

Repulsion: A how-to guide (part IV)

Simon L. Rydin Myerson

April 2021

Hausdorff School: "The Circle Method", Hausdorff Center for Mathematics, Bonn

A theorem of Birch

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ are R lin. indep. forms in n variables of degree $d \geq 2$.
- We set $N_{\vec{f}}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{f}(\vec{x}) = \vec{0}, |\vec{x}| \leq P\}$.
- $\vec{f}(\vec{x})$ is nonsingular if the Jacobian matrix $(\partial f_i(\vec{x}) / \partial x_j)_{ij}$ has rank R at every complex solution $\vec{x} \in \mathbb{C}^n \setminus \{\vec{0}\}$ to $\vec{f}(\vec{x}) = \vec{0}$.

Theorem (Birch, Proc. R. Soc. Lond. A 1962)

If \vec{f} is nonsingular and $n \geq n_0(d, R)$ where

$$n_0 = (d - 1)2^{d-1}R(R + 1) + R \quad (1)$$

then $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$ as $P \rightarrow \infty$ for some real constant $\sigma \geq 0$. If $\vec{f}(\vec{x}) = \vec{0}$ has solutions in $\mathbb{R}^n \setminus \{\vec{0}\}$ and $\mathbb{Q}_p^n \setminus \{\vec{0}\}$ then $\sigma > 0$.

Theorem (SLRM 2021)

We can replace (1) with

$$n_0 = d2^dR + R. \quad (2)$$

Slightly more already known if $d < 4$. For $R \geq \max\{6 - d, 2\}$ this beats (1).

Improved major arcs?

Notation

- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).
- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{a} \in \mathbb{Z}^R, \vec{m}^{(\vec{a}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}.$
- $\mathfrak{M}(Q) = \{\vec{\alpha} \in [0, 1]^R : |\vec{\alpha} - \frac{\vec{a}}{q}| \leq \frac{Q}{qP^d} \text{ } (q \leq Q, \vec{a} \in \mathbb{Z}^R)\}.$

Lemma C (Birch's major arcs)

- If $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$ then $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$.
- Moreover we have $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR+R(R+1)(d-1)\theta}$.
- If \vec{f} is nonsingular then $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f})/P^{(d-1)n\theta} \ll_{\vec{f}} P^{-(n-R+1)\theta}$.

Proof of Lemma C part (i).

If $(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ is in \mathcal{N} but not $\mathcal{N}_{\text{sing}}$ then $\vec{\alpha} \mapsto \vec{m}^{(\vec{\alpha}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ is a nonsingular matrix M of integers $\ll_{\vec{f}} P^{(d-1)\theta}$, with $|M\vec{\alpha} - \vec{v}| \leq P^{(d-1)\theta-d}$. Let M' be a nonsingular $R \times R$ submatrix. Then $|\vec{\alpha} - M'^{-1}\vec{v}'| \leq P^{R(d-1)\theta-d}$. \square

For most \vec{v} , different $R \times R$ matrices should give contradictory approximations to $\vec{\alpha}$. So room to decrease $R(R+1)(d-1)$. If $\vec{\alpha}$ is close to $\vec{0}$, the picture is different. Most $R \times R$ submatrices give the same region for $\vec{\alpha}$, and it's much smaller than the major arc at $\vec{0}$. This makes possible Müller's $d = 2, n \geq 9R$ (J. Th. Nombres Bordeaux '05).

$$\vec{\alpha} \approx \frac{\vec{\alpha}}{pe}$$

$$S(\vec{\alpha}) S(\vec{\alpha} + \vec{\beta})$$

$$\approx S(\vec{\beta})$$

$$\sum e(\vec{\beta} \cdot \vec{x} +$$

lower
order)

What good is repulsion?

Notation

- $S(\vec{\alpha} \cdot \vec{f}) = \sum_{\vec{x} \in \mathbb{Z}^n, |\vec{x}| \leq P} e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$.
- If $F(\vec{x})$ is a degree d form, $\vec{m}^{(F)}(\vec{x}, \dots, \vec{x}) = \frac{\vec{\nabla}_{\vec{x}}}{d!} F(\vec{x})$.
- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).
- $\mathcal{N}_{aux,p}(F) = \{(\vec{x}^{(i)})_i : |\vec{x}^{(i)}| \leq p, \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \equiv \vec{0} (p)\}$.

Lemma A (level-set formulation of Birch, after Bentkus-Götze)

If $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}$ for some $c < 1$, all $k \in [0, dR/c]$, then $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f}, c, C}(P^{n-dR-\delta(c, d, R, n)+\epsilon})$.

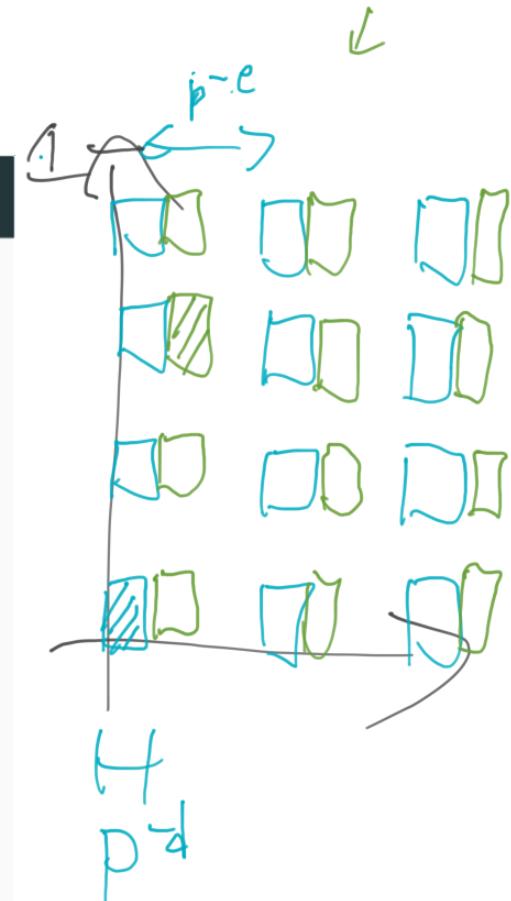
Lemma B' (One more Weyl differencing step)

$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\beta} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f}) / P^{2n+\epsilon}|^{2^{d-1}}.$$

Lemma D (Improved bounds near $\vec{0}$, in p -adic sense)

Suppose $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{aux,p}(\vec{\delta} \cdot \vec{f})$, $e > 0$, $p^{e+d-1} \leq \frac{1}{3}P^d$, and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3}p^{1-d}$. Then $\vec{b} \equiv \vec{0} (p^e)$.

If f is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{aux,p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$.



$$\vec{b} / pe$$

$$\left(\frac{\vec{b}}{p^k} \cdot \vec{f} \right)$$

What good is repulsion?

Notation

- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).
- $\mathcal{N}_{\text{aux}, p}(F) = \{(\vec{x}^{(i)})_i : |\vec{x}^{(i)}| \leq p, \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \equiv \vec{0} (p)\}$.

Want $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| \geq C_{\vec{f}, \epsilon} P^{n-k+\epsilon}\} \leq P^{ck-dR}$ with $c < 1$.

Lemma B' (One more Weyl differencing step)

$$\#\mathcal{N}(P, P^{(d-1)\theta-d}; \vec{\beta} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f}) / P^{2n+\epsilon}|^{2^{d-1}}.$$

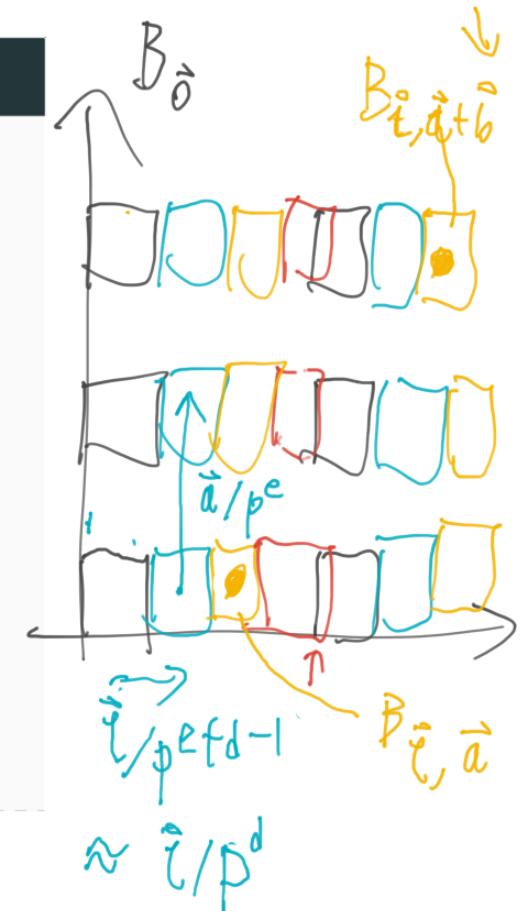
Lemma D (Improved bounds near $\vec{0}$, in p -adic sense)

Suppose $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$, $e > 0$, $p^{e+d-1} \leq \frac{1}{3}P^d$, and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3}p^{1-d}$. Then $\vec{b} \equiv \vec{0} (p^e)$.

If \vec{f} is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$.

Let $B_{\vec{\ell}} = \bigcup_{0 \leq a_i < p^e} B_{\vec{\ell}, \vec{a}}$ where $B_{\vec{\ell}, \vec{a}} = \left[0, \frac{1}{3p^{e+d-1}}\right]^R + \frac{\vec{a}}{p^e} + \frac{\vec{\ell}}{3p^{e+d-1}}$. So $[0, 1]^R \subset \bigcup_{0 \leq \ell_i < 3p^{d-1}} B_{\vec{\ell}}$. Let $A_{\vec{\ell}} = \{\vec{\alpha} \in B_{\vec{\ell}} : |S(\vec{\alpha} \cdot \vec{f})| \gg_{\vec{f}, \epsilon} CP^{n-k+\epsilon}\}$.

Want to show $\text{meas } A_{\vec{\ell}}$ is small. Let $\vec{\alpha}, \vec{\alpha} + \vec{\beta} \in A_{\vec{\ell}}$ with $\vec{\alpha} \in B_{\vec{\ell}, \vec{a}}$, $\vec{\alpha} + \vec{\beta} \in B_{\vec{\ell}, \vec{a} + \vec{b}}$. Suppose $\frac{1}{4}P^{2^d k / (n-R+1)} \leq p \leq P$ and $e > 0$.



What good is repulsion?

Want $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| \geq C_{\vec{f}, \epsilon} P^{n-k+\epsilon}\} \leq P^{ck-dR}$ with $c < 1$.

Lemma B' (One more Weyl differencing step)

$$\#\mathcal{N}(P, P^{(d-1)\theta-d}; \vec{\beta}, \vec{f})/P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha}, \vec{f})S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f})/P^{2n+\epsilon}|^{2^{d-1}}.$$

Lemma D (Improved bounds near $\vec{0}$, in p -adic sense)

Suppose $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^d \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$, $e > 0$, $p^{e+d-1} \leq \frac{1}{3}P^d$, and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3}p^{1-d}$. Then $\vec{b} \equiv \vec{0} \pmod{p^e}$.

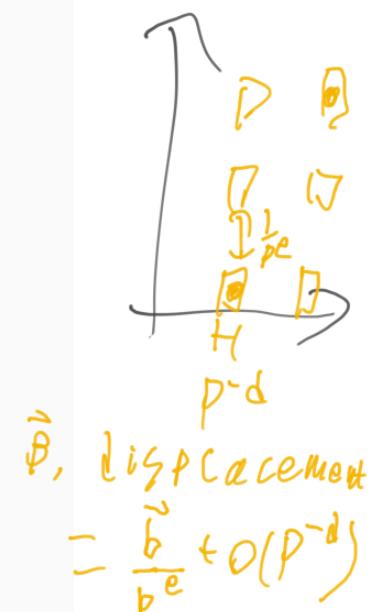
If \vec{f} is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_q^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$.

Let $B_{\vec{\ell}} = \bigcup_{0 \leq a_i < p^e} B_{\vec{\ell}, \vec{a}}$ where $B_{\vec{\ell}, \vec{a}} = [0, \frac{1}{3p^{e+d-1}}]^R + \frac{\vec{a}}{p^e} + \frac{\vec{\ell}}{3p^{e+d-1}}$. So $[0, 1]^R \subset \bigcup_{0 \leq \ell_i < 3p^{d-1}} B_{\vec{\ell}}$. Let $A_{\vec{\ell}} = \{\vec{\alpha} \in B_{\vec{\ell}} : |S(\vec{\alpha} \cdot \vec{f})| \gg_{\vec{f}, \epsilon} CP^{n-k+\epsilon}\}$.

Want to show meas $A_{\vec{\ell}}$ is small. Let $\vec{\alpha}, \vec{\alpha} + \vec{\beta} \in A_{\vec{\ell}}$ with $\vec{\alpha} \in B_{\vec{\ell}, \vec{\alpha}}$, $\vec{\alpha} + \vec{\beta} \in B_{\vec{\ell}, \vec{\alpha} + \vec{\beta}}$. Suppose $\frac{1}{4}P^{2^d k/(n-R+1)} \leq p \leq P$ and $e > 0$.

Then $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) \gg_{\vec{f}, \epsilon} Cp^{(d-2)n+R-1}$ and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3p^{d-1}}$ with $b_i < p^e$. If $C \gg_{\vec{f}, \epsilon} 1$, $p^{e+d-1} \leq \frac{1}{3} P^d$ then $p^e \parallel \vec{b}$ by D, so $\vec{b} = \vec{0}$.

Thus $A_{\vec{\ell}} \subset B_{\vec{\ell}, \vec{a}}$ for some $\vec{a} = \vec{a}(\vec{\ell}) \in \mathbb{Z}^R$, so $\text{meas } \bigcup_{\vec{\ell}} A_{\vec{\ell}} \ll \left(\frac{3p^{d-1}}{3p^{e+d-1}} \right)^R$.



What good is repulsion?

Lemma D (Improved bounds near $\vec{0}$, in p -adic sense)

Suppose $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$, $e > 0$,

$p^{e+d-1} \leq \frac{1}{3}P^d$, and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3}p^{1-d}$. Then $\vec{b} \equiv \vec{0} \pmod{p^e}$.

If \vec{f} is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$.

Let $B_{\vec{\ell}} = \bigcup_{0 \leq a_i < p^e} B_{\vec{\ell}, \vec{a}}$ where $B_{\vec{\ell}, \vec{a}} = [0, \frac{1}{3p^{e+d-1}}]^R + \frac{\vec{a}}{p^e} + \frac{\vec{\ell}}{3p^{e+d-1}}$. So $[0, 1]^R \subset \bigcup_{0 \leq \ell_i < 3p^{d-1}} B_{\vec{\ell}}$. Let $A_{\vec{\ell}} = \{\vec{a} \in B_{\vec{\ell}} : |S(\vec{a} \cdot \vec{f})| \gg_{\vec{f}, \epsilon} CP^{n-k+\epsilon}\}$.

Want to show meas $A_{\vec{\ell}}$ is small. Let $\vec{a}, \vec{a} + \vec{\beta} \in A_{\vec{\ell}}$ with $\vec{a} \in B_{\vec{\ell}, \vec{a}}$,

$\vec{a} + \vec{\beta} \in B_{\vec{\ell}, \vec{a} + \vec{b}}$. Suppose $\frac{1}{4}P^{2^d k / (n-R+1)} \leq p \leq P$ and $e > 0$.

Then $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) \gg_{\vec{f}, \epsilon} Cp^{(d-2)n+R-1}$ and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3p^{d-1}}$

with $b_i < p^e$. If $C \gg_{\vec{f}, \epsilon} 1$, $p^{e+d-1} \leq \frac{1}{3}P^d$ then $p^e \mid \vec{b}$ by D, so $\vec{b} = \vec{0}$.

Thus $A_{\vec{\ell}} \subset B_{\vec{\ell}, \vec{a}}$ for some $\vec{a} = \vec{a}(\vec{\ell}) \in \mathbb{Z}^R$, so $\text{meas } \bigcup_{\vec{\ell}} A_{\vec{\ell}} \ll p^{-eR}$. For

$e = \lceil d(\frac{n-R+1}{2^d k} - 1) \rceil$, $p \leq \frac{1}{2}P^{\frac{d}{e+d-1}} \leq 2p$ we have $p^{-eR} \ll P^{\frac{d2^d Rk}{n-R+1} - dR}$.

Corollary

$\text{meas}\{\vec{a} \in [0, 1]^R : |S(\vec{a} \cdot \vec{f})| \gg_{\vec{f}, \epsilon} P^{n-k+\epsilon}\} \ll P^{\frac{d2^d Rk}{n-R+1} - dR} (\forall k < \frac{n-R+1}{2^d})$.

Several forms in many variables: a theorem

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ are R lin. indep. forms in n variables of degree $d \geq 2$.
- We set $N_{\vec{f}}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{f}(\vec{x}) = \vec{0}, |\vec{x}| \leq P\}$.
- The exponential sum $S(\vec{\alpha} \cdot \vec{f}) = \sum e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$, where $\vec{\alpha} \in \mathbb{R}^R$, $e(t) = e^{2\pi it}$ and the sum is over $\vec{x} \in \mathbb{Z}^n$ with $|\vec{x}| \leq P$.

Corollary

$$\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| \gg_{\vec{f}, \epsilon} P^{n-k+\epsilon}\} \ll P^{\frac{d2^d R k}{n-R+1} - dR} (\forall k < \frac{n-R+1}{2^d}).$$

Theorem (SLRM 2021)

If \vec{f} is nonsingular and

$$n - R \geq d2^d R \quad (2)$$

then $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$ as $P \rightarrow \infty$ for some real constant $\sigma \geq 0$. If $\vec{f}(\vec{x}) = \vec{0}$ has solutions in $\mathbb{R}^n \setminus \{\vec{0}\}$ and $\mathbb{Q}_p^n \setminus \{\vec{0}\}$ then $\sigma > 0$.

Slightly more already known if $d < 4$. For $R \geq \max\{6-d, 2\}$ this beats

$$n_0 = (d-1)2^{d-1}R(R+1) + R. \quad (1)$$

$$\frac{d2^d R}{n-R+1} \underbrace{k}_{< 1}$$

Browning - Heath-Brown

$$\tilde{f}(\tilde{\alpha}) = \tilde{o}$$

Let $\vec{f} = (\vec{f}_2, \dots, \vec{f}_D)$ where $\deg \vec{f}_d = d$ and $\vec{f}_d \in \mathbb{Z}[\vec{x}]^{r_d}$.

Let $\vec{\alpha} \cdot \vec{f} = \sum_{i=2}^D \vec{\alpha}_d \cdot \vec{f}_d$. Then, if $\underline{\vec{\alpha}_d} = \frac{\vec{\alpha}_d}{q} + \frac{\vec{\gamma}_d}{P^d}$ with $q, |\vec{\gamma}_d| \leq P^\nu$ we have

$$S(\vec{\alpha} \cdot \vec{f}) = \frac{P^n}{q^n} \sum_{\vec{b}(q)} e\left(\frac{\vec{a}}{q} \cdot \vec{f}(\vec{b})\right) \int_{|\vec{u}| \leq 1} e(\vec{\gamma} \cdot \vec{f}(\vec{u})) d\vec{u} + O_{\vec{f}}(P^{n-1+2\nu}).$$

Lemma (Several more Weyl differencing steps)

$$\begin{aligned} & \rightarrow \mathcal{N}_D(P^\theta, P^{(D-1)\theta-D}; \vec{\beta}_D \cdot \vec{f}_D) / P^{(D-1)n\theta} \\ & \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f}) / P^{2n+\epsilon}|^{2^{D-1}} \quad (\forall \theta \in (0, 1]). \end{aligned}$$

and if $|\vec{\beta}_{d'}| \leq P^{-d'}$ for $d < d' \leq D$ then

$$\begin{aligned} & \rightarrow \mathcal{N}_d(P^\theta, P^{(d-1)\theta-d}; \vec{\beta}_d \cdot \vec{f}_d) / P^{(d-1)n\theta} \\ & \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f}) / P^{2n+\epsilon}|^{2^{d-1}} \quad (\forall \theta \in (0, 1]). \end{aligned}$$

$$\begin{aligned} & \rightarrow \sum_{d=1}^D \frac{d2^d r_d}{n - B_d} < 1 \\ & \rightarrow \sum_{d=1}^D \frac{d2^d r_d}{n+1 - \sum_{d'=d}^D r_{d'}} < 1 \end{aligned} \quad B_d = \max_{\vec{\delta} \in \mathbb{C}^{r_d} \setminus \{0\}} \dim \{ \vec{v}_{\vec{\delta}}(\vec{\delta} \cdot \vec{f}_d) = 0 \}$$

Skinner

V^R

Let $\vec{f} \in K[\vec{x}]^R$ be lin. indep degree d forms. For $V = K \otimes_{\mathbb{Q}} \mathbb{R}$ and $x \in V$, let $|x| = \max_{v \mid \infty} v(x)$. For prime $\mathfrak{p} \subset \mathcal{O}_K$, set $\rho = (\#\mathbb{F}_{\mathfrak{p}})^{\frac{1}{m}}$ where $[K : \mathbb{Q}] = m$. For $F(\vec{x}) \in V[\vec{x}]^R$ put $S(F) = \sum_{\vec{x} \in \mathcal{O}_K^n, |\vec{x}| \leq P} e(\text{Tr}_{K/\mathbb{Q}} F(\vec{x}))$.

Lemma D (Improved bounds near $\vec{0}$, in \mathfrak{p} -adic sense)

Define \mathcal{N} as before, but now it counts $(\vec{x}^{(i)}) \in (\mathcal{O}_K^n)^{d-1}$. Suppose $\#\mathcal{N}((\#\mathbb{F}_{\mathfrak{p}})^{\frac{1}{m}}, (\#\mathbb{F}_{\mathfrak{p}})^{\frac{d-1}{m}} P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_{\mathfrak{p}}^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, \mathfrak{p}}(\vec{\delta} \cdot \vec{f})$, $e > 0$, $(\#\mathbb{F}_{\mathfrak{p}})^{e+d-1} \ll_m P^{dm}$, $\mu_i \in \mathfrak{p}^{-\mathfrak{L}}$ and $|\vec{\beta} - \vec{\mu}| \ll_m (\#\mathbb{F}_{\mathfrak{p}})^{-\frac{e}{m}}$. Then $\vec{\mu} \in \mathcal{O}^R$.

If \vec{f} is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_{\mathfrak{p}}^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, \mathfrak{p}}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} (\#\mathbb{F}_{\mathfrak{p}})^{(d-2)n+R-1}$.

Proof.

Let $\min_i v_p(\mu_i)_p = -k$, $v_p(\pi) = 1$ and $(\vec{x}^{(i)})_i \in \mathcal{N} \setminus \mathcal{N}_{\text{aux}, \mathfrak{p}}(\pi^k \vec{\mu} \cdot \vec{f})$

Now $|\vec{v} - \vec{m}^{(\vec{\beta} \cdot \vec{f})}| \leq (\#\mathbb{F}_{\mathfrak{p}})^{\frac{d-1}{m}} P^{-d} \leq \frac{c_m}{(\#\mathbb{F}_{\mathfrak{p}})^{e/m}}$. Also

$|\vec{m}^{(\vec{\beta} \cdot \vec{f})} - \vec{m}^{(\vec{\mu} \cdot \vec{f})}| \leq \frac{c_m}{(\#\mathbb{F}_{\mathfrak{p}})^{e/m}}$. So $|\vec{v} - \vec{m}^{(\vec{\mu} \cdot \vec{f})}| < (\#\mathbb{F}_{\mathfrak{p}})^{-\frac{e}{m}}$. Now

$\min_{\lambda \in \mathfrak{p} \setminus \{0\}} |\lambda| \leq (\#\mathbb{F}_{\mathfrak{p}})^{\frac{1}{m}}$ so $\min_{\mu \in \mathfrak{p}^{-1} \setminus \{0\}} |\mu| \gg_m (\#\mathbb{F}_{\mathfrak{p}})^{-\frac{1}{m}}$ and as

$c_m \ll_m 1$ we deduce $\vec{m}^{(\vec{\mu} \cdot \vec{f})} = \vec{v}$. But $(\vec{x}^{(i)})_i \notin \mathcal{N}_{\text{aux}, \mathfrak{p}}(\pi^k \vec{\mu} \cdot \vec{f})$. \clubsuit

$\#\mathcal{N}_{\text{aux}, \mathfrak{p}}(\vec{\delta} \cdot \vec{f}) \ll p^{\dim\{\vec{m}^{(\vec{\delta} \cdot \vec{f})} = \vec{0}\}} \ll (\#\mathbb{F}_{\mathfrak{p}})^{(d-2)n+R-1}$ for $p \gg 1$. \square

$$\vec{x} = \sum \vec{x}^{(i)} w_i$$

$$\vec{x}^{(i)} \in \mathbb{Z}^n$$

$$\int S(\vec{x} \circ \vec{g}) d\vec{x}$$