Aspects of Equivariant Loop Spaces

by

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Contents

List of Figures

Acknowledgements

The life of a PhD student can be thought of as a journey crossing the sea in a sailing boat. You start motivated then lose motivation; you find it again and lose it again. There are bigger waves, then smaller ones, the weather is better, then worse, the wind starts at 2 on the Beaufort Scale, then reaches 10. However, finally you reach your destination.

There is no expedition that can be made without the help of many people. In the first place I would like to thank my guide on this journey, my supervisor Professor John Greenlees. I would not have been able to "reach my destination" without his continuous support, huge knowledge and intuition, which he was always keen to share with me. I am grateful for being his student, which is an honour and privilege.

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Finally a huge thank you goes to my family. Without the support and encouragement of my Mother, Father and Brother, none of this would have been possible. A final heartfelt thanks goes to my wife Agata not only for her support but for her belief in me on this journey.

Declaration

I declare that:

- this thesis was written by myself;
- that it contains my own work;
- that none of the material have been submitted for other degree or professional qualification at other university;
- that the material contained in Chapter 3 has been submitted for publication;
- and that, except for Chapter 3, none of the material of the thesis is published or has been submitted for publication.

Abstract

The non-equivariant theory of iterated loop spaces has a long and rich history in algebraic topology. Nowadays most of the questions of this theory are resolved. However, the situation is different if we move to equivariant topology - there are still many questions to answer. The aim of this thesis is to contribute to the theory of equivariant loop spaces, especially in the area of their homology.

The main tool used in the description of the homology of iterated loop spaces are homology operations, which are parametrised by the Σ_n-equivariant homology of little discs operads. We study an analogue of this homology in the equivariant homotopy theory, working over cyclic group of order 2. By using universal coefficient spectral sequence we aim to compute Bredon homology of simplest equivariant little discs operads. For this purpose we need to describe the homology of fixed points of these operads. To this end, we will use the connection between these fixed points and configuration spaces of orbits described by Hill in [**27**] and the language of planetary systems, as described by Sinha in [**56**].

The equivariant homology is naturally graded over the ring of representations. In order to make calculations in such theories, we need to know the homology of a point. This is surprisingly non-trivial. Using the method based on the Tate diagram described by Greenlees and May in [**22**], we compute this homology with coefficients in any Mackey functor over the cyclic group of order 2. Due to the properties of the Tate diagram we are able to take the multiplicative structure of this homology into account.

Basing on this computation we provide a framework to define homology operations in algebras over equivariant little discs operads. We define these operations in the case of basic representations over the cyclic group of order 2. This part generalises Wilson's work from [**62**] and [**63**] to the case of non-stable operations.

General introduction

Investigation of a structure of loop spaces has a long and rich history in algebraic topology. Its origins reach 1930's and nowadays many of the questions in the non-equivariant theory of loop spaces are answered. However, the situation is different when we move to equivariant topology - the knowledge of the structure of equivariant loop spaces is rather small. The goal of this thesis is to contribute to the theory of equivariant loop spaces, especially to the investigation of their homology. In this introductory chapter we will discuss the history of the research in the area, both on non-equivariant and equivariant side, and explain how this thesis contributes to the existing research.

Before we start the actual discussion, there is an important remark which needs to be done at the beginning - we denote the cyclic group of order 2 by Q. This notation is not standard. However, it makes the group distinguished by saying that this is the group to study, rather than just a member of a big family of cyclic groups. Also, the author got used to this notation, and since the thesis should reflect the author's work, we allow this small idiosyncrasy in the thesis.

Overview of the history and problematics of the non-equivariant theory of loop spaces

Loop spaces, n-fold loop spaces. Our main object of interest in this subsection are *iterated loop spaces*, defined as spaces of based maps from a sphere to some based topological space. The *loop space* ΩX is a space of based maps from the circle to X, and by *n-fold loop space* $\Omega^n X$ we mean a space of based maps out of the n-dimensional sphere S^n to X. We can also consider *infinite loop spaces*, which have the property of being an n-fold loop space for every n.

An observation that one can make about loop spaces is that they possess a multiplication given by the concatenation of loops. This multiplication is not associative on the nose. However, it is as close to being associative as homotopically possible - in other words, a space ΩX is homotopy equivalent to a topological monoid. This means more than being homotopy associative. We do not only require to have a homotopy between any two given choices of bracketings of a product, but also we need that any two such homotopies are homotopic, and that the process of finding 'homotopies between homotopies' can be iterated infinitely. The difference between being 'homotopy associative' and 'homotopy equivalent to a topological monoid', as well as different levels of associativity 'up to higher homotopies' were investigated by Stasheff in [**57**]. These phenomena led him to a notion of an A_∞*-space*, which is defined as a space homotopy equivalent to a topological monoid. Therefore all iterated loop spaces are A_∞-spaces.

The other part of the multiplicative structure of an iterated loop space to consider is its commutativity. For 1-fold loop spaces the multiplication is not homotopy commutative. However, 2-fold loop spaces are homotopy commutative. The level of commutativity up to higher homotopies is the feature which actually distinguishes a degree of folding of an iterated loop space - the closer it is to be homotopy equivalent to a commutative monoid, the higher is its degree of folding.

Operads and little discs operads. The multiplicative properties of iterated loop spaces, as discussed above, were encoded by May in the notion of an *operad* in [**47**]. Similar ideas appeared earlier in the work of Boardman and Vogt [**6**], [**7**], also Adams and MacLane [**44**, Chapter V].

A prototypical example of an operad is an *endomorphism operad* [**47**, Definition 1.2]. For a topological space X its endomorphism operad End_X is defined as a sequence of spaces $\mathrm{Map}(X^n, X)$ together with data encoding the composition of functions and the action of the symmetric group. An operad is an abstraction of this notion - it consists of a sequence of spaces endowed with action of symmetric groups and connected by maps mimicking the composition data. For an operad $\mathbb{O}$ one says that X is an $\mathbb{O}$-algebra if there is a morphism of operads $\mathbb{O} \to \mathrm{End}_X$, i.e., if for every n there is a map $\mathbb{O}(n) \times X^n \to X$ and the sequence of these maps commutes with the composition data of the operad.

The main reason of defining operads was to describe multiplications and homotopies between them in iterated loop spaces. There is a family of operads designed for this purpose - *little n-discs operads.* Every space $\Omega^n X$ is an algebra over the little n-discs operad $\mathcal{D}_n$, thus this operad encompasses data on homotopy associativity and commutativity up to higher homotopies of multiplications appearing in the n-fold loop space structure.

Using little n-discs operads, May proved two main results in the theory of loop spaces: *the recognition principle* and *the approximation theorem*.

Recognition principle and approximation theorem. The first and the most natural question concerning iterated loop spaces which comes into mind is the following: given a topological space X, how we can say that it is weakly equivalent to an n-fold loop space?

Before discussing the answer to this question, we will make a constraint here - for the clarity and brevity of the presentation, we will only consider connected spaces.

The answer to the question posed above for connected spaces was given by the recognition principle, stating that X is weakly equivalent to an n-fold loop space if and only if it admits a structure of an algebra over the operad of little n-discs. The recognition principle in this form was given by May as [**47**, Theorem 1.3]. Earlier formulations working for particular n's appear in the work of Beck [**3**], Boardman [**5**], Boardman-Vogt [**6**], and Stasheff [**57**].

Consider an n-fold loop space as a functor from the category of based topological spaces to itself, which is right adjoint to the *n-th suspension*. This in turn provides a natural map from the space X to $\Omega^n\Sigma^n X$ given by the unit of the adjunction. However, spaces of the form $\Omega^n\Sigma^n X$ are usually hard to work with. Therefore there is a need of good combinatorial models of such spaces.

In the case $n = 1$ the model was first obtained by James [**36**]. For higher n's (including ∞) some models were provided by Milgram [**50**] and Baratt [**2**]. However, all of these models were unified in May's *approximation theorem* [**47**, Theorem 2.7]. It states that if X is a connected space, then there is a weak equivalence between $\Omega^n\Sigma^n X$ and the *free* $\mathcal{D}_n$*-algebra on* X.

Homology of loop spaces. The rich structure of iterated loop spaces has a reflection in their homology. The action of little discs operads endows the homology of iterated loop spaces with *power operations*, which are very similar to Steenrod operations in cohomology. These operations were firstly defined by Araki and Kudo in [**37**] for the homology with coefficients in $\mathbb{F}_2$. This definition was later generalised for other prime fields by Dyer and Lashof in [**16**], and thus these operations are known as *Dyer-Lashof operations*.

Other type of homology operations coming from the action of the little n-discs operad is a binary operation defined by Browder in [**11**] and hence known as the *Browder bracket*. These operations may be seen as obstructions to be a higher loop space, for the Browder bracket coming from an n-fold loop structure disappear if the space is also $n + 1$-fold loop space.

These two types of homology operations allow to give a description of the homology of the space $\Omega^n\Sigma^n X$ in terms of the homology of X. If $n = 1$, one do not even need any operations - the description is provided by Bott-Samelson theorem, which states that under mild assumptions the homology of $\Omega\Sigma X$ is a free tensor algebra over the homology of X.

The approach used by Araki, Kudo, Dyer and Lashof to define power operations did not involve operads, but rather iterated joins of symmetric groups (treated as discrete topological spaces). Although more intuitive, this approach had its drawbacks. For example, with their approach it was not possible to define the Browder bracket. This was later amended by Cohen [**14**], who using little discs operads provided a full description of homology operations in the homology of iterated loop spaces. Having a good insight into the structure of homology operations was important for the aforementioned description of the homology of $\Omega^n\Sigma^n X$. In the full range it was also provided by Cohen in [**14**], where he states that the homology of $\Omega^n\Sigma^n X$ with coefficients in $\mathbb{F}_p$ is a free object in the category of $\mathbb{F}_p$-algebras with Dyer-Lashof operations and Browder brackets.

Equivariant loop spaces. We now turn to the equivariant topology. Many notions of the theory of iterated loop spaces have their equivariant analogues. However, the first striking difference is that in equivariant topology we need to consider more spheres. For a compact Lie group G every G-representation gives a *representation sphere* S^V. Therefore we obtain a notion of a *V-fold loop space* $\Omega^V X$, which is defined as a space of (not necessarily equivariant) maps from S^V to a G-space X. Taking V-fold loop space is a functor, which is right adjoint to the *V-th suspension* $\Sigma^V X$.

There are analogous notions of *little V-discs operads* $\mathcal{D}_V$, which further provides the approximation theorem and the recognition principle. The first one was proven by Hauschild **[25]** and by Rourke and Sanderson in **[52]**. Basing on the work of Rourke-Sanderson, Guillou and May gave the recognition principle in **[24]**.

However, there is a need to describe the homology of equivariant loop spaces. Before we discuss this, we need to say something about the equivariant homology.

Bredon homology. The role of ordinary homology in equivariant topology is played by *Bredon homology*. It has two important features distinguishing it from its non-equivariant analogue. Firstly, the coefficients of Bredon theories are in the form of functors from the orbit category of the group to the category of abelian groups. If the coefficients of a Bredon homology theory can be extended to the structure of *Mackey functor*, then we see the second feature - G-equivariant Bredon homology is naturally graded over the ring of representations rather than graded over the integers. The grading over the ring of G-representations is called $RO(G)$-grading.

These two properties make computations in Bredon homology quite hard in general. One can see the complications of computations in Bredon theories when computing the homology of a point - it is far from being concentrated in degree 0!

Signed loop spaces. From now on, we are going to focus on the following problem:

Problem. Let G be a finite group, V a G-representation and X a G-space. How can we describe the Bredon homology of $\Omega^V \Sigma^V X$ in terms of the homology of X?

On the contrary to the non-equivariant theory, the answer to this problem in its full generality is not known. However, the first case where V is a non-trivial representation was described by Hill in **[27]**. He discusses the case when G is the cyclic group of order 2 and V is its sign representation - hence the name *signed loop spaces.* There is a nice combinatorial model of $\Omega^\sigma \Sigma^\sigma X$, known as the *signed James construction* and described by Rybicki in **[53]**. In his work Hill provided a splitting of this construction, which allowed him further to deduce an answer to the problem for a particular Mackey functor.

Stable power operations in Bredon homology with coefficients in $\underline{\mathbb{F}}_2$. A part of the problem of describing Bredon homology of $\Omega^V \Sigma^V X$ is defining power operations and establishing relations between them. In **[62]** and **[63]** Wilson defines *stable power operations* which appear in the homology of equivariant analogues of infinite loop spaces. He also proved Cartan, Adem and Nishida relations for them.

Content and the organisation of the thesis

As described in the previous section, investigation of the structure of the Bredon homology of equivariant iterated loop space is only touched upon in the literature. The main goal of this thesis is to contribute to this area of research. The biggest part, Chapter 2 is devoted to the study of the equivariant homology of simplest little discs operads over Q, taking also the action of the symmetric group into the consideration.

Basing on the work of Wilson in [**62**] in Chapter 4 we present a way of defining and investigating power operations and Browder bracket in the homology of $\mathcal{D}_\sigma$-algebras and $\mathcal{D}_{2\sigma}$-algebras. This would not be possible without the knowledge of the structure of the coefficients of considered Bredon homology theories. Therefore we compute these coefficients in Chapter 3, using the method based on the Tate diagram, as described by Greenlees and May in [**22**] and Greenlees in [**21**].

Organisation of the thesis. The thesis consists of this Introduction and four chapters, with the first one being the background chapter. Each of the chapters has its own bibliography and, except of the first one, its own introduction. In this section we give an informal review of the content of the thesis, explaining how each chapter fits the general picture of the homology of equivariant loop-suspension spaces.

Chapter 2 - $Q \times \Sigma_n$-equivariant homology of little V-discs operads. The second chapter is devoted to the study of $Q \times \Sigma_n$-equivariant homology of little V-discs operads $\mathcal{D}_V$. Power operations given by the action of the operad $\mathcal{D}_V$ are parametrised by the homology of free $\mathcal{D}_V$-algebras on representation spheres. Such a free algebra is given as a wedge of *extended power constructions*, defined as a Σ_n-balanced product of the n-th space of the operad with the n-fold smash product of a sphere.

If we consider S^0, the n-th extended power construction over $\mathcal{D}_V$ becomes the space $\mathcal{D}_V(n)/\Sigma_n$. Therefore computing Bredon homology of the spaces of the form $\mathcal{D}_V(n)/\Sigma_n$ can be seen as a first step in studying power operations in the homology of equivariant iterated loop spaces. Since spaces $\mathcal{D}_V(n)$ are Σ_n-free, computing the Q-equivariant homology of $\mathcal{D}_V(n)/\Sigma_n$ is equivalent to computing $Q \times \Sigma_n$-equivariant homology of $\mathcal{D}_V(n)$. This chapter focuses on studying this homology.

The main tool is the universal coefficient spectral sequence, which takes as an input the homology of diagrams of fixed points of a space and converges to the Bredon homology:

$$E^2_{pq} = \operatorname{Tor}_p^{\mathcal{O}_G}(H_q(X^\bullet), N) \Rightarrow H^G_{p+q}(X, N).$$

In order to use this spectral sequence, we need to be able to compute Tor's appearing on the second page. So the strategy of calculations is as follows:

(1) We start with a description of homology of fixed points of operads in the category of coefficient systems over *graph subgroups* of $Q \times \Sigma_n$.

(2) We write down projective resolutions of homology coefficient systems using representable coefficient systems.
(3) We compute Tor groups, thus the entries of the E_2-page of the spectral sequence.
(4) We run the spectral sequence.

As shown by Hill in **[27]**, the fixed points spaces of operads may be seen as equivariant analogues of configuration spaces. Therefore in order to describe a homology of fixed points of the operad $\mathcal{D}_V$ as required by point (1) in the strategy above, we need to investigate the homology of such equivariant configuration spaces. To this end we are going to use the theory of *planetary systems* described by Sinha in **[56]**. A planetary system gives an embedding of a torus into a configuration space, and therefore by pushing forward the fundamental class of the torus it gives a homology class. The language of planetary systems allows to describe all generators of homology of configuration spaces. By studying possible actions of $Q \times \Sigma_n$'s on planetary systems we aim to use this language to understand diagrams of homology of fixed points for the simplest little discs operads over Q - such as $\mathcal{D}_{2\sigma}$ and $\mathcal{D}_{1+\sigma}$.

However, the results of this chapter are inconclusive. The major obstruction to realising this strategy is the complicated nature of the Σ_n-equivariant structure of the operads. Therefore it was possible to perform the full calculations only in the simplest cases, such as computations of $Q \times \Sigma_2$-equivariant Bredon homology of $\mathcal{D}_{1+\sigma}(2)$. For more complicated cases, only the first step is discussed. Nevertheless, we believe that the results of this chapter on the $Q \times \Sigma_n$-equivariant structure of operads $\mathcal{D}_{1+\sigma}$ and $\mathcal{D}_{2\sigma}$ are interesting on their own and will provide a basis for future work.

Chapter 3 - On the $RO(Q)$-graded coefficients of Eilenberg-MacLane spectra. Chapter III consists of the paper "On the $RO(Q)$-graded coefficients of Eilenberg-MacLane spectra" **[55]**, submitted for publication. The numbering of pages and layout is changed to be consistent with the rest of the thesis.

As mentioned in the description of the Chapter II, power operations in homology of algebras over an operad are parametrised by the homology of extended power constructions on spheres. However, the prerequisite for computing this homology is good understanding of the structure of coefficients of the homology theory. This is usually simple in non-equivariant topology and non-trivial in equivariant homology.

By 'coefficients of a homology theory' in non-equivariant topology we mean the homology of a point with coefficients in a given abelian group. Due to the dimension axiom, this homology is concentrated in degree 0. Moving to the equivariant world, recall that Bredon homology is naturally graded over the ring of representations - and the equivariant dimension axiom describes only the part of the homology of a point which is graded over trivial representations. This gives a striking difference between non-equivariant and equivariant topology - the $RO(G)$-graded Bredon homology of a point is not concentrated in degree zero!

Bredon homology is represented by *equivariant Eilenberg-MacLane spectra*, thus homology of a point is the same thing as homotopy groups of such spectrum. These homotopy groups are called *coefficients of the spectrum*. In this chapter we compute the $RO(G)$-graded coefficients of Eilenberg-MacLane spectra in the case when G is the cyclic group of order two. This is done by the method based on the Tate diagram, developed by Greenlees and May in [**22**].

The method based on the Tate diagram allows us to compute the $RO(Q)$-graded coefficients of Eilenberg-MacLane spectra on three levels of structure:

(1) as $RO(Q)$-graded abelian groups;
(2) as modules over the coefficients of the Eilenberg-MacLane spectrum associated to the Burnside Mackey functor;
(3) and, if the underlying Mackey functor is a ring Mackey functor, as $RO(Q)$-graded rings.

It is worth to emphasize here that this method is algorithmic, in that the steps are feasible and fully described and work for every Eilenberg-MacLane spectrum over Q.

The main results of this chapter are:

(1) Theorem 3.6.1 which describes the $RO(Q)$-graded abelian structure of the coefficients.
(2) Theorem 3.7.7 describing the $RO(Q)$-graded ring structure of the coefficients of the Eilenberg-MacLane spectrum associated to the Burnside Mackey functor and Observation 3.9.5 which describes the module structure over these coefficients of the coefficients of any other Eilenberg-MacLane spectrum.
(3) Theorem 3.9.1 proving that the coefficients of any Q-Eilenberg-MacLane spectrum are strictly commutative, i.e., the sign coming from the graded commutativity rule is always trivial.
(4) Observation 3.9.6 describing the way of obtaining the $RO(Q)$-graded ring structure of coefficients of Eilenberg-MacLane spectrum associated to a ring Mackey functor.

Chapter 4 - Homology operations in the algebras over Q-equivariant little discs operads. Homology operations play the crucial role in the theory of homology of loop spaces, as discussed in the Introduction. In the last chapter we discuss homology operations appearing in the homology of algebras over operads $\mathcal{D}_\sigma$ and $\mathcal{D}_{2\sigma}$ with coefficients on the constant Mackey functor $\underline{\mathbb{F}}_2$. We define Dyer-Lashof operations in these algebras and prove some of their properties. In particular, we prove the following:

THEOREM. *Power operations in $\mathcal{D}_\sigma$-algebras consists of the norming operation and multiplication by the elements of coefficients of Eilenberg-MacLane spectrum associated to $\underline{\mathbb{F}}_2$.*

In the second part of the fourth chapter we discuss an analogue of the Browder bracket in the homology of $\mathcal{D}_{n\sigma}$-algebras. We start by computations of the homology of spheres with the antipodal Q-action. Basing on this, we define the Browder bracket and prove that it commutes with the multiplication by the elements of coefficients of Eilenberg-MacLane spectrum associated to $\underline{\mathbb{F}}_2$.

This chapter at its current state is aimed to be an example of an application of the knowledge obtained in Chapter 3 to study homology operations. The results are not extensive, so this chapter should be treated as a framework for studying power operations. We hope to extend the results in future work.

Notation and conventions

In general each chapter will have its notation and conventions discussed. Nonetheless, there are some global conventions which we will follow in the whole thesis.

Unless indicated otherwise, we are going to work over finite groups. A generic group will be denoted by letters around G, its trivial element by e and its trivial subgroup by 1. The symmetric group on n letters will be denoted by Σ_n.

As mentioned in the beginning, in general we denote the cyclic group of order 2 by Q. Its non-trivial element will be denoted by γ. In Chapter II we will need to treat this group also as a member of the family of symmetric groups, then it will be denoted by Σ_2.

In the whole thesis we will going to work only with *based loop spaces*, which we will just call *loop spaces*.

CHAPTER 1

Background

Introduction

In this chapter we are going to collect all of the background material from equivariant algebraic topology needed for the thesis.

We begin with a discussion of the category of G-spaces in Section 1.1. In Section 1.2 we discuss the notion of a coefficient system in order to introduce Bredon homology in Section 1.3. Mackey functors and $RO(G)$-graded homology are discussed in Sections 1.4 and 1.5. The background material on G-spectra is in Section 1.6. We also provide some background material on spectral sequences in Section 1.7. This chapter is closed with Section 1.8, where we take a closer look on the theory of loop spaces, described in the introduction.

Notation and conventions. Throughout this chapter we assume that all topological spaces are compactly generated and weak Hausdorff. We denote the category of such spaces by $\mathcal{T}$.

1.1. Category of G-spaces and basic adjunctions

Let G be a finite group. In this section we will provide basic definitions concerning the category of G-spaces in order to unify the terminology in the thesis. For the proofs the reader is referred to [**48**].

Definition 1.1.1.

(1) A *G-space* X is a topological space together with a continuous map $G \times X \to X$. A *based G-space* is a based topological space Y together with a continuous map $G \times Y \to Y$ such that $(g, -): Y \to Y$ preserves the basepoint for every $g \in G$.
(2) Let X and Y be G-spaces. A continuous map $f: X \to Y$ is said to be *G-equivariant* if it satisfies $f(gx) = gf(x)$ for all $x \in X$. If X, Y are based G-spaces, f is additionally assumed to preserve the basepoint.

Definition 1.1.2. The *category of G-spaces* is the category having G-spaces as objects and G-equivariant maps as morphisms. We denote this category by $\mathcal{T}^G$. The *category of based G-spaces* is the category consisting of based G-spaces as objects and based G-equivariant maps as morphisms. We denote this category by $\mathcal{T}_*^G$.

Definition 1.1.3.

(1) Let X be a G-space and $H \leq G$. Then *H-fixed points of X* are defined to be:
$$X^H := \{x \in X \mid hx = x \text{ for all } h \in H\}.$$
(2) For $H \leq G$ denote by $N_G(H)$ the normaliser of H in G and define the *Weyl group of H in G* to be
$$W_G(H) := \frac{N_G(H)}{H}.$$
Note that for every G-space X its H-fixed points X^H are a $W_G(H)$-space.

Remark 1.1.4. Let $X, Y \in \mathcal{T}^G$. We recall here two standard constructions in the category of (based) G-spaces.

- The cartesian product $X \times Y$ becomes a G-space by G acting diagonally - i.e., $g \cdot (x, y) = (gx, gy)$. In the based context this also applies to the smash product.
- The (based) mapping space $\mathcal{T}(X, Y)$ becomes a (based) G-space by G acting by conjugation, i.e., $(g \cdot f)(x) = gf(g^{-1}x)$.

We also have that $\mathcal{T}(X, Y)^G = \mathcal{T}^G(X, Y)$. By $\mathcal{T}_G$ we are going to denote the category consisting of G-spaces as objects and the spaces $\mathcal{T}(X, Y)$ as hom-sets. Note that in this way the category $\mathcal{T}_G$ is enriched over $\mathcal{T}^G$ and $\mathcal{T}_G(X, Y)^G = \mathcal{T}^G(X, Y)$. By $\mathcal{T}_{*G}$ we will denote the analogous category of based G-spaces.

Proposition 1.1.5 (see [**48**, Section I.1]). *Let $X, Y \in \mathcal{T}_G$. There is a G-homeomorphism*
$$\mathcal{T}_G(X \times Y, Z) \cong \mathcal{T}_G(X, \mathcal{T}_G(Y, Z)).$$
In other words, functors
$$- \times Y \colon \mathcal{T}_G \to \mathcal{T}_G$$
$$\mathcal{T}_G(Y, -) \colon \mathcal{T}_G \to \mathcal{T}_G$$
form an adjoint pair.

Similarly, if $X, Y \in \mathcal{T}_{*G}$ there is a G-homeomorphism
$$\mathcal{T}_{*G}(X \wedge Y, Z) \cong \mathcal{T}_{*G}(X, \mathcal{T}_{*G}(Y, Z)).$$

Now we provide a definition of the equivariant analogues of spheres.

Definition 1.1.6. Let V be a representation of G. Then the *representation sphere S^V* is the one-point compactification of V. If $V = \mathbb{R}^n$ is the trivial n-dimensional representation, we write S^n for the sphere $S^{\mathbb{R}^n}$.

Definition 1.1.7. Let V be a representation of G and let $X \in \mathcal{T}_{*G}$. We define:

(1) the *V-th suspension of X* as $\Sigma^V X := S^V \wedge X$;
(2) the *V-th loop space of X* to be $\Omega^V X := \mathcal{T}_{*G}(S^V, X)$.

Remark 1.1.8. By Proposition 1.1.5 we have that the functor Σ^V is left adjoint to Ω^V.

1.2. Categories of coefficient systems and dual coefficient systems

In this section we are going to provide the definitions and properties of coefficient systems over a finite group G. These will serve us in the next section to define Bredon homology theories.

Definition 1.2.1. The *orbit category of* G, denoted by $\mathcal{O}_G$, is the category consisting of

- G-sets of the form G/H for $H \leq G$ as objects;
- G-equivariant maps as morphisms.

We recall here the classic result describing G-maps between orbit spaces G/H.

Proposition 1.2.2 ([**59**, Proposition 1.14]). *Let H, K be subgroups of G.*

(i) There exists a G-equivariant map $G/H \to G/K$ if and only if H is conjugate to a subgroup of K.
(ii) If $a \in G$ and $a^{-1}Ha \subset K$ then we obtain a G-map

$$\begin{aligned} R_a \colon G/H &\to G/K \\ gH &\mapsto gaK. \end{aligned}$$

(iii) Each G-map $G/H \to G/K$ is of the form R_a for some a such that $a^{-1}Ha \subset K$.
(iv) $R_a = R_b$ if and only if $ab^{-1} \in K$.

Definition 1.2.3. A *coefficient system* is a functor $\mathcal{O}_G^{op} \to \mathcal{A}b$, i.e., a contravariant functor from the orbit category of G to the category of abelian groups. The category of coefficient systems over G, denoted by $\mathcal{CS}_G$, is given as a functor category. It has coefficient systems as objects and natural transformations as maps.

Definition 1.2.4. A *dual coefficient system* is a functor $\mathcal{O}_G \to \mathcal{A}b$, i.e. a covariant functor from the orbit category of G to the category of abelian groups. The category of dual coefficient systems over G is given as a functor category and is denoted by $\mathcal{DCS}_G$.

Before giving examples for the definitions above we need to make a notational remark.

Remark 1.2.5. We note here that there is no well-established notation for dual coefficient systems in the literature, and therefore we will not create it ad hoc in this thesis. Examples of dual coefficient systems which we give here are usually part of a bigger structure (Mackey functor, see Subsection 1.4) and will be treated as such, so we will not give them any particular symbols.

Example 1.2.6. Let A be an abelian group. The coefficient system $\underline{A} \colon \mathcal{O}_G^{op} \to \mathcal{A}b$ such that

- $\underline{A}(G/H) = A$ for all $H \leq G$;
- all maps are given as identities;

is called the *constant coefficient system on A*. There is an analogous example for dual coefficient systems.

EXAMPLE 1.2.7. Let M be a $\mathbb{Z}[G]$-module. The *fixed points coefficient system $\underline{M}$* is given by

- $\underline{M}(G/H) = M^H$;
- maps are inclusions and conjugations.

Note that constant coefficient systems are particular cases of fixed points coefficient systems. If A is an abelian group, we can treat it as a $\mathbb{Z}[G]$-module with trivial action. Then $A^H = A$ for all subgroups H of G and inclusions of fixed points are actually identities.

EXAMPLE 1.2.8. Let M be a $\mathbb{Z}[G]$-module. The *fixed points dual coefficient system* is defined on the objects as its coefficient system analog, but the maps are different. For H being subconjugate to K, the maps are given by

$$\begin{aligned} &M^H \to M^K \\ &m \mapsto \textstyle\sum_{t \in W_K(H)} tm. \end{aligned}$$

EXAMPLE 1.2.9. Let X be a G-space. We define a functor

$$\begin{aligned} &X^\bullet \colon \mathcal{O}_G^{op} \to \mathcal{T}op \\ &G/H \mapsto X^H. \end{aligned}$$

Then by the functoriality of homology $H_p(-, A)$ we get a coefficient system $H_p(X^\bullet, A)$ for any p and abelian group A. If $A = \mathbb{Z}$, we will refer to $H_p(X^\bullet, \mathbb{Z})$ as to the *p-th homology coefficient system of X*.

LEMMA 1.2.10. *Categories $\mathcal{CS}_G$ and $\mathcal{DCS}_G$ are abelian categories.*

PROOF. Both categories are functor categories between $\mathcal{O}_G$ and $\mathcal{A}b$. Thus they are abelian by [**45**, Section VIII.3]. □

EXAMPLE 1.2.11. The coefficient systems $H_p(X^\bullet, A)$ from Example 1.2.9 may be obtained as the homology of the chain complex in $\mathcal{CS}_G$. Let $C_p(X^\bullet)$ be a coefficient system given by taking singular p-chains on X^H. The differential $d\colon C_p(X) \to C_{p-1}(X)$ induces a natural transformation $d\colon C_p(X^\bullet) \to C_{p-1}(X^\bullet)$ that satisfies $d \circ d = 0$. Thus

$$H_p(X^\bullet; A) = H_p\left(C_*(X^\bullet) \otimes A\right),$$

where $C_*(X^\bullet) \otimes A$ is defined as:

$$\left(C_*(X^\bullet) \otimes A\right)(G/H) = C_*(X^H) \otimes A.$$

We continue with the definition of representable coefficient systems, which allows us to show that $\mathcal{CS}_G$ and $\mathcal{DCS}_G$ have enough projectives.

DEFINITION 1.2.12. Let $H \leq G$. The *representable coefficient system F^H* is defined by

$$F^H(G/K) = \mathbb{Z}\left[\mathrm{Hom}_{\mathcal{O}_G}(G/K, G/H)\right].$$

Analogously, the *representable dual coefficient system* F_H is defined by

$$F_H = \mathbb{Z}\left[\mathrm{Hom}_{O_G}(G/H, G/K)\right].$$

Lemma 1.2.13. *Functors of the form F^H are projective objects in $\mathcal{CS}_G$. Functors of the form F_H are projective in $\mathcal{DCS}_G$.*

Proof. We prove the statement only for the case of F^H, since the case of F_H follows by the dual argument.

By Lemma 1.2.10 the category $\mathcal{CS}_G$ is abelian, so we need to show that the functor

$$\mathrm{Hom}_{\mathcal{CS}_G}(F^H, -)\colon \mathcal{CS}_G \to \mathcal{A}b$$

preserves epimorphisms.

By Yoneda lemma we have that $\mathrm{Hom}_{\mathcal{CS}_G}(F^H, A) = A(G/H)$. Note that epimorphisms in $\mathcal{CS}_G$ are termwise: $F\colon A_1 \to A_2$ is an epimorphism if and only if for every $K \leq G$ the homomorphism

$$F(G/K)\colon A_1(G/K) \to A_2(G/K)$$

is an epimorphism. In particular if $A_1 \to A_2 \to 0$ is an epimorphism of coefficient systems, then after applying the functor $\mathrm{Hom}_{\mathcal{CS}_G}(F^H, -)$ we obtain that $A_1(G/H) \to A_2(G/H)$ is an epimorphism in $\mathcal{A}b$. □

Proposition 1.2.14. *The categories $\mathcal{CS}_G$ and $\mathcal{DCS}_G$ have enough projectives.*

Proof. As above, we prove the statement only for the category $\mathcal{CS}_G$. The argument for $\mathcal{DCS}_G$ is dual.

Let A be a coefficient system. By Yoneda lemma we have that for every subgroup H of G we have that

$$\mathrm{Hom}_{\mathcal{CS}_G}(F^H, A) = A(G/H).$$

We define a map

$$\bigoplus_{\mathrm{Hom}_{\mathcal{CS}_G}(F^H, A)} F^H \to A$$

which on every component is a corresponding morphism from $\mathrm{Hom}_{\mathcal{CS}_G}(F^H, A)$. By definition, this map is an epimorphism on the G/H level. So if we sum these maps together

$$\bigoplus_{H \leq G}\left(\bigoplus_{\mathrm{Hom}_{\mathcal{CS}_G}(F^H, A)} F^H\right) \to A$$

we obtain an epimorphism from a projective object to A. □

1.3. Bredon homology and cohomology

In this section we introduce Bredon homology theories, which play the role of ordinary homology theories in equivariant topology.

Definition 1.3.1. Let A be a coefficient system and B a dual coefficient system. We define the *tensor product of A and B over $\mathcal{O}_G$* to be the coend (see [**45**, Section IX.6]) of A and B over $\mathcal{O}_G$:

$$A \otimes_{\mathcal{O}_G} B := \int^{G/H \in \mathcal{O}_G} A(G/H) \otimes B(G/H).$$

Observation 1.3.2.

(1) An explicit formula for the tensor product over $\mathcal{O}_G$ may be given as follows:
$$A \otimes_{\mathcal{O}_G} B = \frac{\bigoplus_{G/H \in \mathcal{O}_G} A(G/H) \otimes B(G/H)}{\simeq}$$
where $\simeq$ is a relation given by $f^*\alpha \otimes \beta \simeq \alpha \otimes f_*\beta$ for $f\colon G/K \to G/H$ in $\mathcal{O}_G$, $\alpha \in A(G/H)$ and $\beta \in B(G/K)$. See [**48**, Section I.4].

(2) From the formula above it is easy to see that $F^H \otimes_{\mathcal{O}_G} B \cong B(G/H)$. Indeed, an element of $f \in F^H(G/K) = \mathbb{Z}\left[\mathrm{Hom}_{\mathcal{O}_G}(G/K, G/H)\right]$ is a $\mathbb{Z}$-linear combination of maps $f_1, \ldots, f_n \colon G/K \to G/H$. Each of these elements may be identified with $f_i^* \operatorname{id}_{G/H}$, and by the formula from (1) if $f_i \otimes b \in F^H(G/K) \otimes B(G/K)$ then $f_i \otimes b \simeq id_{G/H} \otimes f_*b$. Therefore a map $F^H \otimes_{\mathcal{O}_G} B \to B(G/H)$ which is defined on the basis elements by $f \otimes b \simeq id_{G/H} \otimes f_*b \mapsto f_*b \in B(G/H)$ is an isomorphism.

Construction 1.3.3. Let X be a G-CW-complex (see [**59**, Section II.1]), A be a coefficient system and B be a dual coefficient system. We define a coefficient system $\underline{C}_n(X)$ by:

$$\underline{C}_n(X)(G/H) = H_n\left((X^{(n)})^H, (X^{(n-1)})^H; \mathbb{Z}\right)$$

with differentials defined by the connecting homomorphisms for triples

$$\left((X^{(n)})^H, (X^{(n-1)})^H, (X^{(n-2)})^H\right).$$

Coefficient systems $\underline{C}_n(X)$ form a chain complex in $\mathcal{CS}_G$. We further define:

$$C_n^G(X; B) := \underline{C}_n(X) \otimes_{\mathcal{O}_G} B$$

and

$$C^n_G(X; A) := \mathrm{Hom}_{\mathcal{CS}_G}(\underline{C}_n(X), A).$$

Definition 1.3.4. The *Bredon homology of X with coefficients in B* is the homology of the chain complex $C_*^G(X; B)$. The *Bredon cohomology of X with coefficients in A* is the cohomology of the cochain complex $C^*_G(X, A)$.

Remark 1.3.5. Bredon cohomology was defined in [**10**] by Bredon (as the name suggests). His goal was to develop an equivariant setting for the obstruction theory. Bredon's work was expanded by Illman in [**35**], where he defines the appropriate homology theory and generalizes Bredon's work to singular complexes.

1.4. Mackey functors

In this section we will introduce Mackey functors and discuss the symmetric monoidal structure on them.

1.4.1. Definitions of Mackey functors. We are going to provide three equivalent definitions of Mackey functors:

(1) in terms of axioms - see Definition 1.4.1;
(2) in terms of contravariant functors from the Burnside category - see Definition 1.4.5;
(3) in terms of bivariant functors on the orbit category - see Definition 1.4.8.

Let G be a finite group.

DEFINITION 1.4.1. A *Mackey functor* $\underline{M}$ over G consists of the following data:

(1) an abelian group $\underline{M}(H)$ for every $H \leq G$;
(2) morphisms
$$\mathrm{res}^H_K\colon \underline{M}(H) \to \underline{M}(K)$$
$$\mathrm{tr}^H_K\colon \underline{M}(K) \to \underline{M}(H)$$
for all subgroups $K, H \leq G$ such that $K \leq H$;
(3) and morphisms
$$c_g\colon \underline{M}(H) \to \underline{M}({}^gH)$$
for all $g \in G$ and $H \leq G$. Here ${}^gH = g^{-1}Hg$.

These data are subject to the following conditions:

(i) The morphisms res^H_H, tr^H_H and c_h are identity morphisms for all $H \leq G$ and $h \in H$.
(ii) The following transitivity relations hold:
$$\mathrm{res}^K_J \,\mathrm{res}^H_K = \mathrm{res}^H_J$$
$$\mathrm{tr}^H_K \,\mathrm{tr}^K_J = \mathrm{tr}^H_J$$
for all subgroups of $J, K, H \leq G$ such that $J \leq K \leq H$.
(iii) For all $g, h \in G$ we have $c_g c_h = c_{gh}$.
(iv) The following relations hold:
$$\mathrm{res}^{{}^gH}_{{}^gK}\, c_g = c_g \,\mathrm{res}^H_K$$
$$\mathrm{tr}^{{}^gH}_{{}^gK}\, c_g = c_g \,\mathrm{tr}^H_K$$
for all $g \in G$ and subgroups K, H such that $K \leq H$.
(v) The following *Mackey decomposition formula* holds:
$$\mathrm{res}^H_J \,\mathrm{tr}^H_K = \sum_{x \in [J\backslash H/K]} \mathrm{tr}^J_{J\cap {}^xK}\, c_x \,\mathrm{res}^K_{J^x \cap K}$$

for all subgroups $J, K \leq H$.

EXAMPLE 1.4.2. Mackey functors over cyclic groups of prime order C_p consists of:

(1) two groups $\underline{M}(C_p)$ and $\underline{M}(e)$;
(2) two morphisms

$$\mathrm{res}_e^{C_p} : \underline{M}(C_p) \to \underline{M}(e)$$
$$\mathrm{tr}_e^{C_p} : \underline{M}(e) \to \underline{M}(C_p);$$

(3) and conjugation morphisms $c_g : \underline{M}(e) \to \underline{M}(e)$.

For groups C_p the Mackey decomposition formula reduces to the following identity:

$$\mathrm{res}_e^{C_p} \mathrm{tr}_e^{C_p}(x) = N \cdot x$$

for $x \in \underline{M}(e)$. Here N denotes the norm element in $\mathbb{Z}[C_p]$, i.e., the sum of all elements of C_p.

Mackey functors over cyclic groups of prime order are usually presented by *Lewis diagrams*:

$$\begin{tikzcd} \underline{M}(C_p) \arrow[d, bend right, "\mathrm{res}_e^{C_p}"'] \\ \underline{M}(e). \arrow[u, bend right, "\mathrm{tr}_e^{C_p}"'] \end{tikzcd}$$

Now we will provide the second definition of a Mackey functor over G. However, we need to start with the definition of the *Burnside category for G* (see [**48**, Section IX.4]).

DEFINITION 1.4.3. The category $\mathfrak{B}_G^+$ is the category having finite G-sets as objects and equivalence classes of spans of G-equivariant maps as morphisms:

$$\begin{tikzcd} & \mathbf{u} \arrow[dl] \arrow[dr] & \\ \mathbf{b} & & \mathbf{c}. \end{tikzcd}$$

Here $\mathbf{u}$, $\mathbf{b}$ and $\mathbf{c}$ are finite G-sets. The composition of spans is given by pullback. Two spans $\mathbf{b} \leftarrow \mathbf{u} \rightarrow \mathbf{c}$ and $\mathbf{b} \leftarrow \mathbf{v} \rightarrow \mathbf{c}$ are equivalent if there exists a commutative diagram of the form

$$\begin{tikzcd} & \mathbf{u} \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ \mathbf{b} & & \mathbf{c} \\ & \mathbf{v} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Each hom-set in $\mathfrak{B}_G^+$ has the structure of an abelian monoid. We take the sum of $\mathbf{b} \leftarrow \mathbf{u} \rightarrow \mathbf{c}$ and $\mathbf{b} \leftarrow \mathbf{v} \rightarrow \mathbf{c}$ to be the span

$$\begin{tikzcd} & \mathbf{u} \amalg \mathbf{v} \arrow[dl] \arrow[dr] & \\ \mathbf{b} & & \mathbf{c}. \end{tikzcd}$$

Therefore we may apply the Grothendieck construction to the hom-sets of $\mathfrak{B}_G^+$.

Definition 1.4.4. The *Burnside category of G*, denoted by $\mathfrak{B}_G$, is the category consisting of:

- finite G-sets as objects;
- and the Grothendieck construction on $\mathfrak{B}_G^+(\mathbf{b}, \mathbf{c})$ as hom-sets.

Definition 1.4.5. A *Mackey functor $\underline{M}$ over G* is an additive functor $\underline{M}\colon \mathfrak{B}_G^{op} \to \mathcal{A}\mathrm{b}$.

Remark 1.4.6. Let $\underline{M}_1$ be a Mackey functor defined according to Definition 1.4.1 and $\underline{M}_2$ a Mackey functor defined as in Definition 1.4.5. Then these two definitions coincide by putting $\underline{M}_1(H) = \underline{M}_2(G/H)$ and defining the restriction and transfer as follows. Let $K \leq H$ be subgroups of G and let $\pi\colon G/K \to G/H$ be the canonical projection. Then we define the restriction res_K^H from Definition 1.4.1 to be the image of the span

$$\begin{tikzcd} & G/K \arrow[dl, equal] \arrow[dr, "\pi"] & \\ G/K & & G/H. \end{tikzcd}$$

Analogously we define the transfer tr_K^H to be the image of the span

$$\begin{tikzcd} & G/K \arrow[dl, "\pi"'] \arrow[dr, equal] & \\ G/H & & G/K. \end{tikzcd}$$

Definition 1.4.7. The category of Mackey functors $\mathcal{M}_G$ is the category of additive functors $\mathfrak{B}_G^{op} \to \mathcal{A}\mathrm{b}$.

For the completeness we will provide here the third definition of Mackey functors.

Definition 1.4.8. A Mackey functor over G consists of a pair of functors $(\underline{M}_*, \underline{M}^*)$, where

$$\underline{M}_*\colon \mathcal{O}_G \to \mathcal{A}\mathrm{b}$$
$$\underline{M}^*\colon \mathcal{O}_G^{op} \to \mathcal{A}\mathrm{b}$$

subject to the following conditions:

(1) for all $\mathbf{u} \in \mathcal{O}_G$ we have that $\underline{M}_*(\mathbf{u}) = \underline{M}^*(\mathbf{u})$. Thus we write $\underline{M}(\mathbf{u}) := \underline{M}^*(\mathbf{u})$.

(2) For every pullback diagram in $\mathcal{O}_G$

$$\begin{tikzcd} \mathbf{t} \arrow[r, "a"] \arrow[d, "b"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{u} \arrow[d, "c"] \\ \mathbf{v} \arrow[r, "d"] & \mathbf{w} \end{tikzcd}$$

we have that $\underline{M}^*(d)\underline{M}_*(c) = \underline{M}_*(b)\underline{M}^*(a)$. In other words, after applying $\underline{M}_*$ and $\underline{M}^*$ in a coherent way the resulting diagram is commutative.

(3) For every $\mathbf{u}, \mathbf{v} \in \mathcal{O}_G$ the two embeddings $\mathbf{u} \to \mathbf{u} \amalg \mathbf{v} \leftarrow \mathbf{v}$ to the disjoint union induce an isomorphism

$$\underline{M}(\mathbf{u}) \oplus \underline{M}(\mathbf{v}) \cong \underline{M}(\mathbf{u} \amalg \mathbf{v}).$$

Proposition 1.4.9. *Definitions 1.4.1 and 1.4.8 are equivalent.*

Proof. See [**60**, Section 2]. □

1.4.2. Box product and symmetric monoidal structure on $\mathcal{M}_G$. The category $\mathcal{M}_G$ (see Definition 1.4.7) has a symmetric monoidal structure with respect to the *box product*. We are going to define it now.

Definition 1.4.10. Let $\overline{\Box}$ denote the external product of Mackey functors given by

$$\underline{M}\overline{\Box}\underline{N} \colon \mathfrak{B}_G \times \mathfrak{B}_G \to \mathcal{A}\mathrm{b}$$

$$(\mathbf{b}, \mathbf{c}) \mapsto \underline{M}(\mathbf{b}) \otimes \underline{N}(\mathbf{c}).$$

Then the *box product of $\underline{M}$ and $\underline{N}$*, denoted by $\underline{M}\Box\underline{N}$ is defined as the left Kan extension of $\underline{M}\overline{\Box}\underline{N}$ along the Cartesian product functor $\times \colon \mathfrak{B}_G \times \mathfrak{B}_G \to \mathfrak{B}_G$:

$$\begin{tikzcd} \mathfrak{B}_G \times \mathfrak{B}_G \arrow[r, "\underline{M}\overline{\Box}\underline{N}"] \arrow[d, "\times"'] & \mathcal{A}\mathrm{b} \\ \mathfrak{B}_G. \arrow[ur, dashed, "\underline{M}\Box\underline{N}"'] & \end{tikzcd}$$

Definition 1.4.11. The *Burnside Mackey functor* $\mathbb{A}_G$ is defined by letting $\mathbb{A}_G(H)$ to be the Grothendieck group of finite G-sets over G/H. The transfer tr_K^H and the restriction res_K^H are defined by composition with $\pi \colon G/K \to G/H$ and pullback along π respectively.

Proposition 1.4.12. *The box product endows the category $\mathcal{M}_G$ with a symmetric monoidal structure with the unit given by $\mathbb{A}_G$.*

Proof. See [**41**, Section 1]. □

Definition 1.4.13. Commutative monoids in $\mathcal{M}_G$ with respect to the symmetric monoidal structure given by the box product are called *Green functors*.

Remark 1.4.14. There is an axiomatic definition of a Green functor, extending Definition 1.4.1. To the conditions of Definition 1.4.1 we need to add:

(vi) For each $H \leq G$, the group $\underline{M}(H)$ is a commutative ring.

(vii) The maps res^H_K and c_g are ring homomorphisms for all subgroups $K \leq H$ and $g \in G$.

(viii) The following *Frobenius axiom* is satisfied: for all subgroups $K \leq H$, $a \in \underline{M}(K)$ and $b \in \underline{M}(H)$ we have that

$$\mathrm{tr}^H_K(a \cdot \mathrm{res}^H_K(b)) = \mathrm{tr}^H_K(a) \cdot b.$$

Remark 1.4.15. Since we are going to use Mackey functors mostly over Q, we will use Lewis diagrams to present them (see Example 1.4.2). Therefore a Mackey functor $\underline{M}$ will be presented using the following scheme:

$$\begin{tikzcd} \underline{M}(Q/Q) \arrow[d, bend right, "\mathrm{res}_{\underline{M}}"'] \\ \underline{M}(Q/e). \arrow[u, bend right, "\mathrm{tr}_{\underline{M}}"'] \end{tikzcd}$$

The same convention will be used for the notation of coefficient systems and dual coefficient systems over Q. Thus the value on Q/Q will be always presented on top and the value on Q/e will be on the bottom; and depending if we work with a coefficient system or a dual coefficient system, the arrow will point up or down.

Example 1.4.16 (Fixed points Mackey functor). Let N be a $\mathbb{Z}[G]$-module. We define the *fixed points Mackey functor* $\underline{N}$ of N by $\underline{N}(G/H) = N^H$ and with maps given as in Examples 1.2.7 for the restriction and 1.2.8 for the transfer.

In particular, if A is any abelian group, we may consider it as having trivial G-action. Then the resulting Mackey functor $\underline{A}$ is the constant Mackey functor. Over Q two important examples of constant Mackey functors are $\underline{\mathbb{Z}}$:

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$$

and $\underline{\mathbb{F}_2}$:

$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "1"'] \\ \mathbb{F}_2. \arrow[u, bend right, "0"'] \end{tikzcd}$$

Both examples above are also Green functors over Q, which can be seen from Remark 1.4.14.

1.5. $RO(G)$-graded Bredon homology

In the equivariant homotopy theory the concept of Mackey functors is closely connected with the extension of Bredon homology to the grading over the ring of G-representations. We are going to describe this in this section.

Definition 1.5.1. Let $\underline{M} \in \mathcal{M}_G$ and define $RO(G)$ to be the free abelian group on the irreducible representations of G (see Remark 1.5.2 below). The *$RO(G)$-graded Bredon homology* is a family of functors

$$H_V^G(-, \underline{M})\colon \mathcal{T}_*^G \to \mathcal{A}\mathrm{b}$$

for $V \in RO(G)$, and natural morphisms

$$\sigma_W \colon H_V^G(X, \underline{M}) \to H_{V+W}^G(\Sigma^W X, \underline{M})$$

for $V, W \in RO(G)$. This data is subject to the following conditions:

(i) If $f\colon X \to Y$ is a weak G-equivalence (i.e., $f^H\colon X^H \to Y^H$ is a weak homotopy equivalence for all $H \leq G$), then the induced map in homology

$$f_V \colon H_V^G(X, \underline{M}) \to H_V^G(Y, \underline{M})$$

is an isomorphism for all $V \in RO(G)$.

(ii) If $A \to X$ is a cofibration of based G-spaces, then the following sequence is exact:

$$H_V^G(A, \underline{M}) \to H_V^G(X, \underline{M}) \to H_V^G(X/A, \underline{M})$$

for every $V \in RO(G)$.

(iii) If $X = \bigvee_i X_i$ is a wedge of based G-spaces, then the inclusions of the wedge summands induce isomorphisms:

$$\bigoplus_i H_V^G(X_i, \underline{M}) \cong H_V^G(X, \underline{M})$$

for all $V \in RO(G)$.

(iv) The maps σ_V are isomorphisms and satisfy $\sigma_0 = \mathrm{id}$ and $\sigma_W \circ \sigma_V = \sigma_{V \oplus W}$.

(v) Let $\tilde{V}$ be an element of $RO(H)$, where $H \leq G$, such that $\tilde{V} + n$ can be identified with an actual representation. Note that n in $RO(H)$ corresponds to the trivial representation $\mathbb{R}^n$. Then there is a natural isomorphism:

$$H_{\tilde{V}+k}^G(G_+ \wedge_H S^{\tilde{V}+n}, \underline{M}) \cong \begin{cases} \underline{M}(G/H) & \text{if } k = n \\ 0 & \text{else.} \end{cases}$$

Remark 1.5.2. We note here that usually $RO(G)$ is defined as the Grothendieck construction of the commutative semiring of *isomorphism classes* of finite dimensional real G-representations, where the sum is given by $\oplus$ and the product by $\otimes$. However, this approach leads to sign issues, since it is not possible to to think of representations as isomorphisms classes and keep track of signs. If we choose an isomorphism class in $RO(G)$ to be $[V]-[W]$ and we want to associate a homology group of degree $[V] - [W]$, we need to choose a representant of this isomorphism class. Moreover, if $V' - W'$ is another representant of the same class, then the isomorphism

$$V - W \to V' - W'$$

may induce an automorphism of the homology group of degree $[V] - [W]$ different than identity, for example multiplication by a unit of the Burnside ring. Therefore it is important to keep track of representatives of isomorphism classes. For in-depth discussion of this issue, see [**1**, Section 6]

The usual way of circumventing this problem is defining the $RO(G)$ to be a free abelian group on irreducible G-representations, as in Definition 1.5.1. Note that we can identify any G-representation with a unique element of $RO(G)$-defined in such a way. Defining $RO(G)$ in this way gives us preferred representatives of every isomorphism class of G-representations.

1.6. G-spectra

1.6.1. Orthogonal G-spectra. The $RO(G)$-graded homology and cohomology theories are represented by G-spectra. We give a brief overview of these in this section.

As a model of the category of G-spectra we choose the orthogonal spectra as defined by Mandell and May in [**46**]. The reason for this choice is that we will need to discuss the symmetric monoidal structure given by the smash product and, building on this, the norm construction. Our exposition of orthogonal G-spectra here is based on Appendices A and B in [**26**]. Other useful reference on the subject are Schwede's lecture notes [**54**].

Recall that G is a finite group.

Definition 1.6.1 (See [**46**, Def.1.1]).

- A *universe for G* is a countably infinite dimensional orthogonal representation of G containing infinitely many copies of the trivial representation.
- The universe $\mathcal{U}$ is *complete* if it contains infinitely many copies of all irreducible representations.
- The *trivial universe* is the universe containing only trivial representations.

Example 1.6.2.

(1) For any group G the vector space $\mathbb{R}^\infty$ treated as a trivial representation is a trivial universe.
(2) For a finite group G the sum of infinitely many copies of the regular representation of G is an example of a complete universe.

Definition 1.6.3 (See [**26**, Section A.2.3]).

(1) Let V be an orthogonal representation of G. Then we define $O(V)$ to be the group of non-equivariant linear isometric maps $V \to V$.
(2) Let V and W be orthogonal representations of G. Then we define $O(V, W)$ to be the G-space of linear isometric embeddings of V into W, where the action of G is given by the conjugation.

REMARK 1.6.4. Note that if we choose an embedding $\phi\colon V \to W$, then there is a G-homeomorphism:

$$O(V, W) \cong \frac{O(W)}{O(W - \phi(V))},$$

where $W - \phi(V)$ denotes the orthogonal complement of $\phi(V)$ in W.

DEFINITION 1.6.5. Let V and W be orthogonal representations of G. Then we define the *complementary vector bundle* E_{W-V} over $O(V, W)$ to be the vector bundle with the total space:

$$E_{W-V} := \{(\psi, w) \in O(V, W) \times W \mid w \perp \psi(V)\}.$$

The projection is given by the canonical projection on the first factor.

DEFINITION 1.6.6 (See [**26**, Def. A.10]). The category $\mathcal{J}_{G,\mathcal{U}}$ is the category consisting of:

- finite dimensional orthogonal subrepresentations of $\mathcal{U}$ as objects;
- and the morphism spaces from V to W given by the Thom complex:

$$\mathcal{J}_{G,\mathcal{U}}(V, W) := \mathrm{Thom}\,(O(V, W); E_{W-V}).$$

If the universe $\mathcal{U}$ is clear from the context, we will omit it in the notation and write $\mathcal{J}_G$.

REMARK 1.6.7.

(1) Note that the category $\mathcal{J}_{G,\mathcal{U}}$ is enriched over $\mathcal{T}^G$.
(2) The G-space $\mathcal{J}_{G,\mathcal{U}}(V, W)$ is equivalent to the wedge

$$\bigvee_{\phi \in O(V,W)} S^{W-\phi(V)}.$$

DEFINITION 1.6.8 (See [**26**, Def. A.13]).

(1) An *orthogonal G-spectrum indexed by $\mathcal{U}$* is a functor

$$\mathcal{J}_{G,\mathcal{U}} \to \mathcal{T}^G$$

of categories enriched over $\mathcal{T}^G$.
(2) Orthogonal G-spectra indexed by the trivial universe are called *naive G-spectra*.
(3) A *genuine G-spectrum* is an orthogonal G-spectrum indexed by a complete universe. If the universe is clear from the context, we will refer to these objects as to *G-spectra*.
(4) The category of orthogonal G-spectra indexed by $\mathcal{U}$, denoted by $\mathcal{S}_{G,\mathcal{U}}$, is the category of enriched functors $\mathcal{J}_{G,\mathcal{U}} \to \mathcal{T}^G$. Note that $\mathcal{S}_{G,\mathcal{U}}$ is enriched over G-topological spaces.

REMARK 1.6.9.

(1) An orthogonal G-spectrum indexed by $\mathcal{U}$ consists of a collection of based G-spaces $\{X_V\}$ indexed by all finite dimensional orthogonal subrepresentations of $\mathcal{U}$ and G-maps

$$S^{W-\phi(V)} \wedge X_V \to X_W$$

for every $\phi \in O(V, W)$. These maps are required to be compatible with composition in $\mathcal{J}_{G,\mathcal{U}}$.

(2) A naive G-spectrum may be seen as a non-equivariant spectrum with a G-action. Therefore we can treat the non-equivariant spectra as naive G-spectra with the trivial G-action.

(3) We further define the category $\mathcal{S}_{\mathcal{U}}^{G}$ to be the category consisting of the same objects as $\mathcal{S}_{G,\mathcal{U}}$ and morphisms given by

$$\mathcal{S}_{\mathcal{U}}^{G}(X, Y) := \mathcal{S}_{G,\mathcal{U}}(X, Y)^{G}.$$

The category $\mathcal{S}_{\mathcal{U}}^{G}$ is enriched over topological spaces.

1.6.2. Basic constructions. In this subsection we are going to discuss basic constructions involving orthogonal G-spectra. We start with a brief discussion of change of universe functors. For more details on this topic, see [**40**, Section II.1] or [**22**, Section I.0].

Let $\mathcal{U}$ be a G-universe and let $i\colon \mathcal{U}^{G} \hookrightarrow \mathcal{U}$ be the inclusion of G-fixed points. Note that $\mathcal{U}^{G}$ is a trivial G-universe (see Definition 1.6.1), since $\mathcal{U}$ contains countably many copies of the trivial representation of G.

Observation 1.6.10. The map i induces the forgetful functor $i^{*}\colon \mathcal{S}_{G,\mathcal{U}} \to \mathcal{S}_{G,\mathcal{U}^{G}}$ from the category of orthogonal G-spectra indexed by $\mathcal{U}$ to the category of naive G-spectra. The functor i^{*} has a left adjoint $i_{*}\colon \mathcal{S}_{G,\mathcal{U}^{G}} \to \mathcal{S}_{G,\mathcal{U}}$ given by the left Kan extension. Thus we obtain the adjoint pair

$$i_{*}\colon \mathcal{S}_{G,\mathcal{U}^{G}} \rightleftarrows \mathcal{S}_{G,\mathcal{U}} : i^{*}.$$

Definition 1.6.11. Let $X \in \mathcal{S}_{G,\mathcal{U}}$. Then the spectrum $i^{*}X$ is called the *underlying naive G-spectrum of X*. If we additionally forget the G-action, this becomes the *underlying spectrum of X*.

Using the adjoint functors given above, we will also define the *categorical fixed points* functor.

Definition 1.6.12. Let $Y \in \mathcal{S}_{G,\mathcal{U}^{G}}$. We define its *categorical fixed points* Y^{Q} by

$$(Y^{Q})_{\mathbb{R}^{n}} := (Y_{\mathbb{R}^{n}})^{Q}$$

where $\mathbb{R}^{n}$ is the n-dimensional trivial G-representation. If $X \in \mathcal{S}_{G,\mathcal{U}}$ then we define $X^{Q} := (i^{*}X)^{Q}$.

From now on, we fix a complete G-universe $\mathcal{U}$ and we will omit it from the notation.

Definition 1.6.13. Let A be a based G-space and X a G-spectrum. We define:

(1) the *smash product* $A \wedge X$ by $(A \wedge X)_{V} := A \wedge X_{V}$ and smash product of id_{A} with structural maps of X;

(2) the *function spectrum* $F(A, X)$ by $F(A, X)_V := \mathrm{Map}(A, X_V)$ with structural maps given by the composite:

$$\Sigma^W \mathrm{Map}(A, X_V) \to \mathrm{Map}(A, \Sigma^W X_V) \to \mathrm{Map}(A, X_{V+W})$$

where the first map is the adjoint of the evaluation map $\Sigma^V \mathrm{Map}(A, X_V) \wedge A \to \Sigma^V X_V$ and the second map comes from applying $\mathrm{Map}(A, -)$ to the structural map $\Sigma^V X_W \to X_{V+W}$.

Definition 1.6.14. Let $H \leq G$.

(1) Let A be a based space with left G-action and right H-action, and let X be an H-spectrum. Then we define $A \wedge_H X$ to be the G-spectrum $A \wedge X$ divided levelwise by the diagonal action of H;
(2) Let A be a based space with left H-action and right G-action, and let X be an H-spectrum. Then we define $F_H(A, X)$ to be the G-spectrum given by $F_H(A, X)_V := \mathrm{Map}(A, X_V)^H$.

Using the construction given above we can define the *change of group* functors.

Definition 1.6.15. Let X be a G-spectrum. If $H \leq G$, we can naturally see X as an H-spectrum as follows. We restrict the universe $\mathcal{U}$ to become an H-universe and restrict the action of G on all spaces X_V to the action of H. The resulting H-spectrum is *the restriction of X to H*. This gives a functor:

$$\mathrm{res}^G_H \colon G\mathcal{S} \to H\mathcal{S}.$$

The restriction has both left and right adjoint, which for $Y \in H\mathcal{S}$ are respectively:

(1) *induction*, given by $G_+ \wedge_H Y$;
(2) *coinduction*, given by $F_H(G_+, H)$.

Definition 1.6.16.

(1) Let $X \in \mathcal{T}_*^G$. Then its *suspension spectrum* $\Sigma^\infty X$ is defined by $(\Sigma^\infty X)_V = \Sigma^V X$ and the structure maps given by canonical isomorphisms. If Y is a non-based G-space, we put $\Sigma^\infty_+ Y := \Sigma^\infty(Y_+)$.
(2) The G-spectrum S^V is defined to be the suspension spectrum of the representation sphere S^V.
(3) For every G-representation V we define also S^{-V} to be the G-spectrum characterised by the functorial equivariant isomorphism (see [**26**, Section A.2.4])

$$\mathcal{S}_G(S^{-V}, X) \cong X_V.$$

1.6.3. Symmetric monoidal structure and norm constructions. The category of orthogonal G-spectra can be endowed with a symmetric monoidal product. Similarly as in the case of non-equivariant spectra, its construction takes a bit of time and needs to be done carefully; therefore we are going to take its existence for granted and refer the reader to the literature for the construction - see [**46**, Section II.3] or [**26**, Section A.2.5].

Recall that we fixed a complete G-universe $\mathcal{U}$.

PROPOSITION 1.6.17. *There exists a* smash product of G-spectra, *denoted by* $- \wedge -$, *which makes the category* $\mathcal{S}_G$ *into a symmetric monoidal category with the unit given by* S^0 *(Definition 1.6.16).*

DEFINITION 1.6.18. Monoids in $\mathcal{S}_G$ with respect to the smash product are called *ring G-spectra*. Commutative monoids are called *commutative ring G-spectra*.

DEFINITION 1.6.19.

(1) Let R be a commutative ring G-spectrum and M be a G-spectrum. We say that M *is a module over* R if there is a map of G-spectra

$$\rho_M \colon R \wedge M \to M$$

satisfying associativity and unitality conditions (see [**26**, Section A.1.2]).

(2) The category of R-modules $\mathcal{M}\mathrm{od}_R$ consists of R-modules as objects and maps of G-spectra preserving associativity and unitality as morphisms.

PROPOSITION 1.6.20. *Let* R *be a commutative ring* G*-spectrum. There exists a* smash product of R-modules $M \wedge_R N$ *such that* $(\mathcal{M}od_R, \wedge_R, R)$ *is a symmetric monoidal category.*

PROOF. Consider the following maps:

$$\rho \wedge N \colon M \wedge R \wedge N \to M \wedge N$$

and

$$M \wedge \rho \colon M \wedge R \wedge N \to M \wedge N.$$

Then the smash product $M \wedge_R N$ is defined as the coequalizer of these two maps:

$$M \wedge R \wedge N \rightrightarrows M \wedge N \to M \wedge_R N.$$

The fact that it makes the category $\mathcal{M}\mathrm{od}_R$ into a symmetric monoidal category comes from [**30**, Lemma 2.2.2]. □

Now we will define the norm construction. We restrict our attention to the particular case of norming from the trivial group to the finite group G, for this will be sufficient for our purposes. For more details the reader is referred to the foundational work on this construction - Hill-Hopkins-Ravenel's article [**26**]. Recall that we treat non-equivariant spectra as orthogonal spectra for the trivial group.

Firstly we need to make the following observation.

OBSERVATION 1.6.21. If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category, then the symmetric group Σ_n acts on any n-fold symmetric monoidal product of objects. For details, see [**45**, Theorem IX.1.1].

Definition 1.6.22. Let X be a (non-equivariant) spectrum. We define the *smash product of X indexed by G* to be the naive G-spectrum

$$X^{\wedge G} := \bigwedge_{g \in G} X_g$$

where X_g are copies of X and G acts as a subgroup of the symmetric group by the action given in Observation 1.6.21. We further define *the norm of X* to be

$$N^G X := i_*(X^{\wedge G}).$$

Proposition 1.6.23 (See [**26**, Subsection 2.2.3]). *The norm satisfies the following properties:*

(1) it forms a functor from the category of spectra to the category of G-spectra.

(2) It is symmetric monoidal.

(3) It is left adjoint to the restriction as a functor from the category of commutative ring spectra to the category of commutative ring G-spectra.

(4) Let X be a based topological space and denote by $N^G X$ analogous construction of the indexed smash product on spaces. Then

$$N^G(\Sigma^\infty X) \simeq \Sigma^\infty(N^G X).$$

In particular if $V \subset \mathcal{U}$ is an Euclidean subspace and ρ_G is a regular G-representation, then

$$N^G(S^V) \simeq S^{V \otimes \rho_G}.$$

1.6.4. Equivariant stable homotopy category. In this subsection we are going to discuss the homotopy theory of the category of orthogonal G-spectra. Similarly as in the case of the smash product, defining the model structure properly takes time - therefore we will take its existence for granted and refer the reader to the literature - [**46**, Section III.5] or [**26**, Section B.4].

However, firstly we will define the class of maps which will become weak equivalences in the model structure on orthogonal spectra. To this end, we follow [**26**, Section 2.2.4]. Recall that we fixed a complete G-universe $\mathcal{U}$.

Definition 1.6.24.

(1) Let V_1, V_2 be two non-zero orthogonal G-representations. We write $V_1 < V_2$ if for every irreducible orthogonal G-representation U we have that

$$\dim \hom^G(U, V_1) < \dim \hom^G(U, V_2) - 1.$$

This relation makes the set of orthogonal G-representations into partially ordered set. If, in particular, V_1 is the trivial representation of dimension k, then the condition $V_1 < V_2$ gives that

$$\dim V_2^G > k + 1.$$

We will abbreviate this by $V_2 > k$.

(2) Let $X \in \mathcal{S}^G$, $k \in \mathcal{T}^G$ and let V_1, V_2 be orthogonal G-representations such that $V_1 < V_2$. Choose an equivariant isometric embedding $t\colon V_1 \to V_2$ and let W be the orthogonal complement of $t(V_1)$ in V_2. We define the map

$$[S^{V_1} \wedge K, X_{V_1}]^G \to [S^{V_2} \wedge K, X_{V_2}]^G$$

by using the identification $S^W \wedge S^{V_1} \cong S^{V_2}$ and the structure map $S^W \wedge X_{V_1} \to X_{V_2}$ to form the composite

$$[S^{V_1} \wedge K, X_{V_1}]^G \to [S^W \wedge S^{V_1}, S^W \wedge X_{V_1}] \to [S^{V_2} \wedge K, X_{V_2}]^G.$$

(3) Let $K \in \mathcal{T}^G$ and let V be a finite dimensional orthogonal representation of G and let $k \in \mathbb{N}$. For $H \leq G$ we define

$$\pi_V^H(K) = [S^V, K]^H$$

to be the set of H-equivariant homotopy classes of maps from the representation sphere S^V (see Definition 1.1.6) to K. For details of the homotopy theory of G-spaces see [**26**, Section 2.1].

(4) Let $X \in \mathcal{S}^G$ and $k \in \mathbb{Z}$. For $H \leq G$ we define the *H-equivariant k-th stable homotopy group of X* to be

$$\pi_k^H(X) := \operatorname*{colim}_{V > -k} \pi_{V+k}^H X_V.$$

(5) A map of G-spectra $f\colon X \to Y$ is a *stable weak equivalence* if it induces isomorphisms

$$\pi_k^H(X) \cong \pi_k^H(Y)$$

for all $k \in \mathbb{Z}$ and $H < G$.

Proposition 1.6.25. *There exists a model structure on the category of orthogonal G-spectra which has as weak equivalences the class of stable weak equivalences as defined in Definition 1.6.24. Together with the smash product this model structure makes $\mathcal{S}_G$ into a symmetric monoidal model category. We are going to call the homotopy category of this model structure the* G-equivariant stable homotopy category. *The set of homotopy classes of G-equivariant maps between G-spectra X, Y in this model structure will be denoted by $[X, Y]^G$.*

Now we are in the position to define the main invariant of G-spectra - the homotopy groups.

Definition 1.6.26. Let X be a G-spectrum. Define the *V-th G-homotopy group of X* to be:

$$\pi_V^G(X) := [S^V, X]^G.$$

Remark 1.6.27. Note that if V is the trivial representation of dimension k, the homotopy groups defined above coincide with the homotopy groups defined in Definition 1.6.24.

Remark 1.6.28. The usual way of indicating the $RO(G)$-grading is by using the five-pointed star in the subscript, e.g., $\pi_\star^G(X)$. It is sometimes useful to restrict attention to the

$RO(G)$-grading restricted to the trivial representations, which we will sometimes call the $\mathbb{Z}$-grading. It is denoted by using the asterisk, as in $\pi_*^G(X)$.

The G-homotopy groups of spectra may be structured to become Mackey-functor valued.

Definition 1.6.29. Let X be a G-spectrum, V be a G-representation and $H \leq G$. We define the *V-th H-homotopy group of* X to be:

$$\pi_V^H(X) := [S^V \wedge G/H_+, X]^G.$$

These groups form together $\pi_V^\bullet(X)$, the *Mackey functor valued homotopy groups of* X with:

(1) the transfer tr_K^H given by the map

$$[S^V, G/K_+ \wedge X]^G \to [S^V, G/H_+ \wedge X]^G$$

induced by applying the functor $[S^V, - \wedge X]^G$ to canonical projection $G/K_+ \to G/H_+$ and using the fact that orbits G/H_+ are self-dual in the equivariant stable homotopy category by *Wirthmüller isomorphism* (see [**40**, Corollary II.6.3]);

(2) and the restriction res_K^H given by

$$[S^V \wedge G/H_+, X]^G \to [S^V \wedge G/K_+, X]^G$$

obtained by the precomposition with the projection $G/K \to G/H$.

As in the non-equivariant topology, every G-spectrum E defines $RO(G)$-graded E-homology and E-cohomology:

Definition 1.6.30. Let E and X be G-spectra. For all $V \in RO(G)$ we define:

(1) the *E-homology of* X to be $E_V^G(X) := \pi_V^G(E \wedge X)$;
(2) the *E-cohomology of* X to be $E_G^V := [\Sigma^{-V} X, E]^G$.

In particular there is an important class of G-spectra which represent $RO(G)$-graded Bredon homology.

Definition 1.6.31. Let $\underline{M}$ be a Mackey functor over G. Then *Eilenberg-MacLane G-spectrum associated to* $\underline{M}$ is the spectrum $H\underline{M}$ defined by the property:

$$\pi_*^\bullet(H\underline{M}) = \begin{cases} \underline{M} & \text{for } * = 0 \\ 0 & \text{else.} \end{cases}$$

For the discussion of the existence, see [**48**, Section XIII.4].

Remark 1.6.32. Note that the condition given in the definition above is only on the $\mathbb{Z}$-graded Mackey valued homotopy groups. This defines Eilenberg-MacLane spectra uniquely up to weak G-homotopy equivalence, but it does not say much about $RO(G)$-graded homotopy groups of $H\underline{M}$. As will be discussed later in the thesis, the homotopy groups of $H\underline{M}$ outside trivial representations are non-trivial.

1.7. Spectral sequences

In this section we are going to review the construction and basic theory of spectral sequences. As we are going to use mostly the homological spectral sequences, we will focus our attention on these. This section is entirely based on [**61**, Chapter 5] with the notation adjusted to this thesis.

Throughout this section, we fix an abelian category $\mathcal{A}$.

1.7.1. Basic definitions.

Definition 1.7.1. A *homology spectral sequence (starting with E^a)* in $\mathcal{A}$ consists of the following data:

(1) A family $\{E^r_{pq}\}$ of objects of $\mathcal{A}$ defined for all $p, q \in \mathbb{Z}$ and $r \geq a$.
(2) Maps $d^r_{pq} \colon E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$ that satisfy $d^r d^r = 0$.
(3) Isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the spot E^r_{pq}:

$$E^{r+1}_{pq} \cong \frac{\ker(d^r_{pq})}{\operatorname{im}(d^r_{p+r,q-r+1})}.$$

Definition 1.7.2. A homology spectral sequence is *bounded* if for each $n \in \mathbb{N}$ there are only finitely many nonzero terms of total degree n in E^a_{**}. Then for every $p, q \geq 0$ we have that $E^r_{pq} = E^{r+1}_{pq}$ for r large enough. We write E^∞_{pq} for this stable value of E^r_{pq}.

In particular, if the spectral sequence satisfies $E^r_{pq} = 0$ unless $p, q \geq 0$, we say that $\{E^r\}$ is a *first quadrant spectral sequence*.

Definition 1.7.3. Let $\{E^r_{pq}\}$ be a bounded spectral sequence. We say that $\{E^r_{pq}\}$ *converges to H_** if there are:

- a family of objects H_n of $\mathcal{A}$;
- a finite filtration on every H_n:

$$0 = F_s H_n \subset \ldots \subset F_{p-1}H_n \subset F_p H_n \subset F_{p+1}H_n \subset \ldots \subset F_t H_n = H_n;$$

- isomorphisms

$$E^\infty_{pq} \cong \frac{F_p H_{p+q}}{F_{p-1}H_{p+q}}.$$

If $\{E^r_{pq}\}$ converges to H_*, we denote it by

$$E^r_{p+q} \Rightarrow H_{p+q}.$$

1.7.2. Constructing spectral sequences. In this section we are going to construct and discuss two ways of constructing spectral sequences: filtered chain complexes and exact couples.

1.7.2.1. *Spectral sequences associated to filtered complexes.*

Definition 1.7.4. Let C be a chain complex in $\mathcal{A}$. A *filtration* F of C is an ordered family of subcomplexes

$$\ldots \subset F_{p-1}C \subset F_pC \subset F_{p+1}C \subset \ldots.$$

The filtration F is *bounded* if for each $n \in \mathbb{Z}$ there are integers $s < t$ such that $F_sC_n = 0$ and $F_tC_n = C_n$.

Theorem 1.7.5 (Spectral sequence associated to a filtered chain complex). *Let C be a filtered chain complex with filtration F. Then F determines a spectral sequence starting with*

$$E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$$

and such that

$$E^1_{pq} = H_{p+q}(E^0_{p*}).$$

If the filtration F is bounded, then the spectral sequence converges to $H_*(C)$

$$E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C).$$

Proof. See [**61**, Construction Theorem 5.4.1]. □

1.7.2.2. *Spectral sequences associated to exact couples.* Another classical way of constructing spectral sequences of modules over some ring are exact couples. So in this subsection the category $\mathcal{A}$ is assumed to be the category of modules over a ring.

Definition 1.7.6. An *exact couple* $\mathcal{E}$ is a pair (D, E) of bigraded modules together with three morphisms

- $\iota\colon D \to D$;
- $\lambda\colon D \to E$;
- $\kappa\colon E \to D$

satisfying the exactness condition:

$$\begin{aligned} \operatorname{im}(\iota) &= \ker(\lambda) \\ \operatorname{im}(\lambda) &= \ker(\kappa) \\ \operatorname{im}(\kappa) &= \ker(\iota). \end{aligned}$$

The defining morphisms are required to have the following bidegrees:

$$\begin{aligned} \text{bidegree}(\iota) &= (1, -1) \\ \text{bidegree}(\lambda) &= (-a, a) \text{ for some } a \in \mathbb{N} \\ \text{bidegree}(\kappa) &= (-1, 0). \end{aligned}$$

Exact couples are often presented in a graphical way as:

$$\begin{tikzcd} D \arrow[rr, "\iota"] & & D \arrow[dl, "\lambda"] \\ & E. \arrow[ul, "\kappa"] & \end{tikzcd}$$

Definition 1.7.7. Let $\mathcal{E} = (D, E)$ be an exact couple. Define

$$H(E) = \ker(\lambda\iota\kappa)/\mathrm{im}(\lambda\iota\kappa).$$

Then the *derived couple of* $\mathcal{E}$ is a pair of bigraded modules $\mathcal{E}' = (\iota(D), H(E))$ with three morphisms, as in the following diagram:

$$\begin{tikzcd} \iota(D) \arrow[rr, "\iota'"] & & \iota(D) \arrow[dl, "\lambda'"] \\ & H(E) \arrow[ul, "\kappa"] & \end{tikzcd}$$

where:

- ι' is a restriction of ι to $\iota(D)$;
- $\lambda'(\iota(d)) := [\lambda(d)]$;
- $\kappa'([e]) = \kappa(e)$.

Proposition 1.7.8. *Let $\mathcal{E}$ be an exact couple and $\mathcal{E}'$ its derived couple. Then $\mathcal{E}'$ is an exact couple.*

This proposition justifies the following definition.

Definition 1.7.9. Let $\mathcal{E}$ be an exact couple such that bidegree$(\lambda) = (-a, a)$. Then its *r-th derived couple* $\mathcal{E}^r$ is defined inductively by $\mathcal{E}^a = \mathcal{E}$ and $\mathcal{E}^r$ is the derived couple of $\mathcal{E}^{r-1}$ for $r \geq a$. We will denote the bigraded modules forming the couple $\mathcal{E}^r$ by D^r and E^r.

Proposition 1.7.10. *An exact couple $\mathcal{E} = (D, E)$ in which* bidegree$(\lambda) = (-a, a)$ *determines a homology spectral sequence $\{E^r_{pq}\}$ starting with E^a.*

Proof. See [**61**, Proposition 5.9.2]. □

Construction 1.7.11 (Exact couple coming from filtered complex). Let C_* be a chain complex with filtration F. Define the bigraded modules:

$$D^1_{pq} = H_{p+q}(F_pC_*)$$
$$E^1_{pq} = H_{p+q}(F_pC_*/F_{p-1}C_*).$$

Consider for each p the following short exact sequence of chain complexes:

$$(*) \qquad 0 \longrightarrow F_{p-1}C_* \longrightarrow F_pC_* \longrightarrow F_pC_*/F_{p-1}C_* \longrightarrow 0.$$

Define maps:

- $\iota_{pq} \colon D^1_{pq} \longrightarrow D^1_{p+1,q-1}$ to be the map induced in homology by the inclusion $F_{p-1}C_* \longrightarrow F_pC_*$.
- $\lambda_{pq} \colon D^1_{pq} \longrightarrow E^1_{p-1,q+1}$ to be the map induced in homology by the projection $F_pC_* \longrightarrow F_pC_*/F_{p-1}C_*$.
- $\kappa_{pq} \colon E^1_{pq} \longrightarrow D^1_{p-1,q}$ to be the connecting homomorphism in homology induced by the sequence $(*)$.

Then the maps ι_{pq} form a map of bigraded modules $\iota\colon D^1 \to D^1$ of bidegree $(1,-1)$. Similarly, maps λ_{pq} form a map $\lambda\colon D^1 \to E^1$ of bidegree $(-1,1)$ and maps κ_{pq} give the map $\kappa\colon E^1 \to D^1$ of bidegree $(-1,0)$.

These data form together an exact couple $\mathcal{E} = (D^1, E^1)$.

Proposition 1.7.12. *Let C_* be a chain complex with filtration F. Then the spectral sequence given by the exact couple $\mathcal{E}$ described in Construction 1.7.11 is naturally isomorphic to the spectral sequence given by Theorem 1.7.5.*

1.7.3. Spectral sequences associated to a double complex. In this subsection we discuss double complexes and associated spectral sequences. This data will be later used for the construction of the universal coefficient spectral sequence.

We return to the assumption that $\mathcal{A}$ is an arbitrary abelian category.

Definition 1.7.13.

- A *double complex* in $\mathcal{A}$ is a family $\mathbf{C} = \{\mathbf{C}_{pq}\}$ for $p, q \in \mathbb{Z}$ of objects $\mathcal{A}$, together with maps:
$$d^h\colon \mathbf{C}_{pq} \to \mathbf{C}_{p-1,q}$$
$$d^v\colon \mathbf{C}_{pq} \to \mathbf{C}_{p,q-1}$$
such that $d^hd^h = d^vd^v = d^vd^h + d^hd^v = 0$. The maps d^h are called *horizontal differentials* and d^v are called *vertical differentials*.
- The *total complex of a double complex* C is the chain complex $\mathrm{Tot}(\mathbf{C})$ given by:
$$\mathrm{Tot}(\mathbf{C})_n := \prod_{p+q=n} \mathbf{C}_{pq}$$
and with the differential $d = d^v + d_h$.
- A double complex $\mathbf{C}$ is said to be *first quadrant double complex* if $\mathbf{C}_{pq} = 0$ if $p < 0$ or $q < 0$.

Construction 1.7.14. Let $\mathbf{C}$ be a double complex in $\mathcal{A}$. Then the double complex structure of $\mathbf{C}$ induces two filtrations of the total complex $\mathrm{Tot}(\mathbf{C})$.

- The first one is the filtration by columns ${}^cF$:
$${}^cF_k(\mathrm{Tot}(\mathbf{C}))_n = \prod_{\substack{p+q=n \\ p\leq k}} \mathbf{C}_{pq}.$$
- The second is the filtration by rows ${}^rF$:
$${}^rF_k(\mathrm{Tot}(\mathbf{C}))_n = \prod_{\substack{p+q=n \\ q\leq k}} \mathbf{C}_{pq}.$$

These filtrations give us two spectral sequences, denoted respectively ${}^cE$ and ${}^rE$. We can identify the first and the second page of these spectral sequences. Denote by H^v the homology of $\mathbf{C}$ with respect to the vertical differential d^v; respectively for H^h and d^h. Then we have that:

$$
\begin{aligned}
{}^cE^1_{pq} &= H^v_q(\mathbf{C}_{p*}) \\
{}^cE^2_{pq} &= H^h_p(H^v_q(\mathbf{C})) \\
{}^rE^1_{pq} &= H^h_q(\mathbf{C}_{*p}) \\
{}^rE^2_{pq} &= H^v_p(H^h_q(\mathbf{C})).
\end{aligned}
$$

PROPOSITION 1.7.15. *Let* $\mathbf{C}$ *be a first quadrant double complex. Then both spectral sequences given in Construction 1.7.14 converge to* $H_*(\mathrm{Tot}(\mathbf{C}))$*:*

$$
\begin{aligned}
{}^cE^2_{pq} &= H^h_p(H^v_q(\mathbf{C})) \Rightarrow H_*(\mathrm{Tot}(\mathbf{C})) \\
{}^rE^2_{pq} &= H^h_p(H^v_q(\mathbf{C})) \Rightarrow H_*(\mathrm{Tot}(\mathbf{C}))
\end{aligned}
$$

PROOF. See [**61**, Section 5.6]. □

1.8. An overview of the theory of loop spaces

In this section we are going to give more detailed discussion of the theory of loop spaces. As is usual in equivariant topology, the equivariant concepts are generalisations of the non-equivariant ones - therefore we will start with an overview of the non-equivariant theory of loop spaces.

1.8.1. Loop spaces and operads.

DEFINITION 1.8.1. Let X be a based topological space.

(1) The *loop space of* X is the space of continuous maps preserving basepoint $\Omega X := \mathrm{Map}_*(S^1, X)$.
(2) Let $n \in \mathbb{N}$. The *n-fold loop space of* X is the space $\Omega^n X$ defined inductively by setting $\Omega^0 X := X$ and $\Omega^n X := \Omega(\Omega^{n-1}X)$.
(3) Moreover, we define an *n-fold loop space* to be a topological space X with a sequence of spaces $\{X_i\}_{0\le i\le n}$ such that $X_0 = X$ and $X_j \simeq \Omega X_{j+1}$ for $0 \le j < n$. If X is an n-fold loop space, then any choice of the spaces $\{X_i\}$ is called a *delooping of* X.
(4) An *infinite loop space* is a topological space X with an infinite delooping, i.e., a sequence of spaces $\{X_i\}_{i\in\mathbb{N}}$ such that $X \simeq X_0$ and $X_i \simeq \Omega X_{i+1}$.

OBSERVATION 1.8.2. By the tensor-hom adjunction in based topological spaces we have that

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y)$$

and more generally

$$\mathrm{Map}_*(\Sigma^n X, Y) \cong \mathrm{Map}_*(X, \Omega^n Y).$$

The unit of this adjunction gives the canonical embedding

$$X \to \Omega^n \Sigma^n X.$$

Example 1.8.3. For any abelian group π its associated Eilenberg-MacLane spaces are infinite loop spaces, since there is a homotopy equivalence $K(\pi, n) \cong \Omega K(\pi, n+1)$.

An important feature of iterated loop spaces is a multiplication which is commutative "up to higher homotopies". The precise meaning of this statement is packed into the notion of an *operad* and *operadic actions*.

Definition 1.8.4 ([**47**, Definition 1.1]). An *operad* $\mathbb{O}$ (in topological spaces) consists of the following data:

- spaces $\mathbb{O}(k)$ for $k \in \mathbb{N}$, with $\mathbb{O}(0) = *$;
- right action of the symmetric group Σ_k on $\mathbb{O}(k)$;
- maps $\phi : \mathbb{O}(k) \times \mathbb{O}(j_1) \times \ldots \times \mathbb{O}(j_k) \to \mathbb{O}(j_1 + \ldots + j_k)$ for $k, j_1, \ldots, j_k \geq 0$ called *structural maps*.

This data is subject to the following axioms:

(1) For all $c \in \mathbb{O}(k)$, $d_s \in \mathbb{O}(j_s)$ and $e_t \in \mathbb{O}(i_t)$ the associativity formula holds:

$$\phi\left(\phi(c; d_1, \ldots, d_k); e_1, \ldots, e_j\right) = \phi\left(c; \phi(d_1, e_1, \ldots, e_{j_1}), \ldots,\right.$$
$$\left.\ldots \phi(d_i, e_{j_1+\ldots+j_{i-1}+1}, \ldots, e_{j_1+\ldots+j_i}), \ldots\right)$$

where $j = \sum_{i=1}^{k} j_i$.

(2) There is an identity element $1 \in \mathbb{O}(1)$ such that for any $j, k \in \mathbb{N}$ the following conditions hold:
   (a) $\phi(1, d) = d$ for all $d \in \mathbb{O}(j)$;
   (b) and $\phi(c, 1^k) = c$ for any $c \in \mathbb{O}(k)$, where $1^k = (1, \ldots, 1) \in \mathbb{O}(1)^k$.

(3) For all $c \in \mathbb{O}(k)$, $d_s \in \mathbb{O}(j_s)$, $\tau \in \Sigma_k$ and $\tau_s \in \Sigma_{j_s}$ the equivariance formulas hold:
   (a)
   $$\phi(c\tau, d_1, \ldots, d_k) = \phi(c, d_{\tau^{-1}(1)}, \ldots, d_{\tau^{-1}(k)})\tau(j_1, \ldots, j_k)$$
   where $\tau(j_1, \ldots, j_k)$ denotes the block permutation of $j_1 + \ldots + j_k$ letters induced by the partition given by $(j_1, \ldots, j_k)$.
   (b)
   $$\phi(c, d_1\tau_1, \ldots, d_k\tau_k) = \phi(c, d_1, \ldots, d_k)(\tau_1 \oplus \ldots \tau_k)$$
   where $\tau_1 \oplus \ldots \oplus \tau_k$ denotes the image of $(\tau_1, \ldots, \tau_k)$ under the inclusion $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k} \to \Sigma_{j_1+\ldots+j_k}$.

Definition 1.8.5 (See [**47**, Section 1]). Let $\mathbb{O}_1$ and $\mathbb{O}_2$ be operads. A *morphism of operads* $\psi \colon \mathbb{O}_1 \to \mathbb{O}_2$ consists of a sequence of Σ_j-equivariant maps $\psi_j \colon \mathbb{O}_1(j) \to \mathbb{O}_2(j)$ such that

$\psi_1(1) = 1$ and the following diagram commutes:

$$\begin{tikzcd} \mathbb{O}_1(k) \times \mathbb{O}_1(j_1) \times \ldots \times \mathbb{O}_1(j_k) \arrow[r, "\phi"] \arrow[d, "\psi_k \times \psi_{j_1} \times \ldots \times \psi_{j_k}"'] & \mathbb{O}_1(j_1 + \ldots + j_k) \arrow[d, "\psi_{j_1+\ldots+j_k}"] \\ \mathbb{O}_2(k) \times \mathbb{O}_2(j_1) \times \ldots \times \mathbb{O}_2(j_k) \arrow[r, "\phi"] & \mathbb{O}_2(j_1 + \ldots + j_k). \end{tikzcd}$$

Definition 1.8.6 ([**47**, Definition 1.2]). Let X be a topological space. Then we define $\mathcal{E}nd_X$, the *endomorphism operad of* X, by:

- $\mathcal{E}nd_X(n) = \mathrm{Map}(X^n, X)$;
- the identity map as the element $1 \in \mathcal{E}nd_X(1)$;
- and composition of functions as structural maps.

Definition 1.8.7 (See [**47**, Section 1]). Let $\mathbb{O}$ be an operad. An $\mathbb{O}$*-algebra* is a topological space X with a morphism of operads $\psi\colon \mathbb{O} \to \mathcal{E}nd_X$. If X has a structure of an $\mathbb{O}$-algebra, we say that $\mathbb{O}$ *acts on* X.

Definition 1.8.8 (See [**47**, Construction 2.4] or [**38**, Section 19.3.2]). Let X be a topological space and $\mathbb{O}$ be an operad. We define the *k-th extended power construction on X over $\mathbb{O}$* to be

$$\mathrm{Sym}^k_{\mathbb{O}}(X) := \mathbb{O}(k) \times_{\Sigma_k} X^k.$$

Furthermore, we define *the free $\mathbb{O}$-algebra on X* to be

$$F_{\mathbb{O}}(X) := \coprod_{k \geq 0} \mathrm{Sym}^k_{\mathbb{O}}(X).$$

Observation 1.8.9. Note that the structure of an $\mathbb{O}$-algebra is equivalent to the existence of *structural maps* $\tilde{\theta}_k\colon \mathbb{O}(k) \times X^k \to X$. Since these maps come from the morphism of operads $\mathbb{O} \to \mathcal{E}nd_X$, they are Σ_k equivariant, and thus their existence is equivalent to existence of maps $\theta_k\colon \mathrm{Sym}^k_{\mathbb{O}}(X) \to X$. Therefore the $\mathbb{O}$-algebra structure is equivalent to the map $\theta\colon F_{\mathbb{O}}(X) \to X$ (see [**47**, Proposition 2.8]).

Moreover, the free algebra $F_{\mathbb{O}}(X)$ has itself a structure of an $\mathbb{O}$-algebra. By the way how it is defined, the functor $F_{\mathbb{O}}\colon \mathcal{T} \to \mathcal{T}$ is a *monad*, i.e., there is a natural transformation $F_{\mathbb{O}}F_{\mathbb{O}} \to F_{\mathbb{O}}$ ([**47**, Construction 2.4]). Therefore, by the statement above, for any topological space X the space $F_{\mathbb{O}}(X)$ has a structure of an $\mathbb{O}$-algebra.

Remark 1.8.10. Note that, for the clarity of the presentation, we gave here the definition of an operad only in the category of topological spaces. However, Definitions 1.8.4-1.8.8 can be easily generalised to any symmetric monoidal category. In particular, we can discuss operads in categories of G-spaces (discussed below), spectra and G-spectra.

Definition 1.8.11 ([**47**, Definition 4.1]). Let $D(n)$ denote the open unit disc in $\mathbb{R}^n$. A little n-disc is a rectilinear map $f : D(n) \to D(n)$, i.e., a map admitting a translation and a change of the size, but not rotation. We define the *little n-discs operad* $\mathcal{D}_n$ by taking:

- $\mathcal{D}_n(k)$ to be the space of k-tuples of little n-discs with disjoint images, topologised as a subspace of a space of continuous functions $\amalg_k D(n) \to D(n)$;
- structural maps to be composition maps. A symbolic depiction of the composition is given in Figure 1.1.

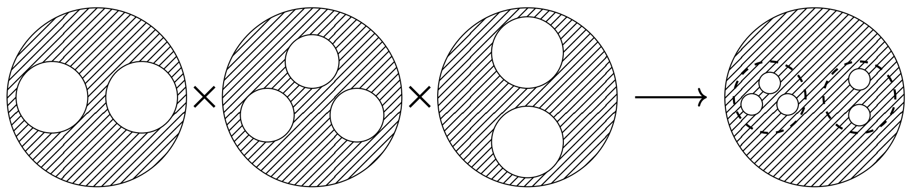


FIGURE 1.1. A composition in little discs operads.

REMARK 1.8.12. Note that by definition it is clear that for any n and k the spaces $\mathcal{D}_n(k)$ are Σ_k-free, where Σ_k acts on $\mathcal{D}_n(k)$ by permuting little discs.

The operadic spaces $\mathcal{D}_n(k)$ are closely related to configuration spaces in $\mathbb{R}^n$.

DEFINITION 1.8.13. Let X be a topological space and n a natural number. We define the *configuration space of n points in X* to be:

$$\mathrm{Conf}_n(X) := \left\{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } 1 \leq i < j \leq n\right\}.$$

PROPOSITION 1.8.14. *Sending a little disc to its centre induces a homotopy equivalence*

$$\mathcal{D}_n(k) \cong \mathrm{Conf}_k(\mathbb{R}^n).$$

PROOF. See [**47**, Theorem 4.8]. □

1.8.2. Recognition principle and approximation theorem. The main goal of defining little discs operads (Definition 1.8.11) is that they "recognize" loop spaces and allow to build homotopical models of these.

PROPOSITION 1.8.15. *Let $n \in \mathbb{N} \cup \{\infty\}$. If X is an n-fold loop space, then the operad $\mathcal{D}_n$ acts on X.*

PROOF. See [**47**, Theorem 5.1]. □

One of two key results in the theory of loop spaces is a reverse of the proposition above.

THEOREM 1.8.16 (Recognition principle). *If X is a connected $\mathcal{D}_n$-algebra, then X is weakly homotopy equivalent to an n-fold loop space.*

PROOF. See [**47**, Chapter 13]. □

The other key result is that free algebras over little disc operads are homotopical models for iterated loop-suspension spaces.

THEOREM 1.8.17 (Approximation Theorem). *There is a natural map of $\mathcal{D}_n$-algebras*

$$\alpha_n \colon F_{\mathcal{D}_n}(X) \to \Omega^n \Sigma^n X,$$

which is a weak homotopy equivalence if X is connected.

PROOF. See [**47**, Chapter 6]. □

1.8.3. Homology of iterated loop spaces. After defining the iterated loop spaces, we would like to study their homology. This has a very rich structure coming from the action of little discs operads. In this subsection we are going to discuss the following question - how one can express the homology of $\Omega^n \Sigma^n X$ in terms of the homology of X?

The description for the case $n = 1$ was given by Bott and Samelson in [**9**].

DEFINITION 1.8.18. Let R be a commutative ring and M be an R-module. Then the *free tensor algebra on M* is the graded algebra $T(M)$ given by

$$T(M)_n := M^{\otimes n},$$

where $M^{\otimes 0} = R$.

THEOREM 1.8.19 (Bott-Samelson theorem). *Let R be a PID an let X be a connected space such that $H_*(X, R)$ is torsion free. Then there is an isomorphism*

$$T(\tilde{H}_*(X, R)) \cong H_*(\Omega\Sigma X, R)$$

and the map $X \to \Omega\Sigma X$ induces the canonical embedding

$$\tilde{H}_*(X, R) \to T(\tilde{H}(X, R)).$$

Using this theorem one can provide a homotopical model for the space $\Omega\Sigma X$, given by James in [**36**].

DEFINITION 1.8.20. Let X be a based connected space and denote its base point by $*$. We define the space $J_n(X)$ to be

$$J_n(X) := \frac{\coprod_{1 \le i \le n} X^i}{\sim},$$

where the relation $\sim$ is defined by

$$(x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_n) \sim (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Then the *James construction on X*, denoted by $J(X)$, is defined as the colimit of

$$J_1(X) \subset J_2(X) \subset \dots \subset J_n(X) \subset J_{n+1}(X) \subset \dots.$$

PROPOSITION 1.8.21. *There is a weak homotopy equivalence*

$$J(X) \simeq \Omega\Sigma X.$$

The description of the homology of n-fold loop spaces for $n > 1$ is more complicated. It involves two kinds of homology operations:

- power operations, known as *Araki-Kudo-Dyer-Lashof* operations;
- and binary operations, known as *Browder bracket*.

We will recall here the description for the homology with coefficients in $\mathbb{F}_2$. We are going to abbreviate "Araki-Kudo-Dyer-Lashof operations" to widely used "Dyer-Lashof operations" - however, it should be remembered that Araki and Kudo were the first to define these operations in [**37**]. Dyer and Lashof extended their results for odd primes in [**16**].

Our exposition below is based on [**38**, Section 19.5].

THEOREM 1.8.22. *Let X be a $\mathcal{D}_n$-algebra in spaces. Then the graded algebra $H_*(X,\mathbb{F}_2)$ has* Dyer-Lashof operations

$$P_i\colon H_m(X,\mathbb{F}_2) \to H_{2m+i}(X,\mathbb{F}_2)$$

for $0 \leq i \leq n-1$. These operations satisfy the following properties:

Additivity: $P_r(x+y) = P_r(x) + P_r(y)$ *for* $r < n-1$.
Squaring: $P_0(x) = x^2$.
Cartan formula: $P_r(xy) = \sum_{p+q=r} P_p(x)P_q(y)$ *for* $r < n-1$.
Adem relations: $P_rP_s(x) = \sum \binom{j-s-1}{2j-r-s} P_{r+2s-2j}P_j(x)$ *for* $r > s$.
Stability: $\sigma P_0 = 0$ *and* $\sigma P_r = P_{r-1}$ *for* $r > 0$*, where* σ *is the* homology suspension.
Extension: *If a $\mathcal{D}_n$-algebra structure extends to a $\mathcal{D}_{n+1}$-structure, then the Dyer-Lashof operations for $\mathcal{D}_{n+1}$-algebras coincide with the operations for $\mathcal{D}_n$-algebras.*

THEOREM 1.8.23. *Let X be a $\mathcal{D}_n$-algebra in spaces. Then the graded algebra $H_*(X,\mathbb{F}_2)$ has a* Browder bracket

$$[-,-]\colon H_r(X,\mathbb{F}_2) \otimes H_s(X,\mathbb{F}_2) \to H_{r+(n-1)+s}(X,\mathbb{F}_2)$$

satisfying the following properties:

Antisymmetry: $[x,y] = [y,x]$ *and* $[x,x] = 0$.
Unit: $[x,1] = 0$.
Leibniz rule: $[x,yz] = [x,y]z + y[x,z]$.
Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.
Dyer-Lashof vanishing: $[x,P_ry] = 0$ *for* $r < n-1$.
Top additivity: $P_{n-1}(x+y) = P_{n-1}x + P_{n-1}y + [x,y]$.
Top Cartan formula: $P_{n-1}(xy) = \sum_{p+q=n-1} P_p(x)P_q(y) + x[x,y]y$.
Adjoint identity: $[x,P_{n-1}y] = [y,[y,x]]$.
Extension: *If a $\mathcal{D}_n$-algebra structure on X extends to a $\mathcal{D}_{n+1}$-algebra structure, the Browder bracket for $\mathcal{D}_n$-algebra structure on X become zero map.*
$\mathcal{D}_1$**-bracket:** $[x,y] = xy + yx$ *if* $n = 1$.

These two types of operations allow to provide a description of the homology of iterated loop-suspension space in terms of the homology of the base space.

Theorem 1.8.24 ([**14**, Theorem III.3.1]). *Let X be a based topological space. Then $H_*(F_{\mathcal{D}_n}(X), \mathbb{F}_2)$ is the free object $\mathfrak{D}_n(H_*(X, \mathbb{F}_2))$ in the category of graded $\mathbb{F}_2$-algebras with Dyer-Lashof operations and Browder brackets.*

Remark 1.8.25.

(1) Note that due to Approximation Theorem 1.8.17 the theorem above give us a full description of the $\mathbb{F}_2$-homology of the space $\Omega^n\Sigma^n X$ if X is connected.
(2) Theorem 1.8.24 is an analogue of calculations of the cohomology of Eilenberg-MacLane spaces as free objects in the category of algebras with Steenrod operations.
(3) As mentioned earlier, a similar description in terms of algebras with power operations and Browder bracket of the $\mathbb{F}_p$-homology for p being an odd prime in terms of algebras with power operations and Browder bracket exists due to Dyer and Lashof [**16**], who gave this description in a certain range, and later Cohen [**14**, Part III], who provided the full description.

1.8.4. Equivariant loop spaces. In this subsection we are going to discuss the theory of equivariant loop spaces.

Throughout this section we fix a finite group G. Recall that for a G-representation V and a G-space X we defined in Definition 1.1.7 the representation sphere S^V, V-th suspension $\Sigma^V X$ and V-fold loop space $\Omega^V X$. Recall also that functors

$$\Sigma^V : \mathcal{T}_{*G} \rightleftarrows \mathcal{T}_{*G} : \Omega^V$$

form an adjoint pair.

1.8.5. Little V-discs operads and approximation theorem. Similarly as in the non-equivariant case, there is a notion of little V-discs operads.

Definition 1.8.26 (See [**24**, Definition 1.1]). Let V be an orthogonal G-representation and let $D(V)$ denote the unit disc in V. A *little V-disc* is a rectilinear map $f : D(V) \to D(V)$. We define the *little V-discs operad $\mathcal{D}_V$* by:

- $\mathcal{D}_V(k)$ is the G-space of k-tuples of little V-discs with disjoint images, where G acts by conjugation;
- structural maps as in the non-equivariant case, see Figure 1.1.

There are also appropriate versions of the recognition principle and the approximation theorem. We will need only the approximation theorem in the thesis, so only this theorem will be discussed. The recognition principle may be found in [**24**].

Firstly we need to expand the notion of connectedness.

Definition 1.8.27. Let X be a G-space. We say that X is *G-connected* if for every subgroup $H \leq G$ the fixed points space X^H is connected.

Example 1.8.28.

- Let ρ be the regular representation of Q. Then S^{ρ} is Q-connected.
- Let σ be the sign representation of Q. Then S^{σ} is not Q-connected, because $(S^{\sigma})^{Q} = S^{0}$, which is not connected.

Theorem 1.8.29 (Approximation theorem, see [**24**, Theorem 1.11]). *Let V be an orthogonal G-representation and X a G-space. Then there is a natural map of $\mathcal{D}_V$-algebras*

$$\alpha_V \colon F_{\mathcal{D}_V}(X) \to \Omega^V \Sigma^V X$$

which is a weak homotopy equivalence if X is G-connected.

Remark 1.8.30. We note here that the property of being G-connected is very strongt. For example, not all representation spheres are G-connected. If we omit the G-connectedness assumption, the approximation theorem states that the map α_V is a group completion. We are not going to discuss this further - for our purposes it is enough to have the statement that free algebra on X over the little V-discs operad gives a good homotopical model of the space $\Omega^V \Sigma^V X$.

1.8.6. Example: $\mathcal{D}_{\sigma}$-algebras. In this subsection we are going to show two structural results from Hill's work in [**27**] concerning $\mathcal{D}_{\sigma}$-algebras over Q, also known as "signed loop spaces". The first one is the splitting of the signed James construction, which is a Q-equivariant analogue of the James construction, as given in Definition 1.8.20. The second result uses this splitting to show the structure of the Bredon homology of the signed James construction with coefficients in a particular Mackey functor.

We start with a definition of the signed James construction, given by Rybicki in [**53**]. This serves as a combinatorial model for the space $\Omega^{\sigma}\Sigma^{\sigma}X$.

Definition 1.8.31 (See [**53**, Section 2] or [**27**, Def.4.1]). Let X be a based Q-space. We define

$$J_n^{\sigma}(X) := \frac{\coprod_{k=0}^{n} X^{\times k}}{\sim}$$

where:

- $\sim$ is the equivalence relation which omits the basepoint (see Definition 1.8.20);
- and the Q-action is given by

$$\gamma(x_1, \ldots, x_k) = (\gamma x_k, \ldots, \gamma x_1).$$

We further define

$$J^{\sigma}(X) := \operatorname*{colim}_n J_n^{\sigma}(X).$$

Proposition 1.8.32 (See [**53**, Theorem 2.2]). *Let X be a Q-connected space. Then there is a weak equivalence*

$$J^{\sigma}(X) \simeq \Omega^{\sigma}\Sigma^{\sigma}X.$$

Now we are in the position to discuss the results mentioned in the introduction to this subsection.

THEOREM 1.8.33 (See [**27**, Theorem 4.3]). *Let X be a based Q-space of the homotopy type of a Q-CW complex. Then there is a natural weak equivalence*

$$\Sigma^{\rho}_{+} J^{\sigma}(X) \simeq \bigvee_{k\geq 0} \left(\Sigma^{\rho}\left((N^{Q}\operatorname{res}^{Q}_{e} X)^{\wedge k}\right) \vee \Sigma^{\rho}\left((N^{Q}\operatorname{res}^{Q}_{e} X)^{\wedge k} \wedge X\right)\right)$$

where ρ denotes the regular representation of Q and σ denotes its real sign representation.

From this theorem one can deduce the structure of $RO(Q)$-graded Bredon homology of spaces $J^{\sigma}(X)$ with coefficients in a particular Mackey functor.

DEFINITION 1.8.34. Let $\underline{B}$ be the following Mackey functor:

$$\underline{B} := \begin{tikzcd} \mathbb{Z}/4 \arrow[d, bend right, "1"'] \\ \mathbb{Z}/2. \arrow[u, bend right, "2"'] \end{tikzcd}$$

REMARK 1.8.35. The Mackey functor $\underline{B}$ is actually defined as a norm in Mackey functors of $\mathbb{F}_2$. It is also a Green functor, which can be seen from Remark 1.4.14.

The norm in Mackey functors will be again used in Theorem 1.8.36. However, we are not going to define it - the construction can be found in [**28**] and [**31**].

Now we give the second main result. Recall from Definition 1.6.29 that the homotopy groups of Q-spectra can be formed to be Mackey functor valued - therefore the same can be done for the $RO(Q)$-graded Bredon homology of any Q-space. We are going to denote $RO(Q)$-graded Mackey functor valued Bredon homology of the Q-space X with coefficients in a Mackey functor $\underline{M}$ by $\underline{H}_{\star}(X; \underline{M})$. More information on graded Mackey functors can be found in [**42**].

THEOREM 1.8.36 (See [**27**, Theorem 4.6]). *Let X be a (not necessarily based) Q-space and denote by $H\underline{B}^{\bullet}_{\star}$ the Mackey functor valued homotopy groups $\pi^{\bullet}_{\star}(H\underline{B})$. Then there is an isomorphism of $RO(Q)$-graded Mackey functors*

$$\underline{H}_{*}(J^{\sigma}(X); \underline{B}) \cong \left(\bigoplus_{i=0}^{\infty} N^{Q}\left(\tilde{H}_{*}(\operatorname{res}^{Q}_{e} X; \mathbb{F}_2)^{\otimes i}\right) \square (H\underline{B}^{\bullet}_{\star} \oplus \underline{\tilde{H}}_{\star}(X; \underline{B}))\right).$$

REMARK 1.8.37. Note that by this theorem and Theorem 1.8.33 we obtain a description of the homology of the space $\Omega^{\sigma}\Sigma^{\sigma}X$ in terms of the homology of X, provided that X is Q-connected.

CHAPTER 2

$Q \times \Sigma_n$-equivariant homology of little V-discs operads

Introduction

Σ_m-equivariant homology of operads as the source of power operations. Finding a description of the structure present in the homology of loop spaces is a classical problem in algebraic topology. In the non-equivariant theory this is achieved by means of *power operations*, defined by Araki and Kudo in [**37**] for the homology with coefficients in $\mathbb{F}_2$ and generalised for coefficients of odd characteristic by Dyer and Lashof in [**16**]. This approach was unified by Cohen in [**14**].

Dyer-Lashof operations in homology of $\mathcal{D}_n$-algebras are parametrised by the homology of *extended power contructions on spheres* $\mathrm{Sym}^m_{\mathcal{D}_n}(S^k) = \mathcal{D}_n(m)_+ \wedge_{\Sigma_m} (S^k)^m$ (see Definition 1.8.8). If we choose a representant of the homology class $\kappa\colon S^l \to H\mathbb{F}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_n}(S^k)$ and let $x\colon S^k \to H\mathbb{F}_p \wedge X$ to be a representative of a homology class of a $\mathcal{D}_n$-algebra X , then the operation corresponding to κ on the class x is defined as the following composite:

$$S^l \xrightarrow{\kappa} H\mathbb{F}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_n}(S^k) \xrightarrow{H\mathbb{F}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_n}(x)} H\mathbb{F}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_n}(H\mathbb{F}_p \wedge X) \xrightarrow{(\mu\wedge X)\circ(H\mathbb{F}_p\wedge\theta)} H\mathbb{F}_p \wedge X$$

where θ is the map coming from the operadic action on $H\mathbb{F}_p \wedge X$ (see Observation 1.8.8) and μ is the multiplication in the spectrum $H\mathbb{F}_p$. For details, see [**38**, Section 19.4].

On the equivariant side, homology operations in the Bredon homology of $\mathcal{D}_V$-algebras are also parametrised by extended power constructions on representation spheres (Definition 1.8.8). If we choose a representative of a class in $H^G_\star(\mathrm{Sym}^m_{\mathcal{D}_V}(S^W), \underline{\mathbb{F}}_p)$ to be

$$\kappa\colon S^U \to H\underline{\mathbb{F}}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_V}(S^W)$$

we define a corresponding operation on a homology class $x\colon S^W \to H\underline{\mathbb{F}}_p \wedge X$ by the composite similar to the non-equivariant case:

$$S^U \xrightarrow{\kappa} H\underline{\mathbb{F}}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_V}(S^W) \xrightarrow{H\underline{\mathbb{F}}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_V}(x)} H\underline{\mathbb{F}}_p \wedge \mathrm{Sym}^m_{\mathcal{D}_V}(X) \xrightarrow{(\mu\wedge X)\circ(H\underline{\mathbb{F}}_p\wedge\theta)} H\underline{\mathbb{F}}_p \wedge X.$$

From this point of view we can see that the homology of extended power constructions on spheres $H^G_\star(\mathrm{Sym}^m_{\mathcal{D}_V}(S^W); \underline{\mathbb{F}}_p)$ is the fundamental object in studying equivariant Dyer-Lashof operations.

The main goal of this chapter is to compute groups $H^G_\star(\mathrm{Sym}^m_{\mathcal{D}_V}(S^W); \underline{\mathbb{F}}_p)$ in simple cases. We restrict our attention to $G = Q$, $p = 2$ and $W = 0$, so we will study the Bredon homology of spaces $\mathcal{D}_V(m)/\Sigma_m$. Further on, we will consider only $\mathbb{Z}$-graded Bredon homology (i.e., the Bredon homology graded over trivial representations). In this case we note that spaces $\mathcal{D}_V(m)$ are Σ_m-free, and thus there is an isomorphism

$$H^Q(\mathcal{D}_V(m)/\Sigma_m, \underline{\mathbb{F}_2}) \cong H_*^{Q\times\Sigma_m}(\mathcal{D}_V(m), \underline{\mathbb{F}_2}).$$

Therefore the aim is to compute groups $H_*^{Q\times\Sigma_m}(\mathcal{D}_V(m), \underline{\mathbb{F}_2})$ for simple representations.

This goal has not been achieved due to computational issues described later on in the Introduction. However, the results obtained on the way to computing the aforementioned homology seem interesting in their own right. In particular, the theory of configuration spaces of orbits in a representation seems like a field with potentially rich theory; in the second part of this chapter some results concerning homology of such spaces are presented.

$Q \times \Sigma_m$**-equivariant Bredon homology of operads.** Recall that by Q we denote the cyclic group of order 2. As mentioned above, the main goal of this chapter is to compute $Q\times\Sigma_m$-equivariant Bredon homology of little $(1+\sigma)$-discs operad and little 2σ-discs operad. The choice of operads is not random. Operads of little intervals and little σ-discs are homotopically discrete and Σ_m-free, thus after passing to Σ_m-orbits they reduce to a point. Operads $\mathcal{D}_{1+\sigma}$ and $\mathcal{D}_{2\sigma}$ are the simplest ones which do not have this property.

Our main computational device will be the *universal coefficient spectral sequence*. It takes as the input diagrams of homology of fixed points of $\mathcal{D}_V(m)$ with the action of respective Weyl group, and computes the Bredon homology with coefficients in the given dual coefficient system N. Precisely it is given as follows:

$$E^2_{pq} = \mathrm{Tor}_p^{O_{Q\times\Sigma_m}}\left(H_q(\mathcal{D}_V(m)^\bullet), N\right) \Rightarrow H_{p+q}^{Q\times\Sigma_m}(\mathcal{D}_V(m), N).$$

The method of computing the groups $H_*^{Q\times\Sigma_m}(\mathcal{D}_V(m), \underline{N})$ using this spectral sequence should proceed as follows:

(1) We start with a description of the diagrams $H_*(\mathcal{D}_V(m)^\bullet)$, together with the action of respective Weyl groups on all entries.
(2) We compute the Tor groups $\mathrm{Tor}_*^{O_{Q\times\Sigma_m}}\left(H_*(\mathcal{D}_n(m)^\bullet), \underline{N}\right)$, for example by writing a projective resolutions of $H_*(\mathcal{D}_n(m)^\bullet)$ in the category of coefficient systems over $Q\times\Sigma_m$.
(3) We run the universal coefficient spectral sequence.

Moreover, $Q\times\Sigma_n$-spaces which are $1\times\Sigma_n$-free have empty fixed points under action of all subgroups which are not *graph subgroups* - i.e., subgroups Γ such that $\Gamma \cap (1\times\Sigma_m) = 1$. Note that spaces $\mathcal{D}_V(n)$ have the property of being $1\times\Sigma_n$-free. This fact allows us to consider only diagrams of homology of fixed points under graph subgroups when computing the $Q\times\Sigma_m$-equivariant Bredon homology.

We note here that the Bredon homology is generally expected to be graded over the ring of G-representations. However, we focus our attention in this work on the $\mathbb{Z}$-graded

Bredon homology, as computing it is quite often a first step for performing the $RO(G)$-graded calculations.

Connection with the equivariant theory of configuration spaces. The spaces $\mathcal{D}_n(m)$ of the (non-equivariant) operad of little n-discs are homotopy equivalent to spaces of configurations of points in the Euclidean space $\mathbb{R}^n$. This fact was used by Cohen in [**14**], where he computes the Σ_m-equivariant homology of $\mathcal{D}_n(m)$.

In the equivariant context the situation is slightly more complicated. As described by Hill in [**27**], spaces of fixed points of little discs operads $\mathcal{D}_V$ can be seen as spaces of equivariant embeddings of n copies of an orbit in V with disjoint images, which can be further interpreted as *configuration spaces of orbits in* V. These may be understood as variants of non-equivariant configuration spaces with additional constraints; for example, the spaces $\mathcal{D}_{1+\sigma}(m)$ split as disjoint unions of spaces of *signed* configurations of points in the plane.

The language of configuration spaces has a big advantage - there is already developed and well described theory which can be used. However, it also indicates one of the problems which we are going to encounter. The non-equivariant theory of configuration spaces is manageable, but if we want to take the action of the symmetric group into account then the theory becomes much harder.

Problems with using the universal coefficient spectral sequence for calculations. During the application of the framework discussed above one may encounter several major issues. First problem was mentioned above - that while the non-equivariant theory of configuration spaces is well-understood, this is not the case when one take the action of the symmetric group into account. This makes the application of the point (1) quite hard in the full generality.

Second issue is connected with the point (2). One of the methods of finding projective resolutions, applied during work on this project, is using representable coefficient systems as the projective objects. This method is simple in theory and one can easily find the required projective objects. However, since the resource of projective objects is small, writing a projective resolution using them is computationally complex. This suggests that probably such computations should be supported by a computational engine - we leave it for the future research.

These issues made this framework only applicable to the simplest cases - such as $\mathcal{D}_{1+\sigma}(2)$. The complicated $Q \times \Sigma_m$-equivariant nature of the coefficient systems may be already seen in the case of spaces $\mathcal{D}_{1+\sigma}(3)$ or $\mathcal{D}_{2\sigma}(3)$ - see Propositions 2.6.14 and 2.7.23.

Content. Due to the problems described above it was possible to apply the framework based on the universal coefficient spectral sequence in its full range only for the simplest cases. However, in this chapter we are going to describe the method in details, show the basic calculations and present the complexity which may be encountered on the way.

In first three sections we give a detailed description of the framework. We start with the construction of universal coefficient spectral sequence in Section 2.1, proving its basic properties and giving an example of calculations. In Section 2.2 we show that if we want to compute the $Q \times \Sigma_m$-equivariant Bredon homology of spaces $\mathcal{D}_V(m)$, we can consider only diagrams over the orbit category restricted to the graph subgroups. This is shown in a bigger generality - i.e., in the case of G_2-free $G_1 \times G_2$-spaces.

Further on in Section 2.3 we describe the representable objects in the category of coefficient systems over graph subgroups, which will be our main device for writing projective resolutions.

The remaining four sections contains discussion and analysis concerning operads $\mathcal{D}_{1+\sigma}$ and $\mathcal{D}_{2\sigma}$. We start with the description of homology classes of configuration spaces of points in the plane in Section 2.4. This is done using the language of planetary systems and S-trees, following Sinha's paper [**56**].

It is followed by the explanation of the connection between fixed points of operadic spaces and configuration spaces of orbits in representations in Section 2.5. This section follows Hill's work from [**27**].

Then we proceed to analysis of the first operad in Section 2.6, namely the operad of little $1+\sigma$-discs, $\mathcal{D}_{1+\sigma}$. Using the framework based on the universal coefficient spectral sequence, we are going to compute the $Q \times \Sigma_2$-equivariant Bredon homology of the space $\mathcal{D}_{1+\sigma}(2)$ with coefficients in the constant dual coefficients system $\underline{\mathbb{F}_2}$ in Theorem 2.6.13. Afterwards, in Subsection 2.6.4, we present the first step for the framework for the space $\mathcal{D}_{1+\sigma}(3)$ - i.e., description of the homology graph coefficient systems. This is the first place where writing a projective resolution "by hand" is very complex and therefore some computational engine should be used in the future.

The chapter is closed by Section 2.7, containing a discussion of configuration spaces of free orbits in 2σ and their $Q \wr \Sigma_m$-equivariant structure present in homology. Focus on this topic is justified by the fact that these spaces appear as fixed points of $\mathcal{D}_{2\sigma}(m)$ under the action of one of graph subgroups of $Q \times \Sigma_m$.

Notation and conventions. Throughout the whole chapter we will denote the cyclic group of order 2 by Q and its non-trivial element by γ. The real sign representation of Q will be denoted by σ.

The orbit category of a finite group G will be denoted by $\mathcal{O}_G$. If M is a (dual) coefficient system, we will abbreviate $M(G/H)$ to $M(H)$ for $H \leq G$.

The tensor product of a coefficient system M and a dual coefficient system N over the orbit category will be denoted by $M \otimes N$. Note that we are omitting the symbol of the orbit category in the subscript - it should be always clear over which category we are taking the tensor product. Analogously, this convention applies to $\mathrm{Tor}(M, N)$.

2.1. Universal Coefficient Spectral Sequence

In this chapter we are going to discuss the construction and basic properties of the universal coefficient spectral sequence. Throughout this chapter, let G denote a finite group.

2.1.1. Construction of universal coefficients spectral sequence. We begin with the construction of the universal coefficient spectral sequence. It will be provided in the proof of Theorem 2.1.3.

Before we give the main theorem, we need to provide several technical results.

Lemma 2.1.1 (See [**10**, Proposition 10.1]). *Let S be a G-set. Then the coefficient system F^S defined by*

$$F^S(G/H) = \mathbb{Z}[S^H]$$

is a projective object in $\mathcal{CS}_G$.

Proof. Define the category Set^G to be the category consisting of finite G-sets as objects and G-equivariant maps between them. Note that there is an embedding $\mathcal{O}_G \to \mathrm{Set}^G$.

Every finite G-set can be written as a disjoint union of orbits. We can extend our definition of coefficient systems as follows: if $S = \coprod_n G/H_n$ and $M \in \mathcal{CS}_G$, then we put $M(S) = \bigoplus_n M(G/H_n)$.

Therefore we can define $\mathcal{CS}_G$ as a category of contravariant functors $\mathrm{Set}^G \to \mathcal{A}\mathrm{b}$ preserving finite products. With such definition we see that functors $F^S = \mathbb{Z}\left[\mathrm{Hom}_{\mathrm{Set}^G}(-, S)\right]$ are representable, thus by the argument used in the proof of Lemma 1.2.13 they are projective. □

Corollary 2.1.2. *Let X be a G-space. The coefficient system $C_i(X^\bullet)$ is projective for every i.*

Proof. Follows from Lemma 2.1.1 where we put S to be the set of singular i-simplices in X. □

Theorem 2.1.3. *Let X be a G-space and N a dual coefficient system over G. There exists the following spectral sequence:*

$$E^2_{pq} = \mathrm{Tor}_p^{\mathcal{O}_G}(H_q(X^\bullet), N) \Rightarrow H^G_{p+q}(X, N).$$

The name of this spectral sequence will be abbreviated as UCSS.

Proof. Let $P_* \to N \to 0$ be a projective resolution of N. Let $\mathbf{D}$ be the double complex

$$\mathbf{D}_{pq} = C_p(X^\bullet) \otimes_{\mathcal{O}_G} P_q$$

with the total complex $D_p = \bigoplus_{i+j=p} \mathbf{D}_{ij}$. Recall from Definition 1.7.13 that d^h denotes the horizontal differential in D and d^v denotes the vertical differential. By abuse of notation we will use the following:

(1) C_p to denote $C_p(X^\bullet)$;

(2) $\mathcal{H}_p$ to denote $H_p(X^\bullet)$;

(3) and we will omit $\mathcal{O}_G$ form the notation of the tensor product; i.e., all tensor products in this proof are taken over the orbit category of G. With this notation $M \otimes N$ denotes the tensor product of $M \in \mathcal{CS}_G$ and $N \in \mathcal{DCS}_G$ given as the coend, see Definition 1.3.1.

Consider two spectral sequences associated to the double complex $\mathbf{D}_{**}$ given in Construction 1.7.14. Recall that both spectral sequences converge to $H_*(D)$. If we consider the spectral sequence associated to the filtration by columns, using Corollary 2.1.2 we observe that it has the E^2-page of the form:

$${}^cE^2_{pq} = \begin{cases} H_p(\mathcal{C}_* \otimes N) & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

Therefore this spectral sequence degenerates on the E^2-page to yield $H_p(D) = H_p(\mathcal{C}_* \otimes N) = H_p^G(X, N)$.

Since the dual coefficient systems P_i are projective, we have that $H_{q+n}(\mathcal{C}_* \otimes P_n) = \mathcal{H}_q \otimes P_n$. Therefore the E^2-page of the spectral sequence associated to the row filtration has the following form:

$${}^rE^2_{pq} = H_p(\mathcal{H}_q \otimes P_*) = \operatorname{Tor}_p^{\mathcal{O}_G}(\mathcal{H}_q, N).$$

This concludes the proof. □

Remark 2.1.4. As discussed in Subsection 1.7.2, we can obtain the spectral sequence from Theorem 2.1.3 as a spectral sequence of an exact couple. In this case this is the exact couple associated to the row filtration, which takes the following form:

$$\begin{tikzcd}
H\left({}^r\mathbf{F}_k(D)_*\right) \arrow[rr, "\iota"] & & H\left(\mathbf{F}^r_{k+1}(D)_*\right) \arrow[dl, "\lambda"] \\
& H_*\left({}^r\mathbf{F}_{k+1}(D)/{}^r\mathbf{F}_k(D)\right) \cong \mathcal{H}_{*-k} \otimes P_k. \arrow[ul, dashed, "\kappa"] &
\end{tikzcd}$$

2.1.2. Differentials in universal coefficient spectral sequence. In this subsection we are going to prove a useful result which describes differentials in universal coefficient spectral sequence in terms of connecting maps in Tor groups.

Proposition 2.1.5. *Let X be a G-CW-complex (see* [**59**, Section II.1]*) such that for some $i, j \geq 0$:*

(a) it does not have any cells in degrees $i-1$ and $i+j+1$;

(b) the following sequence is exact:

$$0 \to H_{i+j}(X^\bullet) \to C_{i+j}(X^\bullet) \xrightarrow{d} C_{i+j-1}(X^\bullet) \xrightarrow{d} \ldots \xrightarrow{d} C_i(X^\bullet) \to H_i(X^\bullet) \to 0.$$

Then the differentials in UCSS satisfy the following properties:

(1) *If $m > j-1$, then the differentials starting at E^n_{mi} are zero maps for $n \leq j$ and are isomorphisms on E^{j+1}-page.*
(2) *The differentials starting at $E^n_{(j-1)i}$ are zero maps for $n \leq j$ and the differential d^{j+1} starting at $E^{j+1}_{(j-1)i}$ is injective.*

Example 2.1.6. Note that for $G = Q$ and $a, b > 0$ representation spheres $S^{a+b\sigma}$ can be given a Q-CW structure such that they satisfy the conditions of Proposition 2.1.5. This structure is given as follows:

(1) we take the basepoint as the only 0-cell of type Q/Q (fixed cell);
(2) we attach one cell of dimension a and type Q/Q;
(3) we attach one cell of type Q/e (free cell) in each degree from $a+1$ to $a+b$. We will describe the attaching map only for the cell of dimension $a+1$, the rest of cells will be attached by analogous attaching maps. Attaching the free $a+1$-cell needs a Q-equivariant map $\phi\colon Q/e \times S^a \longrightarrow S^a$, which by tensor-hom adjunction in Q-spaces is equivalent to the map $\psi\colon S^a \longrightarrow S^a$. We take the identity as the map ψ, and thus the attaching map is given as its adjunct.

If we consider the reduced homology, the sphere $S^{a+b\sigma}$ satisfies the assumptions of Proposition 2.1.5 with $i = a$ and $j = b$. It has no cells in dimensions $a-1$ and $a+b+1$, so it satisfies assumption (a).

The assumption (b) requires that the space X has $H_n(X^\bullet) = 0$ for $i < n < i+j$. Representation spheres with the given Q-CW-structure satisfy this property, because $H_n((S^{a+b\sigma})^Q) = H_n(S^a) = 0$ for $n > a$ and $H_n((S^{a+b\sigma})^e) = H_n(S^{a+b}) = 0$ if $n < a+b$.

Proof of Proposition 2.1.5. I learned this fact from MathOverflow, from Tyler Lawson's comments to one of my questions - see **[32]**.

Let N be a dual coefficient system, let $P_* \longrightarrow N \longrightarrow 0$ be a projective resolution of N and let

$$F_n := {}^r\mathbf{F}_n(C_*(X^\bullet) \otimes P_*).$$

We are going to use the notation from Proof of Theorem 2.1.3, i.e., we put $C_n := C_n(X^\bullet)$, similarly for homology. We are also going to denote n-th boundaries and n-th cycles coefficients systems by respectively $\mathcal{B}_n$ and $\mathcal{Z}_n$.

Let $\tilde{x} \in \mathrm{Tor}_a(\mathcal{H}_i, N)$ and choose a representing cycle $\bar{x} \in \mathcal{H}_i \otimes P_a$. Note that

$$\mathcal{H}_i \otimes P_a \simeq H_{a+i}(C_* \otimes P_a) \simeq H_{a+i}(F_i/F_{i-1}).$$

We are going to use the following fragment of the unravelled exact couple for the universal coefficient spectral sequence (see Remark 2.1.4):

(∗)

$$\begin{tikzcd}[column sep=small] & H_{a+i-1}(F_{a-j-1}) \arrow[r] \arrow[dl, "\lambda"] & \dots \arrow[r] & H_{a+i-1}(F_{a-2}) \arrow[r, "\iota"] & H_{a+i-1}(F_{a-1}) & \\ \mathcal{H}_{i+j} \otimes P_{a-j-1} & & & & & \mathcal{H}_i \otimes P_a . \arrow[ul, "\kappa"] \end{tikzcd}$$

Our main goal is to show that the consecutive differentials d^n have the same effect as a composition of connecting homomorphisms in Tor associated with the following sequences:

(1) $$0 \to \mathcal{B}_i \to C_i \to \mathcal{H}_i \to 0;$$

(2) $$0 \to \mathcal{Z}_{i+k} = \mathcal{B}_{i+k} \to C_{i+k} \to \mathcal{B}_{i+k-1} \to 0 \quad \text{for } 0 < k < j;$$

(3) $$0 \to \mathcal{H}_{i+j} \to C_{i+j} \to \mathcal{B}_{i+j-1} \to 0.$$

To this end, we are going to track the element $\bar{x} \in \mathcal{H}_i \otimes P_a$ in the exact couple (∗). We will prove association between differentials in the spectral sequence and the connecting homomorphisms as three separate lemmas below.

Recall that $\bar{x}$ is a representing cycle of the element $\tilde{x} \in \mathrm{Tor}_a(\mathcal{H}_i, N)$.

Lemma A. *The element $\kappa(\bar{x}) \in H_{a+i-1}(F_{a-1})$ defines a class in* $\mathrm{Tor}_{a-1}(\mathcal{B}_i, N)$*. Moreover, this class is the same as the image of $\tilde{x}$ under connecting homomorphism for* $\mathrm{Tor}_*(-, N)$ *induced by the sequence* (1).

Proof. Recall from Construction 1.7.11 that the map κ is defined to be the connecting homomorphism in the homology associated to the short exact sequence of the chain complexes:

$$0 \longrightarrow F_{a-1} \longrightarrow F_a \longrightarrow F_a/F_{a-1} \longrightarrow 0.$$

Therefore in order to compute $\kappa(\bar{x})$, we need to:

(1) choose a representing cycle in the chain complex F_a/F_{a-1};
(2) lift it to an element of F_a;
(3) apply the differential of F_a. This element actually comes from the inclusion $F_{a-1} \to F_a$, thus it defines an element in $H_{a+i-1}(F_{a-i})$.

Choose a representative of the class $\bar{x}$ to be $x \in C_i \otimes P_a$. Note that this is the representative from point (1) as $C_* \otimes P_a = F_a/F_{a-1}$. Since x is a representative of an element in $\mathcal{H}_i \otimes P_a$, it is a horizontal cycle and we have that $d^h(x) = 0$. However, x is also a representative of a class in $\mathrm{Tor}_a(\mathcal{H}_i, N)$, so we have that $d^v(x)$ is a horizontal boundary (image under d^h) of an element $y_{i+1,a-1} \in C_{i+1} \otimes P_{a-1}$.

If we apply the differential of the chain complex F_a to x we get $d^h(x) + d^v(x)$. We have that $d^h(x) = 0$ and thus $d^v(x)$ defines a cycle in F_{a-1}, as $(d^v + d^h)(d^v(x)) = d^h(d^v(x)) = -d^v(d^h(x)) = 0$. Therefore $\kappa(\bar{x}) = [d^v(x)]$. Since $d^v(x) = d^h(y_{i+q,a-1})$, we get that $[d^v(x)]$ defines a class in $\mathrm{Tor}(\mathcal{B}_i, N)$. By the definition of the connecting homomorphism for Tor the class defined by the element $\kappa(\bar{x})$ is the same as the value of the connecting homomorphism in Tor associated to the sequence (1) and applied to the element $\tilde{x}$. □

LEMMA B. *The element $\kappa(\bar{x})$ is in the image of ι^j and thus it can be lifted to the element of $H_{a+i-1}(F_{a-j-1})$. Moreover, this lift defines a class in* $\mathrm{Tor}_{a-j}(\mathcal{B}_{i+j-1}, N)$*.*

PROOF. Recall from the proof of Lemma A that $\kappa(\bar{x}) = [d^h(y_{i+1,a-1})]$ for some element $y_{i+1,a-1} \in C_{i+1} \otimes P_{a-1}$. The element $d^h(y_{i+1,a_1})$ is homologous in F_{a-1} to $-d^v(y_{i+1,a-1})$. Recall also that we chose a representing cycle of $\bar{x}$ to be $x \in C_i \otimes P_a$ with the property that $d^v(x)) = d^h(y_{i+q,a-1})$ for some element $y_{i+1,a-1} \in C_{i+1} \otimes P_{a-1}$.

The element $-d^v(y_{i+1,a-1})$ belongs to F_{a-2}. Moreover, it is a cycle in this chain complex. To see this, we apply the differential in F_{a-2} to this element:

$$d_{F_{a-2}}(-d^v(y_{i+1,a-1})) = -d^h(d^v(y_{i+1,a-1})) - d^v(d^v(y_{i+1,a-1})) = -d^h(d^v(y_{i+1,a-1})).$$

To see that $d^h(d^v(y_{i+1,a-1}))$, note that by the properties of differentials in the double complex we have that:

$$-d^h(d^v(y_{i+1,a-1})) = d^v(d^h(y_{i+1,a-1})) = d^v(d^v(x)) = 0.$$

Here x is the representative of the class $\bar{x}$ - see the proof of the previous lemma.

Therefore $-d^v(y_{i+1,a-1})$ defines a class in $H_{a+i-1}(F_{a-2})$. It is in particular a d^h-cycle, so it gives a class in $\mathrm{Tor}_{a-2}(\mathcal{Z}_{i+1}, N)$. By the assumption we have that $\mathcal{H}_{i+1} = 0$, so $-d^v(y_{i+1,a-1}) = d^h(y_{i+2,a-2})$ for some element $y_{i+2,a-2} \in C_{i+2} \otimes P_{a-2}$. Thus we obtain a class in $\mathrm{Tor}_{a-2}(\mathcal{B}_{i+1}, N)$ which is a result of connecting homomorphism associated to the sequence (2). We can repeat this process up to $k = j - 1$. □

LEMMA C. *The element $\lambda\iota^{-j}\kappa(\bar{x})$ defines a class* $\mathrm{Tor}_{a-j-1}(\mathcal{H}_{i+j}, N)$*. This class is the same as the class obtained by applying the connecting homomorphism associated to the exact sequence* (3) *to the class defined by $\iota^{-j}\kappa(\bar{x})$.*

PROOF. Analogous to previous lemmas. □

Now we return to the proof of Proposition 2.1.5. For the first part of the point (1), assume that $n \leq j$. By definition we have that $d^n(\tilde{x}) = [\lambda\iota^{-n+1}\kappa(\bar{x})]$. However, $\lambda\iota^{-n+1}\kappa(\bar{x}) \in \mathcal{H}_{n+i-1} \otimes P_{a-n}$ and by the assumption we have that $\mathcal{H}_{n+i-1} = 0$, so the resulting differential is zero.

For the second claim of the point (1), by lemmas proven in this proof we obtain that the element $d^{j+1}(\tilde{x})$ is the same as the result of applying consecutive connecting homomorphisms associated to sequences (1), (2) and (3). However, all of these sequences have in the middle a coefficient system of the form C_i, which is projective. Therefore the connecting homomorphisms are isomorphisms and so is their composite. So d^{j+1} is an isomorphism.

Analogously we obtain the point (2). □

2.1.3. Universal coefficient spectral sequence in action - Bredon homology of Q-representation spheres. In this section we demonstrate how to calculate using the UCSS. We are going to compute the Bredon homology of representation spheres over Q with coefficients in arbitrary dual coefficient system. We note here that it might be easier to compute the homology of these spheres directly from the definition of Bredon homology. However, the aim here is to show how UCSS can be used for computations.

As a first step we calculate the E_2-page of the UCSS for Q-representation spheres. This can be achieved as follows:

(1) We describe the homology coefficient systems of S^V in Lemma 2.1.8.
(2) We compute Tor groups using explicit projective resolutions by representable objects in $\mathcal{CS}_Q$. See Lemmas 2.1.9, 2.1.10 and 2.1.11.

2.1.3.1. *Homology coefficient systems.* We begin with a description of the coefficient systems which appear in the homology of representation spheres. Recall the notation convention described in Remark 1.4.15 - coefficient systems over Q will be presented by diagrams, where the group on the top is the value of the coefficient system on Q/Q and the group on the bottom is the value on Q/e.

Definition 2.1.7. Define the coefficient systems F^Q, A, B and C as follows:

$$\begin{array}{cccc} F^Q & A & B & C \\ \mathbb{Z} & \mathbb{Z} & 0 & 0 \\ \big\downarrow \mathrm{id} & \big\downarrow & \big\downarrow & \big\downarrow \\ \mathbb{Z} & 0 & \mathbb{Z} & \tilde{\mathbb{Z}}. \end{array}$$

Here $\tilde{\mathbb{Z}}$ is the abelian group $\mathbb{Z}$ with the action of Q given by multiplication by -1.

Note that by Lemma 2.1.1 the first one is projective as a coefficient system of the form F_S with S taken to be a point.

Lemma 2.1.8. *The homology coefficient systems $H\left((S^{a+b\sigma})^\bullet\right)$ can be described as follows:*

(1) If $b = 0$, then

$$H_i\left((S^{a+b\sigma})^\bullet\right) = \begin{cases} F^Q & \text{if } i = 0 \text{ or } i = a \\ 0 & \text{else.} \end{cases}$$

(2) If b is even, then

$$H_i\left((S^{a+b\sigma})^\bullet\right) = \begin{cases} F^Q & \text{if } i = 0 \\ A & \text{if } i = a \\ B & \text{if } i = a+b \\ 0 & \text{else.} \end{cases}$$

In particular, if $a = 0$ then $H_0\left((S^{a+b\sigma})^\bullet\right) = F^Q \oplus A$.

(3) If b is odd, then

$$H_i\left((S^{a+b\sigma})^\bullet\right) = \begin{cases} F^Q & \text{if } i = 0 \\ A & \text{if } i = a \\ C & \text{if } i = a+b \\ 0 & \text{else.} \end{cases}$$

If $a = 0$ then $H_0\left((S^{a+b\sigma})^\bullet\right) = F^Q \oplus A$.

Proof. Follows from $(S^{a+b\sigma})^Q = S^a$ and from the fact that γ acts on $H_{a+b}((S^{a+b\sigma})^e)$ as the degree of the reflection of S^{a+b} fixing the subsphere S^a. □

2.1.3.2. *Computations of Tor groups.* Having described the homology coefficient systems of representation spheres, in this subsection we compute its Tor groups against any dual coefficient system. To this end we describe projective resolutions of coefficient systems A, B and C. We will then use these explicit resolutions to deduce the Tor groups.

Lemma 2.1.9.

$$\mathrm{Tor}_i^{O_Q}(A,N) = \begin{cases} N(Q)/N(e) & \text{if } i = 0 \\ \ker(\mathrm{tr}_N)/(1-\gamma)N(e) & \text{if } i = 1 \\ H_i(Q,N(e)) & \text{if } i > 1 \end{cases}$$

Proof. Recall that groups $\mathrm{Tor}_*^{O_Q}(M,N)$ are calculated by choosing a projective resolution $P \to M$ in $\mathcal{CS}_Q$ and computing homology groups of $P \otimes_{O_Q} N$.

Projective resolution of A in $\mathcal{CS}_Q$ is given by:

$$\begin{tikzcd} 0 \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\ 0 & 0 \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z}[Q] \arrow[l, "\epsilon"'] & \mathbb{Z}[Q] \arrow[l, "1-\gamma"'] & \dots . \arrow[l] \end{tikzcd}$$

Note that from the third place in the bottom row we have a 2-periodic $\mathbb{Z}[Q]$-resolution of $\mathbb{Z}$. Therefore this resolution written in the language of representable coefficient systems has the following form:

$$0 \longleftarrow A \longleftarrow F^Q \xleftarrow{\epsilon} F^e \xleftarrow{1-\gamma} F^e \longleftarrow \dots .$$

Since all of the objects in the resolution are representables, they are projectives and thus this resolution is indeed a projective resolution of A. After applying $- \otimes N$ we obtain the following chain complex:

$$0 \longleftarrow N(Q) \xleftarrow{\mathrm{tr}_N} N(e) \xleftarrow{1-\gamma} N(e) \xleftarrow{1+\gamma} N(e) \longleftarrow \dots .$$

By computing the homology of this chain complex we obtain the statement. □

Lemma 2.1.10.

$$\mathrm{Tor}_i^{O_Q}(B,N) = H_i(Q,N(e)).$$

Proof. The projective resolution of B is given by:

$$0 \longleftarrow B \xleftarrow{\epsilon} F^e \xleftarrow{1-\gamma} F^e \xleftarrow{1+\gamma} F^e \longleftarrow \dots.$$

Similarly as in the proof of Lemma 2.1.9, the statement follows. □

Lemma 2.1.11.

$$\operatorname{Tor}_i^{O_Q}(C,N) = \begin{cases} N(e)/(1+\gamma)N(e) & \textit{if } i = 0 \\ H_{i+1}(Q, N(e)) & \textit{if } i > 0. \end{cases}$$

Proof. The projective resolution of C is given by:

$$0 \longleftarrow B \xleftarrow{\tilde{\epsilon}} F^e \xleftarrow{1+\gamma} F^e \xleftarrow{1-\gamma} F^e \longleftarrow \dots,$$

where $\tilde{\epsilon}$ is the $\mathbb{Z}[Q]$-modules map $\mathbb{Z}[Q] \to \tilde{\mathbb{Z}}$ sending γ to -1. Note that this resolution is the resolution of B shifted by one degree. The statement follows. □

2.1.3.3. *Computations.* We are going to conclude the computations using Lemma 2.1.5. We will use this Lemma for spheres $S^{a+b\sigma}$, where $a, b > 0$, with the Q-CW-structure give in Example 2.1.6. Let N be a dual coefficient system over Q.

Fixed spheres S^a. Since by Lemma 2.1.8 all homology coefficient systems of fixed representation spheres are projective, the E^2-page of UCSS takes the form as in Figure 2.1.

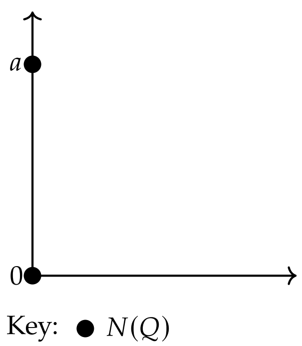


Figure 2.1. E^2-page for S^a.

Therefore we deduce that

$$H_i^Q(S^a, N) = \begin{cases} N(Q) & \text{if } i = 0, a \\ 0 & \text{else.} \end{cases}$$

Spheres with even twisted dimension b. By Lemmas 2.1.9 and 2.1.10 we get that the E^2-page of UCSS for $S^{a+b\sigma}$ with b even takes the form as presented in Figure 2.2. Since representation spheres satisfy the assumptions of Lemma 2.1.5, we deduce that:

$$H_i^Q(S^{a+b\sigma}, N) = \begin{cases} N(Q) & i = 0 \\ N(Q)/N(e) & i = a \\ \ker(\mathrm{tr})/(1+\gamma)N(e) & i = a+1 \\ H_{i-a-1}(Q, N(e)) & a+1 < i < a+b \\ E & i = a+b \\ 0 & \text{else.} \end{cases}$$

Here E is an extension of abelian groups $\frac{N(e)}{I}$ and $H_{a+b}(Q, N(e))$, where I denotes the submodule $\{a \in N(e) \mid (1+\gamma)a = 0\}$. For $i = a+b$ we can compute the homology only up to the extension; using our methods we cannot verify if the extension splits.

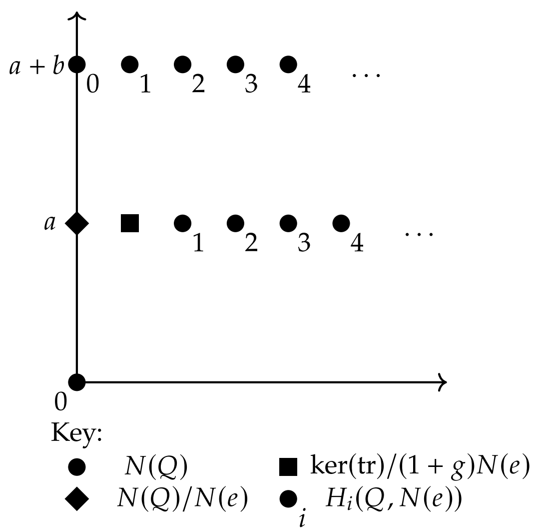


FIGURE 2.2. E^2-page for $S^{a+b\sigma}$ when b is even.

Spheres with odd twisted dimension b. By Lemmas 2.1.9 and 2.1.11 we get that the E^2-page of UCSS takes the form as presented in Figure 2.3. Similarly as in the case of the even twisted dimension, by Lemma 2.1.5 we get that:

$$H_i^Q(S^{a+b\sigma}, N) = \begin{cases} N(Q) & i = 0 \\ N(Q)/N(e) & i = a \\ H_{i-a-1}(Q, N(e)) & a < i < a+b \\ E & i = a+b \\ 0 & \text{else.} \end{cases}$$

Here, as in the case with b even, the abelian group E is an extension of groups $H_{a+b-1}(Q, N(e))$ and $\frac{N(e)}{N(e)^Q}$.

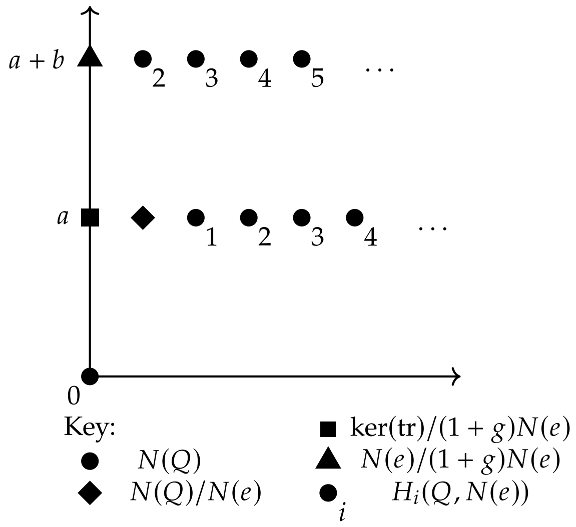


FIGURE 2.3. E^2-page for $S^{a+b\sigma}$ when b is odd.

2.2. Reduction of calculations to graph subgroups

In this section we are going to prove two results. Firstly, we are going to show that the Q-equivariant Bredon homology of spaces $\mathcal{D}_V(n)/\Sigma_n$ is isomorphic to the $Q \times \Sigma_n$-equivariant homology of $\mathcal{D}_V(n)$. Secondly, we will show that in order to compute the $Q \times \Sigma_n$-equivariant homology of spaces $\mathcal{D}_V(n)$ we can restrict our attention to the family of graph subgroups of $Q \times \Sigma_n$.

In fact, we will show that if G_1 and G_2 are finite groups and X is a G_2-free $G_1 \times G_2$-space, then the $G_1 \times G_2$-equivariant Bredon homology of X and G_1-equivariant Bredon homology of X/G_2 are isomorphic to the homology computed by restricting coefficient systems to graph subgroups. These facts are given in Propositions 2.2.17 and 2.2.26. Since the spaces $\mathcal{D}_V(n)$ are Σ_n-free $Q \times \Sigma_n$-spaces, both propositions apply to them.

2.2.1. Graph subgroups. Throughout this section we fix two finite groups G_1 and G_2.

DEFINITION 2.2.1. The subgroup $\Gamma \leq G_1 \times G_2$ is a *graph subgroup* if $\Gamma \cap (1 \times G_2) = 1$.

EXAMPLE 2.2.2.

- For any groups G_1 and G_2 the subgroup $G_1 \times 1$ is a graph subgroup of $G_1 \times G_2$.
- For a finite group G, the diagonal $\{(g, g) \in G \times G \mid g \in G\}$ is a graph subgroup of $G \times G$.

REMARK 2.2.3. The name "graph subgroup" comes from the fact that if Γ is a graph subgroup, then there exists a group homomorphism $f: H \to G_2$ from a subgroup $H \leq G_1$ such that Γ is its graph:

$$\Gamma = \{(g_1, g_2) \in H \times G_2 \mid f(g_1) = g_2\}.$$

Indeed, let Γ be a graph subgroup of $G_1 \times G_2$. To show that it is a graph of a group homomorphism we need to prove that if $(g_1, g_2) \in \Gamma$ and $(g_1, g_2') \in \Gamma$, then $g_2 = g_2'$. But since Γ is a subgroup we have that $(e, g_2(g_2')^{-1}) \in \Gamma$ and by the defining property of graph subgroups we have that $g_2 = g_2'$.

Therefore if Γ is a graph subgroup we define the homomorphism f_Γ as follows: if $(g_1, g_2) \in \Gamma$ then we put $f_\Gamma(g_1) = g_2$. Above we have shown that it is well-defined, and it is a homomorphism by the fact that Γ is a subgroup.

REMARK 2.2.4. Note that if $G_2 = \Sigma_n$ then a graph subgroup Γ endows the set of n points with a structure of H-set via homomorphism f_Γ, where $H \leq G_1$. We will denote this H-set by T_Γ.

PROPOSITION 2.2.5. *Let $\Gamma \leq G_1 \times G_2$. Then the $G_1 \times G_2$-set $\frac{G_1 \times G_2}{\Gamma}$ is G_2-free if and only if Γ is a graph subgroup.*

PROOF. Firstly assume that $\frac{G_1 \times G_2}{\Gamma}$ is $1 \times G_2$-free. Assume that $g \in \Gamma \cap (1 \times G_2)$ and let $(g_1, g_2) \in G_1 \times G_2$. Then since $g \in \Gamma$ we have that

$$g(g_1, g_2)\Gamma = (g_1, g_2)\Gamma.$$

Therefore from the $1 \times G_2$-freeness condition we obtain that $g = e$. So Γ is a graph subgroup.

For the second implication, assume that Γ is a graph subgroup. Let $g, g' \in 1 \times G_2$ and let (g_1, g_2) be such that $g(g_1, g_2)\Gamma = g'(g_1, g_2)\Gamma$. This means that $g'g^{-1} \in \Gamma$. But since $g, g' \in 1 \times G_2$ and Γ is a graph subgroup we get that $g'g^{-1} = e$. Thus $g' = g$ and the action of $1 \times G_2$ on $\frac{G_1 \times G_2}{\Gamma}$ is free. □

PROPOSITION 2.2.6. *Graph subgroups form a family of subgroups in $G_1 \times G_2$, i.e., they are closed under taking subgroups and conjugates.*

PROOF. Let Γ be a graph subgroup and $g = (g_1, g_2) \in G_1 \times G_2$. Then the conjugate ${}^g\Gamma$ is also a graph subgroup. Indeed, let $(e, g_2 g_2' g_2^{-1}) \in {}^g\Gamma \cap G_2$. Since Γ is a graph subgroup, we have that $g_2 g_2' g_2^{-1} = 1$ and so $g_2' = e$. Thus ${}^g\Gamma$ is also a graph subgroup.

Let $\Delta \leq \Gamma$. Then Δ satisfies the graph subgroup property, since

$$\Delta \cap (1 \times G_2) \subset \Gamma \cap (1 \times G_2) = 1.$$

The claim follows. □

DEFINITION 2.2.7.

- The *graph orbit category* $O^\Gamma_{G_1 \times G_2}$ is a full subcategory of $O_{G_1 \times G_2}$ consisting of orbits of the form $(G_1 \times G_2)/\Gamma$, where Γ is a graph subgroup.

- A *graph coefficient system* is a contravariant functor $\mathcal{O}^{\Gamma}_{G_1\times G_2} \to \mathcal{A}\mathrm{b}$. A *graph dual coefficient system* is a covariant functor $\mathcal{O}^{\Gamma}_{G_1\times G_2} \to \mathcal{A}\mathrm{b}$.

EXAMPLE 2.2.8. An important example of graph coefficient system is given by the following functor, where X is a $G_1 \times G_2$-space:

$$\begin{aligned} H_q(X^{\bullet})\colon \mathcal{O}^{\Gamma} &\to \mathcal{A}\mathrm{b} \\ \tfrac{G_1\times G_2}{\Gamma} &\mapsto H_q(X^{\Gamma}). \end{aligned}$$

We will refer to graph coefficient systems of this form as *homology graph coefficient systems*.

2.2.2. Category of G_2-free $G_1 \times G_2$-spaces. In this section we will rephrase the Elmendorf Theorem to show that the homotopy category of G_2-free $G_1 \times G_2$-spaces is equivalent to the homotopy category of diagrams of topological spaces modelled on $\mathcal{O}^{\Gamma}_{G_1\times G_2}$. We will base our exposition on [**48**, Section V.3] and [**58**].

Let $\mathcal{T}^{G_1\times G_2}_{\Gamma}$ denote the full subcategory of $\mathcal{T}^{G_1\times G_2}$ consisting of G_2-free $G_1 \times G_2$-spaces.

DEFINITION 2.2.9.
- A $\mathcal{O}^{\Gamma}_{G_1\times G_2}$*-space* is a contravariant functor $\mathcal{O}^{\Gamma}_{G_1\times G_2} \to \mathcal{T}$.
- *The category of $\mathcal{O}^{\Gamma}_{G_1\times G_2}$-spaces* denoted by $\mathcal{T}^{\mathcal{O}^{\Gamma}_{G_1\times G_2}}$, consists of $\mathcal{O}^{\Gamma}_{G_1\times G_2}$-spaces as objects and natural transformations between them as morphisms.

REMARK 2.2.10. From now on we are going to fix groups G_1 and G_2 and omit them in the notation of the graph orbit category. Thus by $\mathcal{O}^{\Gamma}$ we mean the graph orbit category of $G_1 \times G_2$ and by $\mathcal{T}^{\mathcal{O}^{\Gamma}}$ we mean the category of $\mathcal{O}^{\Gamma}_{G_1\times G_2}$-spaces.

DEFINITION 2.2.11. We define two functors

$$\Theta\colon \mathcal{T}^{\mathcal{O}^{\Gamma}} \rightleftarrows \mathcal{T}^{G_1\times G_2}_{\Gamma} \colon \Phi$$

as follows:

$$\begin{aligned} \Theta(\mathcal{X}) &= \mathcal{X}\,(e\times e) \\ \Phi(X)\left(\tfrac{G_1\times G_2}{\Gamma}\right) &= X^{\Gamma}. \end{aligned}$$

OBSERVATION 2.2.12. $\Theta\Phi(X) = X$.

PROPOSITION 2.2.13. *Θ is left adjoint to Φ.*

PROOF. The map

$$\frac{G_1\times G_2}{e\times e} \to \frac{G_1\times G_2}{\Gamma}$$

induces a map

$$\mathcal{X}(\Gamma) \to \mathcal{X}(e\times e)^{\Gamma}.$$

This gives us a map of $\mathcal{O}^{\Gamma}_{G_1\times G_2}$-spaces

$$\mathcal{X} \to \mathcal{X}(e\times e)^{\bullet}.$$

Let $X \in \mathcal{T}^{\Gamma}$ and let $\phi\colon \mathcal{X} \to \Phi X$ be a map of $\mathcal{O}^{\Gamma}$-spaces. Then the passage from ϕ to $\Theta\phi\colon \Theta\mathcal{X} \to \Theta\Phi X = X$ is a bijection whose inverse sends $f\colon \Theta\mathcal{X} \to X$ to $\Phi(f) \circ \eta$. □

Remark 2.2.14. There is a model structure on $\mathcal{T}^{G_1 \times G_2}$ taking as equivalences maps f such that $\pi_n^H(f)$ is an isomorphism for every $n \in \mathbb{Z}$ and graph subgroup $H \leq G_1 \times G_2$ (see [**58**, Definition 2.3]). We will call this model structure a *graph model structure.*

Proposition 2.2.15. *Functors Θ and Φ constitute a Quillen equivalence between $\mathcal{T}^{\mathcal{O}^{\Gamma}}$ with projective model structure (i.e., such that weak equivalences and fibrations are taken levelwise) and $\mathcal{T}_{\Gamma}^{G_1 \times G_2}$ with graph model structure.*

Proof. See [**58**, Section 3.2]. □

2.2.3. Reduction of $G_1 \times G_2$-equivariant homology to homology over graph subgroups. In this section we describe the isomorphism between $G_1 \times G_2$-equivariant Bredon homology and the *graph Bredon homology* - see Definition 2.2.16.

Let $i\colon \mathcal{O}^{\Gamma}_{G_1 \times G_2} \to \mathcal{O}_{G_1 \times G_2}$ be the inclusion as a full subcategory. Note that if $N \in \mathcal{DCS}_{G_1 \times G_2}$ then we can define the graph dual coefficient system (see Definition 2.2.7) i^*N by $i^*(N)(\Gamma) = N(i(\Gamma))$.

Definition 2.2.16. Let N be a graph dual coefficient system. We define the *graph Bredon homology* $H_*^{\Gamma}(X, N)$ of a $G_1 \times G_2$-space X to be the homology of the chain complex $S_*(X^{\bullet}) \otimes_{\mathcal{O}^{\Gamma}_{G_1 \times G_2}} N$, where $S_*(-)$ denotes the singular chains.

Proposition 2.2.17. *Let X be a $G_1 \times G_2$-space which is G_2-free and let $N \in \mathcal{DCS}_{G_1 \times G_2}$. Then we have that*

$$H_*^{\Gamma}(X, i^*N) = H_*^{G_1 \times G_2}(X, N).$$

Proof. We are going to show that under the given assumption on X we have that

$$S_*(X^{\bullet}) \otimes_{\mathcal{O}_{G_1 \times G_2}} N = S_*(X^{\bullet}) \otimes_{\mathcal{O}^{\Gamma}_{G_1 \times G_2}} i^*N.$$

Recall the definition of the tensor product over the orbit category:

$$S_*(X^{\bullet}) \otimes_{\mathcal{O}_{G_1 \times G_2}} N = \frac{\bigoplus_{H \leq G_1 \times G_2} S_*(X^H) \otimes N(H)}{I}$$

where

$$I = \left\langle f^*x \otimes y - x \otimes f_*y \;\middle|\; f \in \mathrm{Map}_{\mathcal{O}_{G_1 \times G_2}}\left(\frac{G_1 \times G_2}{K}, \frac{G_1 \times G_2}{H}\right), x \in S_*(X^H), y \in N(K) \right\rangle.$$

Since X is G_2-free $G_1 \times G_2$-space, it may have non-trivial fixed points only under graph subgroups. Therefore we have

$$\bigoplus_{H \leq G_1 \times G_2} S_*(X^H) \otimes N(H) = \bigoplus_{\substack{\Gamma \leq G_1 \times G_2 \\ \Gamma \cap (1 \times G_2) = 1}} S_*(X^{\Gamma}) \otimes N(\Gamma).$$

For the same reason we can restrict in the definition of the subgroup I only to graph subgroups. Thus the statement follows. □

2.2.4. Graph Bredon homology vs. the homology of orbits under G_2-action. The remainder of this section is devoted to showing that the G_1-equivariant Bredon homology of the space X/G_2, where $X \in \mathcal{T}^{\Gamma}_{G_1 \times G_2}$, is isomorphic to graph Bredon homology. This is given in Proposition 2.2.26. The proof of this fact is based on the MathOverflow answer given by Gregory Arone to my question **[33]**. In this section, to ensure Hausdorffness, we will assume that X is a $G_1 \times G_2$-CW-complex. Note that in the graph model structure (see 2.2.14) every $G_1 \times G_2$-space which is G_2-free is weakly equivalent to a G_2-free $G_1 \times G_2$-CW-complex.

Before stating the main result of this section, we need to give several facts concerning adjunctions between diagram categories and coends of functors.

Definition 2.2.18. Let $\pi \colon G_1 \times G_2 \to G_1$ be the projection on the first factor. Then we define a functor $P \colon \mathcal{O}^{\Gamma} \to \mathcal{O}_{G_1}$ by:

$$P\left(\frac{G_1 \times G_2}{\Gamma}\right) = \frac{G_1}{\pi(\Gamma)}.$$

Proposition 2.2.19. *The functor $P \colon \mathcal{O}^{\Gamma} \to \mathcal{O}_{G_1}$ can be identified by a natural isomorphism with taking quotient by $1 \times G_2$.*

Proof. Consider the group homomorphism $\pi|_{\Gamma} \colon \Gamma \to \pi(\Gamma)$. Its kernel consists of all elements of the form $(e, g_2) \in \Gamma$. However, since Γ is a graph subgroup, we need to have that $g_2 = e$, so the kernel of $\pi|_{\Gamma}$ is trivial and $\pi|_{\Gamma}$ is an isomorphism. Since $\Gamma \cap (1 \times G_2) = 1$, the statement follows for objects.

Let $f \colon \frac{G_1 \times G_2}{\Delta_1} \to \frac{G_1 \times G_2}{\Delta_2}$ be a map in $\mathcal{O}^{\Gamma}$. By Proposition 2.2.5 both domain and codomain are $1 \times G_2$-free and thus f induces a map on orbits. We take $P(f)$ to be this induced map. □

Observation 2.2.20. The functor P induces the following adjoint pair:

$$P_! \colon \mathcal{T}^{\mathcal{O}^{\Gamma}} \rightleftarrows \mathcal{T}^{\mathcal{O}_{G_1}} \colon P^*$$

where the right adjoint P^* is the precomposition with P and $P_!$ is the left Kan extension along P.

Recall the pair of functors

$$\Theta \colon \mathcal{T}^{\mathcal{O}^{\Gamma}} \rightleftarrows \mathcal{T}^{G_1 \times G_2}_{\Gamma} \colon \Phi$$

from Definition 2.2.11 and define

$$\tilde{\Theta} \colon \mathcal{T}^{\mathcal{O}_{G_1}} \rightleftarrows \mathcal{T}^{G_1} \colon \tilde{\Phi}$$

to be the pair of functors constituting Elmendorf's correspondence for G_1.

Proposition 2.2.21. *The composite functor*

$$\tilde{\Theta} P_! \Phi \colon \mathcal{T}^{G_1 \times G_2}_{\Gamma} \to \mathcal{T}^{G_1}$$

can be identified by a natural isomorphism with the functor $X \mapsto \frac{X}{1 \times G_2}$.

Proof. Note that for an $\mathcal{O}^\Gamma$-space $\mathcal{X}$ the functor $P_!$ is defined as the left Kan extension of $\mathcal{X}$ along P. In such a case, there is an explicit formula for $P_!\mathcal{X}(G_1/H)$ in terms of coends (see [**45**, Section X.4], also [**43**, Proposition 2.3.6]):

$$P_!\mathcal{X}(G_1/H) = \int^{\frac{G_1\times G_2}{\Gamma}\in\mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)}\right) \times \mathcal{X}\left(\frac{G_1\times G_2}{\Gamma}\right).$$

Therefore, by definitions of functors $\tilde{\Theta}$ and Φ, we get that

$$\tilde{\Theta}P_!\Phi(X) = \int^{\frac{G_1\times G_2}{\Gamma}\in\mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{1}, \frac{G_1}{\pi(\Gamma)}\right) \times X^\Gamma.$$

Note that $\mathcal{O}_{G_1}(G_1/1, G_1/H) \cong G_1/H$. Therefore this coend can be written as follows:

$$\int^{\frac{G_1\times G_2}{\Gamma}\in\mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{1}, \frac{G_1}{\pi(\Gamma)}\right) \times X^\Gamma = \frac{\coprod \frac{G_1}{\pi(\Gamma)} \times X^\Gamma}{\simeq},$$

where the disjoint union in the numerator is taken over all graph subgroups $\Gamma \le G_1 \times G_2$ and $\simeq$ is the equivalence relation described as follows: for a map in $\mathcal{O}^\Gamma$ given by an element $g = (g_1, g_2) \in G_1 \times G_2$:

$$g\colon \frac{G_1\times G_2}{\Delta} \to \frac{G_1\times G_2}{\Gamma}$$

and elements $x \in X^\Gamma$ and $h\pi(\Delta) \in G_1/\pi(\Delta)$ we put that

$$(g_1 h\pi(\Gamma), x) \simeq (h\pi(\Delta), g^{-1}x).$$

Note that by this equivalence relation we have that

$$\frac{G_1}{\pi(\Gamma)} \times X^\Gamma \ni (h\pi(\Gamma), x) \simeq (e, (h, e)^{-1}x) \in \frac{G_1}{1} \times X^{\{e\times e\}}.$$

Now we proceed as in the Observation 1.3.2. Define the map

$$\phi\colon \int^{\frac{G_1\times G_2}{\Gamma}\in\mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{1}, \frac{G_1}{\pi(\Gamma)}\right) \times X^\Gamma \to \frac{X}{1\times G_2}$$

by

$$\phi : (h\pi(\Gamma), x) \simeq (e, (h^{-1}, e)x) \mapsto [(h^{-1}, e)x].$$

This map is well-defined in that its codomain is indeed $\frac{X}{1\times G_2}$. Indeed, let $g_2 \in G_2$. Then

$$\phi((e, g_2)(e, x)) = \phi((e, g_2 x)) = [g_2 x] = [x].$$

Define the map

$$\psi\colon \frac{X}{1\times G_2} \to \int^{\frac{G_1\times G_2}{\Gamma}\in\mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{1}, \frac{G_1}{\pi(\Gamma)}\right) \times X^\Gamma$$

by

$$\psi\colon [x] \mapsto [(e, x)],$$

where $(e, x) \in \frac{G_1}{1} \times X^{e\times e}$.

Note that this map is also well-defined. Indeed, let $g_2 \in G_2$. Then we have $[x] = [(e, g_2)x]$ and

$$\psi([(e, g_2)x]) = [(e, (e, g_2)x)] = [(e, g_2)].$$

We also have that $\psi\phi = \mathrm{id}$ and $\phi\psi = \mathrm{id}$. Therefore this maps constitutes homeomorphism and the claim follows. □

REMARK 2.2.22. Recall that by Proposition 2.2.15 we can identify (up to homotopy) the category $\mathcal{T}_\Gamma^{G_1\times G_2}$ with the category of O^Γ-spaces. Therefore, by Proposition 2.2.21, the functor $P_!$ corresponds via Elmendorf's correspondence to the quotient by $1 \times G_2$.

LEMMA 2.2.23. *Let $F\colon C_0 \to C$ be a functor between locally small categories and let $G\colon C^{op} \to \mathcal{D}$ be a functor. Then*

$$\int^{y\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(Fz, y) \otimes G(y) \cong GF(z)$$

for any $z \in \mathcal{D}$. Here "$\otimes$" stands for the copower over $\mathcal{S}et$ in the category $\mathcal{D}$. If $\mathcal{D}$ is closed monoidal, then it is enriched over itself and the copower over $\mathcal{S}et$ can be extended to coincide with the monoidal product.

PROOF. See [**43**, Proposition 2.2.1]. □

PROPOSITION 2.2.24 (See [**29**, Proposition 19.6.6]). *Let $F\colon C_1 \to C_2$ be a functor between two small categories C_1, C_2 and let $\mathcal{D}$ be a locally small category. Let further $F_!$ and F^* be the induced adjoint pair of functors between functor categories:*

$$F_!\colon\ [C_1, \mathcal{D}] \rightleftarrows [C_2, \mathcal{D}] : F^*,$$

where F^* is a precomposition with F and $F_!$ is the left Kan extension along F. Let $M\colon C_1 \to \mathcal{D}$ and $N\colon C_2^{op} \to \mathcal{D}$. Then there is an isomorphism of coends (see 1.3.1):

$$M \otimes_{C_1} F^*N \cong F_!M \otimes_{C_2} N.$$

PROOF. Firstly, let us recall that similarly as in the proof of Proposition 2.2.21 there is an explicit formula for the left Kan extension:

$$F_!M\colon y \mapsto \int^{z\in C_1} \mathrm{Hom}_{C_2}(Fz, y) \otimes M(z). \tag{$*$}$$

We now derive the isomorphism from the "Fubini formula for coends" (see [**45**, Section IX.4], [**43**, Theorem 1.3.1]):

$$\begin{aligned}
F_! M \otimes_{C_2} N &= \int^{y \in C_2} F_! M(y) \otimes N(y) \overset{(1)}{\cong} \\
&\cong \int^{y \in C_2} \left(\int^{z \in C_1} \mathrm{Hom}_{C_2}(Fz, y) \otimes M(z) \right) \otimes N(y) \overset{(2)}{\cong} \\
&\cong \int^{z \in C_1} M(z) \otimes \left(\int^{y \in C_2} \mathrm{Hom}_{C_2}(Fz, y \otimes N(y)) \right) \overset{(3)}{\cong} \\
&\cong \int^{z \in C_1} M(z) \otimes NF(y) = M \otimes_{C_1} NF = M \otimes_{C_1} F^* N.
\end{aligned}$$

Here the isomorphism (1) is application of formula (*). The isomorphism (2) is an application of the Fubini formula applied to the following functor:

$$\begin{aligned}
&C_1^{op} \times C_1 \times C_2^{op} \times C_2 \to \mathcal{D} \\
&\qquad (y, y', z, z') \mapsto \mathrm{Hom}_{C_2}(Fy', z') \otimes M(y') \otimes N(z)
\end{aligned}$$

together with the fact that $M(z) \otimes -$, as a left adjoint, preserves colimits and thus commutes with coends.

Finally, the isomorphism (3) comes from Lemma 2.2.23. □

Lemma 2.2.25. *Let*

$$P_! \colon \left[O^\Gamma, Ch(\mathcal{A}b) \right] \rightleftarrows [O_{G_1}, Ch(\mathcal{A}b)] : P^*.$$

be the adjunction between functor categories induced by P. Then there is a natural isomorphism:

$$P_!(S_*(X^\bullet)) \cong S_* \left(\left(\frac{X}{1 \times G_2} \right)^\bullet \right).$$

Proof. By the coend formula for the left Kan extension ([**45**, Section X.4],[**43**, Proposition 2.3.6]) we have that

$$P_!(S_*(X^\bullet))(G_1/H) = \int^{\frac{G_1 \times G_2}{\Gamma} \in O^\Gamma} O_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)} \right) \times S_*(X^\Gamma).$$

Define the equivalence relation $\simeq$ on the set

$$\bigoplus_{\substack{\Gamma \leq G_1 \times G_2 \\ \Gamma \cap (1 \times G_2) = 1}} O_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)} \right) \times S_*(X^\Gamma)$$

as follows. Let

$$g \colon \frac{G_1 \times G_2}{\Gamma} \to \frac{G_1 \times G_2}{\Gamma'}$$

be a map in O^Γ given by the element $g = (g_1, g_2) \in G_1 \times G_2$. For elements $\iota \in O_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)} \right)$ and $f \in C_n(X^{\Gamma'})$ treated as a continuous map from the standard n-simplex $f \colon \Delta^n \to X^{\Gamma'}$ we put

$$(g_1\iota, f) \simeq (\iota, g^* f).$$

Here $g^* f$ if the following composite map:

$$\Delta^n \xrightarrow{f} X^{\Gamma'} \xrightarrow{X^g} X^{\Gamma}.$$

Using the relation $\simeq$, we can write that

$$\int^{\frac{G_1\times G_2}{\Gamma}\in \mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)}\right)\times S_*(X^\Gamma) \cong \frac{\bigoplus \mathcal{O}_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)}\right)\times S_*(X^\Gamma)}{\simeq}.$$

Firstly note that for $\iota \in \mathcal{O}_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)}\right)$ and $f\colon \Delta^n \to X^\Gamma$ we have that

$$(\iota, f) \simeq (e, (\iota^{-1}, e)^* f).$$

Indeed, since ι is a G_1-equivariant map from $\frac{G_1}{H}$ to $\frac{G_1}{\pi(\Gamma)}$, it can be identified with the element $\iota \in G_1$ such that $\iota H \iota^{-1} \leq \pi(\Gamma)$. Let ϕ_Γ be the defining homomorphism of Γ (see Remark 2.2.3). Then $\phi_\Gamma|_{\iota H \iota^{-1}}$ defines a subgroup $\tilde{\Gamma} \leq \Gamma$ such that $\pi(\tilde{\Gamma}) = H$ and element (ι^{-1}, e) defines a $G_1 \times G_2$-equivariant map

$$\frac{G_1\times G_2}{(\iota^{-1}, e)\tilde{\Gamma}(\iota, e)} \to \frac{G_1\times G_2}{\Gamma}.$$

In particular, every element (ι, f) is equivalent by $\simeq$ to the element (e, h) such that $h \in S_*(X^\Gamma)$ with $\pi(\Gamma) = H$.

We now define the map

$$\psi\colon \int^{\frac{G_1\times G_2}{\Gamma}\in \mathcal{O}^\Gamma} \mathcal{O}_{G_1}\left(\frac{G_1}{H}, \frac{G_1}{\pi(\Gamma)}\right)\times S_*(X^\Gamma) \to S_*\left(\left(\frac{X}{1\times G_2}\right)^H\right)$$

by $\psi([(e,f)]) = \frac{f}{1\times G_2}$, where $\frac{f}{1\times G_2}$ is obtained by passing to $1\times G_2$-orbits on the $G_1 \times G_2$-equivariant map

$$f\colon \Delta^n \times \frac{G_1\times G_2}{\Gamma} \to X.$$

Recall from Proposition 2.2.19 that

$$\frac{s\frac{G_1\times G_2}{\Gamma}}{1\times G_2} \cong \frac{G_1}{\pi(\Gamma)}.$$

We firstly prove that ψ is injective. Let $f_1\colon \Delta^n \times \frac{G_1\times G_2}{\Gamma} \to X$ and $f_2\colon \Delta^n \times \frac{G_1\times G_2}{\Gamma'} \to X$ be $G_1\times G_2$-equivariant maps such that $\psi([(e,f_1)]) = \psi([(e,f_2)])$. This means that $\frac{f_1}{1\times G_2} = \frac{f_2}{1\times G_2}$.

This condition is equivalent to the following: for every $x \in \Delta^n$ there exists $g \in G_2$ such that

$$f_1(x, e\Gamma) = (e, g) f_2(x, e\Gamma').$$

Thus we obtain a function $\alpha\colon \Delta^n \to G_2$ defined by the equation

$$f_1(x, e\Gamma) = \alpha(x) f_2(x, e\Gamma').$$

If we endow G_2 with the discrete topology, then from the continuity of maps f_1 and f_2 we can derive that α is continuous, and thus there exists $g_2 \in G_2$ such that for all $x \in \Delta^n$

$$f_1(x, e\Gamma) = (e, g_2) f_2(x, e\Gamma').$$

Therefore we obtain that

$$(e, f_2) = (e, (e, g_2)^* f_1) \simeq (e, f_1).$$

So ψ is injective.

Now we will show that ψ is surjective. To this end, we will use the homeomorphism

$$(*) \qquad \frac{X^\Gamma}{1 \times G_2} \cong \left(\frac{X}{1 \times G_2}\right)^{\pi(\Gamma)}.$$

Therefore we obtain that the map

$$X^\Gamma \to \left(\frac{X}{1 \times G_2}\right)^{\pi(\Gamma)}$$

obtained by composing the homeomorphism $(*)$ with the canonical projection on the orbit space is a covering. Note that this is the place where Hausdorffness of X is used, as action of a finite group on a Hausdorff space is always properly discontinuous and this makes projection onto space of orbits a covering map. Now let

$$f\colon \Delta^n \to \left(\frac{X}{1 \times G_2}\right)^H$$

be a generator of $C_n\left(\left(\frac{X}{1\times G_2}\right)^H\right)$. If we put $\Gamma = H \times 1 \leq G_1 \times G_2$, then we have the covering

$$X^\Gamma \to \left(\frac{X}{1 \times G_2}\right)^H.$$

Since Δ^n is contractible space, we can lift the map f to a map $\tilde{f}\colon \Delta^n \to X^\Gamma$. Then we have that $[(e, \tilde{f})] = f$. So ψ is surjective.

The differentials in the coend are given by $d[(\iota, f)] = [(\iota, df)]$. With this definition we have that ψ preserves differentials, therefore it is an isomorphism of chain complexes.

Since the choice of the subgroup H was arbitrary, we get the desired natural isomorphism of functors. $\square$

Proposition 2.2.26. *Let $N \in \mathcal{DCS}_{G_1}$ and let $X \in \mathcal{T}^\Gamma$. Then we have that*

$$H_*^{G_1}\left(\frac{X}{1 \times G_2}, N\right) \cong H_*^\Gamma(X, P^*N).$$

Proof. Note that we can embed the category $\mathcal{CS}_{G_1}$ in $[\mathcal{O}_{G_1}, \mathcal{C}h(\mathcal{A}b)]$ as chain complexes concentrated in degree zero. Therefore using Proposition 2.2.24 and Lemma 2.2.25 we get the following chain of isomorphisms:

$$S_*(X^\bullet) \otimes_{\mathcal{O}^\Gamma} P^*N \cong P_! S_*(X^\bullet) \otimes_{\mathcal{O}_{G_1}} N \cong S_*\left(\left(\frac{X}{1 \times G_2}\right)^\bullet\right) \otimes_{\mathcal{O}_{G_1}} N.$$

The isomorphism of chain complexes induces isomorphism of homology groups and the statement follows. □

2.3. Representable objects in the category of graph coefficient systems over $Q \times \Sigma_n$

In this section we are going back to the case of the main interest and we will describe the representable objects in the category of graph coefficient systems over $Q \times \Sigma_n$.

2.3.1. Graph subgroups of $Q \times \Sigma_n$. In this section we describe the graph subgroups of $Q \times \Sigma_n$.

Remark 2.3.1. Note that any homomorphism $f : Q \to \Sigma_n$ is determined by the value on the generator γ. Since γ is an element of order 2, the order of $f(\gamma)$ needs to divide 2. Thus $f(\gamma)$ has to be either the trivial element or a product of disjoint transpositions.

Therefore we can deduce that a subgroup $\Gamma \subset Q \times \Sigma_n$ is a graph subgroup if and only if it is of one of the following forms:

(1) $e \times e$;
(2) it is a graph of a homomorphism $f : Q \to \Sigma_n$ such that $f(\gamma)$ is either the trivial element or a product of disjoint transpositions.

Proposition 2.3.2. *Let Γ and Γ' be non-trivial graph subgroups of $Q \times \Sigma_n$ corresponding to homomorphisms f, f' such that $f(\gamma)$ has the same cycle type as $f'(\gamma)$. Then Γ is conjugate to Γ'.*

Proof. Let $\sigma = f(\gamma)$ and $\sigma' = f'(\gamma)$. Since both σ and σ' have the same cycle type, they are conjugate. Let τ be such that $\tau\sigma\tau^{-1} = \sigma'$. Then $\Gamma' = (1, \tau)\Gamma(1, \tau^{-1})$. □

Remark 2.3.3. By the proposition above we see that a conjugacy class of graph subgroups in $Q \times \Sigma_n$ is formed of all graph subgroups with the property that the element γ hits by the defining homomorphism either the trivial element or a product of i disjoint transpositions. Therefore all subgroups satisfying this property are isomorphic in the orbit category. Let $\sigma_i := (1\ 2) \ldots (2i-1\ 2i) \in \Sigma_n$ be the product of i disjoint transpositions. We choose the representative of each of conjugacy classes of graph subgroups to be $\Gamma_0 := \{(e, e), (\gamma, e)\}$ and

$$\Gamma_i := \{(e, e), (\gamma, \sigma_i)\}$$

for $i \geq 1$.

These two propositions give us the description of the objects of $\mathcal{O}^\Gamma_{Q \times \Sigma_2}$. We proceed now to the description of morphisms in this category.

Lemma 2.3.4. *The centraliser of the element σ_i in Σ_n can be described as follows:*

$$C_{\Sigma_n}(\sigma_i) \cong (\Sigma_2 \wr \Sigma_i) \times \Sigma_{n-2i}.$$

Proof. Let $\tau \in \Sigma_n$ be such that $\tau\sigma_i\tau^{-1} = \sigma_i$. By properties of the conjugation in the symmetric group we have that:

$$(\tau(1)\ \tau(2))\,(\tau(3)\ \tau(4))\dots(\tau(2i-1)\ \tau(2i)) = (1\ 2)(3\ 4)\dots(2i-1; 2i).$$

This means that the permutation τ can act in the following ways on the set $\{1, 2, \dots, n\}$:

(1) independently swap elements in blocks corresponding to transpositions;
(2) permute blocks corresponding to transpositions;
(3) act in any way on the set $\{2i+1, \dots, n\}$.

Therefore τ is an element of the wreath product $\Sigma_2 \wr \Sigma_i \times \Sigma_{n-2i}$. The statement follows. □

Proposition 2.3.5. *Recall the subgroup $\Gamma_i = \{(e, e), (\gamma, \sigma_i)\}$ in $Q \times \Sigma_n$. Then the group of automorphisms of the orbit given by Γ_i can be described as follows:*

$$\mathrm{Aut}_{O_{Q\times\Sigma_n}}\left(\frac{Q \times \Sigma_n}{\Gamma_i}\right) \cong (Q \wr \Sigma_i) \times \Sigma_{n-2i}.$$

Proof. We need to show that

$$\frac{N_{Q\times\Sigma_n}(\Gamma_i)}{\Gamma_i} \cong Q \wr \Sigma_i \times \Sigma_{n-2i}.$$

Firstly we have to describe the normalizer $N_{Q\times\Sigma_n}(\Gamma_i)$. Since the element γ is of the order two, we have that $N_{Q\times\Sigma_n}(\Gamma_i) = Q \times C_{\Sigma_n}(\sigma_i)$. By Lemma 2.3.4 we obtain further that $N_{Q\times\Sigma_n}(\Gamma_i) = Q \times (Q \wr \Sigma_i \times \Sigma_{n-2i})$.

Since Γ_i has 2 elements, it can be identified with the outer Q in the product above. Thus we obtain the claim. □

Observation 2.3.6. By Proposition 2.3.2 we see that there are no maps between orbits corresponding to different non-trivial graph subgroups of $Q \times \Sigma_n$. The only possible maps are

$$\mathrm{Map}_{O_{Q\times\Sigma_n}}\left(\frac{Q \times \Sigma_n}{e \times e}, \frac{Q \times \Sigma_n}{\Gamma_i}\right) \cong \frac{Q \times \Sigma_n}{\Gamma_i}.$$

Therefore we can describe the category $\mathcal{O}^{\Gamma}_{Q\times\Sigma_n}$ by the following diagram:

$$
\begin{tikzcd}
\frac{Q\times\Sigma_n}{\Gamma_0} \arrow[loop above, "\Sigma_n"] & \frac{Q\times\Sigma_n}{\Gamma_1} \arrow[loop above, "Q\times\Sigma_{n-2}"] & \dots & \frac{Q\times\Sigma_n}{\Gamma_i} \arrow[loop above, "(Q\wr\Sigma_i)\times\Sigma_{n-2i}"] & \dots \\
 & & \frac{Q\times\Sigma_n}{e\times e} \arrow[ull, "\frac{Q\times\Sigma_n}{\Gamma_0}"] \arrow[ul, "\frac{Q\times\Sigma_n}{\Gamma_1}"] \arrow[u, dashed] \arrow[ur, "\frac{Q\times\Sigma_n}{\Gamma_i}"'] \arrow[urr, dashed] \arrow[loop below, "Q\times\Sigma_n"] & &
\end{tikzcd}
$$

2.3.2. Representable objects in $\mathcal{CS}^{\Gamma}_{Q\times\Sigma_n}$. In this section we describe the representable coefficient systems. Since our goal is to use the universal coefficients spectral sequence, we need to compute Tor's over the graph orbit category - and to this end we are going to use the representable objects for finding projective resolutions.

We are going to use the following convention for the notation of coefficient systems over $Q \times \Sigma_n$. For a coefficient system $\mathcal{X}$, in the top row we are going to write values of $\mathcal{X}$ on consecutive graph orbits corresponding to Γ_i (see Remark 2.3.3), and the only spot in the bottom row will be the value of $\mathcal{X}$ on $e \times e$:

$$
\begin{tikzcd}
\mathcal{X}(\Gamma_0) \arrow[drr] & \mathcal{X}(\Gamma_1) \arrow[dr] & \dots \arrow[d, dashed] & \mathcal{X}(\Gamma_i) \arrow[dl] & \dots \arrow[dll, dashed] \\
 & & \mathcal{X}(e\times e) & &
\end{tikzcd}
$$

Observation 2.3.7. In $\mathcal{CS}^{\Gamma}_{Q\times\Sigma_n}$ we have two types of the representable objects:

(1) $F_{e\times e} = \mathbb{Z}\left[\mathrm{Map}_{\mathcal{O}^{\Gamma}_{Q\times\Sigma_n}}\left(-, \frac{Q\times\Sigma_n}{e\times e}\right)\right]$

$$
\begin{tikzcd}
0 \arrow[dr] & \dots & 0 \arrow[dl] \\
 & \mathbb{Z}[Q\times\Sigma_n] &
\end{tikzcd}
$$

(2) $F_{\Gamma_i} = \mathbb{Z}\left[\mathrm{Map}_{\mathcal{O}^{\Gamma}_{Q\times\Sigma_n}}\left(-, \frac{Q\times\Sigma_n}{\Gamma_i}\right)\right]$

$$
\begin{tikzcd}
0 \arrow[drr] & \dots \arrow[dr, dashed] & \mathbb{Z}[(Q\wr\Sigma_i)\times\Sigma_{n-2i}] \arrow[d] & \dots \arrow[dl, dashed] & 0 \arrow[dll] \\
 & & \mathbb{Z}[\frac{Q\times\Sigma_n}{\Gamma_i}] & &
\end{tikzcd}
$$

2.4. Homology of configuration spaces

In this section we are going to recall some basic facts concerning configuration spaces of points in the plane. In particular, we are going to provide a hands-on description of generators of homology groups. This is based on the Dev Sinha's article [**56**]. For the general theory of configuration spaces the reader is referred to [**17**].

Recall Definition 1.8.13:

Definition 2.4.1. Let X be a topological space and n a natural number. We define the *configuration space of n points in X* to be:

$$\mathrm{Conf}_n(X) := \left\{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } 1 \leq i < j \leq n\right\}.$$

Since the case $\mathrm{Conf}_n(\mathbb{R}^2)$ will be of our main interest, from now on we will focus mostly on the case when $X = \mathbb{R}^2$.

Proposition 2.4.2. *The configuration space* $\mathrm{Conf}_2(\mathbb{R}^d)$ *is homotopy equivalent to* S^{d-1}. *Thus the homology of* $\mathrm{Conf}_2(\mathbb{R}^d)$ *is free and of rank one in degrees* 0 *and* $d-1$ *and zero otherwise.*

Proof. See [**56**, Proposition 2.2]. □

Definition 2.4.3 ([**56**, Definition 2.3]).

(1) Let S be a subset of $\{1, \ldots, n\}$. An *S-tree* is an acyclic graph whose vertices are either trivalent or univalent, with a distinguished univalent vertex called *the root*. Univalent vertices other than the root are called *leaves* and are labelled by elements of S. Trivalent vertices are called *internal vertices*. We embed this tree in the upper half-plane with the root at the origin.

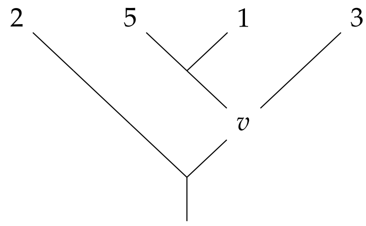


Figure 2.4. An example of S-tree. The vertex v will be used in the next example.

(2) The *height of a vertex v* in an S-tree, denoted by $h(v)$, is the number of edges between v and the root. Edges which connect a vertex to higher vertices are called outgoing.
(3) Let v be a vertex. The *subtree associated to v*, denoted by T_v, is defined as a subtree consisting of all edges lying above v with additional root and edge connecting the root and v.
(4) We say that v *is above/over* w if w lies in the shortest path from v to the root. Define a total order on internal vertices of T so that $v < w$ if v lies over the left outgoing

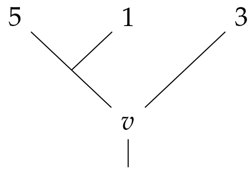


FIGURE 2.5. The subtree T_v of the tree from Figure 2.4.

edge of w and $v > w$ if it lies over the right outgoing edge of w. This total ordering can be realized as a left to right ordering of an appropriate planar embedding.

(5) Let v be an internal vertex of T. Then the T_v^L is the subtree associated to the closest left vertex over v, analogously T_v^R is the subtree associated to the closest right vertex over v.

REMARK 2.4.4. Note that the embedding in the upper half-plane in the point (1) of the definition above is used only for making sense of notions of "left" and "right" in the rest of the definition.

Now we are going to describe how S-trees give embeddings of tori in configuration spaces. These embeddings will be later on used for describing elements of homology groups of configuration spaces.

DEFINITION 2.4.5 ([**56**, Definition 2.4]). Let T be an S-tree and $\mathbf{x} \in \mathrm{Conf}_n(\mathbb{R}^d)$. The *center* $c(\mathbf{x}, T)$ *of configuration* $\mathbf{x}$ *with respect to the tree* T is defined inductively as follows:

(1) If T consists of only one leaf labeled by i, then $c(\mathbf{x}, T) = x_i$.

(2) If v has at least one internal vertex v, then

$$c(\mathbf{x}, T_v) = \frac{1}{2}\left(c(\mathbf{x}, T_v^L)\right) + \frac{1}{2}\left(c(\mathbf{x}, T_v^R)\right).$$

Finally, we define the key notion which will establish the connection between S-trees and homology of configuration spaces of points in the Euclidean space.

DEFINITION 2.4.6 ([**56**, Definition 2.5]). Let $\delta > 0$. Given an S-tree T, the *planetary system* P_T is the submanifold of $\mathrm{Conf}_n(\mathbb{R}^d)$ consisting of all configurations $\mathbf{x}$ such that:

(1) $c(\mathbf{x}, T) = 0$.

(2) For any vertex v of T, the following distance condition is satisfied:

$$\rho\left(c(\mathbf{x}, T_v^L), c(\mathbf{x}, T_v))\right) = \delta^{h(v)} = \rho\left(c(\mathbf{x}, T^v), c(\mathbf{x}, T_v^R))\right).$$

Here ρ is the Euclidean metric in $\mathbb{R}^d$ and $h(v)$ is the height of the vertex v - see Definition 2.4.3.

(3) If $i \notin S$, x_i is fixed as some point "at infinity".

An example of the planetary system is given in Figure 2.6.

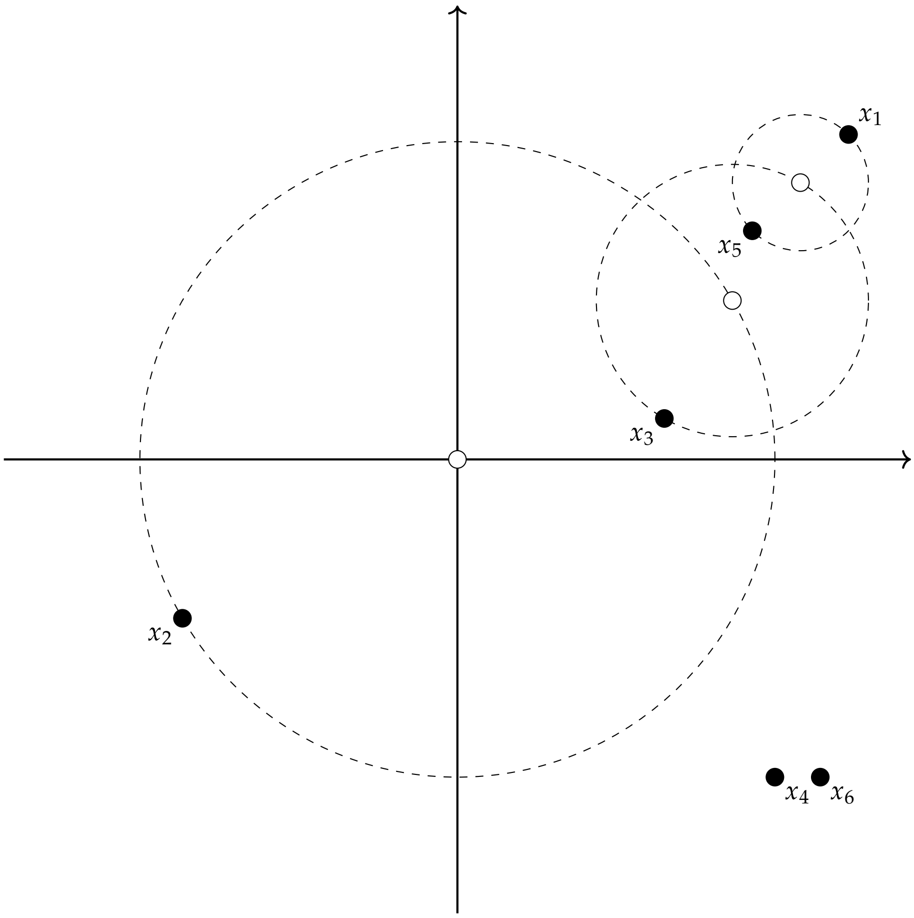


FIGURE 2.6. The planetary system in $\mathrm{Conf}_6(\mathbb{R}^2)$ associated to the tree from Figure 2.4. Empty dots are the centers of mass and dashed circles represent "orbits" on which points rotate around centers of mass.

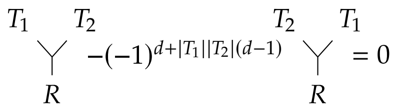
REMARK 2.4.7. Note that the planetary system associated to an S-tree T with i internal vertices gives an embedding of the i-fold product of spheres $(S^{d-1})^i$ in the $\mathrm{Conf}_n(\mathbb{R}^d)$ (see [**56**, Definition 2.6]). This embedding, which we will denote by ϵ_T, sends the point $(u_1, \dots, u_i) \in (S^{d-1})^i$ to the configuration $(x_1, \dots, x_n)$ such that

$$x_k = \sum_{v_j \text{ below leaf } k} \pm \delta^{h(v_j)} u_j.$$

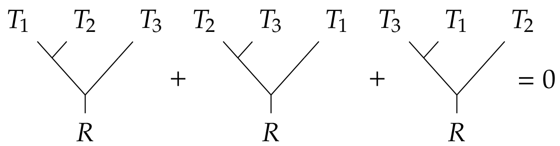
The sum is taken over all internal vertices v_j lying on the path from the leaf i to the root. The sign of a summand is + if the path from the leaf i to the root goes through the left edge over v_j and $-$ if the path goes through the right edge over v_j.

The embedding ϵ_T gives rise to degree i homology class of $\mathrm{Conf}_n(\mathbb{R}^d)$ associated to the tree T. This class is defined as an $\epsilon_{T*}([(S^{d-1})^i])$, where $[(S^{d-1})^i]$ is the fundamental class.

PROPOSITION 2.4.8. *The classes in $H_*(\mathrm{Conf}_n(\mathbb{R}^d))$ given by S-trees satisfy the following relations:*

(1)  *(anti-symmetry)*

Here T_1 and T_2 are any S-trees and R is the root. By $|T_i|$ we mean the number of internal vertices.

(2)  *(Jacobi identity)*

As above, T_1, T_2 and T_3 are any S-trees and R is the root.

PROOF. See [**56**, Proposition 2.7]. □

DEFINITION 2.4.9 ([**56**, Definition 2.8]).

(1) An n-forest is a collection of S-trees which have in total n leaves and each integer from 1 to n labels exactly one leaf. We embed this collection in the upper half-plane such that the root vertices are in points $(0, 0), (1, 0), \ldots$.
(2) If $F = \bigcup T_i$ is a forest, we define its associated planetary systepm P_F as in Definition 2.4.6 with the condition (1) replaced by $c(\mathbf{x}, T_i) = (i, 0, 0, \ldots)$.

Now we are in the position to state the key result.

THEOREM 2.4.10. *Let Φ_i^n be a free abelian group generated by n-forests in which $n - i$ trees consist of a single leaf. Then*

$$H_i\left(\mathrm{Conf}_n(\mathbb{R}^d)\right) \cong \frac{\Phi_i^n}{I},$$

where I is a submodule generated by elements expressing the anti-symmetry, Jacobi identity (see Proposition 2.4.8) and the following commutativity relation: if F_1 and F_2 consist of the same trees, then $F_1 = sgn(\sigma)^{d-1}F_2$, where σ is the permutation which relates the ordering of internal vertices of F_1 with that of F_2.

PROOF. See [**56**, Theorem 2.10]. □

2.5. Fixed points of equivariant little discs operads under graph subgroups

In this short section we are going to present the connection between fixed points of little discs operads and equivariant analogues of configuration spaces. This chapter is based on Mike Hill's work in [**27**].

Throughout this section, let G be a finite group and V a G-representation.

Proposition 2.5.1. *Let Γ be a graph subgroup of $G \times \Sigma_n$ corresponding to the H-set T_Γ (see Remark 2.2.4). Then the map taking a disc to its center gives a weak equivalence*

$$\mathcal{D}(V)^\Gamma \simeq \mathrm{Emb}^H(T_\Gamma, V),$$

where $\mathrm{Emb}^H(T_\Gamma, V)$ is the space of H-equivariant embeddings of T_Γ into V. Moreover, this equivalence is equivariant with respect to the Weyl group action:

$$W_{G\times\Sigma_n}(\Gamma) \cong \mathrm{Aut}^H(T_\Gamma).$$

Proof. See [**27**, Lemma 2.4]. □

Definition 2.5.2 ([**27**, Definition 2.5]). Let $n \geq 1$, X be a G-space and H be a subgroup of G. Define the *configuration space of nG/H in X* to be:

$$\mathrm{Conf}_{nG/H}(X) := \left\{ (x_1, \ldots, x_n) \in X^n \;\middle|\; \begin{array}{c} \forall_i \ \mathrm{Stab}(x_i) = H \text{ and } x_i \neq gx_j \\ \text{for all } 1 \leq i < j \leq n \text{ and } g \in G \end{array} \right\}.$$

Proposition 2.5.3. *If X is any G-space, then there is a homeomorphism*

$$\mathrm{Emb}^G(nG/H, X) \cong \mathrm{Conf}_{nG/H}(X).$$

This homeomorphism is equivariant with respect to

$$\mathrm{Aut}^G(nG/H) \cong W_G(H) \wr \Sigma_n.$$

Proof. See [**27**, Proposition 2.6]. □

Theorem 2.5.4. *Let $T = n_1G/H_1 \coprod \ldots \coprod n_kG/H_k$, where if $i \neq j$ then H_i and H_j are not conjugate. Then there is an $\mathrm{Aut}^G(T)$-equivariant homeomorphism*

$$\mathrm{Emb}^G(T, V) \cong \prod_{i=1}^{k} \mathrm{Conf}_{n_iG/H_i}(V).$$

Proof. See [**27**, Theorem 2.9]. □

2.6. The operad $\mathcal{D}_{1+\sigma}$

From now on we are going to work over the group Q and recall that by σ we denote its real sign representation. In this chapter we are going to investigate equivariant and non-equivariant structure appearing in the homology of the operad of little $1+\sigma$-discs.

We are going to use *Stirling numbers of the first kind* for the description, so we start with a background on these.

2.6.1. Stirling numbers of the first kind.

Definition 2.6.1. The *Stirling number of the first kind* $\left[{n \atop k}\right]$ is defined as the number of permutations on n elements with k disjoint cycles. Equivalently, it can be defined as the coefficient in front of x^k in the following polynomial:

$$x(x+1)\ldots(x+n-1).$$

PROPOSITION 2.6.2. *The Stirling numbers of the first kind satisfy the following properties:*

(1) $\left[{0 \atop 0}\right] = 1$ *and* $\left[{n \atop 0}\right] = 0$ *if* $n > 0$.
(2) $\left[{n \atop k}\right] = 0$ *if* $k > n$.
(3) $\left[{n \atop n}\right] = 1$.
(4) $\left[{n \atop 1}\right] = (n-1)!$ *if* $n > 0$.
(5) $\left[{n \atop n-1}\right] = \binom{n}{2}$.
(6) $\left[{n \atop n-k}\right] = \sum_{0 \leq i_1 < \ldots < i_k < n} i_1 i_2 \ldots i_k$.

PROOF. See [**20**, Section 6.1]. □

2.6.2. Non-equivariant structure of homology graph coefficient systems for $\mathcal{D}_{1+\sigma}$. In this subsection we are going to describe the graph homology coefficient systems of $\mathcal{D}_{1+\sigma}$ as diagrams of abelian groups.

Recall from Section 2.3.1 that Γ_i is the graph subgroup of $Q \times \Sigma_n$ which is the graph of the group homomorphism such that γ hits a product of i disjoint transpositions. The main result connecting Stirling numbers of the first kind with homology of fixed points of $\mathcal{D}_{1+\sigma}(n)$ under graph subgroups is the following:

PROPOSITION 2.6.3. *There is a (non-equivariant) isomorphism:*

$$H_p(\mathcal{D}_{1+\sigma}(n)^{\Gamma_i}) \cong \left(\mathbb{Z}^{\left[{i \atop i-p}\right]}\right)^{\oplus 2^i (n-2i)!}.$$

Before proving this proposition we need to provide a couple of technical lemmas and observations.

OBSERVATION 2.6.4. Note that Proposition 2.5.1 and Theorem 2.5.4 applied to the case $V = 1 + \sigma$ give us that

$$(\mathcal{D}_{1+\sigma}(n))^{\Gamma_i} \simeq \mathrm{Conf}_{iQ/e}(1+\sigma) \times \mathrm{Conf}_{(n-2i)Q/Q}(1+\sigma).$$

Note that the second factor of this product is homotopically discrete. Indeed, by Definition 2.5.2 we have that:

$$\mathrm{Conf}_{(n-2i)Q/Q}(1+\sigma) = \left\{ (x_1, \ldots, x_{n-2i}) \in 1 + \sigma \;\middle|\; \begin{array}{c} \forall_{1 \leq j \leq n-2i}\ \mathrm{Stab}(x_j) = Q \text{ and} \\ \forall_{1 \leq j < k \leq n-2i}\ x_j \neq x_k \end{array} \right\}.$$

If a point $x \in 1+\sigma$ is such that $\mathrm{Stab}(x) = Q$, then x lies in the copy of the trivial representation in $1+\sigma$. Therefore $\mathrm{Conf}_{(n-2i)Q/Q}(1+\sigma)$ is the configuration space of $n-2i$ points in the real line, which is homotopy equivalent to Σ_{n-2i} treated as a topological space with the discrete topology.

LEMMA 2.6.5 (See [**14**, Lemma III.6.3], [**17**, Section V.1].). *The homology of* $\mathrm{Conf}_n(\mathbb{R}^2)$ *is free and its Poincaré polynomial is*

$$P(t) = (1+t)(1+2t)\ldots(1+(n-1)t).$$

PROOF. We will use the fibration given by omitting the last point in a configuration (see [**17**, Section I.1]):

$$\begin{gathered}\mathrm{Conf}_n\left(\mathbb{R}^2\right) \to \mathrm{Conf}_{n-1}\left(\mathbb{R}^2\right)\\ (x_1, \ldots, x_{i-1}, x_i) \mapsto (x_1, \ldots, x_{i-1}).\end{gathered}$$

The fibre of this fibration is homotopy equivalent to the wedge of $i-1$ circles $\bigvee_{i-1} S^1$. The fundamental group of the base acts trivially on the homology of the fibre (see [**14**, Lemma III.6.3]), and thus the Serre spectral sequence for this fibration takes the following form:

$$E^2_{pq} = H_p\left(\mathrm{Conf}_{n-1}(\mathbb{R}^2),\ H_q\left(\bigvee_{n-1} S^1\right)\right) \Rightarrow H_{p+q}\left(\mathrm{Conf}_n(\mathbb{R}^2)\right).$$

Now we proceed by induction. The statement is true for $n = 2$, since by Proposition 2.4.2 we have that $\mathrm{Conf}_2(\mathbb{R}^2) \simeq S^1$. Assume it is true for some $n-1$. Then the E^2-page of the Serre spectral sequence consists of two columns:

$$\begin{gathered}E^2_{p0} = \left(H_p(\mathrm{Conf}_{n-1}(\mathbb{R}^2))\right)\\ E^2_{p1} = \left(H_{p-1}(\mathrm{Conf}_{n-1}(\mathbb{R}^2))\right)^{\oplus(n-1)}.\end{gathered}$$

Therefore we have that

$$H_p(\mathrm{Conf}_n(\mathbb{R}^2)) = \left(H_p(\mathrm{Conf}_{n-1}(\mathbb{R}^2))\right) \oplus \left(H_{p-1}(\mathrm{Conf}_{n-1}(\mathbb{R}^2))\right)^{\oplus(n-1)}.$$

From this we deduce that the Poincaré polynomial for $H_*(\mathrm{Conf}_n(\mathbb{R}^2))$ is equal to

$$P_n(t) = P_{n-1}(t) + (n-1)P_{n-1}(t) = (1 + (n-1)t)P_{n-1}(t).$$

□

LEMMA 2.6.6. *The coefficient in the front of t^k in $P_n(t)$, i.e., the rank of $H_k(\mathrm{Conf}_n(\mathbb{R}^2))$, is equal to $\left[{n \atop n-k}\right]$.*

PROOF. The coefficient in the front of t^k in $P_n(t)$ is equal to

$$\sum_{1\le i_1<\ldots<i_k\le n} i_1 i_2 \ldots i_k$$

which, by Proposition 2.6.2, is equal to $\left[{n \atop n-k}\right]$. □

PROOF OF PROPOSITION 2.6.3. By Observation 2.6.4 we have that

$$\left(\mathcal{D}_{1+\sigma}(n)\right)^{\Gamma_i} \simeq \mathrm{Conf}_{iQ/e}(1+\sigma) \times \mathrm{Conf}_{(n-2i)Q/Q}(1+\sigma),$$

where the second factor is homotopically discrete and equivalent to Σ_{n-2i} understood as a discrete topological space of permutations of $n-2i$ letters.

The first factor, $\mathrm{Conf}_{iQ/e}(1+\sigma)$, is a configuration space of i free orbits in $1+\sigma$. Define the space of signed configurations $\widetilde{\mathrm{Conf}}_i(\mathbb{R}^2)$ of i points in the right half-plane to be the following subspace of $\left(\mathbb{R}^2 \times \{-1, +1\}\right)^i$:

$$\left\{ ((x_1, y_1, s_1), \ldots, (x_i, y_i, s_i)) \in \left(\mathbb{R}^2 \times \{-1, +1\}\right)^i \;\middle|\; \begin{array}{l} x_a > 0 \text{ for all } 1 \leq a \leq i \text{ and} \\ x_a \neq x_b \text{ for all } 1 \leq a < b \leq i \end{array} \right\}.$$

Define the map $f\colon (1+\sigma) \setminus (0 \times \mathbb{R}) \longrightarrow \mathbb{R}^2 \times \{-1, +1\}$ as follows:

$$f(x, y) = \begin{cases} (x, y, +1) & \text{if } x > 0 \\ (-x, y, -1) & \text{if } x < 0. \end{cases}$$

Then the map f induces a homeomorphism:

$$\tilde{f}\colon \operatorname{Conf}_{iQ/e}(1+\sigma) \cong \widetilde{\operatorname{Conf}_i(\mathbb{R}^2)}$$

by $\tilde{f}(x_1, \ldots, x_i) = (f(x_1), \ldots, f(x_i))$ for $(x_1, \ldots, x_i) \in \operatorname{Conf}_{iQ/e}(1+\sigma)$. The inverse of $\tilde{f}$ is induced by the map $g\colon \mathbb{R}_+ \times \mathbb{R} \times \{+1, -1\}$ given by

$$g(x, y, +1) = (x, y)$$
$$g(x, y, -1) = (-x, y).$$

The space $\widetilde{\operatorname{Conf}_i(\mathbb{R}^2)}$ splits as a disjoint union of copies of $\operatorname{Conf}_i(\mathbb{R}^2)$ indexed over all possible combinations of signs:

$$\widetilde{\operatorname{Conf}_i(\mathbb{R}^2)} \cong \coprod_{(s_1, s_2, \ldots, s_i) \in \{-1,+1\}^i} \operatorname{Conf}_i(\mathbb{R}^2) \cong \coprod_{2^i} \operatorname{Conf}_i(\mathbb{R}^2).$$

Thus we obtain that

$$(\mathcal{D}_{1+\sigma}(n))^{\Gamma_i} \simeq \left(\coprod_{2^i} \operatorname{Conf}_i(\mathbb{R}^2) \right) \times \Sigma_{n-2i}.$$

Therefore by Künneth theorem and Lemmas 2.6.5 and 2.6.6 we obtain that

$$H_p(\mathcal{D}_{1+\sigma}(n)^{\Gamma_i}) \cong \left(\mathbb{Z}^{\left[{i \atop i-p} \right]} \right)^{\oplus 2^i (n-2i)!}.$$

□

By Proposition 2.6.3 we can describe the homology graph coefficient system of $\mathcal{D}_n(1+\sigma)$ without action of Weyl groups as in Figure 2.7.

2.6.3. Computations of $Q \times \Sigma_2$-equivariant Bredon homology of $\mathcal{D}_{1+\sigma}(2)$. The actual challenge with using the framework based on the universal coefficient spectral sequence is describing the action of respective Weyl groups on entries of the homology graph coefficient systems. In general, this is a complex task. This complexity made an application of the framework possible only in the simplest cases, such as $H^{Q \times \Sigma_2}(\mathcal{D}_{1+\sigma}(2), \underline{\mathbb{F}_2})$, which is discussed in this subsection.

We begin with the description of homology graph coefficient systems of $\mathcal{D}_{1+\sigma}(2)$.

$$\begin{tikzcd}
n!\,\mathbb{Z}^{\genfrac{[}{]}{0pt}{}{0}{0-p}} \arrow[drr] & 2(n-2)!\,\mathbb{Z}^{\genfrac{[}{]}{0pt}{}{1}{1-p}} \arrow[dr] & \dots \arrow[d, dashed] & 2^i(n-2i)!\,\mathbb{Z}^{\genfrac{[}{]}{0pt}{}{i}{i-p}} \arrow[dl] & \dots \arrow[dll, dashed] \\
 & & \mathbb{Z}^{\genfrac{[}{]}{0pt}{}{n}{n-p}} & &
\end{tikzcd}$$

FIGURE 2.7. The graph coefficient system $H_p(\mathcal{D}_{1+\sigma}(n)^\bullet)$ without action of Weyl groups.

2.6.3.1. *Homology graph coefficient systems $H_*(\mathcal{D}^\bullet_{1+\sigma}(2))$.* By Proposition 2.6.3 and Figure 2.7 we see that $H_*(\mathcal{D}_{1+\sigma}(2)^\bullet)$ is nontrivial only in degrees 0 and 1.

PROPOSITION 2.6.7. *Homology graph coefficient systems for $\mathcal{D}_{1+\sigma}(2)$ are given as follows:*

(1) $\mathcal{H}_0 := H_0(\mathcal{D}_{1+\sigma}(2)^\bullet) =$

$$\begin{tikzcd}
\mathbb{Z}\left[\frac{Q\times\Sigma_2}{\Gamma_0}\right] \arrow[dr] & & \mathbb{Z}[Q] \arrow[dl] \\
 & \mathbb{Z} &
\end{tikzcd}$$

(2) $\mathcal{H}_1 := H_1(\mathcal{D}_{1+\sigma}(2)^\bullet) =$

$$\begin{tikzcd}
0 \arrow[dr] & & 0 \arrow[dl] \\
 & \tilde{\mathbb{Z}}_Q &
\end{tikzcd}$$

Here $\tilde{\mathbb{Z}}_Q$ is the $\mathbb{Z}[Q\times\Sigma_2]$-module with the underlying abelian group $\mathbb{Z}$, the action of Q given by sign and trivial Σ_2-action.

PROOF.

(1) The space $\mathcal{D}_{1+\sigma}(2)^{\Gamma_0}$ is homotopy equivalent to $\mathrm{Conf}_{2Q/Q}(1+\sigma)$ (see Observation 2.6.4) and thus it is further Σ_2-homotopy equivalent to $\frac{Q\times\Sigma_2}{\Gamma_0}$. By the analogous argument the space $\mathcal{D}_{1+\sigma}(2)^{\Gamma_1}$ is Q-homotopy equivalent to $\frac{Q\times\Sigma_2}{\Gamma_1}$. So the graph coefficient system $\mathcal{H}_0$ has the form as presented.

(2) By Proposition 2.6.3 we see that the values of $\mathcal{H}_1$ on Γ_0 and Γ_1 are 0. We see also that the rank of $\mathcal{H}_1(e\times e) = H_1(\mathcal{D}_{1+\sigma}(2)^{e\times e})$ as a free abelian group is equal to 1. Therefore we have that the underlying abelian group of $\mathcal{H}_1(e\times e)$ is $\mathbb{Z}$, and we are

left with the description of the action of $Q \times \Sigma_2$ on it. To this end we use the fact that $\mathcal{D}_{1+\sigma}(2) \simeq S(\rho \otimes \upsilon)$ as $Q \times \Sigma_2$-space. Here υ is the sign Σ_2-representation.

The group Q acts on $S(\rho \otimes \tau)$ by the axis symmetry, and thus the degree of γ as a continuous map is -1. The group Σ_2 acts on $S(\rho \otimes \tau)$ by the antipodal map, so its degree is equal to 1. Therefore we can write that $H_1(S(\rho \otimes \tau)) = \tilde{\mathbb{Z}}_Q$, i.e., the $Q \times \Sigma_2$-module with the underlying abelian group is $\mathbb{Z}$ and on which Q acts by sign and Σ_2 trivially.

□

2.6.3.2. *Computations of Tor groups* $\mathrm{Tor}_*(\mathcal{H}_0, \underline{\mathbb{F}_2})$ *and* $\mathrm{Tor}_*(\mathcal{H}_1, \underline{\mathbb{F}_2})$. In this subsection we compute the entries of the Universal Coefficient Spectral Sequence for $\mathcal{D}_{1+\sigma}(2)$, which are given by the Tor groups $\mathrm{Tor}_*(\mathcal{H}_0, \underline{\mathbb{F}_2})$ and $\mathrm{Tor}_*(\mathcal{H}_1, \underline{\mathbb{F}_2})$.

THEOREM 2.6.8. *We have that:*

$$\mathrm{Tor}_p(\mathcal{H}_0, \underline{\mathbb{F}_2}) = \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 & \text{if } p = 0 \\ \mathbb{F}_2 & \text{if } p = 1 \\ \mathbb{F}_2^{p-1} & \text{if } p > 1 \end{cases}$$

and

$$\mathrm{Tor}_p(\mathcal{H}_1, \underline{\mathbb{F}_2}) = \mathbb{F}_2^{p+1}.$$

The rest of this subsection is devoted to the proof of this theorem.

Recall the representable graph coefficient systems in $\mathcal{CS}^{\Gamma}_{Q\times\Sigma_2}$ from Observation 2.3.7:

$$F^{\Gamma_0} \qquad\qquad\qquad\qquad F^{\Gamma_1}$$

$$\begin{tikzcd} \mathbb{Z}[\Sigma_2] \arrow[dr] & & 0 \arrow[dl] & 0 \arrow[dr] & & \mathbb{Z}[Q] \arrow[dl] \\ & \mathbb{Z}\left[\frac{Q\times\Sigma_2}{\Gamma_0}\right] & & & \mathbb{Z}\left[\frac{Q\times\Sigma_2}{\Gamma_1}\right]. & \end{tikzcd}$$

OBSERVATION 2.6.9. The $\mathbb{Z}[Q\times\Sigma_2]$-module $\tilde{\mathbb{Z}}_Q$ can be seen as the tensor product $\tilde{\mathbb{Z}}_Q \otimes_{\mathbb{Z}} \mathbb{Z}_{\Sigma_2}$, where the first factor is the Q-module $\tilde{\mathbb{Z}}$ and the second is the trivial Σ_2-module $\mathbb{Z}$. Therefore for the $\mathbb{Z}[Q \times \Sigma_2]$-projective resolution of $\tilde{\mathbb{Z}}_Q$ we can choose the tensor product over $\mathbb{Z}$ of the free resolutions of $\tilde{\mathbb{Z}}$ and $\mathbb{Z}$ respectively over $\mathbb{Z}[Q]$ and $\mathbb{Z}[\Sigma_2]$ given below:

$$R_1\colon 0 \longleftarrow \tilde{\mathbb{Z}}_Q \xleftarrow{\tilde{\epsilon}} \mathbb{Z}[Q] \xleftarrow{1+\gamma} \mathbb{Z}[Q] \xleftarrow{1-\gamma} \mathbb{Z}[Q] \longleftarrow \dots$$

$$R_2\colon 0 \longleftarrow \mathbb{Z}_{\Sigma_2} \xleftarrow{\epsilon} \mathbb{Z}[\Sigma_2] \xleftarrow{1-\gamma} \mathbb{Z}[\Sigma_2] \xleftarrow{1+\gamma} \mathbb{Z}[\Sigma_2] \longleftarrow \dots .$$

Here τ denotes the non-trivial element of Σ_2, ϵ is the augmentation map and $\tilde{\epsilon}$ is "twisted augmentation" - i.e., the $\mathbb{Z}[Q]$-module map which is given by $1 \mapsto 1$ and $\gamma \mapsto -1$. By

Künneth formula for chain complexes (see [**61**, Theorem 3.6.3]) we obtain that $H_*(R_1 \otimes_{\mathbb{Z}} R_@)$ is zero in all non-zero degrees and thus $R_1 \otimes_{\mathbb{Z}} R_2$ gives a free resolution of $\tilde{\mathbb{Z}}_Q$. We will denote this resolution by $P_{\tilde{\mathbb{Z}}_Q}$.

Observation 2.6.10. The projective resolution $P_{\tilde{\mathbb{Z}}_Q}$ actually gives also a projective resolution of $\mathcal{H}_1$. This is given by:

$$0 \longleftarrow \mathcal{H}_1 \longleftarrow F^{e\times e} \longleftarrow F^{e\times e} \longleftarrow \dots,$$

where on the $e\times e$-level we have $P_{\tilde{\mathbb{Z}}_Q}$ and 0 maps on the other levels. By abuse of notation, we denote this resolution also by $P_{\tilde{\mathbb{Z}}_Q}$.

Remark 2.6.11. From now on we are going to identify $\frac{Q\times\Sigma_2}{\Gamma_0}$ with Σ_2 and $\frac{Q\times\Sigma_2}{\Gamma_1}$ with Q.

Lemma 2.6.12. *Let $\delta = (\gamma, \tau)$ in $\mathbb{Z}[Q\times\Sigma_2]$. A projective resolution in $\mathcal{CS}^{\Gamma}_{Q\times\Sigma_2}$ of $\mathcal{H}_0$ is given by:*

$$0 \longleftarrow \mathcal{H}_0 \xleftarrow{\epsilon} F^{\Gamma_0}\oplus F^{\Gamma_1} \xleftarrow{\phi} F^{e\times e} \xleftarrow{e+\tau-\gamma-\delta} P_{\tilde{\mathbb{Z}}_Q},$$

where ϵ is the map given by the augmentation on $e\times e$-level and identity on Γ_0 and Γ_1 levels and ϕ is given by sending 1 to $(1,-1)$.

Proof. Since the map ϵ is surjective, we need to find its kernel. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ in $\mathbb{Z}[\Sigma_2]\oplus\mathbb{Z}[Q]$. The kernel of ϵ is given by the submodule $\langle e_1, -e_2\rangle$. Let ϕ be the $\mathbb{Z}[Q\times\Sigma_2]$-module map given by

$$\begin{aligned}\phi\colon \mathbb{Z}[Q\times\Sigma_2] &\to \mathbb{Z}[\Sigma_2]\oplus\mathbb{Z}[Q]\\ 1 &\mapsto e_1 - e_2.\end{aligned}$$

Direct calculations show that the kernel of ϕ is the submodule $\langle 1+\tau-\gamma-\delta\rangle$. Since this submodule is isomorphic to $\tilde{\mathbb{Z}}_Q$, the result follows. □

Proof of Theorem 2.6.8. We start by proving the second statement, i.e., that

$$\mathrm{Tor}_p(\mathcal{H}_1, \underline{\mathbb{F}}_2) = \mathbb{F}_2^{p+1}.$$

Since the graph coefficient system $\mathcal{H}_1$ takes zero values on Γ_0 and Γ_1, we need only to find the resolution of the $e\times e$-level - as in Observation 2.6.10. This is given by $P_{\tilde{\mathbb{Z}}_Q}$ - see Observation 2.6.9. By this Observation we also see that

$$\left(P_{\tilde{\mathbb{Z}}_Q}\right)_n = \bigoplus_{n+1}\mathbb{Z}[Q\times\Sigma_n].$$

Using the fact (see Observation 1.3.2) that $F^{e\times e}\otimes\underline{\mathbb{F}}_2 = \mathbb{F}_2$, we get that

$$\left(P_{\tilde{\mathbb{Z}}_Q}\otimes\underline{\mathbb{F}}_2\right)_n = \bigoplus_{n+1}\mathbb{F}_2$$

Since the only possible action of Q on $\mathbb{F}_2$ is the trivial action, all differentials in the chain complex
$$\left(P_{\tilde{\mathbb{Z}}_Q} \otimes \underline{\mathbb{F}}_2\right)_n$$
are zero. Thus the claim follows.

Now we proceed to the first statement, i.e., that
$$\mathrm{Tor}_p(\mathcal{H}_0, \underline{\mathbb{F}}_2) = \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 & \text{if } p = 0 \\ \mathbb{F}_2 & \text{if } p = 1 \\ \mathbb{F}_2^{p-1} & \text{if } p > 1 \end{cases}$$

After tensoring the projective resolution of $\mathcal{H}_0$ given in Lemma 2.6.12 with $\underline{\mathbb{F}}_2$ we obtain the following chain complex:

$$(*) \qquad 0 \longleftarrow \mathbb{F}_2 \oplus \mathbb{F}_2 \xleftarrow{\phi\otimes\underline{\mathbb{F}}_2} \mathbb{F}_2 \longleftarrow P_{\tilde{\mathbb{Z}}_Q} \otimes \mathbb{F}_2.$$

We firstly show that $\phi \otimes \underline{\mathbb{F}}_2$ is zero. Since $\mathcal{CS}_{Q\times\Sigma_n}$ is an abelian category, we obtain that $\mathrm{Hom}_{\mathcal{CS}_{Q\times\Sigma_n}}(F^{e\times e}, F^{\Gamma_0} \oplus F^{\Gamma_1}) \cong \mathrm{Hom}_{\mathcal{CS}_{Q\times\Sigma_n}}(F^{e\times e}, F^{\Gamma_0}) \oplus \mathrm{Hom}_{\mathcal{CS}_{Q\times\Sigma_n}}(F^{e\times e}, F^{\Gamma_1})$. By the Yoneda embedding we obtain that
$$\mathrm{Hom}_{\mathcal{CS}_{Q\times\Sigma_n}}(F^{e\times e}, F^{\Gamma_0}) \cong \mathrm{Hom}_{\mathcal{O}^{\Gamma}_{Q\times\Sigma_2}}\left(\frac{Q\times\Sigma_2}{e\times e}, \frac{Q\times\Sigma_2}{e\times e}\right).$$

Therefore any map $F^{e\times e} \otimes \underline{\mathbb{F}}_2 \to F^{\Gamma_0} \otimes \underline{\mathbb{F}}_2$ corresponds to a map between $e \times e$ and Γ_0 levels in the dual coefficient system $\underline{\mathbb{F}}_2$. However, in $\underline{\mathbb{F}}_2$ all maps between different levels are zero. We use analogous argument for Γ_1 to obtain that $\phi \otimes \underline{\mathbb{F}}_2$ is zero.

By the similar reasoning as in the proof of the $\mathrm{Tor}_p(\mathcal{H}_1, \underline{\mathbb{F}}_2)$ case we show that all remaining differentials in the chain complex $(*)$ are zero. Thus the result follows. □

2.6.3.3. *Homology of $\mathcal{D}_{1+\sigma}(2)$.* Finally, using the computations given in the previous subsections we can compute the $Q \times \Sigma_2$-Bredon homology of $\mathcal{D}_{1+\sigma}(2)$.

THEOREM 2.6.13.
$$H_i^{Q\times\Sigma_2}(\mathcal{D}_{1+\sigma}(2)) = \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 & \textit{if } i = 0 \\ \mathbb{F}_2 & \textit{if } i = 1 \\ 0 & \textit{else.} \end{cases}$$

PROOF. By Theorem 2.6.8 we have that the E^2-page of the the universal coefficient spectral sequence for $\mathcal{D}_{1+\sigma}(2)$ looks as in Figure 2.8. Since the E^2-page consists of two rows, we see that the spectral sequence will degenerate on this page. Therefore we only need to describe the d^2-differentials.

To this end, we note that the homology graph coefficient systems of $\mathcal{D}_{1+\sigma}(2)$ fit into the following exact sequence:
$$0 \longrightarrow \mathcal{H}_1 \longrightarrow C_1(\mathcal{D}_{1+\sigma}(2)^\bullet) \longrightarrow C_0(\mathcal{D}_{1+\sigma}(2)^\bullet) \longrightarrow \mathcal{H}_0 \longrightarrow 0.$$

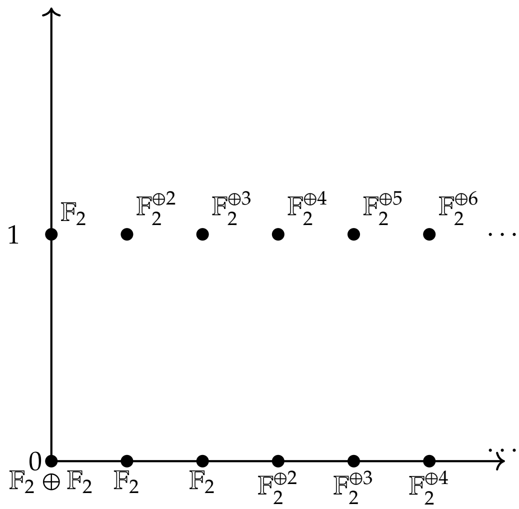


FIGURE 2.8. E^2-page of the universal coefficient spectral sequence for $\mathcal{D}_{1+\sigma}(2)$. Note that the vertical axis has been rescaled for better presentation.

Therefore by Proposition 2.1.5 the differentials starting at entries E^2_{p0} are isomorphisms for $p \geq 2$ and zero otherwise. All remaining differentials are zero and the result follows. $\square$

2.6.4. Computations for the space $\mathcal{D}_{1+\sigma}(3)$. In this subsection we are going to present the computations of homology graph coefficient systems of the space $\mathcal{D}_{1+\sigma}(3)$. This will be the first place where we are going to use the theory of S-trees and planetary systems from Section 2.4. This is also the first case where the growing complexity of projective resolutions makes the computations very hard. Possibly these calculations should be done using computing software - we leave it to further investigation in the future.

2.6.4.1. *Homology graph coefficient systems for $\mathcal{D}_{1+\sigma}(3)$.*

PROPOSITION 2.6.14. *The homology graph coefficient systems of $\mathcal{D}_{1+\sigma}(3)$ are given by:*

(1) $\mathcal{H}_0 := H_0(\mathcal{D}_{1+\sigma}(3)^\bullet) =$

$$\begin{tikzcd}
\mathbb{Z}\left[\frac{Q\times\Sigma_3}{\Gamma_0}\right] \arrow[dr] & & \mathbb{Z}[\Sigma_2] \arrow[dl] \\
& \mathbb{Z} &
\end{tikzcd}$$

(2) $\mathcal{H}_1 := H_1(\mathcal{D}_{1+\sigma}(3)^\bullet) =$

$$\begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ & \mathbb{Z}\left[\frac{Q\times\Sigma_3}{Q\times\Sigma_2}\right] & \end{tikzcd}$$

(3) $\mathcal{H}_2 := H_2(\mathcal{D}_{1+\sigma}(3)^\bullet) =$

$$\begin{tikzcd} 0 \arrow[dr] & & 0 \arrow[dl] \\ & \dfrac{\mathbb{Z}[Q\times\Sigma_3]}{\begin{pmatrix} 1-\gamma \\ 1-(1\ 2) \\ 1+(1\ 2\ 3)+(1\ 3\ 2)\end{pmatrix}} & \end{tikzcd}$$

Proof.

(1) The space $\mathcal{D}_{1+\sigma}(3)$ is path connected, thus $\mathcal{H}_0(e\times e) = \mathbb{Z}$. By similar reasoning as in the proof of Proposition 2.6.7, Point (1) we get that $\mathcal{H}_0(\Gamma_0) = \mathbb{Z}\left[\frac{Q\times\Sigma_3}{\Gamma_0}\right]$.

By Observation 2.6.4 we have that

$$\mathcal{D}_{1+\sigma}(3)^{\Gamma_1} \simeq \mathrm{Conf}_{Q/e}(1+\sigma)\times\mathrm{Conf}_{Q/Q}(1+\sigma) \simeq \mathrm{Conf}_{Q/e}(1+\sigma).$$

The last equivalence comes from the fact that $\mathrm{Conf}_{Q/Q}(1+\sigma)$ is contractible.

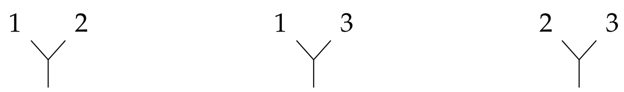
The space $\mathrm{Conf}_{Q/e}(1+\sigma)$ is a free space under the action of

$$W_{Q\times\Sigma_3}(\Gamma_1) \cong \Sigma_2$$

and so its homology is free $\mathbb{Z}[\Sigma_2]$-module. Since by Proposition 2.6.3 we see that this is a free abelian group of rank 2, it has to be $\mathbb{Z}[\Sigma_2]$. Note that we can also deduce this homology from the fact that it $\mathrm{Conf}_{Q/e}(1+\sigma)$ is Σ_2-homotopy equivalent to Σ_2.

(2) By Proposition 2.6.3 we have that the only nontrivial entry of $\mathcal{H}_1$ is on the $e\times e$-level, where it is a free abelian group of rank 3 - thus we need to find the action of $Q\times\Sigma_3$ on it.

The space $\mathcal{D}_{1+\sigma}(3)$ is non-equivariantly homotopy equivalent to $\mathrm{Conf}_3(\mathbb{R}^2)$. Therefore by the theory given in Chapter 2.4 we see that the group $H_1(\mathcal{D}_{1+\sigma}(3))$ as a free abelian group has the basis given by the following trees:



Denote this set of trees by F. Let

$$\epsilon_{ij} \colon S^1 \to \mathcal{D}_{1+\sigma}(3)$$

denote the embedding of a circle as the planetary system with i and j orbiting, for $1 \le i < j \le 3$. For the example of such embedding, see Figure 2.9.

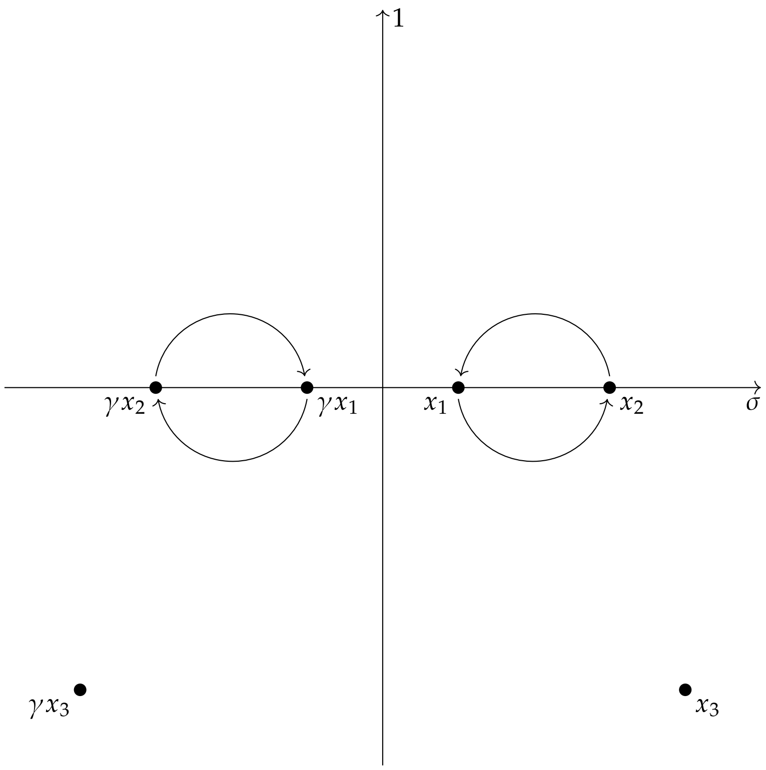


FIGURE 2.9. Depiction of the embedding ϵ_{12} and the effect of the action of γ.

We are going to describe the set F as a $Q \times \Sigma_3$-set. Firstly, note that since the generators of $H_1(\mathcal{D}_{1+\sigma}(3))$ are given by embeddings of circles, to describe the action of Q it is enough to provide the degree of the map $\gamma \circ \epsilon_{ij}$. The element γ acts on $1 + \sigma$ by symmetry about the vertical axis, thus acts on embedded circles as the antipodal maps. So its degree is 1 and thus Q acts trivially on elements of S.

Now note that for each tree in F there is exactly one non-trival element which acts on it trivially - namely, the transposition $(i\ j)$. Therefore we have that

$$F = \frac{Q \times \Sigma_3}{Q \times \Sigma_2}$$

as a $Q \times \Sigma_3$-set, where Σ_2 is embedded in Σ_3 as a subgroup generated by a transposition. Thus the claim follows.

(3) The action of $Q \times \Sigma_3$ on the generating set of the $H_2(\mathcal{D}_{1+\sigma}(3))$ is the same as on $H_1(\mathcal{D}_{1+\sigma}(3))$. However, we need to take one more relation into account - namely Jacobi identity (see Proposition 2.4.8). Therefore we obtain

$$H_2(\mathcal{D}_{1+\sigma}(3)^{e\times e}) \cong \frac{\mathbb{Z}\left[\frac{Q\times\Sigma_3}{Q\times\Sigma_2}\right]}{(1+(1\ 2\ 3)+(1\ 3\ 2))} \cong \frac{\mathbb{Z}[Q\times\Sigma_3]}{\begin{pmatrix} 1-\gamma \\ 1-(1\ 2) \\ 1+(1\ 2\ 3)+(1\ 3\ 2)\end{pmatrix}}.$$

By Proposition 2.6.3 we get that the entries of $\mathcal{H}_2$ on Γ_0 and Γ_1 are zero. Thus the result follows.

□

2.7. The operad $\mathcal{D}_{2\sigma}$ and spaces of configurations of free orbits

In this section we are going to describe the connection of the operad $\mathcal{D}_{2\sigma}$ and spaces of the form $\mathrm{Conf}_{kQ/e}(2\sigma)$ and investigate the equivariant structure of the homology of the latter.

2.7.1. Spaces of configurations of free orbits in 2σ**.** The first subsection is devoted to describing the non-equivariant structure of the homology of the operad $\mathcal{D}_{2\sigma}$. We start with two observations reducing the investigation of the homology graph coefficient systems of $\mathcal{D}_{2\sigma}(n)$ to the studies of spaces of configurations of free orbits in 2σ.

Recall from Definition 2.5.2 that the space $\mathrm{Conf}_{nQ/e}(2\sigma)$ is given as the following subspace of $(\mathbb{R}^2 \setminus \{0\})^n$:

$$\left\{((x_1,y_1),\dots,(x_n,y_n)) \in (\mathbb{R}^2\setminus\{0\})^n \;\middle|\; \begin{matrix}(x_i,y_i)\neq(x_j,y_j) \\ (x_i,y_i)\neq(-x_j,-y_j)\end{matrix} \quad \text{if } 1\leq i<j\leq n\right\}.$$

Observation 2.7.1. The representation 2σ has only one fixed point and thus if a Q-set T may be embedded in 2σ, then T has at most one fixed point. Therefore, by Proposition 2.5.1, we see that the only non-trivial graph subgroup of $Q\times\Sigma_n$ which has non-trivial fixed points on $\mathcal{D}_{2\sigma}(n)$ are the subgroup $\Gamma_{\lfloor \frac{n}{2}\rfloor}$ and its conjugates. Denote the subgroup $\Gamma_{\lfloor \frac{n}{2}\rfloor}$ by $\Gamma_{\mathfrak{m}}$ (here $\mathfrak{m}$ stands for "maximal").

Observation 2.7.2. By Theorem 2.5.4 we get that

$$\mathcal{D}_{2\sigma}(n)^{\Gamma_{\mathfrak{m}}} \simeq \mathrm{Conf}_{\lfloor \frac{n}{2}\rfloor Q/e}(2\sigma).$$

This equivalence is true regardless of the parity of n. If n is odd, by Theorem 2.5.4 we see that there appears additional factor of the cartesian product having the form $\mathrm{Conf}_{Q/Q}(2\sigma)$. However, since 2σ has only one fixed point, this space is just a point.

By the observation above we see that the key objects in studying the homology graph coefficient systems of $\mathcal{D}_{2\sigma}(n)$ are spaces of the form $\mathrm{Conf}_{kQ/e}(2\sigma)$ and the action of $Q \wr \Sigma_k$ on them. Therefore we will focus our attention on these spaces.

Remark 2.7.3. The space $\mathrm{Conf}_{kQ/e}(2\sigma)$ may be seen as a complement of a hyperplane arrangement in $\mathbb{C}^k$ given by the set of hyperplanes:

$$x_i = x_j$$
$$x_i = -x_j$$

This is a complexification of the analogous hyperplane arrangement in $\mathbb{R}^n$.

This hyperplane arrangement is connected to the *root systems of type D*. Therefore, by general theory of root systems and hyperplane arrangements, the fundamental group of $\mathrm{Conf}_{kQ/e}(2\sigma)$ is a *pure braid group of type D*. Details may be found in [**51**]. However, we are not going to use this fact.

Lemma 2.7.4. *There is a fibration:*

$$p\colon \mathrm{Conf}_{kQ/e}(2\sigma) \longrightarrow \mathrm{Conf}_{(k-1)Q/e}(2\sigma)$$

given by the projection on first $k-1$ factors, i.e.,

$$p\colon (x_1, \dots, x_{k-1}, x_k) \mapsto (x_1, \dots, x_{k-1}).$$

Moreover, the fibre of p is homotopy equivalent to a wedge of $2k-1$ circles.

Proof. This is analogous to the classical fibrations in the theory of configuration spaces, see [**17**, Section I.1] □

Lemma 2.7.5. *The action of $\pi_1\left(\mathrm{Conf}_{(k-1)Q/e}(2\sigma)\right)$ on the homology of the fibre of p is trivial.*

Proof. The proof is analogous to the non-equivariant case, see [**14**, Lemma III.6.3]. □

Proposition 2.7.6. *All homology groups of $\mathrm{Conf}_{kQ/e}(2\sigma)$ are free and its Poincaré polynomial is given by:*

$$P_k(t) = (1+t)(1+3t)\dots(1+(2k-3)t)(1+(2k-1)t).$$

Proof. Using Lemma 2.7.5 we get that the E^2-page of the Serre spectral sequence yields:

$$E^2_{pq} = H_p\left(\mathrm{Conf}_{(k-1)Q/e}(2\sigma),\ H_q(\bigvee_{2k-1} S^1)\right) \Rightarrow H_{p+q}\left(\mathrm{Conf}_{kQ/e}(2\sigma)\right).$$

Since the homology of the fibre is concentrated in degrees 0 and 1, the second page of this spectral sequence consists of 2 rows. Now we are going to show that all d^2-differentials are zero and thus the spectral sequence degenerates on the E^2-page.

Note that the fibration p has a section. It can be given for example by:

$$s\colon (x_1, \dots, x_{k-1}) \mapsto (x_1, \dots, x_{k-1}, (||x_1|| + \dots + ||x_{k-1}||, 0)).$$

Denote the fibre $\bigvee_{2k-1} S^1$ by X_k. Since all of the involved groups are free, by Universal Coefficient Theorem we get that $E^2_{pq} \cong H_*\left(\mathrm{Conf}_{(k-1)Q/e}(2\sigma)\right) \otimes H_*(X_k)$. Therefore it is enough to show that the differential

$$d^2_{20}\colon E^2_{20} = H_2\left(\mathrm{Conf}_{(k-1)Q/e}(2\sigma)\right) \longrightarrow H_1(X_k) \cong E^2_{01}$$

is zero. This differential is the edge morphism, and for the Serre spectral sequence this morphism is well-known - this is the trangression (see [**49**, Theorem 6.6]). So the image of d^2_{20} is included in the kernel of the map $H_1(X_k) \longrightarrow H_1\left(\mathrm{Conf}_{kQ/e}(2\sigma)\right)$ induced from the fibre sequence. However, this map is injective by the existence of the section. Therefore we obtain that all d^2-differentials are zero and the spectral sequence collapses on the E^2-page.

By this fact we obtain that

$$H_p(\mathrm{Conf}_{kQ/e}(2\sigma)) \cong H_p(\mathrm{Conf}_{(k-1)Q/e}(2\sigma)) \oplus \left(H_1(X_k) \otimes H_{p-1}(\mathrm{Conf}_{(k-1)Q/e}(2\sigma))\right).$$

Firstly note that $\mathrm{Conf}_{Q/e}(2\sigma) \simeq S(2\sigma)$, so its homology is free. By induction on k we obtain that all homology groups of $\mathrm{Conf}_{kQ/e}(2\sigma)$ are free.

Moreover, we have that

$$\begin{aligned}\mathrm{rank}\left(H_p(\mathrm{Conf}_{kQ/e}(2\sigma))\right) = \mathrm{rank}\left(H_p(\mathrm{Conf}_{(k-1)Q/e}(2\sigma))\right) + \\ + (2k-1)\,\mathrm{rank}\left(H_{p-1}(\mathrm{Conf}_{(k-1)Q/e}(2\sigma)\right).\end{aligned}$$

Since the Poincaré polynomial of $\mathrm{Conf}_{Q/e}(2\sigma) \cong S(2\sigma)$ is $1+t$, the result follows by induction. □

2.7.2. Signed S-trees. From now on, we are going to investigate equivariant structure of the homology of spaces $\mathrm{Conf}_{kQ/e}(2\sigma)$. For this purpose we are going to extend the language of S-trees given in Chapter 2.4 in this section.

Definition 2.7.7. Let $S \subset \{1, \dots, k\}$. Let *signed S-tree* be a graph defined as in Definition 2.4.3, point (1) but with leaves labeled by the elements of the set $-S \cup S$ with the assumption that if a leaf is labeled by $i \in -S \cup S$, no other leaf can be labeled by $-i$.

The *underlying S-tree* of a signed S-tree T is given by the same tree labeled by absolute values of the labels of T.

Define for a subset $A \subset \{1, \dots k\}$ the map

$$\begin{aligned} s_A\colon \mathrm{Conf}_{kQ/e}(2\sigma) &\longrightarrow \mathrm{Conf}_{kQ/e}(2\sigma) \\ i \in A &\Rightarrow s(x_i) = \gamma x_i \\ i \notin A &\Rightarrow s(x_i) = x_i. \end{aligned}$$

Definition 2.7.8. Let T be a signed S-tree and let $\bar{T}$ be its underlying S-tree. Let A be a subset of S given by labels of T which are negative integers. Then the *planetary system P_T* is a submanifold $s_A(P_{\bar{T}})$, where $P_{\bar{T}}$ is a planetary system corresponding to $\bar{T}$ (see Definition 2.4.6) with its center of mass set to be $(1, 0)$, i.e., the point (1) of Definition 2.4.6 replaced by $c(\mathbf{x}, \bar{T}) = (1, 0)$. An example of a planetary system associated to a signed S-tree may be found in Figure 2.10.

Remark 2.7.9. Similarly as in the case of S-trees, if T is a signed S-tree with i internal vertices, then it defines an embedding ϵ_T of the i-dimensional torus in $\mathrm{Conf}_{kQ/e}(2\sigma)$ as the planetary system P_T.

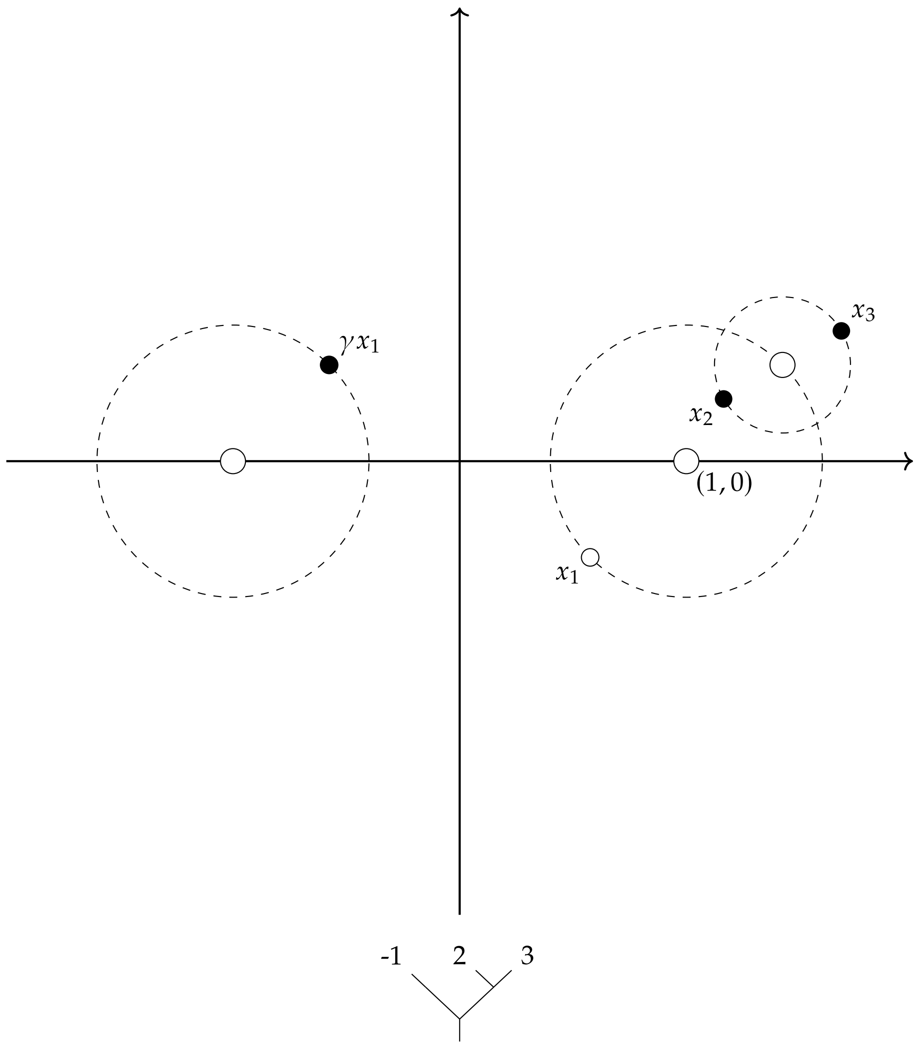


FIGURE 2.10. Example of a planetary system in $\mathrm{Conf}_{3Q/e}(2\sigma)$ associated to a signed S-tree.

REMARK 2.7.10. Signed S-trees are subject to anti-symmetry and Jacobi identity relations given in Proposition 2.4.8.

OBSERVATION 2.7.11. Let $\Theta_n^{(k)}$ be the set of signed S-trees with k leaves and labeled by elements of $\{1, \dots, n\}$. Then $Q \wr \Sigma_k$ acts on $\Theta_n^{(k)}$ by acting on labels, in the sense that Σ_k permutes the labels and every copy of Q changes the sign of the corresponding label.

PROPOSITION 2.7.12. *Let T be a signed S-tree and denote by $\bar{T}$ its corresponding homology class in $H_i(\mathrm{Conf}_{kQ/e}(2\sigma))$. Then $\bar{T}$ satisifies the following relation:*

$$\bar{T} = (\gamma, \gamma, \dots, \gamma, \mathrm{id})\bar{T},$$

where $(\gamma, \gamma, \dots, \gamma, \mathrm{id}) \in Q \wr \Sigma_k$.

Proof. Let P_T be the planetary system associated to T. Then the action of the element $(\gamma, \gamma, \dots, \gamma, \mathrm{id})$ has the effect of the antipodal map of all circles in P_T. Therefore it is a composition of maps of degree 1, thus it has degree 1 itself. So the statement follows. □

2.7.3. $Q \wr \Sigma_k$-equivariant structure of the homology.

Construction 2.7.13. Let $\tilde{\eta}\colon \mathbb{R}^2 \to \mathbb{R}^2_+$ be the following homeomorphism between the plane and the right half-plane:

$$(x, y) \mapsto (e^x, y).$$

Note that $\tilde{\eta}$ is homotopic to the identity by the following homotopy:

$$\begin{aligned} h\colon \mathbb{R}^2 \times I &\to \mathbb{R}^2_+ \\ (x, y, t) &\to (tx + (1-t)e^x, y). \end{aligned}$$

Then $\tilde{\eta}$ induces a Σ_k-equivariant embedding:

$$\eta\colon \operatorname{Conf}_k(\mathbb{R}^2) \to \operatorname{Conf}_{kQ/e}(2\sigma).$$

by

$$(x_1, x_2, \dots, x_k) \mapsto (\tilde{\eta}(x_1), \tilde{\eta}(x_2), \dots, \tilde{\eta}(x_k)).$$

Proposition 2.7.14. *The induced map in homology*

$$\eta_*\colon H_*(\operatorname{Conf}_k(\mathbb{R}^2)) \to H_*(\operatorname{Conf}_{kQ/e}(2\sigma))$$

is injective.

Proof. Note that non-equivariantly $\operatorname{Conf}_{kQ/e}(2\sigma)$ is a subspace of $\operatorname{Conf}_k(\mathbb{R}^2 \setminus \{0\})$. Let

$$\eta'\colon \operatorname{Conf}_{kQ/e}(2\sigma) \to \operatorname{Conf}_k(\mathbb{R}^2)$$

be the embedding induced by $\tilde{\eta}$.

Denote also by $\bar{\eta}\colon \operatorname{Conf}_k(\mathbb{R}^2) \to \operatorname{Conf}_k(\mathbb{R}^2)$ the map given by

$$(x_1, x_2, \dots, x_k) \mapsto (\tilde{\eta}(x_1), \tilde{\eta}(x_2), \dots, \tilde{\eta}(x_k)).$$

Note that since $\tilde{\eta}$ is homotopic to identity, so is $\bar{\eta}$.

The map induced by the composite in the homology

$$(\eta' \circ \eta)_*\colon : H_*(\operatorname{Conf}_k(\mathbb{R}^2)) \to H_*(\operatorname{Conf}_k(\mathbb{R}^2))$$

is the identity. Indeed, we have that

$$(\eta' \circ \eta)(x_1, \dots, x_k) = (\tilde{\eta}^s(x_1), \dots, \tilde{\eta}^s(x_k)) = \bar{\eta}^2(x_1, \dots, x_k).$$

Thus $\eta' \circ \eta = \bar{\eta}^2$ and since $\bar{\eta}$ is homotopic to identity, so is $\eta' \circ \eta$. So $(\eta' \circ \eta)_*$ is the identity. Thus we can conclude that η_* has a left section, so it is injective. □

Definition 2.7.15. Let $A_n^{(k)}$ be the $\mathbb{Z}[Q \wr \Sigma_k]$-submodule of $H_n(\mathrm{Conf}_{kQ/e}(2\sigma))$ generated by $\eta_*\left(H_n(\mathrm{Conf}_k(\mathbb{R}^2))\right)$.

Denote by Q_n^{Δ} the graph of the diagonal homomorphism $Q \to Q^{\oplus n}$.

Proposition 2.7.16. *Let Σ_2 be embedded in Σ_k as a subgroup generated by a transposition τ and let Σ_{k-2} be embedded in Σ_k as a subgroup of all permutations leaving the elements of τ fixed.*

Then there is an isomorphism of $\mathbb{Z}[Q \wr \Sigma_k]$-modules:

$$A_1^{(k)} \cong \mathbb{Z}\left[\frac{Q \wr \Sigma_k}{\left(Q_2^{\Delta} \oplus \bigoplus_{k-2} Q\right) \times (\Sigma_2 \oplus \Sigma_{k-2})}\right],$$

where the subgroup in the denominator should be understood as a subgroup of $Q \wr \Sigma_k$ generated by elements of the given subgroup.

Proof. We need to show that the basis of $A_1^{(k)}$ as free abelian group is of the form

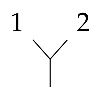
$$\frac{Q \wr \Sigma_k}{\left(Q^{\Delta} \oplus \bigoplus_{k-2} Q\right) \times (\Sigma_2 \oplus \Sigma_{k-1})}.$$

Identify 2σ with the complex sign representation. Recall the map s_A defined for $A \subset \{1, \ldots k\}$ by:

$$\begin{gathered} s_A \colon \mathrm{Conf}_{kQ/e}(2\sigma) \to \mathrm{Conf}_{kQ/e}(2\sigma) \\ i \in A \Rightarrow s(x_i) = -x_i \\ i \notin A \Rightarrow s(x_i) = x_i. \end{gathered}$$

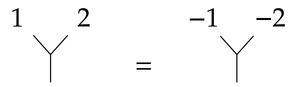
Recall that the basis of $H_1(\mathrm{Conf}_k(\mathbb{R}^2))$ is given by S-trees having one internal vertex:

By the definition of $A_1^{(k)}$, its basis of consists of all signed S-trees having as underlying S-trees the elements of basis of $H_1(\mathrm{Conf}_k(\mathbb{R}^2))$. In order to describe the structure of this basis as a $Q \wr \Sigma_k$-set we need to choose one element and describe its stabilizer. Without loss of generality, let this element be given by the class associated to the signed S-tree



By the antisymmetry relation, we see that this element is fixed by the transposition (1 2) and by any permutation which leaves 1 and 2 fixed.

The elements of $\bigoplus_k Q \leq Q \wr \Sigma_k$ act on the given tree by changing signs of labels. Recall from Proposition 2.7.12 that we have the following relation in $H_1(\mathrm{Conf}_{kQ/e}(2\sigma))$:



Thus we obtain that the element $(g_1, g_2, \ldots, g_k) \in \bigoplus_k Q$ acts trivially on the given tree if and only if $g_1 = g_2$, i.e. if $(g_1, g_2, \ldots, g_k) \in Q_2^\Delta \oplus \bigoplus_{k-2} Q$.

Therefore the stabilizer of the given tree is of the form

$$\left(Q_2^\Delta \oplus \bigoplus_{k-2} Q\right) \times (\Sigma_2 \oplus \Sigma_{k-2}).$$

Thus the orbit of the given tree is of the form

$$\frac{Q \wr \Sigma_k}{(Q_2^\Delta \oplus \bigoplus_{k-2} Q) \times (\Sigma_2 \oplus \Sigma_{k-2})}.$$

This orbit is the whole basis of $A_1^{(k)}$ - as its basis as a free abelian group has $2\binom{k}{2}$ elements, and

$$\left|\frac{Q \wr \Sigma_k}{(Q_2^\Delta \oplus \bigoplus_{k-2} Q) \times (\Sigma_2 \oplus \Sigma_{k-2})}\right| = \frac{2^k k!}{2^{k-1} \times 2 \times (k-2)!} = 2\binom{k}{2}.$$

□

PROPOSITION 2.7.17. *Let Σ_{k-1} be embedded in Σ_k as a subgroup of permutations leaving an element of $\{1, \ldots, k\}$ fixed. Then the $\mathbb{Z}[Q \wr \Sigma_k]$-module $H_1(\mathrm{Conf}_{kQ/e}(2\sigma))$ has a submodule of the form:*

$$B^{(k)} = \mathbb{Z}\left[\frac{Q \wr \Sigma_k}{\bigoplus_k Q \times \Sigma_{k-1}}\right].$$

PROOF. Note that $\mathrm{Conf}_{kQ/e}(2\sigma)$ is a subset of $\mathrm{Conf}_k(\mathbb{R}^2)$. For easiness of presentation, identify both with spaces of configurations of points in $\mathbb{C}$, with additional conditions in the case of the former.

Identify the circle S^1 with the set of complex numbers with the norm 1. Choose a point $(x_1, \ldots, x_k) \in \mathrm{Conf}_{kQ/e}(2\sigma)$. Define for every number $i \in \{1, \ldots, k\}$ the following map:

$$\begin{aligned} s_i \colon S^1 &\to \mathrm{Conf}_{kQ/e}(2\sigma) \\ z &\mapsto (x_1, \ldots, e^z x_i, \ldots, x_k). \end{aligned}$$

The effect of a map s_i is winding a point x_i once around the origin while leaving all other points in the configuration fixed. For a depiction, see Figure 2.11.

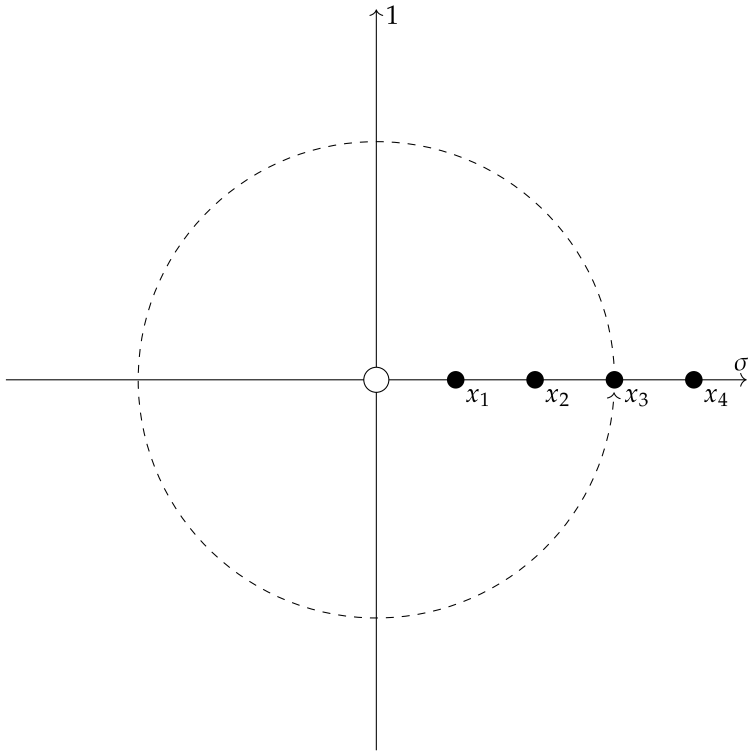


FIGURE 2.11. The effect of map s_3 in $\mathrm{Conf}_{4Q/e}(2\sigma)$.

By abuse of notation, let s_i also denote the homology class in $H_1(\mathrm{Conf}_{kQ/e}(2\sigma))$ given by the circle embedded via s_i.

Define $B^{(k)}$ to be a free abelian group on s_i. We are going to show that as a $\mathbb{Z}[Q \wr \Sigma_k]$-module $B^{(k)}$ has the form as stated. We are going to use the similar method as in the proof of previous proposition, i.e., we are going to describe the stabilizer of the class s_1.

Firstly, note that all copies of Q in $Q \wr \Sigma_k$ act trivially on s_1. The element $(\gamma, 1, \ldots, 1)$ defines an antipodal map on the sphere embedded by s_1, thus it has degree 1. The other elements leave the point x_1 untouched, so they act trivally as well.

Secondly, note that s_1 is fixed by any permutation which leaves 1 fixed. Therefore the stabilizer of s_1 has the form

$$\bigoplus_k Q \times \Sigma_{k-1}.$$

By comparison of numbers of elements we see that the set $\frac{Q \wr \Sigma_k}{\bigoplus_k Q \times \Sigma_{k-1}}$ is whole additive basis of $B^{(k)}$. Thus the result follows. □

COROLLARY 2.7.18. *As $\mathbb{Z}[Q \wr \Sigma_k]$-modules we have that*

$$H_1(\mathrm{Conf}_{kQ/e}(2\sigma)) \cong A_1^{(k)} \oplus B^{(k)}.$$

Proof. Firstly, note that the intersection $A_1^{(k)} \cap B^{(k)}$ is trivial by the way how these submodules are defined.

The statement now follows from the rank argument. By Proposition 2.7.6 we have that the rank of $H_1(\mathrm{Conf}_{kQ/e}(2\sigma))$ is equal to the sum of k first odd numbers, which is equal to k^2. On the other hand, we have that

$$\left|A_1^{(k)} \oplus B^{(k)}\right| = k + 2\binom{k}{2} = k^2.$$

□

Proposition 2.7.19. *If $i < k$, there is an $Q \wr \Sigma_k$-equivariant embedding:*

$$\rho\colon A_i^{(k)} \to H_{i+1}(\mathrm{Conf}_{kQ/e}(2\sigma)).$$

Proof. Let T be an S-tree with i internal vertices and let T also denote the corresponding homology class in $H_i(\mathrm{Conf}_k(\mathbb{R}^2))$. Let P_T be the planetary system given by T and ϵ_T a corresponding embedding of an i-dimensional torus. (see Definition 2.4.6 and Remark 2.4.7).

Recall the map η from Construction 2.7.13. Let T' be the planetary system being the image of $(i+1)$-dimensional torus by the following map:

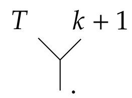
$$\begin{aligned} \epsilon_{T'}\colon (S^1)^i \times S^1 &\to \mathrm{Conf}_k(2\sigma) \\ ((z_1,\ldots,z_i),z) &\mapsto (e^z\eta(\epsilon_T(z_1)),\ldots,e^z\eta(\epsilon_T(z_i))). \end{aligned}$$

Thus T' defines a homology class in $H_{i+1}(\mathrm{Conf}_{kQ/e}(2\sigma))$, being the image of a class from $H_i(\mathrm{Conf}_k(\mathbb{R}^2))$. Define $\rho(T) = T'$.

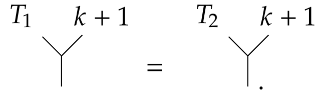
We will firstly show that ρ is injective. Note that $\mathrm{Conf}_{kQ/e}(2\sigma)$ can be (non-equivariantly) embedded in $\mathrm{Conf}_{k+1}(\mathbb{R}^2)$:

$$\begin{aligned} \chi\colon\ \mathrm{Conf}_{kQ/e}(2\sigma) &\to \mathrm{Conf}_{k+1}(\mathbb{R}^2) \\ \chi(x_1,\ldots,x_k) &\mapsto (x_1,\ldots,x_k,0). \end{aligned}$$

The map $\chi_* \circ \rho\colon H_i(\mathrm{Conf}_k(\mathbb{R}^2)) \to H_{i+1}(\mathrm{Conf}_{k+1}(\mathbb{R}^2))$ sends the class T to the class of



Therefore $\chi_* \circ \rho$ is injective on generators. Indeed, assume that T_1 and T_2 are S-trees representing homology classes in $H_i(\mathrm{Conf}_k(\mathbb{R}^2))$ such that in $H_{i+1}(\mathrm{Conf}_{k+1}(\mathbb{R}^2))$ we have that



Since the rightmost leaf is fixed with the label $k+1$, there is no relation given in Theorem 2.4.10 connecting these trees, thus we have that $T_1 = T_2$.

Since the map $\chi_* \circ \rho$ is injective on generators and both groups $H_i(\mathrm{Conf}_k(\mathbb{R}^2))$ and $H_{i+1}(\mathrm{Conf}_{k+1}(\mathbb{R}^2))$ are free, this map is injective. Therefore ρ is injective. Moreover, it sends generators to generators.

Note that ρ is Σ_k-equivariant. Since we defined ρ on the basis elements, we can extend it $Q \wr \Sigma_k$-equivariantly to whole $A_i^{(k)}$. Thus the statement follows. □

Proposition 2.7.20. *The top homology $H_k(\mathrm{Conf}_{kQ/e}(2\sigma))$ has a trivial $\mathbb{Z}[Q \wr \Sigma_k]$-submodule $\mathbb{Z}$.*

Proof. Choose a point $(x_1, \dots, x_k)$. Define a map

$$t\colon (S^1)^k \to \mathrm{Conf}_{kQ/e}(2\sigma)$$
$$(z_1, \dots, z_k) \mapsto (e^{z_1}x_1, \dots, e^{z_k}x_k).$$

Let $\bar{t}$ be the homology class defined as $t_*([(S^1)^k])$, the image of the fundamental class of the k-dimensional torus. Note that $Q \wr \Sigma_k$ acts on this class trivially; every copy of Q acts antipodally on one circle, thus its action is trivial. Elements of Σ_k permute the circles in the image of t, thus they do not change the homology class $\bar{t}$. The stated submodule is a submodule generated by $\bar{t}$. □

2.7.4. Examples. In this section we will present the equivariant structure of homology for basic configuration spaces of orbits in 2σ.

2.7.4.1. *The homology of the space* $\mathrm{Conf}_{Q/e}(2\sigma)$.

Proposition 2.7.21. *The $\mathbb{Z}[Q]$-module structure of the homology of $\mathrm{Conf}_{Q/e}(2\sigma)$ is given as follows:*

$$H_i(\mathrm{Conf}_{Q/e}(2\sigma)) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = 1 \\ 0 & \text{else.} \end{cases}$$

Proof. By Proposition 2.7.6 we see that homology of $\mathrm{Conf}_{Q/e}(2\sigma)$ is concentrated in degrees 0 and 1 and in both degrees it is a free abelian group of rank 1. The trivial Q-action for $i = 0$ is clear. The trivial action on the degree 1 homology group comes from the fact that Q acts antipodally on the generating planetary system - thus acts trivially. □

Remark 2.7.22. The proposition above may be proven in different way. The space $\mathrm{Conf}_{Q/e}(2\sigma)$ is the same as the space $\mathrm{Conf}_1(2\sigma \setminus \{0\})$, thus it is homotopy equivalent to $S(2\sigma)$, the unit sphere in 2σ. So the underlying abelian groups in homology are these of the circle S^1. Since Q acts on $S(2\sigma)$ as the antipodal map on the circle, it is of degree 1. The statement follows.

2.7.4.2. *The homology of space* $\mathrm{Conf}_{2Q/e}(2\sigma)$.

PROPOSITION 2.7.23. *The* $\mathbb{Z}[Q \wr \Sigma_2]$*-module structure of the homology of* $\mathrm{Conf}_{2Q/e}(2\sigma)$ *is given as follows:*

$$H_i(\mathrm{Conf}_{2Q/e}(2\sigma)) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}\left[\frac{Q\wr\Sigma_2}{\bigoplus_2 Q}\right] \oplus \mathbb{Z}\left[\frac{Q\wr\Sigma_2}{Q_2^{\Delta}\times\Sigma_2}\right] & \text{if } i = 1 \\ \mathbb{Z} \oplus \mathbb{Z}\left[\frac{Q\wr\Sigma_2}{Q_2^{\Delta}\times\Sigma_2}\right] & \text{if } i = 2 \\ 0 & \text{else.} \end{cases}$$

PROOF. The case $i = 0$ follows from Proposition 2.7.6.

The second case, $i = 1$ follows from Corollary 2.7.18.

Finally, the description of $H_2(\mathrm{Conf}_{2Q/e}(2\sigma))$ follows from Propositions 2.7.19 and 2.7.20. By these propositions we see that $H_2(\mathrm{Conf}_{2Q/e}(2\sigma))$ has a submodule of the form $\mathbb{Z} \oplus A_1^{(2)}$. By the rank argument, this submodule is the whole module $H_2(\mathrm{Conf}_{2Q/e}(2\sigma))$.

Other homology groups being zero follows from Proposition 2.7.6. □

The generators of the homology groups of $\mathrm{Conf}_{2Q/e}(2\sigma)$ are depicted in Figures 2.12 and 2.13.

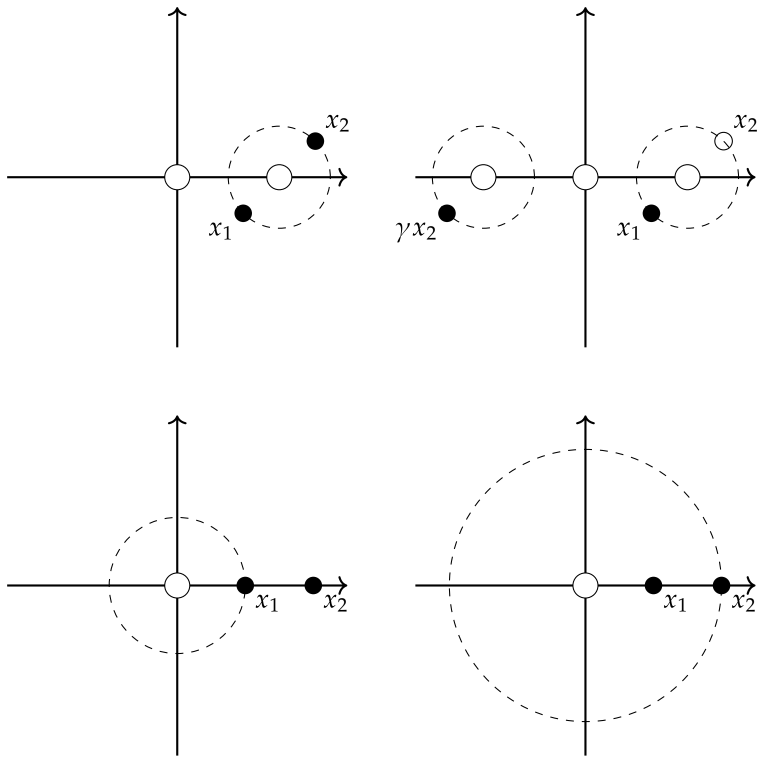


FIGURE 2.12. Generators of $H_1(\mathrm{Conf}_{2Q/e}(2\sigma))$. The rows correspond to different direct summands.

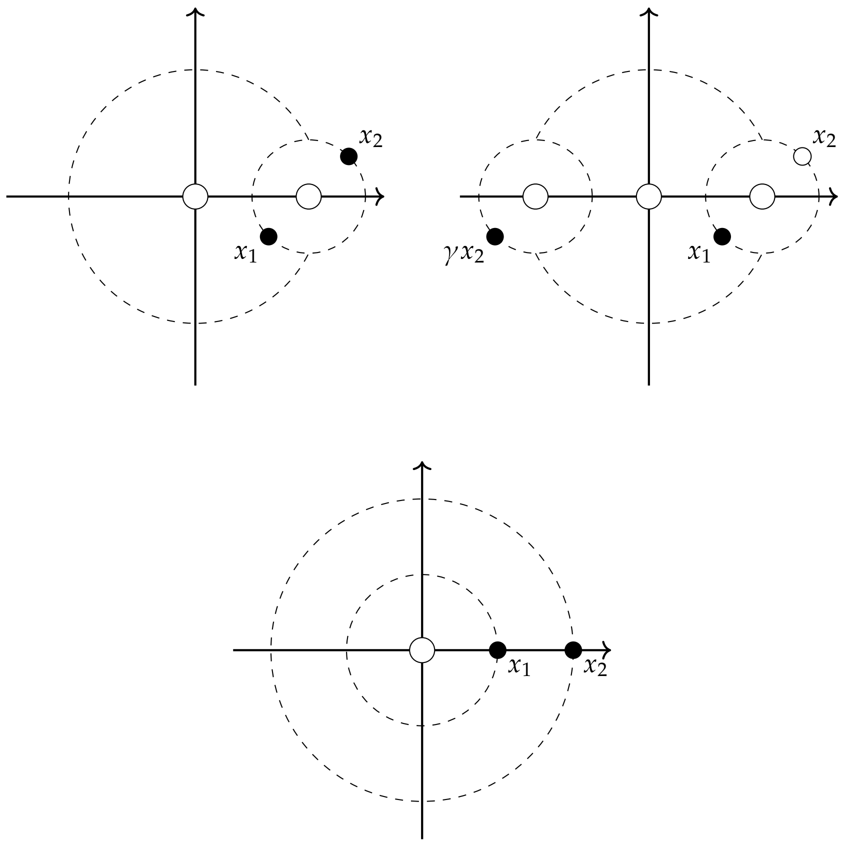


FIGURE 2.13. Generators of $H_2(\mathrm{Conf}_{2Q/e}(2\sigma))$. The rows correspond to different direct summands.

2.7.4.3. *The homology of space* $\mathrm{Conf}_{3Q/e}(2\sigma)$.

PROPOSITION 2.7.24. *The* $\mathbb{Z}[Q \wr \Sigma_3]$*-module structure of the homology of* $\mathrm{Conf}_{3Q/e}(2\sigma)$ *is given as follows:*

$$H_i(\mathrm{Conf}_{3Q/e}(2\sigma)) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}\left[\frac{Q\wr\Sigma_3}{\bigoplus_3 Q\times\Sigma_2}\right] \oplus \mathbb{Z}\left[\frac{Q\wr\Sigma_3}{(Q_3^\Delta\oplus Q)\times\Sigma_2}\right] & \text{if } i = 1 \\ \mathbb{Z}\left[\frac{Q\wr\Sigma_3}{\bigoplus_3 Q\times\Sigma_2}\right] \oplus \frac{\mathbb{Z}\left[\frac{Q\wr\Sigma_3}{Q_3^\Delta\times\Sigma_2}\right]}{(1+(1\ 2\ 3)+(1\ 3\ 2))} \oplus \left(\mathbb{Z}\left[\frac{Q\wr\Sigma_3}{(Q_2^\Delta\oplus Q)\times\Sigma_2}\right]\right)^{\oplus 2} & \text{if } i = 2 \\ \mathbb{Z} \oplus \frac{\mathbb{Z}\left[\frac{Q\wr\Sigma_3}{Q_3^\Delta\times\Sigma_2}\right]}{(1+(1\ 2\ 3)+(1\ 3\ 2))} \oplus \mathbb{Z}\left[\frac{Q\wr\Sigma_3}{(Q_2^\Delta\oplus Q)\times\Sigma_2}\right] & \text{if } i = 3 \\ 0 & \text{else.} \end{cases}$$

PROOF. The cases $i = 0$ and $i = 1$ follows from Proposition 2.7.6 and Corollary 2.7.18.

In the case $i = 2$ we are going to see the submodules A_2^3 and $\rho(A_1^{(3)})$ as direct summands. The former has the form

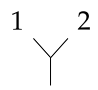
$$\frac{\mathbb{Z}\left[\frac{Q \wr \Sigma_3}{Q_3^{\Delta} \times \Sigma_2}\right]}{(1 + (1\ 2\ 3) + (1\ 3\ 2))}.$$

This comes from the definition of $A_2^{(3)}$. It is spanned by signed S-trees with underlying S-trees being the generators of $H_2(\mathrm{Conf}_3(\mathbb{R}^2))$, subject to the relation given in Proposition 2.7.12 and the Jacobi identity, see Proposition 2.4.8.

There are two direct summands of $H_2(\mathrm{Conf}_{3Q/e}(2\sigma))$ of the form

$$\mathbb{Z}\left[\frac{Q \wr \Sigma_3}{(Q_2^{\Delta} \oplus Q) \times \Sigma_2}\right].$$

One is $\rho(A_1^{(3)})$ (see Proposition 2.7.19), which we already mentioned above. The second direct summand of this form comes from another embedding of $A_1^{(3)}$ in $H_2(\mathrm{Conf}_{3Q/e}(2\sigma))$. Without loss of generality, we will define this embedding on the class corresponding to the tree



This can be easily generalised for other classes coming from signed S-trees. Denote the tree above by T.

Choose a point $(x_1, x_2, x_3) \in \mathrm{Conf}_{3Q/e}(2\sigma)$ such that $(x_1, x_2, x_3) \notin P_T$ and define the following map:

$$\begin{aligned} \phi_T \colon S^1 \times S^1 &\to \mathrm{Conf}_{3Q/e}(2\sigma) \\ (z_1, z_2) &\mapsto (\epsilon_T^{(1)}(z_1), \epsilon_T^{(2)}(z_1), e^{z_2} x_3). \end{aligned}$$

Here $\epsilon^{(i)}$ denotes the i-th factor of the map ϵ_T. This map defines an embedding of a two-dimensional torus, thus a class of degree 2 in $H_*(\mathrm{Conf}_{3Q/e}(2\sigma))$. The assignment $\epsilon_T \mapsto \phi_T$ consititutes the embedding of $A_1^{(3)}$ in $H_2(\mathrm{Conf}_{3Q/e}(2\sigma))$.

The submodule of the form

$$\mathbb{Z}\left[\frac{Q \wr \Sigma_3}{\bigoplus_3 Q \times \Sigma_2}\right]$$

is a submodule coming from embeddings of the two-dimensional torus by the following maps:

$$\begin{aligned} s_{ij} \colon S^1 \times S^1 &\to \mathrm{Conf}_{3Q/e}(2\sigma) \\ (z_i, z_j) &\mapsto (\ldots, e^{z_i} x_i, \ldots, e^{z_j} x_j) \end{aligned}.$$

By the similar reasoning as in the proof of Proposition 2.7.20 we obtain its $\mathbb{Z}[Q \wr \Sigma_3]$-module structure. By the rank argument, we see that these are all submodules of $H_2(\mathrm{Conf}_{3Q/e}(2\sigma))$.

Finally in the degree 3 we see the trivial submodule coming from Proposition 2.7.20 and the submodule $\rho(A_2^{(3)})$ by Proposition 2.7.19. The submodule of the form

$$\mathbb{Z}\left[\frac{Q \wr \Sigma_3}{(Q_2^{\Delta} \oplus Q) \times \Sigma_2}\right]$$

is defined as an image of $\rho(A_1^{(3)})$ under similar embedding to that of $A_1^{(3)}$ in the module $H_2(\mathrm{Conf}_{3Q/e}(2\sigma))$ via ϕ. A depiction of planetary systems obtained by these embeddings may be found in Figure 2.14. The rank argument shows that these are all submodules of $H_3(\mathrm{Conf}_{3Q/e}(2\sigma))$.

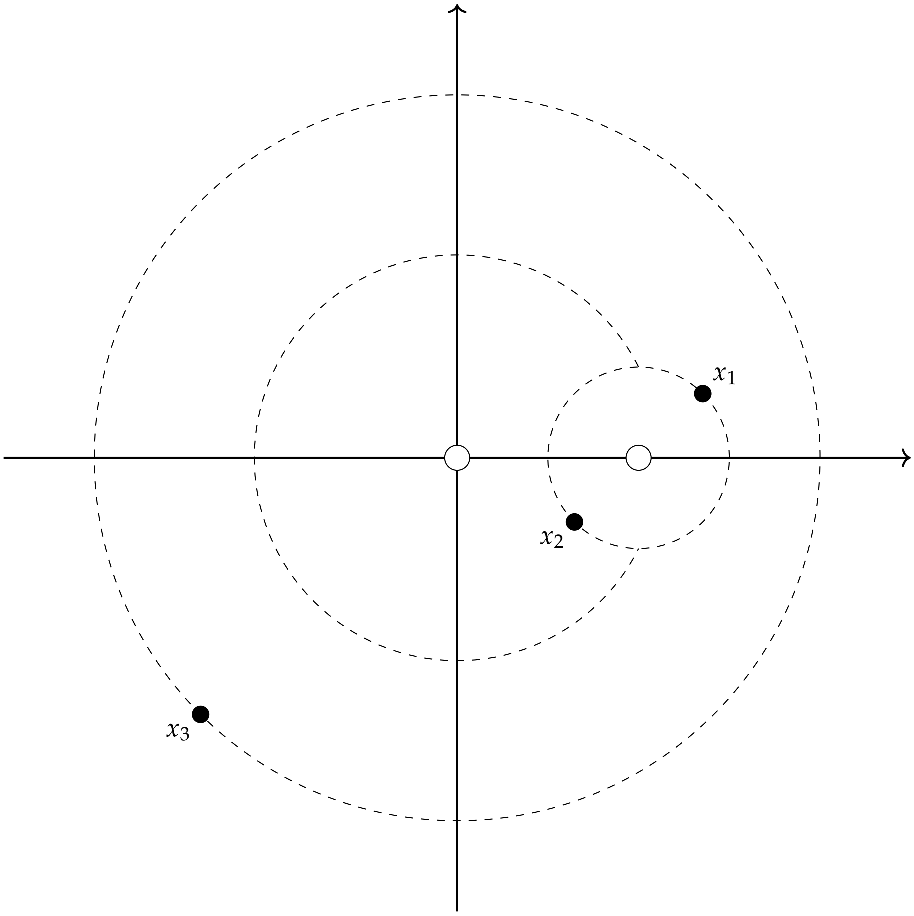


FIGURE 2.14. Embedding of $\rho(A_1^{(3)})$ in $H_3(\mathrm{Conf}_{3Q/e}(2\sigma))$.

□

CHAPTER 3

On the $RO(Q)$-graded coefficients of Eilenberg-MacLane spectra

3.1. Introduction

Let G be a finite group. In G-equivariant topology the role of ordinary cohomology is played by *Bredon cohomology* [**10**]. Whilst easy to define, making computations in Bredon theories is more complicated than in their non-equivariant analogues. Firstly, the coefficients of Bredon theories are of the form of a functor from the orbit category of G to the category of abelian groups. Secondly, Bredon theories are more naturally $RO(G)$*-graded* (graded over the representation ring of G) rather than graded over the integers.

As shown in [**39**], a $\mathbb{Z}$-graded Bredon theory extends to an $RO(G)$-graded one if its coefficients are of the form of a *Mackey functor*. Similarly as in non-equivariant topology, the Bredon homology/cohomology with coefficients in a Mackey functor $\underline{M}$ is represented by the *Eilenberg-MacLane G-spectrum $H\underline{M}$*. Spectra of this form appear in various contexts in equivariant topology. For example, equivariant Eilenberg-MacLane spectra are 0-slices in the slice spectral sequence [**26**].

The difficulties of computations in $RO(G)$-graded Bredon theories may be seen in calculations of $H\underline{M}^G_\star := \pi^G_\star(H\underline{M})$, the $RO(G)$-graded G-homotopy groups of $H\underline{M}$ (five-pointed star indicates $RO(G)$-grading). This is equivalent to computing the $RO(G)$-graded Bredon homology and cohomology of a point with coefficients in $\underline{M}$ and thus we will refer to $H\underline{M}^G_\star$ as the *coefficients of* $H\underline{M}$. The groups $H\underline{M}^G_n$ are zero for $n \in \mathbb{Z}$ unless $n = 0$, which resembles the non-equivariant case. However, if V is not a trivial representation then $H\underline{M}^G_V$ might be non-zero.

In this paper we use the Tate diagram to compute the $RO(Q)$-graded coefficients of Eilenberg-MacLane Q-spectra, where Q is the cyclic group of order 2. We do this in three instances: as an $RO(Q)$-graded abelian group, as a module over the coefficients of the Eilenberg-MacLane spectrum associated to the Burnside Mackey functor and finally as an $RO(Q)$-graded ring, when appropriate.

Tate diagram. The idea behind the Tate diagram is to decompose a Q-spectrum X into computationally simpler pieces:

(1) Borel completion X^h;
(2) free Q-spectrum X_h;

(3) singular spectrum X^{Φ};
(4) Tate spectrum X^t;

which are connected by the following commutative diagram:

$$\begin{tikzcd} X_h \arrow[r] \arrow[d, "\simeq"] & X \arrow[r] \arrow[d] & X^{\Phi} \arrow[d] \\ X_h \arrow[r] & X^h \arrow[r] & X^t, \end{tikzcd}$$

What makes computations by the Tate diagram feasible is that the rows are cofibre sequences and the right-hand square (known as the *Tate square*) is a homotopy pullback. Moreover, the coefficients of the spectra appearing in the bottom row may be computed by the *homotopy orbits* and *homotopy fixed points spectral sequences*. The foundational work on the Tate diagram is **[22]**, where all of the details are discussed.

The computational strength of the Tate diagram has been proven in various contexts - for example Greenlees use it in **[21]** to compute the coefficients of the Eilenberg-MacLane Q-spectrum $H\underline{\mathbb{Z}}$ as an $RO(Q)$-graded ring, Greenlees and Meier compute the coefficients of K-theory with reality $K\mathbb{R}$ in **[23]** and Hu and Kriz use it to compute the Q-equivariant Steenrod algebra in **[34]**. It was also used to compute the coefficients of $H\underline{\mathbb{Z}}$ over groups C_{p^2} with p prime by Zeng in **[64]**.

$RO(Q)$-graded abelian group structure. The first step is describing the $RO(Q)$-graded abelian group structure of $H\underline{M}^Q_\star$. We show that it is fully determined by the underlying Mackey functor $\underline{M}$. This structure is given in Theorem 3.6.1, which can be informally stated as follows:

THEOREM. *The $RO(Q)$-graded abelian group structure of $H\underline{M}^Q_\star$ may be presented by Figure 3.1, where:*

1. *every lattice point represents a Q-representation, the horizontal axis describes multiplicity of the trivial Q-representation and the vertical axis describes multiplicity of the sign representation.*
2. *The empty circle at the position $(0, 0)$ is the module $\underline{M}(Q/Q)$.*
3. *The values of $H\underline{M}^Q_\star$ lying on the $x = 0$ axis are submodules of $\underline{M}(Q/Q)$ given by the kernel of the restriction and the cokernel of the transfer.*
4. *The full dots in positions $(1, -1)$ and $(-1, 1)$ are submodules of $\underline{M}(Q/e)$ given by the kernel of the transfer and the cokernel of the restriction, whereas the values lying on the red/blue lines above/below them are their subquotients.*
5. *The values lying in blue and red areas are respectively the group cohomology and homology with coefficients in $\underline{M}(Q/e)$.*
6. *All other values are zero.*

$H\underline{\mathbb{A}}^Q_\star$-module structure. The category of Q-Mackey functors has a symmetric monoidal structure and commutative monoids with respect to this structure are called *Green functors*.

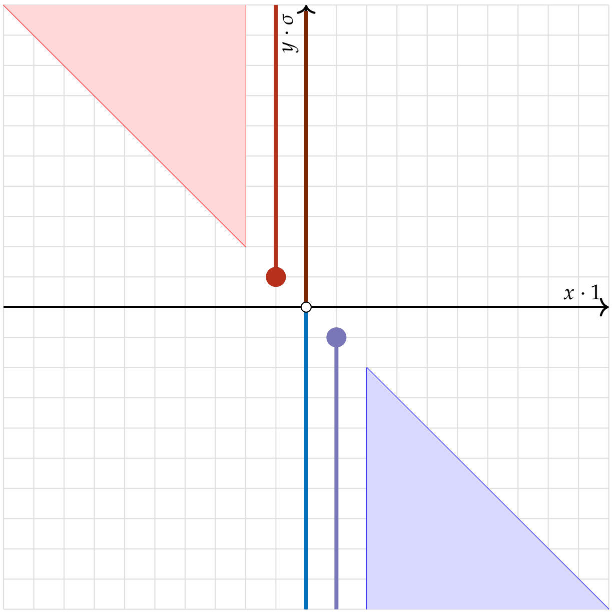


FIGURE 3.1. $RO(Q)$-graded coefficients of $H\underline{M}$.

If $\underline{M}$ is a Green functor, the Q-spectrum $H\underline{M}$ is a (naive) commutative ring Q-spectrum and its homotopy groups form an $RO(Q)$-graded commutative ring. The most fundamental example of a Green functor is the *Burnside Mackey functor* $\underline{\mathbb{A}}$.

Every Q-Mackey functor $\underline{M}$ is a module over $\underline{\mathbb{A}}$. Therefore $H\underline{M}$ is a module over $H\underline{\mathbb{A}}$ and $H\underline{M}^Q_\star$ is a module over the $RO(Q)$-graded commutative ring $H\underline{\mathbb{A}}^Q_\star$.

We describe $H\underline{\mathbb{A}}^Q_\star$ in Section 3.7. Its multiplicative structure is determined by two elements - a and u. The first is an Euler class associated to the inclusion $S^0 \to S^\sigma$, whereas the second is the generator of $H\underline{\mathbb{A}}^Q_{2-2\sigma}$ and corresponds to the generator of $H\underline{\mathbb{A}}^Q_2(S^{2\sigma})$.

The action of $H\underline{\mathbb{A}}^Q_\star$ on $H\underline{M}^Q_\star$ may be informally stated as follows:

OBSERVATION. The $H\underline{\mathbb{A}}^Q_\star$-module structure of $H\underline{M}^Q_\star$ is determined by the action of 3 elements - a, u and ω, where the latter is the class of Q/e in the Burnside ring.

The action by a on $H\underline{M}^Q_\star$ may be easily derived from the cofibre sequence

$$Q_+ \to S^0 \to S^\sigma.$$

However, a description of the action of u requires more work. The detailed analysis of this action is possible due to the Tate diagram and the connection between $H\underline{M}$ and its Borel completion. The details are given in Section 3.8. The action of the element ω is discussed in Section 3.9.

$RO(Q)$**-graded ring structure.** Finally, we describe the multiplicative structure of $H\underline{M}$ when $\underline{M}$ is a Green functor.

In Sections 3.8 and 3.9 we show that most of this structure may be derived from the $H\underline{\mathbb{A}}^Q_\star$-module structure. First issue that we encounter is the graded commutativity. The sign rule for commutativity in $RO(Q)$-graded rings involves units in the Burnside ring - thus not only -1, but also $1-\omega$, where ω is the class of Q/e.

In Section 3.9 we show the following:

THEOREM. *If $\underline{M}$ is a Green functor then $H\underline{M}^Q_\star$ is a strictly commutative ring, i.e., all signs coming from the graded commutativity rule are trivial.*

Finally, at the end of Section 3.9 in Observation 3.9.6, we give a recipe for describing a multiplicative structure of $H\underline{M}^Q_\star$ for any Green functor $\underline{M}$.

OBSERVATION. If $\underline{M}$ is a Q-Green functor then the multiplicative structure of $H\underline{M}^Q_\star$ is fully determined by its $H\underline{\mathbb{A}}^Q_\star$-module structure and relations between elements of degrees $1-\sigma$, $\sigma-1$, $3-3\sigma$ and $3\sigma-3$. These relations may be derived from the induced map of $H\underline{M} \to H\underline{M}^h$.

The procedure of obtaining multiplicative structure is illustrated with a wide array of examples in Sections 3.7 and 3.10.

Contribution of the paper and related work. The computations of coefficients of Eilenberg-MacLane spectra over Q already have a long history. Based on unpublished work of Stong, Lewis computed $H\underline{\mathbb{A}}^Q_\star$, where $\underline{\mathbb{A}}$ is the Burnside Mackey functor for Q in [**41**]. Calculations for the constant Mackey functor $\underline{\mathbb{F}}_2$ by Caruso can be found in [**13**]. The computations for the constant Mackey functor $\underline{\mathbb{Z}}$ may be found in Dugger's work [**15**].

The full $RO(Q)$-graded Mackey functor valued coefficients of $H\underline{M}$ for any Mackey functor $\underline{M}$ was given by Ferland in [**19**] and may also be found in Ferland-Lewis [**18**]. Their computations are based on the cofibre sequence

$$Q_+ \to S^0 \to S^\sigma.$$

This sequence allows one to describe attaching maps of free cells, and thus to compute the homology of any representation sphere. Therefore this method may be described as the "cell method". While being intuitive and allowing the computation of the $RO(Q)$-graded abelian group structure, this approach does not provide efficient methods of describing the multiplicative structure. We note that Ferland's computations work for all cyclic groups of prime order, but in this paper we will restrict our attention to the group of order 2. From Ferland's work we can derive the $RO(Q)$-graded abelian group structure and $H\underline{\mathbb{A}}^Q_\star$-module structure of $H\underline{M}^Q_\star$.

The aim of this paper is to show the computational strength of the Tate diagram on the example of computations of coefficients of equivariant Eilenberg-MacLane spectra. As demonstrated by Greenlees in his computations of $H\underline{\mathbb{Z}}^Q_\star$ in [**21**], the method based on the Tate diagram gives not only a good insight into the abelian group structure, but also into the multiplicative structure. In this paper we generalise Greenlees's results to all Q-Mackey and Green functors. Therefore the main contribution of this paper is the description of the multiplicative structure of $H\underline{M}^Q_\star$ and showing that the Tate diagram gives an algorithmic way to describe this structure.

The other strong computational technique in equivariant homotopy theory is the *slice spectral sequence* of Hill-Hopkins-Ravenel [**26**]. The idea behind it is a decomposition of a G-spectrum into a tower similar to the Postnikov tower, but retaining more equivariant information. However, the main challenge is usually to determine the filtration quotients. Therefore it needs additional techniques to support the calculations. An example of this is the fact that for any Q-spectrum X the starting input for the slice spectral sequence is given by the spectrum $\Sigma^{-1}H\underline{\pi_{-1}}(X)$, the desuspension of Eilenberg-MacLane spectrum of $\underline{\pi_{-1}}(X)$. In this paper we demonstrate that the computations based on the Tate diagram for simple equivariant objects, such as $H\underline{M}$, may be carried out in a neat and algorithmic way and thus provide a useful method of doing auxiliary calculations for the slice spectral sequence.

Additionally, it is worth noting that equivariant homotopy theory has a rather small repository of computational examples and is in a constant need for developing methods of doing calculations. Therefore an auxiliary contribution of this paper is adding another piece to this development.

We note here that the "full information" is given by the *Mackey functor valued $RO(Q)$*-graded coefficients $H\underline{M}^\bullet_\star$, as given for example in [**41**]. However, the main computational effort is computing the Q/Q level of the $RO(Q)$-graded Mackey functor $H\underline{M}^\bullet_\star$ and this is where the Tate method gives a hand. Therefore we focus our attention in this paper on computing $H\underline{M}^Q_\star$. The rest of the Mackey functor structure of $H\underline{M}^\bullet_\star$ may be easily deduced.

3.1.1. Notation and conventions. Throughout the whole paper Q denotes the group of order 2 and γ its non-trivial element. We denote the norm element of the group ring $\mathbb{Z}[Q]$ by N, i.e., $N = 1 + \gamma$. The Burnside ring of Q is denoted by $\mathbb{A}(Q)$ and we write ω for the the class of Q/e in this ring.

If M is a $\mathbb{Z}[Q]$-module, we denote its Q-fixed points by M^Q. We also put

$$M_Q = {}^{M}\!/\!_{(1-\gamma)M}.$$

We denote N-torsion elements of M by ${}_NM$, i.e., ${}_NM = \{x \in M \mid Nx = 0\}$. Two important $\mathbb{Z}[Q]$-modules are:

- $\mathbb{Z}$ - the integers with trivial Q-action;
- $\tilde{\mathbb{Z}}$ - the integers with sign action.

We denote the n-th Tate cohomology group of Q with coefficients in M by $\hat{H}^n(Q;M)$. These are defined as [**61**, Definition 6.2.4]:

$$\hat{H}^n(Q;M) := \begin{cases} M^Q/NM & \text{if } n \in \mathbb{Z} \text{ even} \\ {}_N M/(\gamma-1)M & \text{if } n \in \mathbb{Z} \text{ odd} \end{cases}$$

For details see also [**12**, Chapter VI].

Underlined capital letters are used to denote Mackey functors. If we need to show the structure of some particular Mackey functor we use the Lewis diagram:

$$\begin{tikzcd} \underline{M}(Q/Q) \arrow[d, bend right, "\mathrm{res}_{\underline{M}}"'] \\ \underline{M}(Q/e). \arrow[u, bend right, "\mathrm{tr}_{\underline{M}}"'] \end{tikzcd}$$

The map $\underline{M}(Q/Q) \to \underline{M}(Q/e)$ is the *restriction map* and will be denoted by $\mathrm{res}_{\underline{M}}$. The map $\underline{M}(Q/e) \to \underline{M}(Q/Q)$ is called the *transfer map* and will be denoted by $\mathrm{tr}_{\underline{M}}$. We will drop the subscripts if it is clear from the context which Mackey functor the notation refers to. For brevity we use V to denote the $\mathbb{Z}[Q]$-module $M(Q/e)$.

Throughout whole chapter we work in the category of genuine Q-spectra, and as preferred model we choose orthogonal spectra (see Section 1.6). By "commutative ring Q-spectrum" we mean an E_∞-ring in Q-spectra. In the literature these are also called naive commutative ring Q-spectra.

If X is a Q-spectrum we put $X^Q_\star = \pi^Q_\star(X)$. The five-pointed star is used to indicate the grading over $RO(Q)$ - the representation ring of Q. This means that our grading is "two-dimensional" - all representations of Q may be written in the form $x\mathbb{R} + y\sigma$, where x, y are integers, $\mathbb{R}$ is a real trivial one-dimensional representation and σ is a real sign representation. We will abbreviate $x\mathbb{R} + y\sigma$ to $x + y\sigma$. We sometimes refer to x in the gradation as a *fixed degree* and to y as a *twisted degree*. By the *antidiagonal* we mean the line $y = -x$. Whenever we want to restrict our attention to $\mathbb{Z}$-grading, we will use an asterisk $*$ instead.

We put $X^e_\star := \pi^Q_\star(F(Q_+, X))$. Note that by the induction-restriction adjunction $X^e_{x+y\sigma}$ is the same as a $\mathbb{Z}[Q]$-module $\pi_{x+y}(\mathrm{res}^Q_e(X))$, which we abbreviate as $\pi_{x+y}(X)$. A map $\phi\colon X \to Y$ that induces an isomorphism $X^e_* \cong Y^e_*$ will be called a *nonequivariant equivalence*.

Note that in some papers a different grading convention for $X^Q_\star$ is used, e.g., by Dugger in [**15**]. Our grading is related to this by $\mathbb{R}^{x+y,y} = x + y\sigma$.

3.1.2. Acknowledgments. I would like to thank John Greenlees for the idea and continuous supervision of this work. I am also very grateful to Luca Pol and Jordan Williamson for numerous discussions, comments and corrections. The final version of this paper owes its readability to their countless suggestions and patient reading of previous versions.

3.2. The Tate method

In this section we recall the Tate diagram method, see [**22**].

Let EQ be the free contractible Q-space and let us define the *isotropy separation sequence* to be the cofibre sequence:

$$EQ_+ \longrightarrow S^0 \longrightarrow \widetilde{EQ}.$$

Let X be a Q-spectrum. Let ϵ be the map

$$\epsilon \colon X \cong F(S^0, X) \to F(EQ_+, X).$$

By smashing ϵ with the isotropy separation sequence we obtain the following diagram:

$$(*) \qquad \begin{tikzcd} EQ_+ \wedge X \arrow[r] \arrow[d, "EQ_+ \wedge \epsilon"] & X \arrow[r] \arrow[d, "\epsilon"] & \widetilde{EQ} \wedge X \arrow[d, "\widetilde{EQ} \wedge \epsilon"] \\ EQ_+ \wedge F(EQ_+, X) \arrow[r] & F(EQ_+, X) \arrow[r] & \widetilde{EQ} \wedge F(EQ_+, X). \end{tikzcd}$$

The left-hand vertical map $EQ_+ \wedge \epsilon$ is a Q-equivalence (i.e., stable weak equivalence in the sense of Definition 1.6.24) by the following [**22**, Proposition 1.1]:

Lemma 3.2.1. *Let $\phi \colon X \to Y$ be a map of Q-spectra which is a nonequivariant equivalence. Then the induced maps*

$$\phi \wedge EQ_+ \colon X \wedge EQ_+ \to Y \wedge EQ_+$$
$$F(EQ_+, \phi) \colon F(EQ_+, X) \to F(EQ_+, Y)$$

are Q-equivalences.

Since the map ϵ is a nonequivariant equivalence, the map $EQ_+ \wedge \epsilon$ is a Q-equivalence. Therefore the right-hand square is a homotopy pullback square.

We define

$$X^h := F(EQ_+, X),$$
$$X_h := EQ_+ \wedge X,$$
$$X^t := F(EQ_+, X) \wedge \widetilde{EQ} = X^h \wedge \widetilde{EQ},$$
$$X^\Phi := \widetilde{EQ} \wedge X.$$

We also put $X_{hQ} := (X_h)^Q$ (see Definition 1.6.12), respectively for X^{hQ}, $X^{\Phi Q}$ and X^{tQ}. Note that if X is a naive spectrum, then X_{hQ} defined as above coincides with *homotopy orbits* $(EQ_+ \wedge X)/Q$ by the *Adams isomorphism* (see [**40**, Theorem II.7.1] or [**22**, Theorem 5.3]).

After renaming entries in the diagram $(*)$ we obtain the following commutative diagram, called the *Tate diagram*:

$$\begin{tikzcd} X_h \arrow[r, "f"] \arrow[d] & X \arrow[r, "g"] \arrow[d, "\epsilon"] & X^\Phi \arrow[d, "\epsilon_t"] \\ X_h \arrow[r, "\nu"] & X^h \arrow[r, "\theta"] & X^t. \end{tikzcd}$$

The right-hand square in the Tate diagram is a homotopy pullback square of Q-spectra and is known as the *Tate square*. If additionally X is a ring Q-spectrum, all of the corners of the Tate square are also ring Q-spectra and the square is a homotopy pullback of ring Q-spectra. The analogous statement holds for X-module Q-spectra.

We will need the following fact about X_h:

PROPOSITION 3.2.2. *If X is a ring Q-spectrum, X_h is a module over X^h in the homotopy category.*

PROOF. By Proposition 3.2.1 the map $EQ_+ \wedge \epsilon$ is a Q-equivalence, so it has an inverse in the homotopy category. Denote this inverse by $\bar{\epsilon}$. Let $\mu\colon X \wedge X \to X$ be the multiplication in X. The X^h-module structure is given by the following composite:

$$\begin{tikzcd} F(EQ_+, X) \wedge (EQ_+ \wedge X) \arrow[r, "\bar{\epsilon}\wedge 1"] & X \wedge EQ_+ \wedge X \arrow[r, "twist"] & EQ_+ \wedge X \wedge X \arrow[d, "EQ_+\wedge\mu"] \\ & & EQ_+ \wedge X. \end{tikzcd}$$

□

Now we will describe an important multiplicative property of the spectra X^Φ and X^t. Let a be the element of $\pi^Q_{-\sigma}(S^0)$ corresponding to the inclusion $S^0 \to S^\sigma$. The map $a \wedge 1\colon X \to S^\sigma \wedge X$ gives a multiplication by a in X. If a acts as an isomorphism on $X^Q_\star$ we say that X is *a-periodic*.

LEMMA 3.2.3. *If X is a Q-spectrum, then*

$$\pi^Q_\star(X \wedge \widetilde{EQ}) = a^{-1}X^Q_\star.$$

So the spectrum $X \wedge \widetilde{EQ}$ is a-periodic. In particular X^Φ and X^t are a-periodic.

PROOF. The space $\widetilde{EQ}$ has a model $S^{\infty\sigma}$. So $X \wedge \widetilde{EQ}$ may be seen as a homotopy colimit of the sequence:

$$X \xrightarrow{a} S^\sigma \wedge X \xrightarrow{a} S^{2\sigma} \wedge X \xrightarrow{a} \dots.$$

After applying $\pi^Q_\star$ we obtain the following sequence:

$$X^Q_\star \xrightarrow{a} X^Q_{\star-\sigma} \xrightarrow{a} X^Q_{\star-2\sigma} \xrightarrow{a} \dots$$

from which we get an identification

$$\pi^Q_\star(X \wedge \widetilde{EQ}) \cong a^{-1}X^Q_\star.$$

Thus a acts as an isomorphism on $\pi^Q_\star(X \wedge \widetilde{EQ})$. □

Let $\underline{M}$ be a Q-Mackey functor. To compute $H\underline{M}^Q_\star$ we will use the following method, described in **[21]**:

(1) Firstly we compute $H\underline{M}^{hQ}_\star$ and $\left(H\underline{M}_{hQ}\right)_\star$ using the homotopy fixed points and homotopy orbits spectral sequences - details are in Section 3.4.

(2) Using Lemma 3.2.3 we deduce $H\underline{M}^{tQ}_\star$ from $H\underline{M}^{hQ}_\star$ by inverting a, since $H\underline{M}^t = H\underline{M}^h \wedge \widetilde{EQ}$.
(3) We infer $H\underline{M}^{\Phi Q}_\star$ from $H\underline{M}^{tQ}_\star$. Since both theories are a-periodic, we need only to compute $H\underline{M}^{\Phi Q}_n$ for $n \in \mathbb{Z}$. We calculate them using the fact that fibres of ϵ and ϵ_t in the Tate diagram are equivalent.
(4) Finally we deduce $H\underline{M}^Q_\star$ from the Tate diagram.

3.3. Structure of $H\underline{M}^Q_{*\sigma}$

From now on let $\underline{M}$ be a Mackey functor. We start with a description of the subgroup $H\underline{M}^Q_{*\sigma}$ with fixed degree 0. In this case the computations follow from the cofibre sequence

$$(\dagger) \qquad Q_+ \longrightarrow S^0 \longrightarrow S^\sigma.$$

From this sequence we can deduce the following lemma, which gives a complete description of the multiplication by the element $a \in \pi^Q_{-\sigma}(S^0)$ on Eilenberg-MacLane spectra:

Lemma 3.3.1. *The map $a\colon H\underline{M}^Q_{x+y\sigma} \to H\underline{M}^Q_{x+(y-1)\sigma}$ is:*
(1) a monomorphism if $x-1=-y$;
(2) an epimorphism if $x=-y$;
(3) an isomorphism otherwise.

Proof. After smashing $(\dagger)$ with $H\underline{M}$ and applying $\pi^Q_{x+y\sigma}(-)$ we obtain an exact sequence:

$$H\underline{M}^e_{x+y\sigma} \longrightarrow H\underline{M}^Q_{x+y\sigma} \xrightarrow{a} H\underline{M}^Q_{x+(y-1)\sigma} \longrightarrow H\underline{M}^e_{(x-1)+y\sigma}.$$

We obtain the given form of the outer groups by the Wirthmüller isomorphism (see Definition 1.6.29, point(1)):

$$\pi^Q_{x+y\sigma}(Q_+ \wedge H\underline{M}) = [S^{x+y\sigma}, Q_+ \wedge H\underline{M}]^Q \cong [S^{x+y\sigma} \wedge Q_+, H\underline{M}]^Q = \pi^e_{x+y\sigma}(H\underline{M}).$$

Note that by the induction-restriction adjunction (see Definition 1.6.15) we further get that $H\underline{M}^e_{x+y\sigma} \cong \pi_{x+y}(\mathrm{res}^Q_e H\underline{M})$. The spectrum $\mathrm{res}^Q_e H\underline{M}$ is the Eilenberg-MacLane spectrum associated to the abelian group $V = \underline{M}(Q/e)$. Thus $H\underline{M}^e_{x+y\sigma} = 0$ unless $x=-y$ and the lemma follows. □

Lemma 3.3.1 simplifies the calculations of the groups $H\underline{M}^Q_{y\sigma}$ for $y \in \mathbb{Z}$. These are also derived from the cofibre sequence $(\dagger)$.

Proposition 3.3.2. *The groups $H\underline{M}^Q_{y\sigma}$ are given by:*

$$H\underline{M}^Q_{y\sigma} = \begin{cases} \ker(\mathrm{res}_{\underline{M}}) & \text{if } y>0 \\ \mathrm{coker}(\mathrm{tr}_{\underline{M}}) & \text{if } y<0 \\ \underline{M}(Q/Q) & \text{if } y=0. \end{cases}$$

Moreover, the multiplication $a\colon H\underline{M}^Q_\sigma \to H\underline{M}^Q_0$ *is the inclusion of* $\ker(\mathrm{res}_{\underline{M}})$ *in* $M(Q/Q)$ *and* $a\colon H\underline{M}^Q_0 \to H\underline{M}^Q_{-\sigma}$ *is the projection onto* $\mathrm{coker}(\mathrm{tr}_{\underline{M}})$*.*

Proof. The case $y = 0$ is the definition of an Eilenberg-MacLane spectrum of a Mackey functor. So we need to prove the proposition for $y \neq 0$.

By Lemma 3.3.1 it is enough to prove the claim for $y = 1$ and $y = -1$. If we smash the cofibre sequence (†) with $H\underline{M}$ and apply $\pi^Q_0(-)$ we obtain an exact sequence

$$H\underline{M}^e_0 \xrightarrow{\mathrm{tr}_{\underline{M}}} H\underline{M}^Q_0 \xrightarrow{a} H\underline{M}^Q_{-\sigma} \longrightarrow 0.$$

Thus $H\underline{M}^Q_{-\sigma} = \mathrm{coker}(\mathrm{tr}_{\underline{M}})$ and the multiplication by a is the projection. Analogously, by applying $[-, H\underline{M}]^Q$ to (†) we get that $H\underline{M}^Q_\sigma = \ker(\mathrm{res}_{\underline{M}})$. □

3.4. Homotopy fixed points, homotopy orbits and the Tate spectrum of $H\underline{M}$

In this section we compute the coefficients of the Q-spectra $H\underline{M}^h$, $H\underline{M}_h$ and $H\underline{M}^t$. These are given in Proposition 3.4.2 and Proposition 3.4.5. However, firstly we need to give a brief description of homotopy fixed points and homotopy orbits spectral sequences.

3.4.1. Homotopy fixed points and homotopy orbits spectral sequences. The space EQ used in the definitions of Q-spectra $H\underline{M}^h$, $H\underline{M}_h$ and $H\underline{M}^t$ (see Section 3.2) has a very convenient model of the form $S(\infty\sigma)$. This space has a filtration by skeleta

$$(*) \qquad S(\sigma) \subset S(2\sigma) \subset S(3\sigma) \subset \dots$$

and consequently, if X is a Q-spectrum, we obtain a filtration of the spectra X_h and X^h. The spectral sequences associated to these filtrations take the form

$$E^2_{pq} = H_p(Q; \pi_q(X)) \Rightarrow \pi_{p+q}(X_{hQ})$$

and

$$E^{pq}_2 = H^p(Q; \pi_{-q}(X)) \Rightarrow \pi_{-(p+q)}(X^{hQ}).$$

The first spectral sequence is called *the homotopy orbits spectral sequence* and the second is known as *the homotopy fixed points spectral sequence*. Details of the construction may be found in [**22**, Chapter 10].

Both spectral sequences can be used to compute the $RO(Q)$-graded coefficients of X_h and X^h. To this end we note that

$$X^{hQ}_{x+y\sigma} = [S^{x+y\sigma}, F(EQ_+, X)]^Q \cong [S^x, F(EQ_+, F(S^{y\sigma}, X))]^Q = \pi_x(F(S^{y\sigma}, X)^{hQ}).$$

One may argue similarly for the homotopy orbits, and thus we can arrange the homotopy orbits and homotopy fixed points spectral sequences to be trigraded:

$$E^2_{pq}(y) = H_p\left(Q; \pi_q\left(F(S^{y\sigma}, X)\right)\right) \Rightarrow \pi_{p+q}(F(S^{y\sigma}, X)_{hQ})$$

and

$$E_2^{pq}(y) = H^p\left(Q; \pi_{-q}\left(F(S^{y\sigma}, X)\right)\right) \Rightarrow \pi_{-(p+q)}(F((S^{y\sigma}), X)^{hQ}).$$

Note that even though the spectral sequences are trigraded, the differentials live on the "layers" corresponding to the single value of y. If X is a ring Q-spectrum then the pairings

$$S^{y_1\sigma} \wedge S^{y_2\sigma} \longrightarrow S^{(y_1+y_2)\sigma}$$

give the trigraded homotopy fixed points spectral sequence a multiplicative structure.

It will prove useful to have a description of the E_1-page. Since we will only use it for the homotopy fixed points spectral sequence, we omit here the description of the E^1-page of the homotopy orbits spectral sequence.

The filtration (∗) gives a filtration of $X^{hQ} \cong F_Q(EQ_+, X)$, the spectrum of Q-equivariant maps from EQ_+ to X. Therefore the E_1-page has the form

$$E_1^{pq} = \pi_{-q}(X).$$

However, there is more useful description of E_1^{pq} given in [**22**, Chapter 9] which allows us also to determine differentials.

Proposition 3.4.1. *There is an isomorphism*

$$E_1^{pq} = [Q_+ \wedge S^p, \Sigma^{p+q}X]^Q \cong \mathrm{Hom}_{\mathbb{Z}[Q]}\left(H_p(Q_+ \wedge S^p), X^e_{-q}\right).$$

The differential

$$d_1 : E^{p,q} \longrightarrow E^{p+1,q}$$

is induced by a map $\partial_* : H_{p-1}(Q_+ \wedge S^p) \to H_p(Q_+ \wedge S^{p-1})$, which is further induced in homology by the *geometric boundary*:

$$\partial : Q_+ \wedge S^p \longrightarrow \Sigma S\left((p-1)\sigma\right)_+ \longrightarrow \Sigma(Q_+ \wedge S^{p-1}).$$

Note that the complex

$$\ldots \longrightarrow H_{p+1}(Q_+ \wedge S^{p+1}) \xrightarrow{\partial_*} H_p(Q_+ \wedge S^p) \xrightarrow{\partial_*} H_{p-1}(Q_+ \wedge S^{p-1}) \longrightarrow \ldots$$

is the cellular complex of $S(\infty\sigma)$ and thus it is exact ($S(\infty\sigma)$ is contractible) with differentials given by the degrees of attaching maps. Therefore, since $H_p(Q_+ \wedge S^p) \cong \mathbb{Z}[Q]$, the complex above gives us a 2-periodic $\mathbb{Z}[Q]$-resolution of $\mathbb{Z}$. Thus along the q-th row on the E_1-page we have the complex computing the group cohomology with coefficients in X^e_{-q}. This concludes the description of the E_1-page.

3.4.2. Calculations for Eilenberg-MacLane spectra. Now we specialise to the case $X = H\underline{M}$. Recall the notation $V := \underline{M}(Q/e)$.

Proposition 3.4.2. *The $RO(Q)$-graded coefficients of $H\underline{M}_h$ and $H\underline{M}^h$ are given by:*

- $\left(H\underline{M}^Q_{hQ}\right)_{(y\sigma-y)+p} = H_p(Q; H^y(S^{y\sigma}, V))$

- $H\underline{M}^{hQ}_{(y\sigma-y)-p} = H^p(Q; H^y(S^{y\sigma}, V))$.

Proof. The proposition follows from the fact that the trigraded homotopy orbits and homotopy fixed points spectral sequences collapse on the second page. The coefficients of the group homology/cohomology on the second page are given by

$$\pi_q(F(S^{y\sigma}, H\underline{M})) \cong H^{-q}(S^{y\sigma}, V).$$

The (reduced) singular cohomology $H^{-q}(S^{y\sigma}, V)$ is 0 unless $-q = y$. Thus for every y the $E_2(y)$-page consists of one row and all differentials are 0. $\square$

Note that since Q acts on $H^y(S^{y\sigma}, \mathbb{Z})$ by degree of γ as a map, it is isomorphic to $\mathbb{Z}$ if y is even and $\tilde{\mathbb{Z}}$ if y is odd. So by the Universal Coefficient Theorem $H^y(S^{y\sigma}, V)$ is isomorphic to V if y is even and $\tilde{V} := \tilde{\mathbb{Z}} \otimes V$ if y is odd.

The projective $\mathbb{Z}[Q]$-resolution of $\tilde{\mathbb{Z}}$ is given by:

$$\ldots \longrightarrow \mathbb{Z}[Q] \xrightarrow{1-\gamma} \mathbb{Z}[Q] \xrightarrow{1+\gamma} \mathbb{Z}[Q] \xrightarrow{\tilde{\epsilon}} \tilde{\mathbb{Z}} \longrightarrow 0.$$

Here $\tilde{\epsilon}$ is the ring map defined by $\tilde{\epsilon}(\gamma) = -1$. From this resolution we can see that there are isomorphisms $H^i(Q; \tilde{V}) \cong H^{i+1}(Q; V)$ and $H_i(Q; \tilde{V}) \cong H_{i+1}(Q; V)$ for $i \geq 1$. This gives us four potentially non-zero values for $(H\underline{M}_{hQ})_\star$ and four for $H\underline{M}^{hQ}_\star$, as depicted in Figures 3.2 and 3.3.

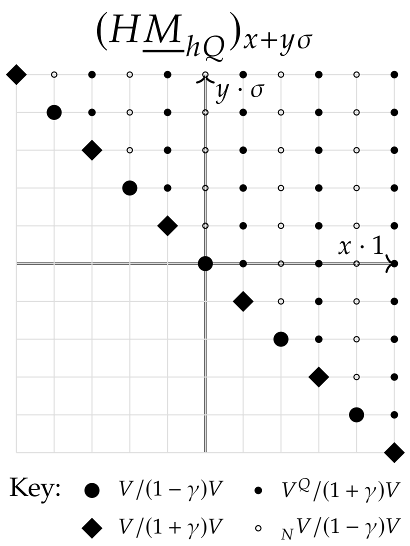


Figure 3.2. Coefficients of $H\underline{M}_h$.

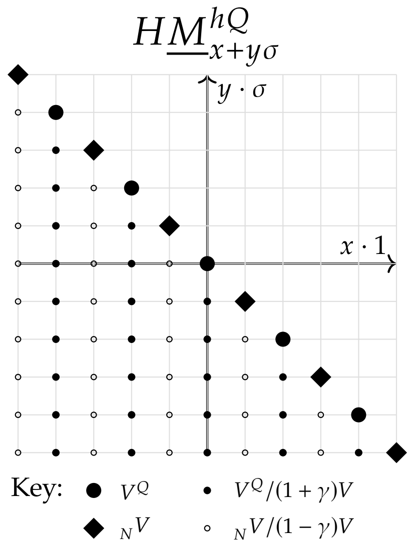


Figure 3.3. Coefficients of $H\underline{M}^h$.

Remark 3.4.3. Note that from Figures 3.2 an 3.3 it is easy to see that both $\left(H\underline{M}_{hQ}\right)_\star$ and $H\underline{M}^{hQ}_\star$ are $(2-2\sigma)$-periodic. We will attribute this phenomenon to the multiplication by the generator of $H\underline{\mathbb{A}}^Q_{2-2\sigma}$ in Section 3.8.

Before proceeding to the description of $H\underline{M}^{tQ}_\star$ we give a proposition that identifies maps induced in homotopy by $f\colon X_h \to X$ and $\epsilon\colon X \to X^h$ from the Tate diagram with particular maps induced by the structure of Mackey functor.

Proposition 3.4.4. *The map $f_0\colon V_Q = \left(H\underline{M}_{hQ}\right)_0 \to H\underline{M}^Q_0 = \underline{M}(Q/Q)$ is the map induced by $\mathrm{tr}_{\underline{M}}$ on V_Q. The map $\epsilon_0\colon \underline{M}(Q/Q) = H\underline{M}^Q_0 \to H\underline{M}^{hQ}_0 = V^Q$ is the restriction of $\mathrm{res}_{\underline{M}}$ to the codomain V^Q.*

Proof. Let $i\colon Q_+ \to EQ_+$ be the inclusion of the 0-skeleton. Note that the map $Q_+ \to S^0$ extends to the filtration $(*)$ of EQ_+ and thus it factors as

$$Q_+ \xrightarrow{\ i\ } EQ_+ \longrightarrow S^0.$$

By smashing this sequence with $H\underline{M}$ and taking $\pi^Q_0(-)$ we obtain the following commutative diagram:

$$\begin{tikzcd} H\underline{M}^e_0 \arrow[rr, "\mathrm{tr}_{\underline{M}}"] \arrow[dr] & & H\underline{M}^Q_0 \\ & \left(H\underline{M}_{hQ}\right)_0 \arrow[ur, "f_0"'] & \end{tikzcd}$$

By Proposition 3.4.2 this diagram is isomorphic to the following:

$$\begin{tikzcd} V \arrow[rr, "\mathrm{tr}_{\underline{M}}"] \arrow[dr] & & \underline{M}(Q/Q) \\ & V/(1-\gamma)V \arrow[ur, "f_0"'] & \end{tikzcd}$$

The left diagonal arrow is the canonical projection. Thus the map $f_0\colon \left(H\underline{M}_{hQ}\right)_0 \to H\underline{M}^Q_0$ is the map induced by the transfer on V_Q. Note that by the properties of Mackey functors $\mathrm{tr}_{\underline{M}}$ always factors via V_Q. We obtain the second assertion by using the dual argument. □

Proposition 3.4.5. *The coefficients of the Q-spectrum $H\underline{M}^t$ are given by*

$$H\underline{M}^{tQ}_{x+y\sigma} = \hat{H}^{-x}(Q;V),$$

where $\hat{H}^x(Q;V)$ denotes the x-th Tate cohomology of Q with coefficients in V (see Section 3.1.1). Note that since $H\underline{M}^t$ is a-periodic, $H\underline{M}^{tQ}_{x+y\sigma}$ does not depend on y.

Proof. By definition $X^t = X^h \wedge \widetilde{EQ}$. So by Lemma 3.2.3 we have that

$$H\underline{M}^{tQ}_\star = a^{-1}H\underline{M}^{hQ}_\star.$$

From this isomorphism we see in particular that $H\underline{M}^{tQ}_{x+y\sigma} = H\underline{M}^{hQ}_{x+y\sigma}$ below the antidiagonal, i.e., when $y < -x$. So if x is even we obtain by Proposition 3.4.2 and a-periodicity of $H\underline{M}^t$ that

$$H\underline{M}^{tQ}_{x+y\sigma} \cong H\underline{M}^{tQ}_{x-(x+1)} = H\underline{M}^{hQ}_{x-(x+1)\sigma} \cong V^Q/NV.$$

Analogously, if x is odd we obtain that $H\underline{M}^{tQ}_{x+y\sigma} \cong {}_NV/(1-\gamma)V$. □

3.5. Geometric fixed points

In this section we describe the coefficients of the Q-spectrum $H\underline{M}^{\Phi}$.

THEOREM 3.5.1. *The coefficients of $H\underline{M}^{\Phi}$ are given by:*

$$H\underline{M}^{\Phi Q}_{x+y\sigma} = \begin{cases} \operatorname{coker}(\mathrm{tr}_{\underline{M}}) & \text{if } x = 0 \\ \big(\ker(\mathrm{tr}_{\underline{M}})\big)_Q & \text{if } x = 1 \\ H\underline{M}^{tQ}_{x+y\sigma} = \hat{H}^x(Q;V) & \text{if } x \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the values of $H\underline{M}^{\Phi Q}_{\star}$ depend only on the fixed degree, which comes from the Lemma 3.2.3. We prove Theorem 3.5.1 by a series of lemmas.

LEMMA 3.5.2. $H\underline{M}^{\Phi Q}_{y\sigma} = \operatorname{coker}(\mathrm{tr}_{\underline{M}})$ *and* $H\underline{M}^{\Phi Q}_{1+y\sigma} = \big(\ker(\mathrm{tr}_{\underline{M}})\big)_Q$.

PROOF. By Lemma 3.2.3 we have that $H\underline{M}^{\Phi Q}_{y\sigma} \cong H\underline{M}^{\Phi Q}_0$, so we need to prove the claim only in this case. Analogously for $x = 1$. From the top row of the Tate diagram we obtain the following exact sequence:

$$H\underline{M}^Q_1 \xrightarrow{g_1} H\underline{M}^{\Phi Q}_1 \longrightarrow \Big(H\underline{M}_{hQ}\Big)_0 \xrightarrow{f_0} H\underline{M}^Q_0 \xrightarrow{g_0} H\underline{M}^{\Phi Q}_0 \longrightarrow \Big(H\underline{M}_{hQ}\Big)_{-1}$$

However, the outer groups are zero: $H\underline{M}^Q_1$ by the definition of an Eilenberg-MacLane spectrum and $\Big(H\underline{M}_{hQ}\Big)_{-1}$ by Proposition 3.4.2. Thus $H\underline{M}^{\Phi Q}_0$ and $H\underline{M}^{\Phi Q}_1$ are respectively cokernel and kernel of the map f_0. Thus by Proposition 3.4.4 we obtain that $\operatorname{coker}(f_0) = \operatorname{coker}(\mathrm{tr}_{\underline{M}})$ and $\ker(f_0) = \ker(\mathrm{tr}_{\underline{M}})_Q$. □

LEMMA 3.5.3. *If $x \geq 2$ then $H\underline{M}^{\Phi Q}_{x+y\sigma} = H\underline{M}^{tQ}_{x+y\sigma}$.*

PROOF. Since both $H\underline{M}^{\Phi}$ and $H\underline{M}^t$ are a-periodic by Lemma 3.2.3, we need only to show that $H\underline{M}^{\Phi Q}_x = H\underline{M}^{tQ}_x$ (i.e., with twisted degree 0) for $x \geq 2$. Note that the fibres of $\epsilon\colon H\underline{M} \to H\underline{M}^h$ and $\epsilon_t\colon H\underline{M}^{\phi} \to H\underline{M}^t$ are equivalent, as they are both of the form $F(\widetilde{EQ}, H\underline{M})$. Let F denotes this fibre. By applying the long exact sequence in homotopy to the fibration $F \to H\underline{M} \to H\underline{M}^h$ we get:

$$\ldots \longrightarrow H\underline{M}^Q_{m+1} \xrightarrow{\epsilon_*} H\underline{M}^{hQ}_{m+1} \longrightarrow F^Q_m \longrightarrow H\underline{M}^Q_m \xrightarrow{\epsilon_*} H\underline{M}^{hQ}_m \longrightarrow \ldots.$$

But by the definition of Eilenberg-MacLane spectrum we have that $H\underline{M}^Q_m = 0$ if $m \neq 0$ and $H\underline{M}^{hQ}_m = 0$ if $m \geq 1$ by Proposition 3.4.2, so we get from this exact sequence that $F^Q_m = 0$ for $m \geq 1$. By applying the analogous long exact sequence to the fibration $F \to H\underline{M}^\Phi \to H\underline{M}^t$ we get that $H\underline{M}^{\Phi Q}_m = H\underline{M}^{tQ}_m$ for $m \geq 2$. □

Lemma 3.5.4. $H\underline{M}^{\Phi Q}_{x+y\sigma} = 0$ *for* $x < 0$.

Proof. As above, by Lemma 3.2.3 we need only to show the statement for the twisted degree $y = 0$. Writing the long exact sequence in homotopy for the upper row of the Tate diagram yields:

$$\dots \to \left(H\underline{M}_{hQ}\right)_m \to H\underline{M}^Q_m \to H\underline{M}^{\Phi Q}_m \to \left(H\underline{M}_{hQ}\right)_{m-1} \to H\underline{M}^Q_{m-1} \to \dots .$$

But $H\underline{M}^Q_m = 0$ for $m \neq 0$ and $\left(H\underline{M}_{hQ}\right)_m = 0$ for $m < 0$ by Proposition 3.4.2, so $H\underline{M}^{\Phi Q}_m = 0$ for $m < 0$. □

Proof of Theorem 3.5.1. Follows from Lemmas 3.5.2, 3.5.3 and 3.5.4. □

3.6. $RO(Q)$-graded abelian group structure of $H\underline{M}^Q_\star$

In this section we describe the structure of $H\underline{M}^Q_\star$ as a $RO(Q)$-graded abelian group. It is given by the following theorem:

Theorem 3.6.1. *The $RO(Q)$-graded abelian group structure of $H\underline{M}^Q_\star$ is given by:*

(1)

$$H\underline{M}^Q_{y\sigma} = \begin{cases} \ker(\mathrm{res}_{\underline{M}}) & \text{if } y > 0 \\ \mathrm{coker}(\mathrm{tr}_{\underline{M}}) & \text{if } y < 0 \\ \underline{M}(Q/Q) & \text{if } y = 0 \end{cases}$$

(2)

$$H\underline{M}^Q_{1+y\sigma} = \begin{cases} \ker(\mathrm{tr}_{\underline{M}}) & \text{if } y = -1 \\ \ker(\mathrm{tr}_{\underline{M}})_Q & \text{if } y < -1 \\ 0 & \text{if } y > -1 \end{cases}$$

(3)

$$H\underline{M}^Q_{y\sigma-1} = \begin{cases} \mathrm{coker}(\mathrm{res}_{\underline{M}}) & \text{if } y = 1 \\ V^Q/\mathrm{im}(\mathrm{res}_{\underline{M}}) & \text{if } y > 1 \\ 0 & \text{if } y < 1 \end{cases}$$

(4) if $x \geq 2$ then $H\underline{M}^Q_{x+y\sigma} = H\underline{M}^{hQ}_{x+y\sigma}$;

(5) if $x \leq -2$ then $H\underline{M}^Q_{x+y\sigma} = \left(H\underline{M}_{hQ}\right)_{x+y\sigma}$.

This data is presented in Figure 3.4.

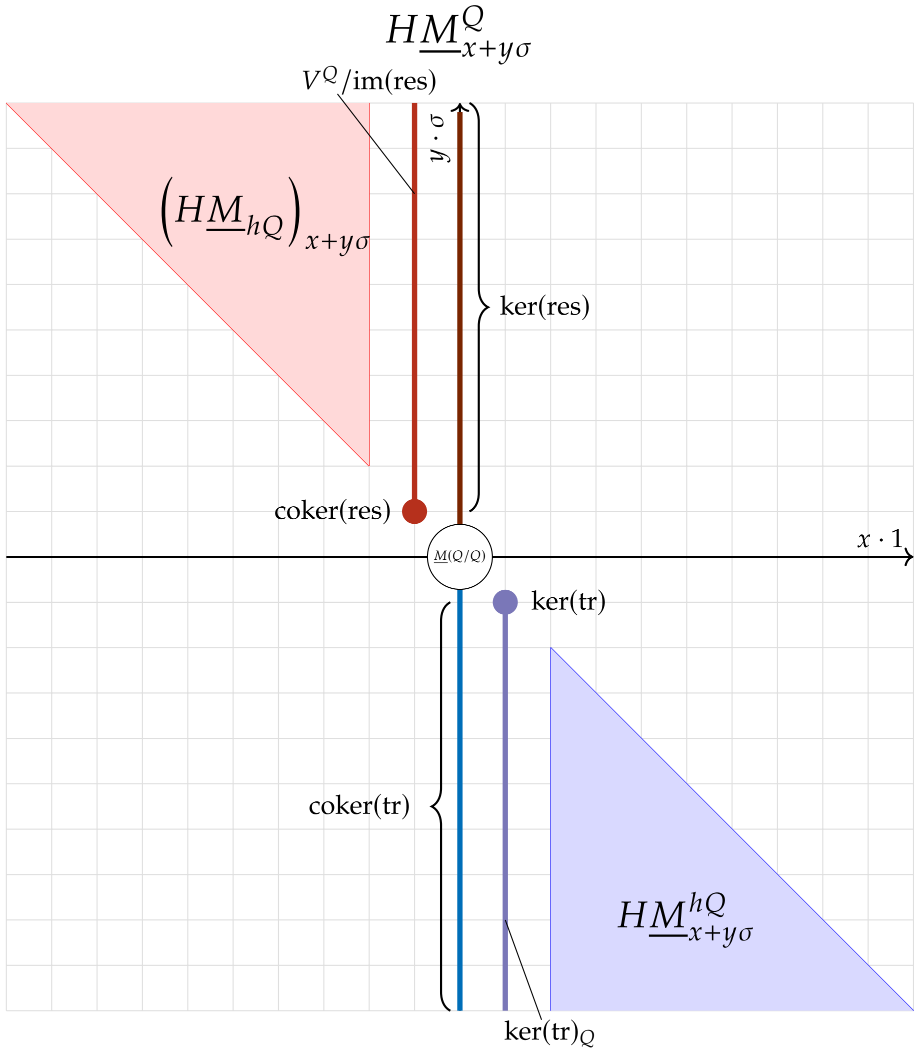


FIGURE 3.4. Symbolic depiction of the coefficients of a general Mackey functor.

REMARK 3.6.2. Recall from the introduction that the same values can be found in [**18**, Theorem 8.1].

We are going to prove Theorem 3.6.1 in a series of lemmas. Firstly, let us note that the point (1) of the theorem is actually Proposition 3.3.2, so it is already proven.

We start with proving a lemma which describes the general shape of $H\underline{M}^{Q}_{\star}$ - i.e., that it is zero below the antidiagonal on the half-plane $x < 0$ and above the antidiagonal on the half-plane $x > 0$.

LEMMA 3.6.3. *For $x < 0$, the groups $H\underline{M}^{Q}_{x+y\sigma}$ are zero if $y < -x$. If $x > 0$ then $H\underline{M}^{Q}_{x+y\sigma}$ is zero for $y > -x$.*

Proof. The groups $H\underline{M}_x^Q$ are zero if $x \neq 0$. Let $x > 0$. By Lemma 3.3.1 iterated multiplication by a gives isomorphisms $H\underline{M}_x^Q = H\underline{M}_{x+y\sigma}^Q$ if $y > -x$. Thus we get that $H\underline{M}_{x+y\sigma}^Q = 0$ if $y > -x$. The case when $x < 0$ follows similarly. □

Lemma 3.6.4. *$H\underline{M}_{1+y\sigma}^Q$ is given by*

$$H\underline{M}_{1+y\sigma}^Q = \begin{cases} \ker(\mathrm{tr}_{\underline{M}}) & \text{if } y = -1 \\ \ker(\mathrm{tr}_{\underline{M}})_Q & \text{if } y < -1 \\ 0 & \text{if } y > -1. \end{cases}$$

Proof. Firstly we note that the case $y > -1$ follows from Lemma 3.6.3, so it is proven.

Let $y = -1$. If we smash the cofibre sequence (†) with $H\underline{M}$ from Section 3.3 we obtain the following cofibre sequence:

$$Q_+ \wedge H\underline{M} \longrightarrow H\underline{M} \longrightarrow S^\sigma \wedge H\underline{M}.$$

This cofibration gives us the following exact sequence in homotopy:

$$H\underline{M}_1^Q \longrightarrow H\underline{M}_{1-\sigma}^Q \longrightarrow H\underline{M}_0^e \xrightarrow{\mathrm{tr}_{\underline{M}}} H\underline{M}_0^Q.$$

By the definition of Eilenberg-MacLane spectrum we have that $H\underline{M}_1^Q = 0$. Thus $H\underline{M}_{1-\sigma}^Q = \ker(\mathrm{tr}_{\underline{M}})$.

Finally, let $y < -1$. By applying the long exact sequence in homotopy to the upper row of the Tate diagram we get

$$\left(H\underline{M}_{hQ}\right)_{1+y\sigma} \longrightarrow H\underline{M}_{1+y\sigma}^Q \longrightarrow H\underline{M}^{\Phi Q} \longrightarrow \left(H\underline{M}_{hQ}\right)_{y\sigma}.$$

By Proposition 3.4.2 the outer terms of the sequence above are 0, so $H\underline{M}_{1+y\sigma}^Q \cong H\underline{M}_{1+y\sigma}^{\Phi Q}$. By Theorem 3.5.1 we get that $H\underline{M}_{1+y\sigma}^Q \cong \ker(\mathrm{tr}_{\underline{M}})_Q$. □

Lemma 3.6.5. *$H\underline{M}_{y\sigma-1}^Q$ is given by*

$$H\underline{M}_{y\sigma-1}^Q \cong \begin{cases} \mathrm{coker}(\mathrm{res}_{\underline{M}}) & \text{if } y = 1 \\ V^Q/\mathrm{im}(\mathrm{res}_{\underline{M}}) & \text{if } y > 1 \\ 0 & \text{if } y < 1. \end{cases}$$

Proof. The case $y < 1$ is proven by Lemma 3.6.3. For $y = 1$ we use a dual argument to the one given in the proof of Lemma 3.6.4. Thus we are left with one case.

Let $y > 1$. By applying the long exact sequences in homotopy to the Tate diagram we obtain the following commutative diagram:

$$(**)\qquad \begin{array}{ccccccc} H\underline{M}^{\Phi Q}_{y\sigma} & \longrightarrow & \left(H\underline{M}_{hQ}\right)_{y\sigma-1} & \longrightarrow & H\underline{M}^{Q}_{y\sigma-1} & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \epsilon_{t*}} & & \Big\downarrow{\scriptstyle =} & & & & \\ H\underline{M}^{tQ}_{y\sigma} & \longrightarrow & \left(H\underline{M}_{hQ}\right)_{y\sigma-1} & & & & \end{array}$$

Here ϵ_{t*} is the map induced by ϵ_t. The zero in the top right corner comes from the fact that $H\underline{M}^{\Phi Q}_{y\sigma-1} = 0$ by Theorem 3.5.1. Note that the bottom arrow is a part of the following sequence:

$$H\underline{M}^{hQ}_{y\sigma} \longrightarrow H\underline{M}^{tQ}_{y\sigma} \longrightarrow \left(H\underline{M}_{hQ}\right)_{y\sigma_1} \longrightarrow H\underline{M}^{hQ}_{y\sigma-1}.$$

In this sequence the outer terms are zero by Proposition 3.4.2 if $y > 1$ and thus the map $H\underline{M}^{tQ}_{y\sigma} \to \left(H\underline{M}_{hQ}\right)_{y\sigma-1}$ in the diagram (**) is an isomorphism. Thus we can deduce from this diagram that $H\underline{M}^{Q}_{y\sigma-1} = \operatorname{coker}(\epsilon_{t*})$. So it remains to describe the map $\epsilon_{t*}\colon H\underline{M}^{\Phi Q}_{y\sigma} \to H\underline{M}^{tQ}_{y\sigma}$.

Since both $H\underline{M}^{\Phi}$ and $H\underline{M}^{t}$ are a-periodic by Lemma 3.2.3, it is enough to identify this map for $y = 0$. It is described by the following diagram:

$$\begin{array}{ccc} H\underline{M}^{Q}_{0} = M(Q/Q) & \longrightarrow & H\underline{M}^{\Phi Q}_{0} = \operatorname{coker}(\mathrm{tr}_{\underline{M}}) \\ \Big\downarrow{\scriptstyle \widetilde{\mathrm{res}_{\underline{M}}}} & & \Big\downarrow{\scriptstyle \epsilon_{t*}} \\ H\underline{M}^{hQ}_{0} = V^{Q} & \longrightarrow & H\underline{M}^{tQ}_{0} = V^{Q}/(1+\gamma)V. \end{array}$$

The map $\widetilde{\mathrm{res}_{\underline{M}}}\colon \underline{M}(Q/Q) \to V^{Q}$ is the map $\mathrm{res}_{\underline{M}}$ with the codomain restricted to V^{Q}. Note that by properties of Mackey functors we have that $\mathrm{im}(\mathrm{res}_{\underline{M}}) \subset V^{Q}$. Both horizontal maps are canonical projections. So ϵ_{t*} is the map induced by $\mathrm{res}_{\underline{M}}$. Thus we have:

$$H\underline{M}^{Q}_{y\sigma-1} \cong \operatorname{coker}(\epsilon_{t*}) = \frac{\left(V^{Q}/NV\right)}{\mathrm{res}_{\underline{M}}\left(\underline{M}(Q/Q)/\mathrm{tr}_{\underline{M}}(V)\right)} \cong V^{Q}/\mathrm{im}(\mathrm{res}_{\underline{M}}).$$

□

PROOF OF THEOREM 3.6.1. Follows from Lemmas 3.6.3, 3.6.4 and 3.6.5. □

3.7. Examples

In this section we present two examples of the structure shown in previous sections. Since both examples are Green functors, we will also describe a multiplicative structure on the coefficients. The main example is the Burnside Mackey functor $\underline{\mathbb{A}}$, as we are going to build the multiplicative structure of other Green functors upon the knowledge of $H\underline{\mathbb{A}}^{Q}_{\star}$. However, a big part of computations of the coefficients of $H\underline{\mathbb{A}}$ is the same as in the case of constant Mackey functor $\mathbb{Z}$, coefficients of which are already computed by Tate method in [**21**]. Thus we begin the examples with computations of $H\underline{\mathbb{Z}}^Q_\star$.

3.7.1. Constant Mackey functor $\underline{\mathbb{Z}}$. The Mackey functor $\underline{\mathbb{Z}}$ has the following form:

$$\begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "1"'] \\ \mathbb{Z} \arrow[u, bend right, "2"'] \end{tikzcd}$$

Computations of $H\underline{\mathbb{Z}}^Q_\star$ as a ring may be found in [**15**] and later on, using the Tate square technique, in [**21**]. From the latter we recall the following two lemmas [**21**, Lemma 2.1, Corollary 2.3, Lemma 2.5]:

LEMMA 3.7.1.

$$H\underline{\mathbb{Z}}^{hQ}_\star = BB[u, u^{-1}]$$
$$\left(H\underline{\mathbb{Z}}_{hQ}\right)_\star = NB[u, u^{-1}]$$
$$H\underline{\mathbb{Z}}^{tQ}_\star = \mathbb{F}_2[a, a^{-1}][u, u^{-1}]$$

where $BB = \mathbb{Z}[a]/2a$, $NB = \mathbb{Z} \oplus \Sigma^{-1+\sigma}\mathbb{F}_2[a^{-1}]$ *and* $|a| = -\sigma$, $|u| = 2 - 2\sigma$.

LEMMA 3.7.2. *The coefficients of* $H\underline{\mathbb{Z}}^\Phi$ *are given by* $H\underline{\mathbb{Z}}^{\Phi Q}_\star = \mathbb{F}_2[a, a^{-1}][u]$.

REMARK 3.7.3. Note that the element a in Lemmas 3.7.1 and 3.7.2 is the same as the element $a \cdot 1$ in the sense of Lemma 3.3.1, so there is no clash of the notation.

Now we are in a position to describe the structure of $H\underline{\mathbb{Z}}^Q_\star$ as a ring.

THEOREM 3.7.4.

(1) *The $RO(Q)$-graded abelian group structure of $H\underline{\mathbb{Z}}^Q_\star$ is given by:*

$$H\underline{\mathbb{Z}}^Q_{x+y\sigma} = \begin{cases} \mathbb{Z} & \text{if } y = -x \text{ and } x = 2n \text{ for } n \in \mathbb{Z} \\ \mathbb{Z}/2 & \text{if } y < -x \text{ and } x = 2n \text{ for } n \in \mathbb{Z}_{\geq 0} \\ \mathbb{Z}/2 & \text{if } y \geq -x \text{ and } x = -2n-1 \text{ for } n \in \mathbb{Z}_{\geq 1} \\ 0 & \text{else.} \end{cases}$$

(2) *The multiplicative structure of $H\underline{\mathbb{Z}}^Q_\star$ is characterized by the following properties:*
   (a) *it is strictly commutative;*
   (b) *red lines in Figure 3.5 represent multiplication by $a \in H\underline{\mathbb{Z}}^Q_{-\sigma}$;*
   (c) *blue dashed lines represent multiplication by $u \in H\underline{\mathbb{Z}}^Q_{2-2\sigma}$, the generator corresponding to 1;*
   (d) *the map $u\colon H\underline{\mathbb{Z}}^Q_{2\sigma-2} \to H\underline{\mathbb{Z}}^Q_0$ is multiplication by 2;*
   (e) *the groups $H\underline{\mathbb{Z}}^Q_{2n\sigma-2n}$ for $n > 0$ are generated by $2u^{-n}$.*

PROOF.

(1) This comes from Theorem 3.6.1:

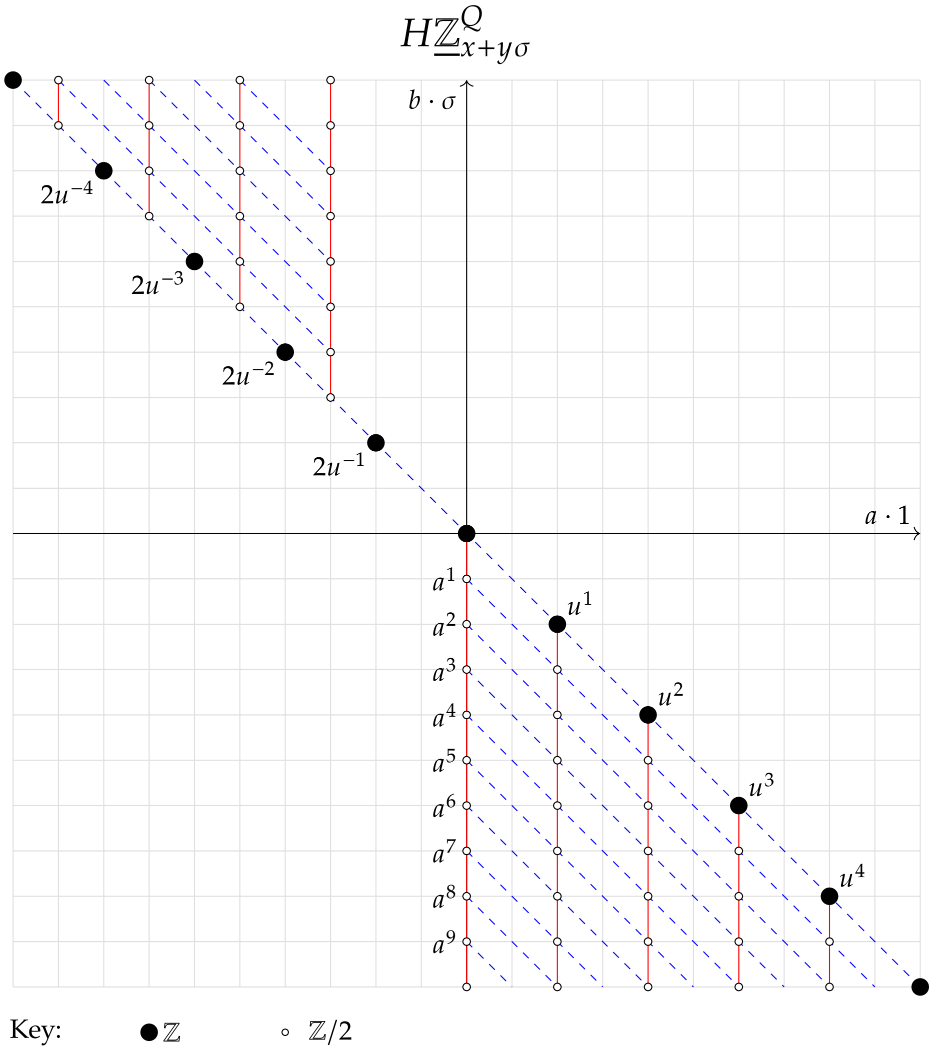


FIGURE 3.5. Coefficients of $H\underline{\mathbb{Z}}$.

- if $x \leq -2$ we have that $H\underline{\mathbb{Z}}^{Q}_{x+y\sigma} = \left(H\underline{\mathbb{Z}}_{hQ}\right)_{x+y\sigma}$, so the structure is described by Lemma 3.7.1;
- if $x \geq 2$ then $H\underline{\mathbb{Z}}^{Q}_{x+y\sigma} = H\underline{\mathbb{Z}}^{hQ}_{x+y\sigma}$, so the structure is also described by Lemma 3.7.1;

- if $-1 \leq x \leq 1$ the structure comes from the following calculations:

$$\begin{aligned} H\underline{\mathbb{Z}}^{Q}_{1-\sigma} &= \ker(\mathrm{tr}_{\underline{\mathbb{Z}}}) = 0 \\ H\underline{\mathbb{Z}}^{Q}_{1-y\sigma} &= \ker(\mathrm{tr}_{\underline{\mathbb{Z}}})_{Q} = 0 \text{ for } y > 1 \\ H\underline{\mathbb{Z}}^{Q}_{\sigma-1} &= \mathrm{coker}(\mathrm{res}_{\underline{\mathbb{Z}}}) = 0 \\ H\underline{\mathbb{Z}}^{Q}_{y\sigma-1} &= V^{Q}/\mathrm{im}(\mathrm{res}_{\underline{\mathbb{Z}}}) = 0 \text{ for } y > 1 \\ H\underline{\mathbb{Z}}^{Q}_{y\sigma} &= \ker(\mathrm{res}_{\underline{\mathbb{Z}}}) = 0 \text{ for } y > 0 \\ H\underline{\mathbb{Z}}^{Q}_{y\sigma} &= \mathrm{coker}(\mathrm{tr}_{\underline{\mathbb{Z}}}) = \mathbb{Z}/2 \text{ for } y < 0 \end{aligned}$$

(2) (a) Since $\underline{\mathbb{Z}}$ is a Green functor, $H\underline{\mathbb{Z}}$ is a commutative ring Q-spectrum and so $H\underline{\mathbb{Z}}^{Q}_{\star}$ is a graded commutative ring. The graded commutativity rule is given as follows: if $\alpha \in H\underline{\mathbb{Z}}^{Q}_{x+y\sigma}$ and $\beta \in H\underline{\mathbb{Z}}^{Q}_{x'+y'\sigma}$ then

$$\alpha\beta = (-1)^{xx'}(1-\omega)^{yy'}\beta\alpha,$$

where ω is the class of Q/e in $\mathbb{A}(Q)$. For details on the graded commutativity rule in equivariant homotopy theory see **[1]**, **[40]** or **[34]** specifically for the case of Q. However, since all the entries except of the antidiagonal are either zero or $\mathbb{Z}/2$, the sign rule might give non-trivial sign only in $\mathbb{Z}$'s on the antidiagonal. But they all lie in even fixed and twisted degrees, so the sign is always 1.

(b) The multiplication by a is described by Lemma 3.3.1.

(c) Firstly recall from Section 3.2 that the map $\epsilon_*\colon H\underline{\mathbb{Z}}^{Q}_{\star} \to H\underline{\mathbb{Z}}^{hQ}_{\star}$ is a ring map. Thus for $x \geq 0$ multiplication by the element u is described by Lemma 3.7.1. By Proposition 3.2.2 we have that $\left(H\underline{\mathbb{Z}}_{hQ}\right)_{\star}$ is a module over $H\underline{\mathbb{Z}}^{Q}_{\star}$, so the claim follows from Lemma 3.7.1.

(d) We have the following commutative diagram:

$$\begin{array}{ccc} \left(H\underline{\mathbb{Z}}_{hQ}\right)_{2\sigma-2} & \xrightarrow{u} & \left(H\underline{\mathbb{Z}}_{hQ}\right)_{0} \\ \downarrow & & \downarrow{\scriptstyle f_0} \\ H\underline{\mathbb{Z}}^{Q}_{2\sigma-2} & \xrightarrow{u} & H\underline{\mathbb{Z}}^{Q}_{0}. \end{array}$$

By Lemma 3.7.1 the upper horizontal arrow is an isomorphism. By Theorem 3.6.1 the left vertical arrow is an isomorphism, so multiplication $u\colon H\underline{\mathbb{Z}}^{Q}_{2\sigma-2} \to H\underline{\mathbb{Z}}^{Q}_{0}$ is up to isomorphism the same as the right vertical map f_0. By Proposition 3.4.4 this map is induced by the transfer. Thus if $\alpha \in H\underline{\mathbb{Z}}^{Q}_{2\sigma-2}$ we obtain that $u \cdot \alpha = 2\alpha \in H\underline{\mathbb{Z}}^{Q}_{0}$.

(e) Let θ be the generator corresponding to 1 in $H\underline{\mathbb{Z}}^{Q}_{2\sigma-2} \cong \mathbb{Z}$. By the previous point $u \cdot \theta = 2$, so $\theta = 2u^{-1}$. Since $H\underline{\mathbb{Z}}^{Q}_{x+y\sigma} \cong \left(H\underline{\mathbb{Z}}_{hQ}\right)_{x+y\sigma}$ if $x \leq -2$ and multiplication by u is an isomorphism on $\left(H\underline{\mathbb{Z}}_{hQ}\right)_\star$, we get that $H\underline{\mathbb{Z}}^Q_{2n\sigma-2n}$ is generated by $2u^{-n}$ for $n > 0$.

□

Remark 3.7.5. The ring $H\underline{\mathbb{Z}}^Q_\star$ has a more concise description as:

$$H\underline{\mathbb{Z}}^Q_\star \cong BB[u] \oplus u^{-1} \cdot NB[u^{-1}]$$

with BB as before and $NB = \mathbb{Z} \oplus \Sigma^{-\rho}\mathbb{F}_2[a^{-1}]$. The structure of BB-module of NB is as suggested by the notation. See [**21**, Corollary 2.6].

3.7.2. Burnside Mackey functor. The second example is the Burnside Mackey functor $\underline{\mathbb{A}}$. It is the most important example - since we will build our understanding of multiplicative structure of Eilenberg-MacLane spectra upon the knowledge of the graded ring structure of $H\underline{\mathbb{A}}^Q_\star$.

The Mackey functor $\underline{\mathbb{A}}$ has the following form:

$$\begin{tikzcd} \mathbb{Z}[\omega]/\omega^2 - 2\omega \arrow[d, bend right] \\ \mathbb{Z} \arrow[u, bend right] \end{tikzcd}$$

with structure maps defined by $\mathrm{tr}_{\underline{\mathbb{A}}}(1) = \omega$ and restriction being a ring map with $\mathrm{res}_{\underline{\mathbb{A}}}(\omega) = 2$. Note that the Q/Q-level of $\underline{\mathbb{A}}$ is in fact the Burnside ring $\mathbb{A}(Q)$ with ω being the class of Q/e. The Mackey functor valued coefficients of $\underline{\mathbb{A}}$ may be found in [**41**] and [**18**].

The coefficients of $H\underline{\mathbb{A}}^h$, $H\underline{\mathbb{A}}_h$ and $H\underline{\mathbb{A}}^t$ depend only on $\underline{\mathbb{A}}(Q/e) = \mathbb{Z}$ (see Propositions 3.4.2 and 3.4.5), which is the same as $\underline{\mathbb{Z}}(Q/e)$. So $H\underline{\mathbb{A}}^{hQ}_\star \cong H\underline{\mathbb{Z}}^{hQ}_\star$, analogously for $\left(H\underline{\mathbb{Z}}_{hQ}\right)_\star$ and $H\underline{\mathbb{A}}^{tQ}_\star$. Thus Lemma 3.7.1 gives a description for this entries of the Tate diagram.

Now we need to compute the coefficients of $H\underline{\mathbb{A}}^\Phi$.

Lemma 3.7.6.

$$H\underline{\mathbb{A}}^{\Phi Q}_\star \cong \mathbb{Z}[a, a^{-1}][u]/2u.$$

Proof. By Theorem 3.5.1 we have that $H\underline{\mathbb{A}}^{\Phi Q}_0 \cong \mathbb{Z}$ and $H\underline{\mathbb{A}}^{\Phi Q}_1 \cong 0$. So by a-periodicity of $H\underline{\mathbb{A}}^\Phi$ (see Lemma 3.2.3) we obtain that $H\underline{\mathbb{A}}^{\Phi Q}_{*\sigma} \cong \mathbb{Z}[a, a^{-1}]$. Since the map $\epsilon_{t\star}\colon H\underline{\mathbb{A}}^{\Phi Q}_{x+y\sigma} \to H\underline{\mathbb{A}}^{tQ}_{x+y\sigma}$ induced by ϵ_t is a ring map and it is an isomorphism if $x \geq 2$, the result follows by Lemma 3.7.1. □

THEOREM 3.7.7.

(1) *The $RO(Q)$-graded abelian group structure of $H\underline{\mathbb{A}}^Q_\star$ is given by:*

$$H\underline{\mathbb{A}}^Q_{x+y\sigma} \cong \begin{cases} \mathbb{A}(Q) & \text{if } x = y = 0 \\ \mathbb{Z} & \text{if } x = 0 \text{ and } y \neq 0 \\ \mathbb{Z} & \text{if } x \text{ even and } y = -x \\ \mathbb{Z}/2 & \text{if } x \text{ odd, } x \leq -3 \text{ and } y > -x \\ \mathbb{Z}/2 & \text{if } x \text{ even, } x \geq 2 \text{ and } y < -x \\ 0 & \text{else .} \end{cases}$$

(2) *The multiplicative structure of $H\underline{\mathbb{A}}^Q_\star$ is given by the following properties:*
   (a) *it is strictly commutative;*
   (b) *red lines on Figure 3.6 represent multiplication by a;*
   (c) *blue dashed lines represent multiplication by u, the generator of $H\underline{\mathbb{A}}^Q_{2-s\sigma}$ corresponding to* 1*;*
   (d) *if τ is a generator of $H\underline{\mathbb{A}}^Q_\sigma$, then $a\tau = \omega - 2$;*
   (e) *$u\colon H\underline{\mathbb{A}}^Q_{2\sigma-2} \to H\underline{\mathbb{A}}^Q_0$ is the transfer map in $\underline{\mathbb{A}}$ and $u\colon H\underline{\mathbb{A}}^Q_0 \to H\underline{\mathbb{A}}^Q_{2-2\sigma}$ is the restriction;*
   (f) *for $n > 0$ the group $H\underline{\mathbb{A}}^Q_{2n\sigma-2n}$ is generated by ωu^{-n}.*

In particular, the subring consisting of entries for $x \geq 0$ and $y \leq 0$ is a truncated polynomial algebra

$$\frac{A(Q)[a,u]}{a\omega, 2au}.$$

The data above is presented on Figure 3.6.

PROOF. We provide here only the proof of points 1 and 2a, since the rest is completely analogous to the case of $H\underline{\mathbb{Z}}$ given in Theorem 3.7.4.

(1) If $-1 \leq x \leq 1$ then the statement comes from the following calculations:

$$\begin{aligned} &\ker(\mathrm{res}_{\underline{\mathbb{A}}}) = (\omega - 2) \cong \mathbb{Z} \text{ as an abelian group} \\ &\mathrm{coker}(\mathrm{res}_{\underline{\mathbb{A}}}) = 0 \\ &\ker(\mathrm{tr}_{\underline{\mathbb{A}}}) = 0 \\ &\mathrm{coker}(\mathrm{tr}_{\underline{\mathbb{A}}}) = \mathbb{Z}. \end{aligned}$$

   The rest follows from Lemma 3.7.1 in the same way as in the proof of Theorem 3.7.4.

(2) (a) Firstly note that $\underline{\mathbb{A}}$ is a Green functor, so $H\underline{\mathbb{A}}$ is a commutative ring Q-spectrum. Recall the graded commutativity rule from the proof of Theorem 3.7.4, part 2a: if $\alpha \in H\underline{\mathbb{A}}^Q_{x+y\sigma}$ and $\beta \in H\underline{\mathbb{A}}^Q_{x'+y'\sigma}$ then

$$\alpha\beta = (-1)^{xx'}(1-\omega)^{yy'}\beta\alpha.$$

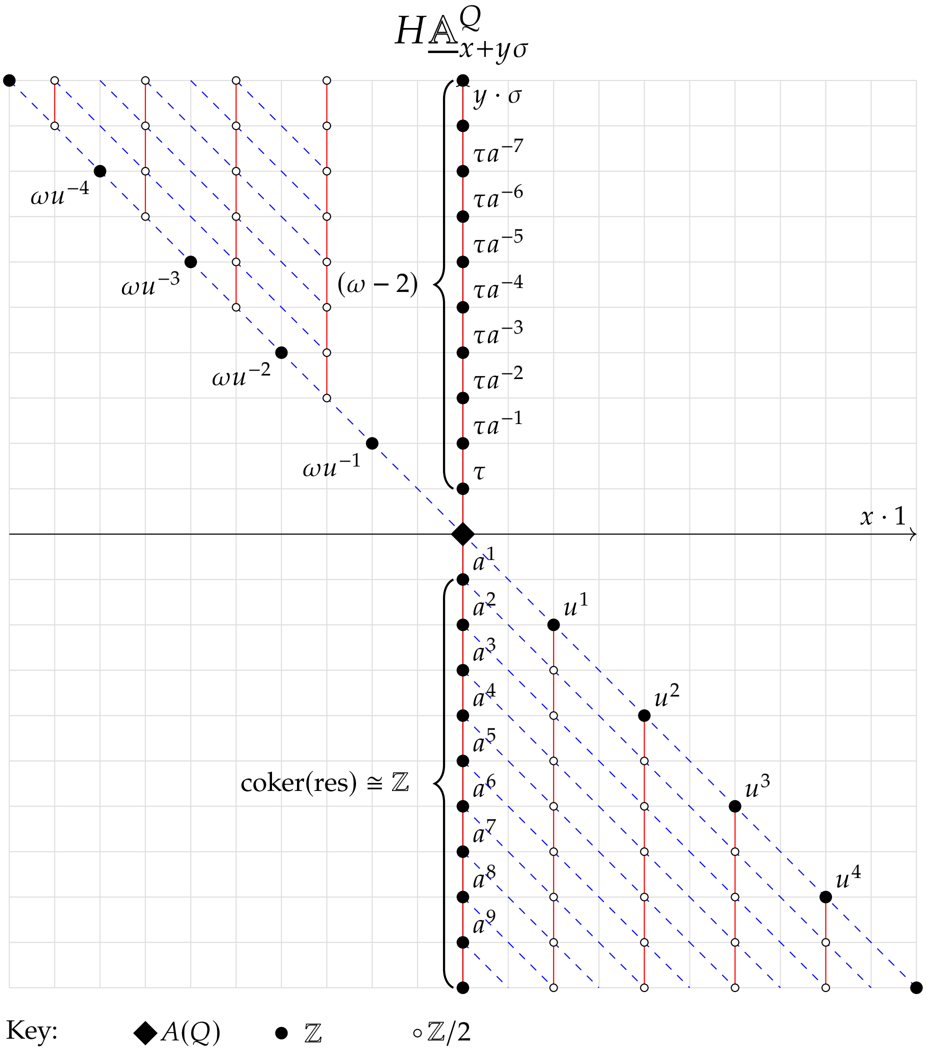


FIGURE 3.6. Coefficients of $H\underline{\mathbb{A}}$.

Since all non-zero entries have even fixed degree, the first unit is always 1. So we need to show that $1-\omega$ also acts as 1 in all cases. To this end we need to show that this claim holds only on the antidiagonal and on y-axis, as all other non-zero entries are $\mathbb{Z}/2$.

On the antidiagonal all non-zero entries are in even twisted degrees and so $1-\omega$ acts as 1. For $y > 0$ we have that $H\underline{\mathbb{A}}^{Q}_{y\sigma} = \ker(\mathrm{res}_{\underline{\mathbb{A}}}) = (\omega - 2)$. Multiplication by ω on this ideal gives zero, since $\omega(\omega-2) = \omega^2 - 2\omega = 0$ in $\mathbb{A}(Q)$. So $1-\omega$ acts as 1 on $H\underline{\mathbb{A}}^{Q}_{y\sigma}$ if $y > 0$. Finally, let $y < 0$. In this case we have that $H\underline{\mathbb{A}}^{Q}_{y\sigma} = \mathrm{coker}(\mathrm{tr}_{\underline{\mathbb{A}}}) = \mathbb{A}(Q)/\omega$, so ω acts as 0 and $1-\omega$ as 1.

$\square$

3.8. Periodicity in the antidiagonal direction

Sections 3.2 and 3.7 suggest that there exist some patterns in the coefficients of Eilenberg-MacLane Q-spectra. One of them is a repetition along the vertical lines, which by Lemma 3.3.1 we can attribute to multiplication by a. The other one may be seen in the antidiagonal direction - we are going to describe it in this section.

Let u be the generator of $H\underline{\mathbb{A}}^Q_{2-2\sigma} = H\underline{\mathbb{A}}^{hQ}_{2-2\sigma} \cong \mathbb{Z}$.

Theorem 3.8.1. *For any Mackey functor $\underline{M}$ the map $u\colon H\underline{M}^Q_{x+y\sigma} \to H\underline{M}^Q_{(x+2)+(y-2)\sigma}$ is:*

1. *the map induced by the transfer $V_Q \to \underline{M}(Q/Q)$ if $x = -2$ and $y = 2$;*
2. *the restriction map $\underline{M}(Q/Q) \to V^Q$ if $x = y = 0$;*
3. *multiplication by $1-\gamma$ if $x = -1$ and $y = 1$;*
4. *the inclusion $\ker(\mathrm{tr}_{\underline{M}}) \to {}_N V$ if $x = 1$ and $y = -1$;*
5. *the projection $V/NV \to \mathrm{coker}(\mathrm{res}_{\underline{M}})$ if $x = -3$ and $y = 3$;*
6. *the map $\ker(\mathrm{tr}_{\underline{M}})_Q \to {}_N V/(1-\gamma)V$ induced by the inclusion from Point 4 if $x = 1$ and $y < -1$;*
7. *the projection $V^Q/NV \to V^Q/\mathrm{im}(\mathrm{res}_{\underline{M}})$ if $x = -1$ and $y > -3$;*
8. *an isomorphism otherwise.*

Before proving this theorem we give a couple of preparatory lemmas. Note that since $H\underline{M}$ is a module over $H\underline{\mathbb{A}}$ we have that $H\underline{M}^h$ is a module over $H\underline{\mathbb{A}}^h$ (see [**22**, Proposition 3.5]). The equivalence

$$\epsilon\colon EQ_+ \wedge X \to EQ_+ \wedge F(EQ_+, X)$$

gives a structure of $H\underline{\mathbb{A}}^h$-module on $H\underline{M}_h$ in the homotopy category by the following composite:

$$EQ_+ \wedge F(EQ_+, H\underline{\mathbb{A}}) \wedge H\underline{M} \xrightarrow{\bar{\epsilon}\wedge H\underline{M}} EQ_+ \wedge H\underline{\mathbb{A}} \wedge H\underline{M} \longrightarrow EQ_+ \wedge H\underline{M}.$$

Here $\bar{\epsilon}$ is the inverse of ϵ in the homotopy category. The domain of the composite is $H\underline{\mathbb{A}}^h \wedge H\underline{M}_h$ after applying appropriate twist map.

Lemma 3.8.2. *For every Mackey functor $\underline{M}$ the modules $H\underline{M}^{hQ}_\star$ and $\left(H\underline{M}_{hQ}\right)_\star$ are u-periodic, i.e., u acts as a unit.*

Proof. By Lemma 3.7.1 this is true for $H\underline{\mathbb{A}}$.

By [**22**, Proposition 8.4] the pairing

$$H\underline{\mathbb{A}}^h \wedge H\underline{M}^h \to H\underline{M}^h$$

gives the pairing in the group cohomology:

$$H^*(Q;\mathbb{Z}) \otimes H^*(Q;V) \to H^*(Q;V).$$

Since $H\underline{\mathbb{A}}^{hQ}_{2-2\sigma} = H^0(Q;\mathbb{Z}) \cong \mathbb{Z}$ and u is a generator, the statement follows for $H\underline{M}^{hQ}_{\star}$. By the discussion preceding the lemma $\left(H\underline{M}_{hQ}\right)_{\star}$ is a module over $H\underline{\mathbb{A}}^{hQ}_{\star}$, so the second part follows analogously. □

Note that if X is a Q-spectrum, its homotopy groups $X^Q_\star$ and $X^e_\star$ form an $RO(Q)$-graded Mackey functor denoted by $X^\bullet_\star$. We investigate here the Mackey functor structure of two entries of $H\underline{M}^\bullet_\star$, namely $H\underline{M}^\bullet_{1-\sigma}$ and $H\underline{M}^\bullet_{\sigma-1}$. Recall the notation $\tilde{V} = \tilde{\mathbb{Z}} \otimes V$.

Lemma 3.8.3. *The Mackey functor structure of $H\underline{M}^\bullet_{\sigma-1}$ is given by:*

$$\begin{tikzcd} \operatorname{coker}(\operatorname{res}_{\underline{M}}) \arrow[d, bend right, "N"'] \\ \tilde{V} \arrow[u, bend right, "\pi"'] \end{tikzcd}$$

where π is the map induced by projection of V onto $\operatorname{coker}(\operatorname{res}_{\underline{M}})$.

Proof. By Proposition 3.4.1 and the untwisting homeomorphism $Q_+ \wedge S^1 \cong Q_+ \wedge S^\sigma$ we have that $H\underline{M}^e_{\sigma-1} = \tilde{V}$. From Theorem 3.6.1 we get that $H\underline{M}^Q_{\sigma-1} = \operatorname{coker}(\operatorname{res}_{\underline{M}})$.

Recall the cofibre sequence (†) from Section 3.3:

$$Q_+ \longrightarrow S^0 \longrightarrow S^\sigma.$$

After smashing this sequence with $H\underline{M}$ and applying $[S^{\sigma-1}, -]^Q$ we obtain the following exact sequence of abelian groups, where tr denotes the transfer in $H\underline{M}^\bullet_{\sigma-1}$:

$$H\underline{M}^e_\sigma \longrightarrow H\underline{M}^Q_\sigma \longrightarrow H\underline{M}^Q_0 \longrightarrow H\underline{M}^e_{\sigma-1} \xrightarrow{\operatorname{tr}} H\underline{M}^Q_{\sigma-1} \longrightarrow H\underline{M}^Q_{-1}.$$

The outer terms are zero, so from this exact sequence and Theorem 3.6.1 we can read that the underlying map of abelian groups of tr is the projection of V onto $\operatorname{coker}(\operatorname{res}_{\underline{M}})$.

Let $\tilde{x} \in \tilde{V}$ and $x \in V$ be the underlying element of V. Then the transfer of $H\underline{M}^\bullet_{\sigma-1}$ is given by

$$\tilde{x} \mapsto x + \operatorname{im}(\operatorname{res}_{\underline{M}}).$$

Note that this map satisfies the condition $\operatorname{tr}(\tilde{x}) = \operatorname{tr}(\gamma\tilde{x})$ for transfer in a Mackey functor.

Finally, by the definition of a Mackey functor we have that $\operatorname{res}(\operatorname{tr}(\tilde{x})) = N\tilde{x}$. Thus the restriction in $H\underline{M}^Q_{\sigma-1}$ has to be the map

$$x + \operatorname{im}(\operatorname{res}_{\underline{M}}) \mapsto N\tilde{x}.$$

An easy calculation shows that this map is well-defined and satisfies the required properties. □

LEMMA 3.8.4. *The Mackey functor structure of $H\underline{M}^{\bullet}_{1-\sigma}$ is given by:*

$$\begin{tikzcd} \ker(\mathrm{tr}_{\underline{M}}) \arrow[d, bend right, "i"'] \\ \tilde{V} \arrow[u, bend right, "N"'] \end{tikzcd}$$

where i is the inclusion of $\ker(\mathrm{tr}_{\underline{M}})$ *in $(\tilde{V})^Q = {}_NV$.*

PROOF. This is analogous to the proof of Lemma 3.8.3. Note that the identification $(\tilde{V})^Q = {}_NV$ follows from the following calculation: $\tilde{x} \in (\tilde{V})^Q$ if and only if $\tilde{x} - \gamma\tilde{x} = 0$. But the last equality on the underlying element in V gives that $x + \gamma x = 0$, i.e., $x \in {}_NV$. □

PROOF OF THEOREM 3.8.1.

(1) Multiplication by u on $\left(H\underline{M}_{hQ}\right)_\star$ and $H\underline{M}^Q_\star$ gives us the following commutative diagram:

$$\begin{tikzcd} \left(H\underline{M}_{hQ}\right)_{2\sigma-2} \arrow[r, "u"] \arrow[d] & \left(H\underline{M}_{hQ}\right)_0 \arrow[d] \\ H\underline{M}^Q_{2\sigma-2} \arrow[r, "u"] & H\underline{M}^Q_0. \end{tikzcd}$$

The left vertical arrow is an isomorphism by Theorem 3.6.1 and the top arrow is an isomorphism by Lemma 3.8.2. Thus the bottom arrow is up to isomorphism the same as the right vertical arrow - which is the map f_0. Thus by Proposition 3.4.4 the bottom arrow is the map induced by transfer on V_Q.

(2) Follows analogously to Point 1.

(3) The proof of this point needs the following diagram:

$$\begin{tikzcd} H\underline{M}^Q_{\sigma-1} \arrow[r, "u"] \arrow[d] & H\underline{M}^Q_{1-\sigma} \arrow[d] \\ H\underline{M}^{hQ}_{\sigma-1} \arrow[r, "u", "\cong"'] & H\underline{M}^{hQ}_{1-\sigma}. \end{tikzcd}$$

By Theorem 3.6.1 and Lemma 3.8.3 the left vertical map is the map

$$\mathrm{coker}(\mathrm{res}_{\underline{M}}) \to (\tilde{V})^Q = {}_NV$$

induced by the restriction of the Mackey functor $H\underline{M}^{\bullet}_{\sigma-1}$. It is the multiplication by $1+\sigma$ in $\tilde{V}$, thus the multiplication by $1-\gamma$ in V. The right vertical map is induced by the restriction of the Mackey functor $H\underline{M}^{\bullet}_{1-\sigma}$, so by Lemma 3.8.4 it is an inclusion $\ker(\mathrm{res}_{\underline{M}}) \to {}_NV$. Since by Lemma 3.8.2 the bottom arrow is an isomorphism, the top arrow is a multiplication by $1-\gamma$.

(4) Note that the three corners of the diagram displaying multiplication by u are isomorphic:

$$\begin{tikzcd} H\underline{M}^{Q}_{1-\sigma} \arrow[r, "u"] \arrow[d] & H\underline{M}^{Q}_{3-3\sigma} \arrow[d] \\ H\underline{M}^{hQ}_{1-\sigma} \arrow[r, "u", "\cong"'] & H\underline{M}^{hQ}_{3-3\sigma}. \end{tikzcd}$$

The right vertical arrow is an isomorphism by Theorem 3.6.1 and the bottom arrow by Lemma 3.8.2. Thus the top arrow is the same as the left vertical arrow, which is induced by the restriction of the Mackey functor $H\underline{M}^{\bullet}_{1-\sigma}$. So it is the inclusion $\ker(\mathrm{tr}_{\underline{M}}) \to {}_N V$.

(5) The proof is analogous to the previous point.

(6) By the same argument as in Point 4 we have that $u\colon H\underline{M}^{Q}_{1+y\sigma} \to H\underline{M}^{hQ}_{3+(y-1)\sigma}$. Now we use the following diagram:

$$(*) \qquad \begin{tikzcd} H\underline{M}^{Q}_{1+y\sigma} \arrow[r] \arrow[d] & H\underline{M}^{\Phi Q}_{1+y\sigma} \arrow[d, "\zeta"] \\ H\underline{M}^{hQ}_{1+y\sigma} \arrow[r] & H\underline{M}^{tQ}_{1+y\sigma}. \end{tikzcd}$$

If we apply the long exact sequence in homotopy to the top and bottom rows of the Tate diagram we obtain:

$$\dots \longrightarrow \left(H\underline{M}_{hQ}\right)_{1+y\sigma} \longrightarrow H\underline{M}^{Q}_{1+y\sigma} \longrightarrow H\underline{M}^{\Phi Q}_{1+y\sigma} \longrightarrow \left(H\underline{M}_{hQ}\right)_{y\sigma} \longrightarrow \dots$$

and

$$\dots \longrightarrow \left(H\underline{M}_{hQ}\right)_{1+y\sigma} \longrightarrow H\underline{M}^{hQ}_{1+y\sigma} \longrightarrow H\underline{M}^{tQ}_{1+y\sigma} \longrightarrow \left(H\underline{M}_{hQ}\right)_{y\sigma} \longrightarrow \dots .$$

Since $\left(H\underline{M}_{hQ}\right)_{1+y\sigma} = \left(H\underline{M}_{hQ}\right)_{y\sigma} = 0$ by Proposition 3.4.2, the top and bottom horizontal arrows in the diagram (*) are isomorphisms. Thus u acts on $H\underline{M}^{Q}_{1+y\sigma}$ in the same way as ζ, so we need to identify this map.

Note that by a-periodicity of $H\underline{M}^{\Phi}$ and $H\underline{M}^{t}$ (see Lemma 3.2.3) it is enough to identify this map for $y = 0$. Recall from the proof of Lemma 3.5.3 that the fibre F of the map $\epsilon_t\colon H\underline{M}^{\Phi} \to H\underline{M}^{t}$ is 0-coconnective, in particular $F^{Q}_{1} = 0$. Thus by Proposition 3.4.5 and Theorem 3.5.1 we have that ζ is an inclusion

$$\ker(\mathrm{tr}_{\underline{M}})_Q \hookrightarrow {}_N V/(1-\gamma)V.$$

(7) The proof is analogous to the previous point.

(8) If $x \leq -2$ or $x \geq 2$ then by Theorem 3.6.1 we have that $H\underline{M}^Q_{x+y\sigma}$ is isomorphic to respectively $\left(H\underline{M}_{hQ}\right)_{x+y\sigma}$ and $H\underline{M}^{hQ}_{x+y\sigma}$. Since by Lemma 3.8.2 both $\left(H\underline{M}_{hQ}\right)_\star$ and $H\underline{M}^{hQ}_\star$ are u-periodic, the statement follows.

□

3.9. Commutativity

In Section 3.7 we have seen that both examples share one feature - the coefficients are strictly commutative ring, i.e., the sign coming from the swap of factors is always trivial. In this section we show that this happens for all Green functors. Throughout this section, let $\underline{M}$ be a Green functor.

The Burnside ring $\mathbb{A}(Q)$ acts on $H\underline{M}^Q_\star$ as a 0-th Q-homotopy group of the sphere spectrum. By the associativity of a smash product this is given by the action on $H\underline{M}^Q_0$:

$$\pi_0^Q(S^0) \otimes (H\underline{M}^Q_0 \otimes H\underline{M}^Q_V) = (\pi_0^Q(S^0) \otimes H\underline{M}^Q_0) \otimes H\underline{M}^Q_V \to H\underline{M}^Q_V.$$

Recall that the class of Q/e in $\mathbb{A}(Q)$ is denoted by ω. By the definition of a box product of Mackey functors we have that ω acts on $\underline{M}(Q/Q)$ as $\mathrm{tr}_{\underline{M}}(1)$:

$$\omega \cdot 1 = \mathrm{tr}_{\underline{A}}(1) \cdot 1 = \mathrm{tr}_{\underline{A}\Box\underline{M}}(1 \otimes (\mathrm{res}_{\underline{M}}(1))) = \mathrm{tr}_{\underline{M}}(1).$$

We used here relations in the box product of Mackey functors. Details may be found in **[41]**. The last equality is obtained by the fact that the restriction in a Green functor is a ring homomorphism, in particular it preserves identity.

Recall that V denotes $\underline{M}(Q/e)$. Note that if $\underline{M}$ is a Green functor then V is a $\underline{M}(Q/Q)$-algebra with action given by $\mathrm{res}_{\underline{M}}$. From this we can deduce that if $v \in V$ then

$$\omega v = \mathrm{res}_{\underline{M}}(\mathrm{tr}_{\underline{M}}(1))v = 2v,$$

i.e., ω acts on V as a multiplication by 2. The last equality here comes from the fact that if $\underline{M}$ is a Green functor, then γ needs to act on V as a unitary ring homomorphism - thus $\mathrm{res}_{\underline{M}}(\mathrm{tr}_{\underline{M}}(1)) = (1+\gamma)1 = 2$.

Theorem 3.9.1. *If $\underline{M}$ is a Green functor, then $H\underline{M}^Q_\star$ is a strictly commutative ring.*

Before giving a proof of Theorem 3.9.1 we prove a couple of preparatory lemmas. Recall the graded sign rule for $H\underline{M}^Q_\star$ from the proof of Theorem 3.7.4, Point 2: if $\alpha \in H\underline{M}^Q_{x+y\sigma}$ and $\beta \in H\underline{M}^Q_{x'+y'\sigma}$ then

$$\alpha\beta = (-1)^{xx'}(1-\omega)^{yy'}\beta\alpha.$$

Lemma 3.9.2. *For any Mackey functor $\underline{M}$ groups $H\underline{M}^Q_{x+y\sigma}$ are 2-torsion if $x \neq 0$ and $x \neq -y$.*

Proof. Note that if the conditions of Lemma 3.6.3 holds (i.e., $x + y\sigma$ lies below the antidiagonal on the half-plane $x < 0$ or above the antidiagonal on the half-plane $x > 0$) then $H\underline{M}^Q_{x+y\sigma}$ is zero, so the statement is trivially satisfied.

We will consider four cases depending on the value of x.

(1) $x \geq 2$ and $y < -x$. Assume firstly that y is even. By Theorem 3.6.1 and Proposition 3.4.2 we have that
$$H\underline{M}^Q_{x+y\sigma} \cong H^{-x-y}(Q;V).$$
Note that the group cohomology $H^p(Q;V)$ is 2-torsion for $p > 0$ (this can be easily deduced from [**61**, Theorem 6.2.2].) By the assumption $-x-y \geq 1$, so the statement holds. If y is odd, we have that
$$H\underline{M}^Q_{x+y\sigma} \cong H^{-x-y}(Q;\tilde{V}).$$
Here we use the fact that $H^p(Q;\tilde{V}) \cong H^{p+1}(Q;V)$ (see Section 3.4). Thus the claim is proven in this case.

(2) $x \leq -2$ and $y > -x$. In this case we proceed analogically to the previous point, using the fact that $H\underline{M}^Q_{x+y\sigma}$ is isomorphic to the group homology.

(3) $x = 1$ and $y < -x$. Then $H\underline{M}^Q_{1+y\sigma} \cong \ker(\mathrm{tr}_{\underline{M}})_Q$. Let $\alpha \in \ker(\mathrm{tr}_{\underline{M}})_Q$. Then $(1+\gamma)\alpha = \mathrm{res}_{\underline{M}}(\mathrm{tr}_{\underline{M}}(\alpha)) = 0$, so $\alpha = -\gamma\alpha$. Thus $2\alpha = \alpha - \gamma\alpha = (1-\gamma)\alpha = 0$.

(4) $x = -1$ and $y > -x$. Then $H\underline{M}^Q_{y\sigma-1} \cong V^Q/\mathrm{im}(\mathrm{res}_{\underline{M}})$. Let $\alpha \in V^Q/\mathrm{im}(\mathrm{res}_{\underline{M}})$. Then $2\alpha = (1+\gamma)\alpha = \mathrm{res}_{\underline{M}}(\mathrm{tr}_{\underline{M}}(\alpha)) = 0$.

□

LEMMA 3.9.3. *The subring $H\underline{M}^Q_{*-*\sigma}$ is strictly commutative.*

PROOF. Note that unless $x = 0$ the groups $H\underline{M}^Q_{x-x\sigma}$ are submodules or subquotients of V, thus ω acts on them as 2. Let $\alpha \in H\underline{M}^Q_{x-x\sigma}$ and $\beta \in H\underline{M}^Q_{x'-x'\sigma}$. By the sign rule the only possibility when a non-trivial sign might occur is when both x and x' are odd. In this case we have
$$\alpha\beta = (-1)^{xx'}(1-2)^{xx'}\beta\alpha = \beta\alpha.$$

□

LEMMA 3.9.4. *The subring $H\underline{M}^Q_{*\sigma}$ is strictly commutative.*

PROOF. Since the fixed degree is zero we need only to check the possible sign coming from the multiplication by $1-\omega$. We are going to prove that ω acts as 0 unless $y \neq 0$. Consider two cases:

- $y > 0$. Then $H\underline{M}^Q_{y\sigma}$ by Theorem 3.6.1. Let $min \ker(\mathrm{res}_{\underline{M}})$. We have that
$$\omega \cdot m = \mathrm{tr}(1) \cdot m = \mathrm{tr}(1 \cdot \mathrm{res}(m)) = 0.$$
So ω acts as 0.
- $y < 0$. Then $H\underline{M}^Q_{y\sigma} \cong \mathrm{coker}(\mathrm{tr}_{\underline{M}})$. Thus $\mathrm{tr}_{\underline{M}}(1) = 0$ and ω acts as 0.

Now let $m \in H\underline{M}^Q_{y\sigma}$ and $n \in H\underline{M}^Q_{y'\sigma}$. If any of y, y' is zero or even then the statement trivially holds. Thus let both y and y' be odd. Then
$$\alpha\beta = (1-\omega)^{yy'}\beta\alpha = ((1-\omega)\beta)\alpha = \beta\alpha.$$

□

Proof of Theorem 3.9.1. Let $\alpha \in H\underline{M}^{Q}_{x+y\sigma}$ and $\beta \in H\underline{M}^{Q}_{x'+y'\sigma}$. Then

$$\alpha\beta \in H\underline{M}^{Q}_{(x+x')+(y+y')\sigma}.$$

Consider the following cases depending on the degree of the product $\alpha\beta$:

(1) $x + x' \neq 0$ and $x + x' \neq -(y + y')$. Then the product $\alpha\beta$ does not lie on either the antidiagonal or the axis $x = 0$. So by Lemma 3.9.2 the product $\alpha\beta$ belongs to a 2-torsion group, thus $\alpha\beta = \beta\alpha$.
(2) $x = x' = 0$. Then we are in the situation of Lemma 3.9.4, so the statement holds.
(3) $x = -y$ and $x' = -y'$. In this case we use Lemma 3.9.3.
(4) $x, x' \neq 0$ and $x + x' = 0$. If both α and β lie on the antidiagonal then the claim is proven by the previous point. Without loss of generality assume that $x \neq -y$, i.e., α does not lie on the antidiagonal. By the proof of Lemma 3.9.4 the possible sign coming from $(1-\omega)^{yy'}$ is trivial since ω acts trivially on groups lying on $x = 0$ axis. Note that α belongs to a 2-torsion group, so $\alpha\beta$ is also 2-torsion and the statement follows.
(5) $y + y' = -(x + x')$, i.e., $\alpha\beta$ lies on the antidiagonal. Then we use the same argument as in the previous point.

□

Observation 3.9.5. By Theorem 3.7.7 we know that in order to describe the $H\underline{\mathbb{A}}^{Q}_{\star}$-module structure of $H\underline{M}^{Q}_{\star}$ for any Mackey functor $\underline{M}$ we need to describe an action of three elements - a, u and ω. Thus we have already described this structure:

(1) action of a is described by Lemma 3.3.1;
(2) action of u is described by Theorem 3.8.1;
(3) action of ω is described in this Section.

Observation 3.9.6. Let $\underline{M}$ be a Green functor. A big part of the multiplicative structure may be derived from the $H\underline{\mathbb{A}}^{Q}_{\star}$-module structure and commutativity described in this Section - we know that there are elements $a_{H\underline{M}} = a \cdot 1 \in H\underline{M}^{Q}_{-\sigma}$ and $u_{H\underline{M}} \in H\underline{M}^{Q}_{2-2\sigma}$ and their multiplicative relations are described by $H\underline{\mathbb{A}}^{Q}_{\star}$-module structure.

This gives a full description of the multiplication in the even fixed degrees. For the odd fixed degrees we proceed as follows:

(1) We need to consider only the elements lying on the antidiagonal, as the relations for other elements follows from Lemma 3.3.1.
(2) Since the multiplication map $u\colon H\underline{M}^{Q}_{x+y\sigma} \to H\underline{M}^{Q}_{(x+2)+(y-2)\sigma}$ is an isomorphism if $x \geq 3$ or $x \leq -4$, we can restrict our attention to the elements of degrees $1-\sigma$, $3-3\sigma$, $\sigma-1$ and $3\sigma-3$.

(3) Relations between these elements can be inferred from the ring maps

$$\epsilon_\star \colon H\underline{M}_\star^Q \to H\underline{M}_\star^{hQ}$$

and

$$g_\star \colon H\underline{M}_\star^Q \to H\underline{M}_\star^{\Phi Q}.$$

Examples of such computations are in Section 3.10.

3.10. Further examples

3.10.1. Constant Mackey functor $\underline{\mathbb{F}}_2$. We start this section with the constant Mackey functor $\underline{\mathbb{F}}_2$. It has the following structure:

$$\begin{tikzcd} \underline{\mathbb{F}}_2 \arrow[d, bend right, "1"'] \\ \underline{\mathbb{F}}_2. \arrow[u, bend right, "0"'] \end{tikzcd}$$

Computations of the coefficients of $H\underline{\mathbb{F}}_2$ which are built on the unpublished work of Stong appear in **[13]** and **[19]**.

Theorem 3.10.1. *The $RO(Q)$-graded abelian group structure and the multiplicative structure of $H\underline{\mathbb{F}}_2$ are given by:*

$$\left(H\underline{\mathbb{F}}_2\right)_\star^Q \cong \mathbb{F}_2[a, \lambda] \oplus \Sigma^{2\sigma-2}\mathbb{F}_2[a^{-1}, \lambda^{-1}]$$

with $|\lambda| = 1 - \sigma$ and $|a| = -\sigma$.

This data is presented in Figure 3.7.

In order to prove this theorem we will need to describe the multiplicative structure of $\left(H\underline{\mathbb{F}}_2\right)_\star^{hQ}$ and coefficients of $\left(H\underline{\mathbb{F}}_2\right)_h$.

Lemma 3.10.2.

$$\left(H\underline{\mathbb{F}}_2\right)_\star^{hQ} \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}].$$

Proof. The $RO(Q)$-graded abelian group structure follows from Proposition 3.4.2. To see the multiplicative structure we are going to describe the E_1-page of the trigraded homotopy fixed points spectral sequence.

Fix y. Recall from Section 3.2 that for $X = H\underline{\mathbb{F}}_2$:

$$E_1^{pq}(y) = \mathrm{Hom}_Q\left(H_p(Q_+ \wedge S^p), \pi_{-q}\left(F\left(S^{y\sigma}, H\underline{\mathbb{F}}_2\right)\right)\right)$$

where $\pi_{-q}\left(F\left(S^{y\sigma}, H\underline{\mathbb{F}}_2\right)\right)$ denotes the $-q$-th homotopy group of the underlying naive spectrum of $F\left(S^{y\sigma}, H\underline{\mathbb{F}}_2\right)$.

We have that $H_p(Q_+ \wedge S^p) \cong \mathbb{Z}[Q]$ as a $\mathbb{Z}[Q]$-module and

$$\pi_{-q}\left(F\left(S^{y\sigma}, H\underline{\mathbb{F}}_2\right)\right) \cong \begin{cases} \mathbb{Z}/2 & \text{if } y = q \\ 0 & \text{else.} \end{cases}$$

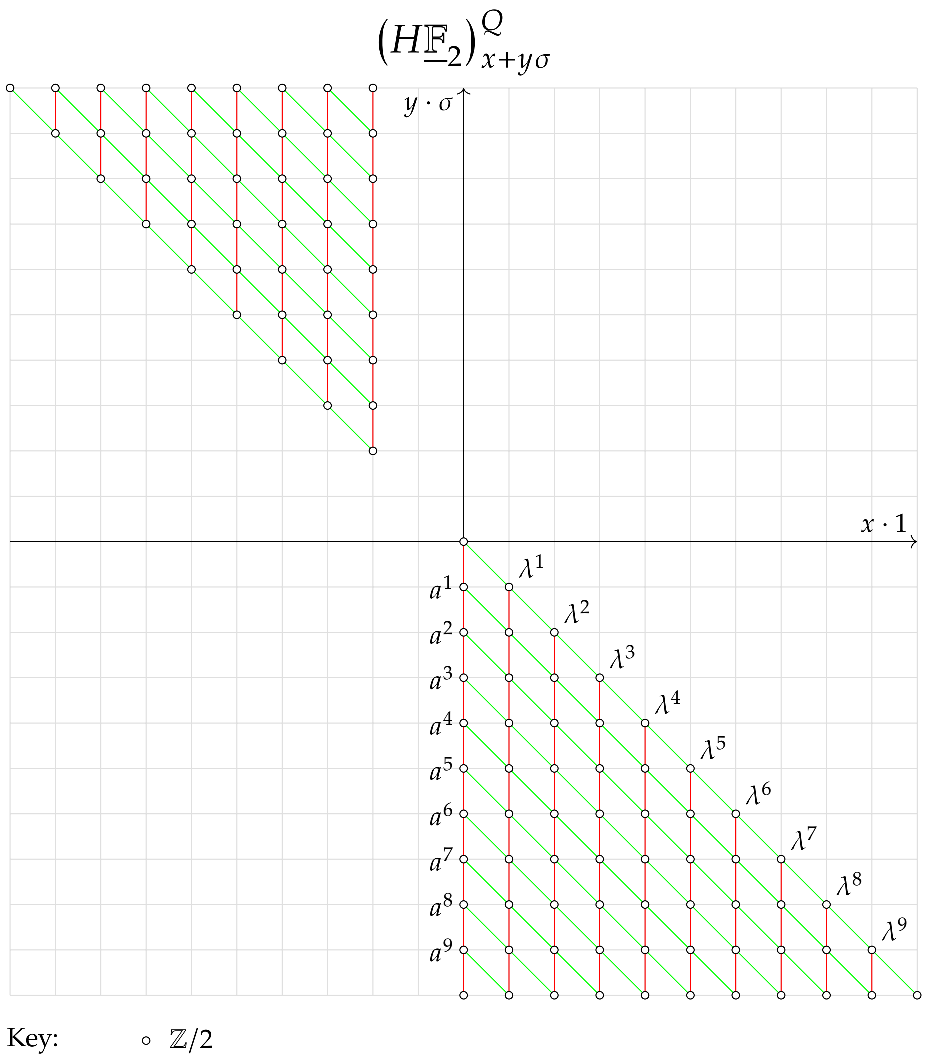


FIGURE 3.7. Coefficients of $H\underline{\mathbb{F}}_2$. The green lines represent multiplication by λ. Red lines, as before, represent multiplication by a.

The differentials on the E_1-page are the differentials computing the group cohomology, thus they are all 0. So the spectral sequence collapses on the E_1-page.

The trigraded homotopy fixed points spectral sequence is multiplicative, so we can describe the E_1-page as an algebra as follows:

$$E_1^{**}(*) \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}]$$

with $|a| = (1, -1, -1)$ and $|\lambda| = (0, -1, -1)$. Since all differentials are 0, we get that

$$\left(H\underline{\mathbb{F}}_2\right)^{hQ}_{\star} \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}]$$

with $|a| = -\sigma$ and $|\lambda| = 1 - \sigma$. □

Corollary 3.10.3.

$$\begin{aligned}(H\underline{\mathbb{F}}_2)^{tQ}_\star &\cong \mathbb{F}_2[a, a^{-1}][\lambda, \lambda^{-1}]\\ \left((H\underline{\mathbb{F}}_2)_{hQ}\right)_\star &\cong \mathbb{F}_2[a^{-1}][\lambda, \lambda^{-1}].\end{aligned}$$

Proof. The first statement follows from the fact that $H\underline{\mathbb{F}}_2^t \simeq H\underline{\mathbb{F}}_2^h \wedge \widetilde{EQ}$ and from Lemma 3.8.2. The second part follows from the long exact sequence in homotopy for the cofibre sequence

$$(H\underline{\mathbb{F}}_2)_h \to H\underline{\mathbb{F}}_2^h \to H\underline{\mathbb{F}}_2^t$$

where the map $\left((H\underline{\mathbb{F}}_2)_{hQ}\right)_\star \to (H\underline{\mathbb{F}}_2)^{hQ}_\star$ is multiplication by N, so zero in this case. □

Proof of Theorem 3.10.1. The $RO(Q)$-graded abelian group structure follows from the Theorem 3.6.1. Following Observation 3.9.6 we know that the multiplicative structure of $(H\underline{\mathbb{F}}_2)^Q_\star$ is described by the following elements:

- $a = a_{H\underline{\mathbb{F}}_2}$ - for the properties of multiplication by a see Lemma 3.3.1;
- $u = u_{H\underline{\mathbb{F}}_2}$ - see Section 3.8;
- $\lambda \in (H\underline{\mathbb{F}}_2)^Q_{1-\sigma}$.

We need to check two relations that do not follow directly from previous sections - i.e., we need to prove that $\lambda^2 = u$ and that if θ is the generator of $H\underline{\mathbb{A}}^Q_{2\sigma-2}$ then $H\underline{M}^Q_{3\sigma-3}$ is generated by $\lambda^{-1}\theta$. The rest of the structure will follow.

We are going to prove that $\lambda^2 = u$. Consider the diagram expressing multiplication by λ:

$$\begin{tikzcd} (H\underline{\mathbb{F}}_2)^Q_{1-\sigma} \arrow[r, "\lambda"] \arrow[d, "\epsilon_{1-\sigma}"] & (H\underline{\mathbb{F}}_2)^Q_{2-2\sigma} \arrow[d, "\epsilon_{2-2\sigma}"] \\ (H\underline{\mathbb{F}}_2)^{hQ}_{1-\sigma} \arrow[r, "\lambda"] & (H\underline{\mathbb{F}}_2)^{hQ}_{2-2\sigma} \end{tikzcd}.$$

By Theorem 3.6.1, the right vertical arrow is an isomorphism, so does the bottom horizontal arrow by Lemma 3.10.2. From Lemma 3.8.4 we deduce that the left vertical arrow is also an isomorphism. Thus the top vertical arrow is an isomorphism and $\lambda^2 = u$.

Let θ be a generator of $(H\underline{\mathbb{F}}_2)^Q_{2\sigma-2}$ and θ' a generator of $(H\underline{\mathbb{F}}_2)^Q_{3\sigma-3}$. Similar argument as above applied to the diagram

$$\begin{tikzcd} \left((H\underline{\mathbb{F}}_2)_{hQ}\right)_{3\sigma-3} \arrow[r, "\lambda"] \arrow[d, "f_{\sigma-1}"] & \left((H\underline{\mathbb{F}}_2)_{hQ}\right)_{2\sigma-2} \arrow[d, "f_{2\sigma-2}"] \\ (H\underline{\mathbb{F}}_2)^Q_{3\sigma-3} \arrow[r, "\lambda"] & (H\underline{\mathbb{F}}_2)^Q_{2\sigma-2} \end{tikzcd}$$

together with Corollary 3.10.3 shows that $\lambda^{-1}\theta = \theta'$. Thus the result follows. □

3.10.2. The norm of $\mathbb{F}_2$, $N_e^Q\mathbb{F}_2$. We continue the examples section with the Mackey functor $N_e^Q\mathbb{F}_2$. It has the form

$$\begin{tikzcd} \mathbb{Z}/4 \arrow[d, bend right, "1"'] \\ \mathbb{Z}/2. \arrow[u, bend right, "2"'] \end{tikzcd}$$

This Mackey functor appears in the work of Hill in **[27]**, where he computes the Bredon homology with coefficients in it of spaces of the form $\Omega^\sigma\Sigma^\sigma X$. We will follow the notation from this paper and denote this Mackey functor by $\underline{B}$. For us $\underline{B}$ is an example of an interesting feature - it has only two zero columns, which are in fixed degrees 1 and -1.

Since $\underline{B}(Q/e) = \underline{\mathbb{F}}_2(Q/e)$, Lemma 3.10.2 and Corollary 3.10.3 describe also $H\underline{B}_\star^{hQ}$, $\left(H\underline{B}_{hQ}\right)_\star$ and $H\underline{B}_\star^{tQ}$. Most of the proofs follow in the analogous way as in the previous examples, so we will only comment on how to get the multiplicative structure of $H\underline{B}_\star^Q$.

Lemma 3.10.4.
$$H\underline{B}_\star^{\Phi Q} \cong \mathbb{F}_2[a,a^{-1}] \oplus \lambda^2\mathbb{F}_2[a,a^{-1},\lambda]$$
with $|a| = -\sigma$ *and* $|\lambda| = 1-\sigma$.

Theorem 3.10.5. *The $RO(Q)$-graded abelian group structure and the multiplicative structure of $H\underline{B}_\star^Q$ is given by:*
$$H\underline{B}_\star^Q \cong \frac{BB[u,u\lambda]}{ua^{-1}} \oplus 2u^{-1}\cdot \mathbb{Z}/4[a^{-1},u^{-1},(u\lambda)^{-1}].$$
Here $|a| = -\sigma$, $|\lambda| = 1-\sigma$, $|u| = 2-2\sigma$, $u=\lambda^2$ *and*
$$BB = \frac{\mathbb{Z}/4[a]}{2a} \oplus 2a^{-1}\cdot\mathbb{Z}/4[a^{-1}].$$
This data is presented in Figure 3.8.

Proof. We are going to comment only on the multiplicative structure. By Observation 3.9.6 we know that it can be described by the following elements:

- $a_{H\underline{B}} \in H\underline{B}^Q_{-\sigma}$;
- $u_{H\underline{B}} \in H\underline{B}^Q_{2-2\sigma}$;
- a generator of $H\underline{B}^Q_{3-3\sigma}$.

and their inverses. From the map $\epsilon_\star\colon H\underline{B}_\star^Q \to H\underline{B}_\star^{hQ}$ we can deduce that the last element may be described as $\lambda^3 = u\lambda$. Thus the result follows. □

3.10.3. The Mackey functor $\underline{\tilde{\mathbb{Z}}}$. We close the examples section with an example of a Mackey functors with a non-trivial action on the Q/e-level. This is the fixed points Mackey

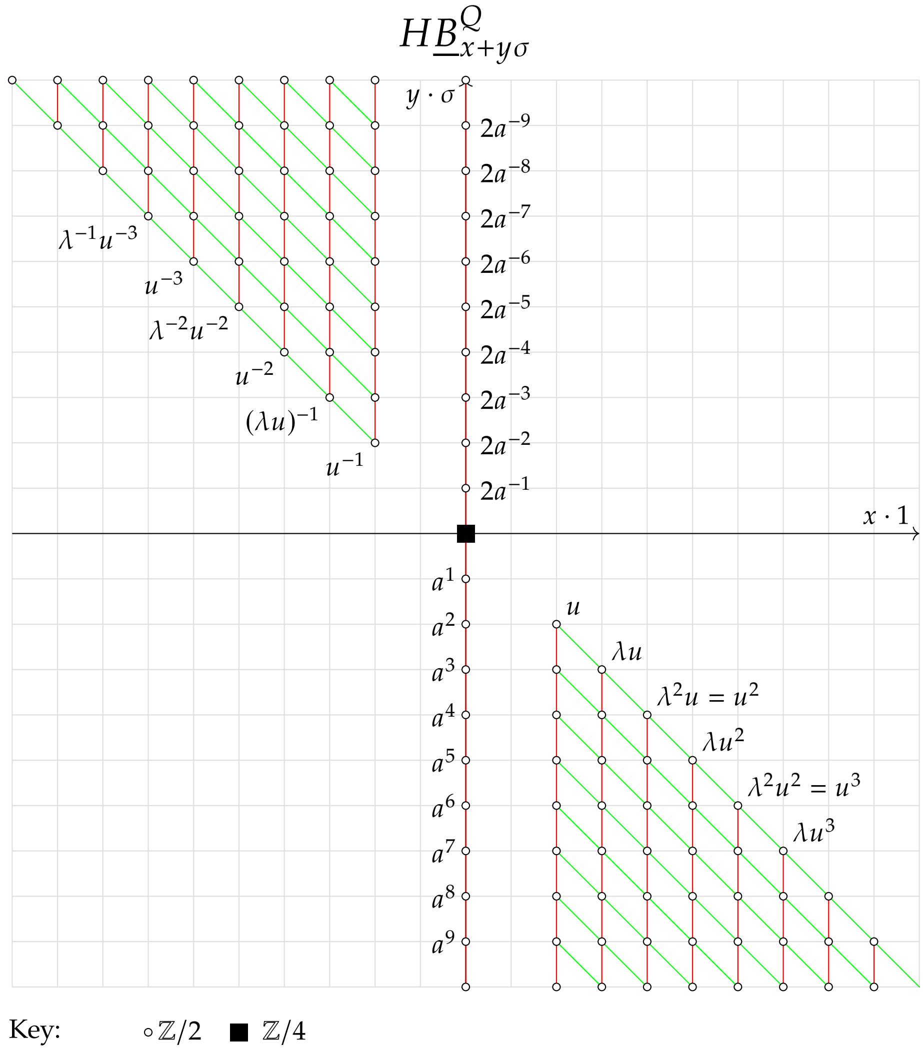


FIGURE 3.8. Coefficients of $H\underline{B}$. Notation as in Figure 3.7.

functor of $\tilde{\mathbb{Z}}$ and its structure is given by:

$$\begin{tikzcd} 0 \arrow[d, bend right] \\ \tilde{\mathbb{Z}}. \arrow[u, bend right] \end{tikzcd}$$

We denote this Mackey functor by $\underline{\tilde{\mathbb{Z}}}$. It is not a Green functor, so we describe only $H\underline{\mathbb{A}}^Q_\star$-module structure of $H\underline{\tilde{\mathbb{Z}}}^Q_\star$.

Lemma 3.10.6.

$$\left(H\underline{\tilde{\mathbb{Z}}}_{hQ}\right)_{\star} \cong \left(H\underline{\mathbb{Z}}_{hQ}\right)_{\star+1-\sigma}$$
$$H\underline{\tilde{\mathbb{Z}}}^{hQ}_{\star} \cong H\underline{\mathbb{Z}}^{hQ}_{\star+1-\sigma}.$$

Proof. Follows from the fact that $\tilde{\tilde{\mathbb{Z}}} \cong \mathbb{Z}$ as $\mathbb{Z}[Q]$-modules and Proposition 3.4.2. □

The $H\underline{\mathbb{A}}^{Q}_{\star}$-module structure of $H\underline{\tilde{\mathbb{Z}}}^{Q}_{\star}$ is depicted on Figure 3.9. Multiplication by u is represented by blue dashed lines, whereas multiplication by a is given by red dashed lines. Note that since the Q-action on Q/e-level of $\underline{\tilde{\mathbb{Z}}}$ is non-trivial, entries on $1-\sigma$ and $\sigma-1$ spots are different then the rest of $x=1$ and $x=-1$ columns respectively.

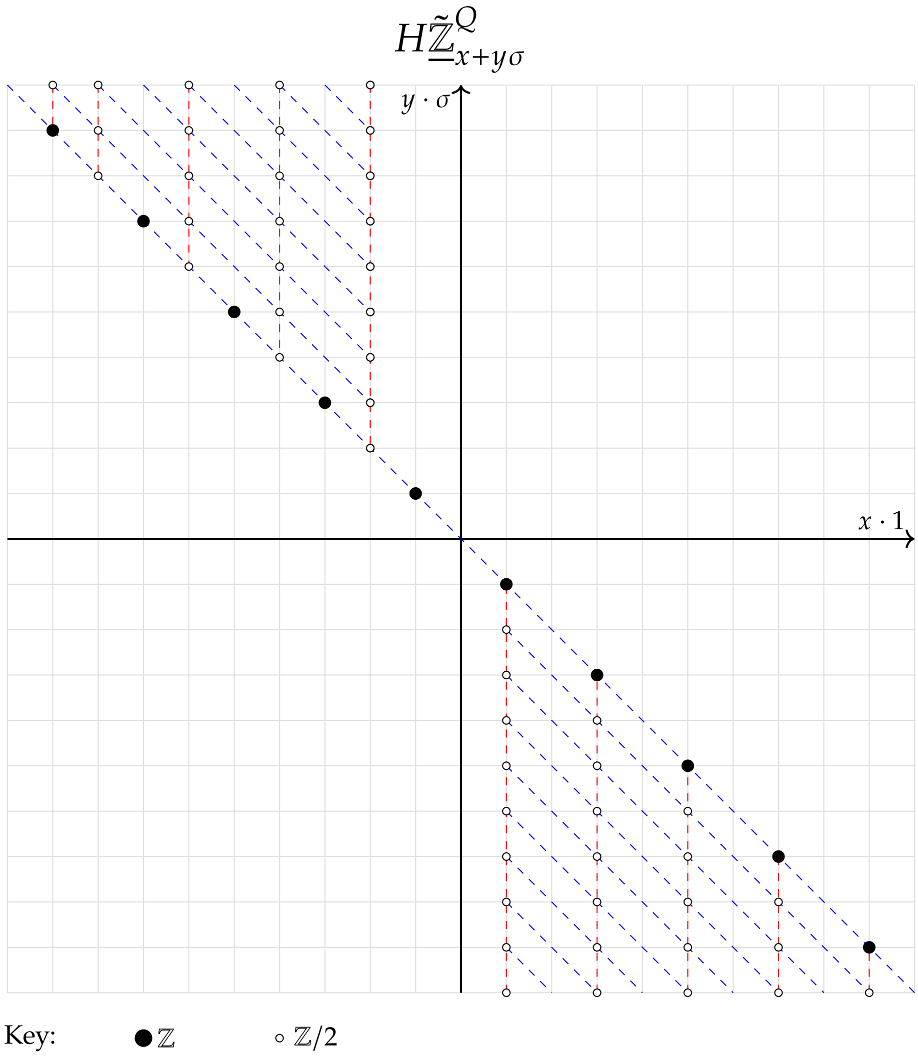


Figure 3.9. Coefficients of $H\underline{\tilde{\mathbb{Z}}}$.

CHAPTER 4

Homology operations in the algebras over Q-equivariant little discs operads

Introduction

As discussed in previous chapters, power operations are crucial in the understanding of the structure of the homology of loop spaces. These operations are well-studied in the case of non-equivariant loop spaces, with foundational work done by Araki and Kudo [**37**], Dyer and Lashof [**16**] and Cohen [**14**].

On the contrary, the situation is far from being understood in the context of equivariant loop spaces. In [**62**] and [**63**], Wilson defined and described stable Dyer-Lashof operations for Bredon homology with coefficients in $\underline{\mathbb{F}}_2$. These can be seen as an analogue of power operations in the homology of infinite loop spaces. However, the power operations for the homology of V-fold loop spaces with V being finite dimensional representation have not been studied in the literature so far.

The goal of this chapter is to contribute to filling this gap. Basing on Wilson's work, we present here a framework in which power operations in homology of V-fold loop spaces over Q can be studied. We focus on two representations - the real sign Q-representation σ and its multiple, 2σ. In these two cases we define the power operations in Definitions 4.2.9 and 4.3.9. For σ-fold loop spaces the power operations occur to be rather simple - they reduce to the norming operation and multiplication by the elements of the coefficients ring, see Propositions 4.2.10 and 4.2.13.

In the case of 2σ-fold loop spaces, we need firstly to compute the homology of the extended power construction on representation spheres - this is done in Theorem 4.3.3. We focus on the basic case - the power operations of weight 2. For 2σ-fold loop spaces we see the first "non-trivial" power operation, where by "non-trivial" we mean different from norming.

The other part of the structure of the homology of loop spaces is the Browder bracket. We define it for the case of $n\sigma$-fold loop spaces in Definition 4.4.3. To this end we need to compute the $RO(Q)$-graded homology of spheres with the antipodal action of Q - this is done in Proposition 4.4.1.

At the current state this chapter of the thesis should be treated as a framework. It provides also a great example of using the knowledge acquired in Chapter 3 , since most of the calculations rely on the structure of the rings of coefficients. The author hopes to extend the results of this chapter in the future work.

Notation and conventions. Throughout the whole chapter we are going to denote an operad in G-spectra by $\mathbb{O}$. Operads are usually denoted by calligraphic font, e.g., $\mathcal{O}$. However, we chose to use the bold font to avoid collision with the notation for the orbit category, appearing in previous chapters.

If $\mathbb{O}$ is an operad and X is an $\mathbb{O}$-algebra, then by θ we are going to denote the operadic action of $\mathbb{O}$ on X.

4.1. Overview of the homology operations

In this section we are going to provide an overview of the theory of the homology operations following [**38**]. We will adapt this theory to the setting of G-spectra, where G is a finite group.

Fix a G-spectrum E and let $\mathbb{O}$ be an operad in G-spectra (see Definition 1.8.4).

Definition 4.1.1 (See [**38**, Definition 19.4.1]). Let $V, W \in RO(G)$ (see 1.5.1). An *E-homology operation* for $\mathbb{O}$-algebras is a natural transformation of functors $E_V(-) \to E_{V+W}(-)$ on the homotopy category of $\mathbb{O}$-algebras.

In order to study these operations we firstly prove that for any $\mathbb{O}$-algebra A the object $E \wedge A$ is an $\mathbb{O}$-algebra in the category of E-modules. This will be done in Lemmas 4.1.2 and 4.1.3.

Lemma 4.1.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal categories and $F\colon \mathcal{C} \to \mathcal{D}$ be lax symmetric monoidal functor and let $\mathbb{O}$ be an operad in $\mathcal{C}$. If A is an $\mathbb{O}$-algebra, then $F(A)$ is $F(\mathbb{O})$-algebra.*

Proof. This follows easily from the properties of operads. Being an $\mathbb{O}$-algebra in $\mathcal{C}$ means that for every $k \in \mathbb{N}$ the object X is endowed with Σ_k maps:

$$\mathbb{O}(k) \otimes_{\mathcal{C}} X^{\otimes_{\mathcal{C}} k} \to X.$$

Since F is lax symmetric monoidal, there are maps

$$F(\mathbb{O}(k)) \otimes_{\mathcal{D}} F(X)^{\otimes_{\mathcal{D}} k} \to F\left(\mathbb{O}(k) \otimes_{\mathcal{C}} X^{\otimes_{\mathcal{C}} k}\right) \to F(X).$$

One can easily verify that the required diagrams commute. Thus the result follows. □

Lemma 4.1.3. *Let $i\colon R \to S$ be a map of commutative ring G-spectra. Then the* extension of scalars

$$i_*\colon \mathcal{M}od_R \to \mathcal{M}od_S$$

given by

$$M \mapsto S \wedge_R M$$

is symmetric monoidal.

Proof. This is a standard fact in the equivariant homotopy theory, so we will provide only a sketch of the proof. We need to show that if M, N are R-modules then there is an isomorphism

$$S \wedge_R (M \wedge_R N) \cong (S \wedge_R M) \wedge_S (S \wedge_R N).$$

To this end, we use the isomorphisms coming from the unitality and associativity of $\wedge_S$ and $\wedge_R$:

$$S \wedge_R (M \wedge_R N) \cong S \wedge_R ((M \wedge_S S) \wedge_R N) \cong (S \wedge_R M) \wedge_S (S \wedge_R N).$$

□

Corollary 4.1.4. *If E is a commutative ring G-spectrum and X is an $\mathbb{O}$-algebra in G-spectra, then $E \wedge X$ is an $\mathbb{O}$-algebra in $\mathcal{M}od_E$.*

By definition we have that

$$E_V(A) = [S^V, E \wedge A]_{G\mathcal{S}\mathrm{p}}$$

so we can construct natural operations on the E-homology of $\mathbb{O}$-algebras by finding natural operations on the G-homotopy groups of $\mathbb{O}$-algebras in $\mathcal{M}\mathrm{od}_E$.

Recall from Definition 1.8.8 the construction of the free monad $F_{\mathbb{O}}$ associated to the operad $\mathbb{O}$.

Definition 4.1.5. Let X be a G-spectrum and $\mathbb{O}$ an operad in G-spectra. We define *extended power construction on X* to be

$$\mathrm{Sym}^k_{\mathbb{O}}(X) := \mathbb{O}(k) \wedge_{\Sigma_k} X^{\wedge k}.$$

Definition 4.1.6. The *associated free $\mathbb{O}$-algebra* functor is defined as

$$F_{\mathbb{O}}(X) := \bigvee_{k \geq 0} \mathrm{Sym}^k_{\mathbb{O}}(X).$$

Remark 4.1.7. Note that the structure of an $\mathbb{O}$-algebra on X is equivalent to the existence of a map (see [**47**, Proposition 2.8])

$$\theta \colon F_{\mathbb{O}}(X) \to X.$$

Proposition 4.1.8 (See [**38**, Proposition 19.4.9]). *Suppose that E is a commutative ring G-spectrum, $\mathbb{O}$ is an operad with associated monad denoted by $F_{\mathbb{O}}$. Then there is a natural isomorphism*

$$\pi^G_V(A) \cong \left[E \wedge F_{\mathbb{O}}(S^V), A\right]_{\mathcal{A}lg_{\mathbb{O}}(\mathcal{M}od_E)}$$

for A in the homotopy category of $\mathcal{A}lg_{\mathbb{O}}(\mathcal{M}od_E)$. In particular, the object $E \wedge F_{\mathbb{O}}(S^V)$ is a corepresenting object for the functor $\pi^G_V \colon \mathcal{A}lg_{\mathbb{O}}(\mathcal{M}od_E) \to \mathcal{A}b$.

Corollary 4.1.9 (See [**38**, Corollary 19.4.11]). *Let $\mathrm{Op}^E_{\mathbb{O}}(V, W)$ denote the group of E-homology operations $E_V \to E_W$. There is an isomorphism*

$$\mathrm{Op}^E_{\mathbb{O}}(V, W) \cong E_W(F_{\mathbb{O}}(S^V)).$$

Proof. Follows easily from the Yoneda lemma, as natural transformations of functors $\pi_V^G \to \pi_{V+W}^G$ are in a bijective correspondence with $\pi_{V+W}^G(E \wedge F_{\mathbb{O}}(S^V)) \cong E_{V+W}(F_{\mathbb{O}}(S^V))$. □

Definition 4.1.10 (See [**38**, Definition 19.4.12.]). Let $k \geq 0$. A *power operation of weight k* is an element of the subgroup

$$\mathrm{Op}_{\mathbb{O}}^E(V,W)^{\langle k\rangle} := E_W(\mathrm{Sym}_{\mathbb{O}}^k(S^V))$$

of $\mathrm{Op}_{\mathbb{O}}^E(V,W)$.

Definition 4.1.11 (See [**38**, Definition 19.4.17.]). Let E be a commutative ring G-spectrum and let $k \geq 0$. The group of *power operations of weight k on degree V* for $\mathbb{O}$-algebras in $\mathcal{M}\mathrm{od}_E$ is the graded abelian group

$$\mathrm{Pow}_{\mathbb{O}}^E(V,k) := \pi_\star^G\left(F(S^V, E \wedge \mathrm{Sym}_O^k(S^V))\right) \cong \bigoplus_{W \in RO(G)} \mathrm{Op}_{\mathbb{O}}^E(V, V+W)^{\langle k\rangle}.$$

Now we are going to prove a proposition, which will be useful in later considerations.

Proposition 4.1.12. *Let X be an $\mathbb{O}$-algebra, $x \in E_V^Q(X)$ be a homology class and let $\kappa_1, \kappa_2 \in E_W^Q(\mathrm{Sym}_{\mathbb{O}}^k(S^V))$. If we denote by $p_\kappa^{\langle k\rangle}$ the power operation of weight k associated to the class κ, then*

$$p_{\kappa_1+\kappa_2}^{\langle k\rangle}(x) = p_{\kappa_1}^{\langle k\rangle}(x) + p_{\kappa_2}^{\langle k\rangle}(x).$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
S^W \arrow[r, "\nabla"] \arrow[d, "\kappa_1+\kappa_2"] & S^W \vee S^W \arrow[d, "\kappa_1 \vee \kappa_2"] \\
E \wedge \mathrm{Sym}_{\mathbb{O}}^k(S^V) \arrow[d, "\mathrm{Sym}_{\mathbb{O}}^k(x)"] & \left(E \wedge \mathrm{Sym}_{\mathbb{O}}^k(S^V)\right) \vee \left(E \wedge \mathrm{Sym}_{\mathbb{O}}^k(S^V)\right) \arrow[l, "\mathrm{fold}"'] \arrow[d, "\mathrm{Sym}_{\mathbb{O}}^k(x) \vee \mathrm{Sym}_{\mathbb{O}}^k(x)"] \\
E \wedge \mathrm{Sym}_{\mathbb{O}}^k(E \wedge X) \arrow[d, "(\mu \wedge X)\circ(E \wedge \theta)"] & \left(E \wedge \mathrm{Sym}_{\mathbb{O}}^k(E \wedge X)\right) \vee \left(E \wedge \mathrm{Sym}_{\mathbb{O}}^k(E \wedge X)\right) \arrow[l, "\mathrm{fold}"'] \\
E \wedge X &
\end{tikzcd}$$

In this diagram the last left vertical map is the composition of the map coming from the action of the operad

$$E \wedge \theta \colon E \wedge \mathrm{Sym}_{\mathbb{O}}^k(E \wedge X) \to E \wedge E \wedge X$$

with the multiplication

$$\mu \wedge X \colon E \wedge E \wedge X \to E \wedge X.$$

The left-hand column in the diagram expresses the operation $p_{\kappa_1+\kappa_2}(x)$ and the right-hand column expresses the class $p_{\kappa_1}(x) + p_{\kappa_2}(x)$. The commutativity of this diagram follows from the definition of the sum of homotopy classes. Therefore the claim follows. □

4.2. Power operations in $\mathcal{D}_\sigma$-algebras

As a direct application of the theory given in the previous section we are going to describe the power operations in the algebras over the $\mathcal{D}_\sigma$-operad.

4.2.1. Extended power constructions over $\mathcal{D}_\sigma$. Recall that by Definition 4.1.11 power operations in E-homology are parametrized by the E-homology of the extended power constructions on representation spheres. We will begin by showing how the extended power over $\mathcal{D}_\sigma$ is connected with the norm construction, see Definition 1.6.22.

Observation 4.2.1. Recall from Observation 2.7.1 that $\Gamma_{\mathfrak{m}}$ stands for the graph subgroup of $Q \times \Sigma_n$ in which γ hits the product of disjoint transpositions of maximal length. Then the $Q \times \Sigma_n$-space $\mathcal{D}_\sigma(n)$ is $Q \times \Sigma_n$-equivalent to the $Q \times \Sigma_n$-set

$$\frac{Q \times \Sigma_n}{\Gamma_{\mathfrak{m}}}.$$

Proposition 4.2.2. *Let X be a Q-spectrum and let*

$$T = a(Q/e) \amalg b(Q/Q)$$

be a Q-set consisting of a free orbits and b fixed points. Let Γ_T be a corresponding graph subgroup of $Q \times \Sigma_{2a+b}$. Then

$$\left(\frac{Q \times \Sigma_{2a+b}}{\Gamma_T}\right)_+ \wedge_{\Sigma_{2a+b}} X^{\wedge 2a+b} \cong X^{\wedge b} \wedge \bigwedge_a N^Q(\mathrm{res}^Q_e X).$$

Proof. It is a particular case of [**4**, Proposition 6.2]. □

Proposition 4.2.3. *Let V be a finite dimensional orthogonal Q-representation. Then there is a Q-homotopy equivalence:*

$$\mathrm{Sym}^k_{\mathcal{D}_\sigma}(S^V) = \mathcal{D}_\sigma(k)_+ \wedge_{\Sigma_k} (S^V)^{\wedge k} \cong \begin{cases} S^{V \otimes (k/2)\rho} & \text{if } k \text{ is even;} \\ S^{V \otimes ((\frac{k-1}{2})\rho + 1)} & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Recall from Proposition 1.6.23 that $N^Q(\mathrm{res}^Q_e(S^V)) \cong S^{V \otimes \rho}$. Therefore the result follows from Proposition 4.2.2. □

Remark 4.2.4. Note that there is an isomorphism of Q-representations

$$V \otimes \rho \cong (\dim V)\rho.$$

This comes from the following fact in representation theory: if U is a representation of a finite group G and W is a representation of a subgroup $H \leq G$, then they are connected by the Frobenius reciprocity:

$$U \otimes \mathrm{ind}^G_H W \cong \mathrm{ind}^G_H(\mathrm{res}^G_H(U) \otimes V).$$

We will abbreviate $V \otimes \rho \cong (\dim V)\rho$ as $V\rho$.

4.2.2. Power operations in $\mathcal{D}_\sigma$-algebras. In this section we are going to describe the power operations in Bredon homology coming from the action $\mathcal{D}_\sigma$. Throughout this section, $\underline{M}$ is a Green functor over Q. If X is a Q-spectrum, we are going to use NX to denote $N^Q(\operatorname{res}^Q_e X)$ (see Definition 1.6.22).

Remark 4.2.5. Note that by Proposition 4.2.2 every $\mathcal{D}_\sigma$-algebra in Q-spectra X is endowed with a *norm map* $\nu\colon NX \to X$ and the action of the norm $\xi\colon NX \wedge X \to X$. See [**27**, Proposition 2.2].

We note here the important subtlety. If X is a commutative ring G-spectrum, there is a map $NX \to X$ being the counit of the adjunction given in Proposition 1.6.23. However, above we observed that in order to be endowed with such a map X does not need to be a commutative ring G-spectrum - it is enough to be a $\mathcal{D}_\sigma$-algebra.

Definition 4.2.6. Let X be a $\mathcal{D}_\sigma$-algebra in Q-spectra. We define the *norming operation in* $\pi^Q_\star(X)$ to be the map

$$n\colon \pi^Q_\star(X) \to \pi^Q_{\star\otimes\rho}(X)$$

where for $x \in \pi^Q_V(X)$ we define $n(x)$ to be the homotopy class of

$$N(S^V) \cong S^{V\rho} \xrightarrow{Nx} N(X) \xrightarrow{\nu} X.$$

Remark 4.2.7. We emphasize here the notational difference: if x is a class in homotopy or homology, by $n(x)$ we mean the norm of element x according to the definition given above. On the other hand, by $N(x)$ we denote the value of the functor N on the map $x\colon S^V \to X$.

Proposition 4.2.8.

$$H\underline{M}^Q_\star(\operatorname{Sym}^k_{\mathcal{D}_\sigma}(S^V)) \cong \begin{cases} H\underline{M}^Q_{\star-(k/2)V\otimes\rho} & \textit{if } k \textit{ is even;} \\ H\underline{M}^Q_{\star-\left((\frac{k-1}{2})V\otimes\rho-V\right)} & \textit{if } k \textit{ is odd.}\end{cases}$$

Proof. Follows from Proposition 4.2.3 and properties of $RO(Q)$-graded homology. □

From this proposition we see that all power operations in $\mathcal{D}_\sigma$-algebras are parametrised by the elements of $H\underline{M}^Q_\star$.

Definition 4.2.9. Let $\kappa \in H\underline{M}^Q_\star$. Then by $p^{\langle k\rangle}_\kappa$ we are going to denote the corresponding power operation of weight k:

$$p^{\langle k\rangle}_\kappa \in \operatorname{Pow}^{H\underline{M}}_{\mathcal{D}_\sigma}(V,k) \cong H\underline{M}^Q_{\star+V-V\rho}.$$

Here V is any element of $RO(Q)$.

In the case of $\mathcal{D}_\sigma$-algebras it is possible to give an explicit description of power operations of any weight.

4.2.2.1. *Power operations of weight 2.* We begin with the description of power operations of weight 2.

Proposition 4.2.10. *Let X be a $\mathcal{D}_\sigma$-algebra in Q-spectra, $x \in H\underline{M}^Q_\star(X)$ and $\kappa \in H\underline{M}^Q_\star$. Then*

$$p^{\langle 2\rangle}_\kappa(x) = \kappa n(x).$$

We are going to prove this proposition in two lemmas.

Lemma 4.2.11. *Let ι be the unit element of $H\underline{M}^Q_0 = \underline{M}(Q/Q)$ and $x \in H\underline{M}^Q_\star(X)$. Then $p^{\langle 2\rangle}_\iota(x) = n(x)$.*

Proof. Note that $\iota \in H\underline{M}^Q_0 \cong H\underline{M}^Q_{V\rho}(S^{V\rho}) \cong H\underline{M}^Q_{V\rho}(N(S^V))$ is the element given by the unit of the ring G-spectrum $\eta\colon S^0 \to H\underline{M}$. Thus the operation $p^{\langle 2\rangle}_\iota$ is given by the following sequence, where μ is a multiplication in $H\underline{M}$:

$$\begin{tikzcd}
S^{V\rho} \arrow[r, "\iota"] & H\underline{M} \wedge \mathrm{Sym}^2_{\mathcal{D}_\sigma}(S^V) \arrow[r, "H\underline{M}\wedge N(x)"] \arrow[d, phantom, "\cong"] & H\underline{M} \wedge \mathrm{Sym}^2_{\mathcal{D}_\sigma}(H\underline{M} \wedge X) \arrow[r] \arrow[d, phantom, "\cong"] & H\underline{M} \wedge H\underline{M} \wedge X \arrow[d, "\mu\wedge\mathrm{id}"] \\
& H\underline{M} \wedge N(S^V) & H\underline{M} \wedge N(H\underline{M} \wedge X) & H\underline{M} \wedge X.
\end{tikzcd}$$

This sequence indeed expresses a norm of element x since it is actually the sequence defining the norm of x from Definition 4.2.6 smashed on the left with the unit of the spectrum $H\underline{M}$. By the properties of ring G-spectra, we have that the map $(\mu \wedge X) \circ (\eta \wedge N(x))$ is homotopic to $N(x)$. Therefore the claim follows. □

Lemma 4.2.12. *Let $\kappa \in H\underline{M}^Q_W$ and $x \in H\underline{M}^Q_\star(X)$. Then*

$$p^{\langle 2\rangle}_\kappa(x) = \kappa p^{\langle 2\rangle}_\iota(x).$$

Proof. The power operation $p^{\langle 2\rangle}_\kappa$ on x is expressed by the following sequence:

$$\begin{tikzcd}
S^{V\rho+W} \arrow[r, "\kappa"] & H\underline{M} \wedge N(S^V) \arrow[r, "N(x)"] & H\underline{M} \wedge N(H\underline{M} \wedge X) \arrow[r, "\mathrm{id}\wedge\nu"] & H\underline{M} \wedge H\underline{M} \wedge X \arrow[d, "\mu\wedge\mathrm{id}"] \\
& & & H\underline{M} \wedge X.
\end{tikzcd}$$

However, this sequence may be equivalently expressed as the following:

$$\begin{tikzcd}
S^W \wedge S^{V\rho} \arrow[r, "\kappa\wedge\iota"] & H\underline{M} \wedge H\underline{M} \wedge N(S^V) \arrow[r, "\mathrm{id}\wedge\mathrm{id}\wedge N(x)"] & H\underline{M} \wedge H\underline{M} \wedge N(H\underline{M} \wedge X) \arrow[d, "\mathrm{id}\wedge\mathrm{id}\wedge\nu"] \\
H\underline{M} \wedge X & H\underline{M} \wedge H\underline{M} \wedge X \arrow[l, "\mu\wedge\mathrm{id}"'] & H\underline{M} \wedge H\underline{M} \wedge H\underline{M} \wedge X \arrow[l, "\mathrm{id}\wedge\mu\wedge\mathrm{id}"']
\end{tikzcd}$$

This sequence expresses the class $p^{\langle 2\rangle}_\kappa(x)$ as $\kappa p^{\langle 2\rangle}_\iota(x)$. Thus the claim follows. □

Proof of Proposition 4.2.10. Follows from Lemmas 4.2.11 and 4.2.12. □

4.2.2.2. *Power operations of higher weights.* Using the same technique as was used to prove Proposition 4.2.10, we can obtain the following general statement.

PROPOSITION 4.2.13. *Let X be a $\mathcal{D}_\sigma$-algebra in Q-spectra, $k \leq 2$, $\kappa \in H\underline{M}_\star^Q$ and $x \in H\underline{M}_\star^Q(X)$. Then*

$$p_\kappa^{\langle k \rangle}(x) = \begin{cases} n(x)^{k/2} & \text{if } k \text{ is even;} \\ n(x)^{\frac{k-1}{2}} x & \text{if } k \text{ is odd.} \end{cases}$$

4.3. Power operations in $\mathcal{D}_{2\sigma}$-algebras

In this chapter we are going to define Dyer-Lashof operations in the $RO(Q)$-graded $H\underline{\mathbb{F}}_2$-homology of $\mathcal{D}_{2\sigma}$-algebras. These are going to be power operations of weight 2. Throughout the whole chapter we abbreviate $H\underline{\mathbb{F}}_2$ to $H\underline{\mathbb{F}}$.

4.3.1. Spectral sequence associated to a filtered spectrum. In the beginning we need to provide a description of a trigraded spectral sequence which will be used to compute the Bredon homology of extended power construction. This section is based on [**22**, Appendix B].

DEFINITION 4.3.1. Let G be a finite group and X be a G-spectrum. Then *a filtration on X* consists of an increasing family of subspectra $\{X^p \mid p \in \mathbb{Z}\}$ such that:

(1) $X = \bigcup_p X^p$;
(2) the map $X^p \to X^{p+1}$ is a cofibration for all $p \in \mathbb{Z}$;
(3) the map $X^p \to X$ is a cofibration for all $p \in \mathbb{Z}$.

CONSTRUCTION 4.3.2 (The spectral sequence associated to a filtered spectrum). Let X be a G-spectrum endowed with a filtration by $\{X^p \mid p \in \mathbb{Z}\}$. Denote by $\bar{X}^p$ the cofibre of the map $X^{p-1} \to X^p$. Then there is the following sequence of cofibre sequences:

$$X^{p-1} \to X^p \to \bar{X}^p \to \Sigma X^{p-1}$$

and the associated exact couple:

$$\begin{aligned} D_{pq}^1 &= \pi_{p+q}^G(X^p \wedge Y) \\ E_{pq}^1 &= \pi_{p+q}^G(\bar{X}^p \wedge Y). \end{aligned}$$

This spectral sequence, called *the spectral sequence associated to the filtered spectrum X*, is relevant for computations of $\pi_*^G(X \wedge Y)$. We will refer to Y as a *coefficient spectrum*. The d^1-differential in the spectral sequence is induced by the geometric boundary map:

$$\partial \colon \bar{X}^p \to \Sigma X^{p-1} \to \Sigma \bar{X}^{p-1}.$$

If X and Y have bounded below G-homotopy groups, then the spectral sequence described above is strongly convergent to $\pi_*^G(X \wedge Y)$. Since all G-spectra which we will consider fall into this case, we are not going to discuss convergence further. For the in-depth discussion of the convergence see [**22**, Appendix B] or [**8**].

4.3.2. Bredon homology of extended power construction over $\mathcal{D}_{2\sigma}$ on representation spheres. In this section we are going to compute the $H\underline{\mathbb{F}}$-homology of $\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V)$ for any finite dimensional orthogonal representation V. This is given by the following Theorem:

THEOREM 4.3.3. *Let V be a finite dimensional orthogonal representation. Then*

$$H_\star(\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V);\underline{\mathbb{F}}_2) \cong H\underline{\mathbb{F}}^Q_\star\{e^{V\rho}\} \oplus H\underline{\mathbb{F}}^Q_\star\{e^{V\rho+1}\},$$

where $e^{V\rho}$ and $e^{V\rho+1}$ are elements of degrees respectively $V\rho$ and $V\rho+1$.

We are going to prove this theorem by a series of lemmas.

OBSERVATION 4.3.4. Note that $\mathcal{D}_{2\sigma}(2) \simeq S(2\sigma\otimes\tau)$ as a $Q\times\Sigma_2$-space, where τ denotes the real sign representation of Σ_2. Therefore $\mathcal{D}_{2\sigma}(2)$ has a filtration given by

$$F_0 = \tfrac{Q\times\Sigma_2}{\Gamma_m}$$
$$F_1 = S(2\sigma\otimes\tau).$$

For any Q-spectrum X the filtration above induces a filtration on $\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(X)$ such that

$$\mathrm{gr}_0(X) = F_0(X) = \tfrac{Q\times\Sigma_2}{\Delta}_+ \wedge_{\Sigma_2} X^{\wedge 2} \simeq N^Q X$$
$$\mathrm{gr}_1 = F_1(X)/F_0(X) = (S^1\wedge \tfrac{Q\times\Sigma_2}{\Delta}_+)\wedge_{\Sigma_2} X^{\wedge 2} \simeq S^1\wedge N^Q X.$$

We are going to prove Theorem 4.3.3 using the spectral sequence associated to the filtered spectrum $\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V)$, taking as a coefficient spectrum $H\underline{\mathbb{F}}\wedge S^{n\sigma}$ for $n\in\mathbb{Z}$. The E^1-page of this spectral sequence consists of two columns:

$$E^1_{0q} = H\underline{\mathbb{F}}^Q_{q-n\sigma}(\mathrm{gr}_0(S^V))$$
$$E^1_{1q} = H\underline{\mathbb{F}}^Q_{q+1-n\sigma}(\mathrm{gr}_1(S^V)).$$

We are going to prove that d_1-differentials in this spectral sequence are 0 and therefore the spectral sequence degenerates on the E^1-page.

LEMMA 4.3.5. *The d^1-differential in the spectral sequence associated to the filtered spectrum $\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V)$ with the coefficient spectrum $H\underline{\mathbb{F}}\wedge S^{n\sigma}$ is zero.*

PROOF. Fix n. Define the following map:

$$\partial_V\colon \left(S^1\wedge \frac{Q\times\Sigma_2}{\Delta}_+\right)\wedge_{\Sigma_2}(S^V)^2 \to S^1\wedge\left(\frac{Q\times\Sigma_2}{\Delta}_+\wedge_{\Sigma_2}(S^V)^2\right)$$

induced by the cofibre sequence:

$$\tfrac{Q\times\Sigma_2}{\Delta}_+\wedge_{\Sigma_2}\left(S^V\right)^{\wedge 2} \longrightarrow S\left(2(\sigma\otimes\tau)\right)_+\wedge_{\Sigma_2}\left(S^V\right)^{\wedge 2} \longrightarrow \left(S^1\wedge\tfrac{Q\times\Sigma_2}{\Delta}_+\right)\wedge_{\Sigma_2}\left(S^V\right)^{\wedge 2}$$

By Construction 4.3.2 we have that the d^1-differential is induced in homology by the map ∂_V:

$$\partial_{V*}\colon H\underline{\mathbb{F}}^Q_{*-1-n\sigma}(N^Q S^V)\to H\underline{\mathbb{F}}^Q_{*-1-n\sigma}(N^Q S^V).$$

The fact that ∂_{V*} is zero is Corollary 4.3.7 of Lemma 4.3.6. □

Lemma 4.3.6. *The map $\partial_0 \colon S^1 \longrightarrow S^1$ defines an element of $\pi_0^Q(\mathbb{S})$ corresponding to* 0.

Proof. The map $\partial_0 \colon S^1 \to S^1$ comes from passing to Σ_2-orbits in the $Q \times \Sigma_2$-map

$$\tilde{\partial} \colon S^1 \wedge \frac{Q \times \Sigma_2}{\Delta}_+ \longrightarrow S^1 \wedge \frac{Q \times \Sigma_2}{\Delta}_+.$$

This is the connecting map in the following $Q \times \Sigma_2$-cofibre sequence:

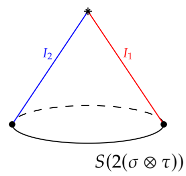
$$\tfrac{Q\times\Sigma_2}{\Delta}_+ \xrightarrow{\ i\ } S\left(2(\sigma \otimes \tau)\right)_+ \longrightarrow S^1 \wedge \tfrac{Q\times\Sigma_2}{\Delta}_+.$$

Consider the $Q \times \Sigma_2$-representation $2(\sigma \otimes \tau) \times \mathbb{R}$. We embed the orbit $\frac{Q\times\Sigma}{\Delta}_+$ as a set

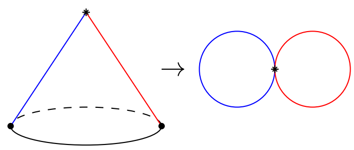
$$\{(1,0,0), (-1,0,0), (0,0,1)\}$$

where the action of Q and Σ_2 is given by the sign and $(0,0,1)$ is taken as the basepoint. Analogically we embed $S\left(2(\sigma \otimes \tau)\right)_+$ as the unit sphere in $2(\sigma \otimes \tau)$ (i.e., vectors of the form $(x, y, 0)$ of the norm 1) with disjoint basepoint $(0, 0, 1)$.

With these identifications we can see the cone of i as $S\left(2(\sigma \otimes \tau)\right)_+$ with added intervals $I_1 = (t, 0, 1-t)$ and $I_2 = (-t, 0, 1-t)$ for $t \in [0, 1]$:



The map $\tilde{\partial}$ is given by collapsing $S\left(2(\sigma \otimes \tau)\right)_+$ in the cone above to the point, so it may be depicted as follows:



Let $H \colon C(i) \wedge I_+ \longrightarrow C(i)$ be a $Q \times \Sigma_2$-homotopy collapsing the intervals I_1 and I_2 to the point, i.e., the homotopy between the identity and the map constant at I_1 and I_2 and properly extended on the rest of $C(i)$.

The homotopy H makes the map $\tilde{\partial}$ nullhomotopic. Since $\partial_0 = \tilde{\partial}/\Sigma_2$, the claim follows. □

Corollary 4.3.7. *The map $H_\star^Q(N^Q S^V; \underline{\mathbb{F}}_2) \to H_\star^Q(N^Q S^V; \underline{\mathbb{F}}_2)$ induced by ∂_V is zero.*

Proof. Let ρ_{Σ_2} be a regular representation of Σ_2 and note that if V is a Q-representation, then $(S^V)^2$ with Σ_2 action given by twisting is $Q \times \Sigma_2$-homeomorphic to $S^{V \otimes \rho_{\Sigma_2}}$.

Recall the map $\tilde{\partial}$ from the proof of Lemma 4.3.6:

$$\tilde{\partial}\colon S^1 \wedge \frac{Q \times \Sigma_2}{\Delta}_+ \to S^1 \wedge \frac{Q \times \Sigma_2}{\Delta}_+.$$

Define the map

$$\tilde{\partial}_V\colon S^1 \wedge \frac{Q \times \Sigma_2}{\Delta}_+ \wedge S^{V \otimes \rho_{\Sigma_2}} \to S^1 \wedge \frac{Q \times \Sigma_2}{\Delta}_+ \wedge S^{V \otimes \rho_{\Sigma_2}}$$

by $\tilde{\partial}_V = \tilde{\partial} \wedge S^{V \otimes \rho_{\Sigma_2}}$.

Note that ∂_V is a map induced on Σ_2-orbits by $\tilde{\partial}_V$. Therefore after passing to Σ_2-orbits ∂_V defines the same element of $\pi_0^Q(\mathbb{S})$ as ∂_0. So the map induced by ∂_V in $\underline{\mathbb{F}}_2$-homology is zero. $\square$

Proof of Theorem 4.3.3. By Lemma 4.3.5 we see that the spectral sequence associated to the filtered spectrum $\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V)$ with coefficient spectra $H\underline{\mathbb{F}} \wedge S^{n\sigma}$ degenerates on the E^1-page, for it has only 2 columns and all differentials are zero. Therefore we get that

$$\begin{aligned} H^Q_{m+n\sigma}(\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V), \underline{\mathbb{F}}_2) &\cong H^Q_{m+n\sigma}\left(\mathrm{gr}_0(S^V), \underline{\mathbb{F}}_2\right) \oplus H^Q_{m+n\sigma}\left(\mathrm{gr}_1(S^V), \underline{\mathbb{F}}_2\right) \cong \\ &\cong H^Q_{m+n\sigma}\left(N^Q(S^V), \underline{\mathbb{F}}_2\right) \oplus H^Q_{m-1+n\sigma}\left(N^Q(S^V), \underline{\mathbb{F}}_2\right). \end{aligned}$$

Since this isomorphism holds for every $n \in \mathbb{Z}$ and $N^Q(S^V) \simeq S^{V\rho}$ by Proposition 1.6.23, we have that

$$H^Q_\star(\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V), \underline{\mathbb{F}}_2) \cong H\underline{\mathbb{F}}^Q_\star\{e^{V\rho}\} \oplus H\underline{\mathbb{F}}^Q_\star\{e^{V\rho+1}\}.$$

$\square$

Corollary 4.3.8. *There is a homotopy equivalence*

$$Sym^2_{\mathcal{D}_{2\sigma}}(S^V) \wedge H\underline{\mathbb{F}} \simeq \Sigma^{V\rho} H\underline{\mathbb{F}} \vee \Sigma^{V\rho+1} H\underline{\mathbb{F}}.$$

4.3.3. Power operations of weight 2 in $H\underline{\mathbb{F}}$-homology for $\mathcal{D}_{2\sigma}$-algebras. In this section we are going to describe the power operations of weight 2 in the $H\underline{\mathbb{F}}$-homology of $\mathcal{D}_{2\sigma}$-algebras.

Definition 4.3.9.

(1) Let $\kappa \in H^Q_\star(\mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V), \underline{\mathbb{F}}_2)$. Then by p_κ we are going to denote the corresponding power operation of weight 2 (compare Definition 4.2.9).
(2) Let X be a $\mathcal{D}_{2\sigma}$-algebra in Q-spectra and $x \in H^Q_V(X, \underline{\mathbb{F}}_2)$. We define

$$P_0(x) := p_{e^{V\rho}}(x)$$
$$P_1(x) := p_{e^{V\rho+1}}(x)$$

Now we are going to discuss several properties of these operations.

Proposition 4.3.10. *Let $x \in H^Q_V(X, \underline{\mathbb{F}}_2)$ for some $\mathcal{D}_{2\sigma}$-algebra X and let*

$$\kappa \in H\underline{\mathbb{F}}^Q_\star\{e^{V\rho}\} \oplus H\underline{\mathbb{F}}^Q_\star\{e^{V\rho+1}\} \cong H^Q_\star(Sym^2_{\mathcal{D}_{2\sigma}}(S^V))$$

be such that $\kappa = \alpha e^{V\rho} \oplus \beta e^{V\rho+1}$. *Then*

$$p_\kappa(x) = \alpha P_0(x) + \beta P_1(x).$$

Proof. The fact that $p_{\alpha e^{V\rho}}(x) = \alpha P_0(x)$ follows from a similar argument to the proof of Lemma 4.2.12, likewise for P_1. The distributivity over sum comes from Proposition 4.1.12. □

Proposition 4.3.11. *Let* $x \in H\underline{\mathbb{F}}^Q_V(X)$. *Then* $P_0(x) = n(x)$, *the norm of the class* x.

Proof. Recall that P_0 is defined as the power operation associated to $e^{V\rho}$. Consider the following commutative diagram:

$$\begin{tikzcd}
S^{V\rho} \arrow[d, "e^{V\rho}"] & & \\
H\underline{\mathbb{F}} \wedge S^{V\rho} \simeq H\underline{\mathbb{F}} \wedge N(S^V) \arrow[r, "i"] \arrow[d, "H\underline{\mathbb{F}}\wedge N(x)"] & H\underline{\mathbb{F}} \wedge \mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V) \arrow[d, "H\underline{\mathbb{F}}\wedge \mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(S^V)(x)"] & \\
H\underline{\mathbb{F}} \wedge N(H\underline{\mathbb{F}} \wedge X) \arrow[r, "i"] & H\underline{\mathbb{F}} \wedge \mathrm{Sym}^2_{\mathcal{D}_{2\sigma}}(H\underline{\mathbb{F}} \wedge X) \arrow[r, "(\mu\wedge X)\circ(H\underline{\mathbb{F}}\wedge\theta)"] & H\underline{\mathbb{F}} \wedge X.
\end{tikzcd}$$

Here i is inclusion induced by the filtration given in Observation 4.3.4. The right-hand bottom map is the composition of the operadic action θ and the multiplication μ in $H\underline{\mathbb{F}}$.

Similarly as in the proof of Lemma 4.2.11, the claim follows from the diagram above. □

4.4. Browder type operations for $\mathcal{D}_{n\sigma}$-algebras

The next type of operations which we will discuss are binary operations, similar to the Browder operations defined for algebras over the non-equivariant little discs operads.

4.4.1. Homology of the sphere with the antipodal action. Before defining operations of Browder type, we need to compute $H^Q_\star(\mathcal{D}_{n\sigma}(2)_+, \underline{\mathbb{F}}_2)$. To this end, we use the following cofibre sequence:

$$(\dagger) \qquad S(n\sigma)_+ \to S^0 \to S^{n\sigma}.$$

The second map in this sequence is iterated multiplication by a, see Section 3.2.

Proposition 4.4.1.

$$H^Q_\star(S(n\sigma)_+, \underline{\mathbb{F}}_2) = \left(\mathbb{F}_2[a^{-1}]/a^{-n}\right)[\lambda, \lambda^{-1}].$$

Proof. Fix n. After smashing the sequence (†) with $H\underline{\mathbb{F}}$ we get the following cofibre sequence:

$$H\underline{\mathbb{F}} \wedge S(n\sigma)_+ \longrightarrow H\underline{\mathbb{F}} \xrightarrow{a^n} \Sigma^{n\sigma} H\underline{\mathbb{F}}.$$

After applying $[S^V, -]^Q$ we get the following long exact sequence:

$$\ldots \longrightarrow H\underline{\mathbb{F}}^Q_{V+n\sigma+1} \longrightarrow H^Q_V\left(S(n\sigma)_+, \underline{\mathbb{F}}_2\right) \longrightarrow H\underline{\mathbb{F}}^Q_V \xrightarrow{a^n} H\underline{\mathbb{F}}^Q_{V+n\sigma} \longrightarrow \ldots.$$

We consider two cases, depending on the fixed dimension of $V = x + y\sigma$:

(1) $x \geq 0$. We further divide this case into the following:
  (a) $y < -x$. In this situation maps $a^n \colon H\underline{\mathbb{F}}_V^Q \to H\underline{\mathbb{F}}_{V-n\sigma}^Q$ and $a^n \colon H\underline{\mathbb{F}}_{V+1}^Q \to H\underline{\mathbb{F}}_{V-n\sigma+1}^Q$ are isomorphisms, so $H_V^Q\left(S(n\sigma)_+, \underline{\mathbb{F}}_2\right) = 0$.
  (b) $x \leq y < x + n$. Then $H\underline{\mathbb{F}}_V^Q = H\underline{\mathbb{F}}_{V+1}^Q = 0$ and $H\underline{\mathbb{F}}_{V-n\sigma}^Q = H\underline{\mathbb{F}}_{V+1-n\sigma}^Q = \mathbb{F}_2$, so $H_V^Q\left(S(n\sigma)_+, \underline{\mathbb{F}}_2\right) = \mathbb{F}_2$.
  (c) $x + n \leq y$. Here $H\underline{\mathbb{F}}_V^Q = H\underline{\mathbb{F}}_{V+1-n\sigma}^Q = 0$, thus $H_V^Q\left(S(n\sigma)_+, \underline{\mathbb{F}}_2)\right) = 0$.

(2) $x = -1$. Here for any twisted dimension we have $H\underline{\mathbb{F}}_V^Q = 0$. Thus the only possibility for $H_V^Q\left(S(n\sigma)_+, \underline{\mathbb{F}}_2\right)$ to be non-zero is if $H\underline{\mathbb{F}}_{V+1}^Q = 0$ and $H\underline{\mathbb{F}}_{V+1-n\sigma}^Q = \mathbb{F}_2$, which happens if $1 \leq y \leq n$.

(3) $x \leq -1$. In this case we proceed as in Point (1).

This gives us the $RO(Q)$-abelian group structure of $H_V^Q\left(S(n\sigma)_+, \underline{\mathbb{F}}_2\right)$. The $H\underline{\mathbb{F}}_\star^Q$-module structure follows from the fact that it is the kernel of a map $a^n \colon H\underline{\mathbb{F}}_\star^Q \to H\underline{\mathbb{F}}_\star^Q$. □

Remark 4.4.2. Note that the same result may be obtained from the simple observation that $S(n\sigma)$ consists of free Q-cells of degrees 0 up to $n-1$.

4.4.2. Operations of Browder type.

Definition 4.4.3. Let ϕ be the generator of $H\underline{\mathbb{F}}_{n-1}^Q(S(n\sigma)) = \mathbb{F}_2$. Let X be a $\mathcal{D}_{n\sigma}$-algebra in Q-spaces and let $\alpha \in H_V^Q(X, \underline{\mathbb{F}}_2)$ and $\beta \in H_W^Q(X, \underline{\mathbb{F}}_2)$. We define the *Browder bracket of classes α and β* to be the class $[\alpha, \beta] := \theta_\star(\phi \otimes \alpha \otimes \beta) \in H_{V+W+n-1}^Q(X, \underline{\mathbb{F}}_2)$.

Remark 4.4.4. Note that here the choice of a class ϕ in the definition of the Browder bracket is completely arbitrary - we may choose any class of the total degree $n-1$. However, we show below that the choice does not matter.

Proposition 4.4.5. *For any classes $\alpha, \beta \in H_\star^Q(X, \underline{\mathbb{F}}_2)$ we have*

$$a[\alpha, \beta] = [a\alpha, \beta] = [\alpha, a\beta]$$
$$\lambda[\alpha, \beta] = [\lambda\alpha, \beta] = [\alpha, \lambda\beta].$$

Proof. The formula for multiplication by a follows from the following commutative diagram:

$$\begin{tikzcd}
\Sigma^{n-1}H\underline{\mathbb{F}} \wedge \Sigma^V H\underline{\mathbb{F}} \wedge \Sigma^W H\underline{\mathbb{F}} \arrow[d, "\phi\wedge\alpha\wedge\beta"] & \\
(H\underline{\mathbb{F}} \wedge \mathcal{D}_{n\sigma}(2)_+) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \arrow[r, "\theta"] \arrow[d, "\mathrm{id}\wedge a\wedge\mathrm{id}"] & H\underline{\mathbb{F}} \wedge X \arrow[ddd, "a"] \\
(H\underline{\mathbb{F}} \wedge \mathcal{D}_{n\sigma}(2)_+) \wedge_{H\underline{\mathbb{F}}} (\Sigma^\sigma H\underline{\mathbb{F}} \wedge X) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \arrow[d, "\mathrm{id}\wedge\beta\wedge\mathrm{id}"] & \\
(H\underline{\mathbb{F}} \wedge \mathcal{D}_{n\sigma}(2)_+) \wedge_{H\underline{\mathbb{F}}} (\Sigma^\sigma H\underline{\mathbb{F}}) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \arrow[d, "\mathrm{twist}"] & \\
(\Sigma^\sigma H\underline{\mathbb{F}}) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge \mathcal{D}_{n\sigma}(2)_+) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X) \arrow[r, "(\mu\wedge X)\circ(\Sigma^\sigma H\underline{\mathbb{F}}\wedge\theta)"] & \Sigma^\sigma H\underline{\mathbb{F}} \wedge X.
\end{tikzcd}$$

Here μ is the multiplication in the Q-spectrum $H\underline{\mathbb{F}}$ and the map β is the following composition:

$$\Sigma^\sigma H\underline{\mathbb{F}} \wedge X \to (\Sigma^\sigma H\underline{\mathbb{F}} \wedge_{H\underline{\mathbb{F}}} H\underline{\mathbb{F}}) \wedge X \to \Sigma^\sigma H\underline{\mathbb{F}} \wedge_{H\underline{\mathbb{F}}} (H\underline{\mathbb{F}} \wedge X).$$

The commutativity of the diagram comes from the fact that the twist map is an isomorphism. Therefore we obtain that $[a\alpha, \beta] = a[\alpha, \beta]$. However, by multiplying the second factor by a and applying an appropriate twist map we get also that $[\alpha, a\beta] = a[\alpha, \beta]$. By similar argument we obtain the statement for λ. $\square$

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