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On Boolean threshold functions with minimum specification number ☆

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A B S T R A C T

A set S of Boolean points is a *specifying set* for a threshold function f if the only threshold function consistent with f on S is f itself. The minimal cardinality of a specifying set for f is the *specification number of* f and it is never smaller than $n+1$ for a function with n relevant variables. In the present paper, we develop an inductive approach to describing the set of Boolean threshold functions with minimum specification number by means of operations that allow us to extend functions of n variables in this set to functions of $n+1$ variables.

1. Introduction

The notion of specification number arises in the context of on-line learning with a helpful teacher [6]. Speaking informally, *teaching* an unknown function f in a given class is the problem of producing its *teaching* (or *specifying*) set, i.e. a set of points in the domain which uniquely specifies f within the class.

In the present paper, we study *threshold Boolean functions*, also known as *linearly separable* or *halfspaces*. A Boolean function is *threshold* if there exists a hyperplane separating true and false points of the function. Threshold functions play fundamental role in the theory of Boolean functions and they appear in a variety of applications such as electrical engineering, artificial neural networks, reliability theory, game theory etc. (see, for example, [4]). Specifying sets of threshold and related (non-Boolean) functions are studied in [14,1,11].

It is known that in the worst case the specifying set of a threshold Boolean function contains all the 2^n points of the Boolean hypercube [2]. However, *positive* (or *increasing*) threshold functions, i.e. functions where an increase of a variable cannot lead to a decrease of the function, can be specified by the set of its *extremal points*, i.e. its maximal false and minimal true points (in the worst case $\binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}$ points [2]). It turns out that this description can be redundant, i.e. sometimes a positive threshold function f can be specified by a proper subset of its extremal points, namely, by the set of *essential* points. The minimum cardinality of a teaching set of f, i.e. the minimum number of points needed to specify f, is the *specification number* of f. For threshold functions this number coincides with the number of essential points of f. The role

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of essential points in the characterization of threshold and related functions was also studied in [12] and later in [13] where small subsets of essential points were used to estimate the number of intersections of threshold functions.

Anthony et al. proved in [2] that the specification number of a threshold function with n relevant variables is at least $n+1$ and that this bound is attained for so-called linear read-once functions. The importance of linear read-once functions in learning theory is evidenced, in particular, by their connection with special types of decision lists [9].

In addition to showing that the lower bound $n+1$ is attained by the linear read-once functions, Anthony et al. [2] also conjectured that for all other threshold functions with n relevant variables the specification number is strictly greater than $n+1$. In [8], we disproved this conjecture by exhibiting an infinite family of threshold functions with n relevant variables, which are not linear read-once and for with the specification number equals $n+1$. However, the problem of describing the set of all such functions remains open. This set is not closed under taking subfunctions, which makes the problem of producing a complete description highly non-trivial. In the present paper, we explore an approach to this problem that allows us to construct Boolean threshold functions with minimum specification number inductively.

We denote the set of all threshold n-variable functions that have the minimum specification number $n+1$ by $\mathcal{T}_n$ and identify a number of operations that allow us to construct a function in $\mathcal{T}_n$ from a function in $\mathcal{T}_{n-1}$. Our operations produce a variety of new Boolean threshold functions with minimum specification number. In particular, these operations provide a complete description of functions in $\mathcal{T}_n$ for $n \le 5$. For larger values of n, the problem remains open and we discuss it in Section 7. All preliminary information related to the topic of the paper is presented in Section 2.

2. Preliminaries

Let n be a natural number and $B = \{0,1\}$. For a point $\mathbf{x} \in B^n$ and an index $i \in [n]$, we denote by

$(\mathbf{x})_i$ the i-th coordinate of $\mathbf{x}$,
$\bar{\mathbf{x}}$ the point in B^n with $(\bar{\mathbf{x}})_i = 1$ if and only if $(\mathbf{x})_i = 0$ for every $i \in [n]$.

Also, we denote by **0** and **1** the points consisting of all 1s and all 0s respectively.

By $\preccurlyeq$ we denote a partial order over B^n, induced by inclusion in the power set lattice of the n-set. In other words, $\mathbf{x} \preccurlyeq \mathbf{y}$ if $(\mathbf{x})_i = 1$ implies $(\mathbf{y})_i = 1$. In this case we will say that $\mathbf{x}$ is *below* $\mathbf{y}$. When $\mathbf{x} \preccurlyeq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ we will sometimes write $\mathbf{x} \prec \mathbf{y}$.

A Boolean function $f = f(x_1,\dots,x_n)$ be on B^n is called *positive* (also known as *positive monotone* or *increasing*) if $f(\mathbf{x}) = 1$ and $\mathbf{x} \preccurlyeq \mathbf{y}$ imply $f(\mathbf{y}) = 1$. If f is positive, then we say that a point $\mathbf{x}$ is a *minimal one* of f if $f(\mathbf{x}) = 1$ and $f(\mathbf{y}) = 0$ for each $\mathbf{y}$ such that $\mathbf{y} \prec \mathbf{x}$. Similarly, a point $\mathbf{x}$ is a *maximal zero* of f if $f(\mathbf{x}) = 0$ and $f(\mathbf{y}) = 1$ for each $\mathbf{y}$ such that $\mathbf{x} \prec \mathbf{y}$. A point will be called an *extremal point of* f if it is either a maximal zero or a minimal one of f.

The *dual* of a function f is the function g defined as follows:

$$g(\mathbf{x}) = \overline{f(\bar{\mathbf{x}})} \text{ for each } \mathbf{x} \in B^n.$$

If f coincides with its dual then it is called *self-dual*.

For an index $k \in [n]$, and $\alpha_k \in \{0,1\}$ we denote by $f_{|x_k=\alpha_k}$ the Boolean function on B^{n-1} defined as follows:

$$f_{|x_k=\alpha_k}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n) = f(x_1,\dots,x_{k-1},\alpha_k,x_{k+1},\dots,x_n).$$

Inductively, for $i_1, \dots, i_k \in [n]$ and $\alpha_1, \dots, \alpha_k \in \{0,1\}$ we denote by $f_{|x_{i_1}=\alpha_1,\dots,x_{i_k}=\alpha_k}$ the function $(f_{|x_{i_1}=\alpha_1,\dots,x_{i_{k-1}}=\alpha_{k-1}})_{|x_{i_k}=\alpha_k}$. We say that $f_{|x_{i_1}=\alpha_1,\dots,x_{i_k}=\alpha_k}$ is the *restriction* of f to $x_{i_1}=\alpha_1,\dots,x_{i_k}=\alpha_k$. We also say that a Boolean function g is a *restriction* (or a *subfunction*) of f if there exist $i_1,\dots,i_k \in [n]$ and $\alpha_1,\dots,\alpha_k \in \{0,1\}$ such that $g = f_{|x_{i_1}=\alpha_1,\dots,x_{i_k}=\alpha_k}$.

A function f is called *canalyzing* if there exists $i \in [n]$ such that $f_{|x_i=0}$ or $f_{|x_i=1}$ is a constant function. It is easy to see that if f is a *positive* canalyzing function, then $f_{|x_i=0} \equiv \mathbf{0}$ or $f_{|x_i=1} \equiv \mathbf{1}$ for some $i \in [n]$.

A variable x_k is called *irrelevant* for f if $f_{|x_k=1} \equiv f_{|x_k=0}$, i.e., $f_{|x_k=1}(\mathbf{x}) = f_{|x_k=0}(\mathbf{x})$ for every $\mathbf{x} \in B^{n-1}$. Otherwise, x_k is called *relevant* for f and we also say that f *depends on* x_k. Two distinct variables x_i and x_j of f are *symmetric* if for every $\mathbf{x} \in B^n$ we have $f(\mathbf{x}) = f(\mathbf{x}')$, where $\mathbf{x}'$ is obtained from $\mathbf{x}$ by swapping the i-th and j-th coordinates.

We denote by $\vee$ and $\wedge$ the logical disjunction and conjunction respectively. We also often omit the operator $\wedge$ and denote conjunction by mere juxtaposition. We say that a Boolean formula is in disjunctive normal form (DNF) if it is a disjunction consisting of one or more conjunctive clauses, each of which is a conjunction of one or more literals (variables or their negations). A Boolean formula is in conjunctive normal form (CNF) if it is a conjunction consisting of one or more disjunctive clauses, each of which is a disjunction of one or more literals. We say that a Boolean formula is *positive* if it does not contain the operation of negation. It is well-known that any positive Boolean function can be represented by a positive Boolean formula.

Two Boolean functions f and g are *congruent*, if they are identical up to renaming (without identification) and/or negation of variables.

2.1. Threshold and linear read-once functions

A Boolean function f on B^n is called *threshold* if there exist n *weights* $w_1,\dots,w_n \in \mathbb{R}$ and a *threshold* $t \in \mathbb{R}$ such that $f(x_1,\dots,x_n) = 0$ if and only if $w_1x_1 + \dots + w_nx_n \le t$. The latter is called *threshold inequality* representing f.

Let $k \in \mathbb{N}, k \geq 2$. A Boolean function f on B^n is *k-summable* if, for some $r \in \{2, \ldots, k\}$, there exist r not necessarily distinct false points $\mathbf{x}_1, \ldots, \mathbf{x}_r$ and r not necessarily distinct true points $\mathbf{y}_1, \ldots, \mathbf{y}_r$ of f such that $\sum_{i=1}^r \mathbf{x}_i = \sum_{i=1}^r \mathbf{y}_i$. A function is *asummable* if it is not k-summable for all $k \geq 2$.

Theorem 1 *([5]). A Boolean function is a threshold function if and only if it is assumable.*

It is known (see e.g. [4]) that the class of threshold functions is closed under taking restrictions, i.e., any restriction of a threshold function is again a threshold function.

A Boolean function f is called *read-once* if it can be represented by a *read-once formula*, i.e., by a Boolean formula involving only the operations of conjunction, disjunction, and negation in which every variable appears at most once. A read-once function f is *linear read-once* (*lro*) if it is either a constant function, or it can be represented by a *nested formula* defined recursively as follows:

1. both literals x and $\overline{x}$ are nested formulas;
2. $x \vee F$, $x \wedge F$, $\overline{x} \vee F$, $\overline{x} \wedge F$ are nested formulas, where x is a variable and F is a nested formula that contains neither x, nor $\overline{x}$.

It is not difficult to see that an lro function f is positive if and only if the nested formula representing f does not contain negations.

2.2. Essential points and specifying sets of Boolean threshold functions

Let $f = f(x_1, \ldots, x_n)$ be a *threshold* function on B^n. A set of points $S \subseteq B^n$ is a *specifying set for f* if the only threshold function consistent with f on S is f itself. The minimal cardinality of a specifying set for f is called the *specification number of f* and denoted $\sigma(f)$.

It was shown in [10] and later in [2] that the specification number of a threshold function of n variables is at least $n+1$. Furthermore, it was shown in [2] that this lower bound is attained on linear read-once functions. More specifically, the specification number of an lro function depending on all its n variables is exactly $n+1$. In fact, the authors of [2] conjectured that lro functions depending on all their n variables are the only threshold functions that have the minimum specification number $n+1$. This conjecture was disproved in [8] by showing that for every $n \geq 4$ the function

$$x_1x_2 \vee x_1x_3 \vee \cdots \vee x_1x_{n-1} \vee x_2x_3 \ldots x_n \tag{1}$$

is not linear read-once, threshold and has specification number $n+1$. However, complete characterization of threshold functions with the minimum specification number remains a tantalizing open problem. In the present paper we make a progress towards such a characterization by broadening the class of known threshold functions with the minimum specification number. For a natural number n, we denote by

$\mathcal{T}_n$ the class of all threshold n-variable functions that have the minimum specification number $n+1$.

The concept of specification number of a threshold function is closely related to so-called essential points of the function. A point $\mathbf{x}$ is *essential* for a threshold function f, if there exists a threshold function g on B^n such that $g(\mathbf{x}) \neq f(\mathbf{x})$ and $g(\mathbf{y}) = f(\mathbf{y})$ for every $\mathbf{y} \in B^n \setminus \{\mathbf{x}\}$. A point that is not essential for f is called *inessential* for f. Clearly, any specifying set for f should include all essential points for f. It turns out that the essential points alone are sufficient to specify f [3], and, in particular, we have the following well-known result

Theorem 2 *([3]). The specification number $\sigma(f)$ of a threshold function f is equal to the number of its essential points.*

Furthermore, it is known that a Boolean threshold function f of n variables has at least $n+1$ essential points in general position:

Theorem 3 *([2]). Let f be a Boolean threshold function of n variables. Then f has at least $n+1$ essential points in general position.*

This fact was used in [2] to show that if f has an irrelevant variable, then it has at least $2n$ essential points. By a similar argument one can show that if f is a self-dual function, then it has at least $2n$ essential points.

Lemma 1. *Let f be a self-dual threshold function. Then f has at least $2n$ essential points.*

Proof. First, notice that if a point $\mathbf{x}$ is an essential point of a self-dual threshold function, then its complement $\overline{\mathbf{x}}$ is also an essential point for the function. Let S be the set of essential points of f, and let S' be a subset of $n+1$ essential points in general position that are guaranteed to exist by Theorem 3. We claim that S' cannot contain more than one pair of complementary points. Assume to the contrary that S' contains at least two pairs of complementary points, i.e. there exist two non-complementary points $\mathbf{x}, \mathbf{y} \in S'$ such that $\overline{\mathbf{x}}, \overline{\mathbf{y}} \in S'$. Then

$$\mathbf{x} + \overline{\mathbf{x}} = \mathbf{y} + \overline{\mathbf{y}},$$

which implies that the four points $\mathbf{x}, \overline{\mathbf{x}}, \mathbf{y}, \overline{\mathbf{y}}$ lie on a 2-flat, i.e. a plane, which contradicts the assumption that the points in S' are in general position. Therefore, S' contains at most one pair of complementary points and at least n points none of which is the complement of another. Consequently, the complements of these n points are in $S \setminus S'$ and hence $|S| \geq 2n$. □

Since we are studying Boolean functions with the minimum number $n+1$ of essential points, due to the above facts, we will often assume that functions under consideration depend on all their variables and are not self-dual. Furthermore, it was observed in [2] that in the study of specification number of threshold functions, one can be restricted to positive functions, because for any threshold function f there exists a *positive* function g that is congruent to f and has the same specifying set. We will therefore consider only positive threshold functions. One useful benefit of this positivity assumption is the fact that for a *positive* threshold function depending on all its variables any essential point is an extremal point. This is a corollary of the following

Theorem 4 *([2]). Suppose f is a positive threshold function depending on all its variables. Then the set of extremal points of f specifies it.*

As any specifying set of a Boolean function contains all its essential points we conclude

Corollary 1. *Suppose f is a positive threshold function depending on all its variables. Then all essential points of f are extremal.*

3. More non-lro threshold functions with minimum specification number

In this section we extend the family of functions (1) that provide a counterexample to the conjecture of Anthony et al. [2].

Theorem 5. *Let n and k be integers such that $3 \leq k \leq n-1$ and let $f_{n,k}(x_1, \ldots, x_n)$ be a Boolean function defined by its DNF*

$$x_1x_2 \vee x_1x_3 \vee \cdots \vee x_1x_k \vee x_2x_3 \ldots x_n.$$

Then $f_{n,k}$ is a positive, non-lro, threshold function, depending on all its variables, and the specification number of $f_{n,k}$ is $n+1$.

Proof. Clearly, $f_{n,k}$ is positive and depends on all its variables. Also $f_{n,k}$ is not canalyzing, and therefore it is non-linear read-once. We show next that $f_{n,k}$ is a threshold function. To this end we will identify all minimal ones and all maximal zeros of $f_{n,k}$, and then present a threshold inequality that separates these sets of points.

It is easy to check that the minimal ones of $f_{n,k}$ are

$$\begin{aligned}
\mathbf{x}_1 &= (1, 1, 0, 0, \ldots, 0, 0, \ldots, 0), \\
\mathbf{x}_2 &= (1, 0, 1, 0, \ldots, 0, 0, \ldots, 0), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mathbf{x}_{k-1} &= (1, 0, 0, 0, \ldots, 1, 0, \ldots, 0), \\
\mathbf{x}_k &= (0, 1, 1, \ldots, 1, 1),
\end{aligned}$$

and the maximal zeros are

$$\begin{aligned}
\mathbf{y}_1 &= (0, 0, 1, 1, \ldots, 1, 1, 1), \\
\mathbf{y}_2 &= (0, 1, 0, 1, \ldots, 1, 1, 1), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\mathbf{y}_{n-2} &= (0, 1, 1, 1, \ldots, 1, 0, 1), \\
\mathbf{y}_{n-1} &= (0, 1, 1, 1, \ldots, 1, 1, 0), \\
\mathbf{z} &= (z_1, \ldots, z_n), \\
&\text{where } z_i = 0 \text{ iff } i \in \{2, \ldots, k\}.
\end{aligned}$$

Now, if $k = n-1$, then the inequality

$$(2n-5)x_1 + 2(x_2 + x_3 + \cdots + x_{n-1}) + x_n \geq 2n-3$$

holds in all minimal ones and does not hold in all maximal zeros, which can be checked directly for each extremal point by substituting its coordinates in the inequality, and hence $f_{n,n-1}$ is threshold. Similarly, if $k < n-1$, then the inequality

$$((k-1)(n-k+1)-1)x_1 + \sum_{i=2}^{k}(n-k+1)x_i + \sum_{i=k+1}^{n} x_i \geq k(n-k+1)-1 \tag{2}$$

separates minimal ones and maximal zeros witnessing that $f_{n,k}$ is threshold.

It remains to show that $f_{n,k}$ has $n+1$ essential points. Since $f_{n,k}$ depends on all its variables, by Corollary 1, every essential point of $f_{n,k}$ is extremal. Therefore, since $f_{n,k}$ has $n+k$ extremal points, it suffices to prove that $k-1$ of them are not essential. We will show that maximal zeros $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{k-1}$ are not essential. Suppose, towards a contradiction, there exists a threshold function g that differs from $f_{n,k}$ only in point $\mathbf{y}_i$ for some $i \in [k-1]$, i.e., $g(\mathbf{y}_i) = 1$ and $g(\mathbf{x}) = f_{n,k}(\mathbf{x})$ for every $\mathbf{x} \neq \mathbf{y}_i$. From $\overline{\mathbf{y}_{n-1}} \preccurlyeq \mathbf{z}$ and $f_{n,k}(\mathbf{z}) = 0$ we conclude that $f_{n,k}(\overline{\mathbf{y}_{n-1}}) = 0$, and therefore $\mathbf{x}_i + \mathbf{y}_i = \mathbf{y}_{n-1} + \overline{\mathbf{y}_{n-1}}$ implies that g is 2-summable, and hence is not threshold. This contradiction completes the proof. □

We observe that the restriction $f_{n,k|x_{k+1}=1,\dots,x_n=1}$ is equal to the k-variable function $x_1x_2 \vee x_1x_3 \vee \dots \vee x_1x_k \vee x_2x_3\dots x_k$. This function, as was shown in [7], has $2k$ essential points, and therefore we conclude the following

Corollary 2. *The set of threshold functions with minimum specification number is not closed under taking restrictions.*

This corollary also shows that specification number is not monotone with respect to restrictions, i.e., by restricting a function specification number can increase.

4. Extension on a variable

By definition any linear read-once function of $n > 1$ variables can be obtained from a linear read-once function of $n-1$ variables as the conjunction or disjunction of this function with a new variable. We will refer to these operations as *adding a variable*. To prove that all linear read-once functions have minimum specification number, it was shown in [2] that the operation of adding a variable increases specification number by exactly one.

Lemma 2 *([2]). Let $f = f(x_1, \dots, x_n)$ be a threshold function depending on all its variables and let y be a new variable. Then the functions $f' = y \vee f$ and $f'' = y \wedge f$ both have specification number $\sigma(f)+1$.*

Since any linear read-once function can be constructed recursively using the operations of adding a variable starting from a constant function, Lemma 2 implies that any such function depending on all its variables has specification number one more than the number of variables.

It is natural to ask whether the recursive definition of the class of linear read-once functions can be generalized to the whole class $\mathcal{T}_n$. This section is devoted to some results in this direction.

Definition 1. Let $f(x_1, \dots, x_n)$ be a positive Boolean function, $i \in [n]$, and let y be a new variable. The *(x_i, y)-extension of f* is the function

$$f^{(x_i,y)}(x_1, \dots, x_n, y) = x_i(y \vee f_{|x_i=1}) \vee y f_{|x_i=0}.$$

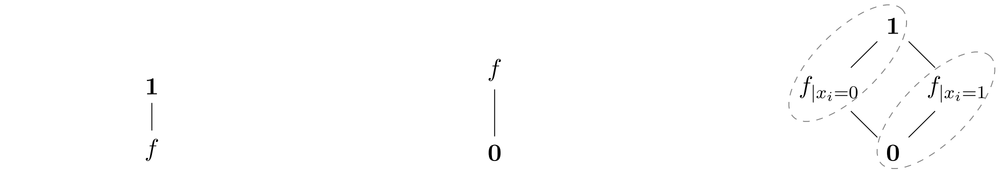
We say that $f^{(x_i,y)}$ is obtained from f *by the extension on the variable x_i*.

To illustrate the relation between the operations of adding a variable and extension on a variable we introduce the notion of restriction graph.

Definition 2. Let $f = f(x_1, \dots, x_n)$ be a Boolean function and let $S = \{x_{i_1}, \dots, x_{i_k}\}$ be a set of variables of f. We say that a graph G is the *S-restriction graph* for f if its vertex set is the set of all restrictions of f to S and for any $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \in \{0,1\}$ two vertices $f_{|x_{i_1}=\alpha_1,\dots,x_{i_k}=\alpha_k}$ and $f_{|x_{i_1}=\beta_1,\dots,x_{i_k}=\beta_k}$ are connected by an edge if and only if vectors $(\alpha_1, \dots, \alpha_k)$ and $(\beta_1, \dots, \beta_k)$ differ in exactly one coordinate.

For a function f and a new variable y, the $\{y\}$-restriction graphs for the functions $f \vee y$ and $f \wedge y$ both contain a vertex corresponding to a constant function (see Fig. 1a, 1b). Similarly, half of the vertices of the $\{x_i, y\}$-restriction graph of $f^{(x_i,y)}$ (Fig. 1c) correspond to constant functions. Moreover, this graph can be split into two subgraphs, which are the $\{y\}$-restriction graphs of the functions $f_{|x_i=1} \vee y$ and $f_{|x_i=0} \wedge y$, i.e., functions that are obtained from $f_{|x_i=1}$ and $f_{|x_i=0}$ via operations of adding variable y.

In this section we reveal more resemblance between the two operations by showing that, similarly to the operation of adding a variable, the operation of extension on a variable, when applied to a function in $\mathcal{T}_n$, preserves thresholdness and increases specification number by exactly one, i.e., the resulting function belongs to $\mathcal{T}_{n+1}$. To prove the former property, we start with an auxiliary statement characterizing functions that are extensions on a variable.



(a) The $\{y\}$-restriction graph for the function $f \vee y$.

(b) The $\{y\}$-restriction graph for the function $f \wedge y$.

(c) The $\{x_i, y\}$-restriction graph for the function $f^{(x_i,y)}$. The dashed ellipses correspond to the $\{y\}$-restriction graphs for $f_{|x_i=0} \wedge y$ and $f_{|x_i=1} \vee y$.

Fig. 1. The restriction graphs for the functions obtained from a given positive Boolean function $f = f(x_1, \ldots, x_n)$ by the operations of adding a variable and extension on the variable x_i for some $i \in [n]$.

Lemma 3. *Let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n, x_{n+1})$ be Boolean functions, and let $i \in [n]$. Then g is the (x_i, x_{n+1})-extension of f if and only if*

$$g(x_1, \ldots, x_{n+1}) = \begin{cases} x_i, & x_{n+1} = x_i \\ f(x_1, \ldots, x_n), & x_{n+1} \neq x_i \end{cases}$$

Proof. Let f' be the (x_i, x_{n+1})-extension of f. To prove the statement, it is enough to show that $f' \equiv g$. The latter follows from the observation that the two functions coincide on all restrictions to the variables x_i and x_{n+1}:

$$\begin{aligned} f'_{|x_i=0,x_{n+1}=0} &= \quad 0 &&= g_{|x_i=0,x_{n+1}=0}, \\ f'_{|x_i=1,x_{n+1}=1} &= \quad 1 &&= g_{|x_i=1,x_{n+1}=1}, \\ f'_{|x_i=1,x_{n+1}=0} &= \quad f_{|x_i=1} &&= g|_{x_i=1,x_{n+1}=0}, \\ f'_{|x_i=0,x_{n+1}=1} &= \quad f_{|x_i=0} &&= g_{|x_i=0,x_{n+1}=1}. \quad \square \end{aligned}$$

Lemma 4. *Let $f(x_1, \ldots, x_n)$ be a positive threshold function on $n \geq 2$ variables. The extension of f on a variable is a positive threshold function.*

Proof. Without loss of generality, we prove the lemma for the extension on variable x_1, i.e., we will show that the (x_1, x_{n+1})-extension of f is a positive threshold function. Denote $f_0 = f_{|x_1=0}$ and $f_1 = f_{|x_1=1}$. Then, by definition,

$$f^{(x_1,x_{n+1})}(x_1, \ldots, x_n, x_{n+1}) = x_1(x_{n+1} \vee f_1) \vee x_{n+1} f_0.$$

The positivity of $f^{(x_1,x_{n+1})}$ clearly follows from the formula above. Towards a contradiction, assume $f^{(x_1,x_{n+1})}$ is not threshold and let k be the minimum number such that $f^{(x_1,x_{n+1})}$ is k-summable. Let $\mathbf{y}_1, \ldots, \mathbf{y}_k$ be k not necessarily distinct zeros and $\mathbf{z}_1, \ldots, \mathbf{z}_k$ be k not necessarily distinct ones of $f^{(x_1,x_{n+1})}$ such that

$$\mathbf{y}_1 + \cdots + \mathbf{y}_k = \mathbf{z}_1 + \cdots + \mathbf{z}_k = (a_1, \ldots, a_{n+1}) \tag{3}$$

for some non-negative integers $a_1, \ldots, a_{n+1}$. Since for any $\alpha_2, \ldots, \alpha_n \in \{0, 1\}$,

$$f^{(x_1,x_{n+1})}(0, \alpha_2, \ldots, \alpha_n, 0) = 0$$

and

$$f^{(x_1,x_{n+1})}(1, \alpha_2, \ldots, \alpha_n, 1) = 1$$

we conclude that for every $i \in [k]$ at least one of $(\mathbf{y}_i)_1$ and $(\mathbf{y}_i)_{n+1}$ is equal to 0, and at least one of $(\mathbf{z}_i)_1$ and $(\mathbf{z}_i)_{n+1}$ is equal to 1. Therefore,

$$k \geq \sum_{i=1}^{k} \big((\mathbf{y}_i)_1 + (\mathbf{y}_i)_{n+1}\big) = a_1 + a_{n+1} = \sum_{i=1}^{k} \big((\mathbf{z}_i)_1 + (\mathbf{z}_i)_{n+1}\big) \geq k,$$

and hence $(\mathbf{y}_i)_1 = \overline{(\mathbf{y}_i)_{n+1}}$ and $(\mathbf{z}_i)_1 = \overline{(\mathbf{z}_i)_{n+1}}$ for every $i \in [k]$. Consequently, by Lemma 3, for every $i \in [k]$,

$$f((\mathbf{y}_i)_1, \ldots, (\mathbf{y}_i)_n) = f^{(x_1,x_{n+1})}(\mathbf{y}_i)$$

and

$$f((\mathbf{z}_i)_1, \dots, (\mathbf{z}_i)_n) = f^{(x_1,x_{n+1})}(\mathbf{z}_i),$$

which together with equation (3) imply that f is k-summable, a contradiction. □

In the rest of the section we show that the operation of extension on a variable increases specification number by at most one. For this we first establish two auxiliary claims.

Claim 1. *Let $f(x_1, \dots, x_n)$ be a positive threshold function and let $\mathbf{x}$ be a zero of f. Then $\mathbf{x}$ is an* inessential *zero of f if and only if for some positive m and $k \geq m$ there exist k not necessarily distinct zeros $\mathbf{z}_1, \dots, \mathbf{z}_k$ and $k - m$ not necessarily distinct ones $\mathbf{z}_{k+1}, \dots, \mathbf{z}_{2k-m}$ of f such that $\mathbf{x} \notin \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ and*

$$\mathbf{z}_1 + \cdots + \mathbf{z}_k = \mathbf{z}_{k+1} + \cdots + \mathbf{z}_{2k-m} + m \cdot \mathbf{x}. \tag{4}$$

Proof. Denote by g the Boolean function equal to f in all points except $\mathbf{x}$. First, if equation (4) holds for some k zeros and $k - m$ ones of f, then g is k-summable, and hence $\mathbf{x}$ is inessential for f.

To prove the claim in the other direction, assume that $\mathbf{x}$ is an inessential point of f, then g is k-summable for some $k \geq 2$, and therefore there exist not necessarily distinct zeros $\mathbf{z}_1, \dots \mathbf{z}_k$ and not necessarily distinct ones $\mathbf{z}_{k+1}, \dots, \mathbf{z}_{2k}$ of g such that

$$\mathbf{z}_1 + \cdots + \mathbf{z}_k = \mathbf{z}_{k+1} + \cdots + \mathbf{z}_{2k}. \tag{5}$$

Since point $\mathbf{x}$ is a one of g, it does not belong to $\{\mathbf{z}_1, \dots, \mathbf{z}_k\}$. Furthermore, $\mathbf{x}$ belongs to $\{\mathbf{z}_{k+1}, \dots, \mathbf{z}_{2k}\}$, as otherwise f would be k-summable by (5). Therefore, equation (5) can be rewritten in the form of (4) for some $m \leq k$. □

The following claim is an analog of Claim 1 for inessential ones and can be proved similarly.

Claim 2. *Let $f(x_1, \dots, x_n)$ be a positive threshold function and let $\mathbf{x}$ be a one of f. Then $\mathbf{x}$ is an* inessential *one of f if and only if for some positive m and $k \geq m$ there exist k not necessarily distinct ones $\mathbf{z}_1, \dots, \mathbf{z}_k$ and $k - m$ not necessarily distinct zeros $\mathbf{z}_{k+1}, \dots, \mathbf{z}_{2k-m}$ of f such that $\mathbf{x} \notin \{\mathbf{z}_1, \dots, \mathbf{z}_k\}$ and*

$$\mathbf{z}_1 + \cdots + \mathbf{z}_k = \mathbf{z}_{k+1} + \cdots + \mathbf{z}_{2k-m} + m \cdot \mathbf{x}.$$

We are now ready to prove the main statement.

Theorem 6. *Let $n \geq 2$ and let $f = f(x_1, \dots, x_n)$ be a positive function from $\mathcal{T}_n$. Then the extension of f on a variable belongs to $\mathcal{T}_{n+1}$.*

Proof. Without loss of generality, we prove the statement for the (x_1, x_{n+1})-extension $f^{(x_1,x_{n+1})}$. Denote $f_0 = f_{|x_1=0}$ and $f_1 = f_{|x_1=1}$. Then

$$f^{(x_1,x_{n+1})}(x_1, \dots, x_n, x_{n+1}) = x_1(x_{n+1} \vee f_1) \vee x_{n+1} f_0.$$

By Lemma 4, the function $f^{(x_1,x_{n+1})}$ is threshold, so we only need to show that its specification number is $n + 2$.

Assume first that at least one of f_0 and f_1 is a constant function. Notice that neither $f_1 \equiv 0$ nor $f_0 \equiv 1$, as otherwise f would be a constant function, contradicting the assumption that $f \in \mathcal{T}_n$. Suppose now that $f_1 \equiv 1$. Then $f = x_1 \vee f_0$. Since f depends on all its variables, f_0 also depends on all its variables, and therefore from Lemma 2 we conclude $\sigma(f_0) = n$. Moreover, $f^{(x_1,x_{n+1})} = x_1 \vee x_{n+1} f_0$, and hence, again by Lemma 2, specification number of $f^{(x_1,x_{n+1})}$ is $\sigma(f_0) + 2 = n + 2$. The case $f_0 \equiv 0$ can be treated in a similar way.

Assume now that both f_0 and f_1 are non-constant functions. We show next that if $\mathbf{y}$ is an inessential zero (respectively an inessential one) of f, then $\mathbf{y}' = ((\mathbf{y})_1, \dots, (\mathbf{y})_n, \overline{(\mathbf{y})_1})$ is an inessential zero (respectively an inessential one) of $f^{(x_1,x_{n+1})}$. We provide the arguments only for the case when $\mathbf{y}$ is an inessential zero, as the case of an inessential one is proved similarly. By Lemma 3, we have $f^{(x_1,x_{n+1})}(\mathbf{y}') = f(\mathbf{y}) = 0$. Since $\mathbf{y}$ is an inessential zero of f, by Claim 1, for some m and k $(0 < m \leq k)$ there exist k not necessarily distinct zeros $\mathbf{z}_1, \dots, \mathbf{z}_k$ and $k - m$ not necessarily distinct ones $\mathbf{z}_{k+1}, \dots, \mathbf{z}_{2k-m}$ of f such that

$$\mathbf{z}_1 + \cdots + \mathbf{z}_k = \mathbf{z}_{k+1} + \cdots + \mathbf{z}_{2k-m} + m \cdot \mathbf{y}.$$

Let $\mathbf{z}_1 + \cdots + \mathbf{z}_k = (a_1, \dots, a_n)$ and let $\mathbf{z}'_i = ((\mathbf{z}_i)_1, \dots, (\mathbf{z}_i)_n, \overline{(\mathbf{z}_i)_1})$ for $i \in [2k - m]$. Then

$$\mathbf{z}'_1 + \cdots + \mathbf{z}'_k = \mathbf{z}'_{k+1} + \cdots + \mathbf{z}'_{2k-m} + m \cdot \mathbf{y}' = (a_1, \dots, a_n, k - a_1).$$

Furthermore, by Lemma 3, we have $f^{(x_1,x_{n+1})}(\mathbf{z}'_i) = f(\mathbf{z}_i)$ for each $i \in [2k - m]$. Therefore, by Claim 1, $\mathbf{y}'$ is an inessential zero of $f^{(x_1,x_{n+1})}$, as desired.

By the above discussion, we conclude that the number of essential points of $f^{(x_1,x_{n+1})}$ with distinct first and last coordinates does not exceed the number of essential points of f, i.e., $n+1$. To complete the proof we will show that $f^{(x_1,x_{n+1})}$ has at most one essential point with the same first and last coordinates. We start noticing that $f^{(x_1,x_{n+1})}$ depends on all its variables and hence, by Corollary 1, only extremal points of f can be essential. We proceed by showing that only two points with the same first and last coordinates are extremal points of $f^{(x_1,x_{n+1})}$, namely, $(0,1,\dots,1,0)$ is a maximal zero and $(1,0,\dots,0,1)$ is a minimal one of $f^{(x_1,x_{n+1})}$. Indeed, the point $(0,1,\dots,1,0)$ is a maximal zero of $f^{(x_1,x_{n+1})}$ because $f^{(x_1,x_{n+1})}(0,1,\dots,1,0)=0$ and

$$f^{(x_1,x_{n+1})}(1,1,\dots,1,0)=f_1(1,\dots,1)=1,$$
$$f^{(x_1,x_{n+1})}(0,1,\dots,1,1)=f_0(1,\dots,1)=1.$$

Both equations hold as f_0 and f_1 are non-constant positive functions. The extremality of $(1,0,\dots,0,1)$ is proved similarly and we omit the details. All other points with the same first and last coordinates are either below the maximal zero $(0,1,\dots,1,0)$ or above the minimal one $(1,0,\dots,0,1)$, and therefore they are not extremal points.

It remains to show that one of the points $(1,0,\dots,0,1)$ and $(0,1,\dots,1,0)$ is inessential. We observe that $f^{(x_1,x_{n+1})}$ is not self-dual. Indeed, otherwise Lemma 3 would imply that f is self-dual, which, by Lemma 1, would contradict the assumption that f has $n+1$ essential points. Now since $f^{(x_1,x_{n+1})}$ is not self-dual, there exists a point $\mathbf{x}$ such that $f^{(x_1,x_{n+1})}(\mathbf{x}) = f^{(x_1,x_{n+1})}(\overline{\mathbf{x}})$. Therefore, combining the equation

$$\mathbf{x}+\overline{\mathbf{x}}=(0,1,\dots,1,0)+(1,0,\dots,0,1)=(1,\dots,1)$$

with Claims 1 and 2, we conclude that regardless of the value of $f^{(x_1,x_{n+1})}$ in the point $\mathbf{x}$, at least one of the points $(1,0,\dots,0,1)$ and $(0,1,\dots,1,0)$ is inessential for $f^{(x_1,x_{n+1})}$. Consequently, $f^{(x_1,x_{n+1})}$ has at most (and, by Theorem 3, at least) $n+2$ essential points and hence belongs to $\mathcal{T}_{n+1}$, as claimed. □

Example 1. The function $f_{n,k}$ from Theorem 5 is obtained from the linear read-once function

$$f(x_1,x_2,x_{k+1},\dots,x_n)=x_2(x_1\vee x_{k+1}x_{k+2}\dots x_n)$$

by applying $k-2$ times the operation of extension on the variable x_1:

$$\begin{aligned}
(x_2(x_1\vee x_{k+1}x_{k+2}\dots x_n))^{(x_1,x_3)} &= x_1(x_2\vee x_3)\vee x_2x_3x_{k+1}\dots x_n,\\
(x_1(x_2\vee x_3)\vee x_2x_3x_{k+1}\dots x_n)^{(x_1,x_4)} &= x_1(x_2\vee x_3\vee x_4)\vee x_2x_3x_4x_{k+1}\dots x_n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
(x_1(x_2\vee x_3\vee\dots\vee x_{k-1})\vee x_2\dots x_{k-1}x_{k+1}\dots x_n)^{(x_1,x_k)} &\\
&= x_1(x_2\vee\dots\vee x_k)\vee x_2x_3\dots x_n=f_{n,k}.
\end{aligned}$$

5. Symmetric variables extension

Inspired by the relation between the operations of adding a variable and extension on a variable, in this section, we develop another operation that can be used to inductively construct new threshold functions with minimum specification number from functions with symmetric variables.

Definition 3. Let $f(x_1,\dots,x_n)$ be a positive Boolean function, let x_i and x_j be its symmetric variables, and let y be a new variable. The (x_i,x_j,y)-*s-extension* (stands for *symmetric variables extension*) of f is denoted $f^{(x_i,x_j,y)}(x_1,\dots,x_n,y)$ and defined as follows:

$$f^{(x_i,x_j,y)}(x_1,\dots,x_n,y)=x_ix_jyf_{|x_i=1,x_j=1}\vee(x_i\vee x_j\vee y)f_{|x_i=1,x_j=0}\vee f_{|x_i=0,x_j=0}.$$

We say that $f^{(x_i,x_j,y)}$ is *obtained from f by the operation of symmetric variables extension* on the variables x_i and x_j.

It is easy to see that if f is a positive function then its symmetric variables extension on any pair of variables is positive. We also observe that the variables x_i,x_j,y are pairwise symmetric in the function $f^{(x_i,x_j,y)}$ and illustrate the new operation with restriction graphs (see 2). The $\{x_i,x_j\}$-restriction graph G_1 for f is composed of vertices of three different types: $f_0=f_{|x_i=0,x_j=0}$, $f_{11}=f_{|x_i=1,x_j=1}$, and $f_{10}=f_{|x_i=1,x_j=0}=f_{|x_i=0,x_j=1}$ (the latter equality is due to the symmetry of x_i and x_j). The same holds for the $\{x_i,x_j,y\}$-restriction graph G_2 for $f^{(x_i,x_j,y)}$. Indeed, it follows from the definition of $f^{(x_i,x_j,y)}$ (and positivity of f) that $f^{(x_i,x_j,y)}_{|x_i=0,x_j=0,y=0}=f_{00}$, $f^{(x_i,x_j,y)}_{|x_i=1,x_j=1,y=1}=f_{11}$,

$$f^{(x_i,x_j,y)}_{|x_i=1,x_j=0,y=0}=f^{(x_i,x_j,y)}_{|x_i=0,x_j=1,y=0}=f^{(x_i,x_j,y)}_{|x_i=0,x_j=0,y=1}=f_{10},$$

and

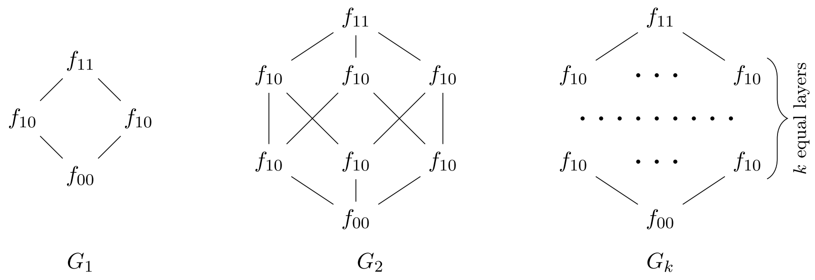


Fig. 2. The graph G_1 is the $\{x_i, x_j\}$-restriction graph for a function f with symmetric variables x_i and x_j, where $f_{00} = f_{|x_i=0,x_j=0}$, $f_{10} = f_{|x_i=1,x_j=0} = f_{|x_i=0,x_j=1}$, $f_{11} = f_{|x_i=1,x_j=1}$. The graph G_2 is the $\{x_i, x_j, y\}$-restriction graph for the (x_i, x_j, y)-s-extension of f. The graph G_k is the $\{x_i, x_j, y_1, \dots, y_k\}$-restriction graph for the function obtained from f by applying k time the symmetric variables extension operations, where $y_1, \dots, y_k$ are the new variables.

$$f^{(x_i,x_j,y)}_{|x_i=1,x_j=1,y=0} = f^{(x_i,x_j,y)}_{|x_i=1,x_j=0,y=1} = f^{(x_i,x_j,y)}_{|x_i=0,x_j=1,y=1} = f_{10}.$$

Furthermore, if we apply the symmetric variables extension operation k times on the same pair of variables x_i, x_j, the $\{x_i, x_j, y_1, \dots, y_k\}$-restriction graph for the resulting function will also be composed of the same three functions: the functions f_{11} and f_{00} will be in the top and the bottom layers of the graph respectively and all k internal "layers" will be the same and consist of vertices corresponding to the function f_{10}.

As the following example demonstrates, in contrast to the operation of extension on a variable, the operation of symmetric variables extension does not necessarily leave the function in the class of threshold functions.

Example 2. Consider function $f(x_1, \dots, x_5) = x_1x_2 \vee (x_1 \vee x_2)x_3x_4x_5$. This function is threshold as witnessed by the following threshold inequality

$$\frac{x_1}{2} + \frac{x_2}{2} + \frac{x_3}{6} + \frac{x_4}{6} + \frac{x_5}{6} < 1.$$

The (x_1, x_2, x_6)-s-extension of f is

$$f^{(x_1,x_2,x_6)} = x_1x_2x_6 \vee (x_1 \vee x_2 \vee x_6)x_3x_4x_5.$$

The function $f^{(x_1,x_2,x_6)}$ is 2-summable as

$$f^{(x_1,x_2,x_6)}(1,1,0,0,0,1) = f^{(x_1,x_2,x_6)}(0,1,1,1,1,0) = 1,$$
$$f^{(x_1,x_2,x_6)}(0,1,1,0,1,1) = f^{(x_1,x_2,x_6)}(1,1,0,1,0,0) = 0,$$

and

$$(1,1,0,0,0,1) + (0,1,1,1,1,0) = (0,1,1,0,1,1) + (1,1,0,1,0,0).$$

Hence $f^{(x_1,x_2,x_6)}$ is not threshold.

Although the operation of symmetric variables extension does not always preserve the property of being threshold, when it does and when it is applied to a function in $\mathcal{T}_n$, it increases specification number by exactly one, i.e., the resulting function belongs to $\mathcal{T}_{n+1}$. To prove this fact, we first provide several auxiliary statements.

Claim 3. *Let $f(x_1, \dots, x_n)$ be a positive Boolean function. If there exist k ones $\mathbf{x}_1, \dots, \mathbf{x}_k$ and k zeros $\mathbf{y}_1, \dots, \mathbf{y}_k$ of f such that*

$$(a_1, \dots, a_n) = \mathbf{x}_1 + \dots + \mathbf{x}_k \preccurlyeq \mathbf{y}_1 + \dots + \mathbf{y}_k = (b_1, \dots, b_n),$$

then f is k-summable. If, in addition, $b_i = k$ or $a_i = 0$ for some $i \in [n]$, then $f_{|x_i=1}$ or $f_{|x_i=0}$ is k-summable respectively.

Proof. To prove the first part of the statement, we first observe that if $(a_1, \dots, a_n) = (b_1, \dots, b_n)$ then $k \geq 2$ and f is k-summable by definition. Now, if $(a_1, \dots, a_n) \prec (b_1, \dots, b_n)$, by changing some coordinates of the points $\mathbf{x}_1, \dots, \mathbf{x}_k$ from zeros to ones we can obtain k points $\mathbf{x}'_1, \dots, \mathbf{x}'_k$ such that

$$\mathbf{x}'_1 + \dots + \mathbf{x}'_k = (b_1, \dots, b_n).$$

Since f is a positive function and $\mathbf{x}_j \preccurlyeq \mathbf{x}'_j$ for each $j \in [k]$, we have $f(\mathbf{x}'_1) = \dots = f(\mathbf{x}'_k) = 1$, and hence, as before, f is k-summable.

To prove the second part of the statement, assume, without loss of generality, that $b_n = k$ and denote $f_1 = f_{|x_n=1}$. The assumption implies that $(\mathbf{y}_j)_n = 1$ for every $j \in [k]$, and hence $f_1((\mathbf{y}_j)_1, \dots, (\mathbf{y}_j)_{n-1}) = f(\mathbf{y}_j) = 0$. Also, since f is positive, we have $f_1((\mathbf{x}_j)_1, \dots, (\mathbf{x}_j)_{n-1}) = f(\mathbf{x}_j) = 1$ for every $j \in [k]$. Therefore, by the first part of the statement, f_1 is k-summable. The case $a_n = 0$ is treated similarly. □

Claim 4. *Let $f(x_1,\dots,x_n)$ be a positive threshold function and let* $\mathbf{x}$ *be a zero of* f. *If for some positive* m *and* $k \geq m$ *there exist* k *not necessarily distinct zeros* $\mathbf{z}_1,\dots,\mathbf{z}_k$ *and* $k-m$ *not necessarily distinct ones* $\mathbf{z}_{k+1},\dots,\mathbf{z}_{2k-m}$ *of* f *such that* $\mathbf{x} \notin \{\mathbf{z}_1,\dots,\mathbf{z}_k\}$ *and*

$$\mathbf{z}_{k+1}+\cdots+\mathbf{z}_{2k-m}+m\cdot\mathbf{x} \preccurlyeq \mathbf{z}_1+\cdots+\mathbf{z}_k,$$

then $\mathbf{x}$ *is an* inessential *zero of* f.

Proof. Let g be the function that coincides with f in all points except $\mathbf{x}$. By Claim 3 the function g is k-summable, and hence $\mathbf{x}$ is not an essential point of f. □

The following claim is an analog of Claim 4 for inessential ones and can be proved similarly.

Claim 5. *Let $f(x_1,\dots,x_n)$ be a positive threshold function and let* $\mathbf{x}$ *be a one of* f. *If for some positive* m *and* $k \geq m$ *there exist* k *not necessarily distinct ones* $\mathbf{z}_1,\dots,\mathbf{z}_k$ *and* $k-m$ *not necessarily distinct zeros* $\mathbf{z}_{k+1},\dots,\mathbf{z}_{2k-m}$ *of* f *such that* $\mathbf{x} \notin \{\mathbf{z}_1,\dots,\mathbf{z}_k\}$ *and*

$$\mathbf{z}_1+\cdots+\mathbf{z}_k \preccurlyeq \mathbf{z}_{k+1}+\cdots+\mathbf{z}_{2k-m}+m\cdot\mathbf{x},$$

then $\mathbf{x}$ *is an* inessential *one of* f.

Lemma 5. *Let $f(x_1,\dots,x_n) \in \mathcal{T}_n$, and let x_i and x_j be symmetric variables of f. Then the set of essential points of f has exactly* 2 *points with different i-th and j-th coordinates and these points only differ in these two coordinates.*

Proof. Without loss of generality we assume $i=1$ and $j=2$. Notice that since f has exactly $n+1$ essential points, by Theorem 3, they all are in general position. This in particular implies that f has at least one essential point with distinct values in the first and second coordinates.

Let $\mathbf{a}=(\alpha_1,\overline{\alpha_1},\alpha_3,\dots,\alpha_n)$ be an essential point of f for some $\alpha_1,\alpha_3,\dots,\alpha_n \in \{0,1\}$. Due to the symmetry of x_1 and x_2, the point $\mathbf{a}'=(\overline{\alpha_1},\alpha_1,\alpha_3,\dots,\alpha_n)$ is also essential for f. We claim that there are no other essential points of f with distinct first two coordinates. Suppose, towards a contradiction, that there exists an essential point $\mathbf{b}=(\beta_1,\overline{\beta_1},\beta_3,\dots,\beta_n)$ for some $\beta_1,\beta_3,\dots,\beta_n \in \{0,1\}$ such that $\mathbf{b} \notin \{\mathbf{a},\mathbf{a}'\}$. As before, the point $\mathbf{b}'=(\overline{\beta_1},\beta_1,\beta_3,\dots,\beta_n)$ is also an essential point of f. However, depending on the values of α_1 and β_1 we have either $\mathbf{a}+\mathbf{b}'=\mathbf{a}'+\mathbf{b}$ or $\mathbf{a}+\mathbf{b}=\mathbf{a}'+\mathbf{b}'$. Consequently, the points $\mathbf{a},\mathbf{a}',\mathbf{b},\mathbf{b}'$ are not in general position, a contradiction. □

For convenience, in the lemma below we will use the following notation. For a Boolean vector $\mathbf{a}=(\alpha_1,\dots,\alpha_m)$ and a set of Boolean numbers $\beta_1,\dots,\beta_n \in \{0,1\}$ we will denote by $(\mathbf{a},\beta_1,\dots,\beta_n)$ the $(m+n)$-dimensional vector $(\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n)$.

Lemma 6. *Let $f(x_1,\dots,x_n)$ be a positive threshold function with symmetric variables x_i and x_j, $i,j \in [n]$, and let f' be its (x_i,x_j,x_{n+1})-s-extension. If f' is threshold, then for any inessential point $\mathbf{a}=(\alpha_1,\dots,\alpha_n)$ of f and any $\alpha_{n+1} \in \{0,1\}$, the point $\mathbf{a}'=(\mathbf{a},\alpha_{n+1})$ is inessential for f' whenever $f(\mathbf{a})=f'(\mathbf{a}')$ and $(\alpha_i,\alpha_j,\alpha_{n+1}) \in \{(0,0,0),(1,1,1),(1,0,0),(0,1,0),(1,0,1),(0,1,1)\}$.*

Proof. Assume, without loss of generality, that $i=n-1$ and $j=n$. Let $\mathbf{a}=(\alpha_1,\dots,\alpha_n)$ be an inessential point of f and let $a_{n+1} \in \{0,1\}$ be such that $f(\mathbf{a})=f'(\mathbf{a}')$ and

$$(\alpha_{n-1},\alpha_n,\alpha_{n+1}) \in \{(0,0,0),(1,1,1),(1,0,0),(0,1,0),(1,0,1),(0,1,1)\}.$$

Suppose first that $f(\mathbf{a})=f'(\mathbf{a}')=0$. Since $\mathbf{a}$ is inessential for f, by Claim 1, for some positive m and $k \geq m$ there exist $k-m$ ones $\mathbf{x}_1,\dots,\mathbf{x}_{k-m}$ and k zeros $\mathbf{y}_1,\dots,\mathbf{y}_k$ of f such that

$$\mathbf{x}_1+\cdots+\mathbf{x}_{k-m}+m\cdot\mathbf{a}=\mathbf{y}_1+\cdots+\mathbf{y}_k=(b_1,\dots,b_n),$$

where $b_1,\dots,b_n \in [0,k]$. Denote the following sets of points

$$\begin{aligned}
&X=\{\mathbf{x}_1,\dots,\mathbf{x}_{k-m}\},\\
&Y=\{\mathbf{y}_1,\dots,\mathbf{y}_k\},\\
&X_{11}=\{(x_1\dots,x_n)\in X \mid x_{n-1}=x_n=1\},\\
&X_{00}=\{(x_1\dots,x_n)\in X \mid x_{n-1}=x_n=0\},\\
&X_{10}=\{(x_1\dots,x_n)\in X \mid x_{n-1}\neq x_n\},\\
&Y_{11}=\{(x_1\dots,x_n)\in Y \mid x_{n-1}=x_n=1\},\\
&Y_{00}=\{(x_1\dots,x_n)\in Y \mid x_{n-1}=x_n=0\},\\
&Y_{10}=\{(x_1\dots,x_n)\in Y \mid x_{n-1}\neq x_n\}.
\end{aligned}$$

Without loss of generality we assume

$$\begin{aligned}X_{00} &= \{\mathbf{x}_1, \dots, \mathbf{x}_{|X_{00}|}\},\\ X_{10} &= \{\mathbf{x}_{|X_{00}|+1}, \dots, \mathbf{x}_{|X_{00}|+|X_{10}|}\},\\ X_{11} &= \{\mathbf{x}_{|X_{00}|+|X_{10}|+1}, \dots, \mathbf{x}_{k-m}\},\\ Y_{00} &= \{\mathbf{y}_1, \dots, \mathbf{y}_{|Y_{00}|}\},\\ Y_{10} &= \{\mathbf{y}_{|Y_{00}|+1}, \dots, \mathbf{y}_{|Y_{00}|+|Y_{10}|}\},\\ Y_{11} &= \{\mathbf{y}_{|Y_{00}|+|Y_{10}|+1}, \dots, \mathbf{y}_k\}.\end{aligned}$$

Since

$$f'(x_1, \dots, x_{n-2}, 1, 1, 1) = f(x_1, \dots, x_{n-2}, 1, 1),$$

by definition of symmetric variables extension and positivity of f, we have

$$f'(\mathbf{x}, 1) = 1 \text{ for all } \mathbf{x} \in X_{11}. \tag{6}$$

Similarly, we obtain

$$\begin{aligned}&f'(\mathbf{x}, 0) = 1 \text{ for all } \mathbf{x} \in X_{00} \cup X_{10},\\ &f'(\mathbf{y}, 0) = 0 \text{ for all } \mathbf{y} \in Y_{00},\\ &f'(\mathbf{y}, 1) = 0 \text{ for all } \mathbf{y} \in Y_{10} \cup Y_{11}.\end{aligned} \tag{7}$$

Using these $k-m$ ones and k zeros of f' we will prove that $\mathbf{a}'$ is an inessential point of f'. First, observe that

$$\begin{aligned}(\mathbf{x}_1, 0) + \dots + (\mathbf{x}_{|X_{00}|+|X_{10}|}, 0) + (\mathbf{x}_{|X_{00}|+|X_{10}|+1}, 1) + \dots + (\mathbf{x}_{k-m}, 1) + m \cdot \mathbf{a}'\\ = (b_1, \dots, b_n, |X_{11}| + \alpha_{n+1} m)\end{aligned}$$

and

$$(\mathbf{y}_1, 0) + \dots + (\mathbf{y}_{|Y_{00}|}, 0) + (\mathbf{y}_{|Y_{00}|+1}, 1) + \dots + (\mathbf{y}_k, 1) = (b_1, \dots, b_n, |Y_{10}| + |Y_{11}|).$$

Hence, the desired conclusion will follow from Claim 4 if $|X_{11}| + \alpha_{n+1} m \le |Y_{10}| + |Y_{11}|$. Notice that since all points in Y_{00} have zero in the $(n-1)$-th and n-th coordinates, we have $|Y_{10}| + |Y_{11}| = |Y| - |Y_{00}| \ge \max(b_{n-1}, b_n)$. Therefore, it suffices to show that

$$|X_{11}| + \alpha_{n+1} m \le \max(b_{n-1}, b_n). \tag{8}$$

To this end we consider three cases:

1. $\alpha_{n+1} = 0$, i.e., $(\alpha_{n-1}, \alpha_n, \alpha_{n+1}) \in \{(0, 0, 0), (1, 0, 0), (0, 1, 0)\}$. Since X_{11} is the set of points where both the $(n-1)$-th and n-th coordinates are ones, we have $|X_{11}| + \alpha_{n+1} m = |X_{11}| \le \min(b_{n-1}, b_n) \le \max(b_{n-1}, b_n)$.
2. $\alpha_n = \alpha_{n+1} = 1$, i.e., $(\alpha_{n-1}, \alpha_n, \alpha_{n+1}) \in \{(0, 1, 1), (1, 1, 1)\}$. Since the n-th coordinate of every point in $X_{11} \cup \{\mathbf{a}\}$ is a one, we conclude $|X_{11}| + \alpha_{n+1} m = |X_{11}| + m \le b_n \le \max(b_{n-1}, b_n)$.
3. $(\alpha_{n-1}, \alpha_n, \alpha_{n+1}) = (1, 0, 1)$. Since the $(n-1)$-th coordinate of every point in $X_{11} \cup \{\mathbf{a}\}$ is a one, we have $|X_{11}| + \alpha_{n+1} m = |X_{11}| + m \le b_{n-1} \le \max(b_{n-1}, b_n)$.

Inequality (8) holds in all three cases, thus, by Claim 4, the point $\mathbf{a}'$ is inessential for f'.

Suppose now that $f(\mathbf{a}) = f'(\mathbf{a}') = 1$. By Claim 2, for some positive m and $k \ge m$ there exist k ones $\mathbf{x}_1, \dots, \mathbf{x}_k$ and $k-m$ zeros $\mathbf{y}_1, \dots, \mathbf{y}_{k-m}$ of f such that

$$\mathbf{x}_1 + \dots + \mathbf{x}_k = \mathbf{y}_1 + \dots + \mathbf{y}_{k-m} + m \cdot \mathbf{a} = (b_1, \dots, b_n),$$

where $b_1, \dots, b_n \in [0, k]$. We denote $X = \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ and $Y = \{\mathbf{y}_1, \dots, \mathbf{y}_{k-m}\}$ and define the sets X_{00}, X_{10}, X_{11}, Y_{00}, Y_{10}, and Y_{11} as before. Also, without loss of generality, we assume

$$\begin{aligned}X_{00} &= \{\mathbf{x}_1, \dots, \mathbf{x}_{|X_{00}|}\},\\ X_{10} &= \{\mathbf{x}_{|X_{00}|+1}, \dots, \mathbf{x}_{|X_{00}|+|X_{10}|}\},\\ X_{11} &= \{\mathbf{x}_{|X_{00}|+|X_{10}|+1}, \dots, \mathbf{x}_k\},\\ Y_{00} &= \{\mathbf{y}_1, \dots, \mathbf{y}_{|Y_{00}|}\},\\ Y_{10} &= \{\mathbf{y}_{|Y_{00}|+1}, \dots, \mathbf{y}_{|Y_{00}|+|Y_{10}|}\},\\ Y_{11} &= \{\mathbf{y}_{|Y_{00}|+|Y_{10}|+1}, \dots, \mathbf{y}_{k-m}\}.\end{aligned}$$

Notice that equations (6) and (7) also hold for these sets, and therefore, similarly to the previous case, we have

$$(\mathbf{x}_1, 0) + \dots + (\mathbf{x}_{|X_{00}|+|X_{10}|}, 0) + (\mathbf{x}_{|X_{00}|+|X_{10}|+1}, 1) + \dots + (\mathbf{x}_k, 1) = (b_1, \dots, b_n, |X_{11}|)$$

and

$$(\mathbf{y}_1, 0) + \cdots + (\mathbf{y}_{|Y_{00}|}, 0) + (\mathbf{y}_{|Y_{00}|+1}, 1) + \cdots + (\mathbf{y}_{k-m}, 1) + m \cdot \mathbf{a}' = (b_1, \ldots, b_n, |Y_{10}| + |Y_{11}| + \alpha_{n+1} m).$$

We will show that $|X_{11}| \le |Y_{10}| + |Y_{11}| + \alpha_{n+1} m$, which together with Claim 5 will imply that $\mathbf{a}'$ is an inessential point for f', as desired. Since $|X_{11}| \le \min(b_{n-1}, b_n)$, it is enough to show that

$$\min(b_{n-1}, b_n) \le |Y_{10}| + |Y_{11}| + \alpha_{n+1} m. \tag{9}$$

To this end we consider two cases:

1. At least one of α_{n-1} and α_n is a zero, i.e., $(\alpha_{n-1}, \alpha_n, \alpha_{n+1}) \in \{(0,0,0), (1,0,0), (0,1,0), (0,1,1), (1,0,1)\}$. Without loss of generality, assume $\alpha_n = 0$. Then $\min(b_{n-1}, b_n) \le b_n \le |Y_{10}| + |Y_{11}| \le |Y_{10}| + |Y_{11}| + \alpha_{n+1} m$.
2. $(\alpha_{n-1}, \alpha_n, \alpha_{n+1}) = (1,1,1)$. Then $\min(b_{n-1}, b_n) \le |Y_{10}| + |Y_{11}| + m = |Y_{10}| + |Y_{11}| + \alpha_{n+1} m$. □

We are now in a position to prove the main result of the section.

Theorem 7. *Let $f(x_1, \ldots, x_n)$ be a positive function in $\mathcal{T}_n$ and let x_i and x_j be symmetric variables of f. Let f' be the (x_i, x_j, y)-s-extension of f. If f' is threshold, then it belongs to $\mathcal{T}_{n+1}$.*

Proof. Without loss of generality, we assume $i = n-1$, $j = n$. Let S be the set of essential points of f and let $S_{\alpha\beta} = \{(x_1, \ldots, x_n) \in S \mid x_{n-1} = \alpha, x_n = \beta\}$. Lemma 5 implies that each of the sets S_{01} and S_{10} consists of a single point, and hence

$$|S_{00}| + |S_{11}| = n + 1 - |S_{01}| - |S_{10}| = n - 1.$$

From this equation and Lemma 6 it follows that the set of essential points of f' has at most $n-1$ points with equal $(n-1)$-th, n-th and $(n+1)$-th coordinates.

We now turn to the points in which the last three coordinates are not all the same. Let $(\alpha_1, \ldots, \alpha_{n-2}, 0, 1)$ denote the unique point in S_{01}. Then, by Lemma 5, we have $S_{10} = \{(\alpha_1, \ldots, \alpha_{n-2}, 1, 0)\}$. It follows from Lemma 6 and symmetry of the variables x_{n-1} and x_n that the only points with non-equal $(n-1)$-th and n-th coordinates which can be essential for f' are

$$(\alpha_1, \ldots, \alpha_{n-2}, 1, 0, 0),$$
$$(\alpha_1, \ldots, \alpha_{n-2}, 0, 1, 0),$$
$$(\alpha_1, \ldots, \alpha_{n-2}, 1, 0, 1),$$
$$(\alpha_1, \ldots, \alpha_{n-2}, 0, 1, 1).$$

Taking into account the symmetry between the variables x_n and x_{n+1} we conclude that the only points with non-equal last three coordinates which can be essential for f' are four points above and the points

$$(\alpha_1, \ldots, \alpha_{n-2}, 0, 0, 1),$$
$$(\alpha_1, \ldots, \alpha_{n-2}, 1, 1, 0).$$

We claim that only three of the six above points can be extremal for f', and therefore essential. Indeed, since f' has the same value in all of them, the first three points cannot be maximal zeros and the last three points cannot be minimal ones. Hence, at most three of them can be extremal.

All in all, f' has at most $n-1$ essential points with equal $(n-1)$-th, n-th and $(n+1)$-th coordinates and at most three other essential points, resulting in at most $n+2$ essential points in total, and therefore $f' \in \mathcal{T}_{n+1}$. □

By definition, the operation of symmetric variables extension is only applicable to functions with symmetric variables. However, we believe that among threshold functions with minimum number of essential points this property is not rare. We support our intuition by the following observations. First, it is easy to see, that after applying the conjunction operation to a Boolean function twice, the new variables in the resulting function are symmetric. Obviously, the same also holds for the disjunction operation. In fact, according to the following easily verifiable observation, almost all positive linear read-once functions have symmetric variables.

Observation 1. *Let $f(x_1, \ldots, x_n)$ be a positive linear read-once function without symmetric variables. Then f is either a constant or a single variable.*

Second, the operation of extension of a function applied twice on the same variable, produces a function with two new variables that are symmetric:

$$\left(f(x_1, \ldots, x_n)^{(x_i, y)}\right)^{(x_i, z)} = x_i(y \vee z \vee f_{|x_i=1}) \vee yz f_{|x_i=0}.$$

6. Characterization of the functions in $\mathcal{T}_n$ with at most six variables

In this section we characterize functions in $\mathcal{T}_n$ for $n \le 6$. We start by noting that without loss of generality we can restrict our consideration to non-canalyzing functions. Indeed, it is easy to see that a canalyzing function f is of the form $x \vee g$, $x \wedge g$, $\bar{x} \vee g$ or $\bar{x} \wedge g$ for some variable x and function g that does not depend on x. Furthermore, g is threshold whenever f is. Hence, it follows from Lemma 2 that a canalyzing function in $\mathcal{T}_n$ is obtained from a function in $\mathcal{T}_{n-1}$ using the operation of adding a variable.

Claim 6. *All functions in $\mathcal{T}_n$ for $n \le 3$ are linear read-once.*

Proof. It was proved in [7] that a non-lro threshold function contains as a restriction a function congruent to $x_1x_2 \vee x_1x_3 \vee \dots \vee x_1x_n \vee x_2 \dots x_n$ for some $n \ge 3$. Since $n \ge 3$ and for $n = 3$ the function does not belong to $\mathcal{T}_n$, this implies the result. □

The following two claims were obtained by a computer-aided enumeration of all functions in $\mathcal{T}_n$ for the corresponding values of n.

Claim 7. *Any non-canalyzing function in $\mathcal{T}_4$ is congruent to $f_{4,3}(x_1, x_2, x_3, x_4) = x_1(x_2 \vee x_3) \vee x_2x_3x_4$ or to its dual function.*

We observe that the function from Claim 7 is congruent to the (x_1, x_4)-extension of the linear read-once function $f(x_1, x_2, x_3) = (x_1 \vee x_2)x_3$.

Claim 8. *Any non-canalyzing function in $\mathcal{T}_5$ is congruent to one of the following seven functions (or their duals), each of which can be obtained from a function in $\mathcal{T}_4$ via the operation of extension on a variable.*

1. $f_{5,4} = x_1(\mathbf{x_2} \vee x_3 \vee x_4) \vee \mathbf{x_2}x_3x_4x_5 = f_{4,2}^{(x_1,x_2)}(x_1, x_3, x_4, x_5)$,
2. $f_{5,3} = x_1(\mathbf{x_2} \vee x_3) \vee \mathbf{x_2}x_3x_4x_5 = (x_3(x_1 \vee x_4x_5))^{(x_1,x_2)}$,
3. $x_1(\mathbf{x_2} \vee x_3 \vee x_4x_5) \vee \mathbf{x_2}x_3x_4 = f_{4,3}^{(x_1,x_2)}(x_3, x_1, x_4, x_5)$,
4. $x_1(\mathbf{x_2} \vee x_3 \vee x_4x_5) \vee \mathbf{x_2}x_3 = (x_1x_4x_5 \vee x_3)^{(x_1,x_2)}$,
5. $x_1(x_2 \vee x_3x_4x_5) \vee x_2x_4x_5 = x_2(\mathbf{x_1} \vee x_4x_5) \vee \mathbf{x_1}x_3x_4x_5 = ((x_2 \vee x_3)x_4x_5)^{(x_2,x_1)}$,
6. $x_1(x_2 \vee x_3x_5) \vee x_2x_5(x_3 \vee x_4) = x_2(\mathbf{x_1} \vee x_5(x_3 \vee x_4)) \vee \mathbf{x_1}x_3x_5 = (x_5(x_2x_4 \vee x_3))^{(x_2,x_1)}$,
7. $x_1(\mathbf{x_2} \vee x_3x_4 \vee x_3x_5 \vee x_4x_5) \vee \mathbf{x_2}x_3(x_4 \vee x_5) = f_{4,3}^{(x_1,x_2)}(x_3, x_4, x_5, x_1)$.

Claims 6, 7, and 8 imply the following

Lemma 7. *The classes $\mathcal{T}_n, n \le 5$ can be defined inductively starting from $\mathcal{T}_1$ and using the operations of adding a variable and extension on a variable.*

A complete description of all functions in $\mathcal{T}_6$ can be found in Appendix. Unfortunately, Lemma 7 cannot be extended to $\mathcal{T}_6$, since this set contains functions that cannot be obtained from any function in $\mathcal{T}_5$ using only the operations of adding a variable, extension on a variable or symmetric variables extension. The following example exhibits such a function, which is not difficult to check:

$$f^*(x_1, \dots, x_6) = x_1(x_2 \vee x_3 \vee x_5 \vee x_6) \vee x_2x_3(x_4 \vee x_5) \vee x_6(x_2 \vee x_3).$$

Nonetheless, the latter function f^* can also be obtained inductively using the following operation

$$\begin{aligned} f_{n,k}(x_1, \dots, x_n) \quad \to \quad & f'(x_1, \dots, x_n, x_{n+1}, x_{n+2}) \\ & = f_{n,k|x_1=1,x_n=0}(x_1 \vee x_{n+2}) \vee f_{n,k|x_1=0,x_n=1}(x_n \vee x_{n+1}) \vee x_1(x_{n+1} \vee x_{n+2}), \end{aligned} \tag{10}$$

which transforms the function $f_{n,k} \in \mathcal{T}_n$ into a function of $n+2$ variables. Indeed, by applying operation (10) to the function $f_{4,3}$ from Claim 7, we obtain f^*. We conjecture that operation (10) applied to functions $f_{n,k}$ always preserves the property of being threshold with minimum specification number and leaves the proof of this conjecture as an open problem.

7. Concluding remarks and open problems

In the present paper, we have introduced a number of operations that allow us to construct inductively Boolean threshold functions with minimum specification number. Although these operations and most of the related statements were formulated for positive threshold functions, they can be extended to the class of all threshold functions as each threshold function is either positive or congruent to some positive threshold function and congruent functions have the same specification number. The provided operations produce all functions with at most 5 variables and a variety of functions with more variables. Whether the set of all Boolean threshold functions with minimum specification number admits an inductive description by means of finitely many operations remains an open problem, but definitely the set of operations presented in the paper must be extended, as the example of the function f^* in the end of Section 6 shows.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

Any non-canalyzing function in $\mathcal{T}_6$ is congruent to one of the following 70 functions (or their duals).

1. $x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
2. $x_1x_2x_3x_6 \vee x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_1x_3x_5x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_2x_3x_5x_6 \vee x_4x_5x_6$
3. $x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
4. $x_1x_2x_3x_4 \vee x_1x_2x_3x_5 \vee x_1x_5x_6 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_2x_4x_6 \vee x_2x_5x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
5. $x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_1x_3x_5x_6 \vee x_1x_4x_5x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_2x_3x_5x_6 \vee x_2x_4x_5x_6 \vee x_3x_4x_5x_6$
6. $x_1x_2x_3x_4 \vee x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_5x_6 \vee x_2x_4x_6 \vee x_2x_5x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
7. $x_1x_2x_3x_4 \vee x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_2x_5x_6 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
8. $x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_1x_3x_5x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_2x_3x_5x_6 \vee x_4x_5x_6$
9. $x_1x_2x_3x_4 \vee x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
10. $x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_1x_3x_5x_6 \vee x_1x_4x_5x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_2x_3x_5x_6 \vee x_2x_4x_5x_6 \vee x_3x_4x_5x_6$
11. $x_1x_2x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
12. $x_1x_2x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
13. $x_1x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_5x_6$
14. $x_1x_2x_3x_6 \vee x_4x_5 \vee x_4x_6 \vee x_5x_6$
15. $x_1x_2x_4x_5 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
16. $x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
17. $x_1x_2x_3x_4x_5 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
18. $x_1x_2x_3x_6 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
19. $x_1x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
20. $x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
21. $x_1x_2x_3x_4x_5 \vee x_2x_3x_6 \vee x_4x_6 \vee x_5x_6$
22. $x_1x_2x_3x_6 \vee x_2x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
23. $x_1x_2x_3x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_6 \vee x_5x_6$
24. $x_1x_2x_4x_6 \vee x_2x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
25. $x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_5x_6$
26. $x_1x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
27. $x_1x_2x_3x_5 \vee x_3x_6 \vee x_4x_5 \vee x_4x_6 \vee x_5x_6$
28. $x_1x_2x_6 \vee x_3x_4x_5 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
29. $x_1x_2x_4x_5 \vee x_3x_4x_5 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
30. $x_1x_2x_3x_4x_5 \vee x_2x_6 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
31. $x_1x_2x_6 \vee x_2x_3x_4x_5 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
32. $x_1x_2x_3x_5 \vee x_2x_3x_6 \vee x_4x_5 \vee x_4x_6 \vee x_5x_6$
33. $x_1x_3x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_6 \vee x_4x_6 \vee x_5x_6$
34. $x_1x_2x_3x_6 \vee x_2x_4x_5 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
35. $x_1x_2x_4x_5 \vee x_2x_3x_6 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
36. $x_1x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_6 \vee x_5x_6$
37. $x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_2x_3x_6 \vee x_4x_6 \vee x_5x_6$
38. $x_1x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
39. $x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_6 \vee x_5x_6$
40. $x_1x_2x_3x_6 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
41. $x_1x_2x_3x_4x_5 \vee x_2x_5x_6 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
42. $x_1x_2x_5x_6 \vee x_2x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
43. $x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
44. $x_1x_2x_3x_5 \vee x_2x_4x_5 \vee x_3x_4x_5 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
45. $x_1x_2x_3x_4 \vee x_2x_3x_5 \vee x_3x_6 \vee x_4x_5 \vee x_4x_6 \vee x_5x_6$

46. $x_1x_2x_3x_5 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
47. $x_1x_2x_3x_6 \vee x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
48. $x_1x_2x_4x_5 \vee x_2x_3x_6 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
49. $x_1x_5x_6 \vee x_2x_3x_4x_5 \vee x_2x_5x_6 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
50. $x_1x_2x_4x_6 \vee x_2x_5x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
51. $x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_2x_5x_6 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
52. $x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
53. $x_1x_2x_4x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_5x_6$
54. $x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
55. $x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
56. $x_1x_2x_4x_6 \vee x_1x_2x_5x_6 \vee x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
57. $x_1x_2x_5x_6 \vee x_1x_3x_4x_6 \vee x_1x_3x_5x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_2x_3x_5x_6 \vee x_4x_5x_6$
58. $x_1x_2x_4x_5 \vee x_1x_2x_6 \vee x_1x_3x_6 \vee x_2x_3x_6 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
59. $x_1x_2x_4x_5 \vee x_1x_2x_6 \vee x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_3x_6 \vee x_4x_6 \vee x_5x_6$
60. $x_1x_2x_3x_5 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
61. $x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_4x_5 \vee x_2x_4x_5 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
62. $x_1x_2x_3x_5 \vee x_1x_2x_4x_5 \vee x_1x_3x_6 \vee x_2x_3x_6 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
63. $x_1x_2x_3x_4 \vee x_2x_3x_5 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_3x_4x_5 \vee x_4x_6 \vee x_5x_6$
64. $x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_2x_5x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
65. $x_1x_2x_3x_5 \vee x_1x_2x_4x_5 \vee x_1x_4x_6 \vee x_2x_3x_6 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_4x_6 \vee x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
66. $x_1x_2x_4x_5 \vee x_1x_2x_4x_6 \vee x_1x_3x_4x_5 \vee x_2x_3x_4x_5 \vee x_2x_5x_6 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
67. $x_1x_2x_4x_6 \vee x_1x_3x_4x_5 \vee x_1x_3x_4x_6 \vee x_2x_3x_4x_5 \vee x_2x_3x_4x_6 \vee x_2x_5x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
68. $x_1x_2x_3x_4 \vee x_1x_2x_3x_5 \vee x_1x_4x_6 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$
69. $x_1x_2x_3x_5 \vee x_1x_2x_3x_6 \vee x_1x_2x_4x_5 \vee x_2x_4x_6 \vee x_2x_5x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_3x_5x_6 \vee x_4x_5x_6$
70. $x_1x_4x_6 \vee x_2x_3x_4 \vee x_2x_3x_5 \vee x_2x_3x_6 \vee x_2x_4x_5 \vee x_2x_4x_6 \vee x_3x_4x_5 \vee x_3x_4x_6 \vee x_5x_6$

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