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Bounds for spectral projectors on the Euclidean cylinder

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Abstract. We prove essentially optimal bounds for norms of spectral projectors on thin spherical shells for the Laplacian on the cylinder $(\mathbb{R}/\mathbb{Z})\times\mathbb{R}$. In contrast to previous investigations into spectral projectors on tori, having one unbounded dimension available permits a compact self-contained proof.

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1. Introduction

1.1. *Spectral projectors on general manifolds and tori*

Given a Riemannian manifold with Laplace–Beltrami operator Δ, consider the spectral projector $P_{\lambda,\delta}$ on (perhaps generalized) eigenfunctions with eigenvalues within $O(\delta)$ of λ. It is defined through functional calculus by the formula

$$P_{\lambda,\delta}=P^{\chi}_{\lambda,\delta}=\chi\left(\frac{\sqrt{-\Delta}-\lambda}{\delta}\right),$$

where χ is a cutoff function, which is irrelevant for our purposes.

An interesting question is to determine the operator norm from L^2 to L^p, with $p>2$, of this operator. A theorem of Sogge [5] gives an optimal answer for any Riemannian manifold if $\delta=1$

$$\left\|P_{\lambda,1}\right\|_{L^2\to L^p}\lesssim\lambda^{\frac{d-1}{2}-\frac{d}{p}}+\lambda^{\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

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While this completely answers the question if $\delta > 1$, the case $\delta < 1$ is still widely open. Understanding the case $\delta < 1$ requires a global analysis on the Riemannian manifold, which makes it very delicate.

In the case of the rational torus $\mathbb{R}^d/\mathbb{Z}^d$, L^p bounds on eigenfunctions attracted a lot of attention; this corresponds to the choice $\delta = 1/\lambda$, since the distance between two consecutive eigenvalues is $\sim \frac{1}{\lambda}$. The best result in this direction is due to Bourgain and Demeter [3]. More recently, the authors of the present paper [4] considered the problem for general values of λ and δ, conjectured the bound for general tori

$$\|P_{\lambda,\delta}\|_{L^2\to L^p} \lesssim \lambda^{\frac{d-1}{2}-\frac{d}{p}}\delta^{1/2} + (\lambda\delta)^{\frac{(d-1)}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \qquad \text{for } \delta > 1/\lambda,$$

and were able to establish this bound for a range of the parameters δ, λ, p.

A full proof of this conjecture seems very challenging in every dimension d. Restricting to the case $d = 2$, consider the case $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R} = \mathbb{T} \times \mathbb{R}$ instead of $\mathbb{T}^2$. The conjecture remains identical, but a short proof, relying on ℓ^2 decoupling, can be provided; this is the main observation of the present paper. Generalizations to higher dimensions are certainly possible.

1.2. *The Euclidean cylinder*

On $\mathbb{T}\times\mathbb{R} = (\mathbb{R}/\mathbb{Z})\times\mathbb{R}$, we choose coordinates (x, y), with $x \in [0,1]$ and $y \in \mathbb{R}$. The Laplacian operator is given by

$$\Delta = \partial_x^2 + \partial_y^2.$$

A function f on $\mathbb{T} \times \mathbb{R}$ can be expanded through Fourier series in x and Fourier transform in y:

$$f(x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{f}(k,\eta) e^{2\pi i(kx+\eta y)}\, d\eta.$$

The spectral projector can then be expressed as

$$P_{\lambda,\delta} f(x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \chi\left(\frac{\sqrt{k^2+\eta^2}-\lambda}{\delta}\right) \widehat{f}(k,\eta) e^{2\pi i(kx+\eta y)}\, d\eta.$$

Theorem 1 (Main theorem). *If $\lambda > 1$ and $\delta < 1$,*

$$\|P_{\lambda,\delta}\|_{L^2\to L^p} \lesssim_\epsilon \lambda^\epsilon \delta^{-\epsilon}\left[\lambda^{\frac{1}{2}-\frac{2}{p}}\delta^{\frac{1}{2}} + (\lambda\delta)^{\frac{1}{4}-\frac{1}{2p}}\right]$$

Furthermore, this estimate is optimal, up to the subpolynomial factor $\lambda^\epsilon\delta^{-\epsilon}$ and the multiplicative constant.

1.3. *Strichartz estimates*

It is interesting to draw a parallel with Strichartz estimates in dimension 2 for the Schrödinger equation, in which case the critical exponent equals 4. It was proved in the foundational paper of Bourgain [2] that

$$\left\|e^{it\Delta} f\right\|_{L^4([0,1]\times\mathbb{T}^2)} \lesssim_s \|f\|_{H^s(\mathbb{T}^2)} \qquad \text{for } s > 0.$$

Takaoka and Tzvetkov [6] proved that the above inequality fails for $s = 0$, but that, on $\mathbb{T}\times\mathbb{R}$,

$$\left\|e^{it\Delta} f\right\|_{L^4([0,1]\times\mathbb{T}\times\mathbb{R})} \lesssim \|f\|_{L^2(\mathbb{T}\times\mathbb{R})}.$$

Finally, Barron, Christ and Pausader [1] determined the correct global (in time) estimate, for which a further summation index is needed. These examples suggest that optimal estimates might differ by subpolynomial factors between $\mathbb{T}^2$ and $\mathbb{T} \times \mathbb{R}$.

2. Proof of the Theorem 1 (main theorem)

Proof. By Plancherel's theorem, it suffices to prove

$$\|f\|_{L^p} \lesssim_\epsilon \lambda^\epsilon \delta^{-\epsilon}\left[\lambda^{\frac{1}{2}-\frac{2}{p}}\delta^{\frac{1}{2}} + (\lambda\delta)^{\frac{1}{4}-\frac{1}{2p}}\right]\|f\|_{L^2} \tag{1}$$

for f a function whose Fourier transform is localized in the corona $\mathscr{C}_{\lambda,\delta}$ of radius λ and with thickness δ/λ:

$$\mathscr{C}_{\lambda,\delta} = \left\{(k,\eta) \text{ such that } \lambda - \delta < \sqrt{k^2+\eta^2} < \lambda + \delta\right\}.$$

By symmetry, one can furthermore assume that $\widehat{f}(k,\eta)$ is localized in the first quadrant $k,\eta \geq 0$.

The function f can be split into two pieces, which will correspond to the two terms on the right-hand side of (1).

$$f(x) = \left[\sum_{|k-\lambda|\leq\frac{1}{\delta}} + \sum_{|k-\lambda|>\frac{1}{\delta}}\right]\int_{\mathbb{R}} \widehat{f}\left(k,\eta\right) e^{2\pi i(kx+\eta y)}\, d\eta = f_1(x) + f_2(x).$$

The Case $|k-\lambda| \leq \frac{1}{\delta}$. The Fourier support of f_1 is made up of a collection of segments. We will see in Lemma 2 below that the added length of these segments can be bounded by

$$\left|\operatorname{Supp}\widehat{f_1}\right| \lesssim \sqrt{\lambda\delta}.$$

Therefore, by the Cauchy–Schwartz inequality,

$$\|f_1\|_{L^\infty} \leq \left|\operatorname{Supp}\widehat{f_1}\right|^{1/2} \cdot \|\widehat{f_1}\|_{L^2}^{1/2} \lesssim (\lambda\delta)^{1/4}\|f_1\|_{L^2}.$$

Interpolating with L^2, this gives

$$\|f_1\|_{L^p} \lesssim (\lambda\delta)^{\frac{1}{4}-\frac{1}{2p}}\|f\|_{L^2}.$$

The Case $|k-\lambda| > \frac{1}{\delta}$. We start by choosing a function $\phi \in \mathscr{S}$ which is $> 1/2$ on $[-1,1]$, and has Fourier support in $[-1,1]$. We use periodicity in the x variable to expand the range of x from $x \in [0,1]$ to $x < \delta^{-1}$, so that

$$\|f_2\|_{L^p(\mathbb{T}\times\mathbb{R})} \lesssim \delta^{1/p}\left\|\phi(\delta x)\sum_{|k-\lambda|>\frac{1}{\delta}}\int_{\mathbb{R}}\widehat{f}(k,\eta)e^{2\pi i(kx+\eta y)}\,d\eta\right\|_{L^p(\mathbb{R}^2)}.$$

We now change variables as follows: $X = \lambda x$, $Y = \lambda y$, $K = k/\lambda$, $H = \eta/\lambda$,

$$f_2(x,y) = \lambda F(X,Y), \qquad F(X,Y) = \phi\left(\frac{\delta X}{\lambda}\right)\sum_{\substack{K\in\mathbb{Z}/\lambda \\ |K-1|>\frac{1}{\delta\lambda}}}\int_{\mathbb{R}}\widehat{f}(\lambda K,\lambda H)e^{2\pi i(KX+HY)}\,dH$$

to obtain

$$\|f_2\|_{L^p(\mathbb{T}\times\mathbb{R})} \lesssim \delta^{1/p}\lambda^{1-\frac{2}{p}}\|F\|_{L^p(\mathbb{R}^2)}.$$

The effect of this change of variables is that the function of (X,Y) whose L^p norm we want to evaluate has Fourier transform supported in the corona $\mathscr{C}_{1,3\delta/\lambda}$ of radius 1 and width δ/λ, and also in the first quadrant $X,Y \geq 0$. This enables us to apply the ℓ^2 decoupling theorem of Bourgain and Demeter [3]: for a smooth partition of unity (χ_θ) corresponding to a suitable almost disjoint covering of $\mathscr{C}_{1,3\delta/\lambda}$ by caps (θ) of size $\sim \frac{\delta}{\lambda}\times\sqrt{\frac{\delta}{\lambda}}$,

$$\|f_2\|_{L^p(\mathbb{T}\times\mathbb{R})} \lesssim \delta^{\frac{1}{p}}\lambda^{1-\frac{2}{p}}(\delta/\lambda)^{-\frac{1}{4}+\frac{3}{2p}-\epsilon}\left(\sum_\theta \|\chi_\theta(D)F\|_{L^p(\mathbb{R}^2)}^2\right)^{1/2},$$

where $\chi_\theta(D)$ is the Fourier multiplier with symbol χ_θ.

We now apply the inequality $\|g\|_{L^p(\mathbb{R}^d)} \lesssim \|g\|_{L^2} |\mathrm{Supp}\, \widehat{g}|^{\frac{1}{2}-\frac{1}{p}}$ (if $p \geq 2$), which follows by applying in turn the Hausdorff–Young and Hölder inequalities, and then the Plancherel theorem. Since by Lemma 2 below

$$\left|\mathrm{Supp}\, \widehat{\chi_\theta(D)F}\right| = \left|\mathrm{Supp}\, \chi_\theta \widehat{F}\right| \lesssim \delta^{5/2} \lambda^{-3/2},$$

it follows that

$$\|f_2\|_{L^p(\mathbb{T}\times\mathbb{R})} \lesssim \delta^{\frac{1}{p}} \lambda^{1-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{4}+\frac{3}{2p}-\epsilon} \left(\delta^{5/2}\lambda^{-3/2}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\sum_\theta \|\chi_\theta(D)F\|^2_{L^2(\mathbb{R}^2)}\right)^{1/2}.$$

By almost orthogonality, this becomes

$$\|f_2\|_{L^p(\mathbb{T}\times\mathbb{R})} \lesssim \delta^{\frac{1}{p}} \lambda^{1-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{4}+\frac{3}{2p}-\epsilon} \left(\delta^{5/2}\lambda^{-3/2}\right)^{\frac{1}{2}-\frac{1}{p}} \|F\|_{L^2(\mathbb{R}^2)}.$$

Finally, undoing the change of variables and using once again periodicity in the x variable gives

$$\begin{aligned}\|f_2\|_{L^p(\mathbb{T}\times\mathbb{R})} &\lesssim \delta^{\frac{1}{p}} \lambda^{1-\frac{2}{p}} (\delta/\lambda)^{-\frac{1}{4}+\frac{3}{2p}-\epsilon} \left(\delta^{5/2}\lambda^{-3/2}\right)^{\frac{1}{2}-\frac{1}{p}} \delta^{-1/2} \|f\|_{L^2} \\ &\lesssim \lambda^{\frac{1}{2}-\frac{2}{p}} \delta^{1/2}\end{aligned}$$

Optimality. The optimality of the statement of the theorem is proved through two examples. The first one is an analog of the Knapp example: assume $\lambda \in \mathbb{N}$, and consider the function g given by its Fourier transform

$$\widehat{g}(k,\eta) = \mathbf{1}_\lambda(k)\chi\left(\frac{\eta}{\sqrt{\lambda\delta}}\right).$$

Here, $\mathbf{1}_\lambda$ is the indicator function of $\{\lambda\}$ and χ is a cutoff function with a sufficiently small support, so that $\mathrm{Supp}\, \widehat{g} \subset \mathscr{C}_{\lambda,\delta}$. In physical space,

$$g(x,y) = \sqrt{\lambda\delta}\, e^{2\pi i \lambda x} \widehat{\chi}\left(\sqrt{\lambda\delta}\, y\right).$$

It has L^p norm $\sim (\lambda\delta)^{\frac{1}{2}-\frac{1}{2p}}$, so that

$$\frac{\|g\|_{L^p}}{\|g\|_{L^2}} \sim (\lambda\delta)^{\frac{1}{4}-\frac{1}{2p}}.$$

We now consider the function h given by its Fourier transform

$$\widehat{h}(k,\eta) = \mathbf{1}_{\mathscr{C}_{\lambda,\delta}}(k,\eta)\mathbf{1}_{[0,\lambda/2]}(k);$$

here, $\mathbf{1}_{\mathscr{C}_{\lambda,\delta}}$ is the indicator function of the annulus, and $\mathbf{1}_{[0,\lambda/2]}$ the indicator function of the interval. It is easy to check that $|\mathrm{Supp}\, \widehat{h}| \sim \lambda\delta$, so that $\|h\|_{L^\infty} \sim \lambda\delta$ and $\|h\|_{L^2} \sim \sqrt{\lambda\delta}$, and finally

$$\frac{\|h\|_{L^\infty}}{\|h\|_{L^2}} \sim \sqrt{\lambda\delta}.$$

By the Bernstein inequality,

$$\frac{\|h\|_{L^p}}{\|h\|_{L^2}} \gtrsim \lambda^{-2/p} \frac{\|h\|_{L^\infty}}{\|h\|_{L^2}} \sim \lambda^{\frac{1}{2}-\frac{2}{p}} \delta^{1/2}.$$

The examples g and h show that the statement of the Theorem 1 is optimal, up to subpolynomial losses. □

3. Bounds for the Fourier support

Lemma 2 (Bound on the size of Fourier support). *With the notations of the proof of Theorem 1,*

(i) *The function f_1 is a function on $\mathbb{T}\times\mathbb{R}$. As such, its Fourier transform is supported on a union of lines, and has one-dimensional measure*

$$\left|\operatorname{Supp}\widehat{f_1}\right| \lesssim \sqrt{\lambda\delta}.$$

(ii) *The function $\chi_\theta(D)F$ is a function on $\mathbb{R}^2$. Its Fourier transform is defined on $\mathbb{R}^2$, and has two-dimensional measure*

$$\left|\operatorname{Supp}\chi_\theta\widehat{F}\right| \lesssim \delta^{5/2}\lambda^{-3/2}.$$

Proof.
(i) Consider f as in the proof of Theorem 1, namely with Fourier support in $\mathscr{C}_{\lambda,\delta}$. Since (k,η) range in $\mathbb{Z}\times\mathbb{R}$ with $k,\eta\geq 0$, the Fourier support of f is contained in $\cup_{k\in\mathbb{Z}}\{k\}\times E_k^\lambda$, where

$$E_k^\lambda = \begin{cases} \emptyset & (k\geq k_+),\\ \left(0,\sqrt{k_+^2-k^2}\right) & (|k-\lambda|<\delta),\\ \left(\sqrt{(\lambda-\delta)^2-k^2},\sqrt{k_+^2-k^2}\right) & (0\leq k\leq \lambda-\delta).\end{cases}$$

Recalling that $\widehat{f}_1$ is just $\widehat{f}$ restricted to $|k-\lambda|\leq\frac{1}{\delta}$, one can then add up these pieces to get the bound

$$\left|\operatorname{Supp}\widehat{f_1}\right| \lesssim \sum_{\max\{0,\lambda-\frac{1}{\delta}\}\leq k<k_+} |E_k| \lesssim \sqrt{\lambda\delta} + \sum_{\substack{k\geq 0\\ \frac{1}{\delta}\geq\lambda-k>\delta}} \frac{\delta\lambda}{\sqrt{\lambda(\lambda-x)}}$$

and as $y\mapsto 1/\sqrt{y}$ is decreasing this is

$$\leq 2\sqrt{\lambda\delta} + \int_\delta^{\min\{\lambda,\frac{1}{\delta}\}} \frac{\delta\lambda}{\sqrt{\lambda y}}\,dy \leq 4\sqrt{\lambda\delta}.$$

(ii) Turning to F, it has Fourier support in

$$\bigcup_{\substack{k\in\mathbb{Z}\\ |k-\lambda|>\frac{1}{\delta}}} \left[\frac{k}{\lambda}-\frac{2\delta}{\lambda},\frac{k}{\lambda}+\frac{2\delta}{\lambda}\right]\times D_k^\lambda, \qquad D_k^\lambda = \left\{H,\ 1-\frac{\delta}{\lambda} < \sqrt{\frac{k^2}{\lambda^2}+H^2} < 1+\frac{\delta}{\lambda}\right\}. \tag{2}$$

Consider $\chi_\theta(D)F$, for a cap θ with dimensions

$$\sim \frac{\delta}{\lambda}\times\sqrt{\frac{\delta}{\lambda}}$$

adapted to the corona $\mathscr{C}_{1,3\delta/\lambda}$. Given such a cap, there is $j\in\mathbb{N}$ with $2^j>\frac{1}{\delta}$ such that every point in the intersection of θ with the set (2) satisfies $|\lambda-k|\sim 2^j$.

There are around $\sqrt{\delta}2^{j/2}$ such values of k for which the vertical strip $[\frac{k}{\lambda}-\frac{2\delta}{\lambda},\frac{k}{\lambda}+\frac{2\delta}{\lambda}]$ intersects the cap θ. For each such k, the size of D_k^λ is $\sim\delta\lambda^{-1/2}2^{-j/2}$. Hence, adding up the contributions in (2),

$$\left|\operatorname{Supp}\chi_\theta(D)F\right| \lesssim \sqrt{\delta}2^{j/2}\cdot\delta\lambda^{-1}\cdot\delta\lambda^{-1/2}2^{-j/2} = \delta^{5/2}\lambda^{-3/2}.$$ □

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