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COMPUTING EULER CHARACTERISTICS USING QUANTUM FIELD THEORY

par

Michael Borinsky & Karen Vogtmann

Résumé. — This paper explains how to use quantum field theory techniques to find formal power series that encode the virtual Euler characteristics of $\text{Out}(F_n)$ and related graph complexes. Finding such power series was a necessary step in the asymptotic analysis of $\chi(\text{Out}(F_n))$ carried out in the authors' previous paper.

1. Introduction

This article is based on the second author's talk at the conference Artin Groups, CAT(0) geometry and related topics (Charneyfest) in July 2021. It is an exposition of some of the techniques that the authors used in their calculation of the Euler characteristic of the group $\text{Out}(F_n)$ [2]. By results of Kontsevich, this Euler characteristic is the same as the orbifold Euler characteristic of the *Lie graph complex* he defined in [9]. Kontsevich also defined *commutative* and *associative* graph complexes, and discussed their orbifold Euler characteristics in Section 7.B of that paper (Section 7.2 in a circulating preprint version).

We will use the term *virtual* Euler characteristic as a uniform term for both Euler characteristics mentioned in the above paragraph. It is different from the alternating sum of the Betti numbers, which we will call the *classical* Euler characteristic. The virtual Euler characteristic is used in group theory partly because it has better properties with respect to short exact sequences and partly because it has deep connections to number theory. It is defined by taking the classical Euler characteristic of a torsion-free subgroup of finite index and dividing by the index. Kontsevich defined the orbifold Euler characteristic of a graph complex on the chain level, by counting each generator (i.e. graph G) with weight $1/|\text{Aut}(G)|$.

In this paper we use the action of $\text{Out}(F_n)$ on Outer space to reduce various virtual Euler characteristic computations to graph-counting problems, then use quantum field theory (QFT) methods to solve those problems. This paper can serve as a tutorial on how this works, and in particular is an explanation of Equations (1) and (3) of [9] Section 7.B. Such QFT methods were studied and further developed in the first author's thesis [1].

As Kontsevich notes, the virtual Euler characteristic is much easier to compute than the classical Euler characteristic. However, the classical Euler characteristic is what one needs to infer information about the dimension of the actual cohomology groups. To compute the classical Euler characteristic one must incorporate information about the automorphism groups of graphs

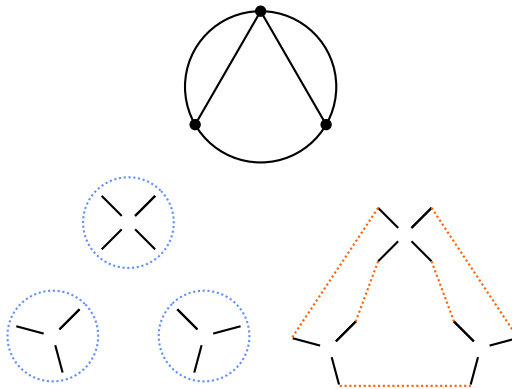


FIGURE 1. A graph G with its representation as triple (H, E, V) below. Half-edges are depicted as short black lines, the partition V is indicated with dotted blue circles and the matching E of half-edges, with dotted orange lines.

into the count. Kontsevich writes that it is reasonable to expect that ‘most’ graphs have no nontrivial automorphisms. From that remark one might expect that the two Euler characteristics of $\text{Out}(F_n)$ are asymptotically the same. However, in [3] we combine techniques described here with asymptotic analysis to prove that the ratio of the two is asymptotically equal to $e^{-\frac{1}{4}}$, so in particular the difference between the two grows as the rank n grows. In particular this solves Problem 6.5 of the paper [10] of Morita, Sakasai and Suzuki.

2. Counting admissible graphs

To a topologist, a graph is a 1-dimensional CW-complex. Finite graphs, all of whose vertices have valence at least three are called *admissible*. Note that a leaf, to a topologist, is an edge terminating in a 1-valent vertex, so an admissible graph does not have leaves. One checks easily that there are only a finite number of admissible graphs with a given Euler characteristic. We remind the reader that only connected admissible graphs occur in Outer space [7].

In this paper, however, we will use the following combinatorial definition of a graph, which is much more convenient for solving counting problems.

Definition 2.1. — A graph is a triple $G = (H, E, V)$, where H is a finite set (the half-edges), E is an involution (whose orbits are the edges and leaves of G), and V is a partition of H (the vertices of G). (See Figure 1.) An isomorphism between two graphs is a bijection between the respective sets of half-edges that commutes with the involution and respects the partitions.

Note that an isolated vertex is not considered to be a graph by this definition, but it does allow for the empty graph. Also notice the difference between univalent vertices and leaves: a univalent vertex is a block of the partition V consisting of exactly one half-edge, whereas a leaf is a half-edge that is fixed by the involution E . In this section and the next we will assume each block of V has at least 3 elements, i.e. the partition is *fat*; this corresponds to assuming all vertices of G have valence at least 3. In Sections 2–3, we will only consider graphs where the involution has no fixed points, so E is a total matching of the half-edges; in other words we only consider admissible graphs for now.

Remark 2.2. — Leaves of graphs are often called *hairs* or *legs*. The latter name is used in QFT and specifically in [1]. The name leaf is standard in more combinatorial contexts and was

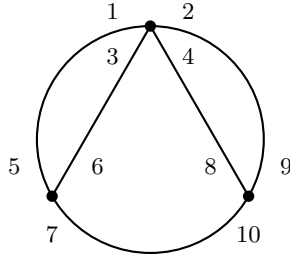


FIGURE 2. A half-edge labeled graph lG .

used in [2]. Here, we will stick to this convention, as trees, connected graphs without cycles, are central to our considerations and trees carry leaves.

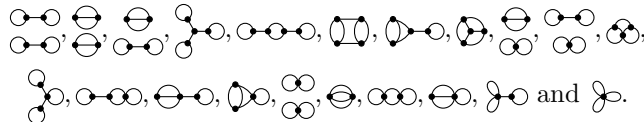
We denote the set of automorphisms of a graph G by $\text{Aut}(G)$, and by $v(G)$, $e(G)$ and $\chi(G) = v(G) - e(G)$ the number of vertices, edges and the Euler characteristic of G .

Let \mathcal{G} denote the set of (isomorphism classes of) admissible graphs, which we redefine to include the empty graph, and $\mathcal{G}(m, k)$ the (finite) subset consisting of admissible graphs with m edges and k vertices. For instance, $\mathcal{G}(2, 1) = \{\circ\circ\}$ and $\mathcal{G}(3, 2) = \{\circ\circ\circ, \ominus\}$. Since the set of admissible graphs with a given Euler characteristic is finite, the coefficient of \hbar in the following formal power series is finite, i.e. this series serves as a generating function for (isomorphism classes of) admissible graphs.

$$\begin{aligned} \mathbf{F}(\hbar) &= \sum_{G \in \mathcal{G}} \frac{\hbar^{-\chi(G)}}{|\text{Aut}(G)|} = 1 + \frac{\hbar}{|\text{Aut}(\circ\circ)|} + \frac{\hbar}{|\text{Aut}(\circ\circ\circ)|} + \frac{\hbar}{|\text{Aut}(\ominus)|} + \dots \\ &= 1 + \frac{\hbar}{8} + \frac{\hbar}{8} + \frac{\hbar}{12} + \dots = 1 + \frac{1}{3}\hbar + \frac{41}{36}\hbar^2 + \dots \end{aligned}$$

We use the symbol \hbar as a formal variable that marks the negative Euler characteristic to conform with typical quantum field theory notation.

All admissible graphs with Euler characteristic -2 contribute to the \hbar^2 coefficients in the expression above. These graphs are



This illustrates that the number of graphs becomes quickly too large to be handled one-by-one.

2.1. Labeled graphs versus isomorphism classes of graphs. — Although we want to count isomorphism classes of graphs, it is convenient to start with *labeled* graphs, where the half-edges are the set of consecutive integers $H = \{1, 2, \dots, |H|\}$ (see Figure 2 for an example). A labeled graph will be denoted lG . The set of labeled admissible graphs with m edges and k vertices will be denoted $\mathcal{LG}(m, k)$.

Two labeled graphs are equal if and only if they are represented by identical triples (H, E, V) . The following proposition relates the count of labeled graphs with the number of isomorphism classes of graphs.

Proposition 2.3. —

$$\sum_{G \in \mathcal{G}(m, k)} \frac{1}{|\text{Aut}(G)|} = \frac{|\mathcal{LG}(m, k)|}{(2m)!}.$$

Démonstration. — The symmetric group Σ_{2m} acts on $\mathcal{LG}(m, k)$ by permuting the labels. The orbit-stabilizer theorem says

$$(2m)! = |(\text{orbit } lG)| \cdot |(\text{stabilizer } lG)|.$$

The orbit of a labeled graph lG is nothing but an isomorphism class of labeled graphs i.e. an unlabeled graph, and the stabilizer of lG is isomorphic to $\text{Aut}(G)$, giving

$$|\mathcal{LG}(m, k)| = \sum_{\text{orbits}} |(\text{orbit})| = \sum_{G \in \mathcal{G}(m, k)} \frac{(2m)!}{|\text{Aut}(G)|}. \quad \square$$

Multiplying by \hbar^{m-k} and summing over all m and k gives the following expression for the generating function \mathbf{F} :

$$\mathbf{F}(\hbar) = \sum_{G \in \mathcal{G}} \frac{\hbar^{-\chi(G)}}{|\text{Aut}(G)|} = \sum_{m, k \geq 0} \frac{1}{(2m)!} |\mathcal{LG}(m, k)| \hbar^{m-k}.$$

2.2. Counting labeled graphs. — Now we want to count labeled graphs. Given $2m$ half-edges, any matching of H and any fat partition V of H corresponds to a unique labeled admissible graph, so to count the number of such graphs we can make with $2m$ half-edges, we need to count the number of matchings and the number of fat partitions. These counts are standard, we record them below.

Lemma 2.4. — *There are $(2m - 1)!! = (2m - 1) \cdot (2m - 3) \cdots 3 \cdot 1$ matchings of a set of $2m$ labeled elements.*

Démonstration. — This is an easy induction on m . □

Lemma 2.5. — *The coefficient of $\lambda^k \frac{x^n}{n!}$ in $\exp(\lambda(e^x - 1 - x - \frac{x^2}{2}))$ is the number of ways of partitioning the set $\{1, \dots, n\}$ into k sets, each of size at least 3.*

Démonstration. — The number of partitions of a set of n elements into non-empty pieces is the coefficient of $\frac{x^n}{n!}$ in the power series expansion of $\exp(e^x - 1)$:

$$\begin{aligned} \exp(e^x - 1) &= \exp\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) \\ &= 1 + \sum_{i \geq 1} \frac{x^i}{i!} + \frac{1}{2!} \sum_{i, j \geq 1} \frac{x^i x^j}{i! j!} + \frac{1}{3!} \sum_{i, j, k \geq 1} \frac{x^i x^j x^k}{i! j! k!} + \dots \end{aligned}$$

If instead we take $\exp(\lambda(e^x - 1))$, the power of λ keeps track of the number of pieces of the partition.

$$\begin{aligned} \exp(\lambda(e^x - 1)) &= \exp\left(\lambda\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\right) \\ &= 1 + \lambda \sum_{i \geq 1} \frac{x^i}{i!} + \frac{\lambda^2}{2!} \sum_{i, j \geq 1} \frac{x^i x^j}{i! j!} + \frac{\lambda^3}{3!} \sum_{i, j, k \geq 1} \frac{x^i x^j x^k}{i! j! k!} + \dots \end{aligned}$$

Replacing $e^x - 1$ by $e^x - 1 - x - \frac{x^2}{2}$ in the above formulas restricts attention to fat partitions, i.e. partitions with at least three elements in each block. □

Since an admissible labeled graph consists of a fat partition of the set $\{1, \dots, |H|\}$ and a matching on H , Lemmas 2.4 and 2.5 give

$$\begin{aligned} |\mathcal{LG}(m, k)| &= (2m-1)!! \left[\lambda^k \frac{x^{2m}}{(2m)!} \right] \exp \left(\lambda \left(e^x - 1 - x - \frac{x^2}{2} \right) \right) \\ &= (2m-1)!! (2m)! [\lambda^k x^{2m}] \exp \left(\lambda \left(e^x - 1 - x - \frac{x^2}{2} \right) \right). \end{aligned}$$

Here the square bracket denotes the *coefficient extraction operator*, which extracts the coefficient of the bracketed monomial expression from the following power series. If \mathbf{F} is a formal power series in commuting variables x, y, z, \dots and M is a monomial in these variables, the notation $[M]\mathbf{F}$ denotes the coefficient of M in \mathbf{F} ; this is itself a power series in the variables that do not occur in M .

Since you can only make a finite number of labeled graphs with $2m$ labeled half-edges, the function

$$p_m(\lambda) = (2m-1)!! [x^{2m}] \exp \left(\lambda \left(e^x - 1 - x - \frac{x^2}{2} \right) \right)$$

is a polynomial in λ ,

$$p_m(\lambda) = \frac{|\mathcal{LG}(m, 1)|}{(2m)!} \lambda + \frac{|\mathcal{LG}(m, 2)|}{(2m)!} \lambda^2 + \dots + \frac{|\mathcal{LG}(m, v_{\max})|}{(2m)!} \lambda^{v_{\max}},$$

where $v_{\max} = \deg p_m \leq \frac{2}{3}m$, as each admissible graph with $2m$ half-edges has at most $\frac{2}{3}m$ vertices. Using this with Proposition 2.3 gives the following formula for our generating function:

$$\mathbf{F}(\hbar) = \sum_{G \in \mathcal{G}} \frac{\hbar^{-\chi(G)}}{|\text{Aut}(G)|} = \sum_{m, k \geq 0} \frac{|\mathcal{LG}(m, k)|}{(2m)!} \hbar^{m-k} = \sum_{m \geq 0} \hbar^m p_m(\hbar^{-1})$$

i.e.

$$(1) \quad \mathbf{F}(\hbar) = \sum_{m \geq 0} \hbar^m (2m-1)!! [x^{2m}] \exp \left(\hbar^{-1} \left(e^x - 1 - x - \frac{x^2}{2} \right) \right).$$

2.3. Connected graphs. — When we counted how many admissible graphs we could make from $2m$ labeled edges, we did not care whether the results were connected or disconnected. If we only want to count connected graphs, we can take the formal logarithm of the generating function. This is standard in the combinatorics literature; see, e.g., Stanley's classic book [11], Chapter 5.1.

The basic principle is that, given an exponential generating function that counts some type of (labeled) objects, formally exponentiating this series counts collections of such objects. We have already used this to count graphs: the generating function $\sum \frac{x^n}{n!}$ counts the number of labeled *corollas* made of one vertex and n half-edges, weighted by the automorphism group of the corolla (the symmetric group Σ_n). An admissible graph is then just a collection of corollas (with at least 3 half-edges each) together with a matching of half-edges. To count collections of corollas, we exponentiated $\lambda \sum_{n \geq 3} \frac{x^n}{n!}$, then multiplied by the number of matchings to get the number of graphs.

2.4. Gaussian integrals and 0-dimensional QFT. — We can replace the expression $\hbar^m (2m-1)!!$ in Formula (1) by a Gaussian integral. We have

$$\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\hbar}} x^n dx = \begin{cases} 0 & n \text{ odd} \\ (2m-1)!! \hbar^m & n = 2m \end{cases}$$

Using this we can ignore the fact that we are only interested in extracting even powers of x in $\exp(\hbar^{-1}(e^x - 1 - x - \frac{x^2}{2}))$, i.e. we can write

$$\mathbf{F}(\hbar) = \sum_{n \geq 3} \left(\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\hbar}} x^n dx \right) [x^n] \exp \left(\hbar^{-1} \left(e^x - 1 - x - \frac{x^2}{2} \right) \right)$$

We now do something that would be unforgivable if we were dealing with power series representatives of actual functions, namely we interchange the summation and integral to get

$$\begin{aligned} \mathbf{F}(\hbar) &= \frac{1}{\sqrt{2\pi\hbar}} \int e^{-\frac{x^2}{2\hbar}} \exp \left(\hbar^{-1} \left(e^x - 1 - x - \frac{x^2}{2} \right) \right) dx \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int \exp \left(\hbar^{-1} (e^x - 1 - x - x^2) \right) dx \end{aligned}$$

The series from which we derived this integral is formally understood as the *path-integral* for the 0-dimensional scalar quantum field theory with potential $e^x - 1 - x - \frac{x^2}{2}$. The generating function $\mathbf{F}(\hbar)$ is the *partition function* and $\log \mathbf{F}(\hbar)$, which counts the associated set of connected graphs, is the *free energy* associated to this scalar QFT. For more information about this perspective, see [1] Chapter 3.

3. The virtual Euler characteristic of the even commutative graph complex

The even commutative graph complex \mathcal{GC}_2 breaks into subcomplexes $\mathcal{GC}_2^{(n)}$ generated by connected admissible graphs with fundamental group F_n , modulo the relation that an automorphism inducing an odd permutation of the edges changes the sign of the generator. In particular, if there is an odd permutation the generator is zero. Sometimes \mathcal{GC}_2 is defined to exclude graphs with loops. We will allow graphs with loops, but this makes no difference in the homology (or the Euler characteristic) (see [12], Proposition 3.4).

We recall the interpretation of $\mathcal{GC}_2^{(n)}$ in terms of Outer space (see [6], Section 5). Let CV_n^* denote the simplicial closure of Outer space CV_n , and $CV_n^\infty = CV_n^* \setminus CV_n$ the subcomplex at infinity. $\text{Out}(F_n)$ acts on CV_n^* preserving CV_n^∞ , and the relative chain complex

$$C_*(CV_n^*/\text{Out}(F_n), CV_n^\infty/\text{Out}(F_n); \mathbb{Q})$$

of the quotient spaces has one generator in dimension m for each connected admissible graph G with $m+1$ edges that has no automorphisms inducing an odd permutation of its edges (see, e.g., [8]). In other words, this relative chain complex is precisely $\mathcal{GC}_2^{(n)}$.

Let $\chi(\mathcal{GC}_2^{(n)})$ denote the orbifold Euler characteristic of this relative quotient (see, e.g., Brown [4], Chapter IX, Section 7). Then

$$\chi(\mathcal{GC}_2^{(n)}) = \sum_{\langle \sigma \rangle} \frac{(-1)^{\dim \sigma}}{|\text{Stab}(\sigma)|},$$

where σ runs over all orbit representatives of the action of $\text{Out}(F_n)$ on $CV_n = CV_n^* \setminus CV_n^\infty$, i.e. all connected admissible graphs with fundamental group F_n . Assembling the $\chi(\mathcal{GC}_2^{(n)})$ into a generating function (with a shift $n \mapsto (n+1)$ to account for the fact that $n = 1 - \chi(G)$) gives

$$\begin{aligned} \mathbf{X}(\hbar) &= \sum_{n \geq 1} \chi(\mathcal{GC}_2^{(n+1)}) \hbar^n = \sum_{G \in \mathcal{G}_c} (-1)^{e(G)} \frac{\hbar^{-\chi(G)}}{|\text{Aut}(G)|} \\ &= \hbar \left(\frac{(-1)^2}{|\text{Aut}(\infty)|} + \frac{(-1)^3}{|\text{Aut}(\text{---}\odot\text{---})|} + \frac{(-1)^3}{|\text{Aut}(\ominus)|} \right) + \dots = -\frac{1}{12} \hbar + \frac{1}{360} \hbar^3 + \dots \end{aligned}$$

where we sum over the set \mathcal{G}_c of all connected graphs in \mathcal{G} .

Exponentiating $\mathbf{X}(\hbar)$, invoking Proposition 2.3 and recalling the definition of $p_m(\lambda)$ from Section 2.2 gives

$$\begin{aligned} \exp(\mathbf{X}(\hbar)) &= \mathbf{E}(\hbar) = \sum_{G \in \mathcal{G}} (-1)^{e(G)} \frac{\hbar^{-\chi(G)}}{|\text{Aut}(G)|} = \sum_{m, k \geq 0} (-1)^m \frac{|\mathcal{LG}(m, k)|}{(2m)!} \hbar^{m-k} \\ &= \sum_{m \geq 0} (-1)^m \hbar^m p_m(\hbar^{-1}) \\ &= \sum_{m \geq 0} (-\hbar)^m (2m-1)!! [x^{2m}] \exp\left(\hbar^{-1} \left(e^x - 1 - x - \frac{x^2}{2}\right)\right) \end{aligned}$$

We are interested in $[\hbar^n] \mathbf{E}(\hbar)$. It is just as good to find a formula for $\mathbf{E}(-\hbar)$, since $[\hbar^n] \mathbf{E}(-\hbar) = (-1)^n [\hbar^n] \mathbf{E}(\hbar)$. Now note

$$\mathbf{E}(-\hbar) = \sum_{m \geq 0} \hbar^m (2m-1)!! [x^{2m}] \exp\left(-\hbar^{-1} \left(e^x - 1 - x - \frac{x^2}{2}\right)\right).$$

If we again replace the factor $\hbar^m (2m-1)!!$ with a Gaussian integral and naively interchange summation and integration as in Section 2.4, we get the path-integral of the 0-dimensional QFT with potential $-(e^x - 1 - x - \frac{x^2}{2})$ and action $-(e^x - 1 - x)$, written

$$(2) \quad \mathbf{E}(-\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-\hbar^{-1}(e^x - 1 - x)} dx.$$

Since $e^x - 1 - x = \sum_{n \geq 2} \frac{x^n}{n!}$ we recover Kontsevich's Equation (3) from [9].

We can call the path-integral above a *topological quantum field theory*, because the partition function $\mathbf{E}(\hbar)$ and the free energy $\log \mathbf{E}(\hbar) = \mathbf{X}(\hbar)$ encode the topological invariants $\chi(\mathcal{GC}_2^{(n)})$.

3.1. Solving the path-integral. — For general potentials such a path-integral expression does not converge, but in the case of (2) it actually does as long as \hbar is positive. To see this, recall Euler's integral formula for the Γ -function: $\Gamma(N) = \int_0^\infty t^{N-1} e^{-t} dt$. The change of variables $t \mapsto Ne^x$ gives

$$\Gamma(N) = \int_{-\infty}^\infty N^N e^{Nx - Ne^x} dx.$$

This looks tantalizingly close to the formula for $\mathbf{E}(-\frac{1}{N})$; if we now divide by $N^N e^{-N} \sqrt{2\pi \frac{1}{N}}$ we in fact get the exact formula:

$$\frac{\Gamma(N)}{N^N e^{-N} \sqrt{2\pi \frac{1}{N}}} = \frac{1}{\sqrt{2\pi \frac{1}{N}}} \int_{-\infty}^\infty e^{N(1+x-e^x)} dx = \mathbf{E}(-\frac{1}{N}).$$

Taking the log of both sides gives

$$\log \Gamma(N) - N \ln N + N - \frac{1}{2}(\log 2\pi - \log N) = \log \mathbf{E}(-\frac{1}{N}) = \mathbf{X}(-\frac{1}{N}) = \sum_{G \in \mathcal{G}_c} \frac{(-1)^{e(G)}}{|\text{Aut}(G)|} \left(-\frac{1}{N}\right)^{-\chi(G)}$$

We now recall that Stirling's asymptotic expansion of $\log \Gamma(N)$ for large N in terms of Bernoulli numbers, which are themselves defined by the generating function $\sum_{n=0}^\infty \frac{B_n}{n!} x^n = x/(e^x - 1)$, (see, e.g., [5], §3.10) is

$$\log \Gamma(N) \sim N \ln N - N + \frac{1}{2}(\log 2\pi - \log N) + \sum_{k=1}^\infty \frac{B_{2k}}{2k(2k-1)} \frac{1}{N^{2k-1}} \text{ for } N \rightarrow \infty,$$

so that the coefficient of $\frac{1}{N^n}$ in $\mathbf{X}(-\frac{1}{N})$ is equal to $\frac{B_{2k}}{2k(2k-1)}$ if $n = 2k - 1$ and vanishes otherwise. As all even coefficients of $\mathbf{X}(\hbar)$ therefore vanish we have $\mathbf{X}(-\hbar) = -\mathbf{X}(\hbar)$.

From a QFT perspective, we can say that we effectively *solved* the QFT encoded in (2) by finding a closed-form expression for the expansion coefficients of the free energy $\mathbf{X}(\hbar)$.

We can summarize all our findings of this section in the following formula,

$$\chi(\mathcal{GC}_2^{(n+1)}) = [\hbar^n] \mathbf{X}(\hbar) = - \sum_{\substack{G \in \mathcal{G}_c \\ \chi(G) = -n}} \frac{(-1)^{e(G)}}{|\text{Aut}(G)|} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{B_{n+1}}{n(n+1)} & \text{if } n \text{ is odd.} \end{cases}$$

The formal interchange of summation and integration while passing to an integral representation can be justified by employing the formalism of asymptotic expansions and Laplace's method as was done in [2, Proposition 5.3].

4. The virtual Euler characteristic of $\text{Out}(F_n)$

To compute the virtual Euler characteristic of $\text{Out}(F_n)$ one uses the spine K_n of Outer space instead of the whole space, since the action on the spine is cocompact. The spine has the structure of a cube complex with one cube for every triple (g, G, F) consisting of a connected admissible graph G , a subforest F and an isomorphism $g: F_n \cong \pi_1(G)$. The action of $\text{Out}(F_n)$ just changes the isomorphism g , so computing the orbifold Euler characteristic reduces to counting pairs (G, F) , weighted by their stabilizers. In this section we show how to do this, using the fact that the stabilizer of the cube (G, F) is isomorphic to $\text{Aut}(G, F)$, the automorphisms of G that send F to itself. By Kontsevich's theorem the virtual Euler characteristic of $\text{Out}(F_n)$ is also the virtual Euler characteristic of the Lie graph complex [9].

A pair (G, F) consisting of an admissible graph and a spanning forest F can be thought of as a collection of trees together with a matching of their leaves. To count such pairs we should count trees, exponentiate to count forests, connect the leaves of forests in pairs, then take logarithm to restrict to connected graphs.

The dimension of the cube in the spine of Outer space corresponding to (G, F) is equal to the number of edges in the forest, so if we are interested in computing Euler characteristic we only need to know the parity of $e(F)$. Using Brown's formula [4] for the virtual Euler characteristic of a discrete group acting properly and cocompactly on a contractible cell complex, we therefore have

$$\chi(\text{Out}(F_n)) = \sum_{(G, F)} \frac{(-1)^{e(F)}}{|\text{Aut}(G, F)|},$$

where the sum is over isomorphism classes of pairs (G, F) with G a connected admissible graph and $\chi(G) = 1 - n$.

As in the previous sections it is convenient to define the generating function

$$\begin{aligned} \mathbf{Y}(\hbar) &= \sum_{n \geq 1} \chi(\text{Out}(F_{n+1})) \hbar^n = \sum_{(G, F)} \frac{(-1)^{e(F)}}{|\text{Aut}(G, F)|} \hbar^{-\chi(G)} \\ &= \hbar \left(\frac{1}{|\text{Aut}(\infty)|} + \frac{1}{|\text{Aut}(\circ \! \! \! \circ)|} + \frac{1}{|\text{Aut}(\ominus)|} + \frac{(-1)}{|\text{Aut}(\omin�)|} + \frac{(-1)}{|\text{Aut}(\circ \! \! \! \circ)|} \right) + \dots \\ &= -\frac{1}{24} \hbar - \frac{1}{48} \hbar^2 - \frac{161}{5760} \hbar^3 + \dots \end{aligned}$$

where we have colored the forest orange in the underlying graph.

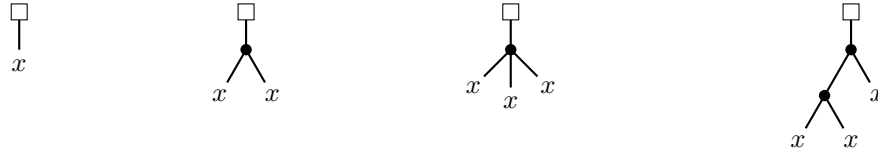
4.1. Counting trees. — A tree is a connected graph with no cycles; it has Euler characteristic equal to 1. For trees we will allow for the involution in its Definition 2.1 as a graph to have fixed points. Each fixed point corresponds to a *leave* of the tree. We will denote the number of leaves of a tree T by $\ell(T)$. A *rooted tree* is a tree with one univalent vertex, called the *root* (note that in our definition of graph, a univalent vertex is not the same as a leaf). To extend our methods to trees, we will require all vertices which are not the root to have degree at least 3, so that there are only a finite number of trees (or rooted trees) with a fixed number of leaves.

It is a classic combinatorial exercise to count rooted trees (see, e.g., [11], Chapter 5.3). In this section we will solve the counting problem that is relevant for the computation of the (virtual) Euler characteristic of $\text{Out}(F_n)$. Let \mathcal{R} denote the set of isomorphism classes of rooted trees and define the generating function

$$(3) \quad \mathbf{R}(\lambda, x) = \sum_{T_0 \in \mathcal{R}} \frac{\lambda^{v(T_0)} x^{\ell(T_0)}}{|\text{Aut}(T_0)|}.$$

Eventually, we will only be interested in the special case $\lambda = -1$, which corresponds to the generating function of alternating sums over rooted trees with fixed number of leaves. But for the benefit of a concrete combinatorial argument, we will start with the general case and specialize later.

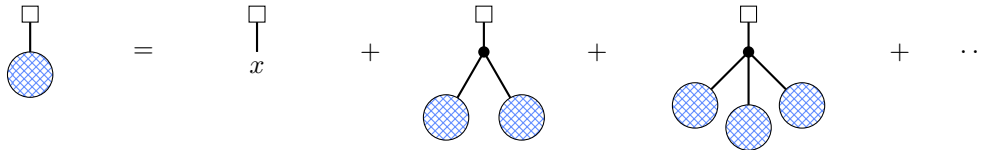
For example, consider the following isomorphism classes of rooted trees



where the root is indicated as a little square and leaves are marked by x . Note that the first rooted tree consists only of the root vertex and a single leaf. In our definition of graph, this is a single half-edge with trivial vertex partition and trivial involution. The trees above contribute the following terms to \mathbf{R} :

$$\mathbf{R}(\lambda, x) = x + \lambda \frac{x^2}{2!} + \lambda \frac{x^3}{3!} + \lambda^2 \frac{x^3}{2!} + \dots$$

Rooted trees have an inherently recursive structure. We can pick such a tree up by the root and attach it to a leaf of another rooted tree. This gives a recursion which starts with the rooted tree that contains only the root vertex and a single half-edge. This recursion is encoded by the following diagram, where the blue crosshatched circle indicates the set of all possibilities for the remainder of the tree:



In terms of the generating function $\mathbf{R}(\lambda, x)$ we have

$$(4) \quad \mathbf{R}(\lambda, x) = x + \lambda \frac{\mathbf{R}(\lambda, x)^2}{2!} + \lambda \frac{\mathbf{R}(\lambda, x)^3}{3!} + \dots = x + \lambda(e^{\mathbf{R}(\lambda, x)} - 1 - \mathbf{R}(\lambda, x)),$$

where the factorials in the denominators account for permuting the blue crosshatched circles. The power of λ keeps track of the number of vertices in the tree, but for the Euler characteristic calculation we only care about the parity of v , i.e. we want to count rooted trees T_0 with sign $(-1)^{v(T_0)}$. To accomplish this we can set $\lambda = -1$, which results in a neat closed expression for the generating function:

Proposition 4.1. — *The generating function for rooted trees counted with sign $(-1)^{v(T_0)}$ is*

$$\mathbf{R}(-1, x) = \sum_{T_0 \in \mathcal{R}} \frac{(-1)^{v(T_0)} x^{\ell(T_0)}}{|\text{Aut}(T_0)|} = \log(1+x) = \sum_{\ell \geq 1} \frac{(-1)^{\ell+1}}{\ell} x^\ell.$$

Démonstration. — By Equation (4) we have

$$\mathbf{R}(-1, x) = x - e^{\mathbf{R}(-1, x)} + 1 + \mathbf{R}(-1, x)$$

i.e. $e^{\mathbf{R}(-1, x)} = x + 1$. □

Let $\mathcal{T}(\ell, k)$ (resp. $\mathcal{R}(\ell, k)$) be the set of isomorphism classes of trees (resp. rooted trees) with ℓ leaves and k vertices and $\mathcal{LT}(\ell, k)$ (resp. $\mathcal{LR}(\ell, k)$) be the corresponding set of leaf-labeled trees (resp. rooted trees), i.e. each leaf is labeled by a number in $1, \dots, \ell$. The symmetric group Σ_ℓ acts on the set of leaf-labels, with orbit corresponding to an isomorphism class of trees T and stabilizer $\text{Aut}(T)$, so the orbit-stabilizer theorem gives

$$(5) \quad \sum_{T \in \mathcal{T}(\ell, k)} \frac{1}{|\text{Aut}(T)|} = \frac{|\mathcal{LT}(\ell, k)|}{\ell!} \quad \text{and} \quad \sum_{T_0 \in \mathcal{R}(\ell, k)} \frac{1}{|\text{Aut}(T_0)|} = \frac{|\mathcal{LR}(\ell, k)|}{\ell!}.$$

Proposition 4.2. — *When counted with sign $(-1)^{v(T_0)}$ there are $(-1)^{\ell+1}(\ell-1)!$ leaf-labeled rooted trees T_0 with $\ell \geq 1$ leaves.*

Démonstration. — Let $\mathcal{LR}(\ell)$ be the set of leaf-labeled rooted trees with $\ell \geq 1$ leaves. We are interested in,

$$\sum_{T_0 \in \mathcal{LR}(\ell)} (-1)^{v(T_0)} = \sum_{k \geq 0} (-1)^k |\mathcal{LR}(\ell, k)|.$$

Combining (5) and Proposition 4.1 gives

$$\sum_{k \geq 0} (-1)^k |\mathcal{LR}(\ell, k)| = \ell! \sum_{k \geq 0} \sum_{T_0 \in \mathcal{R}(\ell, k)} \frac{(-1)^k}{|\text{Aut}(T_0)|} = \ell! \frac{(-1)^{\ell+1}}{\ell} = (-1)^{\ell+1}(\ell-1)!. \quad \square$$

We next count unrooted trees with labeled leaves. Note that an unrooted tree must have at least three leaves, since we do not allow univalent or bivalent vertices.

Proposition 4.3. — *When counted with sign $(-1)^{v(T)}$ there are $(-1)^\ell(\ell-2)!$ leaf-labeled trees T with $\ell \geq 3$ leaves.*

Démonstration. — An (unrooted) tree with ℓ leaves labeled $1, \dots, \ell$ is equivalent to a rooted tree with $\ell-1$ leaves, where the leaf labeled ℓ is taken for the root (or, equivalently, an extra half-edge is added which is matched with the half-edge labeled ℓ and forms its own block of the vertex partition). By the above lemma, there are $(-1)^\ell(\ell-2)!$ such trees, counted with sign $(-1)^{v(T)}$. □

Let $\mathbf{T}(x)$ be the generating function for signed (unrooted) trees T , with x marking leaves, i.e.

$$\mathbf{T}(x) = \sum_{T \in \mathcal{T}} (-1)^{v(T)} \frac{x^{\ell(T)}}{|\text{Aut}(T)|},$$

where the sum is over the set of all isomorphism classes of trees \mathcal{T} .

By (5) and the above Corollary,

$$\mathbf{T}(x) = \sum_{\ell, k \geq 0} (-1)^k |\mathcal{LT}(\ell, k)| \frac{x^\ell}{\ell!} = \sum_{\ell \geq 3} (-1)^\ell (\ell-2)! \frac{x^\ell}{\ell!} = \sum_{\ell \geq 3} (-1)^\ell \frac{x^\ell}{\ell(\ell-1)}$$

It is easy to check that the last expression agrees with the power series expansion of

$$\mathbf{T}(x) = -x - \frac{x^2}{2} + (1+x) \log(1+x).$$

Our passing from the generating function $\mathbf{R}(-1, x)$ of signed rooted trees to the generating function $\mathbf{T}(x)$ of signed trees has an interpretation in terms of a *Legendre transformation* that is also of significance in the QFT context (see, e.g., [1], Chapter 5.8).

4.2. Counting graphs with a marked forest. — A spanning forest F in a graph G has the same vertices and half-edges as G but only some of the edges; in particular it is not allowed to contain a cycle of edges. To count signed pairs (G, F) with G admissible we will first count labeled pairs as before, but we will only label the half-edges that are not contained in edges of F , i.e. these are the leaves of F . Notice that F must have an even number 2ℓ of leaves since G is obtained by matching pairs of them, and G itself has no leaves.

Let $\mathcal{LP}(t, 2\ell, m)$ be the set of labeled pairs (G, F) such that F has t trees with a total of 2ℓ leaves and m edges. The symmetric group $\Sigma_{2\ell}$ acts on such pairs by permuting the labels. The orbit of a labeled pair is the unlabeled pair (G, F) , and the stabilizer is $\text{Aut}(G, F)$, the automorphisms of G that preserve F . We now run the argument exactly as we did when counting graphs in Sections 2.1 and 2.2: The orbit-stabilizer theorem gives

$$|\mathcal{LP}(t, 2\ell, m)| = \sum_{\text{orbits}} |\text{orbit}| = \sum_{(G, F) \in \mathcal{P}(t, 2\ell, m)} \frac{(2\ell)!}{|\text{Aut}(G, F)|},$$

where $\mathcal{P}(t, 2\ell, m)$ is the set of unlabeled pairs (G, F) .

For each tree T we have $e(T) - v(T) = 1$. Therefore, in order to count forests with sign $(-1)^{e(F)}$, we will use the function

$$-\mathbf{T}(x) = - \sum_{T \in \mathcal{T}} (-1)^{v(T)} \frac{x^{\ell(T)}}{|\text{Aut}(T)|} = \sum_{T \in \mathcal{T}} (-1)^{e(T)} \frac{x^{\ell(T)}}{|\text{Aut}(T)|}.$$

In order to keep track of how many trees there are in a forest we mark their number by τ and take $\exp(-\tau \mathbf{T}(x))$. Since a labeled pair is obtained by taking a forest with t trees and matching their 2ℓ leaves, we have

$$\frac{1}{(2\ell)!} \sum_{m \geq 0} (-1)^m |\mathcal{LP}(t, 2\ell, m)| = (2\ell - 1)!! [\tau^t x^{2\ell}] \exp(-\tau \mathbf{T}(x)).$$

For each labeled pair $(G, F) \in \mathcal{LP}(t, 2\ell, m)$ the forest F has t trees, so we have $v(F) - e(F) = t$ and $-\chi(G) = \ell - t$. Therefore

$$(6) \quad \sum_{(G, F)} (-1)^{e(F)} \frac{\hbar^{-\chi(G)}}{|\text{Aut}(G, F)|} = \sum_{\ell \geq 0} \hbar^\ell (2\ell - 1)!! [x^{2\ell}] \exp(-\hbar^{-1} \mathbf{T}(x)),$$

where we sum over all isomorphism classes of pairs (G, F) .

Because $\exp(-\hbar^{-1} \mathbf{T}(x))$ counts leaf-labeled forests, Equation (6) can be seen as a special case of Proposition 4.7 in [2].

Using the exponential formula, we finally obtain an expression for the generating function $\mathbf{Y}(\hbar)$ of $\chi(\text{Out}(F_n))$:

$$\exp(\mathbf{Y}(\hbar)) = \sum_{\ell \geq 0} \hbar^\ell (2\ell - 1)!! [x^{2\ell}] \exp(-\hbar^{-1} \mathbf{T}(x)).$$

As in the previous sections, we can also write this formally as an integral expression:

$$(7) \quad \begin{aligned} \exp(\mathbf{Y}(\hbar)) &= \int \frac{dx}{\sqrt{2\pi\hbar}} \exp(\hbar^{-1}(x - (1+x)\log(1+x))) \\ &= \int \frac{dx}{\sqrt{2\pi\hbar}} \exp\left(-\hbar^{-1} \sum_{n \geq 3} (-1)^n \frac{x^n}{n(n-1)}\right). \end{aligned}$$

In QFT terminology, we have shown that the free energy of the 0-dimensional QFT with action $x - (1+x)\log(1+x)$ encodes the topological invariants $\chi(\text{Out}(F_n))$.

Note that expression (7) agrees with Equation (1) of [9] up to an irrelevant sign in front of x . The reason for this sign difference is that we computed the virtual Euler characteristic of $\text{Out}(F_n)$ using the spine of Outer space, whereas Kontsevich computes the Euler characteristic of the Lie graph complex, which is only quasi-isomorphic to the chain complex of the spine.

4.3. A renormalized topological QFT. — In [2] we obtained different expressions for the invariants $\chi(\text{Out}(F_n))$ (Theorem B and Proposition 8.1) that enabled us to prove that $\chi(\text{Out}(F_n)) < 0$ for all $n \geq 2$ and that their growth rate is given by $\chi(\text{Out}(F_n)) \sim -\frac{1}{\sqrt{2\pi}}\Gamma(n - \frac{3}{2})/\log^2 n$ for $n \rightarrow \infty$ (Theorem A).

Key intuition for proving these facts came from the observation that the invariants $\chi(\text{Out}(F_n))$ are also encoded by a *renormalization* of the QFT described by Equation (2) in Section 3. Specifically, the generating function $\mathbf{Y}(\hbar)$ also respects the implicit path-integral expression

$$1 = \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(-\hbar^{-1}(e^x - x - 1) + \frac{x}{2} + \mathbf{Y}(-\hbar e^{-x})\right) dx.$$

This equation fixes the values of the invariants $\chi(\text{Out}(F_n))$ and enables one to study them. To fully justify this, a precise definition of the renormalization procedure in QFT is needed, but this lies beyond the scope of these expository notes (see [2] Section 6 for a derivation).

The QFT interpretation of the above equation, however, is readily stated: The generating function $\frac{x}{2} + \mathbf{Y}(-\hbar e^{-x})$ encodes *counterterms* or *renormalization constants* that renormalize the QFT with action $-(e^x - x - 1)$, which was the subject of Section 3, in a generalized sense. See [1] Chapters 5–7 for a detailed algebraic and combinatorial treatment of such renormalization operations.

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