

# **Repulsion: A how-to guide (part II)**

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## A theorem of Birch

### Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$  are  $R$  lin. indep. forms in  $n$  variables of degree  $d \geq 2$ .
- We set  $N_{\vec{f}}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{f}(\vec{x}) = \vec{0}, |\vec{x}| \leq P\}$ .
- $\vec{f}(\vec{x})$  is **nonsingular** if the Jacobian matrix  $(\partial f_i(\vec{x}) / \partial x_j)_{ij}$  has rank  $R$  at every complex solution  $\vec{x} \in \mathbb{C}^n \setminus \{\vec{0}\}$  to  $\vec{f}(\vec{x}) = \vec{0}$ .

### Theorem (Birch, Proc. R. Soc. Lond. A 1962)

If  $\vec{f}$  is nonsingular and

$$n - R \geq (d - 1)2^{d-1}R(R + 1) \quad (1)$$

then  $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$  as  $P \rightarrow \infty$  for some real constant  $\sigma \geq 0$ . If  $\vec{f}(\vec{x}) = \vec{0}$  has solutions in  $\mathbb{R}^n \setminus \{\vec{0}\}$  and  $\mathbb{Q}_p^n \setminus \{\vec{0}\}$  then  $\sigma > 0$ .

Want to improve (1) in the case  $R > 1$ .

Müller (J. Théor. Nombres Bordeaux 2005):  $d = 2, n \geq 9R$  for *irrational* systems of forms over  $\mathbb{R}$ . Based on Bentkus-Götze (Acta Arith. 1997, Ann. of Math. 1999).

$\mathbb{Q}[\vec{x}]$

## The circle method: a framework

### Notation

- The exponential sum  $S(\vec{\alpha} \cdot \vec{f}) = \sum e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$ , where  $\vec{\alpha} \in \mathbb{R}^R$ ,  $e(t) = e^{2\pi it}$  and the sum is over  $\vec{x} \in \mathbb{Z}^n$  with  $|\vec{x}| \leq P$ .
- If  $F(\vec{x})$  is a degree  $d$  form, let  $\vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$  be the  $n$ -tuple of symmetric multilinear forms such that  $\vec{m}^{(F)}(\vec{x}, \dots, \vec{x}) = \frac{\vec{\nabla}_{\vec{x}}}{d!} F(\vec{x})$ .
- Let  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$  be the set of integral  $(d-1)$ -tuples with  $|\vec{x}^{(i)}| \leq P^\theta$ ,  $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$  (some  $\vec{v} \in \mathbb{Z}^n$ ).
- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{\alpha} \in \mathbb{Z}^R, \vec{m}^{(\vec{\alpha}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}$ .

### Lemma A (level-set formulation of Birch, after Bentkus-Götze)

If  $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}$  for some  $c < 1$ , all  $k \in [0, dR/c]$ , then  $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f}, c, C}(P^{n-dR-\delta(c, d, R, n)+\epsilon})$ .

### Lemma B (Weyl differencing + shrinking)

$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |\vec{\alpha}|^{-2^{d-1}} \quad (\theta \in (0, 1])$$

### Lemma C (Birch's major arcs)

- If  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$  then  $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$ .
- Moreover we have  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{R(R+1)(d-1)\theta-dR}$ .
- If  $\vec{f}$  is nonsingular then  $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll P^{-(n-R+1)\theta}$ .

$$\vec{\alpha} \cdot \vec{f} = \sum_{i=1}^R \alpha_i f_i$$

$$\sum e^{2\pi i \vec{\alpha} \cdot \vec{f}} = \vec{\nabla}_{\vec{x}} F = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{pmatrix}$$

$$\vec{\alpha} \in [0, 1]^R$$

# The circle method: a framework

## Notation

- If  $F(\vec{x})$  is a degree  $d$  form,  $\vec{m}^{(F)}(\vec{x}, \dots, \vec{x}) = \frac{\vec{\nabla}_{\vec{x}}}{d!} F(\vec{x})$ .
- Let  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$  be the set of integral  $(d-1)$ -tuples with  $|\vec{x}^{(i)}| \leq P^\theta$ ,  $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$  (some  $\vec{v} \in \mathbb{Z}^n$ ).
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- $\mathfrak{M}(Q) = \{\vec{a} \in [0, 1]^R : |\vec{a} - \frac{\vec{a}}{q}| \leq \frac{Q}{qP^d} \text{ } (q \leq Q, \vec{a} \in \mathbb{Z}^R)\}$

## Lemma A (level-set formulation of Birch, after Bentkus-Götze)

If  $\text{meas}\{\vec{\alpha} \in [0, 1]^R : |S(\vec{\alpha} \cdot \vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}$  for some  $c < 1$ , all  $k \in [0, dR/c]$ , then  $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f}, c, C}(P^{n-dR-\delta(c, d, R, n)+\epsilon})$ .

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$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) / P^{n+\epsilon}|^{2^{d-1}} \quad (\theta \in (0, 1])$$

## Lemma C (Birch's major arcs)

i) If  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$  then  $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$ .

ii) Moreover we have  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{R(R+1)(d-1)\theta-dR}$ .

iii) If  $\vec{f}$  is nonsingular then  $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll P^{-(n-R+1)\theta}$ .

$$|S| \gg_{\vec{f}, \epsilon} P^{n-k+\epsilon}, P^\theta \gg P^{\frac{2^{d-1}k}{n-R+1}}, \vec{\alpha} \in \mathfrak{M}(P^{\frac{R(d-1)2^{d-1}k}{(n-R+1)}}), \text{meas} \ll P^{\frac{R(R+1)(d-1)2^{d-1}k}{(n-R+1)} - dR}.$$

$$\begin{aligned} C &\subset \mathfrak{I} \iff \\ n-R &\geq R(R+1) \\ (d-1) \sum k &= \end{aligned}$$

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$$\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) / P^{(d-1)n\theta} \gg_{\vec{f}, \epsilon} |S(\vec{\alpha} \cdot \vec{f}) / P^{n+\epsilon}|^{2^{d-1}} \quad (\theta \in (0, 1]).$$

To outdo Lemma A: don't take the absolute value of  $S$  on the minor arcs (AKA averaging over arcs, AKA a Kloosterman refinement). To outdo Lemma B: clever differencing, see e.g. Browning-Prendiville (Crelle 2017).

$$\int_M S(\vec{\alpha} \cdot \vec{f}) d\alpha$$

## The circle method: a framework

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- i) If  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$  then  $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$ .
- ii) Moreover we have  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR+R(R+1)(d-1)\theta}$ .
- iii) If  $\vec{f}$  is nonsingular then  $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll P^{-(n-R+1)\theta}$ .

To outdo Lemma A: don't take the absolute value of  $S$  on the minor arcs (AKA averaging over arcs, AKA a Kloosterman refinement). To outdo Lemma B: clever differencing, see e.g. Browning-Prendiville (Crelle 2017).

To outdo Lemma C: Change the definition of  $\mathcal{N}_{\text{sing}}$  and maybe  $\mathfrak{M}$  to increase  $\frac{n-R+1}{R(R+1)(d-1)2^{d-1}}$ . Dietmann (Q. J. Math. 2015), Schindler (Adv. Th. Num. 2015). To see what more might be possible, let's look at part of the proof.

## Improved major arcs?

### Notation

- Let  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$  be the set of integral  $(d-1)$ -tuples with  $|\vec{x}^{(i)}| \leq P^\theta$ ,  $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$  (some  $\vec{v} \in \mathbb{Z}^n$ ).
- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{a} \in \mathbb{Z}^R \quad \vec{m}^{(\vec{a}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}.$
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Can we redefine  $\mathcal{N}_{\text{sing}}, \mathfrak{M}$  to increase  $\frac{n-R+1}{R(R+1)(d-1)}$ ?

### Proof of Lemma C part (i).

Let  $(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$  be in  $\mathcal{N}$  but not in  $\mathcal{N}_{\text{sing}}$ .

Then  $\vec{\alpha} \mapsto \vec{m}^{(\vec{\alpha}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$  is a nonsingular matrix  $M$  of integers  $\ll_{\vec{f}} P^{(d-1)\theta}$ , with  $|M\vec{\alpha} - \vec{v}| \leq P^{(d-1)\theta-d}$ . Let  $M'$  be a nonsingular  $R \times R$  submatrix. Then  $|\vec{\alpha} - M'^{-1}\vec{v}'| \leq P^{R(d-1)\theta-d}$ .  $\square$

Heuristically, for most  $\vec{v}$ , the different possible  $R \times R$  submatrices should give contradictory approximations to  $\vec{\alpha}$ . So there is room to reduce the measure. But, we'd have to bound  $\#\{|\vec{x}^{(i)}| \leq P^\theta : \vec{m}^{(\vec{a}, \vec{f})} \equiv \vec{0} (q)\}$ , hard!

$$\tilde{\mathcal{M}}^{(F)}(\tilde{x}_1, \dots, \tilde{x}_d)$$

$$= \frac{\int_{\mathbb{R}^d} F(\tilde{x})}{\downarrow !}$$

$$\tilde{M} \neq \bar{0}$$

$$M \in \mathbb{R}^{n \times R}$$

$$q = \det(M')$$

## A theorem of Birch

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### Theorem (Birch, Proc. R. Soc. Lond. A 1962)

If  $\vec{f}$  is nonsingular and  $n \geq n_0(d, R)$  where

$$n_0(d, R) = (d - 1)2^{d-1}R(R + 1) + R \quad (1)$$

then  $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$  as  $P \rightarrow \infty$  for some real constant  $\sigma \geq 0$ . If  $\vec{f}(\vec{x}) = \vec{0}$  has solutions in  $\mathbb{R}^n \setminus \{\vec{0}\}$  and  $\mathbb{Q}_p^n \setminus \{\vec{0}\}$  then  $\sigma > 0$ .

Want to improve (1) in the case  $R > 1$ .

Müller (J. Théor. Nombres Bordeaux 2005):  $d = 2, n \geq 9R$  for nonsingular *irrational* systems of forms over  $\mathbb{R}$ . Based on Bentkus-Götze (Acta Arith. 1997, Ann. of Math. 1999).

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### Theorem (Dietmann 2015, Schindler 2015)

If  $\vec{f}$  is nonsingular and

$$n > (d-1)2^{d-1}R(R+1) + \sup_{\vec{a} \in \mathbb{Z}^R \setminus \{\vec{0}\}} \dim\{\vec{\nabla}_{\vec{x}}[\vec{a} \cdot \vec{f}(\vec{x})] = \vec{0}\} \quad (1'')$$

then  $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$  as  $P \rightarrow \infty$  for some real constant  $\sigma \geq 0$ . If  $\vec{f}(\vec{x}) = \vec{0}$  has solutions in  $\mathbb{R}^n \setminus \{\vec{0}\}$  and  $\mathbb{Q}_p^n \setminus \{\vec{0}\}$  then  $\sigma > 0$ .

Want to improve (1) in the case  $R > 1$ , at least for fairly typical  $\vec{f}$ .

Munshi (Compos. Math. '15):  $d = R = 2, n = 11$ .

Dietmann (Q. J. Math. 2015), Schindler (Adv. Th. Num. 2015).

## Improved major arcs?

### Notation

- Let  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$  be the set of integral  $(d-1)$ -tuples with  $|\vec{x}^{(i)}| \leq P^\theta$ ,  $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$  (some  $\vec{v} \in \mathbb{Z}^n$ ).
- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{a} \in \mathbb{Z}^R, \vec{m}^{(\vec{a} \cdot \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}.$
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- Moreover we have  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR + R(R+1)(d-1)\theta}$ .
- If  $\vec{f}$  is nonsingular then  $\#\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) / P^{(d-1)n\theta} \ll_{\vec{f}} P^{-(n-R+1)\theta}$ .

Can we redefine  $\mathcal{N}_{\text{sing}}, \mathfrak{M}$  to increase  $\frac{n-R+1}{R(R+1)(d-1)}$ ?

### Proof of Lemma C part (i).

Let  $(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$  be in  $\mathcal{N}$  but not in  $\mathcal{N}_{\text{sing}}$ .

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Heuristically, for most  $\vec{v}$ , the different possible  $R \times R$  submatrices should give contradictory approximations to  $\vec{\alpha}$ . So there is room to reduce the measure. But, we'd have to bound  $\#\{|\vec{x}^{(i)}| \leq P^\theta : \vec{m}^{(\vec{a} \cdot \vec{f})} \equiv \vec{0} (q)\}$ , hard!

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- $\mathcal{N}'_{\text{sing}}(P^\theta; F) = \{|\vec{x}^{(i)}| \leq P^\theta : \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}$ .
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**Lemma C (Dietmann Q. J. Math. 2015, Schindler Adv. Th. Num. 2015 )**

i) If  $\#\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{a} \cdot \vec{f}) > \sup_{\vec{a} \in \mathbb{Z}^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{sing}}(P^\theta; \vec{a} \cdot \vec{f})$  then

$\vec{a} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$ .

ii) Moreover we have  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR+R(R+1)(d-1)\theta}$ .

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Can we redefine  $\mathcal{N}_{\text{sing}}, \mathfrak{M}$  to increase  $\frac{n-R+1}{R(R+1)(d-1)}$ ?

**Proof of Lemma C part (i).**

There is no  $\vec{a} \neq \vec{0}$  with  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{a} \cdot \vec{f}) \subset \mathcal{N}_{\text{sing}}(P^\theta; \vec{a} \cdot \vec{f})$ . Then

$$\vec{a} \mapsto (\vec{m}^{(\vec{a} \cdot \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}))_{(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \in \mathcal{N}}$$

is a nonsingular matrix  $M$  of integers  $\ll_{\vec{f}} P^{(d-1)\theta}$ , with

$|M\vec{a} - \vec{v}^*| \leq P^{(d-1)\theta-d}$ . Let  $M'$  be a nonsingular  $R \times R$  submatrix.

Then  $|\vec{a} - M'^{-1}\vec{v}'| \leq P^{R(d-1)\theta-d}$ . □

For most  $\vec{f}$  this allows us to replace  $n-R+1$  with something bigger.

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- Let  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$  be the set of integral  $(d-1)$ -tuples with  $|\vec{x}^{(i)}| \leq P^\theta$ ,  $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$  (some  $\vec{v} \in \mathbb{Z}^n$ ).
- $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f}) = \{|\vec{x}^{(i)}| \leq P^\theta : \exists \vec{a} \in \mathbb{Z}^R, \vec{m}^{(\vec{a}, \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) = \vec{0}\}.$
- $\mathfrak{M}(Q) = \{\vec{\alpha} \in [0, 1]^R : |\vec{\alpha} - \frac{\vec{a}}{q}| \leq \frac{Q}{qP^d} \text{ } (q \leq Q, \vec{a} \in \mathbb{Z}^R)\}.$

### Lemma C (Birch's major arcs)

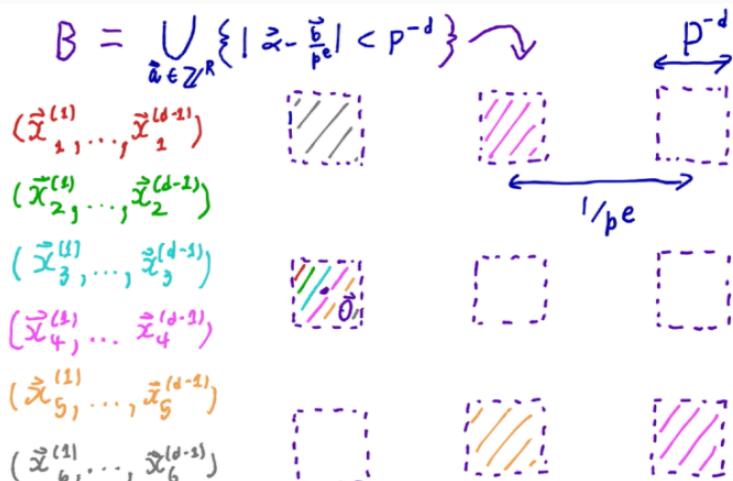
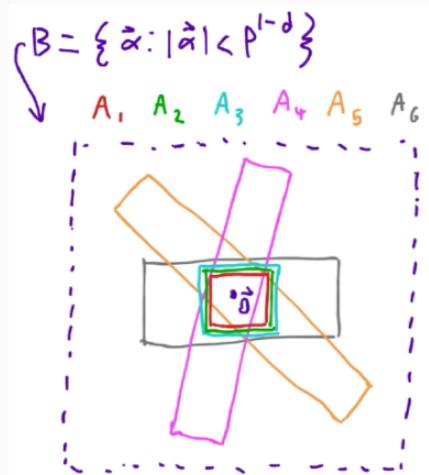
- If  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; \vec{\alpha} \cdot \vec{f}) \not\subseteq \mathcal{N}_{\text{sing}}(P^\theta; \vec{f})$  then  $\vec{\alpha} \in \mathfrak{M}(C_{\vec{f}} P^{R(d-1)\theta})$ .
- Moreover we have  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta}) \ll P^{-dR+R(R+1)(d-1)\theta}$ .
- If  $\vec{f}$  is nonsingular then  $\mathcal{N}_{\text{sing}}(P^\theta; \vec{f})/P^{(d-1)n\theta} \ll_{\vec{f}} P^{-(n-R+1)\theta}$ .

### Proof of Lemma C part (i).

If  $(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$  is in  $\mathcal{N}$  but not  $\mathcal{N}_{\text{sing}}$  then  $\vec{\alpha} \mapsto \vec{m}^{(\vec{\alpha} \cdot \vec{f})}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$  is a nonsingular matrix  $M$  of integers  $\ll_{\vec{f}} P^{(d-1)\theta}$ , with  $|M\vec{\alpha} - \vec{v}| \leq P^{(d-1)\theta-d}$ . Let  $M'$  be a nonsingular  $R \times R$  submatrix. Then  $|\vec{\alpha} - M'^{-1}\vec{v}'| \leq P^{R(d-1)\theta-d}$ .  $\square$

For most  $\vec{v}$ , different  $R \times R$  matrices should give contradictory approximations to  $\vec{\alpha}$ . So room to decrease  $R(R+1)(d-1)$ . If  $\vec{\alpha}$  is close to  $\vec{0}$ , the picture is different. Most  $R \times R$  submatrices give the same region for  $\vec{\alpha}$ , and it's much smaller than the major arc at  $\vec{0}$ . This makes possible Müller's  $d = 2, n \geq 9R$  (J. Th. Nombres Bordeaux '05).

## Improved major arcs?

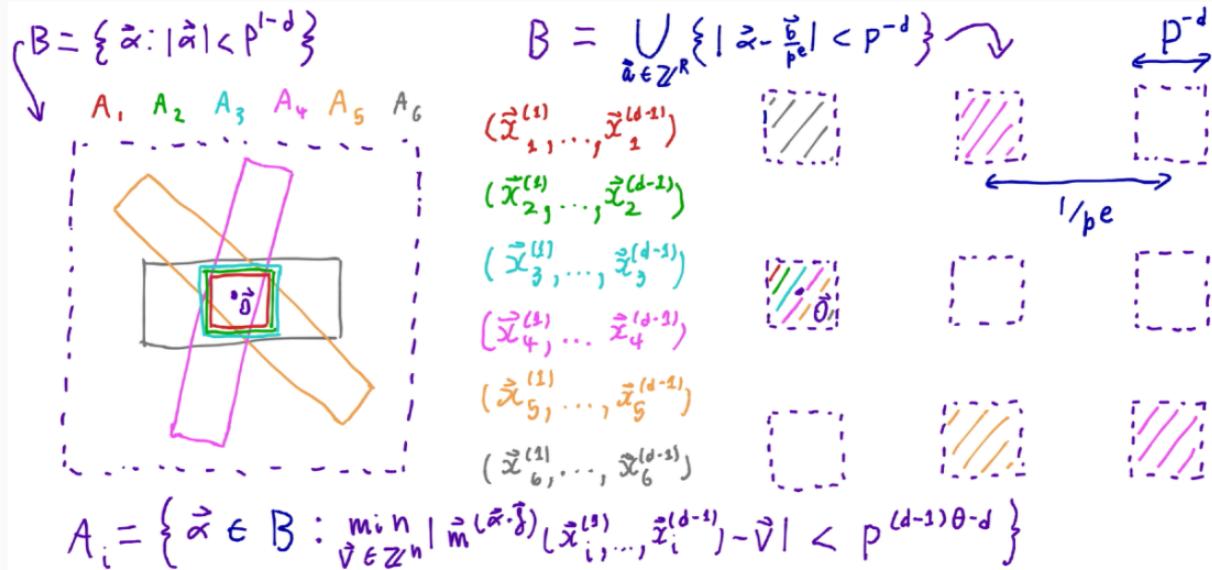


$$A_i = \left\{ \vec{\alpha} \in B : \min_{\vec{v} \in \mathbb{Z}^n} |m^{(\vec{\alpha}, \vec{v})} (\vec{x}_i^{(1)}, \dots, \vec{x}_i^{(d-1)}) - \vec{v}| < p^{(d-1)\theta - d} \right\}$$

If  $N$  is large we get some  $(\vec{x}^{(j)})_j$  such that  $\vec{\alpha}$  belongs to  $A(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})$ . The union  $\bigcup A$  is the “major arcs”, and we want its measure to be smaller than  $\text{meas } \mathfrak{M}(P^{R(d-1)\theta})$ .

If  $B$  is a suitable neighbourhood of  $\vec{0}$ , I claim that the measure of  $B \cap \bigcup A$  is smaller than  $\text{meas } B \cdot \text{meas } \mathfrak{M}(P^{R(d-1)\theta})$ . I haven't yet explained why this is so useful, but let's keep exploring.

## Improved major arcs?



**Lemma D (Improved bounds near  $\vec{0}$ , in  $p$ -adic sense)**

Let  $\mathcal{N}_{\text{aux}, p}(F) = \{(\vec{x}^{(i)})_i : |\vec{x}^{(i)}| \leq p, \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \equiv \vec{0} (p)\}$ .

Suppose  $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\beta} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$ ,  $e > 0$ ,  
 $p^{e+d-1} \leq \frac{1}{3}P^d$ , and  $|p^e \vec{\beta} - \vec{a}| \leq \frac{1}{3}p^{1-d}$ . Then  $\vec{a} \equiv \vec{0} (p^e)$ .

## Improved major arcs?

### Notation

- Let  $\mathcal{N}(P^\theta, P^{(d-1)\theta-d}; F)$  be the set of integral  $(d-1)$ -tuples with  $|\vec{x}^{(i)}| \leq P^\theta$ ,  $|\vec{v} - \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)})| \leq P^{(d-1)\theta-d}$  (some  $\vec{v} \in \mathbb{Z}^n$ ).
- $\mathcal{N}_{\text{aux}, p}(F) = \{(\vec{x}^{(i)})_i : |\vec{x}^{(i)}| \leq p, \vec{m}^{(F)}(\vec{x}^{(1)}, \dots, \vec{x}^{(d-1)}) \equiv \vec{0} \pmod{p}\}.$

### Lemma D (Improved bounds near $\vec{0}$ , in $p$ -adic sense)

Suppose  $\#\mathcal{N}(p, p^{d-1}P^{-d}; \vec{\alpha} \cdot \vec{f}) > \sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f})$ ,  $e > 0$ ,  $p^{e+d-1} \leq \frac{1}{3}P^d$ , and  $|p^e \vec{\alpha} - \vec{a}| \leq \frac{1}{3}p^{1-d}$ . Then  $\vec{a} \equiv \vec{0} \pmod{p^e}$ .

If  $\vec{f}$  is nonsingular,  $\sup_{\vec{\delta} \in \mathbb{F}_p^R \setminus \{\vec{0}\}} \mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll_{\vec{f}} p^{(d-2)n+R-1}$ .

### Proof.

Suppose  $\min_i |a_i|_p = p^{-k}$ , let  $(\vec{x}^{(i)})_i \in \mathcal{N} \setminus \mathcal{N}_{\text{aux}, p}(\frac{\vec{a}}{p^k} \cdot \vec{f})$ . Then

$|\vec{v} - \vec{m}^{(\vec{\alpha} \cdot \vec{f})}| \leq p^{d-1}P^{-d} \leq \frac{1}{3p^e}$ . Also  $|\vec{m}^{(\vec{\alpha} \cdot \vec{f})} - \frac{1}{p^e} \vec{m}^{(\vec{a} \cdot \vec{f})}| \leq \frac{1}{3p^e}$ . Thus

$|\vec{v} - \frac{1}{p^e} \vec{m}^{(\vec{a} \cdot \vec{f})}| < \frac{1}{p^e}$ , so  $\vec{m}^{(\vec{a} \cdot \vec{f})} = p^e \vec{v}$ . But  $(\vec{x}^{(i)})_i \notin \mathcal{N}_{\text{aux}, p}(\frac{\vec{a}}{p^k} \cdot \vec{f})$ .  $\clubsuit$

$\#\mathcal{N}_{\text{aux}, p}(\vec{\delta} \cdot \vec{f}) \ll p^{\dim\{\vec{m}^{(\vec{\delta} \cdot \vec{f})} = \vec{0}\}} \ll p^{(d-2)n+R-1}$  for  $p \gg 1$ .  $\square$