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Mathematisches Forschungsinstitut Oberwolfach

Report No. 53/2016

DOI: 10.4171/OWR/2016/53

Analytic Number Theory

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6 November – 12 November 2016

Abstract.
Invariants of topological spaces of dimension three play a major role in many areas, in particular ...

Mathematics Subject Classification (2010): 11D45,11P55,11J25.

Introduction by the Organisers

The workshop *Invariants of topological spaces of dimension three*, organised by Max Muster (München) and Bill E. Xample (New York) was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds ...

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation

for supporting NAME(S) in the "Simons Visiting Professors" program at the MFO.

Workshop: Analytic Number Theory

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Rydin Myerson, Simon L.

Let $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]^R$ be a system of R homogeneous forms of the same degree $d \geq 2$, with integer coefficients, in n variables. Define the counting function

$$N(P) \overset{\text{def}}{=} \#\left\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0},\, |\boldsymbol{x}|_\infty \leq P\right\}.$$

A classic result of Birch estimates $N(P)$ when the number of variables n is sufficiently large and $\boldsymbol{f}$ is suitably nonsingular. In particular, his work implies:

Theorem 1 (Birch [1])**.** *If $V(\boldsymbol{f}) \subset \mathbb{P}^{n-1}$ is smooth with dimension $n-R-1$, and*

$$(\star) \qquad n \geq R(R+1)(d-1)2^{d-1} + R$$

then the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ satisfies the Hasse principle, and

$$N(P) \sim \nu P^{n-dR}$$

as $P \to \infty$, for some real constant $\nu \geq 0$.

The proof uses the circle method. Birch's work has been very widely generalised, for example to systems of forms with differing degrees by Browning and Heath-Brown [2], to linear spaces of solutions by Brandes [3], to bihomogeneous forms by Schindler [4], and to function fields by Lee [5].

Despite this, improvements in the condition $(\star)$ have until now been confined to the case $R = 1$, with the exception of the case $(d, R) = (2, 2)$ where $(\star)$ has been improved from $n \geq 14$ to $n \geq 11$ by Munshi [6].

In a series of forthcoming papers [7, 8, 9] I prove the following result.

Theorem 2 (RM)**.** *If either $d \leq 3$, or $\boldsymbol{f}$ is in general position, we may replace the condition $(\star)$ in Birch's result with*

$$(\dagger) \qquad n \geq d2^d R + R.$$

This improves on $(\star)$ in each of the following three cases: either $d = 2$ and $R \geq 4$, or $d = 3$ and $R \geq 3$, or $d \geq 4$ and $R \geq 2$. The "general position" condition can be made explicit, and is in some sense a nonsingularity condition.

According to the "square-root cancellation" heuristic, in place of $(\star)$ one would expect the condition $n \geq 2dR + 1$ to suffice. Since $(\star)$ is linear in R, it brings us within a constant factor of square-root cancellation if $d \geq 2$ is held fixed.

I also give a generalisation of Theorem 1 to systems of forms with real coefficients. Let $\boldsymbol{g}(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]^R$ be a system of R forms in n variables of degree $d \geq 2$ with real coefficients. Define

$$M_{\boldsymbol{g}}(P, Q) \overset{\text{def}}{=} \#\left\{\boldsymbol{x} \in \mathbb{Z}^n : |\boldsymbol{g}(\boldsymbol{x})|_\infty \leq Q,\, |\boldsymbol{x}|_\infty \leq P\right\}.$$

We say $\boldsymbol{g}(\boldsymbol{x})$ is *irrational* if there is no $\boldsymbol{\alpha} \in \mathbb{R}^R \setminus \{\boldsymbol{0}\}$ for which $\sum_i \alpha_i g_i(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ has integral coefficients. In [10] I prove

Theorem 3 (RM)**.** *If $d \leq 3$ let $V(\boldsymbol{g})$ be smooth of dimension $n-R-1$. If $d \geq 4$ let $\boldsymbol{g}$ be in general position. Let $\rho \in [0, d-1)$, and if $\rho = 0$ let $\boldsymbol{g}$ be irrational. Suppose $V(\boldsymbol{g})$ has a real point and*

$$n \geq (d-\rho)2^d R + R.$$

Then for some real constant $\nu_\infty > 0$, we have in the limit as $P \to \infty$ that

$$M(P, P^\rho) \sim \text{measure}\{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{g}(\boldsymbol{x})|_\infty \leq P^\rho,\ |\boldsymbol{x}|_\infty \leq P\} \sim \nu_\infty P^{n-(d-\rho)R}.$$

The case $d = 2$ of this result is essentially due to Müller [11], or to Bentkus and Götze [12] when $R = 1$.

Note that when $\rho > 0$, Theorem 3 applies to systems of forms with either real or rational coefficients. Only in the case $\rho = 0$ is it necessary to impose an irrationality condition. This is because an inequality of the form $|f(\boldsymbol{x})| \ll 1$, where f has integral coefficients, would lead to p-adic conditions on the variables $\boldsymbol{x}$ and the asymptotic formula in Theorem 3 would need to be modified accordingly.

The strategy of proof for Theorems 2 and 4 reduces the problem to an upper bound for the number of solutions to a system of multilinear auxiliary inequalities described in [7].

When $d = 2$ these inequalities are linear and the required upper bound is not difficult. When $d = 3$ a strategy of Davenport [13] can be used to treat the problem.

The case $d \geq 4$ seems more difficult, but when $\boldsymbol{f}$ is in general position the auxiliary inequalities cannot be very singular and consequently an upper bound can be obtained by elementary means. It might be hoped that one could generalise Davenport's approach to $d \geq 4$, and so remove the "general position" condition from Theorems 2 and 3.

References

[1] B. J. Birch. Forms in many variables. *Proc. Roy. Soc. Ser. A*, 265:245–263, 1961/1962.

[2] T. Browning and D. Heath-Brown. Forms in many variables and differing degrees. *Journal of the European Mathematical Society*, 12 2014.

[3] J. Brandes. Forms representing forms and linear spaces on hypersurfaces. *Proc. Lond. Math. Soc. (3)*, 108(4):809–835, 2014.

[4] D. Schindler. Bihomogeneous forms in many variables. *J. Théor. Nombres Bordeaux*, 26(1):483–506, 2014.

[5] S.-l. A. Lee. Birch's theorem in function fields. *ArXiv e-prints*, Sept. 2011. arXiv:1109.4953.

[6] R. Munshi. Pairs of quadrics in 11 variables. *Compos. Math.*, 151(7):1189–1214, 2015.

[7] S. L. Rydin Myerson. Systems of quadratic forms. *ArXiv e-prints*, Dec. 2015. arXiv:1512.06003.

[8] S. L. Rydin Myerson. Systems of cubic forms. In preparation.

[9] S. L. Rydin Myerson. Systems of forms of the same degree. In preparation.

[10] S. L. Rydin Myerson. Diophantine inequalities in many variables. In preparation.

[11] W. Müller. Systems of quadratic Diophantine inequalities and the value distribution of quadratic forms. *Monatsh. Math.*, 153(3):233–250, 2008.

[12] V. Bentkus and F. Götze. Lattice point problems and distribution of values of quadratic forms. *Ann. of Math. (2)*, 150(3):977–1027, 1999.

[13] H. Davenport. Cubic forms in sixteen variables. *Proc. Roy. Soc. Ser. A*, 272:285–303, 1963.

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Abstracts