Repulsion: A how-to guide (part I)

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A theorem of Birch

$\vec{f}(\vec{x}) = (f_1(x_1, \dots, x_n), \dots, f_R(x_1, \dots, x_n))$

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ are R lin. indep. forms in n variables of degree $d \geq 2$.
- We set $N_{\vec{f}}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{f}(\vec{x}) = \vec{0},\ |\vec{x}| \leq P\}$.
- $\vec{f}(\vec{x})$ is nonsingular if the Jacobian matrix $(\partial f_i(\vec{x})/\partial x_j))_{ij}$ has rank R at every complex solution $\vec{x} \in \mathbb{C}^n \setminus \{\vec{0}\}$ to $\vec{f}(\vec{x}) = \vec{0}$.

Theorem (Birch, Proc. R. Soc. Lond. A 1962)

If $\vec{f}$ is nonsingular and $n \geq n_0(d, \cancel{n}, R)$ where

$$n_0(d, \cancel{n}, R\cancel{\vec{f}}) = (d-1)2^{d-1}R(R+1) + R \tag{1}$$

then $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$ as $P \to \infty$ for some real constant $\sigma \geq 0$. If $\vec{f}(\vec{x}) = \vec{0}$ has solutions in $\mathbb{R}^n \setminus \{\vec{0}\}$ and $\mathbb{Q}_p^n \setminus \{\vec{0}\}$ then $\sigma > 0$.

Birch actually allows inhomogeneous $\vec{f}$; if $\vec{f}^{[d]}$ is the degree d part he has

$$n_0(\vec{f}) = (d-1)2^{d-1}R(R+1) + 1 + \dim V(\vec{f}), \tag{1'}$$

with $V \subset \mathbb{C}^n$ defined by $\operatorname{rank}(\partial f_i^{[d]}(\vec{x})/\partial x_j))_{ij} < R$. The zeroes of $\vec{f}$ in $\mathbb{R}^n, \mathbb{Q}_p^n$ must be nonsingular. Best lower bound for σ in general: van Ittersum (Acta Arith. 2020).

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Want to improve (1) in the case $R > 1$.

Munshi (Compos. Math. '15): $d = R = 2, n = 11$.

$n_0(d, R)$

$n_0(d, 1)$

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Want to improve (1) in the case $R > 1$.

Munshi (Compos. Math. '15): $d = R = 2, n = 11$. Some $\vec{f}$ with $\boxed{R \leq 3}$: Browning-Munshi (Compos. Math. '13, Forum Math. '15), Heath-Brown & Pierce (Crelle '15), Pierce-Schindler-Wood (Proc. LMS '16).

A theorem of Birch

$|\vec{f}(\vec{x})| \le 1$

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Müller (J. Théor. Nombres Bordeaux 2005): $d = 2$, $\boxed{n \geq 9R}$ for nonsingular *irrational* systems of forms over $\mathbb{R}$. Based on Bentkus-Götze (Acta Arth. 1997, Ann. of Math. 1999).

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Theorem (SLRM 2021)

We can replace (1) with

$$n_0 = d2^d R + R. \tag{2}$$

Slightly more already known if $d < 4$. For $R \geq \max\{6-d, 2\}$ this beats (1).

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Theorem (Dietmann 2015, Schindler 2015)

If $\vec{f}$ is nonsingular and

$$n > (d-1)2^{d-1}R(R+1) + \sup_{\vec{a} \in \mathbb{Z}^R \setminus \{\vec{0}\}} \dim \operatorname{sing} A(\vec{a} \cdot \vec{f}) \qquad (1")$$

then $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$ as $P \to \infty$ for some real constant $\sigma \geq 0$. If $\vec{f}(\vec{x}) = \vec{0}$ has solutions in $\mathbb{R}^n \setminus \{\vec{0}\}$ and $\mathbb{Q}_p^n \setminus \{\vec{0}\}$ then $\sigma > 0$.

Want to improve (1) in the case $R > 1$, at least for fairly typical $\vec{f}$.

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Dietmann (Q. J. Math. 2015), Schindler (Adv. Th. Num. 2015).

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then $N_{\vec{f}}(P) = (\sigma + o(1))P^{n-dR}$ as $P \to \infty$ for some real constant $\sigma \geq 0$. If $\vec{f}(\vec{x}) = \vec{0}$ has solutions in $\mathbb{R}^n \setminus \{\vec{0}\}$ and $\mathbb{Q}_p^n \setminus \{\vec{0}\}$ then $\sigma > 0$.

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The circle method: setup

Notation

- $\vec{f}(\vec{x}) \in \mathbb{Z}[\vec{x}]^R$ are R lin. indep. forms in n variables of degree $d \geq 2$.
- We set $N_{\vec{f}}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{f}(\vec{x}) = \vec{0},\ |\vec{x}| \leq P\}$.
- ! We overuse dot products throughout, e.g. $\vec{\alpha} \cdot \vec{f} = \sum_{i=1}^{R} \alpha_i f_i$.
- The exponential sum $S(\vec{\alpha} \cdot \vec{f}) = \sum e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$, where $\vec{\alpha} \in \mathbb{R}^R$, $e(t) = e^{2\pi i t}$ and the sum is over $\vec{x} \in \mathbb{Z}^n$ with $|\vec{x}| \leq P$.
- major arcs $\mathfrak{M}(Q) = \{\vec{\alpha} \in [0,1]^R : |\vec{\alpha} - \frac{\vec{a}}{q}| \leq \frac{Q}{qP^d}\ (q \leq Q, \vec{a} \in \mathbb{Z}^R)\}$.
- minor arcs $\mathfrak{m}(Q) = [0,1]^R \setminus \mathfrak{M}(Q)$.

- The function $N_{\vec{f}}(P)$ which we want to estimate satisfies

$$N_{\vec{f}}(P) = \int_{[0,1]^R} S(\vec{\alpha}\cdot\vec{f})\, d\vec{\alpha} = \int_{\mathfrak{M}(P^\delta)} S(\vec{\alpha}\cdot\vec{f})\, d\vec{\alpha} + \int_{\mathfrak{m}(P^\delta)} S(\vec{\alpha}\cdot\vec{f})\, d\vec{\alpha}.$$

- We need to show: if n is large then for some $\nu, \delta > 0$, as $P \to \infty$,

$$\int_{\mathfrak{M}(P^\nu)} S(\vec{\alpha}\cdot\vec{f})\, d\vec{\alpha} \sim \sigma P^{n-dR}, \qquad \int_{\mathfrak{m}(P^\nu)} |S(\vec{\alpha}\cdot\vec{f})|\, d\vec{\alpha} \ll P^{n-dR-\delta}.$$

- N.B. $\mathfrak{M}(Q)$ is only truly the "major arcs" if Q is small.

$\sum \alpha_i f_i(\vec{x})$

$= 0$ in

$R[\vec{x}]$

$\Rightarrow \vec{\alpha} = \vec{0}$

$\int_m |S(\vec{\alpha}\cdot\vec{f})|\, d\vec{\alpha} >> P^{n-dR}$ BAD

The circle method: a framework

$|S(\vec{\alpha}\cdot\vec{f})| << P^n$

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$A << B$ means $A = O(B)$

$(P^k)^c$

How might we try to improve on Birch's result? To explore this question I want to break his argument up into four pieces. This is the first one:

Lemma A (level-set formulation of Birch, after Bentkus-Götze)

Suppose $c \in (0,1)$, $C > 1$, and that for each $k \in [0, dR/c)$ we have

$$\mathrm{meas}\{\vec{\alpha} \in [0,1]^R : |S(\vec{\alpha}\cdot\vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}.$$

There is $\delta(c,d,R,n) > 0$ with $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f},c,C}(P^{n-dR-\delta+\epsilon})$.

That is, to run the argument we only need the measure of these level sets.

$|S| > P^{n-dR}$, $k = dR$, $P^{(c-1)dR}$

$cP^{-\delta}$

$k \in \mathbb{R}$

P^n on measure P^{-dR}

NB - probably this formulation is BAD for constants

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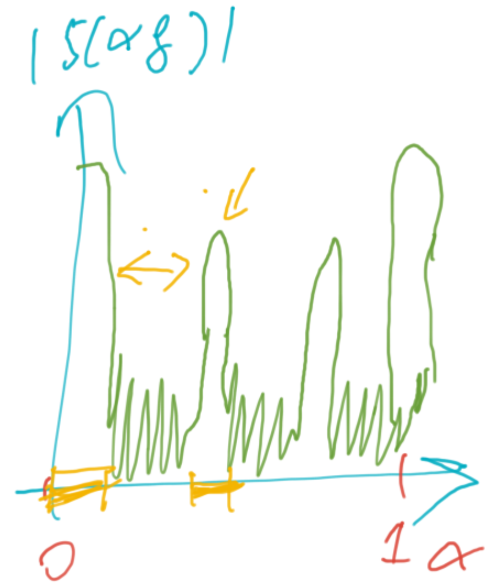

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There is $\delta(c,d,R,n)>0$ with $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f},c,C}(P^{n-dR-\delta+\epsilon})$.

Want ν,δ_1 with $\int_{\mathfrak{m}(P^\nu)} S(\vec{\alpha}\cdot\vec{f})\,d\vec{\alpha} \ll_{\vec{f},c,C} P^{n-dR-\delta_1}$:

i) Dietmann/Schindler: $\sup_{\alpha\in\mathfrak{m}(P^\nu)} |S| \ll_{\vec{f}} P^{n-\delta_0\nu}$ $+\epsilon$ as f_i are lin. indep.

ii) $\int_{\mathfrak{m}(P^\nu)} \ll \sum_{P^k\in 2^{\mathbb{Z}}, k\ge\delta_0\nu} P^{n-k} \operatorname{meas}\{\vec{\alpha}\in[0,1]^R : |S| > P^{n-k+\epsilon}\}$

$\le C\sum_{P^k\in 2^{\mathbb{Z}}, k\ge\delta_0\nu} P^{n-dR-(1-c)k} \ll P^{n-dR-\delta_1 \epsilon}.$

$\int_{\mathfrak{m}(P^\nu)} |S(\vec{\alpha}\cdot\vec{f})|\,d\vec{\alpha}$

$= \sum_{\substack{P^k=2^i \\ i\in\mathbb{Z} \\ k\in\mathbb{R}}} \int_{\substack{\alpha\in\mathfrak{m}(P^\nu) \\ P^{n-k}\le|S(\vec{\alpha}\cdot\vec{f})| \\ \le 2P^{n-k}}} |S(\vec{\alpha}\cdot\vec{f})|\,d\vec{\alpha}$

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If $\operatorname{meas}\{\vec{\alpha}\in[0,1]^R : |S(\vec{\alpha}\cdot\vec{f})| > P^{n-k+\epsilon}\} \le CP^{ck-dR}$ for $k\in[0,dR/c)$, it holds for $k\ge 0$. Want ν,δ_1 with $\int_{\mathfrak{m}(P^\nu)} S(\vec{\alpha}\cdot\vec{f})\,d\vec{\alpha} \ll_{\vec{f},c,C} P^{n-dR-\delta_1}$:

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$\le C\sum_{P^k\in 2^{\mathbb{Z}},k\ge\delta_0\nu} P^{n-dR-(1-c)k} \ll P^{n-dR-\delta_1}$.

Also want $\displaystyle\int_{\mathfrak{M}(P^\nu)} S(\vec{\alpha}\cdot\vec{f})\,d\vec{\alpha} = \sigma P^{n-dR} + O_{\vec{f},c,C}(P^{n-dR-\delta+\epsilon})$:

i) $S((\frac{\vec{a}}{q}+\frac{\vec{\gamma}}{P^d})\cdot\vec{f}) = \frac{P^n}{q^n}\sum_{\vec{b}\,(q)} e(\frac{\vec{a}}{q}\cdot\vec{f}(\vec{b}))\displaystyle\int_{|\vec{u}|\le 1} e(\vec{\gamma}\cdot\vec{f}(\vec{u}))\,d\vec{u} + O_{\vec{f}}(P^{n-1+2\nu})$.

The circle method: a framework

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$\leq C\sum_{P^k\in 2^{\mathbb{Z}}, k\geq\delta_0\nu} P^{n-dR-(1-c)k} \ll P^{n-dR-\delta_1}$.

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γ

i) $$S\left(\left(\frac{\vec{a}}{q}+\frac{\vec{\gamma}}{P^d}\right)\cdot\vec{f}\right) = \frac{P^n}{q^n}\sum_{\vec{b}\,(q)} e\left(\frac{\vec{a}}{q}\cdot\vec{f}(\vec{b})\right)\int_{|\vec{u}|\leq 1} e(\vec{\gamma}\cdot\vec{f}(\vec{u}))\, d\vec{u} + O_{\vec{f}}(P^{n-1+2\nu}).$$

ii) $$\int_{\mathfrak{M}(P^\nu)} = \sum_{q\leq P^\nu}\sum^{*}_{\vec{a}\,(q)} \frac{P^{n-dR}}{q^n}\sum_{\vec{b}\,(q)} e\left(\frac{\vec{a}}{q}\cdot\vec{f}(\vec{b})\right)\boxed{\int_{\substack{|\vec{u}|\leq 1\\ |\vec{\gamma}|\leq \frac{P^\nu}{q}}} e(\vec{\gamma}\cdot\vec{f}(\vec{u}))\, d\vec{u}\, d\vec{\gamma}} + O_{\vec{f}}(P^{n-dR-\delta}).$$

$\int S(\vec{\alpha}\cdot\vec{\gamma})\, d\vec{\alpha}$

iii) By choosing $P = \epsilon\Phi^{1/\nu}$,

$$\int_{\Phi\leq|\vec{\gamma}|\leq 2\Phi}\left|\int_{|\vec{u}|\leq 1} e(\vec{\gamma}\cdot\vec{f}(\vec{u}))\, d\vec{u}\right| d\vec{\gamma} \ll_{\vec{f}} \int_{\mathfrak{m}(P^\nu)} \frac{|S(\vec{\beta}\cdot\vec{f})|}{P^{n-dR}}\, d\vec{\beta} + P^{-1+(2+R)\nu} \ll_{C,c} \Phi^{-\delta_2}.$$

The circle method: a framework

Notation

- Let $S_{a,q} = \sum_{\vec{b}\,(q)} e(\frac{\vec{a}}{q} \cdot \vec{f}(\vec{b}))$ and $I(\vec{\gamma}) = \int_{|\vec{u}| \le 1} e(\vec{\gamma} \cdot \vec{f}(\vec{u}))\, d\vec{u}$

i) Birch showed $S\big((\frac{\vec{a}}{q} + \frac{\vec{\gamma}}{P^d}) \cdot \vec{f}\big) = \frac{P^n}{q^n} S_{a,q} I(\vec{\gamma}) + O_{\vec{f}}(P^{n-1+2\nu})$,

ii) and hence $\displaystyle\int_{\mathfrak{M}(P^\nu)} = \sum_{q \le P^\nu} \sideset{}{^*}\sum_{\vec{a}\,(q)} \frac{P^{n-dR}}{q^n} S_{a,q} \int_{|\vec{\gamma}| \le \cancel{\frac{P^\nu}{q}}} I(\vec{\gamma})\, d\vec{\gamma} + O_{\vec{f}}(P^{n-dR-\delta})$.

iii) By choosing $P = \epsilon \Phi^{1/\nu}$,

$$\int_{\Phi \le |\vec{\gamma}| \le 2\Phi} |I(\vec{\gamma})| d\vec{\gamma} \ll_{\vec{f}} \int_{\mathfrak{m}(P^\nu)} \frac{|S(\vec{\beta} \cdot \vec{f})|}{P^{n-dR}}\, d\vec{\beta} + P^{-1+(2+R)\nu} \ll_{C,c} \Phi^{-\delta_2}.$$

iv) In a similar way $\displaystyle\sum_{Q \le q \le 2Q} \sideset{}{^*}\sum_{\vec{a}\,(q)} |\frac{1}{q^n} S_{a,q}| \ll Q^{-\delta_3}$.

v) (ii)-(iv) give $\displaystyle\int_{\mathfrak{M}(P^\nu)} = \sum_{q \in \mathbb{N}} \sideset{}{^*}\sum_{\vec{a}\,(q)} \frac{P^{n-dR}}{q^n} S_{a,q} \int_{|\vec{\gamma}| \in \mathbb{R}^R} I(\vec{\gamma})\, d\vec{\gamma} + O_{\vec{f},c,C}(P^{n-dR-\delta+\epsilon})$.

The circle method: a framework

Notation

- Let $S_{a,q} = \sum_{\vec{b}\,(q)} e(\frac{\vec{a}}{q} \cdot \vec{f}(\vec{b}))$ and $I(\vec{\gamma}) = \int_{|\vec{u}|\leq 1} e(\vec{\gamma} \cdot \vec{f}(\vec{u}))\, d\vec{u}$
- The exponential sum $S(\vec{\alpha} \cdot \vec{f}) = \sum e(\vec{\alpha} \cdot \vec{f}(\vec{x}))$, where $\vec{\alpha} \in \mathbb{R}^R$, $e(t) = e^{2\pi i t}$ and the sum is over $\vec{x} \in \mathbb{Z}^n$ with $|\vec{x}| \leq P$.
- major arcs $\mathfrak{M}(Q) = \{\vec{\alpha} \in [0,1]^R : |\vec{\alpha} - \frac{\vec{a}}{q}| \leq \frac{Q}{qP^d} \ (q \leq Q, \vec{a} \in \mathbb{Z}^R)\}$.
- minor arcs $\mathfrak{m}(Q) = [0,1]^R \setminus \mathfrak{M}(Q)$

Also want δ with $\displaystyle\int_{\mathfrak{M}(P^\nu)} S(\vec{\alpha} \cdot \vec{f})\, d\vec{\alpha} = \sigma P^{n-dR} + O_{\vec{f},c,C}(P^{n-dR-\delta+\epsilon})$:

ii) [. . .] hence $\displaystyle\int_{\mathfrak{M}(P^\nu)} = \sum_{q \leq P^\nu} \sideset{}{^*}\sum_{\vec{a}\,(q)} \frac{P^{n-dR}}{q^n} S_{a,q} \int_{|\vec{\gamma}| \leq \frac{P^\nu}{q}} I(\vec{\gamma})\, d\vec{\gamma} + O_{\vec{f}}(P^{n-dR-\delta})$.

iii) By choosing $P = \epsilon \Phi^{1/\nu}$,

$$\int_{\Phi \leq |\vec{\gamma}| \leq 2\Phi} |I(\vec{\gamma})| d\vec{\gamma} \ll_{\vec{f}} \int_{\mathfrak{m}(P^\nu)} \frac{|S(\vec{\beta} \cdot \vec{f})|}{P^{n-dR}}\, d\vec{\beta} + P^{-1+(2+R)\nu} \ll_{C,c} \Phi^{-\delta_2}.$$

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Lemma A (level-set formulation of Birch, after Bentkus-Götze)

If $\text{meas}\{\vec{\alpha} \in [0,1]^R : |S(\vec{\alpha} \cdot \vec{f})| > P^{n-k+\epsilon}\} \leq CP^{ck-dR}$ *for some* $c < 1$, *all* $k \in [0, dR/c)$, *then* $N_{\vec{f}}(P) = \sigma P^{n-dR} + O_{\vec{f},c,C}(P^{n-dR-\delta(c,d,R,n)+\epsilon})$.

$N_{\vec{f}}(P)$

$= \int_{\mathfrak{M}} + \int_{\mathfrak{m}}$