

Repulsion: A how-to guide (part III)

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A theorem of Birch

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ are R lin. indep. forms in n variables of degree $d \geq 2$.
- We set $N_{\vec{f}}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{f}(\vec{x}) = \vec{0}, |\vec{x}| \leq P\}$.
- $\vec{f}(\vec{x})$ is **nonsingular** if the Jacobian matrix $(\partial f_i(\vec{x}) / \partial x_j)_{ij}$ has rank R at every complex solution $\vec{x} \in \mathbb{C}^n \setminus \{\vec{0}\}$ to $\vec{f}(\vec{x}) = \vec{0}$.

Theorem (Birch, Proc. R. Soc. Lond. A 1962)

If \vec{f} is nonsingular and

$$n-R \geq (d-1)2^{d-1}R(R+1) \quad (1)$$

then $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$ as $P \rightarrow \infty$ for some real constant $\sigma \geq 0$. If $\vec{f}(\vec{x}) = \vec{0}$ has solutions in $\mathbb{R}^n \setminus \{\vec{0}\}$ and $\mathbb{Q}_p^n \setminus \{\vec{0}\}$ then $\sigma > 0$.

Want to improve (1) in the case $R > 1$.

Müller (J. Théor. Nombres Bordeaux 2005): $d = 2$, $n \geq 9R$ for *irrational* systems of forms over \mathbb{R} . Based on Bentkus-Götze (Acta Arith. 1997, Ann. of Math. 1999).

The circle method: a framework

Notation

- The exponential sum $S(\vec{\alpha} \cdot \vec{f}) = \sum e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$, where $\vec{\alpha} \in \mathbb{R}^R$, $e(t) = e^{2\pi it}$ and the sum is over $\vec{x} \in \mathbb{Z}^n$ with $|\vec{x}| \leq P$.
- If $F(\vec{x})$ is a degree d form, let $\vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ be the n -tuple of symmetric multilinear forms such that $\vec{m}^{(F)}(\vec{x}, \dots, \vec{x}) = \frac{\vec{\nabla}_{\vec{x}}}{d!} F(\vec{x})$.
- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).
- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{\alpha} \in \mathbb{Z}^R, \vec{m}^{(\vec{\alpha} \cdot \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}$.

Lemma A (level-set formulation of Birch, after Bentkus-Götze)

If $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}$ for some $c < 1$, all $k \in [0, dR/c]$, then $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f}, c, C}(P^{n-dR-\delta(c, d, R, n)+\epsilon})$.

Lemma B (Weyl differencing + shrinking)

$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) / P^{n+\epsilon}|^{2^{d-1}} \quad (\theta \in (0, 1])$$

Lemma C (Birch's major arcs)

- i) If $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$ then $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$.
- ii) Moreover we have $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{R(R+1)(d-1)\theta-dR}$.
- iii) If \vec{f} is nonsingular then $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll P^{-(n-R+1)\theta}$.

The circle method: a framework

Notation

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- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{a} \in \mathbb{Z}^R, \vec{m}^{(\vec{a} \cdot \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}$.
- $\mathfrak{M}(Q) = \{\vec{a} \in [0, 1]^R : |\vec{a} - \frac{\vec{a}}{q}| \leq \frac{Q}{qP^d} \text{ } (q \leq Q, \vec{a} \in \mathbb{Z}^R)\}$

Lemma A (level-set formulation of Birch, after Bentkus-Götze)

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Lemma B (Weyl differencing + shrinking)

$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) / P^{n+\epsilon}|^{2^{d-1}} \quad (\theta \in (0, 1])$$

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- ii) Moreover we have $\text{meas} \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{R(R+1)(d-1)\theta-dR}$.
- iii) If \vec{f} is nonsingular then $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll P^{-(n-R+1)\theta}$.

$$|S| \gg_{\vec{f}, \epsilon} P^{n-k+\epsilon}, P^\theta \gg P^{\frac{2^{d-1}k}{n-R+1}}, \vec{\alpha} \in \mathfrak{M}(P^{\frac{R(d-1)2^{d-1}k}{(n-R+1)}}), \text{meas} \ll P^{\frac{R(R+1)(d-1)2^{d-1}k}{(n-R+1)} - dR}.$$

Improved major arcs?

Notation

- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(\vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).
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Lemma C (Birch's major arcs)

- If $\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) > \#\mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$ then $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$.
- Moreover we have $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR + R(R+1)(d-1)\theta}$.
- If \vec{f} is nonsingular then $\#\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll_{\vec{f}} P^{-(n-R+1)\theta}$.

Can we redefine $\mathcal{N}_{\text{sing}}, \mathfrak{M}$ to increase $\frac{n-R+1}{R(R+1)(d-1)}$?

Proof of Lemma C part (i).

Let $(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ be in \mathcal{N} but not in $\mathcal{N}_{\text{sing}}$.

Then $\vec{\alpha} \mapsto \vec{m}^{(\vec{\alpha} \cdot \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ is a nonsingular matrix M of integers $\ll_{\vec{f}} P^{(d-1)\theta}$, with $|M\vec{\alpha} - \vec{v}| \leq P^{(d-1)\theta-d}$. Let M' be a nonsingular $R \times R$ submatrix. Then $|\vec{\alpha} - M'^{-1}\vec{v}'| \leq P^{R(d-1)\theta-d}$. \square

Heuristically, for most \vec{v} , the different possible $R \times R$ submatrices should give contradictory approximations to $\vec{\alpha}$. So there is room to reduce the measure. But, we'd have to bound $\#\{|\vec{x}^{(i)}| \leq P^\theta : \vec{m}^{(\vec{a}, \vec{f})} \equiv \vec{0} (q)\}$, hard!

$$M =$$

$$\bigcup \{\vec{\alpha} \in [0, 1]^R : \vec{m}^{(\vec{\alpha}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}.$$

nonsing,
 $R \times R$ matrix
of integers
 $\ll P^{(d-1)\theta-d}$
 $\vec{f} \in \mathbb{Z}^R$,

but case (\vec{a}, \vec{f})
gcd($M^{(\vec{a}, \vec{f})}$)
 $m_1 \cdots m_n$

Improved major arcs?

Notation

- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).
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- If $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$ then $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$.
- Moreover we have $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR+R(R+1)(d-1)\theta}$.
- If \vec{f} is nonsingular then $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f})/P^{(d-1)n\theta} \ll_{\vec{f}} P^{-(n-R+1)\theta}$.

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For most \vec{v} , different $R \times R$ matrices should give contradictory approximations to $\vec{\alpha}$. So room to decrease $R(R+1)(d-1)$. If $\vec{\alpha}$ is close to $\vec{0}$, the picture is different. Most $R \times R$ submatrices give the same region for $\vec{\alpha}$, and it's much smaller than the major arc at $\vec{0}$. This makes possible Müller's $d = 2, n \geq 9R$ (J. Th. Nombres Bordeaux '05).

if $\vec{v} = 0$ then

$$|\vec{\alpha}| \leq P^{-d}$$

for "most" M

Improved major arcs?

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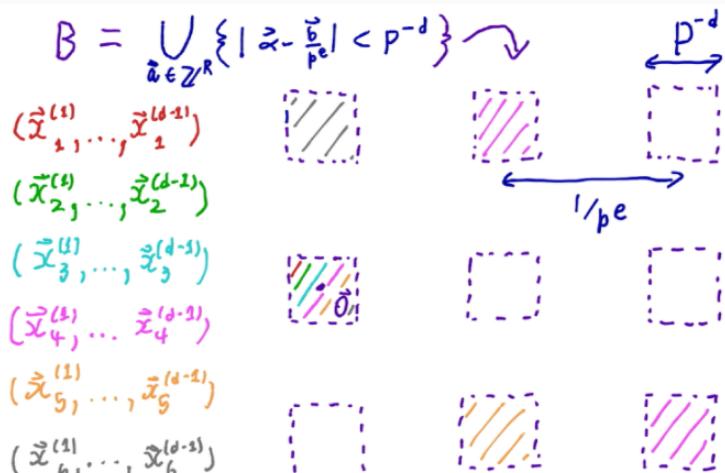
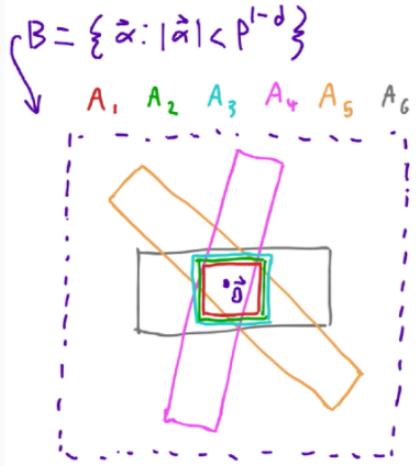
If $(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ is in \mathcal{N} but not $\mathcal{N}_{\text{sing}}$ then

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$\ll_{\vec{f}} P^{(d-1)\theta}$, with $|M\vec{\alpha} - \vec{v}| \leq P^{(d-1)\theta-d}$. Let M' be a nonsingular $R \times R$ submatrix. Then $|\vec{\alpha} - M'^{-1}\vec{v}'| \leq P^{R(d-1)\theta-d}$. \square

For most \vec{v} , different $R \times R$ matrices should give contradictory approximations to $\vec{\alpha}$. So room to decrease $\text{meas } \mathfrak{M}$. If $\vec{\alpha}$ is close to $\vec{0}$, the picture is different: most $R \times R$ submatrices give the same region for $\vec{\alpha}$.

Improved major arcs?



$$A_i = \left\{ \vec{\alpha} \in B : \min_{\vec{v} \in \mathbb{Z}^n} |m(\vec{\alpha} \cdot \vec{v}) (\vec{x}_i^{(1)}, \dots, \vec{x}_i^{(d-1)}) - \vec{v}| < P^{(d-1)\theta - d} \right\}$$

If N is large we get some $(\vec{x}^{(j)})_j$ such that $\vec{\alpha}$ belongs to $A(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$. The union $\bigcup A$ is the “major arcs”, and we want its measure to be smaller than $\text{meas } \mathfrak{M}(P^{R(d-1)\theta})$.

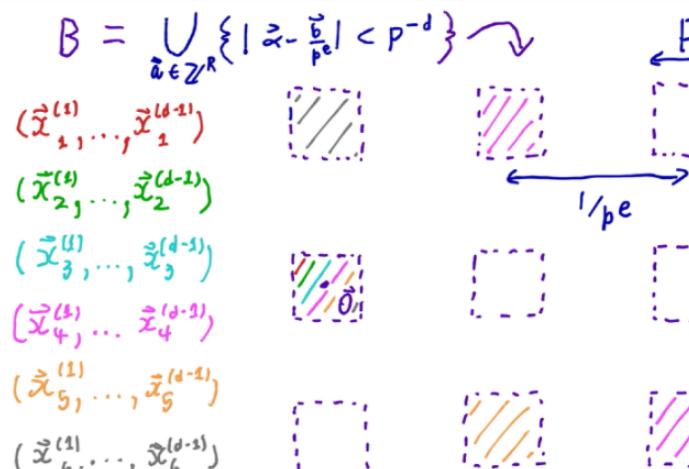
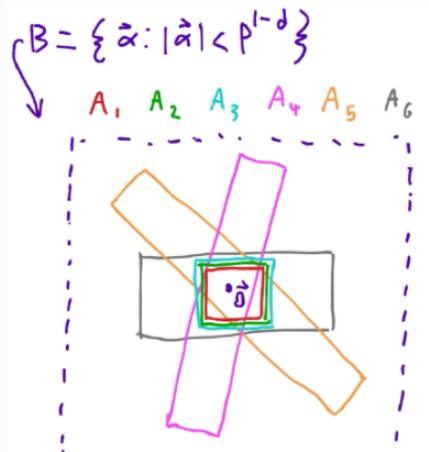
If B is a suitable neighbourhood of $\vec{0}$, I claim that the measure of $B \cap \bigcup A$ is smaller than $\text{meas } B \cdot \text{meas } \mathfrak{M}(P^{R(d-1)\theta})$. I haven't yet explained why this is so useful, but let's keep exploring.

$$A = \{ \vec{\alpha} \in B : \dots \}$$

$$\left| M(\vec{\alpha} - \vec{v}) \right| < P^{\frac{(d-1)\theta}{d}}$$

$$\text{if } \#N(p^e; p^{(d-1)\theta-d}; F) > \#N_{\text{aux}, \infty}(p^\theta; F)$$

Improved major arcs? and if $|F| \leq p^{(l-d)\theta}$ then $|F| <$

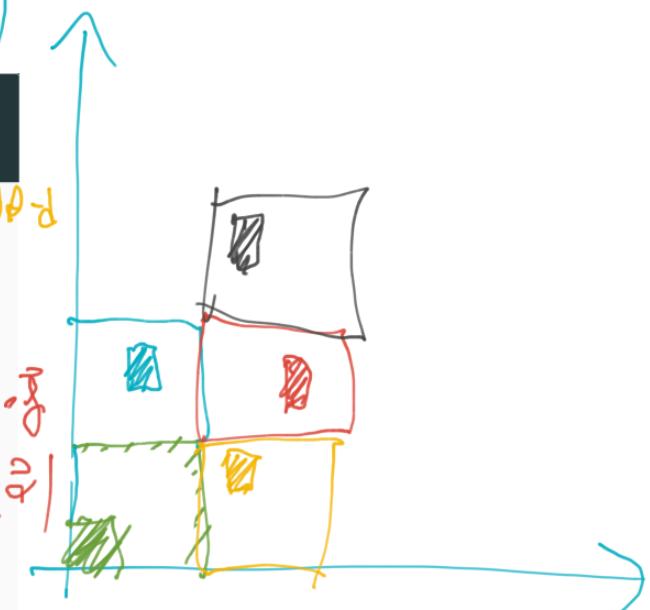


$$A_i = \left\{ \vec{\alpha} \in B : \min_{\vec{v} \in \mathbb{Z}^n} |\vec{m}(\vec{\alpha} \cdot \vec{\delta}) (\vec{x}_i^{(1)}, \dots, \vec{x}_i^{(d-1)}) - \vec{v}| < p^{(d-1)\theta-d} \right\}$$

Lemma D (Improved bounds near $\vec{0}$, in p -adic sense)

Let $\mathcal{N}_{\text{aux}, p}(F) = \{(\vec{x}^{(i)})_i : |\vec{x}^{(i)}| \leq p, \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \equiv \vec{0} \pmod{p}\}$.

Suppose $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$, $e > 0$, $p^{e+d-1} \leq \frac{1}{3}P^d$, and $|p^e \vec{\beta} - \vec{a}| \leq \frac{1}{3}p^{1-d}$. Then $\vec{a} \equiv \vec{0} \pmod{p^e}$.



$$N_{\text{aux}, \infty}(p^\theta; F) =$$

$$\sum \left| \vec{x}^{(i)} \right| \leq p^\theta.$$

$$\begin{aligned} & \left| \vec{m}^{(F)} \left(\underbrace{\vec{\alpha}^{(1)}}_{\vec{\beta}}, \dots, \underbrace{\vec{\alpha}^{(d-1)}}_{\vec{\beta}} \right) \right| \\ & < |F| p^{(d-2)\theta} \end{aligned}$$

Improved major arcs?

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If \vec{f} is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$.

Proof.

Suppose $\min_i |a_i|_p = p^{-k}$, let $(\vec{x}^{(i)})_i \in \mathcal{N} \setminus \mathcal{N}_{\text{aux}, p}(\frac{\vec{a}}{p^k} \cdot \vec{f})$. Then $|\vec{v} - \vec{m}^{(\vec{\alpha} \cdot \vec{f})}| \leq p^{d-1}P^{-d} \leq \frac{1}{3p^e}$. Also $|\vec{m}^{(\vec{\alpha} \cdot \vec{f})} - \frac{1}{p^e} \vec{m}^{(\vec{a} \cdot \vec{f})}| \leq \frac{1}{3p^e}$. Thus $|\vec{v} - \frac{1}{p^e} \vec{m}^{(\vec{a} \cdot \vec{f})}| < \frac{1}{p^e}$, so $\vec{m}^{(\vec{a} \cdot \vec{f})} = p^e \vec{v}$. But $(\vec{x}^{(i)})_i \notin \mathcal{N}_{\text{aux}, p}(\frac{\vec{a}}{p^k} \cdot \vec{f})$. \clubsuit

$\#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll p^{\dim\{\vec{m}^{(\vec{\delta} \cdot \vec{f})} = \vec{0}\}} \ll p^{(d-2)n+R-1}$ for $p \gg 1$. \square

$$\frac{\vec{a}}{p^e} = \frac{\vec{a}'}{p^{e-k}}$$

$$\vec{a}' = \vec{a}/p^k$$

Repulsion

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ are R lin. indep. forms in n variables of degree $d \geq 2$.
- The exponential sum $S(\vec{\alpha} \cdot \vec{f}) = \sum e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$, where $\vec{\alpha} \in \mathbb{R}^R$, $e(t) = e^{2\pi it}$ and the sum is over $\vec{x} \in \mathbb{Z}^n$ with $|\vec{x}| \leq P$.
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- Let $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$ be the set of integral $(d-1)$ -tuples with $|\vec{x}^{(i)}| \leq P^\theta$, $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$ (some $\vec{v} \in \mathbb{Z}^n$).

Lemma B' (One more Weyl differencing step)

Bentkus-Bütze

$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\beta} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f}) / P^{2n+\epsilon}|^{2^{d-1}} \quad (\theta \in (0, 1]).$$

Compare

Lemma B (Weyl differencing + shrinking)

$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) / P^{n+\epsilon}|^{2^{d-1}} \quad (\theta \in (0, 1]).$$

" $S(\vec{\alpha} \cdot \vec{f})$ should be small if $\vec{\alpha} \neq \vec{0}$ "

What good is repulsion?

Notation

- $S(\vec{\alpha} \cdot \vec{f}) = \sum_{\vec{x} \in \mathbb{Z}^n, |\vec{x}| \leq P} e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$.
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- $\mathcal{N}_{\text{aux}, p}(F) = \{(\vec{x}^{(i)})_i : |\vec{x}^{(i)}| \leq p, \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \equiv \vec{0} (p)\}$.

Lemma A (level-set formulation of Birch, after Bentkus-Götze)

If $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}$ for some $c < 1$, all $k \in [0, dR/c]$, then $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f}, c, C}(P^{n-dR-\delta(c, d, R, n)+\epsilon})$.

Lemma B' (One more Weyl differencing step)

$$\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\beta} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) S((\vec{\alpha} + \vec{\beta}) \cdot \vec{f}) / P^{2n+\epsilon}|^{2^{d-1}}.$$

Lemma D (Improved bounds near $\vec{0}$, in p -adic sense)

Suppose $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$, $e > 0$, $p^{e+d-1} \leq \frac{1}{3}P^d$, and $|p^e \vec{\beta} - \vec{b}| \leq \frac{1}{3}p^{1-d}$. Then $\vec{b} \equiv \vec{0} (p^e)$.

If \vec{f} is nonsingular, $\sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$.

