2}g_{\mathbb{S}^2}^{ij}g(\nabla_\Theta^2\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)+\frac{4}{9}g_{\mathbb{S}^2}^{ij}g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\right. \\
&-\frac{4}{9}g_{\mathbb{S}^2}^{il}g_{\mathbb{S}^2}^{kj}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) \\
&+\frac{2}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)-\frac{2}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2 \\
&\left.\left.+\frac{5}{18}\mathrm{Ric}(\Theta,\Theta)^2\right]\rho^4+\mathcal{O}(\rho^5)\right]
\end{aligned}
$$

Inserting $w = w_2\rho^2+\mathcal{O}(\rho^3)$ yields

$$
\begin{aligned}
H^2\sqrt{\det\mathring{g}} = \sin\theta^1 \Bigg[& 4 + \Big[4\Delta_{\mathbb{S}^2} w_2 - 2\mathrm{Ric}(\Theta,\Theta)\Big]\rho^2 + \Big[\frac{4}{3} g^{ij}_{\mathbb{S}^2} g(\nabla_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big]\rho^3 \\
& + \Big[4w_2\Delta_{\mathbb{S}^2} w_2 + (\Delta_{\mathbb{S}^2} w_2)^2 + 2g^{ij}_{\mathbb{S}^2}(w_2)_i(w_2)_j + 4w_2\mathrm{Ric}(\Theta,\Theta) - \frac{4}{3}\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2} w_2 \\
& + \frac{2}{3}(w_2)_k g^{ij}_{\mathbb{S}^2} g^{kn}_{\mathbb{S}^2} g^{ml}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\Big(\partial_i g^{\mathbb{S}^2}_{jl} + \partial_j g^{\mathbb{S}^2}_{il} - \partial_l g^{\mathbb{S}^2}_{ij}\Big) \\
& - \frac{2}{3}(w_2)_k g^{ij}_{\mathbb{S}^2} g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l) + \partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l) - \partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big) \\
& - \frac{4}{3} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla^2_{\mathbb{S}^2} w_2)_{ij} - \frac{2}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2 + \frac{5}{18}\mathrm{Ric}(\Theta,\Theta)^2 \\
& - \frac{4}{9} g^{il}_{\mathbb{S}^2} g^{kj}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k) \qquad (4.43) \\
& + \frac{2}{9} g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) + \frac{1}{2} g^{ij}_{\mathbb{S}^2} g(\nabla^2_\Theta \mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j) \\
& + \frac{4}{9} g^{ij}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau) g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\Big]\rho^4 + \mathcal{O}(\rho^5)\Bigg]
\end{aligned}
$$

Remark 4.4. This is indeed the same as the unperturbed case if we set $w = 0$.

Step 2 - Simplifying the integrand.

In order to simplify the integrand we will use the identities in (3.9), which all follow from the definiiton of the Ricci curvature and the fact that Θ, Θ_1 and $\bar{\Theta}_2$ make an orthonormal basis of T_pM. Throughout, we will use the symmetries of the Riemann tensor to combine as many terms as possible. We will also use some computations for the derivatives of w_2. We have

$$
\begin{aligned}
(w_2)_k &= \partial_k(-\frac{1}{6}\mathrm{Ric}(\Theta,\Theta) + \frac{1}{18}\mathrm{Sc}_p) \\
&= -\frac{1}{6}\partial_k(\mathrm{Ric}(\Theta,\Theta)) \\
&= -\frac{1}{6}(\mathrm{Ric}(\Theta_k,\Theta) + \mathrm{Ric}(\Theta,\Theta_k)) \\
&= -\frac{1}{3}\mathrm{Ric}(\Theta,\Theta_k) \qquad (4.44)
\end{aligned}
$$

$$
\begin{aligned}
(w_2)_{kj} &= \partial_j(-\frac{1}{3}\mathrm{Ric}(\Theta,\Theta_k)) \\
&= -\frac{1}{3}(\mathrm{Ric}(\Theta_j,\Theta_k) + \mathrm{Ric}(\Theta,\Theta_{kj}))
\end{aligned}
$$

which, combined with the fact that

$$
\begin{aligned}
\Theta_{11} &= -\Theta \\
\Theta_{12} &= \Theta_{21} = \cot\theta^1 \Theta_2 \\
\Theta_{22} &= -\sin\theta^1 \cos\theta^1 \Theta_1 - \sin^2\theta^1 \Theta
\end{aligned} \tag{4.45}
$$

shows

$$
\begin{aligned}
(w_2)_{11} &= -\frac{1}{3}(\mathrm{Ric}(\Theta_1,\Theta_1) - \mathrm{Ric}(\Theta,\Theta)) \\
(w_2)_{12} &= (w_2)_{21} = -\frac{1}{3}(\mathrm{Ric}(\Theta_1,\Theta_2) + \cot\theta^1 \mathrm{Ric}(\Theta,\Theta_2)) \\
(w_2)_{22} &= -\frac{1}{3}(\mathrm{Ric}(\Theta_2,\Theta_2) - \sin\theta^1\cos\theta^1 \mathrm{Ric}(\Theta,\Theta_1) - \sin^2\theta^1 \mathrm{Ric}(\Theta,\Theta))
\end{aligned} \tag{4.46}
$$

The first and third order terms are identical to the unperturbed case. The second order term looks different because of the extra $4\Delta_{\mathbb{S}^2} w_2 \rho^2$, but after integration it is also the same (see below). The difference is with the fourth order term, which we will now simplify enough so that it can be integrated using similar methods as before. We begin with the first line and use (4.44) to get

$$
\begin{aligned}
&4w_2\Delta_{\mathbb{S}^2} w_2 + (\Delta_{\mathbb{S}^2} w_2)^2 + 2g^{ij}_{\mathbb{S}^2}(w_2)_i(w_2)_j + 4w_2 \mathrm{Ric}(\Theta,\Theta) - \frac{4}{3}\mathrm{Ric}(\Theta,\Theta)\Delta_{\mathbb{S}^2} w_2 \\
&= (-\frac{2}{3}\mathrm{Ric}(\Theta,\Theta) + \frac{2}{9}\mathrm{Sc}_p)(\mathrm{Ric}(\Theta,\Theta) - \frac{1}{3}\mathrm{Sc}_p) + (\mathrm{Ric}(\Theta,\Theta) - \frac{1}{3}\mathrm{Sc}_p)^2 \\
&\quad + 2g^{ij}_{\mathbb{S}^2}(-\frac{1}{6}\mathrm{Ric}(\Theta,\Theta))_i(-\frac{1}{6}\mathrm{Ric}(\Theta,\Theta))_j + \mathrm{Ric}(\Theta,\Theta)(-\frac{2}{3}\mathrm{Ric}(\Theta,\Theta) + \frac{2}{9}\mathrm{Sc}_p) \\
&\quad - \frac{4}{3}\mathrm{Ric}(\Theta,\Theta)(\mathrm{Ric}(\Theta,\Theta) - \frac{1}{3}\mathrm{Sc}_p) \\
&= -\frac{5}{3}\mathrm{Ric}(\Theta,\Theta)^2 + \frac{4}{9}\mathrm{Sc}_p\mathrm{Ric}(\Theta,\Theta) + \frac{1}{27}\mathrm{Sc}_p^2 + \frac{2}{9}(\mathrm{Ric}(\Theta,\Theta_1)^2 + \mathrm{Ric}(\Theta,\bar{\Theta}_2)^2)
\end{aligned} \tag{4.47}
$$

The second line becomes

$$
\begin{aligned}
&\frac{2}{3}(w_2)_k g^{ij}_{\mathbb{S}^2} g^{kn}_{\mathbb{S}^2} g^{ml}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_n)\Theta,\Theta_m)\left(\partial_i g^{\mathbb{S}^2}_{jl} + \partial_j g^{\mathbb{S}^2}_{il} - \partial_l g^{\mathbb{S}^2}_{ij}\right) \\
&= \sum_k -\frac{2}{3}(w_2)_k g^{22}_{\mathbb{S}^2} g^{kk}_{\mathbb{S}^2} g^{11}_{\mathbb{S}^2} g(\mathcal{R}(\Theta,\Theta_k)\Theta,\Theta_1)(2\sin\theta^1\cos\theta^1) \\
&= \frac{4\cot\theta^1}{9}\mathrm{Ric}(\Theta,\Theta_1) g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) + \frac{4\cot\theta^1}{9}\mathrm{Ric}(\Theta,\bar{\Theta}_2) g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\Theta_1)
\end{aligned} \tag{4.48}
$$

where we have used (4.44) again and the first equality follows because the only non-zero terms occur when $i=j$, $k=n$ and $m=l$ (note that $\partial_i g^{\mathbb{S}^2}_{jk}$ is only non-zero when $j=k=2$ and $i=1$). The third line simplifies to

$$
\begin{aligned}
&-\frac{2}{3}(w_2)_k g^{ij}_{\mathbb{S}^2} g^{kl}_{\mathbb{S}^2}\Big(\partial_i g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_l)+\partial_j g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l)-\partial_l g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big)\\
&=\sum_{i,k}-\frac{2}{3}(w_2)_k g^{ii}_{\mathbb{S}^2} g^{kk}_{\mathbb{S}^2}\Big(2\partial_i g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_k)-\partial_k g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_i)\Big)\\
&=\sum_{i,k}-\frac{2}{3}(w_2)_k g^{ii}_{\mathbb{S}^2} g^{kk}_{\mathbb{S}^2}\Big[2\big(g(\mathcal{R}(\Theta,\Theta_{ii})\Theta,\Theta_k)+g(\mathcal{R}(\Theta,\Theta_i)\Theta_i,\Theta_k)+g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_{ki})\big)\\
&\quad -g(\mathcal{R}(\Theta_k,\Theta_i)\Theta,\Theta_i)-g(\mathcal{R}(\Theta,\Theta_{ik})\Theta,\Theta_i)-g(\mathcal{R}(\Theta,\Theta_i)\Theta_k,\Theta_i)-g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_{ik})\Big]\\
&=-\frac{2}{3}(w_2)_2 g^{11}_{\mathbb{S}^2} g^{22}_{\mathbb{S}^2}\Big[2\big(g(\mathcal{R}(\Theta,\Theta_1)\Theta_1,\Theta_2)+g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_{12})\big)\\
&\quad -g(\mathcal{R}(\Theta_2,\Theta_1)\Theta,\Theta_1)-g(\mathcal{R}(\Theta,\Theta_{12})\Theta,\Theta_1)-g(\mathcal{R}(\Theta,\Theta_1)\Theta_2,\Theta_1)-g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_{12})\Big]\\
&\quad -\frac{2}{3}(w_2)_1 g^{22}_{\mathbb{S}^2} g^{11}_{\mathbb{S}^2}\Big[2\big(g(\mathcal{R}(\Theta,\Theta_{22})\Theta,\Theta_1)+g(\mathcal{R}(\Theta,\Theta_2)\Theta_2,\Theta_1)+g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_{21})\big)\\
&\quad -g(\mathcal{R}(\Theta_1,\Theta_2)\Theta,\Theta_2)-g(\mathcal{R}(\Theta,\Theta_{21})\Theta,\Theta_2)-g(\mathcal{R}(\Theta,\Theta_2)\Theta_1,\Theta_2)-g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_{21})\Big]\\
&\quad -\frac{2}{3}(w_2)_2 g^{22}_{\mathbb{S}^2} g^{22}_{\mathbb{S}^2}\Big[2\big(g(\mathcal{R}(\Theta,\Theta_{22})\Theta,\Theta_2)+g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_{22})\big)\\
&\quad -g(\mathcal{R}(\Theta,\Theta_{22})\Theta,\Theta_2)-g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_{22})\Big]\\
&=\frac{8}{9}\mathrm{Ric}(\Theta,\bar{\Theta}_2)g(\mathcal{R}(\Theta,\Theta_1)\Theta_1,\bar{\Theta}_2)\\
&\quad +\frac{4}{9}\mathrm{Ric}(\Theta,\Theta_1)\big(-\cot\theta^1 g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)+2g(\mathcal{R}(\Theta,\bar{\Theta}_2)\bar{\Theta}_2,\Theta_1)\big)\\
&\quad -\frac{4\cot\theta^1}{9}\mathrm{Ric}(\Theta,\bar{\Theta}_2)g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)
\end{aligned}\tag{4.49}
$$

For the second equality above we have used that $\Theta_{11}=-\Theta$ which means that all the $i=k=1$ terms are zero thanks to the symmetries of the Riemann tensor. The last equality is shown by applying (4.45) and (4.44). Combining (4.48) and (4.49) shows that the second and third lines of the fourth order term become

$$
\begin{aligned}
&\frac{8}{9}\Big(\mathrm{Ric}(\Theta,\bar{\Theta}_2)g(\mathcal{R}(\Theta,\Theta_1)\Theta_1,\bar{\Theta}_2)+\mathrm{Ric}(\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar{\Theta}_2)\bar{\Theta}_2,\Theta_1)\Big)\\
&=\frac{8}{9}\Big(\mathrm{Ric}(\Theta,\bar{\Theta}_2)^2+\mathrm{Ric}(\Theta,\Theta_1)^2\Big)
\end{aligned}\tag{4.50}
$$

where we have used the definition of the Ricci tensor and the symmetries of the Riemann tensor to rewrite $\mathrm{Ric}(\Theta,\bar{\Theta}_2)=g(\mathcal{R}(\Theta,\Theta_1)\Theta_1,\bar{\Theta}_2)$ and $\mathrm{Ric}(\Theta,\Theta_1)=g(\mathcal{R}(\Theta,\bar{\Theta}_2)\bar{\Theta}_2,\Theta_1)$. Turning now to the next term, we will use some more identities, all of which follow from the definitions of Ric and Sc, and the symmetries of the Riemann tensor.

$$\operatorname{Ric}(\Theta,\Theta) = -g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) - g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) \tag{4.51}$$

$$\operatorname{Ric}(\Theta_1,\bar{\Theta}_2) = -g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2) \tag{4.52}$$

$$g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1) = -\operatorname{Ric}(\Theta_1,\Theta_1) - g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2) \tag{4.53}$$

$$g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2) = -\operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) - g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2) \tag{4.54}$$

$$g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2) = -\frac{1}{2}\mathrm{Sc}_p + \operatorname{Ric}(\Theta,\Theta) \tag{4.55}$$

$$\operatorname{Ric}(\Theta_1,\Theta_1) + \operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) = \mathrm{Sc}_p - \operatorname{Ric}(\Theta,\Theta) \tag{4.56}$$

$$\operatorname{Ric}(\Theta_1,\Theta_1)^2 + \operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)^2 = \big(\mathrm{Sc}_p - \operatorname{Ric}(\Theta,\Theta)\big)^2 - 2\operatorname{Ric}(\Theta_1,\Theta_1)\operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) \tag{4.57}$$

We have

$$\begin{aligned}
&-\frac{4}{3}g^{il}_{\mathbb{S}^2}g^{kj}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_l)\Theta,\Theta_k)(\nabla^2_{\mathbb{S}^2}w_2)_{ij} \\
&= -\frac{4}{3}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)\big((w_2)_{11} - \Gamma^i_{11}(w_2)_i\big) - \frac{8}{3\sin^2\theta^1}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_2)\big((w_2)_{12} - \Gamma^i_{12}(w_2)_i\big) \\
&\quad - \frac{4}{3\sin^4\theta^1}g(\mathcal{R}(\Theta,\Theta_2)\Theta,\Theta_2)\big((w_2)_{22} - \Gamma^i_{22}(w_2)_i\big) \\
&= \frac{4}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)\big(\operatorname{Ric}(\Theta_1,\Theta_1) - \operatorname{Ric}(\Theta,\Theta)\big) + \frac{8}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)\operatorname{Ric}(\bar{\Theta}_2,\Theta_1) \\
&\quad + \frac{4}{9}g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\big(\operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) - \operatorname{Ric}(\Theta,\Theta)\big) \\
&= \frac{4}{9}\big(g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)\operatorname{Ric}(\Theta_1,\Theta_1) + g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)\big) + \frac{4}{9}\operatorname{Ric}(\Theta,\Theta)^2 \\
&\quad - \frac{8}{9}\operatorname{Ric}(\bar{\Theta}_2,\Theta_1)^2 \\
&= -\frac{4}{9}\Big[\operatorname{Ric}(\Theta_1,\Theta_1)\big(\operatorname{Ric}(\Theta_1,\Theta_1) + g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)\big) \\
&\quad + \operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)\big(\operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) + g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)\big)\Big] + \frac{4}{9}\operatorname{Ric}(\Theta,\Theta)^2 - \frac{8}{9}\operatorname{Ric}(\bar{\Theta}_2,\Theta_1)^2 \\
&= -\frac{4}{9}\big(\operatorname{Ric}(\Theta_1,\Theta_1)^2 + \operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)^2\big) + \frac{2}{9}\mathrm{Sc}_p\big(\operatorname{Ric}(\Theta_1,\Theta_1) + \operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)\big) \\
&\quad - \frac{4}{9}\operatorname{Ric}(\Theta,\Theta)\big(\operatorname{Ric}(\Theta_1,\Theta_1) + \operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)\big) + \frac{4}{9}\operatorname{Ric}(\Theta,\Theta)^2 - \frac{8}{9}\operatorname{Ric}(\bar{\Theta}_2,\Theta_1)^2 \\
&= -\frac{2}{9}\mathrm{Sc}_p^2 + \frac{2}{9}\mathrm{Sc}_p\operatorname{Ric}(\Theta,\Theta) + \frac{8}{9}\operatorname{Ric}(\Theta_1,\Theta_1)\operatorname{Ric}(\bar{\Theta}_2,\bar{\Theta}_2) + \frac{4}{9}\operatorname{Ric}(\Theta,\Theta)^2 - \frac{8}{9}\operatorname{Ric}(\bar{\Theta}_2,\Theta_1)^2
\end{aligned} \tag{4.58}$$

where for the second equality we used (4.44), (4.46) and the Christoffel symbols for $\mathbb{S}^2$, which are $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta^1$, $\Gamma^1_{22} = -\sin\theta^1\cos\theta^1$ and $\Gamma^i_{11} = \Gamma^1_{22} = \Gamma^1_{12} = \Gamma^1_{21} = 0$. For the third equality we used (4.51) and (4.52), the fourth (4.53) and (4.54), and the fifth (4.55). The final equality follows by (4.56) and (4.57) and collecting like terms. Next, we note that the eleventh and fourteenth fourth order terms cancel because

$$
\begin{aligned}
&\frac{4}{9}g^{ij}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_i)\Theta,E_\tau)g(\mathcal{R}(\Theta,\Theta_j)\Theta,E_\tau)\\
&=\frac{4}{9}g^{ij}_{\mathbb{S}^2}\big[g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_1)+g(\mathcal{R}(\Theta,\Theta_i)\Theta,\bar{\Theta}_2)g(\mathcal{R}(\Theta,\Theta_j)\Theta,\bar{\Theta}_2)\big]\\
&=\frac{4}{9}g^{ij}_{\mathbb{S}^2}g^{lk}_{\mathbb{S}^2}g(\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_l)g(\mathcal{R}(\Theta,\Theta_j)\Theta,\Theta_k)
\end{aligned}
$$

where we have used the orthonormal basis $\{E_1=\Theta,E_2=\Theta_1,E_3=\bar{\Theta}_2\}$. The ninth term becomes

$$
-\frac{2}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\bar{\Theta}_2)^2=-\frac{2}{9}\mathrm{Ric}(\Theta_1,\bar{\Theta}_2)^2 \tag{4.59}
$$

and, finally, the twelfth

$$
\begin{aligned}
&\frac{2}{9}g(\mathcal{R}(\Theta,\Theta_1)\Theta,\Theta_1)g(\mathcal{R}(\Theta,\bar{\Theta}_2)\Theta,\bar{\Theta}_2)\\
&=\frac{2}{9}\big[-\mathrm{Ric}(\Theta_1,\Theta_1)-g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)\big]\big[-\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)-g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)\big]\\
&=\frac{2}{9}\mathrm{Ric}(\Theta_1,\Theta_1)\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)+\frac{2}{9}g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)\big(\mathrm{Ric}(\Theta_1,\Theta_1)+\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)\big)\\
&\quad+\frac{2}{9}g(\mathcal{R}(\Theta_1,\bar{\Theta}_2)\Theta_1,\bar{\Theta}_2)^2\\
&=\frac{2}{9}\mathrm{Ric}(\Theta_1,\Theta_1)\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)+\frac{2}{9}\big(-\frac{1}{2}\mathrm{Sc}_p+\mathrm{Ric}(\Theta,\Theta)\big)\big(\mathrm{Sc}_p-\mathrm{Ric}(\Theta,\Theta)\big)\\
&\quad+\frac{2}{9}\big(-\frac{1}{2}\mathrm{Sc}_p+\mathrm{Ric}(\Theta,\Theta)\big)^2\\
&=\frac{2}{9}\mathrm{Ric}(\Theta_1,\Theta_1)\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)-\frac{1}{18}\mathrm{Sc}_p^2+\frac{1}{9}\mathrm{Sc}_p\mathrm{Ric}(\Theta,\Theta)
\end{aligned} \tag{4.60}
$$

where we have used (4.53) and (4.54) for the first equality, and (4.55) and (4.56) for the third equality. Substituting (4.47), (4.50), (4.58), (4.59) and (4.60) into (4.43) yields

$$
\begin{aligned}
H^2\sqrt{\det\mathring{g}}=\sin\theta^1\Bigg[&4+\Big[4\Delta_{\mathbb{S}^2}w_2-2\mathrm{Ric}(\Theta,\Theta)\Big]\rho^2+\Big[\frac{4}{3}g^{ij}_{\mathbb{S}^2}g(\nabla_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Big]\rho^3\\
&+\Bigg[-\frac{17}{18}\mathrm{Ric}(\Theta,\Theta)^2-\frac{13}{54}\mathrm{Sc}_p^2+\frac{7}{9}\mathrm{Sc}_p\mathrm{Ric}(\Theta,\Theta)+\frac{10}{9}\Big(\mathrm{Ric}(\Theta,\Theta_1)^2+\mathrm{Ric}(\Theta,\bar{\Theta}_2)^2\Big)\\
&+\frac{10}{9}\Big(\mathrm{Ric}(\Theta_1,\Theta_1)\mathrm{Ric}(\bar{\Theta}_2,\bar{\Theta}_2)-\mathrm{Ric}(\Theta_1,\bar{\Theta}_2)^2\Big)+\frac{1}{2}g^{ij}_{\mathbb{S}^2}g(\nabla^2_\Theta\mathcal{R}(\Theta,\Theta_i)\Theta,\Theta_j)\Bigg]\rho^4\\
&+\mathcal{O}(\rho^5)\Bigg]
\end{aligned} \tag{4.61}
$$

Step 3 - Integrate.

We now integrate each term in (4.61) using the same methods as in the unperturbed case.

• **Terms of order ρ^0 - ρ^3.**

The only difference with the unperturbed case is the $4\Delta_{\mathbb{S}^2} w_2 \rho^2$ term. However, since $\partial\mathbb{S}^2 = \emptyset$, by Green's identity we have

$$\int_{\mathbb{S}^2} 4\Delta_{\mathbb{S}^2} w_2 \, dV_{g_{\mathbb{S}^2}} = -4 \int_{\mathbb{S}^2} g_{\mathbb{S}^2}(\mathrm{grad}_{\mathbb{S}^2} 1, \mathrm{grad}_{\mathbb{S}^2} w_2) dV_{g_{\mathbb{S}^2}} = 0$$

and so in fact all the terms of order ρ^0 - ρ^3 are the same as in (3.12).

• **Terms of order ρ^4.**

Most of these terms are also the same as in the unperturbed case, but with different constants. See (3.10). The only completely new term is the one containing $\mathrm{Ric}(\Theta,\Theta_1)^2 + \mathrm{Ric}(\Theta,\bar{\Theta}_2)^2$, which we now integrate. Recall that if

$$\Theta = (\sin\theta^1 \cos\theta^2, \sin\theta^1 \sin\theta^2, \cos\theta^1) = (x,y,z)$$

then

$$\Theta_1 = (\cos\theta^1 \cos\theta^2, \cos\theta^1 \sin\theta^2, -\sin\theta^1) = \left(\frac{xz}{\sqrt{x^2+y^2}}, \frac{yz}{\sqrt{x^2+y^2}}, -\sqrt{x^2+y^2}\right)$$
$$\bar{\Theta}_2 = (-\sin\theta^2, \cos\theta^2, 0) = \left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0\right)$$

Therefore, we can expand the following terms (taking into account that $R_{ij} = R_{ji}$):

$$\mathrm{Ric}(\Theta,\Theta_1)^2 = \Big(R_{11}\frac{x^2 z}{\sqrt{x^2+y^2}} + R_{12}\frac{2xyz}{\sqrt{x^2+y^2}} + R_{13}\frac{z^2 x - x(x^2+y^2)}{\sqrt{x^2+y^2}} + R_{22}\frac{y^2 z}{\sqrt{x^2+y^2}}$$
$$+ R_{23}\frac{z^2 y - y(x^2+y^2)}{\sqrt{x^2+y^2}} - R_{33} z\sqrt{x^2+y^2}\Big)^2$$

$$\mathrm{Ric}(\Theta,\bar{\Theta}_2)^2 = \Big(-R_{11}\frac{xy}{\sqrt{x^2+y^2}} + R_{12}\frac{x^2-y^2}{\sqrt{x^2+y^2}} + R_{22}\frac{xy}{\sqrt{x^2+y^2}} \\ -R_{31}\frac{yz}{\sqrt{x^2+y^2}} + R_{32}\frac{xz}{\sqrt{x^2+y^2}}\Big)^2$$

As in the $w=0$ case, we can ignore any polynomial term in the integrand which has an odd power, since it will integrate to zero. Inspection shows that after the brackets are expanded, the only terms that will consist entirely of even powers of x, y and z are those containing the coefficients of the Ricci tensor displayed below:

$$R_{11}^2,\ R_{11}R_{22},\ R_{11}R_{33},\ R_{12}^2,\ R_{13}^2,\ R_{22}^2,\ R_{22}R_{33},\ R_{23}^2,\ R_{33}^2$$

We compute these terms as follows, using that $x^2+y^2+z^2=1$:

$$\begin{aligned}\Big(R_{11}\frac{x^2z}{\sqrt{x^2+y^2}}\Big)^2 + \Big(-R_{11}\frac{xy}{\sqrt{x^2+y^2}}\Big)^2 &= R_{11}^2\Big(\frac{x^4z^2+x^2y^2}{x^2+y^2}\Big) \\ &= R_{11}^2\Big(\frac{x^2(x^2(1-x^2-y^2)+y^2)}{x^2+y^2}\Big) \\ &= R_{11}^2\Big(\frac{x^2(x^2+y^2)(1-x^2)}{x^2+y^2}\Big) \\ &= R_{11}^2(x^2-x^4)\end{aligned}$$

$$\begin{aligned}&2\Big(R_{11}\frac{x^2z}{\sqrt{x^2+y^2}}\Big)\Big(R_{22}\frac{y^2z}{\sqrt{x^2+y^2}}\Big) - 2\Big(R_{11}\frac{xy}{\sqrt{x^2+y^2}}\Big)\Big(R_{22}\frac{xy}{\sqrt{x^2+y^2}}\Big) \\ &= 2R_{11}R_{22}\Big(\frac{x^2y^2(z^2-1)}{x^2+y^2}\Big) \\ &= -2R_{11}R_{22}x^2y^2\end{aligned}$$

$$-2\Big(R_{11}\frac{x^2z}{\sqrt{x^2+y^2}}\Big)\Big(R_{33}z\sqrt{x^2+y^2}\Big) = -2R_{11}R_{33}x^2z^2$$

$$
\begin{aligned}
\Big(R_{12}\frac{2xyz}{\sqrt{x^2+y^2}}\Big)^2+\Big(R_{12}\frac{x^2-y^2}{\sqrt{x^2+y^2}}\Big)^2 &= R_{12}^2\Big(\frac{4x^2y^2z^2+x^4+y^4-2x^2y^2}{x^2+y^2}\Big)\\
&= R_{12}^2\Big(\frac{4x^2y^2(1-x^2-y^2)+x^4+y^4-2x^2y^2}{x^2+y^2}\Big)\\
&= R_{12}^2\Big(\frac{x^4+y^4+2x^2y^2-4x^4y^2-4x^2y^4}{x^2+y^2}\Big)\\
&= R_{12}^2\Big(\frac{(x^2+y^2)^2-4x^2y^2(x^2+y^2)}{x^2+y^2}\Big)\\
&= R_{12}^2(x^2+y^2-4x^2y^2)\\
&= R_{12}^2(1-z^2-4x^2y^2)
\end{aligned}
$$

$$
\begin{aligned}
\Big(R_{13}\frac{z^2x-x(x^2+y^2)}{\sqrt{x^2+y^2}}\Big)^2+\Big(-R_{31}\frac{yz}{\sqrt{x^2+y^2}}\Big)^2 &= R_{13}^2\Big(\frac{(z^2x-x(1-z^2))^2+y^2z^2}{1-z^2}\Big)\\
&= R_{13}^2\Big(\frac{4z^4x^2-4z^2x^2+x^2+y^2z^2}{1-z^2}\Big)\\
&= R_{13}^2\Big(\frac{4z^4x^2-4z^2x^2+1-y^2-z^2+y^2z^2}{1-z^2}\Big)\\
&= R_{13}^2\Big(\frac{4z^2x^2(z^2-1)+(1-z^2)(1-y^2)}{1-z^2}\Big)\\
&= R_{13}^2(1-y^2-4z^2x^2)
\end{aligned}
$$

$$
-2\Big(R_{22}\frac{y^2z}{\sqrt{x^2+y^2}}\Big)\Big(R_{33}z\sqrt{x^2+y^2}\Big) = -2R_{22}R_{33}y^2z^2
$$

$$
\begin{aligned}
\Big(R_{22}\frac{y^2z}{\sqrt{x^2+y^2}}\Big)^2+\Big(R_{22}\frac{xy}{\sqrt{x^2+y^2}}\Big)^2 &= R_{22}^2\Big(\frac{y^4z^2+x^2y^2}{x^2+y^2}\Big)\\
&= R_{22}^2\Big(\frac{y^2(y^2(1-x^2-y^2)+x^2)}{x^2+y^2}\Big)\\
&= R_{22}^2\Big(\frac{y^2(x^2+y^2)(1-y^2)}{x^2+y^2}\Big)\\
&= R_{22}^2(y^2-y^4)
\end{aligned}
$$

$$\begin{aligned}
\Big(R_{23}\frac{z^2y-y(x^2+y^2)}{\sqrt{x^2+y^2}}\Big)^2+\Big(R_{32}\frac{xz}{\sqrt{x^2+y^2}}\Big)^2 &= R_{23}^2\Big(\frac{(z^2y-y(1-z^2))^2+x^2z^2}{1-z^2}\Big)\\
&= R_{23}^2\Big(\frac{4z^4y^2-4z^2y^2+y^2+x^2z^2}{1-z^2}\Big)\\
&= R_{23}^2\Big(\frac{4z^4y^2-4z^2y^2+1-x^2-z^2+x^2z^2}{1-z^2}\Big)\\
&= R_{23}^2\Big(\frac{4z^2y^2(z^2-1)+(1-z^2)(1-x^2)}{1-z^2}\Big)\\
&= R_{23}^2(1-x^2-4z^2y^2)
\end{aligned}$$

$$\Big(-R_{33}z\sqrt{x^2+y^2}\Big)^2 = R_{33}^2z^2(x^2+y^2) = R_{33}^2z^2(1-z^2) = R_{33}^2(z^2-z^4)$$

Thus, we get

$$
\begin{aligned}
\int_{\mathbb{S}^2} \mathrm{Ric}(\Theta,\Theta_1)^2 + \mathrm{Ric}(\Theta,\bar{\Theta}_2)^2 dV_{g_{\mathbb{S}^2}} &= \int_{\mathbb{S}^2} \Big(R_{11}^2(x^2 - x^4) - 2R_{11}R_{22}x^2y^2 - 2R_{11}R_{33}x^2z^2 \\
&\quad + R_{12}^2(1 - z^2 - 4x^2y^2) + R_{13}^2(1 - y^2 - 4z^2x^2) \\
&\quad + R_{22}^2(y^2 - y^4) - 2R_{22}R_{33}y^2z^2 \\
&\quad + R_{23}^2(1 - x^2 - 4z^2y^2) + R_{33}^2(z^2 - z^4) \Big) dV_{g_{\mathbb{S}^2}} \\
&= R_{11}^2(\frac{4\pi}{3} - \frac{4\pi}{5}) - R_{11}R_{22}\frac{8\pi}{15} - R_{11}R_{33}\frac{8\pi}{15} \\
&\quad + R_{12}^2(4\pi - \frac{4\pi}{3} - \frac{16\pi}{15}) + R_{13}^2(4\pi - \frac{4\pi}{3} - \frac{16\pi}{15}) \\
&\quad + R_{22}^2(\frac{4\pi}{3} - \frac{4\pi}{5}) - R_{22}R_{33}\frac{8\pi}{15} \\
&\quad + R_{23}^2(4\pi - \frac{4\pi}{3} - \frac{16\pi}{15}) + R_{33}^2(\frac{4\pi}{3} - \frac{4\pi}{5}) \\
&= \frac{8\pi}{15}R_{11}^2 - \frac{8\pi}{15}R_{11}R_{22} - \frac{8\pi}{15}R_{11}R_{33} + \frac{8\pi}{5}R_{12}^2 \\
&\quad + \frac{8\pi}{5}R_{13}^2 + \frac{8\pi}{15}R_{22}^2 - \frac{8\pi}{15}R_{22}R_{33} \\
&\quad + \frac{8\pi}{5}R_{23}^2 + \frac{8\pi}{15}R_{33}^2 \\
&= \frac{8\pi}{15}(R_{11}^2 + R_{22}^2 + R_{33}^2 - R_{11}R_{22} - R_{11}R_{33} - R_{22}R_{33}) \\
&\quad + \frac{8\pi}{5}(R_{12}^2 + R_{13}^2 + R_{23}^2) \\
&= \frac{8\pi}{15}(R_{11}^2 + R_{22}^2 + R_{33}^2 + 3R_{12}^2 + 3R_{13}^2 + 3R_{23}^2) \\
&\quad - \frac{8\pi}{15}(R_{11}R_{22} + R_{11}R_{33} + R_{22}R_{33}) \\
&= \frac{4\pi}{5}(R_{11}^2 + R_{22}^2 + R_{33}^2 + 2R_{12}^2 + 2R_{13}^2 + 2R_{23}^2) \\
&\quad - \frac{4\pi}{15}(R_{11}^2 + R_{22}^2 + R_{33}^2 + 2R_{11}R_{22} + 2R_{11}R_{33} + 2R_{22}R_{33}) \\
&= \frac{4\pi}{5}\left[\|\mathrm{Ric}\|^2 - \frac{1}{3}\mathrm{Sc}_p^2\right] \\
&= \frac{4\pi}{5}\|S_p\|^2
\end{aligned} \tag{4.62}
$$

where we have again used

$$
\begin{aligned}
\int_{\mathbb{S}^2} (x^\mu)^2 \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{3} \\
\int_{\mathbb{S}^2} (x^\mu)^2(x^\nu)^2 \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{15} \qquad \mu \neq \nu \\
\int_{\mathbb{S}^2} (x^\mu)^4 \, dV_{g_{\mathbb{S}^2}} &= \frac{4\pi}{5}
\end{aligned}
$$

Step 4 - Obtain Hawking mass expansion.

Combining the calculations in Section 3.2 with (4.62) yields the integral of (4.61).

$$
\begin{aligned}
W(S_{p,\rho}(w)) &= \int_{\mathbb{S}^2} H^2 \sqrt{\det \mathring{g}}\, d\theta^1 d\theta^2 \\
&= 16\pi - \frac{8\pi}{3}\mathrm{Sc}_p\rho^2 + \Bigg[-\frac{34\pi}{135}(2\|\mathrm{Ric}\|^2 + \mathrm{Sc}_p^2) - \frac{26\pi}{27}\mathrm{Sc}_p^2 + \frac{28\pi}{27}\mathrm{Sc}_p^2 \\
&\quad + \frac{8\pi}{9}\|S_p\|^2 - \frac{20\pi}{27}(\|\mathrm{Ric}\|^2 - \mathrm{Sc}_p^2) - \frac{4\pi}{15}\Delta\mathrm{Sc}(p) \Bigg] \rho^4 + \mathcal{O}(\rho^5) \\
&= 16\pi - \frac{8\pi}{3}\mathrm{Sc}_p\rho^2 + \left[-\frac{56\pi}{45}\|\mathrm{Ric}\|^2 + \frac{76\pi}{135}\mathrm{Sc}_p^2 + \frac{8\pi}{9}\|S_p\|^2 - \frac{4\pi}{15}\Delta\mathrm{Sc}(p) \right] \rho^4 + \mathcal{O}(\rho^5) \\
&= 16\pi - \frac{8\pi}{3}\mathrm{Sc}_p\rho^2 + \left[-\frac{56\pi}{45}\big(\|\mathrm{Ric}\|^2 - \frac{1}{3}\mathrm{Sc}_p^2\big) + \frac{4\pi}{27}\mathrm{Sc}_p^2 + \frac{8\pi}{9}\|S_p\|^2 - \frac{4\pi}{15}\Delta\mathrm{Sc}(p) \right] \rho^4 \\
&\quad + \mathcal{O}(\rho^5) \\
&= 16\pi - \frac{8\pi}{3}\mathrm{Sc}_p\rho^2 + \left[\frac{4\pi}{27}\mathrm{Sc}_p^2 - \frac{16\pi}{45}\|S_p\|^2 - \frac{4\pi}{15}\Delta\mathrm{Sc}(p) \right] \rho^4 + \mathcal{O}(\rho^5) \qquad (4.63)
\end{aligned}
$$

Where we have used the traceless Ricci tensor S and the fact that $\|S\|^2 = \|\mathrm{Ric}\|^2 - \frac{1}{3}\mathrm{Sc}^2$ (see Section 2.1).

Finally, substituting (4.63) in to the definition of Hawking mass gives (4.40). □

Remark 4.5. We note that (4.40) actually gives a strictly positive (even though small) lower bound on the Hawking mass of the optimally perturbed geodesic sphere $S_{p,\rho}(w)$ at some p inside a connected, 3D Riemannian manifold, if either $\mathrm{Sc}_p > 0$ or both $\mathrm{Sc} \equiv 0$ and $\|S_p\| \neq 0$. In particular, Schur's lemma (see Corollary 5.4) implies that there will always be such a point if the manifold has $\mathrm{Sc} \geq 0$ and non-constant sectional curvature.

5 Proof of Theorems 1.28 and 1.33

First, we recall some well-known results that will be used in the subsequent proof. See, for example, [Lee18].

Theorem 5.1 (Killing-Hopf [Kil91; Hop26])**.** *Let (M,g) be a complete, simply connected Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature. Then (M, g) is isometric to either $\mathbb{R}^n$, $\mathbb{S}^n_r$ or $\mathbb{H}^n_r$.*

Corollary 5.2. *Let (M,g) be a connected, complete Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature (i.e. a space-form). Then (M, g) is isometric to either*

$$\mathbb{R}^n/\Gamma \quad , \quad \mathbb{S}^n_r/\Gamma \quad \textit{or} \quad \mathbb{H}^n_r/\Gamma$$

where Γ is a discrete group of isometries of the corresponding space which acts freely.

Lemma 5.3 (Schur)**.** *Let (M,g) be a connected Riemannian manifold of dimension $n \geq 3$ and $\mathrm{Ric} = \phi g$ for some $\phi \in C^\infty(M)$. Then ϕ is a constant.*

Corollary 5.4. *Let (M,g) be a connected, 3D Riemannian manifold. If the traceless Ricci tensor is zero, then the sectional curvature K is constant. In particular $K = \frac{\mathrm{Sc}}{6}$, where Sc is the scalar curvature.*

Proof. If $S = 0$ then we have $\mathrm{Ric} = \frac{\mathrm{Sc}}{3}g$ and so by Schur's Lemma we have $\mathrm{Sc} = \mathit{constant}$. Now we can use the following formula for the Riemann curvature tensor in terms of the *Kulkarni-Nomizu product* ([Lee18], Corollary 7.26):

$$\mathrm{Rm} = \mathrm{Ric} \owedge g - \frac{\mathrm{Sc}}{4} g \owedge g$$

which, after substituting $\mathrm{Ric} = \frac{\mathrm{Sc}}{3}g$, becomes

$$\mathrm{Rm} = \frac{\mathrm{Sc}}{12} g \owedge g$$

Unpacking the definitions we see that at any point $p \in M$ and for any plane section S_Π with orthonormal basis (X_p, Y_p) for T_pS_Π, the sectional curvature is

$$\begin{aligned} K(S_\Pi) &= \mathrm{Rm}(X_p, Y_p, Y_p, X_p) \\ &= \frac{\mathrm{Sc}}{12}(g \owedge g)(X_p, Y_p, Y_p, X_p) \\ &= \frac{\mathrm{Sc}}{12}\Big(2g(X_p, X_p)g(Y_p, Y_p) - 2g(X_p, Y_p)g(Y_p, X_p)\Big) \\ &= \frac{\mathrm{Sc}}{6} \end{aligned}$$

□

Proof of Theorem 1.28. We prove that the sectional curvature K vanishes on $\Omega \setminus \partial M$.

Assume (1.14) holds. Firstly, we insert (4.24) into (1.4) to yield the following expansion for the area of optimally perturbed spheres:

$$|S_{p,\rho}(w)|_{\mathring{g}} = |\mathbb{S}^2|_{g_{\mathbb{S}^2}} \rho^2 \left[1 - \frac{1}{18}\mathrm{Sc}_p \rho^2 + \mathcal{O}(\rho^4)\right]$$

Note that the $-2\int_{\mathbb{S}^2} w dV_{\mathbb{S}^2} \rho^2$ term in (1.4) evaluates to zero because

$$w \in \mathrm{Ker}[\Delta_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}+2)]^{\perp} \subset L^2(\mathbb{S}^2)$$

Combining this with the Taylor expansion of $\sqrt{1+x}$ yields

$$\sqrt{|S_{p,\rho}(w)|_{\mathring{g}}} = \sqrt{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}\,\rho \left[1 - \frac{1}{36}\mathrm{Sc}_p \rho^2 + \mathcal{O}(\rho^4)\right] \tag{5.1}$$

Now, substituting (4.40) and (5.1) into (1.14) gives

$$\begin{aligned}
&\limsup_{\rho \downarrow 0} \frac{\sqrt{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}\rho^{-4}}{\sqrt{(16\pi)^3}} \left[1 - \frac{1}{36}\mathrm{Sc}_p \rho^2 + \mathcal{O}(\rho^4)\right] \\
&\times \left[\frac{8\pi}{3}\mathrm{Sc}_p \rho^2 + \left(\frac{4\pi}{15}\Delta\mathrm{Sc}(p) + \frac{16\pi}{45}\|S_p\|^2 - \frac{4\pi}{27}\mathrm{Sc}_p^2\right)\rho^4 + \mathcal{O}(\rho^5)\right] \\
&\leq 0
\end{aligned}$$

for all $p \in \Omega \setminus \partial M$. Simplifying yields

$$\limsup_{\rho \downarrow 0} \left[\frac{8\pi}{3}\mathrm{Sc}_p \rho^{-2} + \left(\frac{4\pi}{15}\Delta\mathrm{Sc}(p) + \frac{16\pi}{45}\|S_p\|^2 - \frac{2\pi}{9}\mathrm{Sc}_p^2\right) + \mathcal{O}(\rho)\right] \leq 0 \tag{5.2}$$

Since Sc is assumed to be non-negative, letting $\rho \downarrow 0$ and looking at the dominating $\frac{8\pi}{3}\mathrm{Sc}_p\rho^{-2}$ term in (5.2), we first infer that

$$\mathrm{Sc} \equiv 0 \quad \text{on } \Omega \setminus \partial M \tag{5.3}$$

Plugging (5.3) into (5.2), the dominant term becomes $\frac{16\pi}{45}\|S_p\|^2$, which is constant in ρ and non-negative. We deduce

$$S \equiv 0 \quad \text{on } \Omega \setminus \partial M \tag{5.4}$$

Then, Corollary 5.4 applied to $\Omega \setminus \partial M$ (or to its connected components if $\Omega \setminus \partial M$ is not connected) implies that $K \equiv 0$ on $\Omega \setminus \partial M$. Equivalently, $\Omega \setminus \partial M$ is locally isometric to $(\mathbb{R}^3, \bar{g})$. $\square$

Proof of Theorem 1.33. In particular, Theorem 1.28 applies when (M,g) is complete and has no boundary. Thus, in this case, the previous proof gives $K \equiv 0$ on M. Now, Corollary 5.2 implies (M,g) is isometric to the space-form $\mathbb{R}^3/\Gamma$, for some Γ.

For the second part of the theorem we also assume that (M^3,g) is ALSC. First, denote with π the covering map.

$$\pi : \mathbb{R}^3 \to \mathbb{R}^3/\Gamma \simeq M$$

The full isometry group of $\mathbb{R}^3$ is isomorphic to the semi-direct product $O(3) \ltimes \mathbb{R}^3$ and so we can write $\gamma = (r,a) \in \Gamma$ where $r \in O(3)$ and $a \in \mathbb{R}^3$. Since Γ is a discrete group of isometries of $\mathbb{R}^3$, acting freely, Theorems 3.3.3 and 3.5.1 in [Wol11] show that each $\gamma \in \Gamma$ can be split uniquely as $\delta \times \psi$ where $\delta = (r|_{\mathbb{R}^n}, a)$ and $\psi = (r|_{\mathbb{R}^{3-n}}, 0)$, and now $a \in \mathbb{R}^n$. Here $n \in \{0,1,2,3\}$ is the rank of the maximal abelian subgroup of Γ (i.e. the subgroup generated by the translations) and we use an orthonormal basis adapted to this subgroup to write $\mathbb{R}^3 = \mathbb{R}^n \times \mathbb{R}^{3-n}$. We can now finish the proof by considering different values of n separately.

• n = 0

In this case we have $\gamma = \psi$ and it consists of purely an orthogonal transformation. Any such transformation has a fixed point and therefore contradicts the fact that Γ acts freely, unless it is the identity transformation. Therefore Γ is trivial and we get $M \simeq \mathbb{R}^3$.

• n = 1

If $n = 1$ then we have $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ and $\delta = (r|_{\mathbb{R}}, a)$ where a is a translation in the $\mathbb{R}$ direction. Now let $s \in [0,1]$ and $(u,v) \in \mathbb{R} \times \mathbb{R}^2$ and consider the curve $\alpha(s) = (u+sa, v) \subset \mathbb{R}^3$. Then for any fixed radius $R > |a|$, the ball $B^g_R(\pi(u,v)) \subset M$ contains the loop $\pi(\alpha)$. For $a \neq 0$, this loop cannot be homotopic to the constant loop $c(s) \equiv \pi(u,v)$. If it was, then by the uniqueness of path liftings in the covering space (see [Mun00, Chapter 9] or [Lee11, Theorem 11.15]), the lifts of $c(s)$ and $\pi(\alpha(s))$ starting at (u,v) must end at the same point. But this is not true because the lift of $c(s)$ is the constant loop in $\mathbb{R} \times \mathbb{R}^2$, so it ends at (u,v), whereas the lift of $\pi(\alpha(s))$ is $\alpha(s)$, which ends at $(u+a,v)$. Therefore $\pi(\alpha(s))$ is a non-trivial loop and so $B^g_R(\pi(u,v))$ is not simply connected. Similarly, $B^g_R(\pi(u,v_i)) \subset M$ contains $\pi(\alpha_i(s))$ where $\alpha_i(s) = (u+sa, v_i)$ and $v_i \in \mathbb{R}^2$ such that $d(\pi(v_i), \pi(v)) \to \infty$. These $B^g_R(\pi(u,v_i))$ are not simply connected, which contradicts the ALSC assumption. Thus $a = 0$ and so γ must be the identity for the same reason as before.

• n = 2

This time we have $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ and $\delta = (r|_{\mathbb{R}}, a)$ where $a \in \mathbb{R}^2$. Now let $s \in [0,1]$ and $(u,v) \in \mathbb{R}^2 \times \mathbb{R}$ and consider the curve $\alpha(s) = (u + sa, v) \subset \mathbb{R}^3$. Then the same argument as for the $n = 1$ case works, but now $R > |a|_{\mathbb{R}^2}$ and $v_i \in \mathbb{R}$.

• n = 3

If $n = 3$ and $a \in \mathbb{R}^3 \setminus \{0\}$ then the quotient $M \simeq \mathbb{R}^3/\Gamma$ is compact, which contradicts the ALSC assumption. Therefore $a = 0$ and we can again conclude that $\Gamma = \{Id\}$. $\square$

6 Bartnik Mass Theorems

In this section we will prove Theorems 1.34 and 1.35. First, recall the variant of the Bartnik mass used in this thesis.

Definition 6.1. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. For a bounded, open set $\Omega \subset M$ with smooth, outer-minimising topological boundary $\partial\Omega$, the Bartnik mass of Ω is*

$$m_B(\Omega) := \inf\{m_{ADM}(N) : N \in \mathcal{A}\}$$

where $\mathcal{A}$ is the set of AF, complete, 3D Riemannian manifolds with non-negative scalar curvature, into which Ω isometrically embeds, where ∂N is the only compact, minimal surface in N and $\partial\Omega \subset N$ is outer-minimising.

As outlined in the summarising Section 1.5, our choice of Bartnik mass allows us to prove the next proposition, which will be the key ingredient in the proof of Theorems 1.34 and 1.35.

Proposition 6.2. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Fix $p \in M \setminus \partial M$ and consider an arbitrary sequence of perturbed geodesic spheres $S_{p,\rho_n}(w_n)$ satisfying $\rho_n \to 0$ and $\|w_n\|_{C^1(S^2)} \to 0$ as $n \to \infty$. Then there exist $N(p) > 0$ such that $S_{p,\rho_n}(w_n)$ is outer-minimising for every $n \geq N$.*

The main proof of Proposition 6.2 will take place in Section 6.3, and then we will use it to complete the proof of the Bartnik mass theorems in Section 6.4. To begin, two preliminary sections provide some results we will need.

6.1 Convergence of Manifolds

Here, we state a few definitions concerning the convergence of manifolds, which will be helpful because they describe properties which apply, in particular, to AF manifolds.

Definition 6.3. *A complete n-dimensional Riemannian manifold (M,g) has bounded geometry if there exists positive constants C and V such that the sectional curvature is bounded*

$$|K| \leq C$$

and the injectivity radius is bounded below

$$\mathrm{inj}_M \geq V > 0$$

Definition 6.4 ([Pet16]). *Let $m \in \mathbb{N}$, $\beta \in [0,1]$, (M,g) be a $C^{m+1,\beta}$-manifold with the $C^{m,\beta}$-metric g and* *let $p \in M$. A sequence of pointed, smooth, complete Riemannian manifolds (M_n, g_n, p_n) converges in the pointed $C^{m,\beta}$-topology to the manifold (M, g, p), written $(M_n, g_n, p_n) \to (M, g, p)$, if for every $R > 0$ we can find a domain Ω_R with $B_R^g(p) \subset \Omega_R \subset M$ together with maps $F_{n,R} : \Omega_R \to M_n$, and a natural number $n_R \in \mathbb{N}$ such that $B_R^{g_n}(p_n) \subset F_{n,R}(\Omega_R)$ and $F_{n,R}$ are $C^{m+1,\beta}$-embeddings for all $n \geq n_R$, and $F_{n,R}^*(g_n) \to g$ on Ω_R in the $C^{m,\beta}$-topology.*

Definition 6.5. *A complete Riemannian manifold (M, g) has $C^{m,\beta}$-bounded geometry if it has bounded geometry and for every diverging sequence of points p_n there exists a subsequence p_{n_l} and a pointed $C^{m+1,\beta}$-manifold $(M_\infty, g_\infty, p_\infty)$ with $C^{m,\beta}$-metric such that $(M, g, p_{n_l}) \to (M_\infty, g_\infty, p_\infty)$ in the pointed $C^{m,\beta}$-topology.*

Lemma 6.6. *If (M^3, g) is AF, then it has $C^{2,\alpha}$-bounded geometry.*

Proof. For the sectional curvature bound, first note that in the AF chart the metric g has the representation $g_{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(|x|^{-\tau})$, where $\tau > \frac{1}{2}$. This implies the same decay on the inverse metric $g^{\mu\nu} = \delta^{\mu\nu} + \mathcal{O}(|x|^{-\tau})$. Using this, together with the decay on the first derivatives $\partial g = \mathcal{O}(|x|^{-\tau-1})$ and the formula for the Christoffel symbols, yields

$$\Gamma^\eta_{\mu\nu} = \frac{1}{2} g^{\eta\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) = \mathcal{O}(|x|^{-\tau-1}) \quad \text{as } |x| \to \infty \tag{6.1}$$

Recalling the formula for the Riemann curvature tensor in coordinates and using the AF condition again, gives

$$R_{\mu\nu\eta\lambda} = g_{\lambda\zeta}\left(\partial_\mu \Gamma^\zeta_{\nu\eta} - \partial_\nu \Gamma^\zeta_{\mu\eta} + \Gamma^\alpha_{\nu\eta}\Gamma^\zeta_{\mu\alpha} - \Gamma^\alpha_{\mu\eta}\Gamma^\zeta_{\nu\alpha}\right) = \mathcal{O}(|x|^{-\tau-2}) \quad \text{as } |x| \to \infty \tag{6.2}$$

Finally, plugging (6.1) and (6.2) in to the formula for the sectional curvature $K(S_\Pi)$ of a plane section S_Π at a point $p \in M$ with respect to an orthonormal basis $X_p, Y_p \in T_p S_\Pi \subset T_p M$, reveals

$$K(S_\Pi) = \mathrm{Rm}(X_p, Y_p, Y_p, X_p) = \mathcal{O}(|x|^{-\tau-2}) \quad \text{as } |x| \to \infty$$

Thus the sectional curvature is bounded.

For the bound on inj_M, consider the complement of the compact set $\mathcal{K} \subset M$ where we have the AF chart. Using (6.1), the geodesic equation for curves $\gamma(t)$ becomes

$$\begin{aligned} 0 &= \ddot{\gamma}^\eta(t) + \dot{\gamma}^\mu(t)\dot{\gamma}^\nu(t)\Gamma^\eta_{\mu\nu}(\gamma(t)) \\ &= \ddot{\gamma}^\eta(t) + \dot{\gamma}^\mu(t)\dot{\gamma}^\nu(t)\mathcal{O}(|x|^{-\tau-1}) \end{aligned}$$

As this is a system of (non-linear) second order ODEs with a smooth function of the perturbation parameter $|x|$, we know that the solutions (i.e. the g-geodesics), depend smoothly on $|x|$, as $|x| \to \infty$. See [Lee13, Appendix D] or [Har02, Chapter 5]. Thus, for a diverging sequence p_n, the g-exponential map, which is defined by the g-geodesics, satisfies

$$\|\exp^g_{p_n} - \exp^{\bar{g}}_{p_n}\|_{C^k} = \mathcal{O}(|p_n|^{-\tau-1}) \qquad \forall k$$

where we consider both maps (for $\exp^g_{p_n}$, via the pullback metric) as maps on the complement of a ball in $\mathbb{R}^3$. Hence the maximal domain where the exponential map at p_n is a diffeomorphism onto its image increases without bound as $n \to \infty$ (i.e. $\mathrm{inj}(p_n) \to \infty$). Enlarging $\mathcal{K}$ as necessary so that $\mathrm{inj}_{M\setminus\mathcal{K}} \geq C > 0$ for some positive constant C, by compactness we have a finite covering of $\mathcal{K}$ made of uniformly normal neighbourhoods [Lee18, Lemma 6.16]. This gives a finite set of positive constants $\{C_1, ..., C_j\}$ such that each point $q \in \mathcal{K}$ satisfies $\mathrm{inj}(q) \geq C_k$ for some $1 \leq k \leq j$. Therefore $\mathrm{inj}_{\mathcal{K}} = \inf_{q\in\mathcal{K}}(\mathrm{inj}(q)) \geq \min(C_1, ..., C_j) > 0$ and $\mathrm{inj}_M \geq \min(C, C_1, ..., C_j) > 0$.

Again, take a diverging sequence of points $p_n \in M$. We will show that $(M, g, p_n) \to (\mathbb{R}^3, \bar{g}, 0)$. Let $R > 0$. Define $\mathcal{T}_a$ and $\mathcal{S}_\lambda$, the translation and scaling maps on $\mathbb{R}^3$, where $a \in \mathbb{R}^3$ and $\lambda > 1$, by

$$\mathcal{T}_a : \mathbb{R}^3 \to \mathbb{R}^3 \qquad x \mapsto x + a$$

$$\mathcal{S}_\lambda : \mathbb{R}^3 \to \mathbb{R}^3 \qquad x \mapsto \lambda \cdot x$$

Since (M, g) is AF, we have the chart ϕ as in Definition 1.2. Thus we can define the following map:

$$\phi_{n,R} := \phi^{-1} \circ \mathcal{T}_{\phi(p_n)} \circ \mathcal{S}_\lambda : B^{\bar{g}}_R(0) \to M$$

Since p_n diverges, so does $\phi(p_n)$, and for $|\phi(p_n)|_{\bar{g}} > \lambda R + 1$ the image $\mathcal{T}_{\phi(p_n)} \circ \mathcal{S}_\lambda(B^{\bar{g}}_R(0))$ is contained in the domain of the diffeomorphism ϕ^{-1}. That is

$$\mathcal{T}_{\phi(p_n)} \circ \mathcal{S}_\lambda(B^{\bar{g}}_R(0)) = B^{\bar{g}}_{\lambda R}(\phi(p_n)) \subset \mathbb{R}^3 \setminus \overline{B^{\bar{g}}_1(0)}$$

Thus we can find an $n'_R \in \mathbb{N}$ such that $\phi_{n,R}$ will be a diffeomorphism onto its image for $n > n'_R$. We also have $\phi_{n,R}(0) = p_n$ and $\phi^*_{n,R} g \to \bar{g}$ on $B^{\bar{g}}_R(0)$ in the $C^{2,\beta}$-topology for all $\beta \in [0, 1]$, because $n \to \infty$ corresponds to $|x| \to \infty$ in the AF chart ϕ. Finally, we need to show that

$$B^g_R(p_n) \subset \phi_{n,R}(B^{\bar{g}}_R(0))$$

for large enough n. Since ϕ is a diffeomorphism, this is equivalent to showing

$$\phi\left(B^g_R(p_n)\right) \subset B^{\bar{g}}_{\lambda R}(\phi(p_n))$$

Using the pull back metric, so that ϕ is an isometry, this means showing

$$B^{(\phi^{-1})^* g}_R(\phi(p_n)) \subset B^{\bar{g}}_{\lambda R}(\phi(p_n))$$

But we know from the above argument that $\exp^g_{p_n} \to \exp^{\bar{g}}_{p_n}$ and $\mathrm{inj}(p_n) \to \infty$. Therefore the metric balls $B^{(\phi^{-1})^* g}_R(\phi(p_n))$ are geodesic balls when n is large enough, and for any small $\varepsilon > 0$ there is an $n''_R \in \mathbb{N}$ such that

$$B^{(\phi^{-1})^* g}_R(\phi(p_n)) \subset B^{\bar{g}}_{R+\varepsilon}(\phi(p_n))$$

for $n > n''_R$. Taking $\varepsilon < \lambda R - R$, we have $\lambda R > R + \varepsilon$, and thus

$$B^{(\phi^{-1})^* g}_R(\phi(p_n)) \subset B^{\bar{g}}_{R+\varepsilon}(\phi(p_n)) \subset B^{\bar{g}}_{\lambda R}(\phi(p_n))$$

for $n > n''_R$. Taking $n_R = \max(n'_R, n''_R)$ finishes the proof.

□

Now we consider the rescaled Riemannian metrics $g_\rho := \rho^{-2} g$.

Lemma 6.7. *The pointed Riemannian manifolds (M^3, g_ρ, p) converge to $(\mathbb{R}^3, \bar{g}, 0)$ in the pointed $C^{m,\beta}$-topology for every m as $\rho \downarrow 0$ (i.e. we have smooth Cheeger-Gromov convergence).*

Proof. Explicitly, consider the (inverse of the) normal coordinate charts with respect to the metrics g_ρ. Lemma 2.6 implies the constant scaling $g_\rho := \rho^{-2} g$ yields $\Gamma^g = \Gamma^{g_\rho}$ and therefore $\nabla^g = \nabla^{g_\rho}$. Thus (M, g) and (M, g_ρ) have the same geodesics and exponential maps. However, using $\exp^g_p$ and $\exp^{g_\rho}_p$ to obtain normal coordinates for (M, g) and (M, g_ρ) respectively requires choosing orthonormal bases of T_pM. To this end, if $\{E_1, E_2, E_3\}$ is a g-orthonormal basis of T_pM, then $\{\rho E_1, \rho E_2, \rho E_3\}$ is a g_ρ-orthonormal basis of T_pM because

$$g_\rho(\rho E_\mu, \rho E_\nu) = \rho^{-2} g(\rho E_\mu, \rho E_\nu) = g(E_\mu, E_\nu) = \delta_{\mu\nu} \quad \text{at } p$$

Using these bases we get the normal coordinate charts (diffeomorphisms) ϕ_g and ϕ_{g_ρ} such that $\phi_g^{-1}(\rho x) = \phi_{g_\rho}^{-1}(x)$. Furthermore, by Lemma 6.6 we know that, for any p, the domain of ϕ_g^{-1} contains $B^{\bar{g}}_{\bar{R}}(0) \subset \mathbb{R}^3$ for some $\bar{R} > 0$. Therefore, for any p, the domain of $\phi_{g_\rho}^{-1}$ contains $B^{\bar{g}}_{\rho^{-1}\bar{R}}(0) \subset \mathbb{R}^3$. Note that the sets $B^{\bar{g}}_{\rho^{-1}\bar{R}}(0)$ exhaust $\mathbb{R}^3$ as $\rho \to 0$.

Let $(g_\rho)_{\mu\nu}$ and $g_{\mu\nu}$ be the components of the metrics g_ρ and g in their respective normal coordinate charts and $q = \phi_{g_\rho}^{-1}(x) = \phi_g^{-1}(\rho x)$ a point near p. From (3.1)

$$g(\partial_\mu, \partial_\nu)(\phi_g^{-1}(x)) = g_{\mu\nu}(x) = \delta_{\mu\nu} + \mathcal{O}(|x|^2)$$

where ∂_μ are the coordinate vector fields induced by the chart ϕ_g. Thus

$$\begin{aligned}
(g_\rho)_{\mu\nu}(x) &= g_\rho(\partial_\mu^\rho, \partial_\nu^\rho)(\phi_{g_\rho}^{-1}(x)) \\
&= \rho^{-2} g(\partial_\mu^\rho, \partial_\nu^\rho)(\phi_{g_\rho}^{-1}(x)) \\
&= g(\partial_\mu, \partial_\nu)(\phi_{g_\rho}^{-1}(x)) \\
&= g(\partial_\mu, \partial_\nu)(\phi_g^{-1}(\rho x)) \\
&= g_{\mu\nu}(\rho x) \\
&= \delta_{\mu\nu} + \mathcal{O}(|\rho x|^2) \qquad (6.3)
\end{aligned}$$

In particular, $(g_\rho)_{\mu\nu}(x) = \delta_{\mu\nu} + \mathcal{O}(\rho^2)$. Therefore, given $R > 0$, any of the diffeomorphisms $\phi_{g_\rho}^{-1}$ with $\rho < \frac{\bar{R}}{R}$ satisfy the requirements for the maps in Definition 6.4, where we can take $\Omega_R = B^{\bar{g}}_R(0)$. Since geodesic balls are also metric balls, the containment requirement is true for $\phi_{g_\rho}^{-1}$. Finally, (6.3) also shows $(\phi_{g_\rho}^{-1})^* g_\rho \to \bar{g}$ in $C^k(B^{\bar{g}}_R(0))$-norm as $\rho \to 0$. Therefore we can find a sequence $\phi_{g_{\rho_n}}^{-1}$, where $\rho_n \to 0$ as $n \to \infty$, such that the lemma is true.

□

6.2 Sets of Finite Perimeter

In this section we prove some properties of finite perimeter sets which will help in the proof of Proposition 6.2. Let (M, g) be a 3D Riemannian manifold. For a Borel subset $E \subset M$, recall that we denote its volume by $|E|_g$ and its perimeter by $\mathrm{P}_g(E)$. Also, recall that $E \Delta F$ is the symmetric difference of the sets E and F.

The *isoperimetric profile function* $\mathcal{I}_{(M,g)} : [0, \infty) \to [0, \infty)$ of (M, g) is defined by

$$\mathcal{I}_{(M,g)}(v) := \inf\{\mathrm{P}_g(E) : E \subset M \text{ is a finite perimeter set with } |E|_g = v\}. \tag{6.4}$$

In the Euclidean case, where $(M,g) = (\mathbb{R}^3, \bar{g})$, we know that the round spheres are the solutions to the related *isoperimetric problem*. Therefore, given a volume v, we can rearrange the volume formula $|B_r^{\bar{g}}(p)|_{\bar{g}} = \frac{4\pi}{3} \cdot r^3$ to find the radius of the round ball of volume v is $r = \left(\frac{3v}{4\pi}\right)^{\frac{1}{3}}$, which implies that (for any $p \in \mathbb{R}^3$) $\mathrm{P}_{\bar{g}}(B_r^{\bar{g}}(p)) = |S_{p,r}(0)|_{g_{S_{p,r}(0)}} = |\mathbb{S}_r^2|_{g_{\mathbb{S}_r^2}} = 4\pi \cdot r^2 = 4\pi \cdot \left(\frac{3v}{4\pi}\right)^{\frac{2}{3}} = (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}}$. Thus $\mathcal{I}_{(\mathbb{R}^3,\bar{g})}(v) = (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}}$ for all $v > 0$. The next result generalises this to AF manifolds.

Lemma 6.8. *If (M,g) is an AF, complete, 3D Riemannian manifold, then*

$$\lim_{v \downarrow 0} v^{-\frac{2}{3}} \mathcal{I}_{(M,g)}(v) = \begin{cases} (36\pi)^{\frac{1}{3}} & \text{if } \partial M = \emptyset \\ (18\pi)^{\frac{1}{3}} & \text{if } \partial M \neq \emptyset \end{cases} \tag{6.5}$$

Proof. First we assume $\partial M = \emptyset$. Thanks to Lemma 6.6, we can apply the following theorem of Nardulli [Nar14, Theorem 1].

Theorem 6.9. *Let (M,g) be a complete, 3D Riemannian manifold with $C^{2,\alpha}$-bounded geometry. There exists $\bar{v}(\alpha) = \bar{v} > 0$ such that for all $0 < v < \bar{v}$*

$$\mathcal{I}_{(M,g)}(v) = \min\left\{|S_{p_\infty,\rho}(w)|_{g_\infty} : (M,g,p_n) \to (M_\infty, g_\infty, p_\infty)\right\} \tag{6.6}$$

Here, $S_{p_\infty,\rho}(w)$ is the unique perturbed sphere in $(M_\infty, g_\infty, p_\infty)$, centered at p_∞, of volume v. Note that the function space used in [Nar14] (see also [Nar09] and [PX09]) for the perturbation w is different to that used in this thesis. They take

$$w \in C^{2,\alpha}(\mathbb{S}^2) \cap \mathrm{Ker}[\Delta_{\mathbb{S}^2} + 2]^\perp \subset L^2(\mathbb{S}^2)$$

together with an additional conditon on the mean curvature of $S_{p,\rho}(w)$. The author proves that such perturbed spheres (called pseudo-bubbles or pseudo-balls) satisfy

$$w = w_2\rho^2 + \mathcal{O}(\rho^3) \qquad \text{and} \qquad \int_{\mathbb{S}^2} w_2 \, dV_{\mathbb{S}^2} = \frac{2\pi}{9} \mathrm{Sc}_p \tag{6.7}$$

In [Nar09, Theorem 1], the same author proves that for small enough v and all $p \in M$ there is a unique $S_{p,\rho}(w)$ such that $|B_{p,\rho}(w)|_g = v$. We also note that Theorem 6.9 is in fact a simplified version of the original in [Nar14, Theorem 1], where the author works with the larger class of manifolds that have bounded norm, in the sense of Petersen [Pet16, Section 11.3].

Case 1: p_n diverges.

In this case we know there is only one limit manifold and $(M_\infty, g_\infty, p_\infty) = (\mathbb{R}^3, \bar{g}, 0)$. In Euclidean space the isoperimetric regions are just the round spheres. Therefore the round spheres are the only pseudo-bubbles we need to consider for diverging sequences in (6.6), and they all yield $|S_{p_\infty,\rho}(w)|_{g_\infty} = |\mathbb{S}^2_\rho|_{g_{\mathbb{S}^2_\rho}} = (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}}$.

Case 2: p_n converges.

For converging sequences, $(M_\infty, g_\infty, p_\infty) = (M, g, p)$ for some $p \in M$. In this case we can apply the formulas (1.4) and (1.5) to the unique pseudo-bubble $S_{p,\rho}(w)$ of volume v (the formulas are true for any small function $w \in C^{2,\alpha}(\mathbb{S}^2)$. See [PX09]). Applying the properties (6.7) of the perturbation function to the formulas yields

$$|S_{p,\rho}(w)|_{\mathring{g}} = |\mathbb{S}^2|_{g_{\mathbb{S}^2}} \left[1 - \frac{1}{6}\mathrm{Sc}_p \rho^2\right] \rho^2 + \mathcal{O}(\rho^6)$$

$$|B_{p,\rho}(w)|_g = \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3} \left[1 - \frac{1}{5}\mathrm{Sc}_p \rho^2\right] \rho^3 + \mathcal{O}(\rho^7)$$

We can now proceed as in the Euclidean case and rearrange to get ρ in terms of $v = |B_{p,\rho}(w)|_g$. We use the same method as in [Nar09, Lemma 3.10]. Setting $A = \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3}$ yields

$$v = A\left[1 + \mathcal{O}(\rho^2)\right]\rho^3$$

Then, with a Taylor expansion

$$\begin{aligned} \rho^3 &= \frac{v}{A}\left[1 + \mathcal{O}(\rho^2)\right]^{-1} \\ &= \frac{v}{A}\left(1 + \mathcal{O}(\rho^2)\right) \end{aligned}$$

Therefore, after another Taylor expansion

$$
\begin{aligned}
\rho^2 &= \left(\frac{v}{A}\right)^{\frac{2}{3}} \left(1+\mathcal{O}(\rho^2)\right)^{\frac{2}{3}} \\
&= \left(\frac{v}{A}\right)^{\frac{2}{3}} \left(1+\mathcal{O}(\rho^2)\right) \\
&= \left(\frac{v}{A}\right)^{\frac{2}{3}} + \mathcal{O}(\rho^4)
\end{aligned}
$$

where we have used that $v = \mathcal{O}(\rho^3)$. Now sub this into the equation for $|S_{p,\rho}(w)|_{\mathring{g}}$, to get

$$
\begin{aligned}
|S_{p,\rho}(w)|_{\mathring{g}} &= |\mathbb{S}^2|_{g_{\mathbb{S}^2}} \rho^2 + \mathcal{O}(\rho^4) \\
&= |\mathbb{S}^2|_{g_{\mathbb{S}^2}} \left[\left(\frac{v}{A}\right)^{\frac{2}{3}} + \mathcal{O}(\rho^4)\right] + \mathcal{O}(\rho^4) \\
&= (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}} + \mathcal{O}(\rho^4)
\end{aligned}
$$

where we substituted $A = \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3} = \frac{4\pi}{3}$.

Therefore, combining both cases, we get

$$
\lim_{v\downarrow 0} v^{-\frac{2}{3}} \mathcal{I}_{(M,g)}(v) = \lim_{v\downarrow 0} v^{-\frac{2}{3}} \min\left\{(36\pi)^{\frac{1}{3}} v^{\frac{2}{3}}, (36\pi)^{\frac{1}{3}} v^{\frac{2}{3}} + \mathcal{O}(\rho^4)\right\} = (36\pi)^{\frac{1}{3}}
$$

Finally, if $\partial M \neq \emptyset$ then the only difference is that the sequence p_n could converge to a point on the boundary. In the limit as $v \to 0$, a sphere centered on the boundary with the same volume as a sphere contained in $M \setminus \partial M$ will have less surface area, by a factor of $2^{-\frac{1}{3}}$. Hence the minimum in (6.6) decreases to $(18\pi)^{\frac{1}{3}} v^{\frac{2}{3}}$. □

Lemma 6.10. *If (M,g) is an AF, complete, 3D Riemannian manifold, then for every $v_0 > 0$ there exists $C = C(v_0) > 0$, such that*

$$
\mathrm{P}_g(E) \geq C|E|_g^{\frac{2}{3}} \quad \textit{for every subset } E \subset M \textit{ of finite perimeter, with } |E|_g \in (0, v_0]. \tag{6.8}
$$

Proof. Thanks to Lemma 6.6, we can apply the following theorem of Nardulli and Flores [MN19, Theorem 2].

Theorem 6.11. *Let (M,g) be a complete Riemannian manifold with bounded geometry. Then $\mathcal{I}_{(M,g)}$ is continuous on the interval $[0, |M|_g)$.*

We note again that Theorem 6.11 is a simplified version of the original in [MN19, Theorem 2] where the authors prove Hölder continuity by using the bounds coming from the bounded geometry assumption. Thus, in our case

$$\mathcal{I} : [0,\infty) \to [0,\infty)$$

is a continuous function. Therefore, considering (6.5), we have

$$v^{-\frac{2}{3}}\mathcal{I}(v) : [0,\infty) \to (0,\infty)$$

is also continuous. As a non-vanishing continuous function on the closed interval $[0,v_0]$, $v^{-\frac{2}{3}}\mathcal{I}(v)$ must attain its minimum $0 < C = C(v_0) \leq (36\pi)^{\frac{1}{3}}$, which depends on v_0. Thus, we have

$$v^{-\frac{2}{3}}\mathcal{I}(v) \geq C \quad \forall v \in [0,v_0]$$

and then

$$\mathcal{I}(v) \geq Cv^{\frac{2}{3}} \quad \forall v \in (0,v_0]$$

Therefore

$$\mathrm{P}_g(E) \geq \mathcal{I}(v) \geq C|E|_g^{\frac{2}{3}} \quad \forall\, E \quad \text{such that} \quad |E|_g = v \in (0,v_0]$$

$\square$

For the rest of this subsection, (M,g) is an arbitrary 3D Riemannian manifold. Once again considering the rescaled metrics $g_\rho := \rho^{-2}g$ on M, we have the following useful lemma.

Lemma 6.12. *Let $E \subset M$ be measurable and $V \subset M$ open. Then $\mathrm{P}_{g_\rho}(E,V) = \rho^{-2}\mathrm{P}_g(E,V)$.*

Proof. First, since $\sqrt{\det g_\rho} = \sqrt{\det \rho^{-2}g} = \rho^{-3}\sqrt{\det g}$, we have $dV_{g_\rho} = \rho^{-3}dV_g$. Second, the conformal factor ρ^{-2} is constant with respect to any coordinates and so, for any vector field $X \in C_c^1(V,TM)$

$$div_{g_\rho}(X) = \frac{1}{\sqrt{\det g_\rho}}\frac{\partial}{\partial x^\mu}(X^\mu\sqrt{\det g_\rho}) = \frac{1}{\rho^{-3}\sqrt{\det g}}\frac{\partial}{\partial x^\mu}(\rho^{-3}X^\mu\sqrt{\det g}) = div_g(X)$$

where we have used the standard formula for its divergence. Therefore

$$\int_{E\cap V} div_{g_\rho}(X)dV_{g_\rho} = \rho^{-3}\int_{E\cap V} div_g(X)dV_g$$

Next, we confirm that the vector field is scaled by ρ^{-1} by computing in normal coordinates at $p \in V$

$$\begin{aligned}
||X_p||_{g_\rho} &= \left[g_\rho(X_p, X_p)\right]^{\frac{1}{2}} \\
&= \left[g_\rho(\rho E_\mu, \rho E_\nu)\cdot X_p(\rho^{-1}\phi_g^\mu)\cdot X_p(\rho^{-1}\phi_g^\nu)\right]^{\frac{1}{2}} \\
&= \left[g(E_\mu, E_\nu)\cdot X_p(\rho^{-1}\phi_g^\mu)\cdot X_p(\rho^{-1}\phi_g^\nu)\right]^{\frac{1}{2}} \\
&= \left[\rho^{-2} g(E_\mu, E_\nu)\cdot X_p(\phi_g^\mu)\cdot X_p(\phi_g^\nu)\right]^{\frac{1}{2}} \\
&= \rho^{-1}||X_p||_g
\end{aligned}$$

where we used $\phi_{g_\rho} = \rho^{-1}\phi_g$.

Thus, for any $p \in V$, $||X_p||_{g_\rho} \leq 1 \iff ||X_p||_g \leq \rho$. Hence $||X||_{\infty,g_\rho} \leq 1 \iff ||X||_{\infty,g} \leq \rho$. Therefore

$$\begin{aligned}
\mathrm{P}_{g_\rho}(E,V) &:= \sup_{||X||_{\infty,g_\rho}\leq 1}\left\{\int_{E\cap V} div_{g_\rho}(X)dV_{g_\rho}\right\} \\
&= \sup_{||X||_{\infty,g}\leq \rho}\left\{\rho^{-3}\int_{E\cap V} div_g(X)dV_g\right\} \\
&= \sup_{||X||_{\infty,g}\leq 1}\left\{\rho^{-3}\int_{E\cap V} div_g(\rho X)dV_g\right\} \\
&= \rho^{-2}\sup_{||X||_{\infty,g}\leq 1}\left\{\int_{E\cap V} div_g(X)dV_g\right\} \\
&= \rho^{-2}\mathrm{P}_g(E,V)
\end{aligned}$$

□

Using (6.3), in g_ρ-normal coordinates at a point q, we have

$$(g_\rho)_{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(\rho^2 r^2) \tag{6.9}$$

where $r \in (0, \mathrm{inj}(q)]$ is the radial distance from q. Recall that this means

$$\|(g_\rho)_{\mu\nu} - \delta_{\mu\nu}\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} \leq C\rho^2 r^2$$

for some suitable $C(q,k) > 0$.

We now look for a relationship between the relative perimeters $\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q))$ and $\mathrm{P}_{\bar{g}}(\phi_{g_\rho}(F), B_r^{\bar{g}}(\phi_{g_\rho}(q)))$, of a finite perimeter set $F \subset M$ and its image $\phi_{g_\rho}(F) \subset \mathbb{R}^3$. We remark that, in such relative perimeter calculations, we can ignore the fact that F may not be contained in the domain of ϕ_{g_ρ} because it is enough that $B_r^{g_\rho}(q)$ is contained there (which, for small ρ, it will be). To achieve the desired relationship we will distinguish between the normal coordinate charts centered at either p or a point q nearby, denoting them $\phi_{g_\rho}^p$ and $\phi_{g_\rho}^q$ respectively. First, a preparatory lemma.

Lemma 6.13. *Let $F \subset (M, g_\rho)$ be a set of finite perimeter and $\phi_{g_\rho}^p$ and $\phi_{g_\rho}^q$ be the normal coordinate charts centered at p and q respectively. Then*

$$\mathrm{P}_{\bar{g}}(\phi_{g_\rho}^p(F), B_r^{\bar{g}}(\phi_{g_\rho}^p(q))) = (1 + \mathcal{O}(\rho^2 r^2))\mathrm{P}_{\bar{g}}(\phi_{g_\rho}^q(F), B_r^{\bar{g}}(\phi_{g_\rho}^q(q))) \tag{6.10}$$

Proof. From (6.9), in the chart $\phi_{g_\rho}^p$, we have both $(g_\rho)_{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(\rho^2 r^2)$ and $(g_\rho)^{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(\rho^2 r^2)$. Therefore, the Christoffel symbols satisfy

$$\begin{aligned}(\Gamma_{g_\rho})^\lambda_{\mu\nu} = \frac{1}{2}(g_\rho)^{\lambda\eta}(\partial_\mu (g_\rho)_{\nu\eta} - \partial_\nu (g_\rho)_{\mu\eta} + \partial_\eta (g_\rho)_{\mu\nu}) &= \frac{1}{2}(\delta_{\lambda\eta} + \mathcal{O}(\rho^2 r^2))\mathcal{O}(\rho^2 r^2) \\ &= \mathcal{O}(\rho^2 r^2)\end{aligned}$$

everywhere in the domain of $\phi_{g_\rho}^p$. Then, for curves $\gamma(t)$, the geodesic equation becomes

$$\begin{aligned}0 &= \ddot{\gamma}^\lambda(t) + \dot{\gamma}^\mu(t)\dot{\gamma}^\nu(t)(\Gamma_{g_\rho})^\lambda_{\mu\nu}(\gamma(t)) \\ &= \ddot{\gamma}^\lambda(t) + \dot{\gamma}^\mu(t)\dot{\gamma}^\nu(t)\mathcal{O}(\rho^2 r^2)\end{aligned}$$

Just as we argued in the proof of Lemma 6.6, this is a system of (non-linear) second order ODEs with a smooth function of the perturbation parameter ρ, we know that the solutions (i.e. the g_ρ-geodesics), depend smoothly on ρ, as $\rho \to 0$. See [Lee13, Appendix D] or [Har02, Chapter 5]. Thus, for any point p' in the domain of $\phi_{g_\rho}^p$, the g_ρ-exponential map, which is defined by the g_ρ-geodesics, satisfies

$$\|\exp_{p'}^{g_\rho} - \exp_{\phi_{g_\rho}^p(p')}^{\bar{g}}\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} \leq C\rho^2 r^2$$

where we consider both maps (for $\exp_{p'}^{g_\rho}$, via the pullback metric) as diffeomorphisms on a ball in $\mathbb{R}^3$. Note that $\exp_{\phi_{g_\rho}^p(p')}^{\bar{g}} = \mathcal{T}_{\phi_{g_\rho}^p(p')}$, where $\mathcal{T}_{\phi_{g_\rho}^p(p')}$ is the translation map by vector $\phi_{g_\rho}^p(p')$. Since the normal coordinate chart is the inverse of the exponential map, this means

$$\|\phi_{g_\rho}^{p'} - \mathcal{T}_{-\phi_{g_\rho}^{p}(p')}\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} \leq C\rho^2 r^2$$

Setting $p' = p$ and $p' = q$ gives $\phi_{g_\rho}^{p} \to \mathcal{T}_{-\phi_{g_\rho}^{p}(p)} = \mathcal{T}_0 = Id$ and $\phi_{g_\rho}^{q} \to \mathcal{T}_{-\phi_{g_\rho}^{p}(q)}$ respectively. Therefore

$$\begin{aligned}
\|\phi_{g_\rho}^{p} - \mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} &= \|\phi_{g_\rho}^{p} - Id + Id - \mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} \\
&\leq \|\phi_{g_\rho}^{p} - Id\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} + \|Id - \mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))} \\
&\leq C\rho^2 r^2 \qquad (6.11)
\end{aligned}$$

In other words, $\phi_{g_\rho}^{p}$ tends towards the identity, where $\phi_{g_\rho}^{q}$ differs from it by a translation. In particular

$$\sup|\phi_{g_\rho}^{p}(x) - \mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}(x)|_{\mathbb{R}^3} = \|\phi_{g_\rho}^{p} - \mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}\|_{C^0} \leq C\rho^2 r^2$$

and so, for any set $\hat{F}$ in the domain of $\phi_{g_\rho}^{p}$ the symmetric difference $\phi_{g_\rho}^{p}(\hat{F})\Delta\mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}(\hat{F})$ is contained in a union of balls of radius $\mathcal{O}(\rho^2 r^2)$. Thus

$$|\phi_{g_\rho}^{p}(\hat{F})\Delta\mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}(\hat{F})|_{\bar{g}} \leq C\rho^2 r^2$$

By (6.11) with $k = 1$ we have $\phi_{g_\rho}^{p} \circ (\mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q})^{-1} = Id + \mathcal{O}(\rho^2 r^2)$. Its jacobian is $1 + \mathcal{O}(\rho^2 r^2)$ and so, by the change of variables formula

$$\begin{aligned}
\int_{\phi_{g_\rho}^{p}(F\cap B_r^{g_\rho}(q))} div_{\bar{g}}(X)\, dV_{\bar{g}} &= \int_{\mathcal{T}_{\phi_{g_\rho}^{p}(q)}\circ\phi_{g_\rho}^{q}(F\cap B_r^{g_\rho}(q))} div_{\bar{g}}(X)(1 + \mathcal{O}(\rho^2 r^2))\, dV_{\bar{g}} \\
&= (1 + \mathcal{O}(\rho^2 r^2)) \int_{\mathcal{T}_{\phi_{g_\rho}^{p}(q)}\circ\phi_{g_\rho}^{q}(F\cap B_r^{g_\rho}(q))} div_{\bar{g}}(X)\, dV_{\bar{g}} \qquad (6.12)
\end{aligned}$$

where the constant in $\mathcal{O}(\rho^2 r^2)$ depends on $\|\phi_{g_\rho}^{p} - \mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}\|_{C^1}$.

To proceed we use the correspondence between vector fields on $\phi_{g_\rho}^{p}(B_r^{g_\rho}(q))$ and their push forward on $\mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q}(B_r^{g_\rho}(q))$ via the transition chart (diffeomorphism) $\mathcal{T}_{\phi_{g_\rho}^{p}(q)} \circ \phi_{g_\rho}^{q} \circ (\phi_{g_\rho}^{p})^{-1} = Id + \mathcal{O}(\rho^2 r^2) := \Psi$. This implies

$$\Psi_* X = X^\mu(\partial_\mu \Psi^\nu)\partial_\nu = X^\mu(\delta^\nu_\mu + \sigma^\nu_\mu)\partial_\nu = X^\mu \partial_\mu + X^\mu \sigma^\nu_\mu \partial_\nu$$

where, for each μ and ν, $\sigma^\nu_\mu = \mathcal{O}(\rho^2 r^2)$. Thus

$$\Psi_* X = X + \mathcal{O}(\rho^2 r^2)||X||_{\infty,\bar{g}} \sum_\mu \partial_\mu$$

Comparing the length of a vector field (at a point) with its push forward via Ψ, shows

$$\bar{g}(\Psi_* X, \Psi_* X) = \bar{g}(X + \mathcal{O}(\rho^2 r^2)\sum_\mu \partial_\mu, X + \mathcal{O}(\rho^2 r^2)\sum_\mu \partial_\mu) = \bar{g}(X,X) + \mathcal{O}(\rho^2 r^2)$$

and therefore $||\Psi_* X||_{\infty,\bar{g}} = (1+\mathcal{O}(\rho^2 r^2))||X||_{\infty,\bar{g}}$.

Now, taking the supremum over all $X \in C^1_c(\phi^p_{g_\rho}(B^{g_\rho}_r(q)), \mathbb{R}^3)$ with $||X||_{\infty,\bar{g}} \leq 1$ in (6.12) yields

$$\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(F), \phi^p_{g_\rho}(B^{g_\rho}_r(q))) = (1+\mathcal{O}(\rho^2 r^2))\mathcal{S}_{p,q,\rho} \tag{6.13}$$

where

$$\mathcal{S}_{p,q,\rho} := \sup_{\substack{||X||_{\infty,\bar{g}} \leq 1 \\ X \in C^1_c(\phi^p_{g_\rho}(B^{g_\rho}_r(q)),\mathbb{R}^3)}} \left\{ \int_{\mathcal{T}_{\phi^p_{g_\rho}(q)} \circ \phi^q_{g_\rho}(F \cap B^{g_\rho}_r(q))} div_{\bar{g}}(X)\, dV_{\bar{g}} \right\}$$

Thanks to the correspondence and norm comparison described above, we have

$$\begin{aligned}
\mathcal{S}_{p,q,\rho} &= \sup_{\substack{||X||_{\infty,\bar{g}} \leq 1+\mathcal{O}(\rho^2 r^2) \\ X \in C^1_c(\mathcal{T}_{\phi^p_{g_\rho}(q)} \circ \phi^q_{g_\rho}(B^{g_\rho}_r(q)),\mathbb{R}^3)}} \left\{ \int_{\mathcal{T}_{\phi^p_{g_\rho}(q)} \circ \phi^q_{g_\rho}(F \cap B^{g_\rho}_r(q))} div_{\bar{g}}(X)\, dV_{\bar{g}} \right\} \\
&= \sup_{\substack{||X||_{\infty,\bar{g}} \leq 1 \\ X \in C^1_c(\mathcal{T}_{\phi^p_{g_\rho}(q)} \circ \phi^q_{g_\rho}(B^{g_\rho}_r(q)),\mathbb{R}^3)}} \left\{ \int_{\mathcal{T}_{\phi^p_{g_\rho}(q)} \circ \phi^q_{g_\rho}(F \cap B^{g_\rho}_r(q))} div_{\bar{g}}((1+\mathcal{O}(\rho^2 r^2))X)\, dV_{\bar{g}} \right\} \\
&= (1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{\bar{g}}(\mathcal{T}_{\phi^p_{g_\rho}(q)} \circ \phi^q_{g_\rho}(F), \mathcal{T}_{\phi^p_{g_\rho}(q)}(B^{\bar{g}}_r(\phi^q_{g_\rho}(q)))
\end{aligned} \tag{6.14}$$

Substituting (6.14) into (6.13) yields

$$
\begin{aligned}
\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(F),\phi^p_{g_\rho}(B^{g_\rho}_r(q))) &= (1+\mathcal{O}(\rho^2r^2))(1+\mathcal{O}(\rho^2r^2))\mathrm{P}_{\bar{g}}(\mathcal{T}_{\phi^p_{g_\rho}(q)}\circ\phi^q_{g_\rho}(F),\mathcal{T}_{\phi^p_{g_\rho}(q)}(B^{\bar{g}}_r(\phi^q_{g_\rho}(q)))\\
&= (1+\mathcal{O}(\rho^2r^2))\mathrm{P}_{\bar{g}}(\mathcal{T}_{\phi^p_{g_\rho}(q)}\circ\phi^q_{g_\rho}(F),\mathcal{T}_{\phi^p_{g_\rho}(q)}(B^{\bar{g}}_r(\phi^q_{g_\rho}(q))) \qquad (6.15)
\end{aligned}
$$

However, we can ignore the translation in the perimeter term on the right hand side of (6.15) because it does not affect $\mathrm{P}_{\bar{g}}$. On the left hand side of (6.15) we can use that $\|\phi^p_{g_\rho}-Id\|_{C^k(B^{g_\rho}_{r\rho^{-1}}(q))}\leq C\rho^2r^2$ implies $|\phi^p_{g_\rho}(B^{g_\rho}_r(q))\Delta B^{g_\rho}_r(\phi^p_{g_\rho}(q))|_{\bar{g}}\leq C\rho^2r^2$ (for the same reason as before) and $|B^{g_\rho}_r(\phi^p_{g_\rho}(q))\Delta B^{\bar{g}}_r(\phi^p_{g_\rho}(q))|_{\bar{g}}\leq C\rho^2r^2$ (by (6.9)), where we again view $B^{g_\rho}_r(q)$ as a subset of $\mathbb{R}^3$. This yields

$$
\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(F),B^{\bar{g}}_r(\phi^p_{g_\rho}(q))) = (1+\mathcal{O}(\rho^2r^2))\mathrm{P}_{\bar{g}}(\phi^q_{g_\rho}(F),B^{\bar{g}}_r(\phi^q_{g_\rho}(q)))
$$

□

Now we have the necessary components for proving the perimeter relationship described earlier. We prove it in the next lemma.

Lemma 6.14. *Let $F\subset(M,g_\rho)$ be a set of finite perimeter and $\phi^p_{g_\rho}$ be the normal coordinate chart centered at p. Then*

$$
\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(F),B^{\bar{g}}_r(\phi^p_{g_\rho}(q))) = (1+\mathcal{O}(\rho^2r^2))\mathrm{P}_{g_\rho}(F,B^{g_\rho}_r(q))+\mathcal{O}(\rho^2r^4) \qquad (6.16)
$$

Proof. Working in g_ρ-normal coordinates, centered at a point q near p, by (6.9) we have $(g_\rho)_{\mu\nu}=\delta_{\mu\nu}+\mathcal{O}(\rho^2r^2)$ in $B^{\bar{g}}_{r\rho^{-1}}(\phi^q_{g_\rho}(q))$. Note that in this proof the notation will suppress the difference between a vector field on M and its image in the normal coordinate chart; we write X for both.

We now compute, for $X\in C^1_c(\phi^q_{g_\rho}(B^{g_\rho}_r(q)),\mathbb{R}^3)$ with $||X||_{\infty,\bar{g}}\leq 1$, again using the formula for the divergence of a vector field.

$$
\begin{aligned}
\int_{F\cap B^{g_\rho}_r(q)} div_{g_\rho}(X)\,dV_{g_\rho} &= \int_{\phi^q_{g_\rho}(F\cap B^{g_\rho}_r(q))}\frac{1}{\sqrt{\det g_\rho}}\frac{\partial}{\partial x^\mu}(X^\mu\sqrt{\det g_\rho})\sqrt{\det g_\rho}\,dV_{\bar{g}}\\
&= \int_{\phi^q_{g_\rho}(F\cap B^{g_\rho}_r(q))}\frac{\partial}{\partial x^\mu}(X^\mu)\sqrt{\det g_\rho}+(X^\mu)\frac{\partial}{\partial x^\mu}\sqrt{\det g_\rho}\,dV_{\bar{g}}\\
&= \int_{\phi^q_{g_\rho}(F\cap B^{g_\rho}_r(q))}\frac{\partial}{\partial x^\mu}(X^\mu)(1+\mathcal{O}(\rho^2r^2))+(X^\mu)\frac{\partial}{\partial x^\mu}(1+\mathcal{O}(\rho^2r^2))\,dV_{\bar{g}}\\
&= \int_{\phi^q_{g_\rho}(F\cap B^{g_\rho}_r(q))}div_{\bar{g}}(X)+\mathcal{O}(\rho^2r^2)div_{\bar{g}}(X)+\mathcal{O}(\rho^2r)\,dV_{\bar{g}}
\end{aligned}
$$

where we have used the formula $\det B=\frac{1}{6}[(\mathrm{tr}(B))^3-3\mathrm{tr}(B)\mathrm{tr}(B^2)+2\mathrm{tr}(B^3)]$ for the determinant of a 3 by

3 matrix, with $B = (g_\rho)_{\mu\nu}$, combined with the Taylor expansion of $\sqrt{1+\ldots}$. Therefore

$$\begin{aligned}
&|\int_{F\cap B_r^{g_\rho}(q)} div_{g_\rho}(X)\, dV_{g_\rho} - \int_{\phi^q_{g_\rho}(F\cap B_r^{g_\rho}(q))} div_{\bar g}(X)\, dV_{\bar g}| \\
&\leq C\rho^2 r^2 \int_{\phi^q_{g_\rho}(F\cap B_r^{g_\rho}(q))} |div_{\bar g}(X)|\, dV_{\bar g} + D\rho^2 r |\phi^q_{g_\rho}(F)\cap B_r^{\bar g}(\phi^q_{g_\rho}(q))|_{\bar g} \\
&\leq C\rho^2 r^2 \left(\int_{\phi^q_{g_\rho}(F\cap B_r^{g_\rho}(q))\cap\{div_{\bar g}(X)>0\}} div_{\bar g}(X)\, dV_{\bar g} + \int_{\phi^q_{g_\rho}(F\cap B_r^{g_\rho}(q))\cap\{div_{\bar g}(X)<0\}} -div_{\bar g}(X)\, dV_{\bar g} \right) \\
&\quad + D\rho^2 r^4 \\
&\leq C\rho^2 r^2 \Big(\mathrm{P}_{\bar g}(\phi^q_{g_\rho}(F), \phi^q_{g_\rho}(B_r^{g_\rho}(q))\cap\{div_{\bar g}(X)>0\}) + \mathrm{P}_{\bar g}(\phi^q_{g_\rho}(F), \phi^q_{g_\rho}(B_r^{g_\rho}(q))\cap\{div_{\bar g}(X)<0\}) \Big) \\
&\quad + D\rho^2 r^4 \\
&\leq C\rho^2 r^2 \mathrm{P}_{\bar g}(\phi^q_{g_\rho}(F), \phi^q_{g_\rho}(B_r^{g_\rho}(q))) + D\rho^2 r^4
\end{aligned}$$

where in the second inequality we used $|\phi^q_{g_\rho}(F)\cap B_r^{\bar g}(\phi^q_{g_\rho}(q))|_{\bar g} \leq |B_r^{\bar g}(\phi^q_{g_\rho}(q))|_{\bar g} = Cr^3$. In the third inequality we have used that X is admissible in (2.8) if and only if $-X$ is, and in the last line we used the monotonicity of perimeter.

Taking the supremum over all $X \in C^1_c(\phi^q_{g_\rho}(B_r^{g_\rho}(q)), \mathbb{R}^3)$ with $||X||_{\infty,\bar g} \leq 1$ yields

$$\sup_{\substack{||X||_{\infty,\bar g}\leq 1 \\ X\in C^1_c(\phi^q_{g_\rho}(B_r^{g_\rho}(q)),\mathbb{R}^3)}} \left\{ \int_{F\cap B_r^{g_\rho}(q)} div_{g_\rho}(X)\, dV_{g_\rho} \right\} = (1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{\bar g}(\phi^q_{g_\rho}(F), B_r^{\bar g}(\phi^q_{g_\rho}(q))) + \mathcal{O}(\rho^2 r^4) \tag{6.17}$$

By the same arguments we used near the end of Lemma (6.13), we have a correspondence between vector fields in $C^1_c(B_r^{g_\rho}(q), TM)$ and $C^1_c(\phi^q_{g_\rho}(B_r^{g_\rho}(q)), \mathbb{R}^3)$ via the chart $\phi^q_{g_\rho}$ such that the norms satisfy $||(\phi^q_{g_\rho})_* X||_{\infty,\bar g} = (1+\mathcal{O}(\rho^2 r^2))||X||_{\infty,g_\rho}$.

Therefore, the left hand side of (6.17) becomes

$$\begin{aligned}
&\sup_{\substack{||X||_{\infty,\bar{g}}\leq 1\\ X\in C_c^1(\phi^q_{g_\rho}(B_r^{g_\rho}(q)),\mathbb{R}^3)}} \left\{\int_{F\cap B_r^{g_\rho}(q)} div_{g_\rho}(X)\, dV_{g_\rho}\right\}\\
&= \sup_{\substack{||X||_{\infty,g_\rho}\leq 1+\mathcal{O}(\rho^2 r^2)\\ X\in C_c^1(B_r^{g_\rho}(q),TM)}} \left\{\int_{F\cap B_r^{g_\rho}(q)} div_{g_\rho}(X)\, dV_{g_\rho}\right\}\\
&= \sup_{\substack{||X||_{\infty,g_\rho}\leq 1\\ X\in C_c^1(B_r^{g_\rho}(q),TM)}} \left\{\int_{F\cap B_r^{g_\rho}(q)} div_{g_\rho}((1+\mathcal{O}(\rho^2 r^2))X)\, dV_{g_\rho}\right\}\\
&= (1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q))
\end{aligned} \tag{6.18}$$

Substituting (6.18) into (6.17) gives

$$(1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q)) = (1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{\bar{g}}(\phi^q_{g_\rho}(F), B_r^{\bar{g}}(\phi^q_{g_\rho}(q))) + \mathcal{O}(\rho^2 r^4)$$

To switch $\phi^q_{g_\rho}$ with $\phi^p_{g_\rho}$, we use (6.10). Thus

$$(1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q)) = (1+\mathcal{O}(\rho^2 r^2))(1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(F), B_r^{\bar{g}}(\phi^p_{g_\rho}(q))) + \mathcal{O}(\rho^2 r^4)$$

which, simplified and rearranged, gives (6.16). □

The next result relies on the theory of rectifiable varifolds, as described in [Sim83].

Lemma 6.15. *Let $F \subset (M, g_\rho)$ be a set of finite perimeter which is stationary for perimeter in a bounded open set $U \subset M$ (i.e. zero first variation and, in particular, zero mean curvature). Then there exists constants $C = C(U, \mathrm{P}_{g_\rho}(F,U))$ and $r_0 = r_0(U, \mathrm{P}_{g_\rho}(F,U)) > 0$ such that, for $r < r_0$, we have*

$$\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q)) \leq Cr^2 \tag{6.19}$$

where $B_r^{g_\rho}(q) \subset U$.

Remark 6.16. The notation in the first part of the proof below matches that of Simon [Sim83] and is not the same as in the rest of this thesis.

Proof. The vital part of the proof is the so-called monotonicity formula for varifolds in Euclidean space. Simon shows [Sim83, Theorem 3.18] that for a rectifiable n-varifold $V = \underline{v}(M', \theta)$ (where $M' \subset \mathbb{R}^{n+k}$ is $\mathcal{H}^n_{\bar{g}}$-measurable and θ is the multiplicity function), with generalised mean curvature $H^{M'\subset\mathbb{R}^{n+k}}$ bounded by some constant Λ inside an open set $U \subset \mathbb{R}^{n+k}$ (in fact an L^p bound suffices), we have that

$$\frac{F(s)\mu_V(B_s^{\bar{g}}[\xi])}{s^n} \tag{6.20}$$

is increasing in s for $0 < s < R$ for some constant R, where $\mu_V(B_s^{\bar{g}}[\xi])$ is the mass of V contained in $B_s^{\bar{g}}[\xi]$, $B_s^{\bar{g}}[\xi] \subset U$ is a closed ball in $\mathbb{R}^{n+k}$ centered at ξ with radius s, and $F(s) \in [e^{-\Lambda R}, e^{\Lambda R}]$. In particular, this holds in the case where M' is stationary for the mass in $U \cap N^{n+l}$, where $N^{n+l} \subset \mathbb{R}^{n+k}$ is a smooth submanifold with bounded second fundamental form, h^N, in $U \cap N^{n+l}$. This is shown in [Sim83] where the author computes the first variation, revealing that $H^{M' \subset \mathbb{R}^{n+k}} = H_{M'}^{N \subset \mathbb{R}^{n+k}}$, where $H_{M'}^{N \subset \mathbb{R}^{n+k}}$ is just $H^{N \subset \mathbb{R}^{n+k}}$ "restricted to M'", defined at a point $p \in M'$ by

$$H_{M'}^{N \subset \mathbb{R}^{n+k}} := \sum_{\mu=1}^{n} h^N(E_\mu, E_\mu)$$

for an orthonormal basis E_μ of T_pM'. By (6.20), we have, for $0 < \sigma < s < R$

$$\mu_V(B_\sigma^{\bar{g}}[\xi]) \leq \frac{F(s)}{F(\sigma)} \frac{\mu_V(B_s^{\bar{g}}[\xi])}{s^n} \sigma^n \leq e^{2\Lambda R} \frac{\mu_V(\mathbb{R}^{n+k})}{s^n} \sigma^n \tag{6.21}$$

Returning to the present work, consider a set of finite perimeter (rectifiable current of codimension 0) $F \subset (M^3, g_\rho)$ which is stationary inside some bounded $U \subset M$. By De Giorgi's Theorem [De 61], the reduced boundary $\partial^* F$ is countably 2-rectifiable with $\theta \equiv 1$ and $\mathrm{P}_{g_\rho}(F, A) = \mathcal{H}^2_{g_\rho}(\partial^* F \cap A)$, for $A \subset M$. Therefore $\partial^* F$ (rectifiable current of codimension 1 in M) plays the role of the rectifiable 2-varifold V above (take $n = 2$ and $l = 1$), by Nash-embedding (M, g_ρ) in to $\mathbb{R}^{2+k}$ for some k (i.e. g_ρ is equal to the induced metric from $(\mathbb{R}^{2+k}, \bar{g})$ under the embedding) [Nas56]. Assuming we have a Nash-embedding for (M, g) given by

$$\varphi : (M, g) \to (\mathbb{R}^{2+k}, \bar{g})$$

then we get one for (M, g_ρ) by using $\varphi_\rho := \rho^{-1}\varphi$, since this gives

$$\begin{aligned} g_\rho(X,Y) = \rho^{-2} g(X,Y) = \rho^{-2}\bar{g}(d\varphi(X), d\varphi(Y)) &= \rho^{-2}\bar{g}(\rho d\varphi_\rho(X), \rho d\varphi_\rho(Y)) \\ &= \bar{g}(d\varphi_\rho(X), d\varphi_\rho(Y)) \end{aligned}$$

at any $p \in M$, where $X, Y \in T_pM$. Equivalently, we have the isometric embedding

$$\varphi : (M, g_\rho) \to (\mathbb{R}^{2+k}, \bar{g}_\rho := \rho^{-2}\bar{g})$$

From this point of view, we can compute the change in the second fundamental form induced by the conformal change of metric in the ambient Euclidean space. First, let E_μ be a g-orthonormal frame and N a unit normal with respect to $\bar{g}$. Then ρE_μ is a g_ρ-orthonormal frame and ρN is the corresponding unit normal with respect to $\bar{g}_\rho$. Thus

$$\begin{aligned} h_{\rho N}^{(M,g_\rho)}(\rho E_\mu, \rho E_\nu) &= \bar{g}_\rho(\nabla^{\bar{g}_\rho}_{\rho E_\mu} \rho E_\nu, \rho N) \\ &= \rho^{-2}\bar{g}(\rho^2 \nabla^{\bar{g}_\rho}_{E_\mu} E_\nu, \rho N) \\ &= \rho \bar{g}(\nabla^{\bar{g}}_{E_\mu} E_\nu, N) \\ &= \rho h_N^{(M,g)}(E_\mu, E_\nu) \end{aligned} \tag{6.22}$$

where in the third line we used (2.9) and the fact that ρ^{-2} is constant.

Since U is contained in a compact set, $h_N^{(M,g)}$ will be bounded on U. Thus $\|h_N^{(M,g)}\| \leq \Lambda$ and so, by (6.22), $\|h_{\rho N}^{(M,g_\rho)}\| \leq \rho\Lambda$. Therefore the constant Λ in (6.21) can be chosen independent of, say, $\rho < \rho_0$.

Thus, the monotonicity formula above applies to $\partial^* F$ where, for $B \subset \mathbb{R}^{2+k}$

$$\mu_{\partial^* F}(B) = \mathcal{H}^2_{\bar{g}}(\partial^* F \cap B) = \mathcal{H}^2_{g_\rho}(\partial^* F \cap B \cap M) = \mathrm{P}_{g_\rho}(F, B \cap M)$$

Thus, we can rewrite (6.21) using our notation and fixing $s = r_0$, to yield, for all $r < r_0$

$$\mathrm{P}_{g_\rho}(F, B^{\bar{g}}_r(\xi) \cap M) \leq e^{2\Lambda R} \frac{\mathrm{P}_{g_\rho}(F)}{r_0^2} r^2 \leq C_1 r^2 \tag{6.23}$$

for some constant $C_1(\Lambda, R, r_0, \rho_0)$. Since M is a smooth submanifold of $\mathbb{R}^{2+k}$, we have

$$\mathrm{P}_{g_\rho}(F, B^{g_\rho}_r(\xi)) \leq \mathrm{P}_{g_\rho}(F, B^{\bar{g}}_{C_2 r}(\xi) \cap M) \tag{6.24}$$

This is true because, using the normal coordinate chart centered at $\xi \in M$, we have

$$\phi^{\xi}_{g_\rho}(B^{g_\rho}_r(\xi)) \subset B^{\bar{g}}_{\hat{C}r}(\xi) \cap T_\xi M$$

for any $\hat{C} \geq 1$, and so

$$B_r^{g_\rho}(\xi) \subset B_{C_2 r}^{\bar{g}}(\xi) \cap M$$

for some $C_2(r_0)$. Combining (6.23) and (6.24), we get

$$\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(\xi)) \leq C_1 C_2^2 r^2$$

which is inequality (6.19), up to relabelling constants and $\xi = q$. $\square$

Finally, we have the following result, whose proof is along the lines of the proof of $(6-9)$ in [MS17].

Lemma 6.17. *Let $B \subset (M, g_\rho)$ be a bounded open set with C^2 boundary. Then, for small enough r, there exists a constant $C = C(B) > 0$ such that, for every $q \in \overline{B}$*

$$\mathrm{P}_{g_\rho}(B) \leq \mathrm{P}_{g_\rho}(G) + Cr^3 \quad \forall\, G \Delta B \subset\subset B_r^{g_\rho}(q)$$

Proof. For q in the interior of B the inequality is clear because, for small r, $B_r^{g_\rho}(q) \cap \partial B = \emptyset$ so in fact $\mathrm{P}_{g_\rho}(B) \leq \mathrm{P}_{g_\rho}(G)$.

Let φ be a chart adapted to the submanifold ∂B [Lee13]. This means the image of a ball centered at a point $q \in \partial B$ will be sent to a ball centered at the origin in $\mathbb{R}^3$ where $\varphi(\partial B) \subset \{x^3 = 0\}$. By a linear change in coordinates we can find a new chart $\tilde{\varphi}$ in which the coefficients of the metric satisfy $(g_\rho)_{\mu\nu}(q) = \delta_{\mu\nu}$. Since the change is linear we will get $\tilde{\varphi}(\partial B) \subset \tilde{\Pi}$, where $\tilde{\Pi}$ is another plane through the origin.

Now, by a Taylor expansnion centered at q we can write the metric as $(g_\rho)_{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(r)$ (note the difference with normal coordinates). Applying the same method as in Lemma 6.14, we get the next two inequalities for a set of finite perimeter G:

$$\mathrm{P}_{g_\rho}(G, B_r^{g_\rho}(q)) \leq (1 + \tilde{C}r)\mathrm{P}_{\bar{g}}(\tilde{\varphi}(G), \tilde{\varphi}(B_r^{g_\rho}(q))) + Dr^3 \tag{6.25}$$

$$\mathrm{P}_{g_\rho}(G, B_r^{g_\rho}(q)) \geq (1 - \tilde{C}r)\mathrm{P}_{\bar{g}}(\tilde{\varphi}(G), \tilde{\varphi}(B_r^{g_\rho}(q))) - Dr^3 \tag{6.26}$$

Let $G = B$ in (6.25). Since $\tilde{\varphi}(\partial B) \subset \tilde{\Pi}$, we have

$$\mathrm{P}_{\bar{g}}(\tilde{\varphi}(B), \tilde{\varphi}(B_r^{g_\rho}(q))) = |\tilde{\varphi}(\partial B) \cap \tilde{\varphi}(B_r^{g_\rho}(q))|_{\bar{g}}$$

Thus

$$\mathrm{P}_{g_\rho}(B, B_r^{g_\rho}(q)) \leq (1+\tilde{C}r)|\tilde{\varphi}(\partial B) \cap \tilde{\varphi}(B_r^{g_\rho}(q))|_{\tilde{g}} + Dr^3 \tag{6.27}$$

Next, restrict to G such that $G \Delta B \subset\subset B_r^{g_\rho}(q)$ in 6.26, which in particular means that $\mathrm{proj}_{\tilde{\Pi}}(G) = \mathrm{proj}_{\tilde{\Pi}}(B)$ where $\mathrm{proj}_{\tilde{\Pi}}$ is the projection map onto the plane $\tilde{\Pi}$. Since this map is 1-Lipschitz, we have

$$\mathrm{P}_{\tilde{g}}(\tilde{\varphi}(G), \tilde{\varphi}(B_r^{g_\rho}(q))) \geq |\mathrm{proj}_{\tilde{\Pi}}(B) \cap \tilde{\varphi}(B_r^{g_\rho}(q))|_{\tilde{g}} = |\tilde{\varphi}(\partial B) \cap \tilde{\varphi}(B_r^{g_\rho}(q))|_{\tilde{g}} \tag{6.28}$$

Therefore, (6.26) becomes

$$\mathrm{P}_{g_\rho}(G, B_r^{g_\rho}(q)) \geq (1-\tilde{C}r)|\tilde{\varphi}(\partial B) \cap \tilde{\varphi}(B_r^{g_\rho}(q))|_{\tilde{g}} - Dr^3 \tag{6.29}$$

Combining (6.27) and (6.29) yields

$$\begin{aligned}
\mathrm{P}_{g_\rho}(B, B_r^{g_\rho}(q)) &\leq \mathrm{P}_{g_\rho}(G, B_r^{g_\rho}(q)) + 2\tilde{C}r|\tilde{\varphi}(\partial B) \cap \tilde{\varphi}(B_r^{g_\rho}(q))|_{\tilde{g}} + 2Dr^3 \\
&\leq \mathrm{P}_{g_\rho}(G, B_r^{g_\rho}(q)) + Cr^3
\end{aligned}$$

□

6.3 Perturbed Geodesic Spheres Are Outer-Minimising

The goal of this section is to prove Proposition 6.2, which will allow us to use the expansion for the Hawking mass of perturbed spheres to get a lower bound on the Bartnik mass in Theorems 1.34 and 1.35.

Recall that we denote by $B_{p,\rho}(w)$ the perturbed geodesic ball enclosed by the perturbed geodesic sphere $S_{p,\rho}(w)$, inside a 3D Riemannian manifold (M, g).

For each $B_{p,\rho}(w)$, consider the set

$$S := \{\Omega \subset M : \mathrm{P}_g(\Omega) < \infty,\ |\Omega|_g < \infty,\ B_{p,\rho}(w) \subset \Omega\}$$

and let $s := \inf\{\mathrm{P}_g(\Omega) : \Omega \in S\}$. Let Ω_i be a sequence such that $\mathrm{P}_g(\Omega_i) \to s$. To get a limit of this sequence we cannot apply compactness directly because it is not necessarily uniformly bounded. Thus, we use a diagonal argument, outlined below.

Take a sequence of balls $B^g_{R_n}(p) \subset M$, where $R_n \to \infty$. For a given n, consider the sequence

$$\Omega^n_i := \Omega_i \cap B^g_{R_n}(p)$$

This new sequence is now uniformly bounded and by Theorem 2.5 there is a limit, say $\Omega^n \subset B^g_{R_n}(p)$, of a subsequence $\Omega^n_{i_j}$. We then use the subsequence Ω_{i_j} to obtain another bounded sequence

$$\Omega^n_{i_j} := \Omega_{i_j} \cap B^g_{R_{n+1}}(p)$$

This has a convergent subsequence, with limit $\Omega^{n+1} \subset B^g_{R_{n+1}}(p)$, such that $\Omega^{n+1} \cap B^g_{R_n}(p) = \Omega^n$. Continuing in this manner we construct a (subsequential) limit of Ω_i, denoted $\Omega_{p,\rho,w}$, where

$$|\Omega_i \Delta \Omega_{p,\rho,w}|_g \to 0 \quad \text{and} \quad B_{p,\rho}(w) \subset \Omega_{p,\rho,w}$$

Furthermore, by lower-semicontinuity (Theorem 2.4) we have $\mathrm{P}_g(\Omega_{p,\rho,w}) = s$.

The following inequalities hold:

$$\mathrm{P}_g(\Omega_{p,\rho,w}) \leq \mathrm{P}_g(B_{p,\rho}(w)), \quad 0 < |B_{p,\rho}(w)|_g \leq |\Omega_{p,\rho,w}|_g \tag{6.30}$$

We will show that, under the conditions of Proposition 6.2, $\partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$ for any $\rho, \|w\|_{C^1}$ small enough, and thus $S_{p,\rho}(w)$ is outer-minimising. As summarized in Section 1.5, this will be achieved in four steps. First we will prove that we have some control over the volume of the sequence $\Omega_{p,\rho,w}$ as $\rho \to 0$. Then we will show that the normal coordinate image of the sequence locally converges to $B^{\bar{g}}_1(0)$ in Euclidean space in the sense of finite perimeter sets. Then, using a regularity result of Tamanini, we show that in fact the boundaries of the elements of the sequence converge as $C^{1,\frac{1}{2}}$-graphs to $\partial B^{\bar{g}}_1(0) = \mathbb{S}^2$. Finally we prove that these boundaries are in fact the corresponding perturbed spheres.

Remark 6.18. We note that in particular $\Omega_{p,\rho,w}$ is a finite perimeter set minimising perimeter amoung all sets containing $B_{p,\rho}(w)$ of fixed volume $|\Omega_{p,\rho,w}|_g$. Therefore we know that, up to measure zero, $\Omega_{p,\rho,w}$ is open and $\partial\Omega_{p,\rho,w}$ is $C^{1,1}$-regular thanks to the result [MS17, Theorem 6.15] (this result requires the obstacle to have C^2-boundary, which is satisfied by $B_{p,\rho}(w)$). This means that in fact the reduced boundary and the topological boundary coincide, i.e. $\partial^*\Omega_{p,\rho,w} = \partial\Omega_{p,\rho,w}$. This will be proven independently in Step 3 below via Theorem 6.24.

Proof. (*of Proposition 6.2*)

Step 1 - Volume control.

In this step we will prove that, for small ρ, we have some control over the volume of the sets $\Omega_{p,\rho,w}$ corresponding to $S_{p,\rho}(w)$. This will be useful in the next step when we consider a scaled metric and the normal coordinate image of the sequence. In particular, it will allow us to apply the Euclidean Isoperimetric Inequality.

Lemma 6.19. *Let (M,g) and $S_{p,\rho_n}(w_n)$ be as in Proposition 6.2. For the corresponding sequence of finite perimeter sets Ω_{p,ρ_n,w_n}, there exists a constant $\hat{C}$ such that*

$$0 < \hat{C}^{-1} \leq \liminf_{n\to\infty} \frac{|\Omega_{p,\rho_n,w_n}|_g}{\rho_n^3} \leq \limsup_{n\to\infty} \frac{|\Omega_{p,\rho_n,w_n}|_g}{\rho_n^3} \leq \hat{C} < \infty \tag{6.31}$$

Proof. First note that (1.4) and (1.5) yield

$$\mathrm{P}_g(B_{p,\rho_n}(w_n)) = \left(|\mathbb{S}^2|_{g_{\mathbb{S}^2}} + \int_{\mathbb{S}^2} w_n^2 dV_{g_{\mathbb{S}^2}} + \frac{1}{2}\int_{\mathbb{S}^2} |\nabla w_n|^2 dV_{g_{\mathbb{S}^2}} - 2\int_{\mathbb{S}^2} w_n dV_{g_{\mathbb{S}^2}} \right) \rho_n^2 + \mathcal{O}(\rho_n^4) \tag{6.32}$$

$$|B_{p,\rho_n}(w_n)|_g = \left(\frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3} + \int_{\mathbb{S}^2} w_n^2 dV_{g_{\mathbb{S}^2}} - \int_{\mathbb{S}^2} w_n dV_{g_{\mathbb{S}^2}} \right) \rho_n^3 + \mathcal{O}(\rho_n^5) \tag{6.33}$$

To get a lower bound in (6.31), we use (6.33) and the second inequality in (6.30), which imply

$$\liminf_{n\to\infty} \frac{|\Omega_{p,\rho_n,w_n}|_g}{\rho_n^3} \geq \liminf_{n\to\infty} \frac{|B_{p,\rho_n}(w_n)|_g}{\rho_n^3} = \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3}$$

since $\rho_n \to 0$ and $\|w_n\|_{C^1(S^2)} \to 0$ as $n \to \infty$. For an upper bound in 6.31, first we use the following lemma, proved in [Cho+21, Theorem C.2].

Lemma 6.20. *Let (M,g) be an AF, complete, 3D Riemannian manifold with non-negative scalar curvature where ∂M is the only compact, minimal surface in M. Then*

$$m_{ADM}(M) = \lim_{v\to\infty} \frac{2}{\mathcal{I}(v)} \left(v - \frac{\mathcal{I}(v)^{\frac{3}{2}}}{6\sqrt{\pi}} \right)$$

In particular the lemma shows that if $\mathcal{I}(v)$ was bounded then $m_{ADM}(M) = \infty$. But we know that, under the conditions of the lemma, $m_{ADM}(M) < \infty$, hence $\lim_{v\to\infty}\mathcal{I}(v) = \infty$. This means that for any sequence Ω_n, we have

$$\lim_{n\to\infty} |\Omega_n|_g = \infty \implies \lim_{n\to\infty} \mathrm{P}_g(\Omega_n) = \infty \tag{6.34}$$

But (6.32) and the first inequality in (6.30) imply

$$\limsup_{n\to\infty} \mathrm{P}_g(\Omega_{p,\rho_n,w_n}) \leq \limsup_{n\to\infty} \mathrm{P}_g(B_{p,\rho_n}(w_n)) = 0$$

Therefore, (6.34) implies the $|\Omega_{p,\rho_n,w_n}|_g$ are eventually bounded by, say v_0. Hence, (6.8) and (6.30) yield

$$\begin{aligned}
\limsup_{n\to\infty} \frac{|\Omega_{p,\rho_n,w_n}|_g}{\rho_n^3} &\leq C^{-\frac{3}{2}} \limsup_{n\to\infty} \frac{\mathrm{P}_g(\Omega_{p,\rho_n,w_n})^{\frac{3}{2}}}{\rho_n^3} \\
&\leq C^{-\frac{3}{2}} \limsup_{n\to\infty} \frac{\mathrm{P}_g(B_{p,\rho_n}(w_n))^{\frac{3}{2}}}{\rho_n^3} \\
&= C^{-\frac{3}{2}} \limsup_{n\to\infty} \left(\frac{\mathrm{P}_g(B_{p,\rho_n}(w_n))}{\rho_n^2} \right)^{\frac{3}{2}} \\
&= C^{-\frac{3}{2}} |\mathbb{S}^2|_{g_{\mathbb{S}^2}}^{\frac{3}{2}}
\end{aligned}$$

where $0 < C = C(v_0) \leq (36\pi)^{\frac{1}{3}}$. Using $|\mathbb{S}^2|_{g_{\mathbb{S}^2}} = 4\pi$, note that

$$C^{-\frac{3}{2}} |\mathbb{S}^2|_{g_{\mathbb{S}^2}}^{\frac{3}{2}} \geq \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3} \quad \text{and} \quad C^{-\frac{3}{2}} |\mathbb{S}^2|_{g_{\mathbb{S}^2}}^{\frac{3}{2}} \times \frac{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}{3} > 1$$

Therefore the lemma is true with $\hat{C} = C(v_0)^{-\frac{3}{2}} |\mathbb{S}^2|_{g_{\mathbb{S}^2}}^{\frac{3}{2}}$. □

Remark 6.21. In fact Lemma 6.19 is true for all w satisfying $\|w\|_{C^1(S^2)} \leq \bar{C}$, for some bound $\bar{C}$. The resulting constant then depends on $\bar{C}$ due to the terms containing w in (6.32) and (6.33).

Step 2. Blow-up and local convergence to a Euclidean ball.

In this step we "blow-up" the Riemannian manifold (M,g) at p with scaling rate ρ^{-1} as $\rho \downarrow 0$, by considering the rescaled metrics $g_\rho = \rho^{-2} g$. We will show that, in g_ρ-normal coordinates, the images $\phi_{g_\rho}(\Omega_{p,\rho,w})$ converge locally to the Euclidean ball of unit radius $B_1^{\bar{g}}(0) \subset \mathbb{R}^3$, in the sense of finite perimeter sets.

Since $\Omega_{p,\rho,w}$ may not be contained in the domain of ϕ_{g_ρ}, we define

$$A_{\rho,w} := \Omega_{p,\rho,w} \cap B^{g_\rho}_{\rho^{-1}\bar{R}}(p) \quad \text{and} \quad B_{\rho,w} := B_{p,\rho}(w) \cap B^{g_\rho}_{\rho^{-1}\bar{R}}(p)$$

where $B^g_{\bar{R}}(p)$ is contained in the domain of ϕ_g. Now consider any sequence $\phi_{g_{\rho_n}}(A_{\rho_n,w_n})$ in $\mathbb{R}^3$, where $\rho_n, \|w_n\|_{C^1(S^2)} \to 0$ as $n \to \infty$. We have the following estimate:

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(A_{\rho_n,w_n}) &= \limsup_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(\Omega_{p,\rho_n,w_n}) \\
&= \limsup_{n\to\infty} \rho_n^{-2} \mathrm{P}_g(\Omega_{p,\rho_n,w_n}) \\
&\leq \limsup_{n\to\infty} \rho_n^{-2} \mathrm{P}_g(B_{p,\rho_n}(w_n)) \\
&= \limsup_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(B_{p,\rho_n}(w_n)) \\
&= \limsup_{n\to\infty} \mathrm{P}_{\bar{g}}(\phi_{g_{\rho_n}}(B_{p,\rho_n}(w_n))) \\
&= \mathrm{P}_{\bar{g}}(B^{\bar{g}}_1(0)) \\
&< \infty
\end{aligned}$$

where the second to last line is true from the definition of $B_{p,\rho_n}(w_n)$ and the fact that $\phi_{g_{\rho_n}} = \rho_n^{-1}\phi_g$. The third to last line follows from (6.16) with $q = p$ and the fact that $\mathrm{P}_{\bar{g}}(\phi_{g_\rho}(B_{p,\rho}(w)))$ is uniformly bounded as $\rho \to 0$. Thus $\mathrm{P}_{g_{\rho_n}}(A_{\rho_n,w_n})$ is uniformly bounded and then, by another application of (6.16)

$$\limsup_{n\to\infty} \mathrm{P}_{\bar{g}}(\phi_{g_{\rho_n}}(A_{\rho_n,w_n})) = \limsup_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(A_{\rho_n,w_n}) < \infty \tag{6.35}$$

This gives a uniform bound on the perimeter, required for compactness and lower semi-continuity. However, we still cannot apply compactness directly because the sequence itself is not necessarily uniformly bounded. Thus we use the same diagonal argument as before, when we found the limit $\Omega_{p,\rho,w}$, to extract a limit of $\phi_{g_{\rho_n}}(A_{\rho_n,w_n})$, denoted $\bar{\Omega}$.

Arguing in a similar way as we did to reach (6.35), we also have

$$\begin{aligned}
\mathrm{P}_{\bar{g}}(\bar{\Omega}) &= \lim_{R\to\infty} \mathrm{P}_{\bar{g}}(\bar{\Omega}, B^{\bar{g}}_R(0)) \\
&\leq \lim_{R\to\infty} \liminf_{n\to\infty} \mathrm{P}_{\bar{g}}(\phi_{g_{\rho_n}}(A_{\rho_n,w_n}), B^{\bar{g}}_R(0)) \\
&= \lim_{R\to\infty} \liminf_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(A_{\rho_n,w_n}, \phi^{-1}_{g_{\rho_n}}(B^{\bar{g}}_R(0))) \\
&= \lim_{R\to\infty} \liminf_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(\Omega_{p,\rho_n,w_n}, \phi^{-1}_{g_{\rho_n}}(B^{\bar{g}}_R(0))) \\
&\leq \lim_{R\to\infty} \liminf_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(\Omega_{p,\rho_n,w_n}) \\
&= \liminf_{n\to\infty} \rho_n^{-2} \mathrm{P}_g(\Omega_{p,\rho_n,w_n}) \\
&\leq \liminf_{n\to\infty} \rho_n^{-2} \mathrm{P}_g(B_{p,\rho_n}(w_n)) \\
&= \liminf_{n\to\infty} \mathrm{P}_{g_{\rho_n}}(B_{p,\rho_n}(w_n)) \\
&= \liminf_{n\to\infty} \mathrm{P}_{\bar{g}}(\phi_{g_{\rho_n}}(B_{p,\rho_n}(w_n))) \\
&= \mathrm{P}_{\bar{g}}(B^{\bar{g}}_1(0)) \qquad (6.36)
\end{aligned}$$

where the second line follows by lower semi-continuity of perimeter, and the fourth line is true because, for a given R

$$(A_{\rho_n,w_n} \Delta \Omega_{p,\rho_n,w_n}) \cap \phi^{-1}_{g_{\rho_n}}(B^{\bar{g}}_R(0)) = \emptyset$$

for n large enough, and so

$$\mathrm{P}_{g_{\rho_n}}(A_{\rho_n,w_n}, \phi^{-1}_{g_{\rho_n}}(B^{\bar{g}}_R(0))) = \mathrm{P}_{g_{\rho_n}}(\Omega_{p,\rho_n,w_n}, \phi^{-1}_{g_{\rho_n}}(B^{\bar{g}}_R(0)))$$

Furthermore, since $B_{\rho_n,w_n} \subset A_{\rho_n,w_n}$, we also have $\phi_{g_{\rho_n}}(B_{\rho_n,w_n}) \subset \phi_{g_{\rho_n}}(A_{\rho_n,w_n})$. Then $\phi_{g_{\rho_n}}(B_{\rho_n,w_n}) \to B^{\bar{g}}_1(0)$ and $\phi_{g_{\rho_n}}(A_{\rho_n,w_n}) \to \bar{\Omega}$ imply

$$B^{\bar{g}}_1(0) \subset \bar{\Omega} \qquad (6.37)$$

By Lemma 6.19 we know that $|\Omega_{p,\rho_n,w_n}|_{g_{\rho_n}}$ is uniformly bounded and therefore so are $|A_{\rho_n,w_n}|_{g_{\rho_n}}$. Using normal coordinates shows that this implies $|\phi_{g_{\rho_n}}(A_{\rho_n,w_n})|_{\bar{g}}$ are uniformly bounded too. Thus $|\bar{\Omega}|_{\bar{g}} < \infty$ and we can appeal to the rigidity of the Euclidean isoperimetric inequality (considering (6.37) and (6.36)) to get $|\bar{\Omega} \Delta B^{\bar{g}}_1(0)|_{\bar{g}} = 0$. Therefore, by the arbitrariness of the sequences ρ_n and w_n, we conclude

$$|\phi_{g_\rho}(A_{\rho,w})\Delta B_1^{\bar{g}}(0)|_{\bar{g}} \to 0 \quad \text{as} \quad \rho, \|w\|_{C^1(S^2)} \to 0$$

Finally, for any compact $\mathcal{K} \subset \mathbb{R}^3$ we have $(A_{\rho,w}\Delta\Omega_{p,\rho,w}) \cap \phi_{g_\rho}^{-1}(\mathcal{K}) = \emptyset$ for ρ small enough. Thus, we also get

$$|(\phi_{g_\rho}(\Omega_{p,\rho,w})\Delta B_1^{\bar{g}}(0)) \cap \mathcal{K}|_{\bar{g}} \to 0 \quad \text{as} \quad \rho, \|w\|_{C^1(S^2)} \to 0$$

giving the desired local convergence.

Step 3. Improving the convergence via regularity theory.

First, we specialise Lemma 6.14 by letting $F = \Omega_{p,\rho,w}$.

Lemma 6.22. *Let (M,g) be a 3D Riemannian manifold and $p \in M$. Then*

$$\mathrm{P}_{\bar{g}}(\phi_{g_\rho}^p(\Omega_{p,\rho,w}), B_r^{\bar{g}}(\phi_{g_\rho}^p(q))) = \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(\rho^2 r^4) \tag{6.38}$$

Proof. First, for any part of $\partial\Omega_{p,\rho,w}$ coinciding with the submanifold $S_{p,\rho}(w)$ we can use normal coordinates to approximate the relative *area* inside a small ball by the area of a disk in the tangent space. Thus, for $q \in S_{p,\rho}(w)$ and small r, we have

$$\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) = \mathrm{A}(S_{p,\rho}(w) \cap B_r^{g_\rho}(q)) \leq Cr^2$$

In g_ρ-normal coordinates $S_{p,\rho}(w)$ is a graph over the unit sphere with graph function $w \to 0$ as $\rho \to 0$, so the constant in the above inequality does not depend on ρ.

Second, away from the intersection points with $S_{p,\rho}(w)$, by construction $\Omega_{p,\rho,w}$ is locally perimeter minimising. Thus, we apply Lemma 6.15 with $F = \Omega_{p,\rho,w}$ to again yield

$$\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) \leq Cr^2 \tag{6.39}$$

The constant coming from Lemma 6.15 is independent of ρ because the $\Omega_{p,\rho,w}$ have uniformly bounded perimeter (by the argument near the beginning of Step 2). Using (6.16) (with $F = \Omega_{p,\rho,w}$) gives

$$\mathrm{P}_{\bar{g}}(\phi_{g_\rho}^p(\Omega_{p,\rho,w}), B_r^{\bar{g}}(\phi_{g_\rho}^p(q))) = (1 + \mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(\rho^2 r^4)$$

Applying (6.39) and simplifying yields (6.38). $\square$

Now we consider the regularity and convergence of the (reduced) boundaries in the sequence $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ by using a result of Tamanini [Tam82, Theorem 1].

Definition 6.23. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and $V \subset \mathbb{R}^n$ open and bounded. Then*

$$\Psi(E,V) := \mathrm{P}_{\bar{g}}(E,V) - \inf\{\mathrm{P}_{\bar{g}}(F,V) | F\Delta E \subset\subset V\}$$

Theorem 6.24 ([Tam82])**.** *Let U be an open subset of $\mathbb{R}^n$, and E a set of finite perimeter satisfying*

$$\Psi(E, B_r(q)) \leq Cr^{n-1+2\alpha} \tag{6.40}$$

for some $\alpha \in (0,1)$ and for all $q \in U$ and $r \in (0,R)$, where C and R are positive constants. Then the reduced boundary $\partial^* E$ is a $C^{1,\alpha}$-hypersurface in U and

$$\mathcal{H}^k((\partial E \setminus \partial^* E) \cap U) = 0 \qquad \forall k > n-8$$

Moreover, assuming that (6.40) holds uniformly for a sequence E_h, L^1-locally convergent to E_∞, then for any sequence of points $q_h \in \partial E_h$ converging to $q_\infty \in \partial^* E_\infty$, there is an h' such that $q_h \in \partial^* E_h$ for $h > h'$ and the unit outer normal to ∂E_h at q_h converges to the unit outer normal to ∂E_∞ at q_∞.

Lemma 6.25. *Let (M,g) be a 3D Riemannian manifold and $p \in M$. Then the sets $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ eventually satisfy the bound (6.40) in Theorem 6.24.*

Proof. Fix $R > 0$ and let $U = B^{\bar{g}}_{R+1}(0)$. Below we will consider ρ small enough, say $\rho < \bar{\rho}(p)$, that we have $U = \phi^p_{g_\rho}(B^{g_\rho}_{R+1}(p))$. For $q \in B^{g_\rho}_{R+1}(p)$, we split the calculation of Ψ into three cases, according to whether $B^{\bar{g}}_R(\phi^p_{g_\rho}(q))$ is completely inside, intersects or is completely outside $\phi^p_{g_\rho}(S_{p,\rho}(w))$. In order to utilise the minimising assumption on $\Omega_{p,\rho,w}$, we will rewrite Euclidean perimeters in terms of their corresponding perimeters in (M,g_ρ) using (6.16) and (6.38). Lemmas 2.3 and 6.17 will also be useful.

Firstly, let $B^{\bar{g}}_R(\phi^p_{g_\rho}(q)) \subset \phi^p_{g_\rho}(B_{p,\rho}(w))$. Then for all $r < R$

$$B^{\bar{g}}_r(\phi^p_{g_\rho}(q)) \subset \phi^p_{g_\rho}(B_{p,\rho}(w)) \subset \phi^p_{g_\rho}(\Omega_{p,\rho,w})$$

so $\mathrm{P}_{\bar{g}}(\phi^p_{g_\rho}(\Omega_{p,\rho,w}), B^{\bar{g}}_r(\phi^p_{g_\rho}(q))) = 0$ and therefore $\Psi = 0$.

Secondly, let $B^{\bar{g}}_R(\phi^p_{g_\rho}(q)) \subset \overline{\phi^p_{g_\rho}(B_{p,\rho}(w))}^c$. Then, for all $r < R$, we can restrict to only F which contain $B_{p,\rho}(w)$ because, otherwise, $\phi^p_{g_\rho}(F)\Delta\phi^p_{g_\rho}(\Omega_{p,\rho,w}) \not\subset B^{\bar{g}}_r(\phi^p_{g_\rho}(q))$. Thus, using (6.16) and (6.38)

$$
\begin{aligned}
&\Psi(\phi^p_{g_\rho}(\Omega_{p,\rho,w}), B^{\bar g}_r(\phi^p_{g_\rho}(q))) \\
&= \mathrm{P}_{\bar g}(\phi^p_{g_\rho}(\Omega_{p,\rho,w}), B^{\bar g}_r(\phi^p_{g_\rho}(q))) \\
&\quad - \inf\{\mathrm{P}_{\bar g}(\phi^p_{g_\rho}(F), B^{\bar g}_r(\phi^p_{g_\rho}(q))) | F \supset B_{p,\rho}(w), \phi^p_{g_\rho}(F)\Delta\phi^p_{g_\rho}(\Omega_{p,\rho,w}) \subset\subset B^{\bar g}_r(\phi^p_{g_\rho}(q))\} \\
&= \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(\rho^2 r^4) - \inf\{(1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(F, B^{g_\rho}_r(q)) + \mathcal{O}(\rho^2 r^4)\} \\
&= \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(\rho^2 r^4) - \big((1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(\rho^2 r^4)\big) \\
&\overset{\rho\le\bar\rho}{=} \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(r^4) - \big((1+\mathcal{O}(r^2))\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(r^4)\big) \\
&= \mathcal{O}(r^2)\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B^{g_\rho}_r(q)) + \mathcal{O}(r^4) \\
&= \mathcal{O}(r^4)
\end{aligned}
$$

where we have used the assumption that $\Omega_{p,\rho,w}$ is minimising among all sets containing $B_{p,\rho}(w)$ to evaluate the infimum, and the inequality (6.39) on the final line.

Finally, we let $B^{\bar g}_R(\phi^p_{g_\rho}(q)) \cap \phi^p_{g_\rho}(S_{p,\rho}(w)) \neq \emptyset$. Note that in this case we can not immediately apply the minimising assumption on $\Omega_{p,\rho,w}$ because it may be that $F \not\supset B_{p,\rho}(w)$. Instead, let $F \subset M$ be such that $F\Delta\Omega_{p,\rho,w} \subset\subset B^{g_\rho}_r(q)$, and define $F' := F \cup B_{p,\rho}(w)$. Then, by the minimising assumption on $\Omega_{p,\rho,w}$, we have

$$
\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}) \le \mathrm{P}_{g_\rho}(F')
$$

Now, recall the inequality in Lemma 2.3

$$
\mathrm{P}_{g_\rho}(F \cup B_{p,\rho}(w)) + \mathrm{P}_{g_\rho}(F \cap B_{p,\rho}(w)) \le \mathrm{P}_{g_\rho}(F) + \mathrm{P}_{g_\rho}(B_{p,\rho}(w))
$$

and the inequality in Lemma 6.17 (with $B = B_{p,\rho}(w)$)

$$
\mathrm{P}_{g_\rho}(B_{p,\rho}(w)) \le \mathrm{P}_{g_\rho}(G) + Cr^3
$$

which is valid when $G\Delta B_{p,\rho}(w) \subset\subset B^{g_\rho}_r(q)$. Letting $G = F \cap B_{p,\rho}(w)$ and combining the three previous inequalities gives

$$
\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}) \le \mathrm{P}_{g_\rho}(F') \le \mathrm{P}_{g_\rho}(F) + Cr^3
$$

for all F such that $F\Delta\Omega_{p,\rho,w} \subset\subset B^{g_\rho}_r(q)$. Since the sets $\Omega_{p,\rho,w}$ and F are equal outside of $B^{g_\rho}_r(q)$, this is equivalent to

$$\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) \le \mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q)) + Cr^3 \tag{6.41}$$

Again using (6.16) and (6.38), we have

$$\begin{aligned}
&\Psi(\phi^p_{g_\rho}(\Omega_{p,\rho,w}), B_r^{\bar g}(\phi^p_{g_\rho}(q)))\\
&= \mathrm{P}_{\bar g}(\phi^p_{g_\rho}(\Omega_{p,\rho,w}), B_r^{\bar g}(\phi^p_{g_\rho}(q)))\\
&\quad - \inf\{\mathrm{P}_{\bar g}(\phi^p_{g_\rho}(F), B_r^{\bar g}(\phi^p_{g_\rho}(q))) | \phi^p_{g_\rho}(F)\Delta\phi^p_{g_\rho}(\Omega_{p,\rho,w}) \subset\subset B_r^{\bar g}(\phi^p_{g_\rho}(q))\}\\
&= \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(\rho^2 r^4) - \inf\{(1+\mathcal{O}(\rho^2 r^2))\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q)) + \mathcal{O}(\rho^2 r^4)\}\\
&\overset{\rho\le\bar\rho}{=} \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(r^4) - \inf\{(1+\mathcal{O}(r^2))\mathrm{P}_{g_\rho}(F, B_r^{g_\rho}(q)) + \mathcal{O}(r^4)\}\\
&\le \mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) + \mathcal{O}(r^4) - \big((1+\mathcal{O}(r^2))(\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}, B_r^{g_\rho}(q)) - Cr^3) + \mathcal{O}(r^4)\big)\\
&= \mathcal{O}(r^3)
\end{aligned}$$

where we have applied (6.41) and (6.39) on the penultimate and final lines respectively. Thus Theorem 6.24 is satisfied with $\alpha = \frac{1}{2}$. □

Remark 6.26. In the case where $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ locally converge, the uniformity requirement is satisfied because the constant implicit in the expansion for Ψ is derived from the one in (6.9) and those from Lemma 6.17 and (6.38), which are independent of ρ.

Step 4. $\partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$

Finally, we can finish the proof of the outer-minimising property for perturbed geodesic spheres and hence of Proposition 6.2.

By Step 2, Lemma 6.25 and Remark 6.26, we can apply Theorem 6.24 to the sequence $\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ and conclude that, up to measure zero, $\partial^*\phi^p_{g_\rho}(\Omega_{p,\rho,w}) = \partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ and the unit outer normal of $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ converges to the unit outer normal of $\partial B_1^{\bar g}(0) = \mathbb{S}^2$, which means that, for $\rho < \bar\rho(p)$, we can eventually view the boundaries $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ as graphs of $C^{1,\frac{1}{2}}$-functions over $\mathbb{S}^2$.

For a given ρ, $\phi^p_{g_\rho}(S_{p,\rho}(w))$ is a graph over $\mathbb{S}^2$ too, and so, for $\rho < \bar\rho(p)$, we can then consider $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ as a graph over $\phi^p_{g_\rho}(S_{p,\rho}(w))$ instead. Thus, $\partial\Omega_{p,\rho,w}$ is parametrised by

$$\partial\Omega_{p,\rho,w} = \{\exp_q^{g_\rho}(u_{\rho,w}(q)\hat N(q)) : q \in S_{p,\rho}(w)\}$$

for unit normal vector $\hat{N}$ of $S_{p,\rho}(w)$ and some function $u_{\rho,w} \in C^1(S_{p,\rho}(w))$, and, because of the assumption that $S_{p,\rho}(w) \subset \Omega_{p,\rho,w}$, we know $u_{\rho,w} \geq 0$. Since both $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w})$ and $\phi^p_{g_\rho}(S_{p,\rho}(w))$ converge to $\mathbb{S}^2$, we also have

$$\|u_{\rho,w}\|_{C^1} \to 0 \quad \text{as} \quad \rho, \|w\|_{C^1} \to 0$$

Thus, $\partial\Omega_{p,\rho,w}$ is a normal graph over $S_{p,\rho}(w)$ in (M, g_ρ) and we have $\mathrm{P}_{g_\rho}(\Omega_{p,\rho,w}) = \mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w})$. With ρ fixed again, consider the Banach space $C^1(S_{p,\rho}(w))$ of graph functions over $S_{p,\rho}(w)$. The area functional

$$\mathrm{A}_{g_\rho} : C^1(S_{p,\rho}(w)) \to \mathbb{R}$$

is Fréchet differentiable at 0, with derivative $d(\mathrm{A}_{g_\rho})_0 \in \mathcal{L}(C^1(S_{p,\rho}(w)), \mathbb{R})$ such that

$$\mathrm{A}_{g_\rho}(0+h) = \mathrm{A}_{g_\rho}(0) + d(\mathrm{A}_{g_\rho})_0(h) + o(\|h\|_{C^1})$$

for any $h \in C^1(S_{p,\rho}(w))$. In particular, setting $h = u_{\rho,w}$ gives

$$\mathrm{A}_{g_\rho}(0+u_{\rho,w}) = \mathrm{A}_{g_\rho}(0) + d(\mathrm{A}_{g_\rho})_0(u_{\rho,w}) + o(\|u\|_{C^1})$$

Comparing this to the first variation (Gateaux derivative) of A_{g_ρ} in (2.16), we see that

$$d(\mathrm{A}_{g_\rho})_0(u_{\rho,w}) = \int_{S_{p,\rho}(w)} H^{g_\rho}_{S_{p,\rho}(w)} u_{\rho,w} \, dV_{g_\rho}$$

Therefore

$$\mathrm{A}_{g_\rho}(0+u_{\rho,w}) = \mathrm{A}_{g_\rho}(0) + \int_{S_{p,\rho}(w)} H^{g_\rho}_{S_{p,\rho}(w)} u_{\rho,w} \, dV_{g_\rho} + o(\|u_{\rho,w}\|_{C^1})$$

The left hand side is just $\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w})$. Furthermore, by (4.4), we know

$$H^g_{S_{p,\rho}(w)} = 2\rho^{-1} + \mathcal{O}(\rho)$$

and so we compute $H^{g_\rho}_{S_{p,\rho}(w)}$ by multiplying with ρ, yielding

$$H^{g_\rho}_{S_{p,\rho}(w)} = 2 + \mathcal{O}(\rho^2) > 0$$

Thus, recalling that $u_{\rho,w} \geq 0$ too, for small ρ we get

$$\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w}) \geq \mathrm{A}_{g_\rho}(0) = \mathrm{A}_{g_\rho}(S_{p,\rho}(w))$$

with equality if and only if $u_{\rho,w} \equiv 0$. But we assumed

$$\mathrm{A}_g(\partial\Omega_{p,\rho,w}) \leq \mathrm{A}_g(S_{p,\rho}(w))$$

which implies (via Lemma 6.12)

$$\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w}) \leq \mathrm{A}_{g_\rho}(S_{p,\rho}(w))$$

Therefore, for small ρ

$$\mathrm{A}_{g_\rho}(\partial\Omega_{p,\rho,w}) = \mathrm{A}_{g_\rho}(S_{p,\rho}(w))$$

and hence $u_{\rho,w} \equiv 0$. Therefore $\partial\phi^p_{g_\rho}(\Omega_{p,\rho,w}) = \phi^p_{g_\rho}(S_{p,\rho}(w))$ and $\partial\Omega_{p,\rho,w} = S_{p,\rho}(w)$. Hence $S_{p,\rho}(w)$ is outer-minimising for small ρ and $\|w\|_{C^1}$ (smallness depending only on p). ☐

6.4 Proof of Theorems 1.34 and 1.35

Proof of Theorem 1.34. For any $p \in \Omega \setminus \partial M$ consider any perturbed geodesic sphere $S_{p,\rho}(w)$ contained in $\Omega \setminus \partial M$. We have

$$m_H(S_{p,\rho}(w)) \leq m_B(B_{p,\rho}(w)) \leq m_B(\Omega) = 0$$

where the first inequality is Lemma 1.20 applied to $B_{p,\rho}(w)$ and the second is by the monotonicity Lemma 1.18, which apply thanks to Proposition 6.2. In particular, this implies

$$m_H(S_{p,\rho}(w)) \leq 0$$

for all optimally perturbed spheres (i.e. w is given by (4.24)) contained in $\Omega \setminus \partial M$. Therefore, we can apply Theorem 1.28 to Ω, which proves that $\Omega \setminus \partial M$ is locally isometric to $(\mathbb{R}^3, \bar{g})$. $\square$

Proof of Theorem 1.35. For $p \in \Omega \setminus \partial M$, we use the inequality from the previous proof. That is

$$m_B(\Omega) \geq m_H(S_{p,\rho}(w))$$

for all optimally perturbed spheres $S_{p,\rho}(w)$ contained in $\Omega \setminus \partial M$. Plugging in the expansion (4.40), we get, for $\rho \in (0, \frac{1}{2} \inf_{q \in \partial M \cup \partial \Omega} \mathrm{d}(p,q))$

$$m_B(\Omega) \geq \sqrt{\frac{|S_{p,\rho}(w)|_{\mathring{g}}}{(16\pi)^3}} \left[\frac{8\pi}{3} \mathrm{Sc}_p \rho^2 + \left(\frac{4\pi}{15} \Delta \mathrm{Sc}(p) + \frac{16\pi}{45} \|S_p\|^2 - \frac{4\pi}{27} \mathrm{Sc}_p^2 \right) \rho^4 + \mathcal{O}(\rho^5) \right]$$

As in the proof of Theorem 1.28 (see (5.1)), we have

$$\sqrt{|S_{p,\rho}(w)|_{\mathring{g}}} = \sqrt{4\pi}\rho \left[1 - \frac{1}{36} \mathrm{Sc}_p \rho^2 + \mathcal{O}(\rho^4) \right]$$

Thus, using $\frac{\sqrt{4\pi}}{\sqrt{(16\pi)^3}} = \frac{1}{32\pi}$

$$\begin{aligned}
m_B(\Omega) &\geq \frac{\rho}{32\pi} \left[1 - \frac{1}{36} \mathrm{Sc}_p \rho^2 + \mathcal{O}(\rho^4) \right] \\
&\quad \times \left[\frac{8\pi}{3} \mathrm{Sc}_p \rho^2 + \left(\frac{4\pi}{15} \Delta \mathrm{Sc}(p) + \frac{16\pi}{45} \|S_p\|^2 - \frac{4\pi}{27} \mathrm{Sc}_p^2 \right) \rho^4 + \mathcal{O}(\rho^5) \right] \\
&= \frac{\rho}{32\pi} \left[\frac{8\pi}{3} \mathrm{Sc}_p \rho^2 + \left(\frac{4\pi}{15} \Delta \mathrm{Sc}(p) + \frac{16\pi}{45} \|S_p\|^2 - \frac{2\pi}{9} \mathrm{Sc}_p^2 \right) \rho^4 + \mathcal{O}(\rho^5) \right] \\
&= \frac{1}{12} \mathrm{Sc}_p \rho^3 + \left(\frac{1}{120} \Delta \mathrm{Sc}(p) + \frac{1}{90} \|S_p\|^2 - \frac{1}{144} \mathrm{Sc}_p^2 \right) \rho^5 + \mathcal{O}(\rho^6)
\end{aligned}$$

$\square$

Remark 6.27. Taking into account Remark 4.5, if Ω is non-flat, then we can choose p such that the lower bound proved above is positive.

7 Rigidity Results Including Non-Zero Cosmological Constants

The standard Hawking mass (1.1) is relevant when the ambient space is a 3D Riemannian manifold with non-negative scalar curvature. Such metrics are natural in the case of a zero cosmological constant. When the cosmological constant Λ is negative (resp. positive), it makes sense to consider metrics with scalar curvature bounded below by a negative (resp. positive) constant. Indeed, the dominant energy condition coupled with the Einstein constraint equations implies that the scalar curvature of a totally geodesic space-like hypersurface (i.e. the so-called time-symmetric case) is bounded below by 2Λ (see Section 9.1). When Λ is negative (resp. positive) it is standard to choose the normalization $\Lambda = -3$ (resp. $\Lambda = 3$) and then compare the geometry of a totally geodesic space-like hypersurface with a space-form of sectional curvature $K = -1$ (resp. $K = 1$).

To account for non-zero cosmological constants, we first generalise the Hawking mass (see, for example, [Nev10]).

Definition 7.1. *The generalized Hawking Mass of an immersed sphere Σ in a 3D Riemannian manifold (M,g) is*

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|_{g_\Sigma}}{(16\pi)^3}}(16\pi - \int_\Sigma H^2 + 4K \, dV_{g_\Sigma}) \tag{7.1}$$

where $K \in \{-1,0,1\}$.

Using this generalised Hawking mass we can straight away obtain a generalisation of Theorem 1.33.

Theorem 7.2. *Let (M,g) be a connected, complete, 3D Riemannian manifold without boundary and with scalar curvature $\mathrm{Sc} \geq 6K$, where $K \in \{-1,0,1\}$. If every $p \in M$ admits a neighbourhood U such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0$$

or, more generally, if

$$\limsup_{\rho \downarrow 0} \rho^{-5} m_H(S_{p,\rho}(w)) \leq 0, \quad \forall p \in M \tag{7.2}$$

where $S_{p,\rho}(w)$ is the optimally perturbed geodesic sphere with w as in (4.24), then (M^3,g) is isometric to a space-form of sectional curvature K.

Proof. We can compute the generalised mass of the optimally perturbed geodesic sphere as before. Since the only difference is the addition of a constant, the variational properties of the functional are the same and we simply need to consider the new term $4K \int_{\mathbb{S}^2} \sqrt{\det \mathring{g}} \, d\theta^1 d\theta^2$. This is easily evaluated, up to fourth order, to be

$$4K \int_{\mathbb{S}^2} \sqrt{\det \mathring{g}} \, d\theta^1 d\theta^2 = 16K\pi\rho^2 - \frac{8K\pi}{9}\mathrm{Sc}_p \rho^4 + \mathcal{O}(\rho^5) \tag{7.3}$$

Inserting (7.3) and (4.40) into (7.1) gives:

$$\begin{aligned} m_H(S_{p,\rho}(w)) = \sqrt{\frac{|S_{p,\rho}(w)|_{\mathring{g}}}{(16\pi)^3}} \Bigg(& \left(\frac{8\pi}{3}\mathrm{Sc}_p - 16K\pi\right)\rho^2 \\ & + \left[\frac{4\pi}{15}\Delta\mathrm{Sc}(p) + \frac{16\pi}{45}\|S_p\|^2 - \frac{4\pi}{27}\mathrm{Sc}_p^2 + \frac{8K\pi}{9}\mathrm{Sc}_p\right]\rho^4 + \mathcal{O}(\rho^5)\Bigg) \end{aligned} \tag{7.4}$$

Now, substitute (7.4) and (5.1) into (7.2), giving

$$\begin{aligned} & \limsup_{\rho\downarrow 0} \frac{\sqrt{|\mathbb{S}^2|_{g_{\mathbb{S}^2}}}\rho^{-4}}{\sqrt{(16\pi)^3}} \left[1 - \frac{1}{36}\mathrm{Sc}_p\rho^2 + \mathcal{O}(\rho^4)\right] \\ & \quad \times \left[\left(\frac{8\pi}{3}\mathrm{Sc}_p - 16K\pi\right)\rho^2 + \left(\frac{4\pi}{15}\Delta\mathrm{Sc}(p) + \frac{16\pi}{45}\|S_p\|^2 - \frac{4\pi}{27}\mathrm{Sc}_p^2 + \frac{8K\pi}{9}\mathrm{Sc}_p\right)\rho^4 + \mathcal{O}(\rho^5)\right] \\ & \leq 0 \end{aligned}$$

for all $p \in M$. Simplifying yields

$$\begin{aligned} & \limsup_{\rho\downarrow 0} \left[\left(\frac{8\pi}{3}\mathrm{Sc}_p - 16K\pi\right)\rho^{-2} + \left(\frac{4\pi}{15}\Delta\mathrm{Sc}(p) + \frac{16\pi}{45}\|S_p\|^2 - \frac{2\pi}{9}\mathrm{Sc}_p^2 + \frac{4K\pi}{3}\mathrm{Sc}_p\right) + \mathcal{O}(\rho)\right] \\ & \leq 0 \end{aligned} \tag{7.5}$$

Since we assumed $\mathrm{Sc} \geq 6K$, letting $\rho \downarrow 0$ and looking at the dominating term $\left(\frac{8\pi}{3}\mathrm{Sc}_p - 16K\pi\right)\rho^{-2}$ in (7.5), we first infer that

$$\mathrm{Sc} \equiv 6K \tag{7.6}$$

Plugging (7.6) into (7.5) and cancelling terms, we see the dominant term becomes $\frac{16\pi}{45}\|S_p\|^2$, which is constant in ρ and non-negative. We deduce

$$S \equiv 0$$

Finally, Corollaries 5.4 and 5.2 imply that (M^3, g) is isometric to a space-form of sectional curvature K. $\square$

Remark 7.3. We note that (7.4) actually gives a positive (even though small) lower bound on the generalised Hawking mass of the optimally perturbed geodesic sphere $S_{p,\rho}(w)$ at some p inside a connected, 3D Riemannian manifold, if either $\mathrm{Sc}_p > 6K$ or both $\mathrm{Sc} \equiv 6K$ and $\|S_p\| \neq 0$. In particular, Schur's lemma implies that there will always be such a point if the manifold has $\mathrm{Sc} \geq 6K$ and non-constant sectional curvature.

Despite the use of the AF assumption in the proof of Proposition 6.2, it may also be true that, for $\rho > 0$ sufficiently small, the surface $S_{p,\rho}(w)$ is outer-minimising when the ambient manifold is asymptotically hyperbolic (see Definition 7.8). We did not push in that direction because it does not seem to be useful for obtaining a lower bound on the hyperbolic analogue of the Bartnik mass by the same methods we used for Theorems 1.34 and 1.35. Indeed, if for $\rho > 0$ sufficiently small the surface $S_{p,\rho}(w)$ is outer-minimising, we can start the (weak) inverse mean curvature flow in the sense of Huisken-Ilmanen [HI01] and the generalised Hawking mass along the flow is monotone non-decreasing also in this setting (see for instance [Nev10] for more details). However, as proved by Neves [Nev10], it may happen that the asymptotic limit of the Hawking mass along the flow exceeds the hyperbolic-ADM mass of the manifold, thus preventing a repeat the proof of Theorems 1.34 and 1.35 in the $K = -1$ case.

7.1 $\mathbb{R}^3$ and $\mathbb{H}^3$ Rigidity in the Homogeneous Setting

In this section we replace the ALSC assumption in the rigidity Theorem 1.33 with the following homogeneity condition.

Definition 7.4. *A Riemannian manifold (M, g) is called homogeneous if its isometry group $\mathrm{Isom}(M, g)$ acts transitively on M. In other words, for all $p, q \in M$ there exists $\gamma \in \mathrm{Isom}(M, g)$ such that $\gamma(p) = q$.*

Even though our spatial universe is not homogenous, at cosmological scales the homogeneity property provides a useful idealisation. Indeed, spatial homogeneity is a standard assumption in Cosmology. For instance, it leads to an exact solution of Einstein's field equations, known as the Robertson-Walker metric for space-time [Wei08; EH73].

Theorem 7.5. *Let (M, g) be a connected, homogeneous, 3D Riemannian manifold with scalar curvature $\mathrm{Sc} \geq 6K$, where $K \in \{-1, 0, 1\}$. If every $p \in M$ admits a neighbourhood U such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0$$

or, more generally, if

$$\limsup_{\rho \downarrow 0} \rho^{-5} m_H(S_{p,\rho}(w)) \leq 0, \quad \forall p \in M$$

where $S_{p,\rho}(w)$ is the optimally perturbed geodesic sphere with w as in (4.24), then (M^3, g) is isometric to one of the following:

- $\mathbb{H}^3$ *(when $K = -1$)*
- $\mathbb{R}^m \times \mathbb{T}^{3-m}$*, for some $0 \leq m \leq 3$, where $\mathbb{T}^{3-m}$ is a flat torus of dimension $3 - m$ (when $K = 0$)*
- $\mathbb{S}^3/\Gamma$ *for some finite subgroup of isometries $\Gamma < \mathrm{Isom}(\mathbb{S}^3)$ acting freely on $\mathbb{S}^3$ (when $K = 1$).*

Proof. Since homogeneity implies completeness and $\partial M = \emptyset$, Theorem 7.2 yields that M has constant sectional curvature K. The conclusion now follows from the classical classification of homogeneous spaces of constant sectional curvature [Wol11, Theorem 2.7.1]. $\square$

Remark 7.6. The proof of [Wol11, Theorem 2.7.1], for $K \leq 0$, relies on the fact that (M,g) must be a quotient of either $\mathbb{R}^n$ or $\mathbb{H}^n$ by a group of isometries Γ and in fact every $\gamma \in \Gamma$ is a so-called Clifford translation. This means that the Riemannian distance between p and $\gamma(p)$ is constant for all $p \in \mathbb{R}^n$ or $\mathbb{H}^n$. In the Euclidean case γ is a usual translation and in the hyperbolic case it turns out that $\gamma = Id_{\mathbb{H}^n}$. When $K > 0$ the proof is different and in fact one can be more precise about which quotients appear [Wol11, Corollary 2.7.2].

7.2 $\mathbb{R}^3$ and $\mathbb{H}^3$ Rigidity Under Global Asymptotic Volume Growth Assumptions

In this section we replace the *ALSC* assumption in the rigidity Theorem 1.33 with a global volume growth assumption. First we recall the framework used in the Riemannian Positive Mass Theorem.

Theorem 7.7 (Riemannian Positive Mass Theorem [SY79; SY81])**.** *If (M,g) is an AF, complete, 3D Riemannian manifold with non-negative scalar curvature, then its total mass $m_{ADM} \geq 0$ and $m_{ADM} = 0$ if and only if it is isometric to $(\mathbb{R}^3, \bar{g})$.*

This setting is supposed to model an isolated gravitational system, so that far enough away from any given point, there is no matter and the space resembles Euclidean space. As mentioned above, non-negative scalar curvature corresponds to the natural physical assumption that the ambient space-time has non-negative local energy density and (M^3, g) is a space-like hypersurface with zero second fundamental form in a universe with zero cosmological constant (see Section 9.1).

We can also consider the hyperbolic case, which means that we instead assume a negative cosmological constant and a negative lower bound on the scalar curvature. Now the model space is $\mathbb{H}^3$ rather than $\mathbb{R}^3$ and we need a hyperbolic version of the AF property. There are various ways to define an asymptotically hyperbolic manifold. For the conformal compactification approach, see Wang [Wan01]. In closer analogy to Definition 1.2, we take the asymptotic chart approach (see also [CH03; Sak21; Her05; SZ21]).

Definition 7.8. *A 3D Riemannian manifold (M,g) is called asymptotically hyperbolic (AH) if there is a compact $C \subset M$ and a diffeomorphism $\phi : M \setminus C \to \mathbb{R}^3 \setminus \overline{B_1(0)}$ such that the metric satisfies*

$$|g_{\mu\nu} - (g_{\mathbb{H}^3})_{\mu\nu}| = \mathcal{O}(r^{-s})$$

for some $s > 0$ in the chart ϕ. Here $\mathbb{H}^3 = (\mathbb{R}^3, g_{\mathbb{H}^3})$ with $g_{\mathbb{H}^3} = \frac{1}{1+r^2} dr^2 + r^2 g_{\mathbb{S}^2}$ in polar coordinates.

In this setting, a natural consideration is how the volume of a metric ball centred at some point $p \in M$ changes with increasing radius. Specifically, we can compare the growth with that of a ball of the same radius in the relevant model space, as the radius goes to infinity.

Definition 7.9. *Let $K \in \{-1, 0\}$ and let (M, g) be a complete, 3D Riemannian manifold without boundary and with scalar curvature* $\mathrm{Sc} \geq 6K$*. Let $p \in M$. We say that (M, g) satisfies the K-Global Asymptotic Volume property (K-GAVP) if*

$$\limsup_{r\to\infty} \frac{|B_r^g(p)|_g}{|B_r^K|_{g_K}} \geq 1, \tag{7.7}$$

where g_K and B_r^K denote the Riemannian metric and metric ball of radius r respectively, in the 3D, simply connected space of constant sectional curvature K.

Remark 7.10. AF and AH manifolds satisfy the 0-GAVP and -1-GAVP respectively. For the AF case, consider a straight line in $\mathbb{R}^3$, $\gamma(t) = \sqrt{r}t\hat{a}$, for some unit vector $\hat{a} \in \mathbb{R}^3$, parametrised on the interval $(0, \sqrt{r})$. Clearly we have $Length_{\bar{g}}(\gamma) = \int_0^{\sqrt{r}} \sqrt{r}dt = r$. But if we compute the length with respect to the AF metric $g = \bar{g} + \sigma$, where $\sigma = \mathcal{O}(r^{-\tau})$ as $r \to \infty$, for some $\tau > \frac{1}{2}$, we get

$$\begin{aligned}
Length_g(\gamma) &= \int_0^{\sqrt{r}} \sqrt{\bar{g}(\dot{\gamma},\dot{\gamma}) + \sigma(\dot{\gamma},\dot{\gamma})}dt \\
&= \int_0^{\sqrt{r}} \sqrt{\bar{g}(\dot{\gamma},\dot{\gamma})}(1 + \frac{1}{2}\bar{g}(\dot{\gamma},\dot{\gamma})^{-1}\sigma(\dot{\gamma},\dot{\gamma}) - \frac{1}{8}\bar{g}(\dot{\gamma},\dot{\gamma})^{-2}\sigma(\dot{\gamma},\dot{\gamma})^2 + \mathcal{O}(r^{-3\tau}))dt \\
&= \int_0^{\sqrt{r}} \sqrt{r}(1 + \frac{1}{2}\sigma(\hat{a},\hat{a}) - \frac{1}{8}\sigma(\hat{a},\hat{a})^2 + \mathcal{O}(r^{-3\tau}))dt \\
&= Length_{\bar{g}}(\gamma)\left(1 + \mathcal{O}(r^{-\tau})\right)
\end{aligned}$$

This shows that $B_r^{\bar{g}}(p) \subset B^g_{r(1+Cr^{-\tau})}(p)$ and therefore $|B_r^{\bar{g}}(p)|_{\bar{g}} \leq |B^g_{r(1+Cr^{-\tau})}(p)|_g$. Sending $r \to \infty$ yields

$$\limsup_{r\to\infty} \frac{|B_r^g(p)|_g}{|B_r^{\bar{g}}(p)|_{\bar{g}}} \geq 1$$

The hyperbolic case is analogous if we replace the straight line with a minimising geodesic in $\mathbb{H}^3$ and apply Definition 7.8.

Theorem 7.11. *Let (M, g) be a connected, complete, 3D Riemannian manifold without boundary, with scalar curvature* $\mathrm{Sc} \geq 6K$ *and satisfying K-GAVP, where $K \in \{-1, 0\}$. If every $p \in M$ admits a neighbourhood U such that*

$$\sup\{m_H(\Sigma) : \Sigma \subset U \text{ is an immersed 2-dimensional surface}\} \leq 0$$

or, more generally, if

$$\limsup_{\rho\downarrow 0} \rho^{-5} m_H(S_{p,\rho}(w)) \leq 0, \quad \forall p \in M$$

where $S_{p,\rho}(w)$ is the optimally perturbed geodesic sphere with w as in (4.24), then (M^3, g) is isometric to either $\mathbb{H}^3$ (when $K = -1$) or $\mathbb{R}^3$ (when $K = 0$).

To prove this rigidity theorem we require a well-known volume comparison result, recalled below [BC64; Gro99].

Theorem 7.12 (Bishop-Gromov)**.** *Let (M,g) be a connected, complete, n-dimensional Riemannian manifold without boundary. Let $p \in M$. If there is a constant K such that* $\mathrm{Ric} \geq (n-1)Kg$*, then*

$$\lim_{r\to 0} \frac{|B_r^g(p)|_g}{|B_r^K|_{g_K}} = 1 \tag{7.8}$$

and the quotient is a non-increasing function of $r \in (0,\infty)$. Furthermore, if the quotient equals 1 for a particular r, then $B_r^g(p)$ is isometric to B_r^K.

Proof of Theorem 7.11. As in the proof of Theorem 7.2, the assumptions on the scalar curvature and Hawking mass imply that $S \equiv 0$ and (M^3,g) has constant sectional curvature K. Therefore $\mathrm{Ric} = 2Kg$ and (M^3,g) satisfies the requirements of the Bishop-Gromov theorem and we have, at $p \in M$

$$\limsup_{r\to\infty} \frac{|B_r^g(p)|_g}{|B_r^K|_{g_K}} \leq 1 \tag{7.9}$$

Combining (7.9) with the K-GAVP assumption, we get

$$\limsup_{r\to\infty} \frac{|B_r^g(p)|_g}{|B_r^K|_{g_K}} = 1 \tag{7.10}$$

In fact, considering (7.8) and that the quotient in (7.10) is non-increasing on $(0,\infty)$, we must have

$$\frac{|B_r^g(p)|_g}{|B_r^K|_{g_K}} = 1 \quad \forall r \in (0,\infty)$$

Therefore, by the final conclusion of the Bishop-Gromov theorem, $B_r^g(p)$ is isometric to B_r^K for every $r \in (0,\infty)$. Hence (M^3,g) is isometric to $\mathbb{H}^3$ (when $K=-1$) or $\mathbb{R}^3$ (when $K=0$). □

8 Further Work

In this section we mention some possible directions for further research into the relationship between mass and geometry, based on the work done in this thesis.

• A new quasi-local mass

Considering the results obtained in this thesis, it seems potentially fruitful to define the following variant of the Hawking mass.

Definition 8.1 (sup-Hawking mass)**.** *Let $\Omega \subset (M^3, g)$ be open. The* sup-Hawking mass *of Ω is defined as*

$$m_{SH}(\Omega) := \sup\{m_H(\Sigma) : \mathbb{S}^2 \cong \Sigma \subset \overline{\Omega} \text{ such that } \Sigma \text{ is smooth and outer-minimising in } \overline{\Omega}\}$$

We note that, similar to the Hawking mass m_H, this quasi-local mass is likely to be most useful in the case where M is AF with non-negative scalar curvature such that ∂M is the only compact, minimal surface in M. It would be interesting to know the connection between this and other quasi-local masses, in particular the various versions of the Bartnik mass. See the appendix of [MT21] where the authors prove some desirable properties of m_{SH}.

• Other variants of Bartnik mass

It would be interseting to see if either of the Bartnik mass theorems proved here are true for other versions of the Bartnik mass [Jau19; McC20]. Any proof using perturbed spheres along the same lines as this thesis, would have to preserve the following key chain of inequalities:

$$m_H(S_{p,\rho}(w)) \leq m_B(B_{p,\rho}(w)) \leq m_B(\Omega)$$

However, as mentioned earlier, these inequalities rely on results which require the conditions we assumed in our version of the Bartnik mass. Therefore, any such proof for a new version would need corresponding new proofs of these results. In particular, Lemma 1.12 is at the heart of the proof in this thesis (via Lemma 1.20) and also in the proof of Lemma 6.20 ([Cho+21, Theorem C.2]) which was used in the proof of Proposition 6.2. Finding an analogue of Lemma 1.12 for a different version of the Bartnik mass would be a big step towards recreating the whole proof. See [Lee19, Theorem 4.53] where the author proves it when Σ is a connected, outer-minimising component of ∂M, the other components are minimal, and M has first Betti number equal to zero. This may complement a choice of Bartnik mass where extensions are not allowed to contain minimal surfaces which enclose the boundary (but others are allowed).

• Hyperbolic rigidity

Note that the rigidity part of Theorem 1.33 was not generalised to the hyperbolic case in Theorem 7.2. This is because the ALSC assumption is not enough to exclude all hyperbolic 3-manifolds other than $\mathbb{H}^3$ using a similar argument to the one for the $K = 0$ case, via the work in [Wol11] (in Theorem 7.11 we replaced the topological ALSC condition with the geometric K-GAVP, and achieved the same rigidity as before). Thus, in order to obtain some kind of rigidity statement in the $K = -1$ case, a deeper understanding of the topology of space-forms with negative sectional curvature (i.e. *hyperbolic 3-manifolds*) is required. This is a topic of much current research.

9 Appendix

9.1 Space-Like Hypersurfaces in a Space-Time

Here, we use the notation of [Lee18, Problems 8-19 and 8-20]. Let $(\tilde{M},\tilde{g})$ be a 4D Lorentzian manifold and $(M,g) \subset (\tilde{M},\tilde{g})$ a 3D Riemannian submanifold. For vector fields W,X,Y,Z on M and future pointing unit normal N along M, we have the Gauss and Codazzi equations

$$\tilde{\mathrm{Rm}}(W,X,Y,Z) = \mathrm{Rm}(W,X,Y,Z) + h(W,Z)h(X,Y) - h(W,Y)h(X,Z) \tag{9.1}$$

$$\tilde{\mathrm{Rm}}(W,X,Y,N) = \nabla h(Y,W,X) - \nabla h(Y,X,W) \tag{9.2}$$

and Einstein's field equation

$$\tilde{\mathrm{Ric}} - \frac{1}{2}\tilde{\mathrm{Sc}}\tilde{g} + \Lambda\tilde{g} = T \tag{9.3}$$

where Λ is the cosmological constant (normalized to -3, 0 or 3) and T is the stress-energy-momentum tensor such that $T(N,N)$ is the energy density and $T(X,N)$ the momentum density in the direction X, on M. Working in an orthonormal basis $\{N,E_1,E_2,E_3\}$ at a point $p \in M$, we take the trace of equation (9.1), giving

$$\tilde{\mathrm{Ric}}(X,Y) + \tilde{\mathrm{Rm}}(N,X,Y,N) = \mathrm{Ric}(X,Y) + Hh(X,Y) - \sum_{i=1}^{3} h(E_i,Y)h(X,E_i)$$

Tracing again yields

$$\tilde{\mathrm{Sc}} + 2\tilde{\mathrm{Ric}}(N,N) = \mathrm{Sc} + H^2 - |h|_g^2 \tag{9.4}$$

Now, taking the trace in equation (9.2) gives

$$\tilde{\mathrm{Ric}}(W,N) = div_g h(W) - \nabla H(W) \tag{9.5}$$

Next, plug in N to equation (9.3) and use $\tilde{g}(N,N) = -1$ to get

$$\tilde{\mathrm{Sc}} + 2\tilde{\mathrm{Ric}}(N,N) = 2\Lambda + 2T(N,N) \tag{9.6}$$

Combining equations (9.6) and (9.4) yields

$$2T(N,N) = \mathrm{Sc} - 2\Lambda - |h|_g^2 + H^2 \tag{9.7}$$

Now, plugging W and N into equation (9.3) and using $\tilde{g}(W,N) = 0$ gives

$$\tilde{\mathrm{Ric}}(W,N) = T(W,N) \tag{9.8}$$

Combining equations (9.8) and (9.5) reveals

$$T(W,N) = div_g h(W) - \nabla H(W) \tag{9.9}$$

Equations (9.7) and (9.9) are called the *Einstein constraint equations*. Now, the Dominant Energy Condition states that the observed energy density is non-negative and that the flow of energy cannot be observed to be faster than the speed of light. In our notation this means that, at $p \in M$

$$T(N,N) \geq |T(\,\cdot\,,N)|_g = \left(\sum_{i=1}^{3} T(E_i,N)^2\right)^{\frac{1}{2}}$$

If (M,g) has $h \equiv 0$, then by using equations (9.7) and (9.9), the condition becomes

$$\mathrm{Sc} - 2\Lambda = 2T(N,N) \geq 2\left(\sum_{i=1}^{3} T(E_i,N)^2\right)^{\frac{1}{2}} = 0$$

which finally yields the scalar curvature condition used throughout this thesis.

9.2 Proof of Lemma 2.12

Proof. Note that, in this proof, the notation will suppress the difference between ∂_t and its push forward by F; we write ∂_t for both.

i) Using the properties of the Levi-Civita connection and the fact that the coordinate vector fields satisfy $[\partial_i, \partial_t] = 0$, we get

$$\partial_t (g_t)_{jk} = \partial_t g(\partial_j, \partial_k) = g(\nabla_{\partial_j}\partial_t, \partial_k) + g(\partial_j, \nabla_{\partial_k}\partial_t)$$

Using the definition of F as a normal variation, the fact that the second fundamental form is symmetric, and $g(N_t, \partial_i) = 0$, yields

$$\begin{aligned}
\partial_t (g_t)_{jk} &= g(\nabla_{\partial_j} f_t N_t, \partial_k) + g(\partial_j, \nabla_{\partial_k} f_t N_t) \\
&= f_t \left(g(\nabla_{\partial_j} N_t, \partial_k) + g(\partial_j, \nabla_{\partial_k} N_t) \right) \\
&= f_t \left((h_t)_{jk} + (h_t)_{kj} \right) \\
&= 2 f_t (h_t)_{jk}
\end{aligned}$$

ii) We have the following Jacobi formula for an invertible matrix $B(t)$:

$$\partial_t(\det B(t)) = \mathrm{tr}[\mathrm{Adj}(B(t)) \cdot \partial_t B(t)] = \det B(t)\mathrm{tr}[B(t)^{-1} \cdot \partial_t B(t)]$$

Applying this with $B(t) = g_t$ gives

$$\begin{aligned}
\partial_t dV_{g_t} &= \partial_t \sqrt{\det(g_t)} dx^1 dx^2 \\
&= \frac{\partial_t \det(g_t)}{2\sqrt{\det(g_t)}} dx^1 dx^2 \\
&= \frac{1}{2}\sqrt{\det g_t}\ \mathrm{tr}[g_t^{ij}\partial_t(g_t)_{jk}]dx^1 dx^2 \\
&= f\sqrt{\det g_t}\ \mathrm{tr}[g_t^{ij}(h_t)_{jk}]dx^1 dx^2 \\
&= f_t H_t dV_{g_t}
\end{aligned}$$

iii) Since $g(\nabla_{\partial_t} N_t, N_t) = \frac{1}{2}\partial_t g(N_t, N_t) = 0$, we know that $\nabla_{\partial_t} N_t$ is tangent to Σ_t and therefore we can write it as

$$\begin{aligned}
\nabla_{\partial_t} N_t &= (\nabla_{\partial_t} N_t)^k \partial_k \\
&= \delta_k^i (\nabla_{\partial_t} N_t)^k \partial_i \\
&= (g_t)^{ij}(g_t)_{kj}(\nabla_{\partial_t} N_t)^k \partial_i \\
&= (g_t)^{ij} g_t(\nabla_{\partial_t} N_t, \partial_j)\partial_i \\
&= (g_t)^{ij} g(\nabla_{\partial_t} N_t, \partial_j)\partial_i
\end{aligned}$$

Since $g(\nabla_{\partial_j} N_t, N_t) = \frac{1}{2}\partial_j g(N_t, N_t) = 0$, we also have

$$\begin{aligned}
0 &= \partial_t g(N_t, \partial_j) \\
&= g(\nabla_{\partial_t} N_t, \partial_j) + g(N_t, \nabla_{\partial_t}\partial_j) \\
&= g(\nabla_{\partial_t} N_t, \partial_j) + g(N_t, \nabla_{\partial_j}\partial_t) \\
&= g(\nabla_{\partial_t} N_t, \partial_j) + g(N_t, \nabla_{\partial_j} f_t N_t) \\
&= g(\nabla_{\partial_t} N_t, \partial_j) + \partial_j f_t
\end{aligned}$$

Substituting this into the previous equation gives

$$\nabla_{\partial_t} N_t = -(g_t)^{ij}\partial_j f_t \partial_i = -\mathrm{grad}_{\Sigma_t} f_t$$

iv)

$$\begin{aligned}
\partial_t (h_t)_{ij} &= \partial_t g(\nabla_{\partial_i} N_t, \partial_j)\\
&= g(\nabla_{\partial_t}\nabla_{\partial_i} N_t, \partial_j) + g(\nabla_{\partial_i} N_t, \nabla_{\partial_t}\partial_j)\\
&= g(\mathcal{R}(\partial_t, \partial_i)N_t + \nabla_{\partial_i}\nabla_{\partial_t} N_t + \nabla_{[\partial_t,\partial_i]} N_t, \partial_j) + g(\nabla_{\partial_i} N_t, \nabla_{\partial_t}\partial_j)\\
&= \mathrm{Rm}(\partial_j, N_t, \partial_t, \partial_i) + g(\nabla_{\partial_i}\nabla_{\partial_t} N_t, \partial_j) + g(\nabla_{\partial_i} N_t, \nabla_{\partial_t}\partial_j)\\
&= f_t \mathrm{Rm}(\partial_j, N_t, N_t, \partial_i) - g(\nabla_{\partial_i}\mathrm{grad}_{\Sigma_t} f_t, \partial_j) + g(\nabla_{\partial_i} N_t, \nabla_{\partial_j}(f_t N_t))\\
&= -f_t \mathrm{Rm}(N_t, \partial_j, N_t, \partial_i) - g(\nabla_{\partial_i}\mathrm{grad}_{\Sigma_t} f_t, \partial_j) + f_t g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} N_t)
\end{aligned}$$

where we have used the formula for $\nabla_{\partial_t} N_t$ and again that $g(\nabla_{\partial_i} N_t, N_t) = 0$.

v) We have the derivative of the inverse of a matrix, $B(t)$:

$$\partial_t (B(t)^{-1}) = -B(t)^{-1}\partial_t(B(t))B(t)^{-1}$$

Using the definition $H_t = (g_t)^{ij}(h_t)_{ij}$, we have

$$\begin{aligned}
\partial_t H_t &= \partial_t((g_t)^{ij})(h_t)_{ij} + (g_t)^{ij}\partial_t (h_t)_{ij}\\
&= -2f_t (g_t)^{il}(h_t)_{lk}(g_t)^{kj}(h_t)_{ij}\\
&\quad + (g_t)^{ij}\left[-f_t \mathrm{Rm}(N_t, \partial_j, N_t, \partial_i) - g(\nabla_{\partial_i}\mathrm{grad}_{\Sigma_t} f_t, \partial_j) + f_t g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} N_t)\right]\\
&= -2f_t |h_t|_g^2 - f_t \mathrm{Ric}(N_t, N_t) - \Delta_{\Sigma_t} f_t + (g_t)^{ij}(g_t)_{kl}(\nabla_{\partial_i} N_t)^k (\nabla_{\partial_j} N_t)^l\\
&= -2f_t |h_t|_g^2 - f_t \mathrm{Ric}(N_t, N_t) - \Delta_{\Sigma_t} f_t + (g_t)^{ij}(g_t)_{kl}(h_t)_i^k (h_t)_j^l\\
&= -2f_t |h_t|_g^2 - f_t \mathrm{Ric}(N_t, N_t) - \Delta_{\Sigma_t} f_t + f_t |h_t|_g^2\\
&= -\Delta_{\Sigma_t} f_t - f_t\left(|h_t|_g^2 + \mathrm{Ric}(N_t, N_t)\right)
\end{aligned}$$

where we have used the fact that g_t is the induced metric, and properties of the curvature tensor.

□

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