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# Characterising rectifiable metric spaces using tangent spaces

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**Abstract** We characterise rectifiable subsets of a complete metric space $X$ in terms of local approximation, with respect to the Gromov–Hausdorff distance, by an $n$-dimensional Banach space. In fact, if $E \subset X$ with $\mathcal{H}^n(E) < \infty$ and has positive lower density almost everywhere, we prove that it is sufficient that, at almost every point and each sufficiently small scale, $E$ is approximated by a bi-Lipschitz image of Euclidean space. We also introduce a generalisation of Preiss's tangent measures that is suitable for the setting of arbitrary metric spaces and formulate our characterisation in terms of tangent measures. This definition is equivalent to that of Preiss when the ambient space is Euclidean, and equivalent to the measured Gromov–Hausdorff tangent space when the measure is doubling.

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## 1 Introduction

A set $E \subset \mathbb{R}^m$ is *$n$-rectifiable* if it can be covered, up to a set of $\mathcal{H}^n$ measure zero, by countably many Lipschitz images of $\mathbb{R}^n$ (throughout this paper, $\mathcal{H}^n$ denotes the $n$-dimensional Hausdorff measure). Rademacher's theorem implies that an $n$-rectifiable $E \subset \mathbb{R}^m$ with $\mathcal{H}^n(E) < \infty$ has a unique $n$-dimensional *approximate tangent plane* at $\mathcal{H}^n$-a.e. $x \in E$: an $n$-dimensional

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affine subspace $W \ni x$ such that, for each $r > 0$, there exists $E_r \subset E \cap B(x, r)$ with

$$\frac{\mathcal{H}^n(E \cap B(x, r) \setminus E_r)}{r^n} \to 0$$

and

$$\frac{d_{\mathrm{H}}(W \cap B(x, r), E_r)}{r} \to 0$$

as $r \to 0$. Here $d_{\mathrm{H}}$ denotes the Hausdorff distance between two subsets of $\mathbb{R}^m$. A central concept of geometric measure theory is to understand sufficient conditions for rectifiability in terms of local approximations by tangent planes.

The simplest such result is the converse to the above statement: if $E \subset \mathbb{R}^m$ with $\mathcal{H}^n(E) < \infty$, the existence of an $n$-dimensional approximate tangent plane at $\mathcal{H}^n$-a.e. point implies that $E$ is $n$-rectifiable (see [1, Theorem 15.19]). A result of Marstrand [2] (for $n = 2$ and $m = 3$) significantly strengthens this by allowing the approximating affine subspace $W$ to depend on $r$ — the approximating tangent plane may "rotate" as one zooms into $E$ — under the additional assumption of the positive lower Hausdorff density of $E$ (see (1) below). This was later generalised by Mattila [3] to all higher dimensions. The case of $n = 1$ is addressed in the work of Besicovitch [4]. This much stronger result is crucial for many results in geometric measure theory, such as the density theorems of Marstrand [2], Mattila [3] and Preiss [5].

Analogously, the work of David and Semmes [6] characterises *uniform* rectifiability in terms of good approximation by (rotating) tangents in the form of Jones's beta numbers [7]. Tolsa and Azzam [8,9] provide similar results for measures.

Recently there has been significant interest in studying analysis on metric spaces. Inevitably, such study leads to questions regarding the geometric nature of rectifiable subsets of a metric space. For a metric space $X$, $E \subset X$ is *n-rectifiable* if it can be covered, up to a set of $\mathcal{H}^n$ measure zero, by countably many Lipschitz images of *subsets* of $\mathbb{R}^n$. Kirchheim [10] gives a precise description of the local structure of rectifiable subsets of a metric space analogous to the description granted by Rademacher's theorem. From this description it is possible to deduce the existence of a tangent structure of a rectifiable metric space akin to an approximate tangent plane after two modifications.

Firstly, one must replace the Hausdorff distance $d_{\mathrm{H}}$ with the *Gromov–Hausdorff* distance $d_{\mathrm{GH}}$. Two metric spaces $M_1$ and $M_2$ satisfy

$$d_{\mathrm{GH}}(M_1, M_2) < \epsilon$$

if there exist a metric space $Z$ and isometric embeddings $\iota_i : M_i \to Z, i = 1, 2$, such that

$$d_{\mathrm{H}}(\iota_1(M_1), \iota_2(M_2)) < \epsilon$$

(see Definition 2.15 for a modification that is more suitable for unbounded metric spaces). Secondly, the affine subspace must be replaced by an $n$-dimensional Banach space; the tangent space may not be equipped with the Euclidean norm.

The question of whether an $E \subset X$ with approximate tangent planes defined in this way is rectifiable has existed since the work of Kirchheim. More recently, this question appeared in the work of Ambrosio, Brué and Semola [11] when studying sets of finite perimeter in RCD metric spaces. To date the question is unknown for any ambient metric space that is not a Euclidean space (in this case $d_{\mathrm{GH}}$ can be replaced by $d_{\mathrm{H}}$ in (3) and so the statement is equivalent to the results of Marstrand and Mattila), even if all of the tangent spaces are assumed to be a fixed $n$-dimensional Banach space.

In this paper we answer this question affirmatively. In fact we prove the following stronger result. For $K \geq 1$ let $\mathrm{biLip}(K)$ be the set of metric spaces $Y = (\mathbb{R}^n, \rho)$ such that $\rho$ is $K$-bi-Lipschitz equivalent to the Euclidean norm, that is

$$\frac{1}{K}\|x - y\|_2 \leq \rho(x, y) \leq K\|x - y\|_2 \quad \forall x, y \in \mathbb{R}^n.$$

Rather than requiring local approximation by a fixed $n$-dimensional Banach space, we allow approximation by an element of $\mathrm{biLip}(K)$ that is allowed to depend on the scale.

**Theorem 1.1** *Let $(X, d)$ be a complete metric space, $n \in \mathbb{N}$ and let $E \subset X$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(E) < \infty$ and*

$$\Theta^n_*(E, x) := \liminf_{r \to 0} \frac{\mathcal{H}^n(B(x, r) \cap E)}{r^n} > 0 \quad \textit{for } \mathcal{H}^n\textit{-a.e. } x \in E. \tag{1}$$

*Suppose that, for $\mathcal{H}^n$-a.e. $x \in E$, there exist $K_x \geq 1$ and, for each $r > 0$, a $Y_r \in \mathrm{biLip}(K_x)$ and $E_r \subset E \cap B(x, r)$ such that*

$$\frac{\mathcal{H}^n(E \cap B(x, r) \setminus E_r)}{r^n} \to 0 \tag{2}$$

*and*

$$\frac{d_{\mathrm{GH}}(Y_r \cap B(0, r), E_r)}{r} \to 0 \tag{3}$$

*as $r \to 0$. Then $E$ is $n$-rectifiable.*

Note that even in the case that $X$ is a Euclidean space, Theorem 1.1 is new since it allows for local approximation by bi-Lipschitz images of $\mathbb{R}^n$.

There is a more subtle aspect in which Theorem 1.1 is stronger than the results of Marstrand and Mattila. Since our notion of a tangent space is defined via Gromov–Hausdorff convergence, we do not rely on any structure of the ambient metric space $X$. On the other hand, the proof of Marstrand crucially relies on tangents that are affine subspaces of $\mathbb{R}^m$. In particular, Marstrand's proof fails for tangents that are isometric copies of $\mathbb{R}^2$ when $X = (\mathbb{R}^3, \|\cdot\|_\infty)$. However, we note that for the special case $n = 1$, the idea of Besicovitch can be modified to work in any metric space; an account of this will appear in [12].

The use of Gromov–Hausdorff tangents may be considered as a further weakening of the concept of the rotating tangent plane. Such a framework is necessary when working in this generality; Even if we assume $X$ is a Banach space, it is not possible to define an approximate tangent plane using ambient linear structure. For example consider

$$\{\chi_{[0,t]} \in L^1([0,1]) : t \in [0,1]\},$$

which is an isometric copy of $[0, 1]$ but does not have an approximate tangent plane at any point.

The results of Marstrand and Mattila are a starting point for the work of Preiss [5]. It is natural to consider generalisations of Preiss's tangent measures when investigating results like Theorem 1.1. Recall that a non-zero Radon measure $\nu$ on $\mathbb{R}^m$ is a *tangent measure* to another Radon measure $\mu$ at $x \in \operatorname{spt}\mu$ if there exist $r_i \to 0$ such that the pushforwards of $\mu$ under the maps $y \mapsto (y - x)/r$, scaled by $1/\mu(B(x, r_i))$, weak* converge to $\nu$.

To define a tangent measure of a measure $\mu$ on a metric space $X$, one requires a definition of a limit of a sequence of pointed metric measure spaces. In this paper, a *pointed metric measure space* $(X, d, \mu, x)$ will consist of a metric space $(X, d)$, a Borel measure $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ (that is, $\mu$ is finite on bounded sets), and a distinguished point $x \in \operatorname{spt}\mu$. Given some notion of convergence of pointed metric measure spaces, we can define a *tangent* of a metric measure space $(X, d, \mu, x)$ as any metric measure space $(Y, \rho, \nu, y)$ for which there exist $r_i \to 0$ such that

$$\left(X, \frac{d}{r_i}, \frac{\mu}{\mu(B(x, r_i))}, x\right) \to (Y, \rho, \nu, y). \tag{4}$$

One such notion of convergence is *measured Gromov–Hausdorff convergence* and the resulting tangent spaces are known as *measured Gromov–Hausdorff tangents*. This convergence can be metrised similarly to the Gromov–Hausdorff distance: In addition to taking the Hausdorff distance between the metric spaces embedded in $Z$, one also considers the distance

between the pushforwards of the measures, using some canonical metric that metrises weak* convergence in $Z$. We construct one such metrisation of measured Gromov–Hausdorff convergence, $d_{\mathrm{pmGH}}$, in Definition 4.1.

However, the requirement that the underlying metric spaces must Gromov–Hausdorff converge is too rigid to study rectifiable sets. Indeed, it is easy to find examples of rectifiable metric spaces with no tangent measures according to this definition (see Example 4.8). Instead, we define a metric, $d_*$, that only considers the distance between the pushforwards of the measures in $Z$ and disregards the Hausdorff distance between the embedded metric spaces (this will be discussed further below). We write $\mathrm{Tan}(X, d, \mu, x)$ for the set of all tangent measures to $(X, d, \mu, x)$ defined using $d_*$ and (4).

Our main result on tangent measures is the following. For $K \geq 1$ let $\mathrm{biLip}(K)^*$ be the set of all pointed metric measure spaces that are supported on an element of $\mathrm{biLip}(K)$.

**Theorem 1.2** *Let $(X, d)$ be a complete metric space, $n \in \mathbb{N}$ and let $E \subset X$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(E) < \infty$. The following are equivalent:*

1. *$E$ is $n$-rectifiable;*
2. *For $\mathcal{H}^n$-a.e. $x \in E$, $\Theta^n_*(E, x) > 0$ and there exists an $n$-dimensional Banach space $(\mathbb{R}^n, \|.\|_x)$ such that*

$$\mathrm{Tan}(X, d, \mathcal{H}^n|_E, x) = \{(\mathbb{R}^n, \|.\|_x, \mathcal{H}^n/2^n, 0)\}.$$

3. *For $\mathcal{H}^n$-a.e. $x \in E$, $\Theta^n_*(E, x) > 0$ and there exists a $K_x \geq 1$ such that $\mathrm{Tan}(X, d, \mathcal{H}^n|_E, x) \subset \mathrm{biLip}(K_x)^*$.*

Once Theorem 1.1 is established, the main steps to proving Theorem 1.2 are to develop the properties of $d_*$ and the properties of tangent measures.

The proof of Theorem 1.1 follows from the combination of two results, Theorem 3.4 and Corollary 6.4. Roughly speaking, Theorem 3.4 shows that, under the hypotheses of Theorem 1.1, for $\mathcal{H}^n$-a.e. $x \in E$, the following is true: for any $\epsilon > 0$ and any sufficiently small $r > 0$, there exists a metric space $\tilde{E} \supset E$ and a continuous (in fact Hölder) map $\iota\colon [0, r]^n \to \tilde{E}$ such that $\iota|_{\partial[0,r]^n}$ is close to having Lipschitz inverse and

$$\mathcal{H}^n_\infty(\iota([0, r]^n) \setminus E) < \epsilon r^n. \tag{5}$$

Here $\mathcal{H}^n_\infty$ denotes the $n$-dimensional Hausdorff content.

Corollary 6.4 is a consequence of a theorem of [13] (see Theorem 6.3). This theorem can be viewed as a replacement for the Besicovitch–Federer projection theorem that is valid in any complete metric space. It considers 1-Lipschitz "non-linear" projections in place of orthogonal projections. Corollary 6.4 states that any $E \subset X$ for which each $E' \subset E$ satisfies the conclusion of Theorem 3.4

is $n$-rectifiable. Note that the hypotheses of Theorem 1.1 are inherited by any subset (by Lemma 2.3).

The construction of Theorem 3.4 is the central result of the paper. Naturally, the idea is to take a set $G \subset E$ for which the approximations given by (3) are in some sense uniform and "glue" together the approximating tangent planes to construct the Hölder surface. Of course, following this approach, one will eventually encounter points in $E \setminus G$ where we halt the gluing process and we begin to define $\tilde{E} \setminus G$. Therefore, at every step of the construction, it is crucial to control the size of the set of points we encounter in $E \setminus G$ to establish (5). The details of this construction are discussed further in Sect. 3.

We will see that the basic theory of tangent measures in our setting follows analogously to Preiss's tangent measures, once the properties of $d_*$ are established.

There are several equivalent ways to define convergence of metric measure spaces that considers only the convergence of measures and not the Gromov–Hausdorff convergence of the metric spaces: one simply has to decide upon a way to metrise weak* convergence in the metric space $Z$. Sturm [14] first considered the $L_2$ transportation distance between probability measures with finite variance. Greven, Pfaffelhuber, and Winter [15] consider the Prokhorov metric between probability measures on compact metric spaces. Gigli, Mondino and Savaré [16] simply define $(X_i, d_i, \mu_i, x_i) \to (X, d, \mu, x)$, for $\mu_i, \mu \neq 0$, if there exist isometric embeddings into $Z$ such that $x_i \to x$ and $\mu_i$ weak* converges to $\mu$.

To define $d_*$, we first define a metric $F$ between $\mu, \nu \in \mathcal{M}_{\mathrm{loc}}(Z)$ by duality with Lipschitz functions. A common approach for *finite* measures is to consider the metric $F^1$ defined as the dual norm to the set of 1-Lipschitz functions on $Z$. It is well known that $F^1$ contains geometric information on the support of $\mu, \nu$. One can define a metric between $\mu$ and $\nu$ by taking the infimal $\epsilon > 0$ for which

$$F^1(\mu|_{B(z,1/\epsilon)}, \nu|_{B(z,1/\epsilon)}) < \epsilon, \tag{6}$$

for some fixed $z \in Z$. Taking this idea further, we let $F^\epsilon$ be the dual norm of the set of $1/\epsilon$-Lipschitz functions, and define $F$ by replacing $F^1$ with $F^\epsilon$ in (6) (see Definition 2.12). Then from $F$ one precisely obtains the Hausdorff distance between two large subsets of the supports of $\mu$ and $\nu$ (see Lemma 4.10). Consequently, by defining $d_*$ using $F$ to metrise weak* convergence in $\mathcal{M}_{\mathrm{loc}}(Z)$, we obtain an explicit relationship between $d_*$ and the $d_{\mathrm{pmGH}}$ distance between two large subsets (see Theorem 4.11). This allows us to use standard Gromov–Hausdorff techniques and Prokhorov's theorem to develop the theory of $d_*$.

However, we do not introduce $d_*$ as a new notion of convergence. Indeed, it is equivalent to any of the previously mentioned notions whenever the latter

is defined: they all correspond to weak* convergence in $Z$ (see Remark 4.14). Rather, we use it as another, convenient description of the established notions.

Note that $\mathrm{Tan}(\mathbb{R}^n, \|\cdot\|_2, \mu, x)$ agrees with (isometry classes of) Preiss's tangent measures. Also, for a sequence of uniformly doubling spaces, convergence in $d_*$ is equivalent to measured Gromov–Hausdorff convergence (by [16, Theorem 3.33] or Corollary 4.18). Consequently, tangent measures of doubling metric measure spaces are precisely measured Gromov–Hausdorff tangents.

To conclude the introduction we discuss results related to this work.

Assuming a set $E \subset \mathbb{R}^m$ has *uniform* approximation by tangents that are isomorphic copies of $\mathbb{R}^n$, a classical construction of Reifenberg [17] constructs a locally bi-Hölder parametrisation of $E$ by an $n$-dimensional ball. This has been generalised by David and Toro [18], using Gromov–Hausdorff tangents, to any metric space. Such a parametrisation can be used to obtain the map $\iota$ in Theorem 3.4 and in fact one may take $\tilde{E} = E$. However, these assumptions are too strong for our situation: they do not allow the exclusion of sets of small measure as in (2) and the convergence in (3) is not uniform in $x$. Moreover, one cannot deduce that such a map exists for subsets of $E$ and so cannot use Corollary 6.4 to deduce rectifiability.

Relationships between tangent spaces and parametrising maps have been explored in specific non-Euclidean metric spaces such as the Heisenberg group (or other Carnot groups) [19,20]. In this case, the additional structure of a particular ambient metric space such as the Heisenberg group enables one to define a much stronger notion of a tangent than a Gromov–Hausdorff tangent.

The outline of the paper is as follows.

In Sect. 2 we recall preliminary notions of geometric measure theory and properties of doubling measures. In Sect. 2.3 we recall facts about weak* convergence of measures in metric spaces and define $F$. In Sect. 2.4 we recall several constructions related to the Gromov–Hausdorff distance, incorporating both $d_{\mathrm{H}}$ and $F$, in a way that the definitions and properties of $d_{\mathrm{pGH}}$, $d_{\mathrm{pmGH}}$ and $d_*$ all follow from the same construction.

In Sect. 3 we prove Theorem 3.4: the construction of a Hölder surface under the hypotheses of sets $X \supset C \supset G$ such that, for each $x \in G$ and each sufficiently small $r > 0$, $B(x, r) \cap C$ is approximated (up to sets of small measure) by bi-Lipschitz images of $\mathbb{R}^n$ (see Definition 3.2).

In Sect. 4 we define $d_{\mathrm{pmGH}}$ and $d_*$ and prove various properties, in particular the relationship between the two in Theorem 4.11. This section may be of independent interest.

Section 5 contains the definition of a tangent measure using $d_*$ and develops the theory of tangent measures in parallel to the basic theory of Preiss [5].

In Sect. 6 we conclude the proof of Theorems 1.1 and 1.2.

## 2 Preliminaries

Throughout this article $n$ will denote a fixed positive integer. For $m \in \mathbb{N}$, $\mathbb{R}^m$ will denote the $m$-dimensional real vector space, $\ell_2^m = (\mathbb{R}^m, \|\cdot\|_2)$ and $\ell_\infty^m = (\mathbb{R}^m, \|\cdot\|_\infty)$. We also write $\ell_\infty$ for the Banach space of all bounded real sequences equipped with the supremum norm.

Let $(X, d)$ be a metric space. If $S \subset X$ and $x \in X$ we will write

$$d(x, S) = \inf\{d(x, s) : s \in S\}$$

and

$$B(S, r) = \{x \in X : d(x, S) \leq r\},$$

the *closed* neighbourhood of $S$ of radius $r$. We write $B(x, r)$ for $B(\{x\}, r)$, the closed ball of radius $r$ centred at $x$. Similarly, we define

$$U(S, r) = \{x \in X : d(x, S) < r\}$$

and $U(x, r) = U(\{x\}, r)$, the open neighbourhood of $S$ and open ball centred on $x$, respectively. We will often omit notation for the metric $d$ when it is not necessary.

By a *measure* on $X$ we mean a non-negative countably sub-additive function $\mu$ defined on the power set of $X$ with $\mu(\emptyset) = 0$. A set $A \subset X$ is *$\mu$-measurable* if

$$\mu(F) = \mu(F \cap E) + \mu(F \setminus E) \quad \forall F \subset X.$$

The set of all $\mu$-measurable subsets of $X$ is a $\sigma$-algebra and $\mu$ is countably additive when restricted to the set of $\mu$-measurable sets. A *Borel measure* on $X$ is a measure for which all Borel subsets of $X$ are $\mu$-measurable. A Borel measure is *Borel regular* if, for any $S \subset X$, there exists a Borel $B \supset S$ with $\mu(B) = \mu(S)$. Note that, for any $Y \subset X$, the monotonicity of $\mu$ implies $\mu(Y \cap S) = \mu(Y \cap B)$. The *support* of $\mu$, denoted $\operatorname{spt} \mu$, is the smallest closed set $C \subset X$ with $\mu(X \setminus C) = 0$.

We write $\mathcal{M}(X)$ for the set of all Borel regular measures on $X$ and define $\mathcal{M}_{\mathrm{loc}}(X)$ to be those $\mu \in \mathcal{M}(X)$ that are finite on all balls in $X$. By [21, Theorem 2.2.2], for any $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$, and any Borel $B \subset X$,

$$\mu(B) = \sup\{\mu(C) : C \subset B \text{ closed}\} = \inf\{\mu(U) : U \supset B \text{ open}\}.$$

Also note that, for any $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$, $\operatorname{spt} \mu$ is separable. Consequently, if $X$ is complete, any $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ is a Radon measure. That is, for every Borel

$B \subset X$,

$$\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}.$$

### 2.1 Rectifiable subsets of a metric space

For $s \geq 0$, $0 \leq \delta \leq \infty$ and $A \subset X$, define

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i\in\mathbb{N}} \operatorname{diam}(A_i)^s : A \subset \bigcup_{i\in\mathbb{N}} A_i,\ \operatorname{diam}(A_i) \leq \delta\right\}$$

and

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A),$$

the $s$-dimensional *Hausdorff measure* of $A$. For any $s \geq 0$ and any metric space $X$, $\mathcal{H}^s$ is a Borel regular measure on $X$, see [1, Section 4]. We note that $\mathcal{H}^s_\infty(A)$, the *$s$-dimensional Hausdorff content* of $A$, is bounded above by $\operatorname{diam}(A)^s$.

For $L \geq 0$, a function $f\colon (X, d) \to (Y, \rho)$ between two metric spaces is $L$-Lipschitz if

$$\rho(f(x), f(y)) \leq Ld(x, y) \quad \forall x, y \in X.$$

Note that, for any $s \geq 0$, $0 \leq \delta \leq \infty$ and $A \subset Y$,

$$\mathcal{H}^s_{L\delta}(f(A)) \leq L^s \mathcal{H}^s_\delta(A).$$

In particular, Lipschitz functions increase $\mathcal{H}^s$ and $\mathcal{H}^s_\infty$ by at most a multiplicative factor of $L^s$. A function $f\colon (X, d) \to (Y, \rho)$ is *$L$-bi-Lipschitz* if it is $L$-Lipschitz and invertible onto its image with $L$-Lipschitz inverse. That is, for any $x, y \in Y$,

$$\frac{1}{L}\rho(x, y) \leq \rho'(f(x), f(y)) \leq L\rho(x, y) \quad \forall x, y \in X.$$

For a $\mathcal{H}^s$-measurable $S \subset X$ and $x \in X$ define the *upper* and *lower Hausdorff densities* of $S$ at $x$ by

$$\Theta^{*,s}(S, x) := \limsup_{r\to 0} \frac{\mathcal{H}^n(B(x, r))}{(2r)^s}$$

and

$$\Theta^s_*(S,x) := \liminf_{r\to 0} \frac{\mathcal{H}^n(B(x,r))}{(2r)^s}$$

respectively.

**Definition 2.1** A $\mathcal{H}^n$-measurable $E \subset X$ is *n-rectifiable* if there exist a countable number of Lipschitz $f_i : A_i \subset \mathbb{R}^n \to X$ such that

$$\mathcal{H}^n\left(E \setminus \bigcup_{i\in\mathbb{N}} f_i(A_i)\right) = 0.$$

A $\mathcal{H}^n$-measurable $S \subset X$ is *purely n-unrectifiable* if every $n$-rectifiable $E \subset X$ satisfies $\mathcal{H}^n(S \cap E) = 0$.

By the same proof as for rectifiable subsets of Euclidean space (see [1, Theorem 15.6]), we have the following decomposition result.

**Lemma 2.2** *Let $Y \subset X$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(Y) < \infty$. There exists a decomposition $Y = E \cup S$ where $E$ is n-rectifiable and $S$ is purely n-unrectifiable.*

Recall the following classical result regarding Hausdorff densities.

**Lemma 2.3** ([21] 2.10.18) *Let $s > 0$ and $S \subset X$ be $\mathcal{H}^s$-measurable with $\mathcal{H}^s(S) < \infty$. Then*

1. *For $\mathcal{H}^s$-a.e. $x \in S$*

$$\Theta^{*,s}(S,x) \le 1;$$

2. *For $\mathcal{H}^s$-a.e. $x \in X \setminus S$*

$$\Theta^{*,s}(S,x) = 0.$$

The following theorem of Kirchheim precisely describes the local structure of rectifiable subsets of a metric space. Note the statements of theorem 2.4 and lemma 2.5 are slightly different to those in [10] due to the choice of normalisation of Hausdorff measure.

**Theorem 2.4** (Theorem 9 [10]) *Let $E \subset X$ be n-rectifiable. For $\mathcal{H}^n$-a.e. $x \in E$ there exists a norm $\|\cdot\|_x$ on $\mathbb{R}^n$, a map $f_x : X \to \mathbb{R}^n$ and a closed set $C_x \subset E$ such that $f_x(x) = 0$,*

$$\lim_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap C_x)}{(2r)^n} = 1 \tag{7}$$

*and*

$$\limsup_{r\to 0}\left\{\left|1-\frac{\|f_x(y)-f_x(z)\|_x}{d(y,z)}\right| : y\neq z\in C_x\cap B(x,r)\right\}=0.$$

*Note that, if* $\mathcal{H}^n(E)<\infty$, (7) *and Lemma* 2.3 *item 1 imply*

$$\frac{\mathcal{H}^n(B(x,r))}{(2r)^n}\to 1 \tag{8}$$

*and*

$$\frac{\mathcal{H}^n(B(x,r)\setminus C_x)}{(2r)^n}\to 0. \tag{9}$$

We also note the following.

**Lemma 2.5** (Lemma 6 [10]) *If* $\|\cdot\|$ *is a norm on* $\mathbb{R}^n$ *then* $\mathcal{H}^n(B(0,r))=(2r)^n$. *Here* $\mathcal{H}^n$ *and* $B(0,1)$ *are defined with respect to* $\|\cdot\|$.

We also require the McShane extension theorem.

**Theorem 2.6** *Let* $S\subset X$ *and* $0<\alpha\leq 1$. *Suppose that* $f: S\to \ell_\infty^m$ *(respectively into* $\ell_\infty$*) is an* $\alpha$*-Hölder map with Hölder constant* $H$. *Then there exists an* $\alpha$*-Hölder extension* $F: X\to \ell_\infty^m$ *(respectively into* $\ell_\infty$*) of* $f$ *with Hölder constant* $H$. *In particular,* $F$ *is* $\alpha$*-Hölder with Hölder constant* $\sqrt{m}H$ *as a map into* $\ell_2^m$.

*Moreover, for each* $1\leq i\leq m$ *(respectively, for each* $i\in\mathbb{N}$*),* $F$ *may be chosen such that*

$$\inf\{f_i(x): x\in S\}\leq F_i\leq \sup\{f_i(x): x\in S\}.$$

*Finally, if* $S\subset B(x,r)$ *for some ball* $B(x,r)\subset X$ *then* $F$ *may be chosen with* spt $F\subset B(x,r')$, *for* $r'=r+(H^{-1}\|f\|_\infty)^{1/\alpha}$.

Here and throughout, spt $F$ denotes the closure of the set

$$\{x\in X: F(x)\neq 0\}.$$

*Proof* The proof of the first part is standard, see [22, Section 6]. For the second part, one simply replaces the $i$th coordinate of $F$ by

$$\max\{\inf\{f_i(x): x\in S\}, \min\{F_i, \sup\{f_i(x): x\in S\}\}\}$$

for each $1\leq i\leq m$ (respectively for each $i\in\mathbb{N}$).

Finally, if $S \subset B(x,r)$ then $f$ is bounded. We first extend $f$ to a map $f'$ defined on

$$S' := S \cup (X \setminus B(x,r'))$$

by defining $f'$ to equal zero on $X \setminus B(x,r')$. Then $f'$ is also $\alpha$-Hölder with Hölder constant $H$. Applying the second part of the theorem to $f'$ gives the final statement. □

### 2.2 Doubling measures

Much of the theory of tangent measures requires the original measure to be asymptotically doubling.

**Definition 2.7** For $M \geq 1$, $\mu \in \mathcal{M}(X)$ is *$M$-doubling* if

$$0 < \mu(B(x,2r)) \leq M\mu(B(x,r)) < \infty$$

for each $x \in X$ and each $r > 0$. A $\mu \in \mathcal{M}(X)$ is *asymptotically doubling* if

$$\limsup_{r \to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \quad \text{for } \mu\text{-a.e. } x \in X.$$

To proceed we need the following standard facts.

**Lemma 2.8** *Let $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$. For any $r > 0$,*

$$x \mapsto \mu(U(x,r)) \text{ is lower semicontinuous} \tag{10}$$

*and*

$$x \mapsto \mu(B(x,r)) \text{ is upper semicontinuous.} \tag{11}$$

*Consequently, for any $R, M, s > 0$ the set*

$$A := \{x \in X : \mu(B(x,2r)) \leq M\mu(B(x,r)) \ \forall 0 < r < R\}$$

*is a Borel set, as are*

$$B := \{x \in X : \mu(B(x,r)) < Mr^s \ \forall 0 < r < R\}$$

*and*

$$C := \{x \in X : \mu(B(x,r)) > Mr^s \ \forall 0 < r < R\}.$$

*Proof* If $x_i \to x$ with $d(x_i, x)$ monotonically decreasing then

$$U(x, r - d(x, x_i))$$

monotonically increases to $U(x, r)$ and is contained in $U(x_i, r)$, proving (10). Similarly,

$$B(x, r + d(x, x_i))$$

monotonically decreases to $B(x, r)$ and contains in $B(x_i, r)$, proving (11). Consequently, for any $R, M > 0$ and $\delta \geq 0$, the set

$$A_{R,M,\delta} := \{x \in X : \mu(U(x, (2+\delta)r)) \leq M\mu(B(x, r)) \ \forall 0 < r < R\}$$

is closed, as is

$$A = \bigcap_{\delta \in \mathbb{Q}^+} A_{R,M,\delta}.$$

The proof that $B$ and $C$ are Borel is similar. □

**Lemma 2.9** *Let $\mu \in \mathcal{M}(X)$. Suppose that for some $M, R > 0$ a set $Y \subset X$ satisfies*

$$0 < \mu(B(x, 2r)) \leq M\mu(B(x, r)) < \infty \quad \forall 0 < r < R, \ \forall x \in Y. \tag{12}$$

*Then for any $x \in Y$ and $r < R/2$, $B(x, r) \cap Y$ is contained in $M^4$ balls of radius $r/2$ centred at points of $B(x, r) \cap Y$. In particular, for any $\epsilon > 0$, $B(x, r) \cap Y$ is contained in $M^{-4\log_2 \epsilon}$ balls of radius $\epsilon r$.*

*Proof* Let $\mathcal{N}$ be a maximal disjoint $r/4$-net of $B(x, r) \cap Y$. Then for any $y \in \mathcal{N}$, (12) gives

$$\mu(B(y, r/4)) \geq \frac{\mu(B(y, r/2))}{M} \geq \frac{\mu(B(y, r))}{M^2} \geq \frac{\mu(B(y, 2r))}{M^3} \geq \frac{\mu(B(x, r))}{M^3}. \tag{13}$$

For $y \in \mathcal{N}$ the $B(y, r/4)$ are disjoint subsets of $B(x, 2r)$. Therefore, if $y_1, \ldots, y_N \in \mathcal{N}$,

$$\mu(B(x, 2r)) \geq \mu\left(\bigcup_{i=1}^{N} B(y_i, r/4)\right) = \sum_{i=1}^{N} \mu(B(y_i, r/4))$$

$$\geq \sum_{i=1}^{N} \frac{\mu(B(x,r))}{M^3} \geq \sum_{i=1}^{N} \frac{\mu(B(x,2r))}{M^4}$$

using (13) for the penultimate inequality and (12) for the final inequality. Thus $N \leq M^4$.

The statement about covering by balls of radius $\epsilon r$ follows by induction. □

**Theorem 2.10** *Any asymptotically doubling $\mu \in \mathcal{M}(X)$ satisfies the Lebesgue density theorem. That is, for any $S \subset X$ and for $\mu$-a.e. $x \in S$,*

$$\lim_{r \to 0} \frac{\mu(S \cap B(x,r))}{\mu(B(x,r))} = 1.$$

*Such an $x \in S$ is a* Lebesgue density point *of $S$.*

*Proof* This is a standard result for doubling measures. First one proves the Vitali covering theorem [22, Theorem 1.6] and uses it to deduce the Lebesgue differentiation theorem [22, Theorem 1.8] and hence the Lebesgue density theorem for Borel sets $S \subset X$.

The proof for asymptotically doubling measures is the same since the proof of the Vitali covering theorem works, with minor modifications, only assuming an almost everywhere countable decomposition into Borel sets, each satisfying (12) for some $M, R > 0$. Assuming $\mu$ is asymptotically doubling, Lemma 2.8 gives such a decomposition of spt $\mu$.

The statement for arbitrary $S \subset X$ follows by considering a Borel $S' \supset S$ with $\mu(S') = \mu(S)$. □

### 2.3 Prokhorov's theorem

A sequence $\mu_i \in \mathcal{M}_{\mathrm{loc}}(X)$ *weak** converges to $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ if, for every bounded and continuous $g\colon X \to \mathbb{R}$ with bounded support,

$$\int g \, \mathrm{d}\mu_i \to \int g \, \mathrm{d}\mu.$$

We write $\mu_i \to \mu$ to denote this convergence.

Recall that, if $\mu_i \to \mu \in \mathcal{M}_{\mathrm{loc}}(X)$ and $x_i \to x \in X$, then for every $r > 0$,

$$\mu(U(x,r)) \leq \liminf_{i \to \infty} \mu_i(U(x_i,r)) \tag{14}$$

and

$$\mu(B(x,r)) \geq \limsup_{i \to \infty} \mu_i(B(x_i,r)). \tag{15}$$

Indeed, these follow by combining the standard proof of lower/upper semi-continuity for a fixed open/closed set and the proof of Eqs. (10) and (11).

Recall Prokhorov's theorem, reformulated to describe convergence in $\mathcal{M}_{\mathrm{loc}}(X)$.

**Theorem 2.11** (Theorem 2.3.4 [23]) *Let $X$ be a complete and separable metric space and $x \in X$. A set $\mathcal{S} \subset \mathcal{M}_{\mathrm{loc}}(X)$ is pre-compact (with respect to weak* convergence) if and only if, for every $r, \epsilon > 0$:*

- *$\{\mu(B(x,r)) : \mu \in \mathcal{S}\}$ is bounded and*
- *There exists a compact $K \subset X$ such that*

$$\mu(B(x,r) \setminus K) \leq \epsilon \quad \forall \mu \in \mathcal{S}. \tag{16}$$

**Definition 2.12** Let $X$ be a metric space and $x \in X$. For $\mu, \nu \in \mathcal{M}_{\mathrm{loc}}(X)$, and $L, r > 0$, define

$$F_x^{L,r}(\mu,\nu) = \sup\left\{\int g \,\mathrm{d}(\mu - \nu) : g\colon X \to [-1,1], \right.$$
$$\left. L\text{-Lipschitz},\ \operatorname{spt} g \subset B(x,r)\right\}.$$

Define $F_x^L(\mu,\nu)$ to be the infimum, over all $0 < \epsilon < 1/2$, for which

$$F_x^{L,1/\epsilon}(\mu,\nu) < \epsilon.$$

If no such $0 < \epsilon < 1/2$ exists, set $F_x^L(\mu,\nu) = 1/2$. Also define $F_x(\mu,\nu)$ to be the infimum, over all $0 < \epsilon < 1/2$, for which

$$F_x^{1/\epsilon,1/\epsilon}(\mu,\nu) < \epsilon.$$

If no such $0 < \epsilon < 1/2$ exists, set $F_x(\mu,\nu) = 1/2$.

Note that $F_x^{L,r}$ is increasing in $L, r$ and

$$F_x^{L',r} \leq \frac{L'}{L} F_x^{L,r}. \tag{17}$$

Note that, for any Borel $S \subset X$,

$$F_x^{L,r}(\mu, \mu|_S) \leq \mu(B(x,r) \setminus S). \tag{18}$$

The reason for allowing the Lipschitz constants of the functions to increase when defining $F_x$ will become clear in Lemma 4.10.

**Proposition 2.13** *Let $X$ be a metric space, $x \in X$ and $L > 0$. Convergence in $F_x$, equivalently $F_x^L$, implies weak* convergence in $\mathcal{M}_{\mathrm{loc}}(X)$ and, if $X$ is complete and separable, the converse is true.*

*Proof* Equivalence of convergence in $F_x$ and $F_x^L$ follows from (17).

Suppose $F_x(\mu_i, \mu) \to 0$ and that $g\colon X \to \mathbb{R}$ is bounded and continuous with $\operatorname{spt} g \subset B(x, r)$. The fact that

$$\int g \,\mathrm{d}\mu_i \to \int g \,\mathrm{d}\mu$$

follows from a standard approximation argument [16, Section 5.1]. Indeed, observe that the sequence

$$g_k(y) := \inf\{g(w) + kd(w, y) : w \in X\}$$

monotonically increases to $g$ and that each $g_k$ is $k$-Lipschitz with

$$\operatorname{spt} g_k \subset B(x, r + \|g\|_\infty).$$

Thus,

$$\begin{aligned}\liminf_{i\to\infty} \int g \,\mathrm{d}\mu_i &\geq \limsup_{k\to\infty} \liminf_{i\to\infty} \int g_k \,\mathrm{d}\mu_i \\ &= \limsup_{k\to\infty} \int g_k \,\mathrm{d}\mu \\ &= \int g \,\mathrm{d}\mu,\end{aligned}$$

using monotonicity of the integral for the inequality, the fact that $F_x(\mu_i, \mu) \to 0$ for the first equality and the monotone convergence theorem for the final equality. The reverse "limsup" equality follows from a similar argument that involves approximating $g$ from above.

Now suppose that $X$ is complete and separable, that $\mu_i \to \mu$ in $\mathcal{M}_{\mathrm{loc}}(X)$ but $F_x(\mu_i, \mu) \not\to 0$. By taking a (non-relabelled) subsequence, there exist an $r > 0$ such that $F_x(\mu_i, \mu) > 1/r$ for all $i \in \mathbb{N}$. That is, for each $i \in \mathbb{N}$ there exists an $r$-Lipschitz $g_i\colon X \to [-1, 1]$ with $\operatorname{spt} g_i \subset B(x, r)$ such that

$$\left|\int g_i \,\mathrm{d}(\mu_i - \mu)\right| \geq 1/r.$$

Set $r' = r + 1/r$. By Theorem 2.11, there exist an $M > 0$ and a compact $K \subset B(x, r')$ such that, for all $i \in \mathbb{N}$, $\mu_i(B(x, r')) \leq M$ and

$$\mu_i(B(x, r') \setminus K) < 1/8r. \tag{19}$$

By increasing $K$ if necessary, we may also suppose $\mu(B(x, r') \setminus K) < 1/8r$.

The Arzelà–Ascoli theorem implies that, after taking a further subsequence, the $g_i$ converge uniformly on $K$ to an $r$-Lipschitz $g\colon K \to [-1, 1]$ with $\operatorname{spt} g \subset B(x, r)$. By Theorem 2.6, we may extend $g$ to an $r$-Lipschitz function defined on the whole of $X$ with $\operatorname{spt} g \subset B(x, r')$. Then, for any sufficiently large $i$ such that

$$\sup\{|g(y) - g_i(y)| : y \in K\}(\mu_i + \mu)(B(x, r')) \leq 1/4r,$$

which exists by (15),

$$\begin{aligned}
\left|\int g \,\mathrm{d}(\mu_i - \mu)\right| &\geq \left|\int_K g \,\mathrm{d}(\mu_i - \mu)\right| - 1/4r \\
&\geq \left|\int_K g_i \,\mathrm{d}(\mu_i - \mu)\right| - 1/2r \\
&\geq \left|\int g_i \,\mathrm{d}(\mu_i - \mu)\right| - 3/4r \geq 1/4r.
\end{aligned}$$

This contradicts $\mu_i \to \mu$. □

**Lemma 2.14** *For any metric space $X$, $x \in X$ and $L > 0$, $F_x$ and $F_x^L$ are metrics on $\mathcal{M}_{\mathrm{loc}}(X)$. If $X$ is complete and separable then so are $(\mathcal{M}_{\mathrm{loc}}(X), F_x)$ and $(\mathcal{M}_{\mathrm{loc}}(X), F_x^L)$.*

*Proof* Certainly $F_x$ are positive, finite on $\mathcal{M}_{\mathrm{loc}}(X)$ and symmetric. If $F_x(\mu, \nu) = 0$, then the monotonicity of $F_x^{L,r}$ implies $F_x^{L,r}(\mu, \nu) = 0$ for every $L, r > 0$. Consequently, the monotone convergence theorem implies $\mu(C) = \nu(C)$ for all closed and bounded $C \subset X$ and hence $\mu(B) = \nu(B)$ for all Borel $B \subset X$. Since $\mu$ and $\nu$ are both Borel regular, this implies $\mu = \nu$.

To see that $F_x$ satisfies the triangle inequality, let $\mu, \nu, \lambda \in \mathcal{M}_{\mathrm{loc}}(X)$. If either of $F_x(\mu, \lambda)$ or $F_x(\lambda, \nu)$ equal 1/2 then the triangle inequality holds since $F_x(\mu, \nu) \leq 1/2$. Otherwise, for $\delta > 0$, let $0 < \epsilon_1 < F_x(\mu, \lambda) + \delta$ and $0 < \epsilon_2 < F_x(\lambda, \nu) + \delta$ satisfy

$$F_x^{\frac{1}{\epsilon_1}, \frac{1}{\epsilon_1}}(\mu, \lambda) < \epsilon_1 \quad \text{and} \quad F_x^{\frac{1}{\epsilon_2}, \frac{1}{\epsilon_2}}(\lambda, \nu) < \epsilon_2.$$

Then the triangle inequality and monotonicity of $F_x^{L,r}$ imply

$$\begin{aligned} F_x^{\frac{1}{\epsilon_1+\epsilon_2},\frac{1}{\epsilon_1+\epsilon_2}}(\mu,\nu) &\leq F_x^{\frac{1}{\epsilon_1+\epsilon_2},\frac{1}{\epsilon_1+\epsilon_2}}(\mu,\lambda) + F_x^{\frac{1}{\epsilon_1+\epsilon_2},\frac{1}{\epsilon_1+\epsilon_2}}(\lambda,\nu) \\ &\leq F_x^{\frac{1}{\epsilon_1},\frac{1}{\epsilon_1}}(\mu,\lambda) + F_x^{\frac{1}{\epsilon_2},\frac{1}{\epsilon_2}}(\lambda,\nu) \\ &\leq \epsilon_1 + \epsilon_2 \\ &\leq F_x(\mu,\lambda) + F_x(\lambda,\nu) + 2\delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, this shows that $F_x$ is a metric.

If $x_i$, $i \in \mathbb{N}$ is a dense subset of $X$, one easily checks that the set

$$\{q_1\delta_{x_1} + \ldots q_m\delta_{x_m} : m \in \mathbb{N},\ q_i \in \mathbb{Q}\}$$

is dense in $\mathcal{M}_{\mathrm{loc}}(X)$.

Finally, the completeness of $F_x$ immediately follows from [23, Corollary 2.3.5] (that convergence of $\int g\,\mathrm{d}\mu_i$ for every bounded Lipschitz $g$ implies (16)), Theorem 2.11 and Proposition 2.13. The proof for $F_x^L$ is similar. □

### 2.4 Gromov distances, embeddings and convergence

Pointed Gromov–Hausdorff *convergence* of metric spaces is prevalent in the literature. However, the metric defining this convergence is less commonly used. In this subsection we recall Gromov's metric for pointed Gromov–Hausdorff convergence and prove some basic facts about it that are missing from the literature (namely that it is a complete metric on the isometry classes of pointed metric spaces). Our presentation of this is sufficiently general so that the corresponding statements for the pointed measured Gromov–Hausdorff distance, and our metric $d_*$, easily follow (see Sect. 4).

We use the following modification of the Hausdorff distance.

**Definition 2.15** *(Section 6* [24]*)* Let $Z$ be a metric space, $z \in Z$ and $X, Y \subset Z$. Define $H_z(X, Y)$ to be the infimum over all $0 < \epsilon < 1/2$ for which

$$X \cap B(z, 1/\epsilon) \subset B(Y, \epsilon) \quad \text{and} \quad Y \cap B(z, 1/\epsilon) \subset B(X, \epsilon). \tag{20}$$

If no such $0 < \epsilon < 1/2$ exists, set $H_z(X, Y) = 1/2$. Note that, if (20) is satisfied for some $\epsilon > 0$ then it is satisfied for all $\epsilon' \geq \epsilon$.

**Lemma 2.16** *For any metric space $Z$ and $z \in Z$, $H_z$ is a pseudometric on the power set of $Z$ and is a metric on the closed subsets of $Z$. If $Z$ is complete then so is the set of closed subsets of $Z$ equipped with $H_z$.*

*Proof* Let $\zeta$ be the metric on $Z$, let $X, Y, W \subset Z$ and let

$$d_{XW} = H_z(X, W) \quad \text{and} \quad d_{WY} = H_z(W, Y).$$

If one of $d_{XW}$ or $d_{WY}$ equals $1/2$ there is nothing to prove. Otherwise, let $0 < \delta < 1/2 - \min\{d_{XZ}, d_{ZY}\}$. If

$$x \in X \cap B(z, 1/(d_{XZ} + d_{ZY} + 2\delta)) \tag{21}$$

then (20) for $X$ and $W$ gives a $w \in W$ with

$$\zeta(w, x) \leq d_{XW} + \delta. \tag{22}$$

In particular, since $d_{XW}, d_{WZ} < 1/2 - \delta$,

$$\begin{aligned}\zeta(w, z) &\leq \zeta(w, x) + \zeta(x, z) \leq d_{XW} + \delta + 1/(d_{XW} + d_{WY} + 2\delta) \\ &\leq 1/(d_{WY} + \delta).\end{aligned}$$

Therefore (20) for $W$ and $Y$ gives a $y \in Y$ with

$$\zeta(w, y) \leq d_{WY} + \delta. \tag{23}$$

In particular,

$$\zeta(x, w) \leq d_{XW} + d_{WY} + 2\delta. \tag{24}$$

Thus Eqs. (21) and (24) show that (20) holds between $X$ and $W$ for $\epsilon = d_{XW} + d_{WY} + 2\delta$. Since $\delta > 0$ is arbitrary, this completes the proof.

If $C \subset Z$ is closed and $H_z(C, S) = 0$, then for any $x \in S$ there exist $x_i \in C$ with $x_i \to x$. Since $C$ is closed, $x \in C$ and so $C \subset S$. Consequently, $H_z$ is a metric on the closed subsets of $Z$.

Completeness follows from the same argument as the regular Hausdorff metric [25, Proposition 7.3.7]. □

In this section we consider isometric embeddings $X \to Z$ of one metric space into another. We identify $X$, its elements and any measure on $X$, with their isometric images in $Z$.

We require the following standard construction.

**Lemma 2.17** *Let $X_i$ be a sequence of metric spaces and, for each $i \leq j \in \mathbb{N}$, suppose that there exist isometric embeddings $X_i, X_j \to (Z_{i,j}, \zeta_{i,j})$ into some metric space. There exists a metric space $(Z, \zeta)$ and, for each $i \in \mathbb{N}$, isometric embeddings $X_i \to Z$ such that, for each $i \leq j \in \mathbb{N}$, $w_i \in X_i$ and $w_j \in X_j$, $\zeta(w_i, w_j) \leq \zeta_{i,j}(w_i, w_j)$, with equality if $j = i + 1$.*

*Proof* Define a function $\zeta$ on

$$\tilde{Z} = \bigsqcup_{i \in \mathbb{N}} X_i$$

as follows. For $i \leq j \in \mathbb{N}$, $w_i \in X_i$ and $w_j \in X_j$, define $\zeta(w_i, w_j)$ to be the infimum of

$$\sum_{p=1}^{m} \zeta_{k_{p-1},k_p}(w_{p-1}, w_p)$$

over all $m \in \mathbb{N}$, $i = k_0 \leq k_1 \leq \ldots \leq k_m = j$ and $w_k \in X_k$ for each $i < k < j$. Then $\zeta$ is a pseudometric on $Z$ such that $\zeta(w_i, w_j) \leq \zeta_{i,j}(w_i, w_j)$ for all $i \leq j \in \mathbb{N}$, $w_i \in X_i$ and $w_j \in X_j$. If $j = i + 1$, the triangle inequality for $\zeta_{i,i+1}$ gives the reverse inequality. Finally, let $(Z, \zeta)$ be the metric space obtained from $(\tilde{Z}, \zeta)$. Since each $X_i \to \tilde{Z}$, the projection to $Z$ gives an isometric embedding $X_i \to Z$. □

**Definition 2.18** A *pointed* metric space $(X, x)$ consists of a metric space $X$ and a distinguished point $x \in X$. An *isometric embedding* $(X, x) \to (Y, y)$ of pointed metric spaces is an isometric embedding of metric spaces with $x = z$. An *isometry* $(X, x) \to (Y, y)$ is a surjective isometric embedding.

A *pointed metric measure space* $(X, \mu, x)$ consists of a complete and separable pointed metric space $(X, x)$ and a $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$.

**Definition 2.19** Let $(X, \mu, x)$ and $(Y, \nu, y)$ be pointed metric measure spaces and $a, b \geq 0$. Define

$$G_{a,b}((X, \mu, x), (Y, \nu, y))$$

to be the infimum, over all pointed metric spaces $(Z, z)$ and all isometric embeddings $(X, x), (Y, y) \to (Z, z)$ of

$$aH_z(X, Y) + bF_z(\mu, \nu).$$

**Proposition 2.20** *For any $a, b \geq 0$, $G_{a,b}$ is a complete pseudometric on the class of pointed metric measure spaces. Moreover,*

$$G_{a,b}((X_i, \mu_i, x_i), (X, \mu, x)) \to 0$$

*if and only if there exists a complete and separable pointed metric space $(Z, z)$ and isometric embeddings $(X_i, x_i) \to (Z, z)$ and $(X, x) \to (Z, z)$ such that*

$$aH_z(X_i, X) + bF_z(\mu_i, \mu) \to 0.$$

*Proof* First observe the following. Let $\zeta, \zeta'$ be metrics on a set $Z$ with $\zeta \leq \zeta'$. Let $z \in Z$ and suppose that $X, Y \subset Z$. Then

$$H_z^{\zeta}(X, Y) \leq H_z^{\zeta'}(X, Y), \tag{25}$$

where the superscript denotes the metric used to define $H_z$. Also, since any Lipschitz function on $X \cup Y \subset Z$ with respect to $\zeta$ is Lipschitz with respect to $\zeta'$, if $\mu \in \mathcal{M}_{\text{loc}}(X)$ and $\nu \in \mathcal{M}_{\text{loc}}(Y)$,

$$F_z^{\zeta}(\mu, \nu) \leq F_z^{\zeta'}(\mu, \nu). \tag{26}$$

To prove the triangle inequality for $G_{a,b}$, for $i = 1, 2, 3$ let $(X_i, d_1, \mu_i, x_i)$ be a pointed metric measure space and suppose there exist isometric embeddings

$$(X_1, d_1, x_1), (X_2, d_2, x_2) \to (Z_{1,2}, \zeta_{1,2}, z_{1,2})$$

and

$$(X_2, d_2, x_2), (X_3, d_3, x_3) \to (Z_{2,3}, \zeta_{2,3}, z_{2,3}).$$

Let $\zeta_{1,3}$ be the metric on $X_1 \sqcup X_3$ that equals $d_1$ and $d_3$ on $X_1$ and $X_3$ respectively and $\zeta(w_1, w_3) = 1 + d_1(w_1, x_1) + d_3(w_3, x_3)$ whenever $w_1 \in X_1$ and $w_3 \in X_3$. Let $(Z, \zeta)$ be the metric space obtained from Lemma 2.17 and note that necessarily $x_1 = x_2 = x_3 =: z$ in $Z$. The triangle inequality for $H_z$ and $F_z$ in $Z$ imply

$$\begin{aligned} aH_z(X_1, X_3) + bF_z(\mu_1, \mu_3) \leq\ & aH_z(X_1, X_2) + bF_z(\mu_1, \mu_2) \\ & + aH_z(X_2, X_3) + bF_z(\mu_2, \mu_3). \end{aligned} \tag{27}$$

Combining Eqs. (25) to (27) gives the triangle inequality for $G_{a,b}$.

Now let $(X_i, \mu_i, x_i)$ be a Cauchy sequence with respect to $G_{a,b}$ and, for each $i \leq j \in \mathbb{N}$, let $(Z_{i,j}, \zeta_{i,j}, z_{i,j})$ be a pointed metric space for which there exist isometric embeddings

$$(X_i, x_i), (X_j, x_j) \to (Z_{i,j}, \zeta_{i,j}, z_{i,j})$$

with

$$aH_{z_{i,j}}(X_i, X_j) + bF_{z_{i,j}}(\mu_i, \mu_j) \leq 2G_{a,b}((X_i, \mu_i, x_i), (X_j, \mu_j, x_j)). \tag{28}$$

Let $(Z, \zeta)$ be the completion of the metric space given by Lemma 2.17 and note that necessarily $x_1 = x_2 = \ldots =: z$ in $Z$. Since each $X_i$ is separable,

so is $Z$. Since each $X_i$ is complete, they are closed subsets of $Z$ and so the $\mu_i \in \mathcal{M}_{\mathrm{loc}}(Z)$. Equations (25), (26) and (28) imply that the $X_i$ and $\mu_i$ are Cauchy sequences with respect to $H_z$ and $F_z$ in $Z$. Since $Z$ is complete, $X_i$ and $\mu_i$ converge to some $X$ and $\mu$. Consequently

$$G_{a,b}((X_i, \mu_i, x_i), (X, \mu, x)) \to 0.$$

Since any convergent sequence is Cauchy, this also proves the moreover statement. □

**Definition 2.21** *(Section 6* [24]*)* The *Gromov–Hausdorff distance* between pointed metric spaces $(X, x)$ and $(Y, y)$ is defined to be

$$d_{\mathrm{pGH}}((X, x), (Y, y)) = G_{1,0}((X, 0, x), (Y, 0, y)).$$

The definition of $d_{\mathrm{pGH}}$ in [24] only requires isometric embeddings into $(Z, \zeta)$ satisfying $\zeta(x, y) \leq \epsilon$, not necessarily $x = y$. However, the next lemma shows that two definitions are bi-Lipschitz equivalent. We will see that fixing the distinguished point simplifies several results.

**Lemma 2.22** *Let $(Z, \zeta)$ be a metric space, $X, Y \subset Z$, $x \in X$ and $y \in Y$ with $x \neq y$.*

1. *The metric $\tilde{\zeta}$ on $X \sqcup Y$ that equals $\zeta$ on each of $X$ and $Y$ and $\tilde{\zeta}(w, w') = \zeta(w, w') + \zeta(x, y)$ for any $w \in X$ and $w' \in Y$ satisfies*

$$\tilde{\zeta}(w, x) \leq \tilde{\zeta}(w, y)\ \forall w \in X \quad \textit{and} \quad \tilde{\zeta}(w, y) \leq \tilde{\zeta}(w, x)\ \forall w \in Y. \tag{29}$$

2. *Let $\tilde{\zeta}$ be* any *metric on $X \sqcup Y$ that equals $\zeta$ on each of $X$ and $Y$ that satisfies* (29)*. If $(Z', \zeta')$ is the quotient metric space of $(X \sqcup Y, \tilde{\zeta})$ with respect to the relation $x \sim y$, there exist isometric embeddings $X, Y \to Z'$ with $x = y$.*
3. *There exists a metric space $(Z', \zeta')$ and isometric embeddings $X, Y \to Z'$ with $x = y$ such that $\zeta' \leq \zeta + \zeta(x, y)$ on $X \cup Y$.*

*Proof* The triangle inequality for $\zeta$ implies the triangle inequality for $\tilde{\zeta}$. Equation (29) is equivalent to the triangle inequality for $\zeta$. This proves item 1.

The quotient metric of any metric $\tilde{\zeta}$ with respect to the relation $x \sim y$ is given by

$$\zeta'(w, w') = \min\{\tilde{\zeta}(w, w'), \tilde{\zeta}(w, x) + \tilde{\zeta}(y, w'), \tilde{\zeta}(w', x) + \tilde{\zeta}(y, z)\}.$$

If $\tilde{\zeta}$ satisfies (29) then $\zeta'$ equals $\zeta$ on each of $X$ and $Y$. Indeed, if $w, w' \in X$,

$$\tilde{\zeta}(w, w') \leq \tilde{\zeta}(w, x) + \tilde{\zeta}(x, w') \leq \tilde{\zeta}(w, x) + \tilde{\zeta}(y, w'),$$

by the triangle inequality and (29). Thus $\zeta'(w, w') = \tilde{\zeta}(w, w')$. The corresponding statement for elements of $Y$ implies item 2.

Combining items 1 and 2 gives item 3. □

The following Lemma is an adaptation of the corresponding concepts of an $\epsilon$-isometry of bounded metric spaces, see [25, Corollary 7.3.28].

**Definition 2.23** Let $(X, d, x)$ and $(Y, \rho, y)$ be pointed metric spaces and $\epsilon > 0$. An *$\epsilon$-isometry* $f : (X, d, x) \to (Y, \rho, y)$ is a Borel map $f : B(x, 1/\epsilon) \to Y$ with $f(x) = y$ such that

$$|\rho(f(z), f(z')) - d(z, z')| \leq \epsilon \quad \forall z, z' \in B(x, 1/\epsilon) \tag{30}$$

and

$$B(y, 1/\epsilon - \epsilon) \subset B(f(B(x, 1/\epsilon)), \epsilon). \tag{31}$$

**Lemma 2.24** *Let $(Z, \zeta, z)$ be a pointed metric space, $X, Y$ separable subsets of $Z$ and $0 < \epsilon < 1/2$. If $H_z(X, Y) < \epsilon$ then there exists a Borel map*

$$f : X \cap B(z, 1/\epsilon) \to Y$$

*with $\zeta(f(x), x) \leq \epsilon$ for all $x \in X \cap B(z, 1/\epsilon)$ and*

$$Y \cap B(z, 1/\epsilon - \epsilon) \subset B(f(X \cap B(z, 1/\epsilon)), 2\epsilon). \tag{32}$$

*In particular, if $d_{\mathrm{pGH}}((X, x), (Y, y)) < \epsilon$, there exists a $2\epsilon$-isometry from $(X, x)$ to $(Y, y)$.*

*Conversely, if there exists an $\epsilon$-isometry from $(X, x)$ to $(Y, y)$ then there exists a pointed metric space $(Z, \zeta, z)$ and isometric embeddings*

$$(X, x), (Y, y) \to (Z, z)$$

*such that, for all $x' \in B(x, 1/\epsilon)$,*

$$\zeta(x', f(x')) \leq 2\epsilon.$$

*In particular,*

$$d_{\mathrm{pGH}}((X, x), (Y, y)) < 2\epsilon.$$

*Proof* Let $\delta > 0$ be such that $H_z(X, Y) < \epsilon - \delta$. Since $X$ is separable, there exist by countably many disjoint and non-empty Borel sets $X_j \subset X$, $j \in \mathbb{N}$, of diameter at most $\delta$ with $B(z, 1/\epsilon) \cap X = \cup_j X_j$. For each $j \in \mathbb{N}$, pick

$x_j \in X_j$. Since $H_z(X, Y) < \epsilon - \delta$, there exists $y_j \in Y$ with $\zeta(x_j, y_j) \leq \epsilon - \delta$. Define $f(w) = y_j$ for each $w \in X_j$. Then $f$ is a Borel map and, by the triangle inequality, for $x$ in any $X_j$,

$$\zeta(f(x), x) \leq \zeta(y_j, x_j) + \zeta(x_j, x) \leq \epsilon - \delta + \delta \leq \epsilon. \tag{33}$$

To see that (32) holds, let

$$y \in B\left(z, \frac{1}{\epsilon} - \epsilon\right)$$

By assumption there exists $x \in X$ with $\zeta(x, y) \leq \epsilon$. Then by the triangle inequality,

$$\zeta(x, z) \leq \zeta(x, y) + \zeta(y, z) \leq \epsilon + (1/\epsilon - \epsilon) \leq 1/\epsilon.$$

Consequently $f(x)$ is defined and, by (33) and the triangle inequality,

$$\zeta(f(x), y) \leq \zeta(f(x), x) + \zeta(x, y) \leq 2\epsilon,$$

proving (32).

For the converse statement, let $f$ be the assumed $\epsilon$-isometry and define $\zeta$ on $X \sqcup Y$ by $\zeta = d$ on $X$, $\zeta = \rho$ on $Y$ and

$$\zeta(z, t) = \inf\{d(z, w) + \rho(f(w), t) + \epsilon : w \in B(x, 1/\epsilon)\}$$

whenever $z \in X$ and $t \in Y$. Then $\zeta$ is a metric: if $z, z'' \in X$ and $w, w'' \in B(x, 1/\epsilon)$, the triangle inequality for $d$, $\rho$ and the fact that $f$ is an $\epsilon$-isometry imply

$$\begin{aligned} d(z, z') &\leq d(z, w) + d(w, w') + d(w', z') \\ &\leq d(z, w) + \rho(f(w), f(w')) + \epsilon + d(w', z') \\ &\leq d(z, w) + \rho(f(w), t) + \epsilon + d(w', z') + \rho(t, f(w')) + \epsilon \end{aligned}$$

for any $t \in Y$. Taking the infimum over all $w, w' \in B(x, 1/\epsilon)$ shows $\zeta(z, z') \leq \zeta(z, t) + \zeta(t, z')$. A similar argument for $t, t' \in Y$ and $z \in X$ shows that $\zeta$ satisfies the triangle inequality. Note that $\zeta$ satisfies $\zeta(x, y) = \epsilon$ and (29). Indeed, for any $w \in B(x, 1/\epsilon)$, since $f$ is an $\epsilon$-isometry,

$$d(z, x) \leq d(z, w) + d(w, x) \leq d(w, z) + \rho(f(w), y) + \epsilon$$

Taking the infimum over all $w \in B(x, 1/\epsilon)$ gives (29).

Let $(Z', \zeta')$ be the metric space obtained from Lemma 2.22 item 2. Observe that for any $x' \in B(x, 1/\epsilon)$, $\zeta'(x', f(x')) \leq \epsilon$, proving and the first containment in (20). If $y' \in Y \cap B(y, 1/\epsilon - \epsilon)$ then there exists $x' \in X \cap B(x, 1/\epsilon)$ with $\rho(y', f(x')) \leq \epsilon$, so that $\zeta'(x', y') \leq 2\epsilon$, proving the second containment in (20), for $\epsilon$ replaced by $2\epsilon$. □

Recall that a metric space $X$ is *proper* if all closed balls in $X$ are compact.

**Corollary 2.25** *The Gromov–Hausdorff distance is a complete and separable metric on the set $\mathbb{M}_p$ of isometry classes of proper pointed metric spaces. That is, if $(X, x)$ and $(Y, y)$ are proper with*

$$d_{\mathrm{pGH}}((X, x), (Y, y)) = 0 \tag{34}$$

*then there exists an isometry $(X, x) \to (Y, y)$.*

*Remark 2.26* Here and below, the issue of considering the set of "all" separable metric spaces can be easily avoided by using the Kuratowski (isometric) embedding of any separable metric space into $\ell_\infty$.

We also make no distinction in notation between spaces and their equivalence classes. It can be easily verified that the concepts we consider to not depend on particular representatives of an equivalence class.

*Proof* By Proposition 2.20 we know that $d_{\mathrm{pGH}}$ is a complete pseudometric. Moreover, if

$$d_{\mathrm{pGH}}((X, x), (Y, y)) < \epsilon$$

and $B(x, 1/\epsilon)$ is covered by $N$ balls of radius $\epsilon$, then $B(y, 1/2\epsilon)$ is covered by $N$ balls of radius $2\epsilon$. Thus, if $(Y, y)$ is a complete pointed metric space and is the $d_{\mathrm{pGH}}$ limit of a sequence of proper pointed metric spaces, then $Y$ is proper too. It is easily verified that the set of finite pointed metric spaces with rational distances is dense in $(\mathbb{M}_p, d_{\mathrm{pGH}})$.

Now let $(X, x)$ and $(Y, y)$ be proper satisfying (34) and fix $k \in \mathbb{N}$. For each $j \in \mathbb{N}$ let $\mathcal{N}_j$ be a finite $1/j$-net of $B(x, k)$. For each fixed $j \in \mathbb{N}$, by taking a convergent subsequence, we may suppose that the $f_i$ converge to an isometry when restricted to $\mathcal{N}_j$ whose image is a $2/j$-net of $B(y, k)$. By taking a diagonal subsequence, we may suppose the $f_i$ converge to an isometry when restricted to each $\mathcal{N}_j$. Let $\iota\colon \cup_j \mathcal{N}_j \to B(y, k)$ be the limiting isometry. Since $\cup_j \mathcal{N}_j$ is dense in $B(x, k)$, $\iota(\cup_j \mathcal{N}_j)$ is dense in $B(y, k)$ and since $B(y, k)$ is complete, $\iota$ extends to an isometry $\iota\colon B(x, k) \to B(y, k)$. Taking a convergent diagonal subsequence over $k \to \infty$ completes the proof. □

The Gromov compactness theorem is an immediate corollary of the completeness of $(\mathbb{M}_p, d_{\mathrm{pGH}})$.

Recall that a set of metric spaces $\mathcal{S}$ is *uniformly totally bounded* if, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that each $X \in \mathcal{S}$ is contained in $N_\epsilon$ balls of radius $\epsilon$.

**Theorem 2.27** (Section 6 [24]) *A set $\mathcal{S} \subset (\mathbb{M}_p, d_{\mathrm{pGH}})$ is pre-compact if and only if, for every $r > 0$,*

$$\{B(x, r) \subset X : (X, x) \in \mathcal{S}\}$$

*is uniformly totally bounded.*

*Proof* The given condition is equivalent to $\mathcal{S}$ being totally bounded in $(\mathbb{M}_p, d_{\mathrm{pGH}})$. Indeed, for $\epsilon > 0$ and $(X, x) \in \mathcal{S}$, suppose that $B(x, 1/\epsilon)$ is covered by $N_\epsilon$ balls of radius $\epsilon$. Let $\mathcal{N}$ be the set of metric spaces of the form

$$(\{1, 2, \ldots, N_\epsilon\}, \rho)$$

with

$$\rho(x, y) \in \{\epsilon j : j \in \mathbb{N},\ 0 < j \leq 1/\epsilon^2\}$$

for each $x \neq y \in Y$ (and such that $\rho$ satisfies the triangle inequality). Then $\mathcal{S}$ is contained in the $2\epsilon$ neighbourhood of $\mathcal{N}$.

Conversely, suppose that $\mathcal{S}$ is totally bounded and $\mathcal{N} \subset \mathcal{S}$ is a finite set such that $\mathcal{S} \subset B(\mathcal{N}, \epsilon)$. Since each $(Y, y) \in \mathcal{N}$ is proper and $\mathcal{N}$ is finite, there exists $N_\epsilon \in \mathbb{N}$ such that, for each $(Y, y) \in \mathcal{N}$, $B(y, 1/\epsilon)$ is covered by $N_\epsilon$ balls of radius $\epsilon$. Then for any $(X, x) \in \mathcal{S}$, $B(x, 1/2\epsilon)$ is contained in $N_\epsilon$ balls of radius $2\epsilon$. □

## 3 Construction of a Hölder surface

Recall that throughout the paper we work with a fixed $n \in \mathbb{N}$. For notational convenience, we define biLip$(K)$ as the set of $K$-bi-Lipschitz images of $\ell_\infty^n$, rather than images of $\ell_2^n$.

**Definition 3.1** For $K \geq 1$ let biLip$(K)$ be the set of isometry classes of all pointed proper metric spaces $(X, x)$ for which there exists a surjective and $K$-bi-Lipschitz $\psi : \ell_\infty^n \to X$.

**Definition 3.2** Let $(X, d)$ be a complete metric space, let $G \subset C \subset X$ be closed subsets and $K \geq 1$, $\eta$, $R_0 > 0$ and $0 < \delta < 1/2$.

The triple $(X, C, G)$ has *good tangential approximation*, written $GTA(\eta, K, \delta, R_0)$, if both of the following conditions hold:

1. For each $x \in C$ and each $0 < r \leq R_0$,

$$\mathcal{H}^n(B(x,r)) \geq \eta r^n; \tag{35}$$

2. For each $x \in G$ and each $0 < r \leq R_0$, there exists a closed

$$C' \subset B(x,r) \cap C$$

containing $x$ with

$$\mathcal{H}^n(B(x,r) \setminus C') < \eta(\delta r)^n \tag{36}$$

and

$$d_{\mathrm{pGH}}((C', d/r, x), \mathrm{biLip}(K)) < \min\{\delta, 1/K(1+2\delta)\}. \tag{37}$$

**Observation 3.3** *For $K \geq 1$ and $0 < \delta < 1/2$ let*

$$\Phi = [-K(1+2\delta), K(1+2\delta)]^n \subset \ell_\infty^n.$$

*If $(X, C, G)$ satisfies $GTA(\eta, K, \delta, R_0)$ then for any $0 < r \leq R_0$ there exists a Borel $\xi\colon \Phi \to C$ with $\xi(0) = x$ such that*

$$\mathcal{H}^n(B(x,r) \setminus B(\xi(\Phi), \delta r)) < \eta(\delta r)^n \tag{38}$$

*and*

$$\frac{\|q - q'\|_\infty}{K} - \delta \leq \frac{d(\xi(q), \xi(q'))}{r} \leq K\|q - q'\|_\infty + \delta \tag{39}$$

*for all $q, q' \in \Phi$. Indeed, let $(Y, \rho, y) \in \mathrm{biLip}(K)$ with*

$$d_{\mathrm{pGH}}((C', d/r, x), (Y, \rho, y)) < \min\{\delta, 1/K(1+2\delta)\} =: \delta'$$

*and let $p\colon \ell_\infty^n \to Y$ be $K$-bi-Lipschitz and surjective with $p(0) = x$. Define $\xi$ by composing $p$ and the $\delta'$-isometry from $(Y, \rho, y)$ to $(C', d/r, x)$ granted by Lemma 2.24.*

For $m \in \mathbb{N}$ and $R > 0$ let

$$\mathcal{D}(R, m) = \{R2^{-m}(j_1, \ldots, j_n) \in [0, R]^n : j_1, \ldots, j_n \in \mathbb{Z}\}.$$

The main result of this section is the following theorem.

**Theorem 3.4** *For any $K \geq 1$, $0 < \gamma < 1$ and $\eta > 0$ there exists $m_0 \in \mathbb{N}$ such that the following is true. Suppose that $m \geq m_0$ and $G \subset C \subset X \subset \ell_\infty$ are closed sets such that $(X, C, G)$ satisfies*

$$GTA(\eta, K, 4^{-m}, R_0). \tag{40}$$

*Then for any $x \in G$ and $0 < r \leq R_0$, there exists a $\gamma$-Hölder map*

$$\iota\colon [0, r]^n \subset \ell_\infty^n \to B(x, 20Kr) \subset \ell_\infty$$

*with $\iota(0) = x$ such that:*

1. *$\iota|_{\mathcal{D}(r,m)}\colon \mathcal{D}(r, m) \to C$ is $(K + 2^{-m})$-bi-Lipschitz onto its image;*
2. *For any $y, z \in [0, r]^n$ with $\|y - z\|_\infty \leq 2^{-m} r$,*

$$\|\iota(y) - \iota(z)\|_\infty \leq 10K2^{-m\gamma/2} r;$$

3.

$$\mathcal{H}^n_\infty(\iota([0, r]^n) \setminus C) \leq \Lambda \mathcal{H}^n(B(\iota(0), 20Kr) \cap X \setminus G),$$

*for some $\Lambda > 0$ depending only upon $K, \gamma, m, n, \eta$.*

Before discussing the proof of Theorem 3.4, we mention how the properties of $\iota$ are used in the proof of Theorem 1.2 (the complete details are given in Theorem 6.5).

Item 1 implies that the map $f := \iota|^{-1}_{\mathcal{D}(r,m)}$ exists and is $L$-Lipschitz for some $L$ independent of $m$ and $r$. After extending $f$ to all of $\ell_\infty$ using Theorem 2.6, Theorem 3.4 items 1 and 2 imply that

$$\|f(\iota(y)) - y\|_\infty \leq A2^{-m} r \tag{41}$$

for some constant $A$ independent of $m$ and $r$. Provided $m$ is sufficiently large, this implies that $f(\iota([0, r]^n))$ contains $[r/4, 3r/4]^n$ and hence

$$\mathcal{H}^n(f(\iota([0, r]^n))) \geq cr^n \tag{42}$$

for an absolute constant $c > 0$ (see Corollary 6.4). Provided

$$\frac{\mathcal{H}^n(B(\iota(0), 20Kr) \cap X \setminus G)}{r^n}$$

is sufficiently small (which can be guaranteed if $\iota(0)$ is a density point of $G$ and $r$ is sufficiently small), item 3 and (42) imply that

$$\mathcal{H}^n(f(\iota([0, r]^n) \cap C)) > 0. \tag{43}$$

Moreover, if $f$ is replaced by any $L$-Lipschitz $g$ with $\|f - g\|_\infty < A2^m r/4$, then (41) and hence (42) and (43) also hold for $g$. The main result of [13] (see Theorem 6.3) then implies that $C$ cannot be purely $n$-unrectifiable. In Proposition 6.2 it is shown that if $X$ satisfies the hypotheses of Theorem 1.2, any subset of $X$ can be covered (up to a $\mathcal{H}^n$ null set) by a countable union of sets which satisfy the hypotheses of $C$. Consequently, $X$ must be $n$-rectifiable.

We now discuss the proof of Theorem 3.4. First note that, by scaling the norm in $\ell_\infty$, it suffices to consider the case $R_0 = r = 1$. The construction of the map $\iota$ involves a multi-scale iteration over sub-cubes of $[0, 1]^n$. To begin we define $\iota_1$ on $\mathcal{D}(1, m)$ to equal the map $\xi$ given by Observation 3.3. By choosing $m_0$ sufficiently large, (39) ensures that item 1 of Theorem 3.4 holds.

We now wish to extend $\iota_1$ to a function $\iota_2$ defined on (a large subset of) $\mathcal{D}(1, 2m)$. The basic idea is the following:

- Suppose that $p \in \mathcal{D}(1, m)$ is such that $\iota_1(p)$ is sufficiently close to a point of $G$. Then the map $\xi$ from Observation 3.3 provides a rich structure to the neighbourhood of $\iota_1(p)$. We wish to use $\xi$ to locally define $\iota_2$ on points in $\mathcal{D}(1, 2m)$ that are close to $p$, whilst maintaining some control on the local Lipschitz constant of $\iota_2$. (See Lemma 3.8 for the fundamental building block of this extension.)
- On the other hand, since $\xi$ takes values in $C$, there may well exist $p \in \mathcal{D}(1, m)$ for which $\iota_1(p)$ lies relatively far away from $G$. At this stage we do not extend $\iota_2$ near to $p$ and this will produce a hole in the domain of $\iota$. Later we will patch over such holes, but we must control the total size of the holes we need to patch. The lower density bound (35) allows us to bound the measure of those points that lie far away from $G$ and consequently on the amount of patching required. (This is addressed much later in Proposition 3.15.)

We then iterate this extension process for each $i \geq 2$, defining $\iota_i$ on a large subset of $\mathcal{D}(1, im)$ as an extension of $\iota_{i-1}$ and defining $\iota$ as the limit of the $\iota_i$. The control on the local Lipschitz constant of each $\iota_i$ leads to a Hölder bound on $\iota$ as in Theorem 3.4 item 2; The control on the total amount of patching over all scales leads to item 3.

Before proceeding with more details, we fix notation for this section.

**Notation 3.5** We fix $K \geq 1$, $0 < \gamma < 1$ and $\eta > 0$.

Let $N \in \mathbb{N}$ be such that

$$(5K^2)^n \leq 2^{N(1-\gamma)}$$

and choose $\gamma \leq \alpha < 1$ such that

$$(5K^2)^n = 2^{N(1-\alpha)}. \tag{44}$$

Write $l = 2^{-N}$ and $\sigma = (5K^2)^n$, so that

$$\sigma l = (5K^2)^n 2^{-N} = 2^{-N\alpha} = l^\alpha. \tag{45}$$

Fix $(X, d)$ a complete metric space and $C, G \subset X$ such that $(X, C, G)$ satisfies $GTA(\eta, K, l/20, 1)$.

We note the following. If $i \in \mathbb{N}$ with $i > 2/\alpha$, Eqs. (44) and (45) give

$$\sigma(\sigma l)^i = l^{\alpha-1} l^{i\alpha} < l^1 \leq 1/2.$$

That is,

$$\sigma^i < \frac{1}{2\sigma l^i}. \tag{46}$$

**Definition 3.6** For $i \in \mathbb{N}$ let

$$\mathcal{D}(i) = \{l^i(j_1, \ldots, j_n) \in [0, 1]^n : j_1, \ldots, j_n \in \mathbb{Z}\}.$$

For an integer $0 \leq m \leq n$, an $m$-dimensional *face* of side length $l^i$ is any set of the form

$$\{x \in [0, 1]^n : p_j \leq x_j \leq p_j + b_j l^i \ \forall 1 \leq j \leq n\},$$

with $p \in \mathcal{D}(i)$, $b \in \{0, 1\}^n$ and exactly $m$ of the $b_i = 1$. We denote by $\mathcal{F}(m, i)$ the set of all such faces. If $1 \leq m \leq n$ and $F \in \mathcal{F}(m, i)$, define the *boundary* of $F$ by

$$\partial F := \bigcup \{F' \in \mathcal{F}(m-1, i) : F' \subset F\}.$$

We also write $\mathcal{Q}(i) = \mathcal{F}(n, i)$, the set of *cubes* of side length $l^i$. For $Q \in \mathcal{Q}(i)$ the $m$-dimensional *skeleton* of $Q$ is

$$\mathrm{skel}(Q, m) := \bigcup \{F \in \mathcal{F}(m, i) : F \subset Q\}.$$

For $\mathcal{Q} \subset \mathcal{Q}(i)$, we define the *corners* of $\mathcal{Q}$ by

$$\mathrm{cor}(\mathcal{Q}) = \bigcup_{Q \in \mathcal{Q}} \mathrm{skel}(Q, 0).$$

We also define the *children* of $\mathcal{Q}$ to be

$$\mathrm{child}(\mathcal{Q}) := \{Q' \in \mathcal{Q}(i+1) : Q' \subset Q, \text{ some } Q \in \mathcal{Q}\}.$$

If $Q \in \mathcal{Q}_i$ we write $\mathrm{child}(Q)$ for $\mathrm{child}(\{Q\})$.

To begin the construction, we define a decomposition of a given collection of cubes in order to implement the idea sketched above.

**Lemma 3.7** *Let $i \in \mathbb{N}$, $\mathcal{Q} \subset \mathcal{Q}(i)$ and, for some $\beta > 0$, suppose that*

$$\iota\colon \mathrm{cor}(\mathcal{Q}) \to C$$

*satisfies*

$$d(\iota(p), \iota(p')) \leq \beta \|p - p'\|_\infty \tag{47}$$

*for each $Q \in \mathcal{Q}$ and each $p, p' \in \mathrm{cor}(Q)$. Then there exists a decomposition $\mathcal{Q} = \mathcal{G} \cup \mathcal{B}$ such that:*

1. *For each $Q \in \mathcal{G}$ there exists $x \in G$ with*

$$\iota(\mathrm{cor}(Q)) \subset B(x, 2\beta l^i).$$

2. *For every $p \in \mathrm{cor}(\mathcal{B})$,*

$$B(\iota(p), \beta l^i) \cap G = \emptyset.$$

*Proof* Let $Q \in \mathcal{Q}$. The triangle inequality and (47) imply that either item 1 or item 2 holds. Decompose $\mathcal{Q}$ accordingly. □

The proof of Theorem 3.4 now consists of two main parts. First, for $i \in \mathbb{N}$ and $\iota$ and $\mathcal{Q}$ satisfying the hypotheses of Lemma 3.7, we define an extension of $\iota$ to

$$D' := \bigcup \mathcal{G} \cap \mathcal{D}(i+1).$$

This extension satisfies the hypotheses of Lemma 3.7 for $i+1$ with a larger value of $\beta$. Note that we do not extend $\iota$ into

$$\bigcup \mathcal{B} \cap \mathcal{D}(i+1).$$

In the second part we iteratively construct such extensions for all $i \in \mathbb{N}$, using the resulting function from one iteration as the input to the next. We

show that the limiting function,

$$\iota\colon S \subset [0,1]^n \to C,$$

is $\alpha$-Hölder continuous. A Hölder extension extends $\iota$ to

$$\iota\colon [0,1]^n \to \ell_\infty.$$

This extension simultaneously patches all holes that were created when we did not extend the function into the cubes in $\mathcal{B}$ after each application of Lemma 3.7. Finally we show how the properties of $\mathcal{B}$ imply Theorem 3.4 item 3.

### 3.1 Constructing the extension to the next scale

The hypothesis that $(X, C, G)$ satisfies $GTA(\eta, K, l/20, 1)$ is used to prove the following extension result. This serves as the fundamental building block that is used construct the extension from $\mathcal{D}(i)$ to a large subset of $\mathcal{D}(i+1)$.

**Lemma 3.8** *For $0 \le m \le n$ and $i \in \mathbb{N}$ let $F \in \mathcal{F}(m, i)$. For $0 < \beta < 1/(2l^i)$ suppose that*

$$\iota\colon \partial F \cap \mathcal{D}(i+1) \to C$$

*satisfies*

$$d(\iota(p), \iota(p')) \le \beta \|p - p'\|_\infty \tag{48}$$

*for each $p, p' \in \partial F \cap \mathcal{D}(i+1)$. Suppose that there exists $x \in G$ with*

$$\iota(\partial F \cap \mathcal{D}(i+1)) \subset B(x, 2\beta l^i). \tag{49}$$

*Then there exists an extension of $\iota$ to*

$$\iota\colon F \cap \mathcal{D}(i+1) \to C$$

*such that*

$$d(\iota(p), \iota(p')) \le 2K^2\beta \|p - p'\|_\infty \tag{50}$$

*for each $p, p' \in F \cap \mathcal{D}(i+1)$.*

*Proof* By hypothesis, we have $r := 2\beta l^i < 1$. Let $\xi\colon \Phi \to C$ be given by Observation 3.3 for $x$ and this value of $r$. Let $\delta = l/20$.

For a moment fix a point $p \in \partial F \cap \mathcal{D}(i+1)$. Since $\iota(p) \in C$, Definition 3.2 item 1 implies

$$\mathcal{H}^n(B(\iota(p), \delta r)) \geq \eta(\delta r)^n.$$

Since $\iota(p) \in B(x, 2\beta l^i)$, (38) implies that there exists $z \in B(\xi(\Phi), \delta r)$ with

$$d(z, \iota(p)) \leq \delta r.$$

That is, there exists a point $\psi(p) \in \Phi$ with

$$d(\iota(p), \xi(\psi(p))) \leq 2\delta r. \tag{51}$$

The function $\psi\colon \partial F \cap \mathcal{D}(i+1) \to \Phi$ is $\frac{3K\beta}{2r}$-Lipschitz. Indeed, for any $p, p' \in \partial F \cap \mathcal{D}(i+1)$, Eq. (39) implies

$$\|\psi(p) - \psi(p')\|_\infty \leq \frac{K}{r} d(\xi(\psi(p)), \xi(\psi(p'))) + \delta K.$$

Combining this with Eq. (51) and using the triangle inequality gives

$$\|\psi(p) - \psi(p')\|_\infty \leq \frac{K}{r} d(\iota(p), \iota(p')) + 5\delta K.$$

Finally, Eq. (48) and the choice of $r$ gives

$$\begin{aligned}
\|\psi(p) - \psi(p')\|_\infty &\leq \frac{K\beta}{r}\|p - p'\|_\infty + 5\delta K \\
&= \frac{K\beta}{r}(\|p - p'\|_\infty + 10\delta l^i) \\
&= \frac{K\beta}{r}(\|p - p'\|_\infty + l^{i+1}/2) \\
&\leq \frac{3K\beta}{2r}\|p - p'\|_\infty.
\end{aligned}$$

This implies that $\psi$ is $\frac{3K\beta}{2r}$-Lipschitz as claimed. Since $\psi$ takes values in $\Phi$, the second conclusion of Theorem 2.6 gives an extension of $\psi$ to a $\frac{3K\beta}{2r}$-Lipschitz function (with respect to $\|\cdot\|_\infty$ in both the domain and image)

$$\psi\colon F \cap \mathcal{D}(i+1) \to \Phi.$$

For each $p \in F \cap \mathcal{D}(i+1) \setminus \partial F$, define $\iota(p) = \xi(\psi(p))$. To check (50), let $p, p' \in F \cap \mathcal{D}(i+1)$. Note that, regardless of whether $p$ or $p' \in \partial F$, Eq.

(51) and the triangle inequality give

$$d(\iota(p), \iota(p')) \leq d(\xi(\psi(p)), \xi(\psi(p'))) + 4\delta r.$$

Combining this with (39) gives

$$d(\iota(p), \iota(p')) \leq Kr\|\psi(p) - \psi(p')\|_\infty + 5\delta r$$

Using the fact that $\psi$ is $\frac{3K\beta}{2r}$-Lipschitz gives

$$d(\iota(p), \iota(p')) \leq \frac{3K^2\beta}{2}\|p - p'\|_\infty + 5\delta r.$$

Finally, substituting in for $r$ and $\delta$ gives

$$d(\iota(p), \iota(p')) \leq \frac{3K^2\beta}{2}\|p - p'\|_\infty + \frac{\beta l^{i+1}}{2} \leq 2K^2\beta\|p - p'\|_\infty,$$

giving (50). □

Suppose we are given $\iota\colon \mathcal{D}(i) \to C$ that we wish to extend to $\mathcal{D}(i+1)$. One way to do this is to consider each $Q \in \mathcal{Q}(i)$ one at a time and apply Lemma 3.8 several times to extend $\iota$ to $Q \cap \mathcal{D}(i+1)$ (for the purposes of this discussion, ignore the requirement given in (49)). Such an extension must agree with the value of $\iota$ at those points in $\partial Q \cap \mathcal{D}(i+1)$ that belong to the boundary of adjacent cubes to which we have previously extended $\iota$. However, each time Lemma 3.8 is applied in this way, the Lipschitz constant of the extension, given by (50), is a multiplicative factor larger than the input Lipschitz constant given in (48). Consequently, if this is repeated for each $Q \in \mathcal{Q}(i)$, the Lipschitz constant of the extension to the whole of $\mathcal{D}(i+1)$ is far too large, and such an extension is useless.

The correct approach is to first extend $\iota$ to all points of

$$S := \mathcal{D}(i+1) \cap \bigcup \mathcal{F}(m, 1)$$

by applying Lemma 3.8 one face at a time. For any $F \in \mathcal{F}(m, 1)$,

$$\partial F \cap \mathcal{D}(i+1) \subset \mathcal{D}(i).$$

Since $\iota$ is defined on $\mathcal{D}(i)$, it can be shown that the Lipschitz constant increases by a fixed amount, independent of the number of $F \in \mathcal{F}(m, 1)$ to which the extension is applied. We then extend to

$$\mathcal{D}(i+1) \cap \bigcup \mathcal{F}(m, 2)$$

one face at a time. Similarly to the previous case, for any $F \in \mathcal{F}(m, 2)$,

$$\partial F \cap \mathcal{D}(i+1) \subset S.$$

Since $\iota$ is now defined on $S$, as in the previous case, the Lipschitz constant only increases once, regardless of the number of $F \in \mathcal{F}(m, 2)$ to which we apply Lemma 3.8. This is repeated $n$ times in total, so that we extended $\iota$ to all of $\mathcal{D}(i+1)$.

To facilitate this iteration we make the following simple observation.

**Observation 3.9** *For $0 \leq m \leq n$ and $i \in \mathbb{N}$, let $Q, Q' \in \mathcal{Q}(i)$, $p \in \operatorname{skel}(Q, m) \cap \mathcal{D}(i+1)$ and $p' \in \operatorname{skel}(Q', m) \cap \mathcal{D}(i+1)$. Suppose that, for some $F \in \mathcal{F}(m, i)$, $p \in F$ and $p' \notin F$. Then there exists $q \in \partial F \cap \mathcal{D}(i+1)$ such that*

$$\max\{\|p - q\|_\infty, \|q - p'\|_\infty\} \leq \|p - p'\|_\infty. \tag{52}$$

*Proof* Without loss of generality, we may suppose that

$$F = \{x \in [0, 1]^n : 0 \leq x_j \leq l^i \ \forall 1 \leq j \leq m\}.$$

If $p \in \partial F$ then we pick $q = p$. Otherwise, we know that

$$0 < p_1, \ldots, p_m < l^i.$$

There are two cases:

1. For some $1 \leq j \leq m$, $p'_j = 0$ or $p'_j \geq l^i$;
2. $0 < p'_1, \ldots, p'_m < l^i$. In this case, since $p' \in \operatorname{skel}(Q', m)$ and hence belongs to some $m$-dimensional face, the other components of $p'$ must be integer multiples of $l^i$. Since $p' \notin F$, one such component, say $p'_j$ differs from $p_j$. Consequently, $|p'_j - p_j| \geq l^i$.

In the first case, define $q_k = p_k$ for each $k \neq j$ and let $q_j$ equal either 0 or $l^i$, such that $q_j$ lies between $p_j$ and $p'_j$. Then $q \in \partial F$, $q_k = p_k$ for all $k \neq j$ and

$$|p_j - p'_j| = |p_j - q_j| + |q_j - p'_j|,$$

which implies (52). In the second case, arbitrarily pick $q \in \partial F \cap \mathcal{D}(i+1)$. Then $\|p - q\|_\infty \leq l^i \leq \|p - p'\|_\infty$, implying one inequality in (52). Also, $|p'_k - q_k| \leq l^i \leq |p'_j - p_j|$ for all $1 \leq k \leq m$ and $|p'_k - q_k| = |p'_k - p_k|$ for all $k > m$. Therefore, $\|p' - q\|_\infty \leq \|p' - p\|_\infty$, proving the other inequality in (52). □

With this observation we demonstrate how to construct the extension from the $m$-dimensional faces to the $m+1$-dimensional faces.

**Lemma 3.10** *Fix $i \in \mathbb{N}$, let $D \subset \mathcal{D}(i+1)$ and let $0 \leq m < n$. Let $0 < \beta < 1/(2l^i)$ and suppose that $\iota\colon D \to C$ satisfies*

$$d(\iota(p), \iota(p')) \leq \beta \|p - p'\|_\infty \tag{53}$$

*for each $p, p' \in D$ with $\|p - p'\|_\infty \leq l^i$. Suppose that $\mathcal{G} \subset \mathcal{Q}(i)$ is such that*

$$\forall Q \in \mathcal{G},\ D \cap Q = \mathrm{skel}(Q, m) \cap \mathcal{D}(i+1) \tag{54}$$

*and*

$$\forall Q \in \mathcal{G},\ \exists x \in G \text{ with } \iota(D \cap Q) \subset B(x, 2\beta l^i). \tag{55}$$

*Then there exists an extension of $\iota$ to*

$$D' := D \cup \bigcup_{Q \in \mathcal{G}} \mathrm{skel}(Q, m+1) \cap \mathcal{D}(i+1)$$

*such that*

$$d(\iota(p), \iota(p')) \leq 5K^2 \beta \|p - p'\|_\infty \tag{56}$$

*for each $p, p' \in D'$ with $\|p - p'\|_\infty \leq l^i$.*

*Proof* Let $Q \in \mathcal{G}$ and $F \in \mathcal{F}(m+1, i)$ with $F \subset Q$. By (54), $\iota$ is defined on $\partial F \cap \mathcal{D}(i+1)$. By (55) and the fact that $\beta < 1/(2l^i)$, the hypotheses of Lemma 3.8 are satisfied. An application of Lemma 3.8 extends $\iota|_{\partial F \cap \mathcal{D}(i+1)}$ to a map

$$\iota\colon F \cap \mathcal{D}(i+1) \to C$$

such that

$$d(\iota(p), \iota(p')) \leq 2K^2 \beta \|p - p'\|_\infty \tag{57}$$

for each $p, p' \in F \cap \mathcal{D}(i+1)$. We combine each of these extensions to form an extension of $\iota$ to a map

$$\iota\colon D' \to C,$$

ignoring any repetitions of faces that appear in multiple cubes.

Now let $p, p' \in D'$ with $\|p - p'\|_\infty \le l^i$. If $p, p' \in D$ then (56) holds for $p, p'$ by hypothesis. Therefore, we may suppose that $p \notin D$. Let $Q \in \mathcal{G}$ with $p \in Q$ and $F \in \mathcal{F}(m+1, i)$ with $p \in F \subset \mathrm{skel}(Q, m+1)$. If $p' \in F$ then (57) implies (56) for $p, p'$. Otherwise, let $q \in \partial F \cap \mathcal{D}(i+1)$ be given by Observation 3.9 such that

$$\max\{\|p - q\|_\infty, \|q - p'\|_\infty\} \le \|p - p'\|_\infty$$

If $p' \notin D$, let $Q' \in \mathcal{G}$ and $F' \in \mathcal{F}(m+1, i)$ with $p' \in F' \subset \mathrm{skel}(Q', m+1)$ and let $q' \in \partial F' \cap \mathcal{D}(i+1)$ be given by Observation 3.9 such that

$$\max\{\|p' - q'\|_\infty, \|q - q'\|_\infty\} \le \|p' - q\|_\infty.$$

If $p' \in D$ let $q' = p'$. In either case, we have $\|q - q'\|_\infty, \|p - p'\|_\infty \le l^i$ and so (53) implies

$$d(\iota(q), \iota(q')) \le \beta \|q - q'\|_\infty.$$

Further, (57) implies

$$d(\iota(p), \iota(q)) \le 2K^2 \beta \|p - q\|_\infty$$

and

$$d(\iota(q'), \iota(p')) \le 2K^2 \beta \|q' - p'\|_\infty.$$

Therefore, the triangle inequality gives

$$\begin{aligned} d(\iota(p), \iota(p')) &\le 2K^2 \beta \|p - q\|_\infty + \beta \|q - q'\|_\infty \\ &\quad + 2K^2 \beta \|q' - p'\|_\infty \\ &\le 5K^2 \beta \|p - p'\|_\infty, \end{aligned}$$

as required. □

Recall $\sigma = (5K^2)^n$ is defined in Notation 3.5. We now iterate the construction in Lemma 3.10 to obtain an extension to $n$-dimensional cubes.

**Lemma 3.11** *Fix $i \in \mathbb{N}$, let $D \subset \mathcal{D}(i)$ and let*

$$0 < \beta < \frac{1}{2\sigma l^i}. \tag{58}$$

*Suppose that $\iota\colon D \to C$ satisfies*

$$d(\iota(p), \iota(p')) \le \beta \|p - p'\|_\infty \tag{59}$$

*for each* $p, p' \in D$ *with* $\|p - p'\|_\infty \leq l^i$. *Suppose also that* $\mathcal{G} \subset \mathcal{Q}(i)$ *is such that* $\mathrm{cor}(\mathcal{G}) \subset D$ *and*

$$\forall Q \in \mathcal{G},\ \exists x \in G \text{ with } \iota(D \cap Q) \subset B(x, 2\beta l^i). \tag{60}$$

*Then there exists an extension of* $\iota$ *to*

$$D' := \left(\bigcup \mathcal{G} \cap \mathcal{D}(i+1)\right) \cup D = \mathrm{cor}(\mathrm{child}(\mathcal{G})) \cup D$$

*such that*

$$d(\iota(p), \iota(p')) \leq \sigma\beta\|p - p'\|_\infty \tag{61}$$

*for each* $p, p' \in D'$ *with* $\|p - p'\|_\infty \leq l^i$.

*Proof* First note that, since $\mathrm{cor}(\mathcal{G}) \subset D$, (54) holds for $m = 0$. Since (60) implies (55), we may apply Lemma 3.10 to obtain an extension of $\iota$ to

$$D_1 := D \cup \bigcup_{Q \in \mathcal{G}} \mathrm{skel}(Q, 1) \cap \mathcal{D}(i+1)$$

such that

$$d(\iota(p), \iota(p')) \leq 5K^2\beta\|p - p'\|_\infty$$

for each $p, p' \in D_1$ with $\|p - p'\|_\infty \leq l^i$.

Now notice that,

$$\forall Q \in \mathcal{G},\ D_1 \cap Q = \mathrm{skel}(Q, 1).$$

Since

$$(5K^2)\beta < \frac{(5K^2)}{2\sigma l^i} \leq \frac{1}{2l^i},$$

we may apply Lemma 3.10 again, but with $\beta$ replaced by $(5K^2)\beta$ and with $m = 1$. This gives an extension of $\iota$ to

$$D_2 := D \cup \bigcup_{Q \in \mathcal{G}} \mathrm{skel}(Q, 2) \cap \mathcal{D}(i+1)$$

such that

$$d(\iota(p), \iota(p')) \leq (5K^2)^2\beta\|p - p'\|_\infty$$

for each $p, p' \in D_2$ with $\|p-p'\|_\infty \le l^i$. Thus, the hypotheses of Lemma 3.10 are satisfied again, with $\beta$ replaced by $(5K^2)^2\beta$ and with $m = 2$.

Since $\beta < 1/(2(5K^2)^n l^i)$, we repeat this for a total of $n$ times, extending $\iota$ to

$$D_n := D \cup \bigcup_{Q\in\mathcal{G}} \operatorname{skel}(Q, n) \cap \mathcal{D}(i+1) = D \cup \left(\bigcup \mathcal{G} \cap \mathcal{D}(i+1)\right) = D'$$

such that

$$d(\iota(p), \iota(p')) \le (5K^2)^n \beta \|p - p'\|_\infty = \sigma\beta\|p - p'\|_\infty$$

for each $p, p' \in D'$ with $\|p - p'\|_\infty \le l^i$. □

### 3.2 Constructing the extension to all scales

In Lemma 3.11 we demonstrated how to extend a function from some $D \subset \mathcal{D}(i)$ to $D' \subset \mathcal{D}(i+1)$. We now iterate this extension over all $i \in \mathbb{N}$.

Note that, in items 4 and 5 of the following lemma, the radius of the ball in item 4 and the upper bound in item 5 are of the form $(\sigma l)^m$, up to some constant multiple. In (44) we chose $\alpha$ so that $(\sigma l) = l^\alpha$, so both of these quantities are comparable to $l^{\alpha m}$. In particular, item 5 implies that each $\iota_i$ is $\alpha$-Hölder with the same Hölder constant. The fact that these two quantities have the same power of $\alpha$ is an essential requirement for the construction given in this section to work; looking ahead to Proposition 3.15, this is exactly the condition required so that we can bound $\mathcal{H}^n_\infty(\iota([0,1]^n) \setminus C)$ in (74). Looking behind, we see that the two powers agree because of Lemma 3.7.

**Lemma 3.12** *Let $M \in \mathbb{N}$ with $M > \alpha/2$, $0 \le L \le \sigma^M$ and suppose that*

$$\iota_M : \mathcal{D}(M) \to C$$

*is $L$-Lipschitz. For each $i \ge M$ there exist $\mathcal{G}_i, \mathcal{B}_i \subset \mathcal{Q}(i)$ such that*

$$[0,1]^n = \bigcup \mathcal{G}_i \cup \bigcup_{M\le j\le i} \bigcup \mathcal{B}_i \tag{62}$$

*and a function*

$$\iota_i : \operatorname{cor}\left(\mathcal{G}_i \cup \bigcup_{M\le j\le i} \mathcal{B}_j\right) \to C \tag{63}$$

*such that:*

1. *$\iota_{i+1}$ is an extension of $\iota_i$;*
2. $\mathrm{child}(\mathcal{G}_i) = \mathcal{G}_{i+1} \cup \mathcal{B}_{i+1}$
3. *For each $Q \in \mathcal{G}_i$, there exists $x \in G$ with*

$$\iota_i(\mathrm{cor}(Q)) \subset B(x, 2L\sigma^{i-M} l^i);$$

4. *For all $p \in \mathrm{cor}(\mathcal{B}_i)$,*

$$B(\iota_i(p), L\sigma^{i-M} l^i) \cap G = \emptyset,$$

5. *For each $p, p' \in \mathrm{dom}\, \iota_i$ with $\|p - p'\|_\infty \leq l^{i-1}$,*

$$d(\iota_i(p), \iota_i(p')) \leq L\sigma^{i-M} \|p - p'\|_\infty.$$

*Here and throughout,* $\mathrm{dom}\, \iota_i$ *denotes the domain of* $\iota_i$.

*Proof* We prove the lemma by induction. To begin, for $\beta := L$, let

$$\mathcal{Q}(M) = \mathcal{G}_M \cup \mathcal{B}_M$$

be given by Lemma 3.7 for $\iota = \iota_M$ and $i = M$. Then $\iota_M, \mathcal{G}_M, \mathcal{B}_M$ satisfy the required conditions.

Now suppose that $\iota_i, \mathcal{G}_i$ and $\mathcal{B}_M, \ldots, \mathcal{B}_i$ exist for some $i \geq M$. We first apply Lemma 3.11 to $\iota_i$ and $\mathcal{G}_i$ and begin by checking that the required hypotheses are satisfied. Since $L \leq \sigma^M$, (46) implies

$$\beta := L\sigma^{i-M} \leq \sigma^i < \frac{1}{2\sigma l^i},$$

and so (58) holds. By induction hypothesis, item 5 implies (59) for this choice of $\beta$. Also, item 3 implies $\mathcal{G}_i$ satisfies (60). An application of Lemma 3.11 constructs an extension of $\iota_i$ to

$$\iota_{i+1} \colon \mathrm{cor}\left(\mathrm{child}(\mathcal{G}_i) \cup \bigcup_{M \leq j \leq i} \mathcal{B}_j\right) \to C$$

such that, for each $p, p' \in \mathrm{dom}\, \iota_{i+1}$ with $\|p - p'\|_\infty \leq l^i$,

$$d(\iota_{i+1}(p), \iota_{i+1}(p')) \leq \sigma\beta \|p - p'\|_\infty = L\sigma^{i+1-M} \|p - p'\|_\infty. \tag{64}$$

Equation (64) implies that $\mathcal{Q} = \mathrm{child}(\mathcal{G}_i)$ satisfies the hypothesis of Lemma 3.7 with $\beta := L\sigma^{i+1-M}$. Let

$$\mathrm{child}(\mathcal{G}_i) = \mathcal{G}_{i+1} \cup \mathcal{B}_{i+1} \tag{65}$$

be the decomposition given by Lemma 3.7, so that item 2 is satisfied and so that $\mathcal{G}_{i+1}$ satisfies item 3 and $\mathcal{B}_{i+1}$ satisfies item 4 for $i = i+1$. Note that (62) for $i$ combined with (65) implies (62) for $i+1$. Thus the required $\iota_{i+1}$, $\mathcal{G}_{i+1}$ and $\mathcal{B}_{i+1}$ exist and so the lemma is true by induction. □

We now prove, by induction on $i \in \mathbb{N}$, a local Lipschitz bound on each $\iota_i$ (with a Lipschitz constant that depends on $i$). Afterwards, this will be upgraded to uniform Hölder continuity.

**Lemma 3.13** *Let $M \in \mathbb{N}$ with $M > \alpha/2$, $0 \le L \le \sigma^M$ and suppose that*

$$\iota_M : \mathcal{D}(M) \to C$$

*is $L$-Lipschitz. For each $i \ge M$ let $\iota_i$, $\mathcal{G}_i$ and $\mathcal{B}_i$ be obtained from Lemma 3.12. Then for any $i \ge M$, $j \in \mathbb{N}$ and $p, q \in \operatorname{dom} \iota_i$ with $p, q \in Q \in \mathcal{Q}(j)$,*

$$d(\iota_i(p), \iota_i(q)) \le 2L\sigma^{-M+1}(\sigma l)^j. \tag{66}$$

*Proof* First note that, if $i = j$, (66) follows from Lemma 3.12 item 5 and if $j > i$ then $p = q$, so that (66) holds. To prove (66) for the case $i > j \ge M$, we will prove that

$$d(\iota_i(p), \iota_i(q)) \le L\sigma^{-M+1} \sum_{k=j}^{i-1} (l\sigma)^k \quad \forall p, q \in \operatorname{dom} \iota_i,\ \ p, q \in Q \in \mathcal{Q}(j) \tag{67}$$

holds for all $i > M$ and all $i > j \ge M$ by induction on $i$. Equation (66) then follows because the geometric series converges by (46).

Note that (67) holds for $i = M+1$ and $j = M$ since $\iota_M$ is $L$-Lipschitz. Now let $i > M+1$, and suppose that (67) holds for $i-1$ and all $i-1 > j \ge M$. Let $i > j \ge M$ and $p, q \in \operatorname{dom} \iota_i$ with $p, q \in Q \in \mathcal{Q}(j)$. If neither of $p$ or $q$ belong to $\operatorname{cor}(\mathcal{G}_i \cup \mathcal{B}_i)$, then they both belong to $\operatorname{dom} \iota_{i-1}$. If $j = i-1$ then Lemma 3.12 item 5 for $i-1$ implies (67) for $i$ since $\iota_i$ extends $\iota_{i-1}$. If $j < i-1$ then the induction hypothesis implies (67) for $i$. Therefore, we may suppose $p \in \operatorname{cor}(Q^*)$ for some $Q^* \in \mathcal{G}_i \cup \mathcal{B}_i$.

Note that Lemma 3.12 item 2 implies that there exists $Q' \in \mathcal{G}_{i-1}$ with $Q^* \subset Q'$. Let $p' \in \operatorname{cor}(Q') \subset \operatorname{dom} \iota_{i-1}$, so that $\|p - p'\|_\infty \le l^{i-1}$. By Lemma 3.12 item 5

$$d(\iota_i(p), \iota_i(p')) \le L\sigma^{i-M} l^{i-1} = L\sigma^{-M+1}(\sigma l)^{i-1}. \tag{68}$$

Note also that, since $Q' \in \mathcal{Q}(i-1)$ and $Q \in \mathcal{Q}(j)$ with $j \leq i-1$, we must have $Q' \subset Q$. Thus, since $p', q \in Q$, the induction hypothesis implies

$$d(\iota_{i-1}(p'), \iota_{i-1}(q)) \leq L\sigma^{-M+1} \sum_{k=j}^{i-2} (l\sigma)^k. \tag{69}$$

Using the fact that $\iota_i$ extends $\iota_{i-1}$, we may combine Eqs. (68) and (69) using the triangle inequality to get

$$d(\iota_i(p), \iota_i(q)) \leq L\sigma^{-M+1} \sum_{k=j}^{i-1} (l\sigma)^k,$$

completing the induction step. □

Recall from (45) that $\sigma l = l^\alpha$. We now show that the limit of the $\iota_i$ is $\alpha$-Hölder. As mentioned before Lemma 3.12, it is essential that the exponent of $l^i$ appearing in item 3 in the following lemma equals $\alpha$ in order to deduce (74).

**Lemma 3.14** *Let $M \in \mathbb{N}$ with $M > 2/\alpha$, $0 \leq L \leq \sigma^M$ and let $\iota \colon \mathcal{D}(M) \to C$ be $L$-Lipschitz. There exist a closed set $S \subset [0,1]^n$ with $S \supset \mathcal{D}(M)$, an extension of $\iota$ to*

$$\iota \colon (S, \|\cdot\|_\infty) \to C \cap B(\iota(0), 5L)$$

*and, for each $i \geq M$, a set $\mathcal{B}_i \subset \mathcal{Q}(i)$ such that:*

1. *$\iota$ is $\alpha$-Hölder continuous with Hölder constant $5Ll^{-\alpha}$.*
2. $$[0,1]^n = S \cup \bigcup_{i \geq M} \bigcup \mathcal{B}_i$$
3. *For each $i \geq M$ and each $p \in \operatorname{cor}(\mathcal{B}_i)$, $p \in S$ and*
$$B(\iota(p), L\sigma^{-M} l^{\alpha i}) \cap G = \emptyset.$$

*Proof* Set $\iota_M = \iota$ and for each $i \geq M$ let $\mathcal{G}_i, \mathcal{B}_i$ and $\iota_i$ be obtained from Lemma 3.12. Set $S' = \cup_{i \geq M} \operatorname{dom} \iota_i$ and define $\iota$ on $S'$ by $\iota(x) = \iota_i(x)$ whenever $x \in \operatorname{dom} \iota_i$. Let $p, p' \in S'$, let $i \in \mathbb{N}$ be sufficiently large such that $p, p' \in \operatorname{dom} \iota_i$ and let $m \in \mathbb{N}$ be such that $l^{m+1} < \|p - p'\|_\infty \leq l^m$.

If $m < M$, let $Q, Q' \in \mathcal{Q}(M)$ be such that $p \in Q$ and $p' \in Q'$ and let $q \in \operatorname{cor}(Q)$ and $q' \in \operatorname{cor}(Q')$ minimise the distance between $\operatorname{cor}(Q)$ and

$\mathrm{cor}(Q')$, so that $\|q - q'\|_\infty \le l^m$. Then by combining (66), the fact that $\iota$ is $L$-Lipschitz, and using the triangle inequality

$$\begin{aligned} d(\iota(p), \iota(p')) &\le d(\iota_i(p), \iota_i(q)) + d(\iota(q), \iota(q')) + \mathrm{d}(\iota_i(q'), \iota_i(p')) \\ &\le 2L\sigma^{-M+1}(\sigma l)^M + Ll^m + 2L\sigma^{-M+1}(\sigma l)^M \\ &\le 4L\sigma l^M + Ll^m \\ &\le 5L(\sigma l)^m. \end{aligned} \tag{70}$$

If $m \ge M$, let $p \in Q \in \mathcal{Q}(m)$ and $p' \in Q' \in \mathcal{Q}(m)$. Since $\|p - p'\|_\infty \le l^m$, $Q$ and $Q'$ have a common corner $q$ and hence $\|p - q\|_\infty, \|p' - q\|_\infty \le l^m$. Thus the triangle inequality and (66) give

$$\begin{aligned} d(\iota(p), \iota(p')) &\le d(\iota_i(p), \iota_i(q)) + d(\iota_i(q), \iota_i(p')) \\ &\le 2L\sigma^{-M+1}(\sigma l)^m + 2L\sigma^{-M+1}(\sigma l)^m \\ &\le 4L(\sigma l)^m, \end{aligned} \tag{71}$$

using the fact that $M \ge 2$ in the final inequality.

Equations (45), (70) and (71) show that

$$d(\iota(p), \iota(p')) \le 5Ll^{\alpha m} \le 5Ll^{-\alpha}\|p - p'\|_\infty^\alpha. \tag{72}$$

That is, $\iota$ is $\alpha$-Hölder with constant $5Ll^{-\alpha}$. This also implies that $\iota$ maps into $B(\iota(0), 5L)$.

Let $S$ be the closure of $S'$ and extend $\iota$ to $S$ maintaining the Hölder bound. Since $C$ is closed, this extension maps into $C \cap B(\iota(0), 5L)$. For each $i \ge M$, (62) implies

$$[0, 1]^n = \bigcup \mathcal{G}_i \cup \bigcup_{j \ge M} \bigcup \mathcal{B}_j. \tag{73}$$

For each $i \in \mathbb{N}$, $\mathrm{cor}(\mathcal{G}_i) \subset \mathrm{dom}\, \iota_i \subset S$ and so $\cup \mathcal{G}_i \subset B(S, l^i)$. Since $S$ is compact this implies

$$\bigcap_{i \ge M} \bigcup \mathcal{G}_i \subset S.$$

Thus, intersecting (73) over all $i \ge M$ gives item 2.

Finally, Eq. (45) and Lemma 3.12 item 4 and imply item 3. □

We now extend the map $\iota$ to $[0, 1]^n$, assuming $X$ is a subset of $\ell_\infty$. This will implicitly construct the metric space $\tilde{E}$ mentioned in the introduction. Since we have constructed $\iota$ to be Hölder continuous, this can be done using

Theorem 2.6. The Hölder condition is also used (in combination with (35)) to deduce (74) from Lemma 3.14 item 3.

**Proposition 3.15** *Let $M \in \mathbb{N}$ with $M > 2/\alpha$, $0 \leq L \leq \sigma^M$ and let $\iota\colon \mathcal{D}(M) \to C$ be $L$-Lipschitz. Suppose that $X$ is a subset of $\ell_\infty$. There exists an $\alpha$-Hölder extension $\iota\colon [0,1]^n \to B(\iota(0), 10L)$ with Hölder constant $5Ll^{-\alpha}$ such that*

$$\mathcal{H}^n_\infty(\iota([0,1]^n) \setminus C) \leq \frac{(50\sigma^M l^{-\alpha})^n}{\eta} \mathcal{H}^n(B(\iota(0), 10L) \cap X \setminus G). \tag{74}$$

*Proof* First apply Lemma 3.14 to find a closed $S \subset [0,1]^n$ and extend $\iota$ to an $\alpha$-Hölder map

$$\iota\colon S \to B(\iota(0), 5L) \cap C. \tag{75}$$

Then apply Theorem 2.6 to extend $\iota$ to an $\alpha$-Hölder map

$$\iota\colon [0,1]^n \to \ell_\infty$$

maintaining the Hölder constant of $5Ll^{-\alpha}$. In particular, since $\mathcal{D}(M) \subset S$ and $M > 2/\alpha$, this extension maps into $B(\iota(0), 10L)$.

Since $\iota(S) \subset C$, we deduce (74) by estimating the Hausdorff content of

$$Y := \iota\left(\bigcup_{i \geq M} \bigcup \mathcal{B}_i\right),$$

where the $\mathcal{B}_i$ are as in Lemma 3.14 item 4. That is, for each $i \geq M$, $\mathcal{B}_i \subset \mathcal{Q}(i)$ and, for each $p \in \operatorname{cor}(\mathcal{B}_i)$, $p \in S$ and

$$B(\iota(p), L\sigma^{-M} l^{\alpha i}) \cap G = \emptyset. \tag{76}$$

Since $\iota$ is $\alpha$-Hölder with constant $5Ll^{-\alpha}$,

$$Y \subset \{B(\iota(p), 5Ll^{-\alpha} l^{i\alpha}) : i \geq M,\ p \in \operatorname{cor}(\mathcal{B}_i)\}.$$

That is, if

$$\mathcal{A} = \{B(\iota(p), L\sigma^{-M} l^{\alpha i}) : i \geq M,\ p \in \operatorname{cor}(\mathcal{B}_i)\},$$

then

$$Y \subset \bigcup_{B \in \mathcal{A}} 5\sigma^M l^{-\alpha} B.$$

Let $\mathcal{A}'$ be a disjoint $5r$-sub-cover of $\mathcal{A}$, so that the elements of $\mathcal{A}'$ are pairwise disjoint and

$$Y \subset \bigcup_{B \in \mathcal{A}'} 25\sigma^M l^{-\alpha} B.$$

In particular,

$$\mathcal{H}^n_\infty(Y) \leq \sum_{B \in \mathcal{A}'} (50\sigma^M l^{-\alpha} \operatorname{rad} B)^n \tag{77}$$

By Eqs. (75) and (76),

$$\bigcup \mathcal{A}' \subset B(\iota(0), 10L) \setminus G. \tag{78}$$

For each $i \geq M$ and $p \in \operatorname{cor}(\mathcal{B}_i)$, $\iota(p) \in C$ and so Definition 3.2 item 1 implies

$$\mathcal{H}^n(B(\iota(p), r) \cap X) \geq \eta r^n$$

whenever $r \leq 1$. In particular, this is true for all balls in $\mathcal{A}'$. Therefore, by (78) and the disjointness of the balls in $\mathcal{A}'$,

$$\begin{aligned} \eta \sum_{B \in \mathcal{A}'} \operatorname{rad}(B)^n &\leq \sum_{B \in \mathcal{A}'} \mathcal{H}^n(B \cap X) \\ &\leq \mathcal{H}^n(B(\iota(0), 10L) \cap X \setminus G). \end{aligned}$$

Combining this with (77) gives

$$\mathcal{H}^n_\infty(Y) \leq \frac{(50\sigma^M l^{-\alpha})^n}{\eta} \mathcal{H}^n(B(\iota(0)), 10L) \cap X \setminus G),$$

as required. □

To conclude the section, we record the main conclusion without the fixed quantities of Notation 3.5. Recall, for $m \in \mathbb{N}$ and $R > 0$, that

$$\mathcal{D}(R, m) = \{R2^{-m}(j_1, \ldots, j_n) \in [0, R]^n : j_1, \ldots, j_n \in \mathbb{Z}\}.$$

**Theorem 3.16** *For any $K \geq 1$, $0 < \gamma < 1$ and $\eta > 0$ there exists $N \in \mathbb{N}$ and $\gamma \leq \alpha < 1$ such that the following is true. For any closed $X \subset \ell_\infty$ and for any $R_0 > 0$, let $(X, C, G)$ satisfy*

$$GTA(\eta, K, 2^{-N}/20, R_0). \tag{79}$$

*Then for any $M \in \mathbb{N}$ with $M > 2/\alpha$, $0 < r < R_0$, $0 \leq L \leq (5K^2)^M$ and any $L$-Lipschitz*

$$\iota\colon \mathcal{D}(r, MN) \subset \ell_\infty^n \to C,$$

*there exists an $\alpha$-Hölder extension*

$$\iota\colon [0, r]^n \subset \ell_\infty^n \to B(\iota(0), 10Lr)$$

*with Hölder constant $5L2^{N\alpha}r^{1-\alpha}$ such that*

$$\mathcal{H}^n_\infty(\iota([0, r]^n) \setminus C) \leq \Lambda \mathcal{H}^n(B(\iota(0), 10Lr) \cap X \setminus G), \tag{80}$$

*for some $\Lambda > 0$ depending only upon $K, \gamma, M, n, \eta$.*

*Proof* Given $K, \gamma, \eta$ as in the hypotheses, let $N \in \mathbb{N}$ and $\gamma \leq \alpha < 1$ be given by (44). For any $(X, C, G)$ satisfying (79) and any $0 < r < R_0$, let $C' = C/r$ and $G' = G/r$, considered as subsets of $X' = X/r \subset \ell_\infty$. Then $(X', C', G')$ satisfies $GTA(\eta, K, 2^{-N}/20, 1)$. Define $\iota'\colon \mathcal{D}(1, MN) \to C'$ by $\iota'(q) = \iota(rq)/r$. Then $\iota'$ is also $L$-Lipschitz. An application of Proposition 3.15 gives an $\alpha$-Hölder extension of $\iota'$ with Hölder constant $5L2^{N\alpha}$ satisfying

$$\mathcal{H}^n_\infty(\iota'([0, 1]^n) \setminus C') \leq \Lambda \mathcal{H}^n(B(\iota(0), 10L) \cap X' \setminus G') \tag{81}$$

for $\Lambda = (20(5K)^{2M}2^{\alpha N})^n/\eta$. This gives an $\alpha$-Hölder extension of $\iota$ defined by $\iota(q) = r\iota'(q/r)$ with Hölder constant $5L2^{N\alpha}r^{1-\alpha}$:

$$\begin{aligned}\|\iota(y) - \iota(z)\|_\infty &= r\|\iota'(y/r) - \iota'(z/r)\|_\infty \\ &\leq r5L2^{N\alpha}\|y/r - z/r\|^\alpha_\infty \\ &= 5L2^{N\alpha}r^{1-\alpha}\|y - z\|^\alpha_\infty.\end{aligned}$$

Equation (81) is equivalent to (80). □

Finally, we conclude the proof of Theorem 3.4

*Proof of Theorem 3.4* The hypotheses of Theorem 3.4 imply the hypotheses of Theorem 3.16. Let $N, \alpha$ be obtained from Theorem 3.16 and let $m_0 = 2N$. If $(X, C, G)$ satisfies (40) and $x \in G$ then by Observation 3.3, there exists a

$(K + 2^{-m})$-bi-Lipschitz $\xi : \mathcal{D}(r, m) \to C$ with $\xi(0) = x$. Since $(X, C, G)$ also satisfies (79), Theorem 3.16 with $M = 2$ gives an $\alpha$-Hölder extension $\iota : [0, r]^n \to B(x, 20Kr)$ with Hölder constant $10K2^{N\alpha}r^{1-\alpha}$ satisfying items 1 and 3 of Theorem 3.4. Since $\gamma \leq \alpha < 1$, $\iota$ is also $\gamma$-Hölder. Moreover, if $y, z \in [0, r]^n$ with $\|y - z\|_\infty \leq 2^{-m}r$ then

$$\begin{aligned}\|\iota(y) - \iota(z)\|_\infty &\leq 10K2^{N\alpha}r^{1-\alpha}\|y - z\|_\infty^\alpha \\ &\leq 10K2^{(N-m)\alpha}r^{1-\alpha+\alpha} \\ &\leq 10K2^{-m\alpha/2}r \\ &\leq 10K2^{-m\gamma/2}r,\end{aligned}$$

giving item 2. □

*Remark 3.17* If, in the hypotheses of Theorem 3.16 or Theorem 3.4, $X$ is assumed to be a subset of a finite dimensional Banach space $(\mathbb{R}^m, \|\cdot\|)$, then the Hölder map $\iota$ can be constructed with values in $(\mathbb{R}^m, \|\cdot\|)$. One simply uses Theorem 2.6 to extend the map into $(\mathbb{R}^m, \|\cdot\|)$, increasing the Hölder constant by a factor that only depends on the norm $\|\cdot\|$.

*Remark 3.18* At this point it would be reasonable to prove Theorem 1.1 using Theorem 3.4 and Corollary 6.4. However, there are various technical steps that are simply easier to deal with using $d_*$. Thus we delay the proof until after the proof of Theorem 1.2.

## 4 Metrics between metric measure spaces

### 4.1 The pointed measured Gromov–Hausdorff metric

Recall the definition of $G_{a,b}$ given in Definition 2.19. We use the following metric to describe pointed measured Gromov–Hausdorff convergence.

**Definition 4.1** For pointed metric measure spaces $(X, \mu, x)$ and $(Y, \nu, y)$ define

$$d_{\mathrm{pmGH}}((X, \mu, x), (Y, \nu, y)) = G_{1,1}((X, \mu, x), (Y, \nu, y)).$$

Note that $d_{\mathrm{pmGH}}$ is bounded below by $d_{\mathrm{pGH}}$ of the corresponding pointed metric spaces.

*Remark 4.2* If one is only interested in metrising pointed measured Gromov–Hausdorff convergence, it is more natural to use $F_z^1$ rather than $F_z$ in the definition of $d_{\mathrm{pmGH}}$. Doing so would not change the results of this subsection, except for minor superficial changes in Corollary 4.4. We use $F_z$ to give the desired relationship to $d_*$ in the next subsection.

We have the following relationship between $\epsilon$-isometries and $F$.

**Lemma 4.3** *Let $(Z, \zeta, z)$ be a pointed metric measure space $X, Y \subset Z$ Borel sets, $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ and $\nu \in \mathcal{M}_{\mathrm{loc}}(Y)$. For $0 < \epsilon < 1/2$ suppose that $f\colon X \cap B(z, 1/\epsilon) \to Y$ is a Borel function that satisfies $\zeta(f(x), x) \leq \epsilon$ for all $x \in X \cap B(z, 1/\epsilon)$. Then for any $0 < r \leq 1/\epsilon - \epsilon$ and $L > 0$,*

$$|F_z^{L,r}(f_\# \mu, \nu) - F_z^{L,r}(\mu, \nu)| \leq \epsilon L \mu(B(z, r + \epsilon)).$$

*Proof* Let $g\colon Z \to [-1, 1]$ be $L$-Lipschitz with $\operatorname{spt} g \subset B(z, r)$. For any $x \in B(z, 1/\epsilon)$,

$$|g \circ f(x) - g(x)| \leq L\zeta(f(x), x) \leq L\epsilon$$

and so, since $r \leq 1/\epsilon - \epsilon$,

$$\left| \int_{B(z,r+\epsilon)} g \circ f \, \mathrm{d}\mu - \int_{B(z,r+\epsilon)} g \, \mathrm{d}\mu \right| \leq L\epsilon\mu(B(z, r + \epsilon)).$$

However, $x \in X \cap B(z, 1/\epsilon) \setminus B(z, r + \epsilon)$ then $x, f(x) \notin \operatorname{spt} g$ and so

$$\left| \int g \circ f \, \mathrm{d}\mu - \int g \, \mathrm{d}\mu \right| \leq L\epsilon\mu(B(z, r + \epsilon)).$$

Therefore,

$$\begin{aligned} \int g \, \mathrm{d}(f_\# \mu - \nu) &= \int g \circ f \, \mathrm{d}\mu - \int g \, \mathrm{d}\nu \\ &\leq L\epsilon\mu(B(z, r + \epsilon) + \int g \, \mathrm{d}(\mu - \nu). \end{aligned}$$

Taking the supremum over all such $g$ gives

$$F_z^{L,r}(f_\# \mu, \nu) \leq L\epsilon\mu(B(z, r + \epsilon)) + F_z^{L,r}(\mu, \nu).$$

Similarly,

$$F_z^{L,r}(\mu, \nu) \leq L\epsilon\mu(B(z, r + \epsilon)) + F_z^{L,r}(f_\# \mu, \nu),$$

as required. □

**Corollary 4.4** *Let $(X, \mu, x)$ and $(Y, \rho, y)$ be pointed metric measure spaces and $0 < \epsilon < 1/2$. If $d_{\mathrm{pmGH}}((X, \mu, x), (Y, \nu, y)) < \epsilon$, there exists a $2\epsilon$-isometry $f\colon (X, x) \to (Y, y)$ such that, for any $0 < r \leq 1/\epsilon - \epsilon$ and $L > 0$,*

$$F_y^{L,r}(f_\# \mu, \nu) \leq \epsilon + \epsilon L \mu(B(x, r + \epsilon)). \tag{82}$$

*Conversely, if there exists an $\epsilon$-isometry $f : (X, \mu, x) \to (Y, \nu, y)$ then*

$$d_{\mathrm{pmGH}}((X, \mu, x), (Y, \nu, y))$$

*is bounded above by the maximum of $2\epsilon$ and*

$$\inf\{1/r + F_y^{r,r}(f_\# \mu, \nu) + \epsilon r \mu(B(x, r + \epsilon)) : 0 < r \leq 2\epsilon\}.$$

*Proof* Let $(Z, \zeta, z)$ be a pointed metric space for which there exist isometric embeddings $(X, x), (Y, y) \to (Z, z)$ with $H_z(X, Y) < \epsilon$ and $F_z(\mu, \nu) < \epsilon$. Lemma 2.24 gives a $2\epsilon$-isometry $f : (X, x) \to (Y, y)$ such that $\zeta(x', f(x')) \leq \epsilon$ for all $x' \in X \cap B(x, 1/\epsilon)$. For any $0 < r \leq 1/\epsilon - \epsilon$, Lemma 4.3 implies (82).

Conversely, Lemma 2.24 gives a pointed metric space $(Z, \zeta, z)$ and isometric embeddings $(X, x), (Y, y) \to (Z, z)$ such that $\zeta(x', f(x')) \leq 2\epsilon$ for all $x' \in X \cap B(x, 1/\epsilon)$ and $H_z(X, Y) \leq 2\epsilon$. Lemma 4.3 implies, for any $0 < r \leq 1/2\epsilon$,

$$F_z^{r,r}(\mu, \nu) \leq F_z^{r,r}(f_\# \mu, \nu) + \epsilon r \mu(B(x, r + \epsilon)).$$

Thus

$$F_z(\mu, \nu) \leq 1/r + F_z^{r,r}(f_\# \mu, \nu) + \epsilon r \mu(B(x, r + \epsilon)).$$

Taking the infimum over all $0 < r \leq 1/\epsilon - \epsilon$ completes the proof. □

**Definition 4.5** An isometry $(X, \mu, x) \to (Y, \nu, y)$ of pointed metric measure spaces is an isometry $(X, x) \to (Y, y)$ with $\mu = \nu$.

**Corollary 4.6** *On the set $\mathbb{M}_{pm}$ of isometry classes of proper pointed metric measure spaces, $d_{\mathrm{pmGH}}$ is a complete and separable metric.*

*Proof* By Proposition 2.20 we know $d_{\mathrm{pmGH}}$ is a complete pseudometric. It is easily verified that the set of finite pointed metric spaces with rational distances and rational combinations of Dirac masses is dense. By Corollary 2.25, any complete limit of proper pointed metric spaces is also proper.

Let

$$d_{\mathrm{pmGH}}((X, \mu, x), (Y, \nu, y)) = 0.$$

By Corollary 4.4, for each $i \in \mathbb{N}$ there exists a $1/i$-isometry $f_i : (X, x) \to (Y, y)$ such that, for any $L, r > 0$,

$$F_{L,r}((f_i)_\# \mu, \nu) \to 0. \tag{83}$$

Fix $r > 0$. As in the proof of Corollary 2.25, for each $j \in \mathbb{N}$ let $\mathcal{N}_j$ be a finite $1/j$-net of $B(x, r)$. By taking a subsequence, we may suppose the $f_i$ converge pointwise to an isometry $\iota$ when restricted to $\cup_j \mathcal{N}_j$ whose image is a dense subset of $B(y, r)$. We extend $\iota$ to an isometry $\iota\colon B(x, r) \to B(y, r)$.

For a moment fix $i, j \in \mathbb{N}$. For any $w \in B(y, r)$ let $z \in \mathcal{N}_j$ with $w \in B(z, 1/j)$. Since $f_i$ is an $\epsilon_i$-isometry and $\iota$ is an isometry,

$$\begin{aligned}\rho(\iota(w), f_i(w)) &\le \rho(\iota(w), \iota(z)) + \rho(\iota(z), f_i(z)) + \rho(f_i(z), f_i(w))\\ &\le d(w, z) + \rho(\iota(z), f_i(z)) + d(w, z) + \epsilon_i\\ &\le \rho(\iota(z), f_i(z)) + 2/j + \epsilon_i.\end{aligned}$$

Thus, for any $r$-Lipschitz $g\colon Y \to [-1, 1]$ with $\operatorname{spt} g \subset B(y, r)$,

$$\int g \,\mathrm{d}((f_i)_\#\mu - \iota_\#\mu) \le r\left(\sup_{z\in\mathcal{N}_j} \rho(f_i(z), \iota(z)) + 2/j + \epsilon_i\right).$$

Consequently,

$$F_y^{r,r}((f_i)_\#\mu, \iota_\#\mu) \le r\left(\sup_{z\in\mathcal{N}_j} \rho(f_i(z), \iota(z)) + 2/j + \epsilon_i\right)$$

and so

$$\limsup_{i\to\infty} F_y^{r,r}((f_i)_\#\mu, \iota_\#\mu) \le 2r/j.$$

Since this is true for all $j \in \mathbb{N}$, (83) gives $F_y^{r,r}(\iota_\#\mu, \nu) = 0$ for all $r > 0$. Thus Lemma 2.14 implies $\iota_\#\mu = \nu$. Consequently, $d_{\mathrm{pmGH}}$ is a metric on $\mathbb{M}_{pm}$. □

**Corollary 4.7** *A set $\mathcal{S} \subset (\mathbb{M}_{pm}, d_{\mathrm{pmGH}})$ is pre-compact if and only if*

1. *$\{(X, x) : (X, \mu, x) \in \mathcal{S}\} \subset (\mathbb{M}_p, d_{\mathrm{pGH}})$ is pre-compact and*
2. *For every $r > 0$, $\{\mu(B(x, r)) : (X, \mu, x) \in \mathcal{S}\}$ is bounded.*

*Proof* Suppose that $\mathcal{S}$ satisfies items 1 and 2 and, for each $i \in \mathbb{N}$, let $(X_i, \mu_i, x_i) \in \mathcal{S}$. After passing to a subsequence, there exists a proper pointed metric space $(X, x)$ such that

$$d_{\mathrm{pGH}}((X_i, x_i), (X, x)) < 1/(2i)$$

for each $i \in \mathbb{N}$. By Lemma 2.24, for each $i \in \mathbb{N}$ there exists a $1/i$-isometry $f_i\colon (X_i, x_i) \to (X, x)$. Item 2 and Theorem 2.11 imply, after passing to a

further subsequence, that $(f_i)_\# \mu_i$ converges to some measure $\mu$ on $X$. Corollary 4.4 then implies

$$d_{\mathrm{pmGH}}((X_i, \mu_i, x_i), (X, \mu, x)) \to 0.$$

The converse statement is immediate. □

### 4.2 Gromov–Hausdorff convergence of large subsets

Gromov–Hausdorff convergence of pointed metric measure spaces is used throughout analysis. However, the requirement that the underlying metric spaces must Gromov–Hausdorff converge is too rigid for some uses in geometric measure theory. For example, it requires that every sequence of null sets must Gromov–Hausdorff converge to some limit.

For a concrete example, consider the following.

*Example 4.8* Let $X = \{0, 1\}$ with $d(0, 1) = 1$ and for each $i \in \mathbb{N}$ let $\mu_i(\{0\}) = 1$ and $\mu(\{1\}) = 1/i$. Then the sequence $(X, d, \mu_i, 0)$ Gromov–Hausdorff converges to $(X, d, \delta_0, 0)$ but *not* to $(\{0\}, \delta_0, 0)$, which is the more natural choice. That is, Gromov–Hausdorff convergence may require one to consider a larger metric space than simply the support of the limit measure.

For each $i \in \mathbb{N}$ let $X_i$ be an $i$-point metric space including a point 0, equipped with the discrete metric $d_i$. Then the sequence $(X_i, d_i, \delta_0, 0)$ has no Gromov–Hausdorff limit, even though a one point metric space with a Dirac measure is the natural candidate from a measure theoretic point of view.

To construct a 1-rectifiable metric space with no Gromov–Hausdorff tangents at any point, one takes an isometric copy of an interval and scatters "dust" around it at all scales as in the previous example.

Rather than Gromov–Hausdorff convergence we will use the following.

**Definition 4.9** For pointed metric measure spaces $(X, \mu, x)$ and $(Y, \nu, y)$ define

$$d_*((X, \mu, x), (Y, \nu, y)) = G_{0,1}((X, \mu, x), (Y, \nu, y)).$$

Note that $d_* \leq d_{\mathrm{pmGH}}$. Also, for any Borel $x \in S \subset X$, (18) implies

$$d_*((X, \mu, x), (S, \mu|_S, x)) \leq \inf\{r > 0 : \mu(B(x, 1/r) \setminus S) < r\}. \quad (84)$$

Although $d_*$ does not require the underlying sets to be close in the sense of $d_{\mathrm{pGH}}$, we can recover large subsets which are.

**Lemma 4.10** *Let $(Z, z)$ be a complete pointed metric space and $\mu, \nu \in \mathcal{M}_{\mathrm{loc}}(Z)$. For any $r, \epsilon, \delta > 0$ there exist compact sets*

$$K_\mu \subset B(z, r) \cap \operatorname{spt} \mu \quad \textit{and} \quad K_\nu \subset B(z, r) \cap \operatorname{spt} \nu$$

*with*

$$\max\{\mu(B(z, r) \setminus K_\mu),\ \nu(B(z, r) \setminus K_\nu)\} < (1 + \delta) F_z^{1/\epsilon, r}(\mu, \nu), \tag{85}$$

*and*

$$H_z(K_\mu, K_\nu) \leq \max\left\{\frac{1}{r - \epsilon}, \epsilon\right\}. \tag{86}$$

*Proof* There exist compact

$$K'_\mu \subset B(z, r) \cap \operatorname{spt} \mu \quad \text{and} \quad K'_\nu \subset B(z, r) \cap \operatorname{spt} \nu$$

with

$$\max\{\mu(B(z, r) \setminus K'_\mu),\ \nu(B(z, r) \setminus K'_\nu)\} < \delta F_z^{1/\epsilon, r}(\mu, \nu) \tag{87}$$

Let

$$\tilde{K}_\mu = (K'_\mu \cap U(z, r - \epsilon)) \setminus B(K'_\nu, \epsilon),$$

a relatively open set in $K'_\mu$, and define $g\colon Z \to [0, 1]$ by

$$g(x) = \begin{cases} 1 & z \in \tilde{K}_\mu \\ 1 - \frac{1}{\epsilon} \min\{d(z, \tilde{K}_\mu), \epsilon\} & \text{otherwise.} \end{cases}$$

Then $g$ is $1/\epsilon$-Lipschitz with $\operatorname{spt} g \subset B(z, r)$. By construction,

$$\mu(\tilde{K}_\mu) \leq \int g \, \mathrm{d}\mu$$

and

$$\int g \, \mathrm{d}\nu \leq \nu(B(z, r) \setminus K'_\nu).$$

Therefore,

$$\mu(\tilde{K}_\mu) \leq F_z^{1/\epsilon, r}(\mu, \nu) + \nu(B(z, r) \setminus K'_\nu)$$

$$\leq (1+\delta) F_z^{1/\epsilon, r}(\mu, \nu), \tag{88}$$

by (87). Define

$$K_\mu := K'_\mu \setminus \tilde{K}_\mu = (K'_\mu \cap B(K'_\nu, \epsilon)) \cup (K'_\mu \setminus U(z, r-\epsilon)), \tag{89}$$

a compact set, and, by Eqs. (87) and (88),

$$\mu(B(z,r) \setminus K_\mu) < (1+2\delta) F_z^{1/\epsilon, r}(\mu, \nu).$$

Also, by (89),

$$K_\mu \cap U(z, r-\epsilon) \subset B(K'_\nu, \epsilon). \tag{90}$$

By the symmetric argument, the compact set

$$K_\nu := (K'_\nu \cap B(K'_\mu, \epsilon)) \cup (K'_\nu \setminus U(z, r-\epsilon))$$

satisfies

$$\nu(B(z,r) \setminus K_\nu) < (1+2\delta) F_z^{1/\epsilon, r}(\mu, \nu)$$

and

$$K_\nu \cap U(y, r-\epsilon) \subset B(K'_\mu, \epsilon). \tag{91}$$

Finally, if $x \in K_\mu \cap U(z, r-\epsilon)$, let $y \in K'_\nu$ with $x \in B(y, \epsilon)$. Then $y \in B(K'_\mu, \epsilon)$ and so $y \in K_\nu$. Thus

$$K_\mu \cap U(x, r-\epsilon) \subset B(K_\nu, \epsilon).$$

Similarly,

$$K_\nu \cap U(y, r-\epsilon) \subset B(K_\mu, \epsilon).$$

Thus $H_z(K_\mu, K_\nu) \leq \max\{1/(r-\epsilon), \epsilon\}$. Therefore, $K_\mu$, $K_\nu$ have the required properties with $\delta$ replaced by $2\delta$. Since $\delta > 0$ is arbitrary, this suffices. □

**Theorem 4.11** *For $\epsilon > 0$ let $(X, \mu, x)$ and $(Y, \nu, y)$ be pointed metric measure spaces with*

$$d_*((X, \mu, x), (Y, \nu, y)) < \epsilon.$$

*There exist compact*

$$x \in K_\mu \subset \operatorname{spt}\mu \cap B(x, 1/\epsilon) \quad \textit{and} \quad y \in K_\nu \subset \operatorname{spt}\nu \cap B(y, 1/\epsilon) \tag{92}$$

*such that*

$$\max\{\mu(B(x, 1/\epsilon) \setminus K_\mu), \nu(B(y, 1/\epsilon) \setminus K_\nu)\} < \epsilon \tag{93}$$

*and*

$$d_{\mathrm{pmGH}}((K_\mu, \mu|_{K_\mu}, x), (K_\nu, \nu|_{K_\nu}, y)) < 3\epsilon. \tag{94}$$

*Conversely, if there exist compact $K_\mu$, $K_\nu$ as in (92) satisfying (93) and*

$$d_*((K_\mu, \mu|_{K_\mu}, x), (K_\nu, \nu|_{K_\nu}, y)) < \epsilon \tag{95}$$

*then*

$$d_*((X, \mu, x), (Y, \nu, y)) < 3\epsilon.$$

*That is, $d_*$ is bi-Lipschitz equivalent to the metric defined by taking the infimum over all $0 < \epsilon < 1/2$ for which Eqs. (93) and (94) hold.*

*Proof* If $\epsilon \geq 1/2$ there is nothing to prove. Otherwise, by the definition of $d_*$, there exists a pointed metric space $(Z, z)$ and isometric embeddings $(X, x), (Y, y) \to (Z, z)$ with $F_z(\mu, \nu) < \epsilon$.

Let $\delta > 0$ be such that $(1 + \delta)F_z(\mu, \nu) < \epsilon$ and let $K_\mu$ and $K_\nu$ be obtained from Lemma 4.10, so that (93) is satisfied. Then (86) implies

$$H_z(K_\mu, K_\nu) \leq 2\epsilon.$$

Finally, Eqs. (18) and (93) and the triangle inequality imply

$$\begin{aligned} F_z^{1/\epsilon,1/\epsilon}(\mu|_{K_\mu}, \nu|_{K_\nu}) &\leq F_z^{1/\epsilon,1/\epsilon}(\mu|_{K_\mu}, \mu) + F_z^{1/\epsilon,1/\epsilon}(\mu, \nu) \\ &\quad + F_z^{1/\epsilon,1/\epsilon}(\nu, \nu|_{K_\nu}) \\ &< 3\epsilon. \end{aligned}$$

Thus (94) is established.

Conversely, by (84) and the triangle inequality,

$$\begin{aligned} d_*((X, \mu, x), (Y, \nu, y)) &\leq d_*((X, \mu, x), (K_\mu, \mu|_{K_\mu}, x)) \\ &\quad + d_*((K_\mu, \mu|_{K_\mu}, x), (K_\nu, \nu|_{K_\nu}, y)) \end{aligned}$$

$$+ d_*((K_\nu, \nu|_{K_\nu}, y), (Y, \nu, y))$$
$$< 3\epsilon.$$

□

Working with measures allows us to take limits of $\epsilon$-isometries (see also [26, Proposition 3.2], which shows the compactness of the set of isometries).

**Lemma 4.12** *Let $(X, d, \mu, x)$ and $(Y, \rho, \nu, y)$ be pointed metric measure spaces. Suppose that, for each $i \in \mathbb{N}$, there exists a Borel set $x \in S_i \subset X$ with*

$$\mu(B(x, i) \setminus S_i) \to 0 \tag{96}$$

*and a $1/i$-isometry $f_i \colon S_i \to S_i' \subset Y$ such that*

$$F_y((f_i)_\#(\mu|_{S_i}), \nu) \to 0. \tag{97}$$

*Then there exists an isometry $\iota \colon (\operatorname{spt} \mu \cup \{x\}, \mu, x) \to (\operatorname{spt} \nu \cup \{y\}, \nu, y)$.*

*Proof* By the inner regularity of $\mu$, we may suppose each $S_i$ is compact. For each $i \in \mathbb{N}$ and $w \in S_i$, let $g_i(w) = (w, f_i(w)) \in X \times Y$ and define

$$\lambda_i = (g_i)_\#(\mu|_{S_i}) \in \mathcal{M}_{\mathrm{loc}}(X \times Y).$$

By Theorem 2.11, after passing to a subsequence, we may suppose that $(g_i)_\#(\mu|_{S_i})$ converges to some measure $\lambda \in \mathcal{M}_{\mathrm{loc}}(X \times Y)$. Note that the pushforward of $\lambda_i$ to $X$ under the projection map is $\mu|_{S_i}$. By (96) the projection of $\operatorname{spt} \lambda$ to $X$ has full $\mu$ measure and by (97) the projection of $\operatorname{spt} \lambda$ to $Y$ has full $\nu$ measure.

If $(x^1, y^1) \in \operatorname{spt} \lambda$ then there exist $x_i^1 \in S_i$ with $g_i(x_i^1) \to (x^1, y^1)$. Indeed, if not then there exist $r > 0$ and $i_k \to \infty$ such that

$$\lambda_{i_k}(B((x^1, y^1), r)) = \mu_{i_k}(S_{i_k} \cap g_{i_k}^{-1}(B((x^1, y^1), r))) = 0 \quad \forall k \in \mathbb{N},$$

contradicting the convergence of $\lambda_i$ to $\lambda$ using (14). In particular, if $(x^1, y^1), (x^2, y^2) \in \operatorname{spt} \lambda$, there exist $x_i^1, x_i^2 \in S_i$ with $(x_i^1, f_i(x_i^1)) \to (x^1, y^1)$ and $(x_i^2, f_i(x_i^2)) \to (x^2, y^2)$. Since each $f_i$ is a $1/i$-isometry, this implies

$$\rho(y^1, y^2) = \lim_{i \to \infty} \rho(f_i(x_i^1), f_i(x_i^2)) = \lim_{i \to \infty} d(x_i^1, x_i^2) = d(x^1, x^2).$$

That is, $\operatorname{spt} \lambda$ lies on the graph of an isometry. This isometry is defined $\mu$ almost everywhere and maps onto a set of full $\nu$-measure. Since $Y$ is complete, $\iota$ can be extended to an isometry of $\operatorname{spt} \mu$ to $\operatorname{spt} \nu$, as required. □

**Corollary 4.13** *Let $\mathbb{M}_*$ be the set of equivalence classes of all pointed metric measure spaces under the relation*

$$(X, \mu, x) \sim (Y, \nu, y) \text{ if } (\operatorname{spt}\mu \cup \{x\}, \mu, x) \text{ is isometric to } (\operatorname{spt}\nu \cup \{y\}, \nu, y).$$

*Then $(\mathbb{M}_*, d_*)$ is a complete and separable metric space.*

*Proof* Proposition 2.20 implies $d_*$ is a complete pseudometric on the set of all pointed metric measure spaces. It is easily verified that the set of finite pointed metric spaces with rational distances and rational combinations of Dirac masses is dense.

We next show that $d_*$ is well defined on $\mathbb{M}_*$. Indeed, if $(X, d, \mu, x) \sim (Y, \rho, \nu, y)$, let

$$\iota\colon \operatorname{spt}\mu \cup \{x\} \to \operatorname{spt}\nu \cup \{y\}$$

be an isometry of pointed metric measure spaces. Define $\zeta$ on $X \sqcup Y$ by $\zeta = d$ on $X$, $\zeta = \rho$ on $Y$ and

$$\zeta(x, y) = \inf\{d(x, w) + \rho(f(w), y) : w \in \operatorname{spt}\mu \cup \{x\}\}$$

whenever $x \in X$ and $y \in Y$. A similar argument to Lemma 2.24 shows that $\zeta$ is a pseudometric on $X \sqcup Y$. Indeed, if $z, z' \in X$ and $t \in Y$ then for any $w, w' \in \operatorname{spt}\mu \cup \{x\}$, the triangle inequality for $d$ and $\rho$ and the fact that $f$ is an isometry implies

$$\begin{aligned}\zeta(z, z') &\le d(z, w) + d(w, w') + d(w', z')\\ &= d(z, w) + \rho(f(w), f(w')) + d(w', z')\\ &\le d(z, w) + \rho(f(w), t) + \rho(t, f(w')) + d(w', z').\end{aligned}$$

Taking the infimum over $w, w' \in \operatorname{spt}\mu \cup \{x\}$ gives $\zeta(z, z') \le \zeta(z, t) + \zeta(t, z')$. A similar argument for $z, z' \in Y$ and $t \in X$ gives the triangle inequality for $\zeta$. Note that $\zeta(x, f(x)) = 0$ for all $x \in \operatorname{spt}\mu \cup \{x\}$.

Let $(Z, \zeta)$ be the metric space obtained from $(X \sqcup Y, \zeta)$ and let $z \in Z$ be the point corresponding to $x = y$. Then $(X, x)$ and $(Y, y)$ isometrically embed into $(Z, z)$ and, since $f$ is an isometry of metric measure spaces, the pushforwards of $\mu$ and $\nu$ under these embeddings agree. In particular, $F_z(\mu, \nu) = 0$ in $Z$ and hence

$$d_*((X, d, \mu, x), (Y, \rho, \nu, y)) = 0.$$

That is, $d_*$ is well defined on $\mathbb{M}_*$.

By combining Corollary 4.4, Lemma 4.12 and Theorem 4.11, any pointed metric measure spaces satisfying

$$d_*((X, \mu, x), (Y, \nu, y)) = 0$$

are isometric. Hence $d_*$ is a metric on $\mathbb{M}_*$. □

When considering $(X, \mu, x) \in \mathbb{M}_*$, the set $X \setminus (\operatorname{spt} \mu \cup \{x\})$ often plays no role. In such cases, to simplify notation, we will write $(\mu, x)$ for the element of $\mathbb{M}_*$.

*Remark 4.14* Sturm [14] defines a metric on the set of metric measure spaces equipped with a probability measure with finite variance. Greven, Pfaffelhuber and Winter [15] define the *Gromov–Prokhorov* metric on the set of compact metric measure spaces equipped with a probability measure. The definitions of the metrics are analogous to that of $d_*$, but use the $L_2$ transportation distance and the Prokhorov metric respectively in place of $F$ and there are no requirements imposed on a distinguished point. In this setting, one can slightly modify the definition of $d_*$ to not make any requirements on a distinguished point, or modify the definitions of [14] or [15] to consider a distinguished point. After doing so, since $F$, the $L_2$ transportation distance and the Prokhorov metric all metrise weak* convergence in the respective settings, the metrics all define the same topology whenever they are defined.

Gigli, Mondino and Savaré [26] define a notion of convergence on the set of pointed metric measure spaces by $(X_i, \mu_i, x_i) \to (X, \mu, x)$ if there exist a complete and separable metric space $Z$ and isometric embeddings $\operatorname{spt} \mu_i, \operatorname{spt} \mu \to Z$ with $x_i \to x$ and $\mu_i \to \mu$ in $\mathcal{M}_{\mathrm{loc}}(Z)$. By Proposition 2.20 we see that this definition agrees with convergence with respect to $d_*$, using Lemma 2.22 to ensure that the distinguished points are mapped to the same point. Several equivalent definitions of this convergence for *non-zero* measures are given in [26, Theorem 3.15].

Shortly after a first version of the present paper appeared I became aware of Pasqualetto and Schultz [27]. Amongst other results, they prove a relationship between pointed measured Gromov–Hausdorff convergence and the convergence of [26]. This relationship corresponds to the relationship between $d_{\mathrm{pmGH}}$ and $d_*$ given by Theorem 4.11.

**Theorem 4.15** *A set $\mathcal{S} \subset (\mathbb{M}_*, d_*)$ is pre-compact if and only if, for every $\epsilon > 0$ and $(\mu, x) \in \mathcal{S}$, there exists a compact $K_{\mu,x} \subset B(x, 1/\epsilon)$ with*

$$\mu(B(x, 1/\epsilon) \setminus K_{\mu,x}) \leq \epsilon \tag{98}$$

*such that*

$$\{(K_{\mu,x}, \mu|_{K_{\mu,x}}, x) : (\mu, x) \in \mathcal{S}\} \subset (\mathbb{M}_{pm}, d_{\mathrm{pmGH}})$$

*is pre-compact.*

*Proof* Let $(\mu_i, x_i)$ be a sequence in $\mathcal{S}$ and, for $i, j \in \mathbb{N}$ let $K_i^j \subset B(x_i, j)$ be as in (98) for $\epsilon = 1/j$. Let $\mathcal{K}(i, j) = (K_i^j, (\mu_i)|_{K_i^j}, x_i)$. First note that (84) implies

$$d_*(\mathcal{K}(i, j), (\mu_i, x_i)) \leq 1/j \quad \forall i, j \in \mathbb{N} \tag{99}$$

and hence

$$d_*(\mathcal{K}(i, j), \mathcal{K}(i, j')) \leq 2/j \quad \forall i \in \mathbb{N}, \ \forall j \leq j' \in \mathbb{N}. \tag{100}$$

By assumption, for each $j \in \mathbb{N}$ there exists a $\mathcal{K}_j \in \mathbb{M}_{pm}$ and a subsequence $i_k^j$ such that

$$d_{\mathrm{pmGH}}(\mathcal{K}(i_k^j, j), \mathcal{K}_j) \to 0 \text{ as } k \to \infty.$$

By taking a diagonal subsequence, we may suppose that

$$d_{\mathrm{pmGH}}(\mathcal{K}(i, j), \mathcal{K}_j) \to 0 \quad \forall j \in \mathbb{N}.$$

Equation (100) then implies

$$d_*(\mathcal{K}_j, \mathcal{K}_{j'}) \leq 2/j \quad \forall j \leq j' \in \mathbb{N}.$$

Since $(\mathbb{M}_*, d_*)$ is complete, $\mathcal{K}_j$ converges to some $(\mu, x)$. Equation (99) and the triangle inequality then imply $(\mu_i, x_i)$ converges to $(\mu, x)$.

Conversely, suppose that $\mathcal{S}$ is pre-compact but the conclusion does not hold. Note that, for any $r > 0$, the set $\{\mu(B(x, r)) : (\mu, x) \in \mathcal{S}\}$ must be bounded. Therefore, there exist $\delta, \epsilon > 0$ and, for each $i \in \mathbb{N}$, a $(\mu_i, x_i) \in \mathcal{S}$ such that, for any $K \subset \operatorname{spt} \mu_i \cap B(x_i, 1/\epsilon)$ with

$$\mu(B(x_i, 1/\epsilon) \setminus K) \leq \epsilon,$$

$K$ contains $i$ points separated by distance at least $\delta$. After passing to a subsequence, we may suppose that $(\mu_i, x_i)$ converges to some $(\mu, x)$. By Proposition 2.20, there exists a complete and separable pointed metric space $(Z, z)$ and isometric embeddings $(\operatorname{spt} \mu_i \cup \{x_i\}, x_i) \to (Z, z)$ with $\mu_i \to \mu$. Theorem 2.11 implies that there exists a compact $K \subset Z$ with $\mu_i(B(z, r) \setminus K) \leq \epsilon$ for all $i$, contradicting our initial assumption. □

*Remark 4.16* Theorem 4.15 implies [15, Proposition 7.1] and [26, Corollary 3.22]. The properties of $d_*$ and its relationship to $d_{\mathrm{pmGH}}$ allow for a simpler proof.

**Lemma 4.17** *For* $0 < \delta < 1$, $0 < \epsilon < 1/2$, *and* $k \geq 1$ *let* $R > 0$ *satisfy*

$$R > (4R(\epsilon/\delta)^{\frac{1}{k}})^{-1}. \tag{101}$$

*Suppose that* $(X, \mu, x)$ *is a* $2^k$*-doubling pointed metric measure space with* $\mu(B(x, R)) \geq \delta$ *and suppose that* $S \subset X$ *satisfies*

$$\mu(B(x, 2R) \setminus S) < \epsilon.$$

*Then*

$$d_{\mathrm{pmGH}}((X, \mu, x), (S, \mu|_S, x)) \leq 4R(\epsilon/\delta)^{\frac{1}{k}} + \epsilon. \tag{102}$$

*Proof* For $0 < r < R$ suppose that $w \in B(x, R) \setminus B(S, r)$. Then

$$\mu(B(w, r)) \leq \mu(B(x, R + r) \setminus S) < \epsilon$$

and so

$$\delta \leq \mu(B(x, R)) \leq \mu(B(w, 2R)) \leq (2^k)^m \mu(B(w, r)) < \epsilon 2^{km},$$

for $m = \lceil \log_2 2R/r \rceil$. Therefore

$$r < 4R(\epsilon/\delta)^{\frac{1}{k}}$$

and so

$$B(x, R) \subset B(S, 4R(\epsilon/\delta)^{\frac{1}{k}}).$$

Combining this with (101) gives

$$B(x, (4R(\epsilon/\delta)^{\frac{1}{k}})^{-1}) \subset B(S, 4R(\epsilon/\delta)^{\frac{1}{k}}),$$

so that

$$H_x(X, S) \leq 4R(\epsilon/\delta)^{\frac{1}{k}}.$$

Since $\mu(B(x, 2R) \setminus S) < \epsilon$, (102) follows from (18). □

We also note the following partial converse to the fact that $d_* \leq d_{\mathrm{pmGH}}$.

**Corollary 4.18** *For $0 < \delta < 1$ and $k \geq 1$ let $\mathcal{S}$ be the equivalence classes of those $2^k$-doubling pointed metric measure spaces $(X, \mu, x)$ with $\mu(B(x, 1)) \geq \delta$. Then, when restricted to $\mathcal{S}$,*

$$d_{\mathrm{pmGH}} \leq 9\delta^{-\frac{1}{k}} d_*^{\frac{1}{2k}}. \tag{103}$$

*Proof* For $0 < \epsilon < 1/2$ suppose that $(X, \mu, x), (Y, \nu, y) \in \mathcal{S}$ with

$$d_*((X, \mu, x), (Y, \nu, y)) < \epsilon.$$

By Theorem 4.11, there exist compact $K_\mu \subset B(x, \epsilon^{-1/2k})$ and $K_\nu \subset B(y, \epsilon^{-1/2k})$ satisfying

$$\max\{\mu(B(x, \epsilon^{-1/2k}) \setminus K_\mu),\ \nu(B(y, \epsilon^{-1/2k}) \setminus K_\nu)\} < \epsilon \tag{104}$$

and Eq. (94). Equation (104) and Lemma 4.17 (applied with $R = \epsilon^{-1/2k}$, which satisfies (101) for $\delta \leq 1$) imply that

$$d_{\mathrm{pmGH}}((X, \mu, x), (K_\mu, \mu|_{K_\mu}, x)) \text{ and } d_{\mathrm{pmGH}}((Y, \nu, y), (K_\nu, \nu|_{K_\nu}, y))$$

are at most $4\delta^{-\frac{1}{k}}\epsilon^{\frac{1}{2k}} + \epsilon$. Therefore (94) and the triangle inequality imply (103). □

## 5 Tangent spaces of metric measure spaces

With the notion of convergence defined in Sect. 4, we may define a tangent space following Preiss [5]. Given $(\mu, d, x) \in \mathbb{M}_*$ with $x \in \operatorname{spt} \mu$ and $r > 0$, let

$$T_r(\mu, d, x) := \left(\frac{\mu}{\mu(B(x, r))}, \frac{d}{r}, x\right)$$

**Definition 5.1** A $(\nu, \rho, y) \in \mathbb{M}_*$ is a *tangent measure* of $(\mu, d, x) \in \mathbb{M}_*$ if there exist $r_k \to 0$ such that

$$d_*(T_{r_k}(\mu, d, x), (\nu, \rho, y)) \to 0.$$

The set of all tangent measures of $(\mu, d, x)$ will be denoted by $\operatorname{Tan}(\mu, d, x)$.

*Remark 5.2* If $X = \ell_2^m$ for some $m \in \mathbb{N}$, any tangent measure is also supported on $\ell_2^m$. In this case, $\operatorname{Tan}(\mu, x)$ consists of the equivalence classes of the tangent measures of Preiss.

By Corollary 4.18, $\mathrm{Tan}(\mu, x)$ agrees with the usual definition of Gromov–Hausdorff tangent spaces for doubling metric measure spaces.

**Proposition 5.3** *Let $X$ be a metric space and let $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$ be asymptotically doubling. For $\mu$-a.e. $x \in \mathrm{spt}\,\mu$ and every $r_0 > 0$,*

$$\{T_r(\mu, x) : 0 < r < r_0\} \text{ is pre-compact.} \tag{105}$$

*In particular, for any $x \in \mathrm{spt}\,\mu$ satisfying (105), $\mathrm{Tan}(\mu, x)$ is a non-empty compact metric space when equipped with $d_*$ and*

$$\forall \delta > 0,\ \exists r_x > 0 \text{ s.t. } d_*(T_r(\mu, x), \mathrm{Tan}(\mu, x)) \le \delta\ \forall 0 < r < r_x. \tag{106}$$

*Proof* By Lemma 2.8, we know that we may cover $\mu$ almost all of $\mathrm{spt}\,\mu$ by

$$X_m := \{x \in \mathrm{spt}\,\mu : \mu(B(x, 2r)) \le m\mu(B(x, r))\ \forall 0 < r < 1/m\}$$

with $m \in \mathbb{N}$. Fix an $m \in \mathbb{N}$. By Theorem 2.10, $\mu$ almost every $x \in X_m$ is a Lebesgue density point of $X_m$. Fix such an $x$. It suffices to prove the conclusion for $x$. To do so, fix $R \ge 2$.

First note that, if $R/(2m) \le r < r_0$ then

$$\frac{\mu(B_{d/r}(x, R))}{\mu(B_d(x, r))} = \frac{\mu(B_d(x, Rr))}{\mu(B_d(x, r))} \le \frac{\mu(B_d(x, Rr_0))}{\mu(B_d(x, R/(2m)))} < \infty.$$

Here the subscripts on the balls indicate the metrics used to define the balls, for $d$ the original metric in $X$. Also, if $0 < r < R/(2m)$ then

$$\frac{\mu(B_{d/r}(x, R))}{\mu(B_d(x, r))} = \frac{\mu(B_d(x, Rr))}{\mu(B_d(x, r))} \le m^{4\log_2 R}.$$

Thus

$$\{\nu(B(x, R)) : (\nu, x) \in T_r(\mu, x),\ 0 < r < r_0\}$$

is bounded.

Secondly, for any $\epsilon > 0$, since $x$ is a Lebesgue density point of $X_m$, there exists $r_1 > 0$ such that

$$\frac{\mu(B_d(x, r) \cap X_m)}{\mu(B_d(x, r))} < \epsilon$$

for all $0 < r < r_1$. Let $r_2 = \min\{r_1, R/(2m)\}$. Also, let $K \subset B(x, Rr_0)$ satisfy

$$\mu(B_d(x, Rr_0) \setminus K) \le \epsilon.$$

Then for any $r_2 \leq r < R_0$,

$$\frac{\mu(B_{d/r}(x, R) \setminus K)}{\mu(B_d(x, r))} = \frac{\mu(B_d(x, Rr) \setminus K)}{\mu(B_d(x, r))} \leq \frac{\epsilon}{\mu(B_d(x, r_2))}.$$

Since $K$ is compact, the metric spaces $(K, d/r)$ with $r_2 \leq r < R_0$ are uniformly totally bounded. On the other hand, by Lemma 2.9, if $0 < r < r_2$ then

$$B_{d/r}(x, R) \cap X_m = B_d(x, Rr) \cap X_m$$

is covered by $m^{-4\log_2 \epsilon}$ balls of the form $B_{d/r}(w_i, \epsilon)$. Thus

$$\{B_{d/r}(x, R) : 0 < r < r_0\}$$

is uniformly totally bounded. Therefore, Theorem 4.15 proves (105).

Now let $x \in \operatorname{spt} \mu$ satisfy (105). By applying (105) to an arbitrary sequence $r_i \to 0$ we see that $\operatorname{Tan}(\mu, x)$ is non-empty. To see that it is compact, for each $j \in \mathbb{N}$ let $(\nu_j, y_j) \in \operatorname{Tan}(\mu, x)$ and let $0 < r_j < 1/j$ be such that

$$d_*(T_{r_j}(\mu, x), (\nu_j, y_j)) < 1/j.$$

By (105) there exists a subsequence $r_{j_k} \to 0$ and a $(\nu, y) \in \mathbb{M}_*$ such that

$$d_*(T_{r_{j_k}}(\mu, x), (\nu, y)) \to 0.$$

In particular, $(\nu, y) \in \operatorname{Tan}(\mu, x)$ and, by the triangle inequality, $(\nu_{j_k}, y_{j_k}) \to (\nu, y)$, as required. Finally, given $\delta > 0$, the existence of such an $r_0$ is given by the contrapositive to (105). □

**Lemma 5.4** *Let $(X, d, \mu)$ be a metric measure space and suppose that $\mathcal{S} \subset \mathbb{M}_*$. Then for any $R, \delta > 0$,*

$$C_{R,\delta}(\mathcal{S}) := \{x \in \operatorname{spt} \mu : d_*((T_r(\mu, d, x), \mathcal{S}) \leq \delta \ \forall 0 < r < R\}$$

*is a closed subset of $X$. In particular, if $S \subset X$ is a Borel set of points satisfying (105) and $\mathcal{C} \subset \mathbb{M}_*$ is closed,*

$$\mathcal{G}(\mathcal{C}) := \{x \in S : \operatorname{Tan}(\mu, d, x) \subset \mathcal{C}\}$$

*is Borel.*

*Proof* Let $x \in X$ and $x_j \in C_{R,\delta}(\mathcal{S})$ with $x_j \to x$. First note that, since $\mu \in \mathcal{M}_{\mathrm{loc}}(X)$, all but countably many $r > 0$ satisfy

$$\mu(B(x,r) \setminus U(x,r)) = 0.$$

For such an $r > 0$, $T_r(\mu, d, x_j) \to T_r(\mu, d, x)$. Indeed, Eqs. (14) and (15) imply that $\mu(B(x_j, r)) \to \mu(B(x,r))$ and consequently Lemma 2.22 implies

$$d_{\mathrm{pmGH}}\left(\left(\frac{\mu}{\mu(B(x_j,r))}, \frac{d}{r}, x_j\right), \left(\frac{\mu}{\mu(B(x,r))}, \frac{d}{r}, x\right)\right) \to 0$$

Therefore

$$d_*(T_r(\mu, x), \mathcal{S}) \leq \delta. \tag{107}$$

For arbitrary $0 < r < R$, let $r_j \downarrow r$ satisfy (107). Then $\mu(B(x, r_j)) \to \mu(B(x,r))$ and so

$$d_{\mathrm{pmGH}}\left(\left(\frac{\mu}{\mu(B(x,r_j))}, \frac{d}{r_j}, x\right), \left(\frac{\mu}{\mu(B(x,r))}, \frac{d}{r}, x\right)\right) \to 0$$

(for any $R > 0$, the identity is a $(r_j - r)R$-isometry on $B(x, R)$). Consequently (107) holds in this case too and hence $C_{R,\delta}(\mathcal{S})$ is closed.

If $x$ satisfies (105) and $\mathrm{Tan}(\mu, d, x) \subset \mathcal{C}$ then by (106),

$$\forall \delta > 0,\ \exists r_x > 0 \text{ s.t. } d_*(T_r(\mu, x), \mathcal{C}) \leq \delta\ \forall 0 < r < r_x. \tag{108}$$

Conversely, if $\mathcal{S}$ is closed, (108) implies $\mathrm{Tan}(\mu, x) \subset \mathcal{C}$. Thus,

$$\mathcal{G}(\mathcal{C}) = S \cap \bigcap_{\delta \in \mathbb{Q}^+} \bigcup_{R \in \mathbb{Q}^+} C_{R,\delta}(\mathcal{C}),$$

a Borel set. □

We now prove three results that correspond results in Preiss [5]. The following statement, and its proof, is essentially that of [5, Theorem 2.12] reformulated in our setting. See also [28,29] for adaptations to Gromov–Hausdorff tangent spaces and the appendix of [11] for an adaptation to tangents of sets of finite perimeter in RCD spaces.

**Proposition 5.5** *Let $(X, d, \mu)$ be an asymptotically doubling metric measure space. Then $\mu$-a.e. $x \in \mathrm{spt}\,\mu$ has the following property:*

$$\forall (\nu, y) \in \mathrm{Tan}(\mu, x) \text{ and } \forall z \in \mathrm{spt}\,\nu,\ (\nu, z) \in \mathrm{Tan}(\mu, x). \tag{109}$$

*Proof* For each $p \in \mathbb{N}$ let $A_p$ be the set of $x \in \operatorname{spt}\mu$ for which there are $(\nu_x, y_x) \in \operatorname{Tan}(\mu, x)$ and $b_x \in \operatorname{spt}\nu_x$ such that

$$d_*((\nu_x, b_x), T_r(\mu, x)) > \frac{1}{p} \quad \forall 0 < r < 1/p.$$

Note that, if $x \notin \cup_{p\in\mathbb{N}} A_p$, then $x$ satisfies the required conclusion.

Suppose that, for some $p \in \mathbb{N}$, $\mu(A_p) > 0$. Then by the separability of $(\mathbb{M}_*, d_*)$, there exists a $E \subset A_p$ with $\mu(E) > 0$ such that

$$d_*((\nu_x, b_x), (\nu_y, b_y)) < 1/2p$$

for every $x, y \in E$. By Theorem 2.10 there exists a Lebesgue density point $x$ of $E$. Let $r_k \to 0$ be such that $T_{r_k}(\mu, x) \to (\nu_x, y_x)$. By Proposition 2.20 there exists a complete and separable pointed metric space $(Z, \zeta, z)$ and isometric embeddings $(\operatorname{spt}\mu, d/r_k, x), (\operatorname{spt}\nu, y_x) \to (Z, \zeta, z)$ with

$$\frac{\mu}{\mu(B(x, r_k))} \to \nu_x.$$

Let $\mu_k$ be the isometric copy of $\mu \in \mathcal{M}_{\mathrm{loc}}(X, d/r_k)$, let $E_k$ be the isometric copy of $(E, d/r_k)$ and let $c_k \in E_k$ be such that

$$\zeta(b_x, c_k) < \zeta(b_x, E_k) + 1/k.$$

Note that

$$\zeta(b_x, E_k) \to 0. \tag{110}$$

Indeed, since $x$ is a density point of $E$, if $R = 2\zeta(y_x, b_x)$ then

$$\lim_{k\to\infty} \frac{\mu_k(E_k \cap B(x, R))}{\mu_k(B(x, R))} = \lim_{k\to\infty} \frac{\mu(E \cap B(x, Rr_k))}{\mu(B(x, Rr_k))} = 1. \tag{111}$$

On the other hand, suppose that there exists $\delta > 0$ such that $B(b_x, \delta) \cap E_k = \emptyset$ for infinitely many $k$. Equations (14) and (15) imply

$$\begin{aligned}\liminf_{k\to\infty} \frac{\mu_k(E_k \cap B(x, R))}{\mu_k(B(x, R))} &\le 1 - \limsup_{k\to\infty} \frac{\mu_k(B(b_x, \delta))}{\mu_k(B(x, R))} \\ &\le 1 - \frac{\nu_x(B(b_x, \delta/2))}{\nu_x(B(y_x, 2R))} < 1,\end{aligned}$$

since $b_x \in \operatorname{spt}\nu_x$. This contradicts (111).

Now, (110) implies $\zeta(b_x, c_k) \to 0$. This and Lemma 2.22 imply

$$d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, c_k\right), (\nu_x, b_x)\right) \to 0.$$

Therefore, there exists $k$ with $r_k < 1/p$ such that

$$d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, c_k\right), (\nu_x, b_x)\right) < 1/2p.$$

Thus, since each $c_k \in E_k$,

$$\begin{aligned} 1/p &< d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, c_k\right), (\nu_{c_k}, b_{c_k})\right) \\ &\le d_*\left(\left(\frac{\mu}{\mu(B(x,r_k))}, \frac{d}{r_k}, c_k\right), (\nu_x, b_x)\right) + d_*((\nu_x, b_x), (\nu_{c_k}, b_{c_k})) \\ &\le 1/2p + 1/2p, \end{aligned}$$

a contradiction. Therefore we must have $\mu(A_p) = 0$ for all $p \in \mathbb{N}$. □

**Lemma 5.6** *Let* $(\mu, x) \in \mathbb{M}_*$ *with* $x \in \operatorname{spt}\mu$ *and*

$$M_x := \limsup_{r\to 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))}. \tag{112}$$

*For every* $(\nu, y) \in \operatorname{Tan}(\mu, x)$ *and* $r > 0$,

$$\nu(B(y,2r)) \le M_x \nu(B(y,r))$$

*and*

$$1 \le \nu(B(y,1)) \le M_x.$$

*In particular, if* $x$ *satisfies* (109), *every* $(\nu, y) \in \operatorname{Tan}(\mu, x)$ *is* $M_x$*-doubling.*

*Proof* By Proposition 2.20 and Eqs. (14) and (15), for any $(\nu, y) \in \operatorname{Tan}(\mu, x)$ there exist $r_i \to 0$ such that, for any $R > 0$ and $\lambda > 1$,

$$\nu(B(y,2R)) \le \liminf_{i\to\infty} \frac{\mu(B(x,\lambda 2Rr_i))}{\mu(B(x,r_i))}$$

and

$$\nu(B(y,\lambda R)) \ge \limsup_{i\to\infty} \frac{\mu(B(x,\lambda Rr_i)}{\mu(B(x,r_i))}.$$

Consequently,

$$\frac{\nu(B(y,2R))}{\nu(B(y,\lambda R))} \leq M_x.$$

Since this is true for all $\lambda > 1$ we also have

$$\frac{\nu(B(y,2R))}{\nu(B(y,R))} \leq M_x. \tag{113}$$

Also note that, by (15),

$$\nu(B(y,1)) \geq \limsup_{i\to\infty} \frac{\mu(B(x,r_i))}{\mu(B(x,r_i))} = 1,$$

and by (14),

$$\nu(B(y,1)) \leq \nu(U(x,2)) \leq \liminf_{i\to\infty} \frac{\mu(U(x,2))}{\mu(B(x,1))} \leq \liminf_{i\to\infty} \frac{\mu(B(x,2))}{\mu(B(x,1))} \leq M_x.$$

□

**Corollary 5.7** *Let $(\mu, d, x) \in \mathbb{M}_*$ and let $S \subset \operatorname{spt}\mu$ be $\mu$-measurable with $x$ a density point of $S$. Then*

$$\operatorname{Tan}(\mu|_S, x) = \operatorname{Tan}(\mu, x).$$

*Proof* Combine (84) and the definition of Tan. □

## 6 Proof of the main theorems

We first demonstrate how tangent measures supported on elements of $\operatorname{biLip}(K)$ lead to a decomposition of the space into sets with good tangential approximation.

Define the projection $\pi_1 : \mathbb{M}_{pm} \to \mathbb{M}_p$ by

$$(X, \mu, x) \mapsto (X, x)$$

and $\pi_2 : \mathbb{M}_{pm} \to \mathbb{M}_*$ by

$$(X, \mu, x) \mapsto (\mu, x).$$

Both of $\pi_1, \pi_2$ are 1-Lipschitz.

For $M \geq 1$ let

$$S_1(M) = \{(X, \mu, x) \in \mathbb{M}_{pm} : M \geq \mu(U(x, 1)),\ \mu(B(x, 1)) \geq 1/M\}$$

and $S_2(M)$ be the set of those $(X, \mu, x) \in \mathbb{M}_{pm}$ with

$$\mu(U(y, r)) \leq M\mu(B(y, 2r)) \quad \forall y \in X \text{ and } r > 0. \tag{114}$$

For $K \geq 1$ let

$$\mathcal{K}(K, M) = \pi_1^{-1}(\mathrm{biLip}(K)) \cap S_1(M) \cap S_2(M).$$

**Lemma 6.1** *For any $K, M \geq 1$,*

1. $\mathrm{biLip}(K)$ *is a compact subset of* $(\mathbb{M}_p, d_{\mathrm{pGH}})$
2. $S_1(M)$ *and* $S_2(M)$ *are closed subsets of* $(\mathbb{M}_{pm}, d_{\mathrm{pmGH}})$
3. $\mathcal{K}(K, M)$ *is a compact subset of* $(\mathbb{M}_{pm}, d_{\mathrm{pmGH}})$ *and hence a compact subset of* $(\mathbb{M}_*, d_*)$.

*Proof* First note that the elements of $\mathrm{biLip}(K)$ are proper and so do belong to $\mathbb{M}_p$. Moreover, for any $r > 0$, the set

$$\{B(x, r) : (X, x) \in \mathrm{biLip}(K)\}$$

is uniformly totally bounded and so, by Theorem 2.27, $\mathrm{biLip}(K)$ is pre-compact.

To see that $\mathrm{biLip}(K)$ is closed, for each $i \in \mathbb{N}$ let $(X_i, x_i) \in \mathrm{biLip}(K)$ and $(X, x) \in \mathbb{M}_p$ with

$$d_{\mathrm{pGH}}((X_i, x_i), (X, x)) < 1/i.$$

For each $i \in \mathbb{N}$ let $f_i : (X_i, x_i) \to (X, x)$ be a $2/i$-isometry given by Lemma 2.24. Also let $\psi_i : \ell_\infty^n \to X_i$ be surjective and $K$-bi-Lipschitz and define

$$\phi_i = f_i \circ \psi_i : \ell_\infty^n \to X.$$

Fix $R > 0$. For any $y, z \in B(0, R) \subset \ell_\infty^n$,

$$\begin{aligned} d(\phi_i(y), \phi_i(z)) &= d(f_i(\psi_i(y)), f_i(\psi_i(z))) \\ &\leq d(\psi_i(y), \psi_i(z)) + 2/i \\ &\leq K\|y - z\|_\infty + 2/i. \end{aligned} \tag{115}$$

Thus the $\phi_i$ are equicontinuous on $B(0, R)$. Since the $\phi_i$ map into $B(x, KR+2)$, a compact set, the Arzelà–Ascoli theorem gives a $\phi\colon B(0, R) \to X$ such that, after passing to a subsequence, $\phi_i \to \phi$ uniformly on $B(0, R)$. Since this is true for all $R > 0$, by taking a diagonal subsequence, we see that there exists a $\phi\colon \ell^n_\infty \to X$ such that $\phi_i \to \phi$ uniformly on bounded sets. A similar estimate to (115) shows that $\phi$ is $K$-bi-Lipschitz. Since each $\phi_i$ is surjective and each $f_i$ is a $2/i$-isometry, $\phi$ is surjective. Thus $\mathrm{biLip}(K)$ is closed, proving item 1.

Item 2 follows from Proposition 2.20 and Eqs. (14) and (15).

Item 3 follows since $\mathcal{K}(K, M)$ is pre-compact and closed in $(\mathbb{M}_{pm}, d_{\mathrm{pmGH}})$ by Corollary 4.7 and items 1 and 2. □

For $K \geq 1$ let

$$\mathrm{biLip}(K)^* = \{(\mu, x) \in \mathbb{M}_* : \mathrm{spt}\,\mu \in \mathrm{biLip}(K)\}.$$

**Proposition 6.2** *Let $X$ be a complete metric space and $S \subset X$ a $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(S) < \infty$. Suppose that, for $\mathcal{H}^n$-a.e. $x \in S$, $\Theta^n_*(S, x) > 0$ and there exists a $K_x \geq 1$ such that*

$$\mathrm{Tan}(\mathcal{H}^n|_S, x) \subset \mathrm{biLip}(K_x)^*.$$

*There exist compact $C_i \subset S$ with*

$$\mathcal{H}^n\left(S \setminus \bigcup_{i\in\mathbb{N}} C_i\right) = 0$$

*and, for each $i \in \mathbb{N}$, an $\eta_i > 0$ such that, for each $x \in C_i$,*

$$\eta_i r^n \leq \mathcal{H}^n(B(x, r) \cap S) < 2(2r)^n \ \forall 0 < r < \eta_i \tag{116}$$

*and*

$$\mathrm{Tan}(\mathcal{H}^n|_S, x) \subset \mathcal{K}(\eta_i^{-1}, \eta_i^{-1}). \tag{117}$$

*Further, for each $i \in \mathbb{N}$ and any $\epsilon > 0$, there exists countably many closed $G_j \subset C_i$ with*

$$\mathcal{H}^n\left(C_i \setminus \bigcup_{j\in\mathbb{N}} G_j\right) = 0,$$

*such that each* $(C_i, G_j)$ *satisfies* $GTA(\eta_i, \eta_i^{-1}, \epsilon, \min\{\eta_i, 1/j\})$.

*Proof* For any $i \in \mathbb{N}$, Lemma 2.8 implies that

$$S_i := \{x \in S : r^n/i \leq \mathcal{H}^n(B(x,r) \cap S) < 2(2r)^n \ \forall 0 < r < 1/i\}$$

is $\mathcal{H}^n$-measurable. By assumption, the $S_i$ monotonically increase to almost all of $S$ as $i \to \infty$.

Fix $i \in \mathbb{N}$. For $j \in \mathbb{N}$, let

$$S_i^j := \{x \in S_i : \mathrm{Tan}(\mathcal{H}^n|_S, x) \subset \mathcal{K}(j, j)\}.$$

If $x \in S_i$ satisfies (109), Lemma 5.6 implies that every $(\nu, y) \in \mathrm{Tan}(\mathcal{H}^n|_S, x)$ is $4^{n+1}i$-doubling with $1 \leq \nu(B(y,1)) \leq 4^{n+1}i$. Therefore, if $\max\{K_x, 4^{n+1}i\} \leq j$ then $x \in S_i^j$ and so $\mathcal{H}^n(S \setminus \cup_{i,j} S_i^j) = 0$. Lemma 6.1 implies that $\mathcal{K}(j, j)$ is compact and so Lemma 5.4 implies each $S_i^j$ is $\mathcal{H}^s$-measurable. Consequently, each can be decomposed as a countable union of compact sets (up to a set of measure zero). Re-indexing completes the proof of the first decomposition.

Now fix $i \in \mathbb{N}$ and let $C = C_i$ and $\eta = \eta_i$. Since $\mathcal{H}^n(C) < \infty$, Lemma 2.3 implies $\Theta^{*,n}(C, x) \leq 1$ for $\mathcal{H}^n$-a.e. $x \in C$ and since $\mathcal{H}^n(S \setminus C) < \infty$, $\Theta^n_*(C, x) \geq \eta$ for $\mathcal{H}^n$-a.e. $x \in C$. Thus $\mathcal{H}^n|_C$ is asymptotically doubling. Therefore, by Corollary 5.7, Theorem 2.10 and Eq. (117),

$$\mathrm{Tan}(\mathcal{H}^n|_C, x) \subset \mathcal{K}(\eta^{-1}, \eta^{-1}) \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in C. \tag{118}$$

For any $\epsilon > 0$ the sets

$$C_{j,\epsilon} := C_{1/j,\epsilon}(\mathcal{K}(\eta^{-1}, \eta^{-1}))$$

defined in Lemma 5.4 are closed and $\cup_j C_{j,\epsilon}$ contains $C$. For any $j \in \mathbb{N}$, any $x \in C_{j,\epsilon}$ and any $0 < r < 1/j$, let $(\nu, y) \in \mathcal{K}(\eta^{-1}, \eta^{-1})$ with

$$d_*(T_r(\mathcal{H}^n|_C, x), (\nu, y)) < 2\epsilon.$$

Apply Theorem 4.11 to obtain a $K \subset C \cap B(x, r/2\epsilon)$ and $K_\nu \subset B(y, 1/2\epsilon)$ with

$$\max\left\{\frac{\mathcal{H}^n(C \cap B(x, r/2\epsilon) \setminus K)}{\mathcal{H}^n(C \cap B(x,r))}, \nu(B(y, 1/2\epsilon) \setminus K_\nu)\right\} < 2\epsilon \tag{119}$$

and

$$d_{\mathrm{pmGH}}(T_r(K, \mathcal{H}^n|_K, x), (K_\nu, \nu|_{K_\nu}, y)) < 6\epsilon. \tag{120}$$

Note that (114) implies $(Y, \nu)$ is $2\eta^{-1}$-doubling. Equation (119) allows us to apply Lemma 4.17 with $k = \log_2 2\eta^{-1}$, $R = \epsilon^{-\frac{1}{2k}}$ and $\delta = 1$ to obtain

$$d_{\mathrm{pmGH}}((K_\nu, \nu|_{K_\nu}, y), (\operatorname{spt}\nu, \nu, y)) \leq 9\epsilon^{\frac{1}{2k}}.$$

The triangle inequality and (120) then give

$$d_{\mathrm{pmGH}}(T_r(K, \mathcal{H}^n|_K, x), (\operatorname{spt}\nu, \nu, y)) \leq 18\epsilon^{\frac{1}{2k}}.$$

In particular,

$$d_{\mathrm{pGH}}((K, d/r, x), \mathrm{biLip}(\eta^{-1})) \leq 18\epsilon^{\frac{1}{2k}}.$$

Also, Eqs. (116) and (119) imply

$$\mathcal{H}^n(B(x,r) \setminus K) < 2\epsilon\mathcal{H}^n(B(x,r)) \leq 4 \cdot 4^n \epsilon r^n = \eta 4(4(\epsilon/\eta)^{1/n} r)^n.$$

That is, $C_{j,\epsilon}$ satisfies $GTA(\eta, \eta^{-1}, \epsilon^*, \min\{\eta, 1/j\})$, for some $\epsilon^*$ depending upon $\eta$ such that $\epsilon^* \to 0$ as $\epsilon \to 0$. Since $\epsilon > 0$ was arbitrary, setting $G_j = C_{1/j,\epsilon'}$ for each $j \in \mathbb{N}$ and appropriate $\epsilon'$ completes the proof. □

To prove that (3) implies (1) in Theorem 1.2 we require the following result [13, Theorem 1.1]. It concerns the set of all 1-Lipschitz functions $f : X \to \ell_2^n$, equipped with the supremum norm.

**Theorem 6.3** *Let $X$ be a complete metric space and $m \in \mathbb{N}$. Suppose that $S \subset X$ is purely $n$-unrectifiable with $\mathcal{H}^n(S) < \infty$ and satisfies $\Theta_*^n(S, x) > 0$ for $\mathcal{H}^n$-a.e. $x \in S$. The set of all 1-Lipschitz $f : X \to \ell_2^m$ with $\mathcal{H}^n(f(S)) = 0$ is residual and hence dense.*

**Corollary 6.4** *Let $X$ be a complete metric space and let $S \subset X$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(S) < \infty$ and $\Theta_*^n(S, x) > 0$ for $\mathcal{H}^n$-a.e. $x \in S$. For $L \geq 1$ and $\delta > 0$ suppose that*

1. *$\iota : [0, r]^n \to X$ is continuous with*

$$\mathcal{H}_\infty^n(\iota([0,r]^n) \setminus S) < \mathcal{H}^n([\delta r, (1-\delta)r]^n)/L.$$

2. *$f : \iota([0, r]^n) \to [0, r]^n \subset \ell_2^n$ is $L$-Lipschitz with*

$$\|f(\iota(x)) - x\|_2 < \delta r \quad \forall x \in \partial[0, r]^n.$$

*Then $S$ is not purely $n$-unrectifiable.*

*Proof* Suppose that $S$ is purely $n$-unrectifiable and let $\epsilon > 0$ be such that

$$\mathcal{H}^n_\infty(\iota([0,r]^n) \setminus S) < \mathcal{H}^n([(\delta+\epsilon)r, (1-(\delta+\epsilon))r]^n)/L.$$

Let $g\colon \iota([0,r]^n) \to \ell_2^n$ be given by Theorem 6.3 such that

- $g$ is $L$-Lipschitz;
- $\|f - g\|_\infty < \epsilon r$;
- $\mathcal{H}^n(g(S)) = 0$.

In particular,

$$\|g(\iota(x)) - x\| < (\delta+\epsilon)r \quad \forall\, x \in \partial[0,r]^n \tag{121}$$

and, since $\mathcal{H}^n_\infty = \mathcal{H}^n$ in $\ell_2^n$,

$$\begin{aligned}\mathcal{H}^n(g(\iota([0,r]^n))) &\le \mathcal{H}^n(g(\iota([0,r]^n) \setminus S)) + \mathcal{H}^n(g(S))\\ &\le L\mathcal{H}^n_\infty(\iota([0,r]^n) \setminus S) + 0\\ &< \mathcal{H}^n([(\delta+\epsilon)r, (1-(\delta+\epsilon))r]^n).\end{aligned} \tag{122}$$

Note that (121) implies that

$$g(\iota([0,r]^n)) \supset [(\delta+\epsilon)r, (1-(\delta+\epsilon))r]^n$$

(this is a consequence of Brouwer's fixed point theorem; See [13, Lemma 7.1] for a proof involving the unit ball). Consequently,

$$\mathcal{H}^n(g(\iota([0,r]^n))) \ge \mathcal{H}^n([(\delta+\epsilon)r, (1-(\delta+\epsilon))r]^n),$$

contradicting (122) □

Recall the definition of $\mathrm{biLip}(K)^*$ preceding Proposition 6.2.

**Theorem 6.5** *Let $(X,d)$ be a complete metric space, $n \in \mathbb{N}$ and $E \subset X$ a $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(E) < \infty$. Suppose that, for $\mathcal{H}^n$-a.e. $x \in E$, $\Theta^n_*(E,x) > 0$ and there exists a $K_x \ge 1$ such that*

$$\mathrm{Tan}(X, d, \mathcal{H}^n|_E, x) \subset \mathrm{biLip}(K_x)^*. \tag{123}$$

*Then $E$ is $n$-rectifiable.*

*Proof* By the Kuratowski embedding, we may identify $X$ with an isometric copy contained in $\ell_\infty$.

Suppose that $E$ is not $n$-rectifiable. Then by Lemma 2.2 there exists a compact purely $n$-unrectifiable $S \subset E$ with $\mathcal{H}^n(S) > 0$. By Lemma 2.3,

$\Theta^n_*(S, x) > 0$ for $\mathcal{H}^n$-a.e. $x \in S$. In particular, $\mathcal{H}^n|_S$ is asymptotically doubling and so by Corollary 5.7,

$$\mathrm{Tan}(\mathcal{H}^n|_S, x) \subset \mathrm{biLip}(K_x)^*$$

for $\mathcal{H}^n$-a.e. $x \in S$. Then $S$ satisfies the hypotheses of Proposition 6.2. Since $\mathcal{H}^n(S) > 0$, Proposition 6.2 implies the existence of a compact $C \subset S$ with $\mathcal{H}^n(C) > 0$ and a $0 < \eta < 1$ such that, for any $\epsilon > 0$, there exist countably many compact $G_j \subset C$ with $\mathcal{H}^n(C \setminus G_j) = 0$ such that each $(C, G_j)$ satisfies $GTA(\eta, \eta^{-1}, \epsilon, \min\{\eta, 1/j\})$.

Fix $0 < \gamma < 1$ and $0 < \delta < \eta/4\sqrt{n}$ and let $m_0 \in \mathbb{N}$ be given by Theorem 3.4 for these parameters and $K = \eta^{-1}$. Let $m \geq m_0$ be such that

$$20\sqrt{n}\eta^{-2}2^{-m\gamma/2} + \sqrt{n}2^{-m} < \delta \tag{124}$$

and set $\epsilon = 4^{-m}$. Since $\mathcal{H}^n(C) > 0$, there exists $j \in \mathbb{N}$ for which $G_j$ obtained in the previous paragraph satisfies $\mathcal{H}^n(G_j) > 0$. Set $G = G_j$ and $R_0 = \min\{\eta, 1/j\}$, so that $(C, G)$ satisfies $GTA(\eta, \eta^{-1}, 4^{-m}, R_0)$. Let $\Lambda > 0$ be obtained from Theorem 3.16. Fix $x$ a density point of $G$ with $\Theta^{*,n}(X, x) \leq 1$ and let $0 < r < R_0$ be such that

$$\Lambda\mathcal{H}^n(B(x, 20\eta^{-1}r) \cap X \setminus G) < \delta r^n. \tag{125}$$

Then Theorem 3.4 gives an $\iota\colon [0, r]^n \subset \ell^n_\infty \to B(x, 20\eta^{-1}r) \subset \ell_\infty$ satisfying Theorem 3.4 items 1 to 3.

Since $\iota|^{-1}_{\mathcal{D}(r,m)}$ is $2\eta^{-1}$-Lipschitz and takes values in $[0, r]^n$, an application of the second conclusion of Theorem 2.6 extends it to a $\sqrt{n}2\eta^{-1}$-Lipschitz map $f\colon \iota([0, r]^n) \to [0, r]^n \subset \ell^n_2$. By Theorem 3.4 item 2, for any $p, q \in [0, r]^n$ with $\|p - q\|_\infty \leq 2^m r$ we have

$$\begin{aligned}\|f(\iota(p)) - f(\iota(q))\|_2 &\leq \sqrt{n}2\eta^{-1}\|\iota(p) - \iota(q)\|_\infty \\ &\leq 20\sqrt{n}\eta^{-2}2^{-m\gamma/2}r.\end{aligned}$$

In particular, for any $p \in [0, r]^n$, pick $q \in \mathcal{D}(r, m)$ with $\|p - q\|_\infty \leq 2^{-m}r$. Then

$$\begin{aligned}\|f(\iota(p)) - p\|_2 &\leq \|f(\iota(p)) - f(\iota(q))\|_2 + \|f(\iota(q)) - p\|_2 \\ &\leq 20\sqrt{n}\eta^{-2}2^{-m\gamma/2}r + \|q - p\|_2 \\ &\leq 20\sqrt{n}\eta^{-2}2^{-m\gamma/2}r + \sqrt{n}2^{-m}r \\ &< \delta r\end{aligned}$$

by (124). Thus, $f$ satisfies Corollary 6.4 item 2.

Combining Theorem 3.4 item 3 and (125) implies that $f$ satisfies Corollary 6.4 item 1. An application of Corollary 6.4 shows that $S$ is not purely $n$-unrectifiable, a contradiction. □

Finally we prove that (1) implies (2) in Theorem 1.2.

**Theorem 6.6** *Let $(X, d)$ be a complete metric space, $n \in \mathbb{N}$ and $E \subset X$ an $n$-rectifiable set with $\mathcal{H}^n(E) < \infty$. For any $x \in E$ that satisfies the conclusion of Theorem 2.4 and Lemma 2.3 item 1,*

$$T_r(X, d, \mathcal{H}^n|_E, x) \to (\mathbb{R}^n, \|\cdot\|_x, \mathcal{H}^n/2^n, 0) \tag{126}$$

*as $r \to 0$. In particular, for $\mathcal{H}^n$-a.e. $x \in E$,*

$$\mathrm{Tan}(X, d, \mathcal{H}^n|_E, x) = \{(\mathbb{R}^n, \|\cdot\|_x, \mathcal{H}^n/2^n, 0)\}.$$

*Proof* Let $x \in E$ be as in the hypotheses and $f_x, \|\cdot\|_x, C_x$ as in the conclusion of Theorem 2.4. For each $r > 0$, let

$$\epsilon_r = \sup\left\{\left|1 - \frac{\|f_x(y) - f_x(z)\|_x}{d(y, z)}\right| : y \neq z \in C_x \cap B(x, \sqrt{r})\right\},$$

so that $\epsilon_r \to 0$ as $r \to 0$. Also let $f_r = f_x/r$ and

$$C_r = C_x \cap B(x, \sqrt{r}).$$

First note that, for any $y, z \in C_r$ with $d(y, z)/r \leq \min\{r^{-1/2}, \epsilon_r^{-1/2}\}$,

$$\begin{aligned}\left|\frac{d(y, z)}{r} - \|f_r(y) - f_r(z)\|_x\right| &= \frac{|d(y, z) - \|f_x(y) - f_x(z)\|_x|}{r} \\ &\leq \frac{\epsilon_r d(y, z)}{r} \\ &\leq \sqrt{\epsilon_r}. \end{aligned} \tag{127}$$

Now observe that $f_r\colon (C_r, d/r) \to (\mathbb{R}^n, \|\cdot\|_x)$ is $L_r$-bi-Lipschitz, for

$$L_r = \frac{1 + \epsilon_r}{1 - \epsilon_r^2} \to 1.$$

Therefore

$$\frac{1}{L_r^n}\mathcal{H}_x^n|_{f_r(C_r)} \leq (f_r)_\#(\mathcal{H}_r^n|_{C_r}) \leq L_r^n \mathcal{H}_x^n|_{f_r(C_r)}, \tag{128}$$

where $\mathcal{H}^n_r$ denotes $\mathcal{H}^n$ on the metric space $(C_r, d/r)$ and $\mathcal{H}^n_x$ denotes $\mathcal{H}^n$ on $(\mathbb{R}^n, \|\cdot\|_x)$. However,

$$(f_r)_\#(\mathcal{H}^n_r|_{C_r}) = (f_r)_\#\left(\frac{\mathcal{H}^n|_{C_r}}{r^n}\right) = \frac{(f_r)_\#(\mathcal{H}^n_d|_{C_r})}{r^n},$$

where $\mathcal{H}^n_d$ denotes $\mathcal{H}^n$ on $(X, d)$. Also observe, for any $R > 0$, by lemma 2.5 and Eq. (9).

$$\begin{aligned}\mathcal{H}^n_x(B(0,R) \setminus f_r(C_r)) &\leq \mathcal{H}^n_x(B(0,R) \setminus f_r(C_r \cap B(x, Rr/L_r))) \\ &\leq (2R)^n - \mathcal{H}^n_x(f_r(C_r \cap B(x, Rr/L_r))) \\ &\leq (2R)^n - \frac{L^n_r}{r^n}\mathcal{H}^n_d(C_r \cap B(x, Rr/L_r)) \\ &\to (2R)^n - (2R)^n = 0 \end{aligned} \tag{129}$$

Let $\|\cdot\|_{\mathcal{M}}$ denote the total variation of elements of $\mathcal{M}((B(0,R), \|\cdot\|_x))$. Then Eqs. (128) and (129) imply

$$\begin{aligned}\left\|\frac{(f_r)_\#(\mathcal{H}^n_d|_{C_r})}{r^n} - \mathcal{H}^n_x\right\|_{\mathcal{M}} &\leq \left\|\frac{(f_r)_\#(\mathcal{H}^n_d|_{C_r})}{r^n} - \mathcal{H}^n_x|_{f_r(C_r)}\right\|_{\mathcal{M}} \\ &\quad + \|\mathcal{H}^n_x|_{f_r(C_r)} - \mathcal{H}^n_x\|_{\mathcal{M}} \\ &\leq (L^n_r - 1)\mathcal{H}^n_x(B(0,R)) \\ &\quad + \mathcal{H}^n_x(B(0,R) \setminus f_r(C_r)) \to 0. \end{aligned}$$

That is,

$$\frac{(f_r)_\#(\mathcal{H}^n_d|_{C_r})}{(2r)^n} \to \frac{\mathcal{H}^n_x}{2^n} \quad \text{strongly on } B(0,R). \tag{130}$$

Equation (130) has two consequences. Firstly, it shows that, for any $R, \delta > 0$,

$$B(f_r(C_r), \delta) \supset B(0,R)$$

for sufficiently small $r > 0$. Together with (127), this shows that there exist $\epsilon'_r \to 0$ such that

$$f_r : (C_r, d/r) \to (\mathbb{R}^n, \|\cdot\|_x) \quad \text{is an } \epsilon'_r\text{-isometry}. \tag{131}$$

Equation (130) also implies that

$$\frac{(f_r)_\#(\mathcal{H}^n|_{C_r})}{r^n} \to \mathcal{H}^n_x \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^n, \|\cdot\|_x). \tag{132}$$

Corollary 4.4 and Eqs. (131) and (132) then imply

$$d_{\mathrm{pmGH}}(T_r(C_r, d, \mathcal{H}^n|_{C_r}, x), (\mathbb{R}^n, \|\cdot\|_x, \mathcal{H}^n/2^n, 0)) \to 0$$

and so Eq. (9) and theorem 4.11 imply (126). □

Theorems 6.5 and 6.6 and Eq. (7) complete the proof of Theorem 1.2. To prove Theorem 1.1, note that, by the translation and scale invariance of $\mathbb{R}^n$, one can replace (3) by

$$\frac{d_{\mathrm{pGH}}((Y_r \cap B(0,r), 0), (E_r, x))}{r} \to 0 \tag{133}$$

and obtain an equivalent hypothesis. Equations (2) and (133) imply (123) (by using Theorem 4.11) and so Theorem 6.5 implies Theorem 1.1.

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## References

1. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces. Cambridge Studies in Advanced Mathematics, vol. 44, p. 343. Cambridge University Press, Cambridge (1995). https://doi.org/10.1017/cbo9780511623813
2. Marstrand, J.M.: Hausdorff two-dimensional measure in 3-space. Proc. Lond. Math. Soc. **3**(11), 91–108 (1961)
3. Mattila, P.: Hausdorff $m$ regular and rectifiable sets in $n$-space. Trans. Am. Math. Soc. **205**, 263–274 (1975)
4. Besicovitch, A.S.: On the fundamental geometrical properties of linearly measurable plane sets of points (II). Math. Ann. **115**(1), 296–329 (1938). https://doi.org/10.1007/BF01448943
5. Preiss, D.: Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability, and densities. Ann. of Math. (2) **125**(3), 537–643 (1987). https://doi.org/10.2307/1971410
6. David, G., Semmes, S.: Analysis of and on Uniformly Rectifiable Sets. Mathematical Surveys and Monographs, vol. 38, p. 356. American Mathematical Society, Providence (1993). https://doi.org/10.1090/surv/038
7. Jones, P.W.: Rectifiable sets and the traveling salesman problem. Invent. Math. **102**(1), 1–15 (1990). https://doi.org/10.1007/BF01233418

8. Azzam, J., Tolsa, X.: Characterization of $n$-rectifiability in terms of Jones' square function: part II. Geom. Funct. Anal. **25**(5), 1371–1412 (2015). https://doi.org/10.1007/s00039-015-0334-7
9. Tolsa, X.: Characterization of $n$-rectifiability in terms of Jones' square function: part I. Calc. Var. Part. Differ. Equ. **54**(4), 3643–3665 (2015). https://doi.org/10.1007/s00526-015-0917-z
10. Kirchheim, B.: Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. Proc. Am. Math. Soc. **121**(1), 113–123 (1994). https://doi.org/10.2307/2160371
11. Ambrosio, L., Brué, E., Semola, D.: Rigidity of the 1-Bakry–Émery inequality and sets of finite perimeter in RCD spaces. Geom. Funct. Anal. **29**(4), 949–1001 (2019). https://doi.org/10.1007/s00039-019-00504-5
12. Bate, D.: On 1-regular and 1-uniform metric measure spaces. (in preparation)
13. Bate, D.: Purely unrectifiable metric spaces and perturbations of Lipschitz functions. Acta Math. **224**(1), 1–65 (2020). https://doi.org/10.4310/acta.2020.v224.n1.a1
14. Sturm, K.-T.: On the geometry of metric measure spaces I. Acta Math. **196**(1), 65–131 (2006). https://doi.org/10.1007/s11511-006-0002-8
15. Greven, A., Pfaffelhuber, P., Winter, A.: Convergence in distribution of random metric measure spaces ($\Lambda$-coalescent measure trees). Probab. Theory Relat. Fields **145**(1–2), 285–322 (2009). https://doi.org/10.1007/s00440-008-0169-3
16. Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. Lectures in Mathematics ETH Zürich, p. 334. Birkhäuser Verlag, Basel (2008)
17. Reifenberg, E.R.: Solution of the Plateau problem for $m$-dimensional surfaces of varying topological type. Acta Math. **104**, 1–92 (1960). https://doi.org/10.1007/BF02547186
18. David, G., Toro, T.: Reifenberg flat metric spaces, snowballs, and embeddings. Math. Ann. **315**(4), 641–710 (1999). https://doi.org/10.1007/s002080050332
19. Antonelli, G., Merlo, A.: On rectifiable measures in Carnot groups: structure theory (2020)
20. Donne, E.L., Young, R.: Carnot rectifiability of sub-Riemannian manifolds with constant tangent (2019)
21. Federer, H.: Geometric Measure Theory, p. 676. Springer, Berlin (1996). https://doi.org/10.1007/978-3-642-62010-2
22. Heinonen, J.: Lectures on Analysis on Metric Spaces. Universitext, p. 140. Springer, New York (2001). https://doi.org/10.1007/978-1-4613-0131-8
23. Bogachev, V.I.: Weak Convergence of Measures. Mathematical Surveys and Monographs, vol. 234, p. 286. American Mathematical Society, Providence (2018). https://doi.org/10.1090/surv/234
24. Gromov, M.: Groups of polynomial growth and expanding maps. Inst. Hautes Études Sci. Publ. Math. **53**, 53–73 (1981)
25. Burago, D., Burago, Y., Ivanov, S.: A Course in Metric Geometry. Graduate Studies in Mathematics, vol. 33, p. 415. American Mathematical Society, Providence (2001). https://doi.org/10.1090/gsm/033
26. Gigli, N., Mondino, A., Savaré, G.: Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. Proc. Lond. Math. Soc. (3) **111**(5), 1071–1129 (2015). https://doi.org/10.1112/plms/pdv047
27. Pasqualetto, E., Schultz, T.: Ultralimits of pointed metric measure spaces (2021)
28. David, G.C.: Tangents and rectifiability of Ahlfors regular Lipschitz differentiability spaces. Geom. Funct. Anal. **25**(2), 553–579 (2015). https://doi.org/10.1007/s00039-015-0325-8
29. Le Donne, E.: Metric spaces with unique tangents. Ann. Acad. Sci. Fenn. Math. **36**(2), 683–694 (2011). https://doi.org/10.5186/aasfm.2011.3636


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