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**Analysis of a Class of Branching Particle
Systems with Spatial Pairwise Interactions**

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degree of Doctor of Philosophy in Mathematics

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Chapter 1

Introduction

1.1 Acknowledgements

Except where otherwise stated the work contained in the following pages is my own and is, to the best of my knowledge, original. All results taken from the work of other authors are clearly referenced, as are those occasions when a calculation or method has been motivated by the work of others.

I am indebted to my supervisor, Roger Tribe, for introducing me to the subject of particle systems, and for his constant encouragement, enthusiasm and support over the last three years.

1.2 Overview and Aims

The main aim of this work is to study a particular type of one-dimensional interacting particle system. The behaviour of the particles in the process is governed

by three mechanisms: diffusive movement, independent single-particle branching and two-particle spatial interaction. It is this third feature, the pairwise interaction, which makes the model new and challenging, and gives rise to some exciting problems. It will be explained later that in most cases the process will not be attractive and hence many of the tools usually used in the analysis of interacting systems will not be applicable.

It is natural to begin by asking how such processes can be constructed, and whether they can in fact be constructed at all. As the number of pairwise interactions can potentially scale as the square of the population, the possibility of explosion in finite time must be considered. The initial work deals with finite systems of particles, but later models consist of infinitely many particles distributed on the real line; in this case it is possible that the model cannot be defined for any time except $t = 0$.

The majority of the following work concentrates on those cases in which the branching forms a 'growth' mechanism whilst the two-particle interactions are 'reductive'. This means that the expected number of offspring from a branching event is greater than one, whilst the expected number of offspring from a pairwise interaction is less than two. These models allow a great deal of intuitive understanding, which is then consolidated through the mathematics.

Once a construction of the process has been provided, attention naturally turns to the existence and uniqueness of associated stationary distributions. In the case of pure branching diffusions, with no interactions between particles, such questions have been fully answered. In these interacting models such questions can be far more difficult.

The motivation for the study of such processes is two-fold. The problems are interesting and beautiful in their own right. Very few examples of non-attractive

interacting systems have been successfully analysed in anything other than a numerical way. Complementing this is the fact that such models appear as dual processes to certain white-noise driven stochastic partial differential equations (SPDEs). This duality will be discussed in detail in this paper and it will be shown that information about the particle system yields information about the corresponding SPDE, and vice-versa.

1.3 Background Material

Before we begin the construction and analysis of these interacting spatial branching processes we present a brief overview of some related areas of branching particle theory. Much of what follows can be considered as direct background material for our work, or at least provides clear motivation for this study. Perhaps more importantly however, this overview allows us to see our work in the context of a far larger field of study, one that is an active and rapidly growing area of probability theory.

1.3.1 Pure Branching Particle Processes

We begin by reviewing some basic facts concerning pure branching diffusion processes. Our interacting systems will be direct generalisations of these, and much of the notation we use here will carry forward to the interacting versions.

Consider a number of particles living in \mathbb{R}^d . These particles have lifetimes which are independently distributed according to an exponential distribution with parameter $\lambda \in (0, \infty)$. Upon its death, a particle is replaced by a random number of offspring, which appear at the position of the death of the parent.

The number of such offspring is randomly distributed according to the random variable B . During their lifetimes particles perform standard Brownian motion on \mathbb{R}^d , and all lifetimes, motion and branching are independent. We consider a right-continuous version of this process, so that if a particle dies at time t_0 then the offspring emerge for $t \geq t_0$. From the Markov property of Brownian motion and the lack-of-memory property of exponential distributions it follows that the process marking the number and positions of the particles is Markovian.

Let I_t be a (possibly infinite) labelling of the particles at time t so that the process η_t is given by

$$\eta_t = \{x_i : i \in I_t\}$$

where $x_i = x_i(t) \in \mathbb{R}^d$ is the position of the i -th particle at time t . If we stipulate finite first moment for the offspring distribution, so that $\beta := \mathbb{E}(B) < \infty$, then the process started from a single particle at x , denoted η_t^x , will not explode almost surely. By this we mean that with probability one the number of branching events in any finite time interval will be finite. Consequently the population at time t will also be finite with probability one. A thorough account of the construction of this single-ancestor process can be found in [20]. Such branching diffusion processes can be constructed from any locally finite initial set of particles $\{x_i : i \in I_0\}$ by super-position, so that

$$\eta_t = \sum_{i \in I_0} \eta_t^{x_i}$$

with the single-ancestor processes $\eta_t^{x_i}$, $i \in I_0$ being independent. This construction remains valid if the initial distribution $\{x_i : i \in I_0\}$ of particles is random. Such models have been well studied and accounts of the general theory can be found in [11], [12] and [16].

As these processes contain no interaction element, the spatial distribution of the particles of η_t has no bearing on the survival of the population when started

from a finite number of particles. In fact the total population of the process at time t , given by

$$N_t := |\{x_i : i \in I_t\}|,$$

can be considered as a continuous-time Galton-Watson branching process. Such non-spatial branching processes have been intensively studied (see, for example, [2], [25]) and this work yields total population size results for the corresponding spatial branching models. It follows that the offspring distribution, B , governs the macroscopic survival behaviour of these processes. Changing the parameter λ can simply be viewed as a re-scaling of time. Let B have the probability generating function $\sum_{k=0}^{\infty} p_k s^k$, so that p_k is the probability that a particle is replaced by k offspring upon its death. With $\beta = \mathbb{E}(B)$ as above, the three regimes $\beta < 1$, $\beta = 1$ and $\beta > 1$ are referred to as *sub-critical*, *critical* and *super-critical* respectively.

In the sub-critical regime the process will die out exponentially quickly from any finite initial state almost surely. In fact we have

$$\mathbb{E}(N_t) = n e^{\lambda(\beta-1)t} \quad (1)$$

where $n = N_0$, the initial number of particles, and $\lambda(\beta - 1) < 0$. We produce estimates analogous to (1) in later chapters, albeit for the more complicated interacting processes. On \mathbb{R}^d the unit point mass on the empty configuration, δ_0 , is the only stationary distribution. For a random infinite initial configuration, η_0 , we have the following: if the law of η_0 has finite intensity, meaning that there exists some $\theta \in (0, \infty)$ such that

$$\mathbb{E}(|\{x_i : i \in I_0\} \cap A|) \leq \theta l(A) \quad \forall \text{ Borel sets } A \subset \mathbb{R}^d$$

where $l(A)$ denotes Lebesgue measure of the set A , then the process converges weakly to δ_0 . Clearly in this infinite case the probability of total extinction on the whole space is zero, although on any finite region extinction will occur

exponentially quickly almost surely. The empty configuration δ_0 is also the only stationary state for the super-critical process, although now it is unstable. From a finite starting configuration the above result (1) for $\mathbb{E}(N_t)$ still holds but with $\lambda(\beta - 1) > 0$, giving exponential growth. Accounts of the results summarised here can be found in [2], [16] and [23].

The critical regime where $\beta = 1$ is by far the most subtle of the three. In the finite case, the population process N_t is now a martingale with $\mathbb{E}(N_t) = n$. Again using results from Galton-Watson branching theory, it is known that the probability that this finite process is extinct by time t tends to 1 as $t \rightarrow \infty$. Consequently $\mathbb{E}(N_t | N_t > 0)$ becomes very large. In the case of an infinite number of initial particles however, it transpires that the behaviour of the system is dependent on the dimension of \mathbb{R}^d .

In the case of a locally-finite infinite initial state in low dimensions ($d \leq 2$), Bramson, Cox and Greven [6] have shown that δ_0 is the only invariant measure for the process. They give necessary and sufficient conditions on η_0 under which the process becomes extinct on finite sets, is stochastically unstable or explodes. For a random initial condition with finite intensity it has been known for some time that the process converges weakly to δ_0 , see [23]. For the critical case on \mathbb{R}^d in higher dimensions ($d \geq 3$) there is a family $\{\nu_\theta : \theta \in [0, \infty)\}$ of extremal invariant measures which are translation invariant and indexed by θ , the spatial intensity of particles. All other invariant measures are convex combinations of these measures, see [7].

In this paper however we concentrate on low dimensions, in particular $d = 1$. The reasons for this, apart from simplicity, will become clear when we introduce interactions into the models. In one dimension there is a very natural zero-range interaction mechanism which allows our processes to directly generalise the pure branching diffusions above. Whilst interactions can of course be added to the

higher dimensional versions, the constructions will in some way be ‘artificial’, either through the introduction of an interaction-range function or a spatial distribution of offspring.

Before moving on from these pure branching diffusions we make two remarks concerning the use of super-position in the above construction. Firstly, although it is stipulated that the initial configuration η_0 must be locally-finite, the super-position construction actually holds for arbitrary initial states. However note that if $\{x_i : i \in I_0\} \cap K = \infty$ for some compact set $K \subset \mathbb{R}^d$, then it follows that $P(\{x_i : i \in I_t\} \cap U = \infty) = 1$ for any open set $U \subset \mathbb{R}^d$ and all $t > 0$, so interesting questions do not really arise. Secondly, the super-position construction works precisely because the families of different initial points are independent. Once two-particle interactions are introduced this will no longer be true and this method of construction will fail.

1.3.2 Interacting Particle Processes

One of the main applications of branching particle systems such as those described above is in the modelling of biological populations. The interpretation of the systems in terms of the propagation and migration of families or individuals has left its legacy in much of the terminology used, for example ‘birth’ and ‘death’, ‘parent’ and ‘offspring’, and ‘survival’ and ‘extinction’. Further applications of particle processes can be found in the fields of chemistry and genetics. Whilst such real-world applications fall outside the scope of this paper, they do serve to highlight the need for an additional element in the models - interaction between particles. Individuals in a biological population compete for resources and habitat, and perhaps inter-breed; chemicals react with each other and may respond to the presence of a catalyst; and genes mutate and combine. Consequently much work has been done on the study of particle processes with

interactions. Whilst little or none of this work deals with models of the type we will study, we do mention a few of the main areas below to highlight the diversity and interest in this subject.

Non-spatial Interacting Branching Processes

Introducing interactions adds an extra dimension of complexity to these particle processes. It is therefore natural to ask which other aspects of the model can be simplified or removed. Perhaps the most obvious such element is the spatial motion of the particles. As seen above, the study of non-spatial branching processes proved useful when discussing their spatial counterparts, and we may have similar hopes for the interacting case. However, whilst much progress has been made in this field, particularly by Russian mathematicians (eg. [27], [4], [28]), this work is not directly applicable to spatial models with local interactions such as ours. The reason for this is that the interactions in a non-spatial model must be, in some sense, 'global'. The particles can be thought of as occupying the same point and each particle interacts with every other. Consequently, comparisons with locally interacting processes, in which a particle is only affected by others which are in some sense 'close', do not naturally arise.

Measure-valued Branching Diffusion Processes

No overview of branching particle systems would be complete without the mention of measure-valued diffusions, or super-processes as they are otherwise known. This is arguably the most active and growing area of particle system research at the current time. These processes arise as the measure-valued re-normalisation limits of ordinary branching processes as the particles become small. Perhaps the most well-known of these processes is super Brownian motion; this arises as

the limit of critical binary branching Brownian motion on \mathbb{R}^d where the particles split into two or die with equal probability. Measure-valued diffusions have been popularised by Dawson *et al.* ([10], [11], [13]) and are often referred to as Dawson-Watanabe super processes.

Interest in such processes has been fuelled by the fact that these models are closely connected to certain non-linear stochastic PDEs, and much progress has been made in their analysis. This success is, according to Perkins [35], mainly due to the independence of the movement and branching of different particles, allowing for the super-position discussed earlier. Perkins remarks that in order for such processes to be applicable as models for ‘real-world’ systems, interaction mechanisms must be introduced. Recent interest has lay in this direction (see for example [35], [21] and [14]), and Dawson and Perkins provide an excellent survey in [15]. None of the interaction mechanisms reviewed in [15] are analogous to that in our models, and more importantly we wish to retain the particle structure rather than re-normalise to a continuum measure-valued limit. Nevertheless, the direction taken in the recent study of super-processes is further motivation for the importance of interacting models.

Discrete Interacting Particle Systems

Many of the most well known examples of interacting particle systems are most naturally defined in discrete space. These include the contact, voter and exclusion processes. Accounts of these and other such models can be found in the two excellent books by Liggett ([31] and [32]) and the work of Griffeath [24] and Durrett [18] amongst many others. Discrete interacting particle systems is a huge area of study encompassing a great number of different models and their associated properties. There are various recipes by which particles can be made to interact, many of which are not comparable with our model. However,

we draw attention briefly to work done by Bramson and Gray [8] on branching annihilating random walks on \mathbb{Z} , since these have much in common with our processes and the questions we wish to ask.

In these models the movement of individual particles is governed by independent continuous-time random walks, which are a discrete space analogue of Brownian motion. The branching of particles occurs according to independent exponential clocks, as in the pure branching diffusion models on \mathbb{R}^d discussed earlier. A branching event involves the splitting of a particle into two offspring, one of which remains at the site of the parent and the other which appears at either of the two neighbouring sites at random. If movement or branching causes two particles to occupy the same point then both are annihilated. Thus the branching mechanism provides growth for the system, whilst the interactions reduce the population. These interactions are clearly spatially local and pairwise, but they occur instantaneously which has two important consequences. Firstly, the offspring of a branching event must be distributed spatially near the place of death of the parent, rather than all at the exact point, otherwise further interactions would occur immediately. Here this distribution is simple since only two offspring are born but would be more difficult for a general number. Secondly, unless a similar spatial distribution is included for particles born from interactions, these interactions are limited to either annihilation (as in [8]) or coalescence (see for example [1]). An interaction resulting in two or more particles would lead instantaneously to further interactions until only one or zero particles remained.

Bramson and Gray state that this model is interesting and difficult precisely because it is non-attractive. An attractive process can be thought of roughly as one in which the introduction of extra particles increases the chance of survival. This no longer holds for models containing interactions which can decrease the

population. For the annihilating branching random walk model above with finite initial state, Bramson and Gray use a contour argument to show that the model dies out with probability one if the branching rate is low compared with the random walk rate. Conversely, if the branching rate is comparatively high then the system will survive with a positive probability. Whilst their contour argument cannot easily be adapted to our Brownian model on \mathbb{R} , and we wish also to consider a system of infinitely many particles, we do wish to study exactly the same questions of extinction and survival.

1.4 The Pairwise Spatial Interacting Model

Having reviewed some related areas of work we return our attention to the class of models which shall be the focus of this study. We wish to add spatially governed interactions between pairs of particles into the pure branching particle systems described earlier. For reasons that will be clarified shortly we allow these interactions to occur according to the local time between particles. An interaction event consists of the interacting pair being removed from the process and replaced with a random number of offspring from a distribution M . It shall be seen that two particles will only interact when occupying the same point, and so the offspring can be placed at the unique point of death of the parents. This is done in a right-continuous manner as with the single-particle branching, and all interactions and offspring numbers are independent.

It is necessary to describe the interaction mechanism in more detail. For $i, j \in I_t$, $i \neq j$, we let $l_t^{i,j}$ be the local time accrued between particles x_i and x_j by time t . In other words $l_t^{i,j}$ is the local time of the process $x_i(s) - x_j(s)$ at zero by time t , where $x_i(s)$ is the position at time s of the particle with label i . Clearly the process $x_i(s)$ is only defined for those times at which the

particle x_i exists. Thus more precisely $l_i^{i,j}$ is the local time accrued by the process $x_i(s) - x_j(s)$ at zero in the time interval $[r, t]$, where $r \geq 0$ is the earliest time at which both particles are in existence. Having defined this inter-particle local-time process we now allow an interaction between particles i and j to occur at rate $dl_i^{i,j}$ independent of all other pairs.

Several observations should be made concerning the selection of the above interaction mechanism:

Intuitively we wish interactions to occur between particles which are 'close' to each other. Such interactions are most prevalent in biological or other 'real' systems. Why then do we not allow instantaneous interactions whereby pairs of particles interact as soon as they touch? The main argument against such a mechanism was mentioned earlier in the discussion of discrete particle processes. The offspring of a branching (or interaction) event are placed at the point of death of the parent (or parents) and this would lead to a sequence of interactions occurring immediately until either one or zero particles remained. It is possible to define models in which offspring are born close to the parent according to some random spatial distribution. However such models would no longer directly generalise the pure branching models of the previous section.

In a similar vein, note that if the particles x_i and x_j meet at time 0, then the set $\{s \leq t : x_i(s) = x_j(s)\}$ contains infinitely many points and has zero Lebesgue measure almost surely for any time $t > 0$ later. Hence neither Lebesgue measure nor a counting measure can be used to formulate an interaction rate. A 'region of influence' could be placed around each particle and the set of times for which two particles are in each others regions of influence could then have positive Lebesgue measure. However a mechanism based around this idea would lead to interactions occurring between particles not occupying the same point, and the question of how to distribute the offspring would become important. Again

some spatial distribution process could be formulated but we wish to avoid such ‘artificial’ constructions.

This brings us to the local time mechanism outlined above. The local time process $l_t^{i,j}$ is increasing in t and has a version which is continuous w.r.t. t . The process is only increasing on the set $\{s : x_i(s) = x_j(s)\}$ and so interactions will only occur when x_i and x_j occupy the same point. Hence offspring particles can be placed at the point previously occupied by the two interacting parent particles. Thus these local time processes $l_t^{i,j}$ seem to recommend themselves as the tool from which to construct the interaction mechanism. The final, and perhaps most important, point to be made is that the duality to certain SPDEs mentioned in the introduction actually dictates interactions of this form, as we shall see later. Accounts of local time, including Trotter’s proof of its existence, can be found in [30], [36] and [19].

A Note on Notation and Indexing the Particles

Thus far the particles of the process alive at time t have been indexed by a set I_t , and we have considered a general point of the process at time t to be x_i for some $i \in I_t$. Whilst such notation will continue to be used throughout the following work, and has the advantage of being simple and uncluttered, it is perhaps useful at this point to give a concrete indexing method. This in turn assigns a precise meaning to this notation.

Firstly consider the following recursively defined sets

$$J_1 = \mathbb{N}, \quad J_n = \left\{ (\{i, j\}, k) : k \in \mathbb{N}, i, j \in \bigcup_{m=1}^{n-1} J_m \right\}$$

and let \mathcal{I} be the union set $\mathcal{I} = \bigcup_{n \in \mathbb{N}} J_n$. Inductively we can see that each J_n is countable and hence so is the set \mathcal{I} . Given any countable initial set of particles

we label them x_1, x_2, \dots in some arbitrary way. Thus

$$I_0 \subset \mathbb{N} = J_1 \subset \mathcal{I}.$$

Now suppose that x_i is some particle at an arbitrary time $t \geq 0$ which is indexed by $i \in \mathcal{I}$. If x_i branches into m offspring then these are given the labels $(\{i, i\}, k)$, $k = 1, \dots, m$ and it is easy to see that each of these labels is in \mathcal{I} also. Similarly, if x_i interacts with x_j , where j is also a label from \mathcal{I} , then the m offspring produced are labelled $(\{i, j\}, k)$, $k = 1, \dots, m$. These labels are also in \mathcal{I} . In this way we see that provided the process can be constructed up to time t , then the index set I_t is defined as all those $i \in \mathcal{I}$ for which a correspondingly labelled particle is alive at time t . As a subset of the countable set \mathcal{I} we see that I_t is also countable.

This indexing set \mathcal{I} is clearly rather unwieldy and we shall usually avoid using it directly. However, in the case of finitely many initial particles, it does allow for a ‘nuts and bolts’ description of the process as follows. Let $\{B_i : i \in \mathcal{I}\}$ be an independent collection of standard one-dimensional Brownian motions, $\{e_i : i \in \mathcal{I}\}$ be a collection of independent outcomes of an exponential random variable of parameter λ , and $\{f_{i,j} : i, j \in \mathcal{I}\}$ be a collection of independent outcomes of an exponential random variable of mean 1. We label the initial particles at $t = 0$ with indices $1, \dots, n \subset \mathcal{I}$. These particles move independently according to the Brownian paths B_i , $i = 1, \dots, n$ relative to their initial positions, until the first time, say s_1 , for which there is some $i = 1, \dots, n$ with either $s_1 \geq e_i$ or $t_{s_1}^{i,j} \geq f_{i,j}$ for some j . At this time the i -th particle will branch (in the first case) or interact with particle j (in the latter) and be removed from the system. Any offspring are given indices from the set \mathcal{I} according to the recipe above. To continue this inductive description of the process, we let τ_i denote the time at which a particle labelled i is first introduced into the system (so $\tau_i = 0$ for $i = 1, \dots, n$). Now from time s_1 the particles will continue to move according to the Brownian paths B_i until the first time, s_2 , for which there is some $i \in I_{s_1}$

with either $s_2 \geq \tau_i + e_i$ or $l_{s_2}^{i,j} \geq f_{i,j}$ for some j . Again we allow a branching or interaction to occur at time s_2 as appropriate and introduce and label any new particles. We then proceed in exactly the same manner to the next time step s_3 , and so on.

Before continuing we use this indexing method to give precise meaning to our notation. When describing the process at time t as $\{x_i : i \in I_t\}$, this is shorthand for

$$\{x_i(t) : i \in \mathcal{I} \text{ and } \exists \text{ particle labelled } i \text{ alive at time } t\}.$$

Here $x_i(t)$ is understood to be the position of the particle labelled i at time t given that it is in existence. Further points concerning this shorthand notation will be explained if and when necessary.

We remark that the model described here retains the Markov property of the pure branching processes. These earlier processes were Markovian due to the corresponding property of Brownian motion and the lack-of-memory property of the exponential branching times. The only additional ingredients here are the interactions occurring at rate $dl_t^{i,j}$. This is equivalent to allowing an interaction to occur once the local time becomes larger than some exponential random variable with mean one. This fact is used in the explicit construction of the process outlined in the note above. As exponential 'clocks' govern the interactions, the models remain Markovian even with this added feature.

The Parameters β and μ

We have seen that the distribution B of offspring from a branching event, and in particular the mean β , govern the macroscopic dynamics of the pure branching particle processes. In these interacting cases that role is now shared with a second offspring distribution, M , for interaction events. Let M have probability

generating function $\sum_{k=0}^{\infty} q_k s^k$, so that q_k is the probability that two interacting particles are replaced with k children. As in the case of B we stipulate finite first moment, so

$$\mu := \mathbb{E}(M) = \sum_{k=0}^{\infty} k q_k < \infty.$$

Whilst $\beta = 1$ was the critical value for the branching mechanism, it is clear that the value $\mu = 2$ plays a similar role for interactions. If $\mu = 2$ then the total population process N_t is a martingale under such interactions. This gives a parameter space of the form

$$(\beta, \mu) \in [0, \infty) \times [0, \infty)$$

divided into distinct regions by the lines $\beta = 1$ and $\mu = 2$. Some of these regimes appear on first thought to be more subtle than others. For example, in the region $\beta < 1, \mu < 2$ we may conjecture similar results to those of sub-critical pure branching, since the interactions are providing a further drive towards extinction. Conversely if $\mu > 2$ we may expect possible explosive behaviour, even for $\beta < 1$, since the interaction rate is potentially quadratic in the population density, whilst the branching rate is linear.

The Main Model

As previously mentioned, the focus of this work will be directed towards the case in which the branching mechanism provides growth and the interaction mechanism is reductive. In terms of the parameters above this means $\beta > 1$ and $\mu < 2$. The first of these conditions is less important and the majority of the following work holds for the case $\beta \leq 1$. The processes are more interesting however when the branching and interacting mechanisms are opposed, so we keep in mind the situation $\beta > 1$. The second condition, that $\mu < 2$, is vital. It

is this which avoids the possibility of explosive behaviour and, as we shall see, admits the use of duality.

Along with the added tractability given by the duality tool there are further reasons to concentrate on this class of processes. The models are open to a simple intuitive understanding which guides our analysis, and they yield non-trivial results. With regards to real world situations we note that interactions between individuals in a biological population are often of a competitive nature, thus limiting growth which would otherwise be exponential. Such situations relate to the region of the parameter space covered by this main model.

It has been stated earlier that these interacting branching processes often do not share the property of attractiveness with the pure branching models that they generalise. An attractive process can be thought of as one in which the addition of extra particles helps the process to survive. More precisely the distribution of the process at time t must exhibit stochastic monotonicity with respect to the initial distribution. Such a mathematical formulation of the concept of attractiveness can be found in [32]. To give a very simple example of why these models are non-attractive, let the branching be given by $p_1 = 1$ (one particle replaced by one, so this is equivalent to no branching) and the interactions be annihilation, $q_0 = 1$. The process started from a single particle will survive forever as a single individual, whilst the process started from a pair of particles will almost surely become extinct.

We can use the notion of coupling to give an alternative definition of an attractive process. Suppose that μ_1 and μ_2 are distributions on the state space E of some process. Suppose further that $\mu_1 \leq \mu_2$, with the stochastic monotonicity as given in [32]. We say that the process can be coupled if we can construct versions η_t and ξ_t on the same probability space such that η_0 has distribution μ_1 , ξ_0 has distribution μ_2 and $\eta_t \leq \xi_t$ a.s. for all $t \geq 0$. It was shown in [31]

that, at least in the case of spin systems, attractive processes are precisely those which can be coupled. As coupling is such an important tool in studying particle processes it is easy to see why so few non-attractive processes have been studied.

1.5 An Associated Stochastic PDE

There are several ways in which particle systems are associated with stochastic partial differential equations (SPDEs). It has already been stated that the measure-valued diffusion processes arising as the small-particle limits of branching systems form solutions to certain SPDEs (again the reader is referred to [15]). The same can occur for other non-branching and even discrete particle processes. For example, Mueller and Tribe [34] have derived white-noise driven SPDEs as the limiting equations of the long-range contact and voter processes. Different re-scaling of the same particle process may lead to more than one corresponding SPDE, as in [5]. Such re-normalisations are just one of many ways in which particle systems and SPDEs can be related. In [40] a system of annihilating Brownian particles is used to approximate the position of interfaces in a solution to the one-dimensional stochastic heat equation. The connection between a particle system and an SPDE can also occur through some form of duality relation. An exposition of the theory of such duality formulae can be found in chapter 4 of [20], and one of the most well-known applications of these ideas is due to Shiga [37]. Duality has proved to be an important and powerful tool in the study of interacting particle systems and, more recently, SPDEs.

One of the main reasons for our interest in these particular spatially interacting branching particle processes is that they occur as dual processes to a certain class of SPDEs. Exploiting such duality allows the behaviour of one system to provide information about the behaviour of the other. Athreya and Tribe [3]

have recently proved a duality relation between the bounded solutions of the one-dimensional SPDE

$$\partial_t u = \frac{1}{2} \Delta u + b(u) + \sqrt{\sigma(u)} \dot{W}_{t,x}, \quad u_0 \in bC(\mathbb{R}) \quad (1)$$

where $\dot{W}_{t,x}$ is space-time white noise, and an interacting diffusion process started from finitely many particles on \mathbb{R} . The functions $b(u)$ and $\sigma(u)$ are required to be analytic and the branching and interaction offspring probabilities are given in terms of the coefficients of their power series. We make extensive use of the duality results outlined by Athreya and Tribe in [3], although the coefficients of the functions in the SPDE are now given in terms of the offspring probabilities p_k and q_k , rather than the other way round. Beginning with the particle system as we do allows the use of this duality in a very simple form.

Consider the SPDE (1) in which

$$b(u) = \lambda \left(\sum_{k=0}^{\infty} p_k u^k - u \right) \quad \text{and} \quad \sigma(u) = \sum_{k=0}^{\infty} q_k u^k - u^2. \quad (2)$$

This is the form of the SPDE with which we wish to work. In particular we wish to consider solutions to this SPDE which are bounded between a and 1, for some $a \in [-1, 0]$. Before proceeding it must be shown that this equation is well-defined and that such solutions can exist. If we can show that $\sigma(u) \geq 0$ on $[a, 1]$ with $\sigma(a) = \sigma(1) = 0$, $b(a) \geq 0$ and $b(1) \leq 0$, then the method of Shiga [38] allows the construction of solutions to (1) which remain bounded between a and 1 with probability 1. This method is used to similar purpose in [33] and [3] for example.

Lemma 1.5.1. *The map σ has at least one root in $[-1, 0]$. If $a \in [-1, 0]$ is the closest of these roots to 0, and iff $\mu \leq 2$, then $\sigma(u) \geq 0$ on $[a, 1]$ with $\sigma(a) = \sigma(1) = 0$, and $b(a) \geq 0$ and $b(1) \leq 0$.*

Proof. Immediately we can write

$$b(1) = \lambda \left(\sum_{k=0}^{\infty} p^k - 1 \right) = 0 \quad \text{and} \quad \sigma(1) = \sum_{k=0}^{\infty} q_k - 1 = 0$$

since the p_k and q_k are probabilities which sum to 1. Next notice that $\sigma(u)$ is continuous and that

$$\sigma(0) = q_0 \geq 0 \quad \text{and} \quad \sigma(-1) = \sum_{k=0}^{\infty} (-1)^k q_k - 1 \leq \sum_{k=0}^{\infty} q_k - 1 = 0$$

so that $\sigma(u)$ has at least one root between -1 and 0. Let $a \in [-1, 0]$ be the closest of these roots to 0.

It remains to show that $\sigma(u) \geq 0$ on $[a, 1]$ and that $b(a) \geq 0$, so we begin with the latter. We write

$$b(u) = \lambda \left(\sum_{k=0}^{\infty} p_k u^k - u \right) = \lambda \left(\sum_{k=0}^{\infty} p_k u^k - \sum_{k=0}^{\infty} p_k u \right) = \lambda \left(\sum_{\substack{k=0 \\ k \neq 1}}^{\infty} (u^k - u) p_k \right)$$

and note that $u^k - u \geq 0$ for all $k \in \mathbb{N}$ and for any $u \in [-1, 0]$. Consequently $b(u) \geq 0$ for all $u \in [-1, 0]$, and so in particular $b(a) \geq 0$.

Thus far the required criteria have been fulfilled by interacting branching processes falling in all regions of the $\beta \times \mu$ parameter space. It is the final condition, that $\sigma(u) \geq 0 \forall u \in [a, 1]$, which places restrictions on the types of models for which the SPDE (1) is sensibly defined. Clearly since a was chosen to be the largest root in $[-1, 0]$ and $\sigma(0) = q_0 \geq 0$ it follows from the continuity of $\sigma(u)$ that $\sigma(u) \geq 0$ on $[a, 0]$. At the opposite end of the interval however we have $\sigma(1) = 0$ and

$$\sigma'(1) = \sum_{k=0}^{\infty} k q_k - 2 = \mu - 2.$$

Thus if $\mu > 2$ this gradient is positive and since $\sigma'(u)$ is continuous there exists $\delta > 0$ such that $\sigma(u) < 0$ for $u \in (1 - \delta, 1)$. As (1) contains a $\sqrt{\sigma(u)}$ term, the SPDE is clearly not well-defined on this region.

To finish we show that when $\mu \leq 2$ we do have $\sigma(u) \geq 0$ on $[a, 1]$. The non-negativity of $\sigma(u)$ on $[a, 0]$ has been shown already, so we restrict our attention to the interval $[0, 1]$. To obtain a contradiction we suppose that the result is false and that $\sigma(u)$ is negative somewhere in $[0, 1]$. As $\sigma'(u)$ is continuous, $\sigma(0) \geq 0$ and $\sigma(1) = 0$, this implies that $\sigma(u)$ has a negative turning point in $(0, 1)$. In other words $\exists u_0 \in (0, 1)$ such that $\sigma(u_0) < 0$ and $\sigma'(u_0) = 0$. We have

$$\begin{aligned}
\sigma'(u_0) = 0 &\implies \sum_{k=0}^{\infty} k q_k u_0^{k-1} - 2u_0 = 0 \\
&\implies \sum_{k=0}^{\infty} k q_k u_0^k - 2u_0^2 = 0 \quad (\text{multiplying by } u_0) \\
&\implies \sum_{k=0}^{\infty} k q_k u_0^k - \sum_{k=0}^{\infty} k q_k u_0^2 \geq 0 \quad (\text{using } \mu \leq 2) \\
&\implies q_1(u_0 - u_0^2) + \sum_{k=2}^{\infty} k q_k (u_0^k - u_0^2) \geq 0 \\
&\implies q_1(u_0 - u_0^2) + \sum_{k=2}^{\infty} q_k (u_0^k - u_0^2) \geq 0 \\
&\hspace{15em} (\text{since } (u_0^k - u_0^2) \leq 0 \text{ for } k \geq 2) \\
&\implies \sum_{k=0}^{\infty} q_k (u_0^k - u_0^2) \geq 0 \quad (\text{since } q_0(1 - u_0^2) \geq 0) \\
&\implies \sum_{k=0}^{\infty} q_k u_0^k - u_0^2 = \sigma(u_0) \geq 0
\end{aligned}$$

and this contradicts the fact that $\sigma(u_0) < 0$. □

It follows from this lemma and the remark preceding it, that if $\mu \leq 2$ and $u_0(x) \in [a, 1]$ for all $x \in \mathbb{R}$, then we can construct solutions to (1) which remain bounded in $[a, 1]$. Thus in the case of our main model, in which the interactions are reductive, the SPDE is well-defined and such bounded solutions exist. The duality relation, presented later, between these bounded solutions to (1) and our particle system give a powerful additional tool for use in our analysis.

1.6 Layout of this Thesis

In the next chapter we present an account of these interacting branching processes living on a unit ring. There are numerous features which make models on such a ring easier to study, not least that the space is bounded, so particle numbers relate directly to particle density. After presenting a construction of these processes we prove one of the main results of our work: the stochastic boundedness of the total population. From this we show the existence of stationary distributions for these systems, which we then categorise fully for our reductive models.

In the third chapter we begin transposing the work of chapter 2 onto the infinite line, beginning with models started from finitely many particles. The addition of a weighting function for the particles helps overcome the problems inherent in the unboundedness of \mathbb{R} , but brings some additional technical difficulties. Nonetheless the construction of the process from the previous chapter still holds on the line, and the main aim is then to prove an analogue of our earlier stochastic boundedness theorem, this time for the weighted population.

Our considerations then turn to processes consisting of infinitely many particles. Path-wise constructions for such systems are difficult to find, although a graphical construction for a certain class of models is given at the end of the fourth chapter. To retain the generality of our work we construct our infinite processes via a soft limiting argument from finite versions. It is this work which forms the bulk of chapter 4, and we make extensive use of the tools which become available to us when we cast our systems in the framework of measure-valued processes.

In the final chapter we can begin to work with our newly constructed infinite processes, and we start by showing that stationary distributions exist for these

processes just as they did on the ring. We also extend and exploit the duality relation, which proves to be an invaluable tool throughout our work. Via this duality we present general theory in which the behaviour of the particle system dictates the behaviour of the associated SPDE, and vice versa. We end chapter 5 with an investigation of a particular example: a process in which single particles branch into two at rate λ and pairs coalesce according to local-time. Do we expect this process to survive or exhibit local extinction? Will it grow arbitrarily for large λ ? An answer to these questions concludes our work.

Chapter 2

The Process on a Ring: Intuition and Construction

2.1 A Very Important Sentence

The ultimate aim of this paper is to analyse these interacting branching processes started from infinitely many particles distributed on \mathbb{R} . It may therefore seem slightly perverse to begin by studying the process started from finitely many particles on a ring. However this case exhibits many of the features of the desired model, whilst avoiding several of the technical difficulties which arise on \mathbb{R} . In this way it provides an ideal test-bed on which to develop both our intuition of the processes, and the tools with which to study them.

We denote by S a one-dimensional ring of unit length, so $S = [0, 1) \bmod 1$. Given a finite set of initial points, $\{x_i : i \in I_0\} \subset S$, the system evolves according to the rules given in the previous section, with the Brownian paths taken *modulo*

1 to give motion on the ring. The process at time t is denoted by ξ_t , so that

$$\xi_t = \{x_i : i \in I_t\}$$

and the total population at time t is $N_t = |I_t|$. It is assumed that the interaction offspring distribution is such that $\mu < 2$, so that the interactions are reductive. Whilst much of what follows holds for any $\beta < \infty$, we have in mind the case $\beta > 1$ so that the interactions act in opposition to the single particle branching.

The remainder of this chapter is devoted to the study of this model. Firstly we show that such a process can in fact be constructed. This is done by proving that only finitely many population events (branchings or interactions) occur in any finite time interval almost surely. Then an intuitive argument is used to show that the total population is stochastically bounded for all t . The intuition here can be summed up in one simple, but very important, sentence:

When the density of particles is sufficiently high, the effect of the pairwise interactions will dominate the effect of the branching.

The idea expressed therein will form the backbone of a large amount our work in studying these processes. In the reductive models specified here this can be paraphrased as saying that as the population grows large, the reductive interactions dominate the branching and drive the population downwards again. We conclude this chapter by examining the stationary distributions of these processes on the ring S .

Remark 2.1.1. What features of the ring S make it easier to study than the infinite line? The main reason is quite simply that the space is bounded. Consequently a large population on S relates directly to a high particle density. Conversely a population increase on \mathbb{R} could lead to a higher particle density, but may also be given by a 'spreading-out' of the process along the line. It is not inconceivable that the population N_t of the process started from finitely

many particles on \mathbb{R} grows arbitrarily large, whilst the particle density remains stochastically bounded. These problems do not arise on the ring. A second topological feature of note is that S is compact. Several of the convergence results used later, particularly in the work on stationary distributions, follow easily because of this fact.

2.2 Constructing the Process

Before we show that our interacting branching processes can be constructed for arbitrarily large t , we require an important result concerning the following abstract situation. Consider some arbitrary stochastic process $\{\xi_t : t \geq 0\}$. Let $A_t \equiv A(\xi_t)$ be some adapted function of ξ_t which is increasing in t . Define a single-step jump process X_t which jumps from 0 at rate A_t . Equivalently X_t jumps at time $\tau = \inf\{s : A_s \geq e_1\}$, where e_1 is an outcome of an exponential random variable with mean 1. The size of the jump at this jump time has distribution $G(\xi_\tau, Z)$, where Z is some random variable entirely independent of ξ_t . Then we have the following result:

Lemma 2.2.1. *If we let $\bar{G}(\eta)$ denote $\mathbb{E}(G(\eta, Z)|\eta)$, then*

$$\mathbb{E}(X_t) = \mathbb{E}\left(\int_0^t 1_{\{s \leq \tau\}} \bar{G}(\xi_s) dA_s\right).$$

Proof. $X_t \equiv X_t(\xi_t, Z, e_1)$ depends on the underlying process ξ , the independent element Z of the jump distribution, and the exponential random variable e_1 . The idea here is to condition on all the information up to time t concerning ξ and Z , so that X_t becomes dependent only on e_1 . As e_1 is simply an exponential random variable of mean 1, this conditional expectation of X_t becomes easy to calculate. Letting $\mathcal{F}_t = \sigma\{\xi_s : s \leq t\}$ be the natural filtration for the underlying

process ξ , we write

$$\begin{aligned}\mathbb{E}(X_t) &= \mathbb{E}(\mathbb{E}(X_t|\mathcal{F}_t, Z)) = \mathbb{E}\left(\int_0^t G(\xi_s, Z)P(\tau \in ds|\mathcal{F}_t, Z)\right) \\ &= \mathbb{E}\left(\int_0^t G(\xi_s, Z)P(e_1 \in dA_s|\mathcal{F}_t, Z)\right) = \mathbb{E}\left(\int_0^t G(\xi_s, Z)e^{-A_s}dA_s\right).\end{aligned}$$

Using a similar conditioning idea from this point we have

$$\begin{aligned}\mathbb{E}(X_t) &= \mathbb{E}\left(\int_0^t G(\xi_s, Z)e^{-A_s}dA_s\right) = \mathbb{E}\left(\mathbb{E}\left(\int_0^t G(\xi_s, Z)e^{-A_s}dA_s|\mathcal{F}_t\right)\right) \\ &= \mathbb{E}\left(\int_0^t \mathbb{E}(G(\xi_s, Z)|\mathcal{F}_t)e^{-A_s}dA_s\right) = \mathbb{E}\left(\int_0^t \bar{G}(\xi_s)e^{-A_s}dA_s\right).\end{aligned}$$

Turning our attention to the right-hand side of the equality stated in the lemma, we have

$$\begin{aligned}\mathbb{E}\left(\int_0^t 1_{\{s \leq \tau\}}\bar{G}(\xi_s)dA_s\right) &= \mathbb{E}\left(\mathbb{E}\left(\int_0^t 1_{\{s \leq \tau\}}\bar{G}(\xi_s)dA_s|\mathcal{F}_t\right)\right) \\ &= \mathbb{E}\left(\int_0^t \bar{G}(\xi_s)\mathbb{E}(1_{\{A_s \leq e_1\}}|\mathcal{F}_t)dA_s\right) = \mathbb{E}\left(\int_0^t \bar{G}(\xi_s)e^{-A_s}dA_s\right).\end{aligned}$$

It follows from the two expressions above that the stated equality holds. \square

Corollary 2.2.2. *If the jump size distribution is independent of the underlying process, so that $G \equiv G(Z)$, then $\mathbb{E}(X_t) = \mathbb{E}(G(Z))\mathbb{E}(1_{\{\tau \leq t\}})$.*

Proof. The function $1_{\{\tau \leq t\}}$ is a single-step jump process which jumps at the same rate as X_t , but then takes the value 1. The rate at which X_t and $1_{\{\tau \leq t\}}$ jump is A_t , so applying lemma 2.2.1 above we have

$$\mathbb{E}(1_{\{\tau \leq t\}}) = \mathbb{E}\left(\int_0^t 1_{\{s \leq \tau\}}dA_s\right).$$

However, if we apply lemma 2.2.1 to X_t , and use the expression above together with the fact that $\bar{G}(\xi_s) = \mathbb{E}(G(Z))$ and can be taken outside the main expectation, then we can write

$$\mathbb{E}(X_t) = \mathbb{E}(G(Z))\mathbb{E}\left(\int_0^t 1_{\{s \leq \tau\}}dA_s\right) = \mathbb{E}(G(Z))\mathbb{E}(1_{\{\tau \leq t\}}).$$

\square

This lemma and corollary are extremely useful. They allow us to represent the expected value of a single-step jump process in terms of the expectation of the compensator term given above. The first application of this is in the lemma below, and we will use the result extensively in future work.

Returning to our construction arguments we suppose that at time $t = 0$ there are $N_0 = n$ particles distributed on S . As stated in the previous chapter it is possible to construct pure branching diffusion processes for arbitrarily large times. We wish to show that the same is true for our interacting processes if the initial population is finite. The process can certainly be constructed up to the first interaction time, say time τ_1 , using a construction such as that outlined in the earlier note on the indexing of particles. In fact in this interval the process behaves exactly as a pure branching model. After the first interaction there are still only finitely many particles almost surely, so the process can now be constructed up to the next interaction time and so on. Thus it follows that we can construct the process up to an arbitrary number of interactions, so up to τ_n for any $n \in \mathbb{N}$. At each stage the expected value of the total population N_t remains finite. Given that the interaction times τ_n tend upwards to ∞ as n gets large we can construct the process for all time $t \geq 0$. The question then is whether these interaction times become arbitrarily large or tend to some finite explosion time.

Now we define a slightly modified version of our process in which interactions and branching no longer occur after the m -th interaction event. These ‘ m -version’ processes are defined in the same way as the original process except that they become pure diffusions after τ_m . Let probabilities with respect to these modified processes be denoted by P_m , and expectations by E_m . Clearly the laws of the original and the m -version of the process are the same until the m -th interaction, so

$$P(\tau_m \leq s) = P_m(\tau_m \leq s).$$

The reason for introducing these new versions of the process is that they are defined for arbitrarily large $t \geq 0$, unlike the original which may only be defined up to some explosion time. We begin with the following lemma:

Lemma 2.2.3. *If π_t denotes the number of interactions occurring in the interval $[0, t]$, then for any finite $N_0 = n$ we have*

$$\mathbb{E}_m(N_t) \leq ne^{\lambda(\beta-1)t} + (\mu - 2)\mathbb{E}_m(\pi_t).$$

Proof. We let Δ_t^B and Δ_t^I be the change in population by time t due to branching and interaction respectively. Further we let $\Delta_t^B(i)$, $i \in \mathcal{I}$ be the change in the population by time t due to the branching (if any) of particle i , and similarly $\Delta_t^I\{i, j\}$, $i \neq j \in \mathcal{I}$, is the change in population due to the interaction of particles i and j . Then clearly we can write

$$\mathbb{E}_m(N_t) = n + \mathbb{E}_m(\Delta_t^B) + \mathbb{E}_m(\Delta_t^I) = n + \mathbb{E}_m\left(\sum_{i \in \mathcal{I}} \Delta_t^B(i)\right) + \mathbb{E}_m\left(\sum_{i \neq j \in \mathcal{I}} \Delta_t^I\{i, j\}\right). \quad (1)$$

We remark that although the above expression contains infinite sums, the majority of the terms will be zero. In most cases a particle with label i will not appear in the process before time t . In the final term above we know that at most m of the terms are non-zero since interactions cease after the m -th interaction event. The number of offspring produced in different interactions are independent and distributed according to M . Thus the change in population due to a given interaction has distribution $M - 2$, and since $|M - 2| \leq M + 2$ we can write

$$\mathbb{E}_m\left(\sum_{i \neq j \in \mathcal{I}} |\Delta_t^I\{i, j\}|\right) \leq m\mathbb{E}(M + 2) = m(\mu + 2) < \infty.$$

This justifies the use of Fubini's theorem, which allows us to write

$$\mathbb{E}_m\left(\sum_{i \neq j \in \mathcal{I}} \Delta_t^I\{i, j\}\right) = \sum_{i \neq j \in \mathcal{I}} \mathbb{E}_m(\Delta_t^I\{i, j\}).$$

Now let θ_t be defined analogously to π_t to be the number of branching events in the interval $[0, t]$. Recalling that all particles in existence branch independently at rate λ up to time τ_m and using Fubini we have

$$\mathbb{E}_m(\theta_t) = \mathbb{E}_m\left(\int_0^{t \wedge \tau_m} \lambda N_s ds\right) = \lambda \int_0^t \mathbb{E}_m(1_{\{s \leq \tau_m\}} N_s) ds \leq \lambda \int_0^t \mathbb{E}_m(N_s) ds < \infty. \quad (2)$$

That the integral is finite follows from our previous remarks that the process behaves as a pure branching diffusion between the interactions.

Now the change in the population due to a given branching event has distribution $B - 1$, and we note that $|B - 1| \leq B + 1$. Letting $\tau(i)$ be the random time at which particle i branches, we use corollary 2.2.2 together with Fubini and expression (2) above to give

$$\begin{aligned} \sum_{i \in \mathcal{I}} \mathbb{E}_m(|\Delta_t^B(i)|) &= \mathbb{E}(|B - 1|) \sum_{i \in \mathcal{I}} \mathbb{E}_m(1_{\{\tau(i) \leq t\}}) \\ &\leq (\beta + 1) \mathbb{E}_m\left(\sum_{i \in \mathcal{I}} 1_{\{\tau(i) \leq t\}}\right) = (\beta + 1) \mathbb{E}_m(\theta_t) < \infty. \end{aligned}$$

This justifies the use of Fubini in the branching term in (1), which allows us to write

$$\mathbb{E}_m(N_t) = n + \sum_{i \in \mathcal{I}} \mathbb{E}_m(\Delta_t^B(i)) + \sum_{i \neq j \in \mathcal{I}} \mathbb{E}_m(\Delta_t^I\{i, j\}). \quad (1')$$

Again using corollary 2.2.2 in a similar way we have

$$\begin{aligned} \sum_{i \in \mathcal{I}} \mathbb{E}_m(\Delta_t^B(i)) &= (\beta - 1) \sum_{i \in \mathcal{I}} \mathbb{E}_m(1_{\{\tau(i) \leq t\}}) \\ &= (\beta - 1) \mathbb{E}_m(\theta_t) = \lambda(\beta - 1) \int_0^t \mathbb{E}_m(1_{\{s \leq \tau_m\}} N_s) ds, \end{aligned}$$

where the final equality comes from (2). Similarly, letting $\tau\{i, j\}$ be the time of any interaction between particles i and j , we have

$$\sum_{i \neq j \in \mathcal{I}} \mathbb{E}_m(\Delta_t^I\{i, j\}) = (\mu - 2) \sum_{i \neq j \in \mathcal{I}} \mathbb{E}_m(1_{\{\tau\{i, j\} \leq t\}}) = (\mu - 2) \mathbb{E}_m(\pi_t).$$

From these we re-formulate (1') as

$$\mathbb{E}_m(N_t) = n + \lambda(\beta - 1) \int_0^t \mathbb{E}_m(1_{\{s \leq \tau_m\}} N_s) ds + (\mu - 2)\mathbb{E}_m(\pi_t). \quad (1'')$$

As $(\mu - 2) < 0$ and $\mathbb{E}_m(\pi_t) \geq 0$ this gives

$$\mathbb{E}_m(N_t) \leq \lambda(\beta - 1) \int_0^t \mathbb{E}_m(1_{\{s \leq \tau_m\}} N_s) ds \leq \lambda(\beta - 1) \int_0^t \mathbb{E}_m(N_s) ds,$$

and as $\mathbb{E}_m(N_t)$ is finite it follows from Gronwall's lemma that $\mathbb{E}_m(N_t) \leq ne^{\lambda(\beta-1)t}$.

Using this upper bound on $\mathbb{E}_m(N_t)$ we write

$$\lambda \int_0^t \mathbb{E}_m(N_s) ds \leq \lambda \int_0^t ne^{\lambda(\beta-1)s} ds = \frac{n}{(\beta - 1)} (e^{\lambda(\beta-1)t} - 1),$$

so now from (1'') we can write

$$\begin{aligned} \mathbb{E}_m(N_t) &\leq n + \lambda(\beta - 1) \int_0^t \mathbb{E}_m(N_s) ds + (\mu - 2)\mathbb{E}_m(\pi_t) \\ &\leq ne^{\lambda(\beta-1)t} + (\mu - 2)\mathbb{E}_m(\pi_t). \end{aligned}$$

□

Lemma 2.2.4. *For any finite $N_0 = n$ and $s \geq 0$ we have*

$$P(\tau_m \leq s) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Proof. From Markov's inequality we have

$$\mathbb{E}_m(\pi_t) \geq mP_m(\pi_t \geq m) = mP_m(\tau_m \leq t),$$

so using lemma 2.2.3 we have

$$\begin{aligned} 0 \leq \mathbb{E}_m(N_s) &\leq ne^{\lambda(\beta-1)s} + (\mu - 2)\mathbb{E}_m(\pi_s) \\ &\leq ne^{\lambda(\beta-1)s} + (\mu - 2)mP_m(\tau_m \leq s). \end{aligned}$$

This can be arranged to give

$$P_m(\tau_m \leq s) \leq \frac{ne^{\lambda(\beta-1)s}}{m(2-\mu)},$$

the right hand side of which tends to 0 as $m \rightarrow \infty$ for any finite n and $s \geq 0$. As has been mentioned previously we have $P_m(\tau_m \leq s) = P(\tau_m \leq s)$, so this is in fact the result we require. \square

Thus almost surely the interaction times tend upwards to ∞ and there does not exist some finite explosion time. This allows us to construct the process for any $t \geq 0$. This in turn allows us to prove that the bound stated in lemma 2.2.3 also holds for the full process.

Lemma 2.2.5. *If the process is started from n particles initially, then*

$$\mathbb{E}(N_t) \leq ne^{\lambda(\beta-1)t} + (\mu - 2)\mathbb{E}(\pi_t).$$

Proof. Letting τ_m be the time of the m -th interaction as before, we see that almost surely

$$N_{t \wedge \tau_m} + (2 - \mu)\pi_{t \wedge \tau_m} \longrightarrow N_t + (2 - \mu)\pi_t \quad \text{as } m \rightarrow \infty.$$

This follows because we know from lemma 2.2.4 that $\tau_m \rightarrow \infty$ almost surely as $m \rightarrow \infty$. Now applying Fatou's lemma we have

$$\mathbb{E}(N_t + (2 - \mu)\pi_t) \leq \liminf_{m \rightarrow \infty} \mathbb{E}(N_{t \wedge \tau_m} + (2 - \mu)\pi_{t \wedge \tau_m}).$$

We remark now that in the m -version processes defined earlier both N_t and π_t become fixed after time τ_m , and until this time the law of the m -version process is the same as the law for the full process. Consequently $\mathbb{E}_m(N_t) = \mathbb{E}(N_{t \wedge \tau_m})$ and $\mathbb{E}_m(\pi_t) = \mathbb{E}(\pi_{t \wedge \tau_m})$. Using this fact in the expression above, together with the bound given in lemma 2.2.3, we have

$$\mathbb{E}(N_t + (2 - \mu)\pi_t) \leq \liminf_{m \rightarrow \infty} \mathbb{E}_m(N_t + (2 - \mu)\pi_t) \leq ne^{\lambda(\beta-1)t},$$

which can be re-arranged to give the stated result. \square

Remark 2.2.1. We should note here that the above results rely in no way on the topology of S . Consequently this proof holds for finitely many particles distributed on \mathbb{R} , so we have shown that the finite process can be constructed there also. Really this is not surprising; the idea underlying the above proof was to find an upper bound for the expected number of interactions in time t . On \mathbb{R} , where the particles are able to spread out, we would perhaps expect this upper bound to be easier to find than on S , where two particles can never be more than a distance $1/2$ apart.

2.3 Some Results on Boundedness

To construct the process as above it was shown that interactions do not occur too quickly, so that the possibility of explosion is avoided. In this section we take the opposite approach; we show that interactions do occur sufficiently quickly to control the growth of the population. To do this we formulate mathematics around the intuition presented in section 2.1.

2.3.1 Basic Results Concerning Brownian Motion

Before proceeding we present some very elementary facts on Brownian motion which will be needed later.

Lemma 2.3.1. *Let X_t be a one-dimensional Brownian motion started at the origin. Fix $a > 0$ and define $T_a = \inf\{t > 0 : X_t = a\}$. Then $\exists c > 0$ which is independent of a such that*

$$P(T_a \leq a^2 c) = \frac{1}{2}.$$

Proof. Recall that if X_t is a standard Brownian motion, then so too is kX_{t/k^2} for any $k > 0$. Clearly there exists some $c > 0$ such that $P(T_1 \leq c) = 1/2$. Now we have:

$$\begin{aligned} T_a &= \inf\{t > 0 : X_t = a\} = \inf\{t > 0 : \frac{1}{a}X_t = 1\} \\ &= \inf\{a^2t > 0 : \frac{1}{a}X_{a^2t} = 1\} \stackrel{D}{=} \inf\{a^2t > 0 : X_t = 1\} \\ &= a^2 \inf\{t > 0 : X_t = 1\} = a^2T_1. \end{aligned}$$

Thus we have shown that T_a is equal in distribution to a^2T_1 and consequently

$$P(T_a \leq a^2c) = P(a^2T_1 \leq a^2c) = P(T_1 \leq c) = \frac{1}{2}.$$

□

Lemma 2.3.2. Let V_t and W_t be independent one-dimensional Brownian motions with $V_0 = m$ and $W_0 = 0$.

Then $X_t = \frac{1}{\sqrt{2}}(V_t - W_t)$ is a one-dimensional Brownian motion with $X_0 = m/\sqrt{2}$.

Proof. It is simple to check that $X_t = \frac{1}{\sqrt{2}}(V_t - W_t)$ fulfils the defining properties of a Brownian motion. □

Corollary 2.3.3. With V_t and W_t as above, define $T'_m = \inf\{t > 0 : V_t = W_t\}$.

Then

$$P(T'_m \leq m^2c/2) = \frac{1}{2}$$

with $c > 0$ as given in lemma 2.3.1.

Proof. Let Y_t be a standard one-dimensional Brownian motion started at zero, and set $X_t = \frac{1}{\sqrt{2}}(V_t - W_t)$ as in lemma 2.3.2. Then

$$T'_m = \inf\{t > 0 : V_t = W_t\} = \inf\{t > 0 : X_t = 0\} \stackrel{D}{=} \inf\{t > 0 : Y_t = \frac{m}{\sqrt{2}}\} = T_{\frac{m}{\sqrt{2}}}$$

and so

$$P(T'_m \leq m^2c/2) = P(T_{m/\sqrt{2}} \leq m^2c/2) = P(T_{m/\sqrt{2}} \leq (m/\sqrt{2})^2c) = \frac{1}{2}$$

using lemma 2.3.1. □

2.3.2 The Main Theorem

The following theorem formalises our intuition concerning the process.

Theorem 2.3.4. *Suppose that n particles are somehow distributed on S at time $t = 0$. Then there exists $n_0 \in \mathbb{N}$, $\alpha \in (0, 1)$ and some $k > 0$ such that*

$$n \geq n_0 \implies \mathbb{E}(N_{t_n}) \leq \alpha n$$

where $t_n = k/n$.

This theorem says that if the number of initial particles is large, then some short time later the expected population will be less, and that the decrease will be proportional to the initial population. The remainder of this section will be devoted to proving this theorem.

What is the significance of the time t_n ? Why is it not possible to have a result which states that

$$\frac{d}{dt} \mathbb{E}(N_t) < 0$$

provided the number of particles N_t at time t is sufficiently large? To explain this consider the situation in which a large number n of particles are equally spaced around S . However large this number n , the initial tendency of the system will be towards growth: the growth mechanism, the single-particle branching, is unaffected by spatial arrangement, whilst the reductive interaction mechanism requires sufficient time for the particles to meet each other. Clearly the larger the number of particles n , the smaller the average inter-particle distance, and the shorter the time required for the diffusions to cause mixing. This notion manifests itself in the scaling of the times t_n , which are inversely proportional to n .

The method of the proof is as follows. In section 2.2 it was shown that the process can be constructed for arbitrary time $t \geq 0$. The expected value of the

total population obeys the bound

$$\mathbb{E}(N_t) \leq ne^{\lambda(\beta-1)t} + (\mu - 2)\mathbb{E}(\pi_t)$$

as stated in lemma 2.2.5, where π_t denotes the number of interactions which have occurred by time t . The idea is to use this expression over a number of small time intervals and to bound $\mathbb{E}(\pi_t)$ below (recalling that $\mu - 2 < 0$) in such a way that the right-hand side of the above inequality is less than n , provided that n is sufficiently large.

The main difficulty in estimating $\mathbb{E}(\pi_t)$ arises from the lack of independence in the system; an interaction between two particles may make further interactions less likely. To overcome this we choose certain special disjoint pairs of particles between which we formulate a kind of sub-independence. In particular we are interested in pairs of particles which are close together at certain time points. The number of such pairs can be bounded below using simple combinatorial arguments, and we then show that the number of interactions involving these pairs is sufficiently high. At each stage particular care must be taken to ensure that no independence or attractiveness properties have been assumed.

2.3.3 Close Pairs

We now give a precise definition of the disjoint pairs discussed above. Suppose that n particles are distributed on S and choose some arbitrary particle. This particle and its 'positive neighbour' form the first disjoint pair, recalling that since $S = [0, 1) \bmod 1$ it is imbued with some orientation. The next particle in the positive direction and its positive neighbour form the next disjoint pair. We continue in this way until we have as many disjoint pairings as possible, noting that if n is odd then there will be a solitary particle remaining which is not part of a pair. Letting $[x]$ denote the integer part of any $x \in \mathbb{R}$, it is easy to see that

there will be $\lfloor n/2 \rfloor$ such pairs.

The sum of the distances between all neighbouring pairs of particles is 1, the length of S . Therefore there must be at least $\lfloor n/2 \rfloor$ neighbouring pairs of particles which are a distance at most $2/n$ apart. This follows because otherwise the distances between those neighbours more than $2/n$ apart would sum to more than 1, which is impossible. Now assume that $n \geq 3$. A brief consideration shows that using the disjoint pair construction above started from each of 3 neighbouring particles will cover all possible pairings. Consequently it is possible to choose disjoint pairs in such a way that at least a third of the neighbours discussed above which are at most $2/n$ apart are included. Thus there are at least $\lfloor n/6 \rfloor$ of these disjoint pairs which are a distance at most $2/n$ apart. These will be referred to as *close pairs* and this notion will prove invaluable over the following pages. We summarise in the following lemma:

Lemma 2.3.5. *However n particles are distributed on S , it is possible to choose $\lfloor n/6 \rfloor$ disjoint pairs of particles which are a distance $\leq 2/n$ apart.*

Proof. If $n \geq 3$ then this follows from the remarks above. If $n < 3$ then $\lfloor n/6 \rfloor = 0$ so the result follows trivially. \square

2.3.4 Some Preliminary Lemmas

One of the main steps in the proof of theorem 2.3.4 involves a comparison between our system and a number of entirely independent pairs of particles which undergo interactions but not branching. To ensure that the argument of the main proof is as uncluttered as possible we present the results we will need now.

The situation we have in mind is as follows. Suppose there are r independent pairs of particles, with the particles in each pair no more than some distance d

apart at some initial time. Each pair evolves independently of all others, and the particles within a pair interact with each other according to the same local-time mechanism as in our processes. There is no branching mechanism. We are only interested in whether or not an interaction has occurred in a given interval so we assume that any interaction causes annihilation.

Define by $m(t)$ the number of these pairs in which the random motion has caused the two particles to meet by time t . We proceed to prove a series of results (2.3.6, 2.3.7 and 2.3.8) concerning this independent pair process culminating in a lower bound on the expected value of π_t . This in turn is then used in the proof of the main result via a careful comparison argument.

Lemma 2.3.6. *Given that $t \geq d^2c/2$, with c given in 2.3.1, then*

$$P(m(t) \geq \lceil r/2 \rceil) \geq \frac{1}{2}.$$

Proof. Each pair is initially separated by a distance of at most d . Thus the probability that any pair meets by time t is greater than the probability of the same event for a pair of Brownian particles started a distance exactly d apart. Corollary 2.3.3 tells us that this probability is at least $1/2$ provided $t \geq d^2c/2$, with $c > 0$ a constant given in lemma 2.3.1. Letting Z be a $Bin(r, 1/2)$ random variable and using the fact that the r pairs evolve independently, we have

$$P(m(t) \geq \lceil r/2 \rceil) \geq P(Z \geq \lceil r/2 \rceil) \geq \frac{1}{2}$$

for any $t \geq d^2c/2$ as required. □

This next lemma gives a bound on the expected number of interactions by time $\tau + t$ given the number of pairs $m(\tau)$ which have met by time τ .

Lemma 2.3.7. *For any $\gamma \in (0, 1)$ and any $m \in \{1, \dots, r\}$ we have*

$$\mathbb{E}(\pi_{\tau+t} | m(\tau) \geq m) \geq m(1 - \gamma) \left\{ 1 - e^{-\gamma\sqrt{\pi t/2}} \right\}$$

where π_t is the number of pairs of particles which have interacted by time t .

Proof. We use F to denote the event $m(\tau) \geq m$. We begin by re-labelling the r independent pairs of particles so that the first m pairs are amongst those which meet by time τ . For $z \in \{1, \dots, r\}$, let e_z be independent outcomes of an exponential random variable of mean 1, and let l_t^z be the local time accumulated by the z -th pair by time t . By construction an interaction occurs between the z -th pair at rate dl_t^z . This is equivalent to letting the pair interact as soon as $l_t^z \geq e_z$. Letting A_t^z be the event that pair z have interacted by time t we have

$$\pi_t = \sum_{z=1}^r 1_{A_t^z} \geq \sum_{z=1}^m 1_{A_t^z}$$

so that

$$\mathbb{E}(\pi_{\tau+t}|F) \geq \mathbb{E}\left(\sum_{z=1}^m 1_{A_{\tau+t}^z}|F\right) = \sum_{z=1}^m P(A_{\tau+t}^z|F) = \sum_{z=1}^m P(l_{\tau+t}^z \geq e_z|F).$$

Now let l_t be the local time at $\tau + t$ of an independent pair of particles which meet for the first time at *exactly* time τ , and let e be another outcome of an exponential random variable of mean 1. As local time is increasing, and since under F each of the pairs labelled $1, \dots, m$ first meet *before* time τ , we have the comparison

$$P(l_{\tau+t}^z \geq e_z|F) \geq P(l_t \geq e) \quad \forall z \in \{1, \dots, m\}.$$

Using this in the above gives

$$\mathbb{E}(\pi_{\tau+t}|F) \geq mP(l_t \geq e) \tag{1}$$

so it remains to show that $P(l_t \geq e)$ can be bounded above by something of the form given in the statement of the lemma.

Firstly we notice that

$$P(l_t \geq e) = \mathbb{E}(P(l_t \geq e|l_t)) = \mathbb{E}(1 - e^{-l_t}) = 1 - \mathbb{E}(e^{-l_t}) \tag{2}$$

since e is an outcome of an exponential random variable of mean 1. Now for any $\delta > 0$ we have

$$\begin{aligned}\mathbb{E}(e^{-l_t}) &= \mathbb{E}(e^{-l_t} | l_t \geq \delta) P(l_t \geq \delta) + \mathbb{E}(e^{-l_t} | l_t < \delta) P(l_t < \delta) \\ &\leq e^{-\delta} P(l_t \geq \delta) + P(l_t < \delta) \\ &= e^{-\delta} + P(l_t < \delta)(1 - e^{-\delta}).\end{aligned}$$

Let X_t be a standard one-dimensional Brownian motion started at the origin. We use the fact that l_t is equal in distribution to $\sup_{0 \leq s \leq t} X_s$ (see [30]) and the reflection principle (see [22]) to write

$$\begin{aligned}P(l_t \geq \delta) &= P(\sup_{0 \leq s \leq t} X_s \geq \delta) \stackrel{\text{refl. prin.}}{=} 2P(X_t \geq \delta) \\ &= 2 \int_{\delta}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx = 1 - \int_{-\delta}^{\delta} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx.\end{aligned}$$

Bounding $\int_{-\delta}^{\delta} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$ above trivially by $2\delta/\sqrt{2\pi t}$ and setting $\delta = \gamma\sqrt{\pi t/2}$ for $\gamma \in (0, 1)$, we see that $P(l_t < \delta) \leq \gamma$ and consequently

$$\mathbb{E}(e^{-l_t}) \leq \gamma + (1 - \gamma)e^{-\gamma\sqrt{\pi t/2}}.$$

Using this in equation (2) and then substituting into (1) gives a bound of the required form. \square

We finish this subsection with a corollary which combines lemmas 2.3.6 and 2.3.7 above. This will give us all the tools we require to prove our main stochastic boundedness theorem on S .

Corollary 2.3.8. *Consider the situation described above: r independent pairs of particles subject to local-time interaction, with the two particles in each pair no more than a distance d apart at some initial time. Then for any $\gamma \in (0, 1)$ we have*

$$\mathbb{E}(\pi_t) \geq \frac{1}{2}[r/2](1 - \gamma) \left\{ 1 - e^{-\frac{1}{2}\gamma\sqrt{\pi t}} \right\}$$

provided $t \geq d^2 c$ with $c > 0$ given by lemma 2.3.1.

Proof. This follows in a fairly straight-forward manner from the work above. Firstly we write

$$\mathbb{E}(\pi_t) \geq \mathbb{E}(\pi_t | m(t/2) \geq \lceil r/2 \rceil) P(m(t/2) \geq \lceil r/2 \rceil).$$

Now we note that $t/2 \geq d^2 c/2$, so we can apply lemma 2.3.6 to the second factor on the right-hand side of the above to get

$$P(E_t) \geq \frac{1}{2} \mathbb{E}(\pi_t | m(t/2) \geq \lceil r/2 \rceil).$$

Finally we apply lemma 2.3.7 to this with $m = \lceil r/2 \rceil$ and replacing both τ and t in that result with $t/2$ here. This gives the expression stated. \square

2.3.5 Proof of Theorem 2.3.4

We now return to our usual interacting branching process on S to prove this important theorem.

Step 1

Suppose that there are at least $\lceil n/2 \rceil$ particles initially distributed on S , and choose a consecutive group g of exactly $\lceil n/2 \rceil$ of them. Applying lemma 2.3.5 from our earlier work on *close pairs* tells us that however these particles are distributed we can choose $\lceil n/12 \rceil$ disjoint pairs of neighbouring particles which are a distance at most $2/\lceil n/2 \rceil \leq 8/n$ apart. The idea of this step of the proof is to use these *close pairs* to formulate a lower bound on the expected number of interactions in this group in the interval $[0, t]$. To do this we make a careful comparison between the *close pairs* in our model, and a system of the type discussed above consisting of $\lceil n/12 \rceil$ independent pairs of particles started a distance at most $8/n$ apart.

Number the *close pairs* from 1 to $\lceil n/12 \rceil$ in some arbitrary way. Now define the following:

$\pi_t(g)$ is the number of interactions occurring by time t involving at least one particle of the group g

π_t^{cl} is the number of interactions occurring by time t in which at least one of the two particles involved is a *close pair* particle

A_t^j is the event that an interaction has occurred by time t involving at least one of the two particles from the j -th *close pair*

B_t^j is the event that an interaction has occurred by time t between a solitary pair of particles started the same distance apart as the j -th *close pair*. This pair of particles is considered on a separate probability space in which there are no other particles and no branching

G_t is the event that none of the $2\lfloor n/12 \rfloor$ exponential, parameter λ random variable outcomes triggering the branching of the *close pair* particles is less than t .

Immediately we write

$$\mathbb{E}(\pi_t(g)) \geq \mathbb{E}(\pi_t^{cl}) \geq \frac{1}{2} \mathbb{E}\left(\sum_{i=1}^{\lfloor n/12 \rfloor} 1_{A_i^j}\right) \geq \frac{1}{2} \mathbb{E}\left(\sum_{i=1}^{\lfloor n/12 \rfloor} 1_{A_i^j} | G_t\right) P(G_t). \quad (1)$$

The factor $1/2$ in the third term follows since a single interaction may occur between two particles from different *close pairs*. Thus although two of the events A_t^j , $j = 1, \dots, \lfloor n/12 \rfloor$ occur, there is only one interaction. However the number of interactions involving *close pair* particles is at least half the number of the events A_t^j which have occurred.

Compared to the separate system containing only two particles beginning in the same position as the j -th *close pair*, the presence of any additional particles can only ever *increase* the probability of an interaction involving one of these two. Thus we have

$$\mathbb{E}\left(\sum_{i=1}^{\lfloor n/12 \rfloor} 1_{A_i^j} | G_t\right) = \sum_{i=1}^{\lfloor n/12 \rfloor} P(A_t^i | G_t) \geq \sum_{i=1}^{\lfloor n/12 \rfloor} P(B_t^i) = \mathbb{E}\left(\sum_{i=1}^{\lfloor n/12 \rfloor} 1_{B_i^j}\right).$$

Each event B_t^j is the event that a solitary pair of non-branching particles on their own probability space interacts by time t , given that they start in the same

position as the j -th *close pair*. Thus we can consider the right-hand side of the expression above as the expected number of interactions by time t in a system of $[n/12]$ independent pairs of particles with no branching, each pair started a distance $\leq 8/n$ apart. This is precisely the situation discussed in the preceding subsection. Consequently we may apply corollary 2.3.8 with $r = [n/12]$ and $d = 8/n$ to give

$$\mathbb{E}\left(\sum_{i=1}^{[n/12]} 1_{B_i^t}\right) \geq \frac{1}{2}[n/24](1-\gamma)\left\{1 - e^{-\frac{1}{2}\gamma\sqrt{\pi}t}\right\}$$

for any $\gamma \in (0, 1)$, provided $t \geq d^2c = 64c/n^2$, with c given in lemma 2.3.1. As branching occurs independently for each particle we have

$$P(G_t) = e^{-2\lambda[n/12]t}$$

so that combining all this together into (1) gives

$$\mathbb{E}(\pi_t(g)) \geq \frac{1}{4}[n/24](1-\gamma)e^{-2\lambda[n/12]t}\left\{1 - e^{-\frac{1}{2}\gamma\sqrt{\pi}t}\right\}$$

provided $t \geq 64c/n^2$. Letting $t = k/n^2$ with $k \geq 64c$ we have

$$\mathbb{E}(\pi_{k/n^2}(g)) \geq \frac{n}{192}(1-\gamma)e^{-k\lambda/6n}\left\{1 - e^{-\frac{1}{2n}\gamma\sqrt{\pi}k}\right\}.$$

We assume n is large enough so that $e^{-k\lambda/6n} \geq 1/2$ and use the fact that $1 - e^{-z} \geq z - z^2/2$ to write

$$\begin{aligned} \mathbb{E}(\pi_{k/n^2}(g)) &\geq \frac{n}{384}(1-\gamma)\left\{\frac{1}{2n}\gamma\sqrt{\pi}k - \frac{1}{8n^2}\gamma^2\pi k\right\} \\ &= \frac{1}{384}(1-\gamma)\left\{\frac{1}{2}\gamma\sqrt{\pi}k - \frac{1}{8n}\gamma^2\pi k\right\}. \end{aligned}$$

It is easy to see that if we assume n is large enough then we have

$$\left\{\frac{1}{2}\gamma\sqrt{\pi}k - \frac{1}{8n}\gamma^2\pi k\right\} \geq \frac{1}{4}\gamma\sqrt{\pi}k,$$

so there is some n'_0 such that if $n \geq n'_0$ then

$$\mathbb{E}(\pi_{k/n^2}(g)) \geq p \tag{*}$$

where $p > 0$ is a constant independent of n .

Step 2

Now assume there are N particles distributed on S at some initial time. We can divide these particles into $r := \lfloor N/\lfloor n/2 \rfloor \rfloor$ disjoint groups of $\lfloor n/2 \rfloor$ consecutive particles, labelled g_1, \dots, g_r . This also holds true when $N < \lfloor n/2 \rfloor$ since then $r = 0$. Letting $\pi_t(g_i)$ denote the number of interactions by time t in which at least one of the particles belongs to group g_i , we have

$$\mathbb{E}(\pi_t) \geq \frac{1}{2} \sum_{i=0}^r \mathbb{E}(\pi_t(g_i)).$$

The factor $1/2$ occurs again because an interaction occurring between a particle in g_i and a particle in g_j , with $i \neq j$, would count as an interaction in $\pi_t(g_i)$ and $\pi_t(g_j)$, but obviously only as one in π_t . In the case when $t = k/n^2$ the work above allows us to use (*) to bound each $\mathbb{E}(\pi_t(g_i))$ to give

$$\mathbb{E}(\pi_{k/n^2}) \geq \frac{1}{2} \sum_{i=0}^r p = \frac{r}{2} p = \frac{p}{2} \left\lfloor \frac{N}{\lfloor n/2 \rfloor} \right\rfloor, \quad (**)$$

provided $n \geq n'_0$, with n'_0 given in step 1.

Step 3

Now we consider precisely the situation described in the theorem, so that there are exactly n particles initially distributed on S . We divide the time interval $[0, t_n] = [0, k/n]$ into n subintervals of length k/n^2 . Thus these intervals have the form $[ik/n^2, (i+1)k/n^2]$ for $i = 1, \dots, n-1$. Letting Δ_i^B and Δ_i^I be the change in population in the i -th subinterval due to branching and interacting respectively, we have

$$\mathbb{E}(N_{k/n}) = n + \sum_{i=0}^{n-1} \mathbb{E}(\Delta_i^B + \Delta_i^I). \quad (2)$$

Let N_i denote the population at the beginning of the i -th interval. We may assume that n is large enough so that $e^{\lambda(\beta-1)k/n} \leq 7/6$. Now it follows from

lemma 2.2.5 and the Markov property that if $\mathbb{E}(N_i) \leq 3n/4$ for any i , then

$$\mathbb{E}(N_{k/n}) = \mathbb{E}(\mathbb{E}(N_{k/n}|N_i)) \leq \mathbb{E}(N_i)e^{\lambda(\beta-1)k/n} \leq \frac{3n}{4} \frac{7}{6} = \frac{7}{8}n.$$

This would give the result stated in the theorem with $\alpha = 7/8$. From this point we assume that this is not the case, so there is no i for which $\mathbb{E}(N_i) \leq 3n/4$.

Letting π_i be the number of interactions occurring in interval i , we use our usual bound on the total population stated in lemma 2.2.5 to write

$$\mathbb{E}(\Delta_i^B + \Delta_i^I) = \mathbb{E}(\mathbb{E}(\Delta_i^B + \Delta_i^I|N_i)) \leq \mathbb{E}(N_i)(e^{\lambda(\beta-1)\frac{k}{n^2}} - 1) + (\mu - 2)\mathbb{E}(\mathbb{E}(\pi_i|N_i)).$$

Using (**) from step 2 above yields

$$\mathbb{E}(\pi_i|N_i) \geq \frac{p}{2} \left\lfloor \frac{N_i}{[n/2]} \right\rfloor$$

so that we have

$$\begin{aligned} & \mathbb{E}(N_i)(e^{\lambda(\beta-1)k/n^2} - 1) + (\mu - 2)\mathbb{E}(\mathbb{E}(\pi_i|N_i)) \\ & \leq \mathbb{E}(N_i)(e^{\lambda(\beta-1)k/n^2} - 1) + \frac{1}{2}p(\mu - 2)\mathbb{E}\left(\left\lfloor \frac{N_i}{[n/2]} \right\rfloor\right) \\ & \leq \mathbb{E}(N_i)(e^{\lambda(\beta-1)k/n^2} - 1) + \frac{1}{2}p(\mu - 2)\mathbb{E}\left(\frac{N_i}{[n/2]} - 1\right) \\ & \leq \mathbb{E}(N_i)(e^{\lambda(\beta-1)k/n^2} - 1) + \frac{1}{2}p(\mu - 2)\mathbb{E}\left(\frac{2N_i}{n} - 1\right) \\ & = \mathbb{E}(N_i)\{e^{\lambda(\beta-1)k/n^2} - 1 + p(\mu - 2)/n\} - p(\mu - 2)/2. \end{aligned}$$

We assume n is large enough so that

$$e^{\lambda(\beta-1)k/n^2} - 1 \leq -p(\mu - 2)/4n$$

and then recalling our assumption that $\mathbb{E}(N_i) \geq 3n/4$ we have

$$\begin{aligned} \mathbb{E}(\Delta_i^B + \Delta_i^I) & \leq \mathbb{E}(N_i)\{3(\mu - 2)/4n\} - p(\mu - 2)/2 \\ & \leq 3n/4\{3(\mu - 2)/4n\} - p(\mu - 2)/2 \\ & \leq p(\mu - 2)/16 = -\tilde{p} < 0. \end{aligned}$$

Using this in expression (2) above gives

$$\mathbb{E}(N_{k/n}) = n + \sum_{i=0}^{n-1} \mathbb{E}(\Delta_i^B + \Delta_i^I) \leq n - \tilde{p}n = n(1 - \tilde{p}) \quad (***)$$

noting that $1 - \tilde{p}$ is a constant less than 1 which is independent of n . Consequently, whatever the expected population $\mathbb{E}(N_i)$ at the subintervals, the statement of the theorem holds with

$$\alpha = \max\{7/8, 1 - \tilde{p}\} \in (0, 1)$$

provided $n \geq n_0$ where n_0 is the smallest integer which fulfils all the size conditions required in this proof.

2.3.6 Two Additional Results

This section is concluded with the presentation of two corollaries to theorem 2.3.4. The first of these replaces the time t_n , which is dependent on n , with a fixed time τ . We can then view the process N_t in fixed, discrete steps. This is then used in our second corollary, which re-formulates our main result into a statement we use directly when investigating the stationary distributions of these interacting branching processes on S .

Corollary 2.3.9. *There exists some $n_1 \in \mathbb{N}$, $\alpha' \in (0, 1)$ and some fixed $\tau > 0$ so that if $N_0 = n \geq n_1$, then*

$$\mathbb{E}(N_\tau) \leq \alpha' n.$$

Proof. With n_0 and α given in theorem 2.3.4, we choose $n_1 \in \mathbb{N}$ large enough so that $n_1 \geq n_0$ and $e^{\lambda(\beta-1)k/n_1} < 1/\sqrt{\alpha}$. We then set $\tau := t_{n_1} = k/n_1$.

For $n \geq n_1$ we have $t_n = k/n \leq \tau$ and so we write

$$\mathbb{E}(N_\tau) = \mathbb{E}(\mathbb{E}(N_\tau | N_{t_n})) \leq \mathbb{E}(N_{t_n} e^{\lambda(\beta-1)(\tau-t_n)}) \leq \mathbb{E}(N_{t_n}) e^{\lambda(\beta-1)\tau}$$

where the first inequality arises from bounding the population by exponential growth. Noting that $n \geq n_1 \geq n_0$ allows us to apply theorem 2.3.4 to give

$$\mathbb{E}(N_\tau) \leq \mathbb{E}(N_{t_n})e^{\lambda(\beta-1)\tau} \leq \alpha n e^{\lambda(\beta-1)\tau} < \alpha n \frac{1}{\sqrt{\alpha}} = \sqrt{\alpha} n.$$

Letting $\alpha' = \sqrt{\alpha} \in (0, 1)$ gives the required result. \square

Proposition 2.3.10. *Provided the initial number N_0 of particles is finite, we have*

$$\sup_{t \geq 0} P(N_t \geq k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof. Markov's inequality tells us that for any k ,

$$\mathbb{E}(N_t) \geq kP(N_t \geq k).$$

Thus

$$\sup_{t \geq 0} P(N_t \geq k) \leq \frac{1}{k} \sup_{t \geq 0} \mathbb{E}(N_t)$$

and the result will hence follow if we show that $\mathbb{E}(N_t)$ is bounded by a constant for all $t \geq 0$.

Let n_1, α' and τ be given by corollary 2.3.9 and choose $M \in \mathbb{N}$ so that

$$(1 - \alpha')M > n_1 e^{\lambda(\beta-1)\tau}. \quad (1)$$

Now we have the following:

$$\begin{aligned} \mathbb{E}(N_{t+\tau}) &= \sum_{n=0}^{\infty} \mathbb{E}(N_{t+\tau} | N_t = n) P(N_t = n) \\ &= \sum_{n=0}^{n_1-1} \mathbb{E}(N_{t+\tau} | N_t = n) P(N_t = n) + \sum_{n=n_1}^{\infty} \mathbb{E}(N_{t+\tau} | N_t = n) P(N_t = n) \\ &\leq \sum_{n=0}^{n_1-1} n e^{\lambda(\beta-1)\tau} P(N_t = n) + \sum_{n=n_1}^{\infty} \alpha' n P(N_t = n) \\ &\leq n_1 e^{\lambda(\beta-1)\tau} + \alpha' \mathbb{E}(N_t). \end{aligned}$$

Hence it follows that if $\mathbb{E}(N_t) > M$ then

$$\begin{aligned}
 \mathbb{E}(N_{t+\tau}) &\leq n_1 e^{\lambda(\beta-1)\tau} + \alpha' \mathbb{E}(N_t) \\
 &\stackrel{\text{using(1)}}{\leq} (1 - \alpha')M + \alpha' \mathbb{E}(N_t) \\
 &< (1 - \alpha')\mathbb{E}(N_t) + \alpha' \mathbb{E}(N_t) \\
 &= \mathbb{E}(N_t).
 \end{aligned}$$

So when $\mathbb{E}(N_t) > M$ the value of $\mathbb{E}(N_{t+\tau})$ is strictly smaller than $\mathbb{E}(N_t)$. Examining $\mathbb{E}(N_t)$ at each of the time points $t \in \{n\tau : n \in \mathbb{N}\}$ and bounding the expected population by exponential growth in between, we see that $\mathbb{E}(N_t)$ is bounded by

$$\max \{ N_0 e^{\lambda(\beta-1)\tau}, M e^{2\lambda(\beta-1)\tau} \}.$$

□

2.4 Stationary Distributions on S - Existence

Thus far it has been shown that the process does indeed behave as predicted by our natural intuition. This intuition, formalised in the main theorem (2.3.4) of the previous section, tells us that if the number of particles becomes sufficiently high then we can expect the interactions to drive the population downwards again. With this theorem and its corollaries in place, we have the tools required to start investigating the stationary distributions for these processes. Of course the most natural question with which to begin is whether such stationary distributions exist at all. This will be the focus of this section.

Our first step is to formulate our models within the framework of random measure-valued processes on S . With each particle represented by a point mass this becomes a natural setting for our systems, and gives us access to many

powerful tools to help with our analysis. Following this we use a coupling argument to prove that our processes fulfil the Feller property; this will be the final ingredient required in the proof that stationary distributions certainly exist.

2.4.1 ξ_t as a Measure-Valued Process

The process ξ_t which we have defined and constructed in the previous sections can easily be viewed as a random measure on S . As stated above we do this by simply placing a unit point mass on each particle of the process. Thus with I_t denoting a labelling of the particles alive at time t as usual, we have

$$\xi_t = \sum_{i \in I_t} \delta_{x_i}.$$

In constructing the process it was shown that from a finite initial condition the process would remain finite almost surely. Consequently the process at time t defines a finite measure on S . The measure $\sum_{i \in I_t} \delta_{x_i}$ contains all information regarding the number and position of the particles, so the Markov property of the process ξ_t is preserved when we consider the measure-valued process.

Letting $\mathcal{M}_f(S)$ denote the set of all finite measures on S , we have that ξ_t is an element of $\mathcal{M}_f(S)$ for any time t . Further we define a subset F of $\mathcal{M}_f(S)$ to be all finite measures consisting of unit masses placed at points on S , so

$$F = \bigcup_{n=0}^{\infty} \left\{ \sum_{i=0}^n \delta_{x_i} : x_1, \dots, x_n \in S \right\}.$$

This is the natural state space for the process considered as a measure.

The space $\mathcal{M}_f(S)$ becomes a topological space under the topology of weak convergence. As S is clearly compact, we can compactify this space $\mathcal{M}_f(S)$ by adding a point at infinity, say $\{\infty_w\}$. The topology of the space $\overline{\mathcal{M}}_f(S) :=$

$\mathcal{M}_f(S) \cup \{\infty_w\}$ is defined by

$$\begin{aligned} x_n \implies x \in \mathcal{M}_f(S) & \text{ iff } \int_S f(s) dx_n(s) \rightarrow \int_S f(s) dx(s) \quad \forall f \in C(S) \\ x_n \implies \infty_w & \text{ iff } \int_S 1 dx_n(s) = x_n(S) \rightarrow \infty \end{aligned}$$

where $C(S)$ is the space of all continuous (and necessarily bounded) real-valued functions on S . This space $\overline{\mathcal{M}}_f(S)$ is then a compact, metrizable topological space called the *Watanabe compactification* of $\mathcal{M}_f(S)$, see [11] and [41]. One such metric is given by

$$d(x, y) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \left| \frac{\langle x, f_n \rangle + \langle x, 1 \rangle}{\langle x, f_n \rangle + \langle x, 1 \rangle + 1} - \frac{\langle y, f_n \rangle + \langle y, 1 \rangle}{\langle y, f_n \rangle + \langle y, 1 \rangle + 1} \right|, \quad x, y \in \overline{\mathcal{M}}_f(S),$$

where $\{f_n : n \in \mathbb{N}\}$ is a countable set of functions on S which are convergence determining in $\mathcal{M}_f(S)$ and for which $f_1 \equiv 1$. One such set of functions is introduced at the start of section 2.4.2. We remark that the quotient terms in the above metric are given the value 1 in the case where x or y is ∞_w . For more information on such random measures, the reader is referred to [29].

These measure spaces afford a very elegant interpretation of our processes, but have been introduced for a particular purpose. We wish to apply the following general tool (see [20] for a proof of this result) to aid in the analysis of our models.

Theorem 2.4.1. *Suppose that some topological space E is compact and metrizable, and let $\mathcal{M}_1(E)$ be the space of all probability measures on E with the topology of weak convergence. Then any sequence $\{\mu_n : n \in \mathbb{N}\}$ in $\mathcal{M}_1(E)$ has a convergent subsequence.*

The natural state space for the process is F rather than the larger space $\mathcal{M}_f(S)$, but F is not compact. Whilst we could apply the above theorem with $E = \overline{\mathcal{M}}_f(S)$ we would rather use $E = F \cup \{\infty_w\}$, and to do this we must show

that this smaller space is metrizable and compact. Metrizability is inherited from the metrizability of the larger space $\overline{\mathcal{M}}_f(S)$, and so it remains to show that $F \cup \{\infty_w\}$ is compact. This follows easily if we know that $F \cup \{\infty_w\}$ is in fact closed, and this can be proved using the following lemma.

Lemma 2.4.2. *Suppose $x \in \mathcal{M}_f(S)$ and let $At(x)$ denote the set of points of S at which x is atomic, so $At(x) = \{s \in S : x(\{s\}) > 0\}$. Then*

$$x \in F \iff x([a, b]) \in \mathbb{N} \text{ for any interval } [a, b] \text{ on } S \text{ with } a, b \notin At(x).$$

Proof. The implication from left to right is clear, so we now concentrate on the inverse.

Firstly we notice that as x is a finite measure, $At(x)$ is countable and so cannot contain an interval. This follows since

$$At(x) = \bigcup_{n=1}^{\infty} \{s \in S : x(\{s\}) > 1/n\}$$

and each of the sets within this union is finite. From this we see that S can be considered as an interval $[0, 1)$ such that, without loss of generality, $0 \notin At(x)$ and we can consider the function

$$g(s) = x([0, s]) \quad \forall s \in (0, 1) \setminus At(x).$$

By assumption this function is increasing and \mathbb{N} -valued, and it is certainly bounded by $x(S) < \infty$. Thus it follows that $g(s)$ must have a finite number of jump discontinuities at which the value of g increases by some positive integer. Between these points the measure x has no mass, whilst at each jump point it has a positive integer mass which can be considered as a number of unit point masses. Consequently x is of the form required to be in F . \square

This characterisation of the elements of F yields a simple proof of the closure of $F \cup \{\infty_w\}$. It is here that it becomes clear why we wished to exclude the atoms of x in the above result.

Lemma 2.4.3. *The space $F \cup \{\infty_w\}$ is closed in $\overline{\mathcal{M}}_f(S)$.*

Proof. Let $\{x_n : n \in \mathbb{N}\}$ be a convergent sequence of elements of $F \cup \{\infty_w\}$, say with $x_n \implies x$ as $n \rightarrow \infty$. To show that $F \cup \{\infty_w\}$ is closed it suffices to show that $x \in F \cup \{\infty_w\}$. If $x = \{\infty_w\}$ then this is certainly true, so we assume $x \neq \infty_w$ and need to show that $x \in F$.

Firstly notice that since the constant function 1 is continuous on S we have

$$x_n(S) = \int_S 1 dx_n(s) \rightarrow \int_S 1 dx(s) = x(S).$$

The sequence on the left-hand side is a convergent sequence of non-negative integers and must hence eventually be some constant integer. Consequently $x(S)$ must also be an integer.

Now we assume that $x \notin F$, so by lemma 2.4.2 above there exists some interval $[a, b]$ on S with $a, b \notin \text{At}(x)$ such that $x([a, b]) = \gamma \notin \mathbb{N}$. As a, b are not atoms of x we notice that $x((a, b)) = x([a, b]) = \gamma$. Now we define continuous approximating super- and sub-functions, f_ϵ and g_ϵ , to the indicator on $[a, b]$ by

$$f_\epsilon(s) = \begin{cases} 1 & s \in [a, b] \\ 0 & s \notin [a - \epsilon, b + \epsilon] \\ \text{linear in-between} & \end{cases} \quad g_\epsilon(s) = \begin{cases} 1 & s \in [a + \epsilon, b - \epsilon] \\ 0 & s \notin [a, b] \\ \text{linear in-between.} & \end{cases}$$

Note that $f_\epsilon(S)$ is defined provided $\epsilon < (1 - (b - a))/2$ and $g_\epsilon(s)$ is defined for $\epsilon < (b - a)/2$, so both are defined for sufficiently small $\epsilon > 0$. As $\gamma \notin \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m < \gamma < m + 1$. Let $\Delta = \min\{\gamma - m, (m + 1) - \gamma\}$. As $\epsilon \rightarrow 0$

$$\int_S f_\epsilon(s)x(ds) \searrow \int_S 1_{[a,b]}(s)x(ds) = x([a, b]) = \gamma \quad \text{and}$$

$$\int_S g_\epsilon(s)x(ds) \nearrow \int_S 1_{(a,b)}(s)x(ds) = x((a, b)) = x([a, b]) = \gamma$$

so we can choose $\delta > 0$ small enough so that

$$\gamma - \frac{\Delta}{2} < \int_S g_\delta(s)x(ds) \leq \gamma \leq \int_S f_\delta(s)x(ds) < \gamma + \frac{\Delta}{2}.$$

Here it is the fact that $a, b \notin At(x)$ that allows the limiting integrals of both f_ϵ and g_ϵ to be γ . It is for this reason that we excluded atoms in lemma 2.4.2. By the definition of $x_n \implies x$ we have

$$\lim_{n \rightarrow \infty} \int_S f_\delta(s) x_n(ds) = \int_S f_\delta(s) x(ds) \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_S g_\delta(s) x_n(ds) = \int_S g_\delta(s) x(ds)$$

and so we choose N large enough so that

$$\max \left\{ \left| \int_S f_\delta(s) x_N(ds) - \int_S f_\delta(s) x(ds) \right|, \left| \int_S g_\delta(s) x_N(ds) - \int_S g_\delta(s) x(ds) \right| \right\} < \frac{\Delta}{2}.$$

Thus we have

$$x_N([a, b]) \leq \int_S f_\delta(s) x_N(ds) < \int_S f_\delta(s) x(ds) + \frac{\Delta}{2} < \gamma + \Delta < m + 1$$

$$x_N([a, b]) \geq \int_S g_\delta(s) x_N(ds) > \int_S g_\delta(s) x(ds) - \frac{\Delta}{2} > \gamma - \Delta > m$$

which together imply that $x_N([a, b]) \in (m, m + 1)$ and so is not an integer. As $x_N \in F$ this is clearly a contradiction and so there exists no such interval $[a, b]$. Consequently x is an element of F and the space $F \cup \{\infty_w\}$ is thus closed. \square

Corollary 2.4.4. *The space $F \cup \{\infty_w\} \subset \overline{\mathcal{M}}_f(S)$ is a compact, metrizable space.*

Proof. As previously remarked, metrizability is inherited from the space $\overline{\mathcal{M}}_f(S)$. By lemma 2.4.3 above, $F \cup \{\infty_w\}$ is a closed subspace of the compact space $\overline{\mathcal{M}}_f(S)$ and so is itself compact. \square

This shows us that the set $E = F \cup \{\infty_w\}$ is suitable for the application of theorem 2.4.1. Before we use this however we require an additional tool, namely the Feller property.

2.4.2 The Feller Property on F

The aim of this subsection is to show that the Feller property holds for our process. In other words we want to show that the map

$$x \mapsto \mathbb{E}_x(f(\xi_t)) := \mathbb{E}(f(\xi_t)|\xi_0 = x)$$

from F to \mathbb{R} is continuous and bounded whenever the function $f : S \rightarrow \mathbb{R}$ is. We remark here that it is natural to return to the space F when considering the Feller property rather than work with the compactified space $F \cup \{\infty_w\}$. This follows since these expectation maps above are not defined at the point ∞_w . This point can be thought of as representing all those configurations on S consisting of infinitely many particles. We have no construction of our processes started from such initial conditions, so the expression $\mathbb{E}(f(\xi_t)|\xi_0 = \infty_w)$ is meaningless.

We begin by defining a metric on F which corresponds to the topology of weak convergence. Recalling that $C(S)$ is the space of continuous functions on S , note that there exists a countable subset $\{f_n : n \in \mathbb{N}\} \subset C(S)$, with each f_n bounded by 1, whose span is dense in $C(S)$. The set of functions $\{1, \cos(2\pi ks), \sin(2\pi ks) : k \in \mathbb{N}\}$ would certainly suffice, where the trigonometric functions are naturally defined on $S = [0, 1) \bmod 1$. However we wish to specify a particular family of maps in order to simplify some of our arguments later on.

There are only countably many intervals of the form $[a, b)$, $a, b \in \mathbb{Q}$ on S . For each such interval we can define functions

$$f_{n,m}^{[a,b]}(s) = \begin{cases} 1 & s \in [a, b) \\ 0 & s \notin [a - (1/n), b + (1/m)] \\ \text{linear inbetween} & \end{cases}$$

for any $n, m \in \mathbb{N}$ for which this definition is sensible. Thus $f_{n,m}^{[a,b]}$ can be defined whenever the length of the interval $[a - (1/n), b + (1/m)]$ is not greater than

1. If the length of $[a, b)$ is 1 then the only possible function is the constant function 1. For each such interval $[a, b)$ the corresponding set of functions is countable and we take the union of such families over all such intervals to be our set $\{f_n : n \in \mathbb{N}\}$. Additionally we assume that $f_1 \equiv 1$. That the span of this set is dense in $C(S)$ is established using the fact that any continuous function on compact S must be uniformly continuous.

Now define a map $d : F \times F \rightarrow \mathbb{R}$ by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, f_n \rangle - \langle y, f_n \rangle| \quad \forall x, y \in F$$

where $\langle x, f \rangle := \int_S f(s)x(ds)$. Alternatively we may sometimes use the notation $x(f) := \int_S f(s)x(ds)$.

Lemma 2.4.5. *The map $d : F \times F \rightarrow \mathbb{R}$ defines a metric on F whose topology is the topology of weak convergence.*

Proof. This is a standard result which holds regardless of the collection of functions used to define d , provided they are bounded by 1 and have a span dense in $C(S)$. An outline of the proof is given here.

All the metric properties are easily checked for d , with perhaps the exception of $d(x, y) = 0 \implies x = y$. To show this we remark that since $\langle x, f_n \rangle = \langle y, f_n \rangle$ for each f_n and the span of $\{f_n : n \in \mathbb{N}\}$ is dense in $C(S)$, then we can use approximations from this span to show that $\langle x, f \rangle = \langle y, f \rangle$ for all $f \in C(S)$. Consequently $x = y$.

A similar method can also be used to show that if $x_i \rightarrow x$ under d , then x_i converges weakly to x . Conversely if x_i converges weakly to x in F , then $\langle x_i, f_n \rangle \rightarrow \langle x, f_n \rangle$ for each f_n . This, together with the fact that x must be a finite measure, can be used to show that $d(x_i, x) \rightarrow 0$. Thus convergence in the

two topologies is equivalent, which shows that both must have the same closed sets and so are equal. \square

Our proof of the Feller property relies on the fact that two independent processes, started from initial conditions which are close under the above metric, have a high probability of coupling quickly. The precise nature of this coupling will be explained after the following technical lemma. This lemma can be thought of as saying that for the distance between $x, y \in F$ to be small, the point masses of the two measures are required to 'line up'.

Lemma 2.4.6. *Fix $x \in F$. Then for any $\epsilon > 0$ there is some $\delta > 0$ so that $d(x, y) < \delta$, for $y \in F$, implies*

- (i) *the total mass of x and the total mass of y are equal, and*
- (ii) *there is a one-to-one correspondence between the point masses of x and y , so that any point mass of y is within an ϵ -ball on S of the corresponding point mass on x .*

Proof. Firstly recall that $f_1 \equiv 1$, so

$$\begin{aligned} d(x, y) < \frac{1}{2} &\implies \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, f_n \rangle - \langle y, f_n \rangle| < \frac{1}{2} \\ &\implies |\langle x, f_1 \rangle - \langle y, f_1 \rangle| < 1 \\ &\implies \left| \int_S x(ds) - \int_S y(ds) \right| < 1 \\ &\implies |x(S) - y(S)| < 1. \end{aligned}$$

Both $x(S)$ and $y(S)$ are integer-valued so we can conclude that $x(S) = y(S)$, so the two measures have the same mass. From this point onwards we assume that $d(x, y) < 1/2$ so that we can write

$$x = \sum_{i=1}^m \delta_{x_i}, \quad y = \sum_{i=1}^m \delta_{y_i}$$

where $m \in \mathbb{N}$ and the x_i and y_i are points on S .

Now we let ϵ_0 be the minimum of the non-zero inter-particle distances in x (inter-particle distances of zero occur if more than one point mass is placed at any point). So

$$\epsilon_0 = \min\{|x_i - x_j| : i, j = 1, \dots, m, |x_i - x_j| > 0\}$$

where $|x_i - x_j|$ is the smallest Euclidean distance between x_i and x_j measured around S , which is at most $1/2$. As there are only finitely many points of x this minimum ϵ_0 is strictly positive, so we can define $\epsilon_1 = \min\{\epsilon/2, \epsilon_0/4\} > 0$. By the construction of the family $\{f_n : n \in \mathbb{N}\}$ earlier, we see that for each point $x_i \in S$ of the measure x there is some function f_{n_i} which is 1 within an ϵ_1 -ball on x_i and is zero outside a $2\epsilon_1$ -ball on x_i . We let $n' = \max\{n_i\}$.

Suppose now that m_{x_i} point masses of the measure x lie on the point x_i of S . By the construction of ϵ_0 and ϵ_1 there are no other points of the measure x within a $2\epsilon_1$ -ball around x_i . Assume that m point masses of the measure y lie within a $2\epsilon_1$ -ball of x_i , with $m < m_{x_i}$. Then

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle x, f_n \rangle - \langle y, f_n \rangle| > \frac{1}{2^{n_i}} |\langle x, f_{n_i} \rangle - \langle y, f_{n_i} \rangle| > \frac{m_{x_i} - m}{2^{n_i}} > \frac{1}{2^{n_i}} > \frac{1}{2^{n'}}.$$

Thus provided $\delta < 1/2^{n'}$ we must have $m \geq m_{x_i}$, so that there are at least m_{x_i} point masses of y within a distance $2\epsilon_1$ of x_i . This holds for any point x_i of x , and by construction the $2\epsilon_1$ -balls around different points of x do not intersect. As the total mass of x and y are the same this is only possible if the number of point masses of y within a $2\epsilon_1$ -ball of x_i is exactly m_{x_i} for each i . It is then simple to form a correspondence between the m_{x_i} points of x at x_i and the equal number of points of y around them. The distance between corresponding points of x and y is at most $2\epsilon_1 < \epsilon$, so we are done. Finally we remark that we can now re-label the masses of y so that the point mass x_i of x corresponds to the point mass y_i of y . \square

With the above lemma in mind we define a *coupling mechanism* for two independent processes ξ_t and η_t started from initial configurations x and y respectively. This coupling mechanism is in fact rather strict and artificial but it does allow us to prove the Feller property. Firstly it is only possible for the two processes to couple if their initial conditions x and y have exactly the same mass. If this is the case then we may assume that the point masses of x and y are labelled $i = 1, \dots, m$ for some finite m . Now we allow the systems to evolve until the first time τ at which a branching or interaction occurs in either system. Up to this time each system consists of m labelled particles undergoing random Brownian motion. If the particles labelled i in each system occupy the same position on S at the same time then we allow them to 'stick together' and follow the same path. If this has happened to all m pairs of particles by time τ then the two systems are coupled; all future evolution of ξ_t and η_t is identical. If a branching or interaction occurs in either of the systems before all pairs are joined then no coupling occurs and the systems continue independently. It is clear that this coupling is dependent on the labelling of particles but we have in mind a correspondence of the type given in the lemma above.

With the coupling thus defined we have the following lemma:

Lemma 2.4.7. *Fix any $T > 0$ and let ξ_t and η_t be as described above, with $x \in F$ fixed. Then for any $\alpha > 0$ there exists $\delta > 0$ so that*

$$d(x, y) < \delta \implies P(\xi_t \text{ and } \eta_t \text{ coupled by time } T) > 1 - \alpha.$$

Proof. Fix $\epsilon > 0$. By lemma 2.4.6 there exists some $\delta > 0$ so that if $d(x, y) < \delta$, then both x and y have the same total mass, say m , and the i -th particle of y is within an ϵ -ball of the i -th particle of x for all i . Let $B_t^\xi(i)$, $i = 1, \dots, m$ and $B_t^\eta(i)$, $i = 1, \dots, m$ be the Brownian processes governing the motion of the initial particles. By this we mean that the particle labelled i in ξ_t (or η_t) follows path $B_t^\xi(i)$ (or $B_t^\eta(i)$) until the first time that any interaction or branching

removes it from the system. The advantage of introducing these underlying motion processes is that they are defined on the entire interval $[0, t]$, regardless of whether the corresponding particle is removed from the system at some earlier time.

Let A_t be the event that none of the initial particles in either of our processes branch before time t .

Let B_t be the event that no interactions occur in either of our processes before time t .

Let C_t be the event that for all $i \in \{1, \dots, m\}$, $B_t^\xi(i)$ and $B_t^\eta(i)$ 'meet' in the interval $[0, t]$. By this we mean that there is some time $s \in [0, t]$ such that $B_s^\xi(i)$ and $B_s^\eta(i)$ occupy the same point on their respective copies of S .

Clearly if there is no branching or interaction in either system, and all the motion processes of corresponding particles meet, then the two processes will become coupled. Thus letting E_t be the event that coupling occurs before time t , we have

$$P(E_t) \geq P(A_t \cap B_t \cap C_t).$$

Consequently by considering the complimentary event and using Boole's inequality, we write

$$P(E_t^c) \leq P((A_t \cap B_t \cap C_t)^c) = P(A_t^c \cup B_t^c \cup C_t^c) \leq P(A_t^c \cup B_t^c) + P(C_t^c). \quad (1)$$

Now let $l_t^{i,j}(\xi)$ and $l_t^{i,j}(\eta)$ be the local time by time t between the i -th and j -th particles in ξ_t and η_t respectively. Let l_t be the local time by time t of an independent generic pair of particles on S which start at the same point. Let $e_{i,j}^\xi$, $e_{i,j}^\eta$ and e be independent outcomes of an exponential random variable of mean 1. Then we have

$$\begin{aligned} P(B_t|A_t) &= P\left(\{l_t^{i,j}(\xi) < e_{i,j}^\xi\} \text{ and } \{l_t^{i,j}(\eta) < e_{i,j}^\eta\} \quad \forall i, j|A_t\right) \\ &= 1 - P\left(\{l_t^{i,j}(\xi) \geq e_{i,j}^\xi\} \text{ or } \{l_t^{i,j}(\eta) \geq e_{i,j}^\eta\} \text{ for any } i, j|A_t\right). \end{aligned}$$

Now using Boole's inequality and comparing each pair to a generic pair as described above we write

$$\begin{aligned} P\left(\{l_t^{i,j}(\xi) \geq e_{i,j}^\xi\} \text{ or } \{l_t^{i,j}(\eta) \geq e_{i,j}^\eta\} \text{ for any } i, j | A_t\right) \\ \leq \sum_{i < j} P\left(\{l_t^{i,j}(\xi) \geq e_{i,j}^\xi\} | A_t\right) + \sum_{i < j} P\left(\{l_t^{i,j}(\eta) \geq e_{i,j}^\eta\} | A_t\right) \\ \leq 2m(m-1)P(l_t \geq e) \end{aligned}$$

so that

$$P(B_t | A_t) \geq 1 - 2m(m-1)P(l_t \geq e). \quad (2)$$

Recalling that $P(l_t \geq e) = P(\sup_{0 \leq s \leq t} X_s \geq e)$ where X_t is a standard one-dimensional Brownian motion started at zero, we see that the right-hand side of (2) tends continuously to 1 as $t \rightarrow 0$.

So for any fixed $\alpha > 0$ we can choose $t_\alpha > 0$ small enough so that

- (i). $t_\alpha \leq T$.
- (ii). $1 - 2m(m-1)P(l_t \geq e) > (1 - \alpha/2)^{1/2}$.
- (iii). $(e^{-\lambda t_\alpha})^{2m} > (1 - \alpha/2)^{1/2}$.

From this we can write

$$\begin{aligned} P(A_{t_\alpha}^c \cup B_{t_\alpha}^c) &= 1 - P(A_{t_\alpha} \cap B_{t_\alpha}) = 1 - P(A_{t_\alpha})P(B_{t_\alpha} | A_{t_\alpha}) \\ &\leq 1 - (e^{-\lambda t_\alpha})^{2m} \{1 - 2m(m-1)P(l_{t_\alpha} \geq e)\} < \alpha/2. \end{aligned} \quad (3)$$

We know from lemma 2.3.2 that if V_t and W_t are Brownian particles started a distance ϵ apart, then $U_t = (V_t - W_t)/\sqrt{2}$ is a Brownian motion with $U_0 = \epsilon/\sqrt{2}$. Letting X_t be an independent standard Brownian motion, then clearly

$$\tau := \inf\{t > 0 : V_t = W_t\} = \inf\{t > 0 : U_t = 0\} \stackrel{D}{=} \inf\{t > 0 : X_t \geq \epsilon/\sqrt{2}\}.$$

Using the reflection principle,

$$P(\tau < t_\alpha) = 2P(X_{t_\alpha} \geq \epsilon/\sqrt{2}) = 2 \int_{\epsilon/\sqrt{2}}^{\infty} \frac{1}{\sqrt{2\pi t_\alpha}} e^{-x^2/2t_\alpha} dx.$$

Thus given this value $t_\alpha > 0$ we can choose $\epsilon > 0$ small enough so that

$$2 \int_{\epsilon/\sqrt{2}}^{\infty} \frac{1}{\sqrt{2\pi t_\alpha}} e^{-x^2/2t_\alpha} dx > (1 - \alpha/2)^{1/m}. \quad (4)$$

As mentioned at the beginning of this proof there is some $\delta > 0$ corresponding to this ϵ in the sense of lemma 2.4.6, and we now assume $d(x, y) < \delta$.

We can now use (4) to bound $P(C_{t_\alpha})$. C_{t_α} is the event that m pairs of particles meet in the interval $[0, t_\alpha]$, when the two particles in each pair start a distance at most ϵ apart. We compare this to m pairs started exactly ϵ apart, and use the independence of the movement processes and (4) to write

$$P(C_{t_\alpha}) \geq \left(2 \int_{\epsilon/\sqrt{2}}^{\infty} \frac{1}{\sqrt{2\pi t_\alpha}} e^{-x^2/2t_\alpha} dx \right)^m > ((1 - \alpha/2)^{1/m})^m = 1 - \alpha/2,$$

so that $P(C_{t_\alpha}^c) < \alpha/2$. Using this along with expression (3) and the fact that $t_\alpha \leq T$ gives

$$P(E_T^c) \leq P(E_{t_\alpha}^c) \leq P(A_{t_\alpha}^c \cup B_{t_\alpha}^c) + P(C_{t_\alpha}^c) < \alpha/2 + \alpha/2 = \alpha.$$

Thus we have shown that for any fixed $T > 0$ and for any $\alpha > 0$, we can choose $\epsilon > 0$, and hence $\delta > 0$, so that

$$d(x, y) < \delta \implies P(\xi_t \text{ and } \eta_t \text{ coupled by time } T) > 1 - \alpha.$$

□

This coupling lemma yields the following simple proof of the Feller property for these spatially interacting branching processes on S .

Proposition 2.4.8. *The Feller Property:*

If $f : F \rightarrow \mathbb{R}$ is bounded and continuous on F , then so is the map $x \mapsto \mathbb{E}_x(f(\xi_t))$.

Proof. Fix some such bounded, continuous f and let $M \in \mathbb{R}$ be such that $|f(x)| \leq M \quad \forall x \in F$. We have

$$|\mathbb{E}_x(f(\xi_t))| = \left| \int_F f(y) P(\xi_t \in dy | \xi_0 = x) \right| \leq \int_F |f(y)| P(\xi_t \in dy | \xi_0 = x) \leq M$$

so this map is certainly bounded.

Now fix $x \in F$ and choose some $\epsilon > 0$. As in the coupling arguments above let ξ_t and η_t be versions of the process started from $\xi_0 = x$ and $\eta_0 = y$ respectively, where y is some other element of F . By lemma 2.4.7 there exists some $\delta > 0$ such that if $d(x, y) < \delta$, then the probability that the two processes have coupled by time t is greater than $1 - \frac{\epsilon}{M}$. Again let E_t be the event that such coupling occurs. Then

$$\begin{aligned} \mathbb{E}(f(\eta_t)) &= \mathbb{E}(f(\eta_t)|E_t)P(E_t) + \mathbb{E}(f(\eta_t)|E_t^c)P(E_t^c) \\ &\leq \mathbb{E}(f(\xi_t)) + \mathbb{E}(f(\eta_t)|E_t^c)\frac{\epsilon}{M} \\ &\leq \mathbb{E}(f(\xi_t)) + M\frac{\epsilon}{M} \\ &= \mathbb{E}(f(\xi_t)) + \epsilon. \end{aligned}$$

Thus by symmetry it follows that $|\mathbb{E}(f(\xi_t)) - \mathbb{E}(f(\eta_t))| \leq \epsilon$. We have shown that for any fixed x

$$\forall \epsilon > 0 \quad \exists \delta > 0 \text{ such that } d(x, y) < \delta \implies |\mathbb{E}_x(f(\xi_t)) - \mathbb{E}_y(f(\xi_t))| \leq \epsilon$$

and so this map is continuous at $x \in F$. The point x was an arbitrary point of F so the map is continuous on F . \square

2.4.3 An Existence Theorem for Stationary Distributions

To finish this section we prove a theorem which tells us that, for each initial condition $x \in F$, we can use Cesaro averages to construct a corresponding stationary distribution for the process. In many cases this distribution will be δ_0 , the unit mass on the empty set. This is always a stationary distribution for these interacting branching processes as there is no immigration of particles from outside the system. In some cases however, the limiting stationary distribution is

non-trivial. A full classification of the stationary distributions will be given in the final section of this chapter, but firstly we must prove our existence theorem.

Let $\mathcal{M}_1(E)$ be the set of all probability measures on the space $E = F \cup \{\infty_w\}$, together with the topology of weak convergence. Now assume some initial condition $x \in F$ for the process so that $\xi_0 = x$. Define the probability measures $\{\mu_t^x : t \geq 0\} \subset \mathcal{M}_1(E)$ by

$$\mu_t^x(A) = P(\xi_t \in A | \xi_0 = x), \quad \forall A \subset E.$$

Now let

$$\nu_n^x = \frac{1}{n} \int_0^n \mu_s^x ds$$

be the Cesaro averages of the μ_s^x up to time $n \in \mathbb{N}$. This gives a sequence $\{\nu_n^x : n \in \mathbb{N}\}$ of measures in $\mathcal{M}_1(E)$.

Theorem 2.4.9. *For each $x \in F$ there is a convergent subsequence $\{\nu_{n'}^x\}$ of these Cesaro averages, such that*

$$\nu_{n'}^x \implies \nu^x \quad \text{weakly in } \mathcal{M}_1(E) \quad \text{as } n' \rightarrow \infty.$$

The limit $\nu^x \in \mathcal{M}_1(F)$ and is a stationary distribution for the process.

Proof. By corollary 2.4.4 we know that $F \cup \{\infty_w\}$ is compact and metrisable. Thus from theorem 2.4.1 any sequence in $\mathcal{M}_1(E)$ has a convergent subsequence. Let $\{\nu_{n'}^x\}$ be such a subsequence for these Cesaro averages and denote the limit of this subsequence by ν^x .

To complete the proof we require the following two facts:

- (i). ν^x gives no mass to $\{\infty_w\}$ and so can be considered as a probability on F .
- (ii). ν^x is stationary under the dynamics of the process.

The proofs of these will be the subject of the following lemmas. □

Lemma 2.4.10. *The measure ν^x given above is supported on $F \subset E$, so gives no mass to $\{\infty_w\}$.*

Proof. We begin by defining some subsets of F (and hence of E). For $n \in \mathbb{N}$ let

$$F_n = \left\{ \sum_{i=1}^n \delta_{x_i} : x_1, \dots, x_n \in S \right\}$$

be the set of measures consisting of exactly n point masses (or equivalently configurations of the process containing exactly n particles). The sets F_n are both open and closed in F , since as seen in the proof of lemma 2.4.6, measures of different masses are at least a distance $1/2$ apart under the metric corresponding to weak convergence. From the nature of the one-point compactification it follows that these sets are open and closed in $E = F \cup \{\infty_w\}$ also.

Define the map $I_k : E = F \cup \{\infty_w\} \rightarrow \mathbb{R}$ by

$$I_k(x) = \begin{cases} 0 & \text{if } x = \sum_{i=1}^n \delta_{x_i} \text{ with } n \leq k \\ 1 & \text{if } x = \sum_{i=1}^n \delta_{x_i} \text{ with } n > k \\ 1 & \text{if } x = \infty_w. \end{cases}$$

This map is clearly bounded on E . We show that it is also continuous on E by proving that for any open $U \subset \mathbb{R}$ then $I_k^{-1}(U)$ is open in E .

If $\{0, 1\} \subset U$ then $I_k^{-1}(U) = E$, which is open in E .

If $0 \in U, 1 \notin U$ then $I_k^{-1}(U) = \bigcup_{n=1}^k F_n$, which is open in E .

If $0 \notin U, 1 \in U$ then $I_k^{-1}(U) = \{\infty_w\} \cup \bigcup_{n>k} F_n = (\bigcup_{n=1}^k F_n)^c$, which is open in E .

If $0 \notin U, 1 \notin U$ then $I_k^{-1}(U) = \emptyset$, which is open in E .

As I_k is both bounded and continuous it follows from the definition of weak convergence in $\mathcal{M}_1(E)$ that

$$\int_E I_k(y) \nu_n^x(dy) \rightarrow \int_E I_k(y) \nu^x(dy).$$

Letting $W_k = \{\infty_w\} \cup \bigcup_{n>k} F_n$, this can be re-written as

$$\nu_{n'}^x(W_k) \rightarrow \nu^x(W_k).$$

Now we note that

$$\begin{aligned} \nu_{n'}^x(W_k) &= \frac{1}{n'} \int_0^{n'} \mu_s^x(W_k) ds \\ &= \frac{1}{n'} \int_0^{n'} P(\xi_s \in W_k | \xi_0 = x) ds \\ &= \frac{1}{n'} \int_0^{n'} P(N_s \geq k+1 | \xi_0 = x) ds \\ &\leq \frac{1}{n'} \int_0^{n'} \sup_{s \geq 0} P(N_s \geq k+1 | \xi_0 = x) ds \\ &= \sup_{s \geq 0} P(N_s \geq k+1 | \xi_0 = x). \end{aligned}$$

This holds for any n' and as $\nu_{n'}^x(W_k) \rightarrow \nu^x(W_k)$ in \mathbb{R} it will also hold for the limit, so we can conclude that

$$\nu^x(W_k) \leq \sup_{s \geq 0} P(N_s \geq k+1 | \xi_0 = x).$$

Using proposition 2.3.10 to control the right-hand side of this expression, we see that

$$\nu^x(W_k) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

and therefore ν^x gives no mass to the point ∞_w as required. \square

Before finalising the proof of theorem 2.4.9 by showing that ν^x is stationary, it is necessary to develop some general topological theory concerning convergence of probability measures. As this theory will not only be useful here, but also in later chapters, we formulate it in an abstract setting.

Let S be a metric space, and let R be some Borel subset of S . Suppose that

$$\mu_n \implies \mu \quad \text{in } \mathcal{M}_1(S),$$

the space of probability measures on S with the topology of weak convergence. Further suppose that the measures $\{\mu_n, n \in \mathbb{N}\}$ and μ are supported on R , so that none of these measures give any mass to $S \setminus R$. We give a sufficient condition on R and S under which this implies that μ_n converges weakly to μ in the space of probability measures on R , $\mathcal{M}_1(R)$. Our work depends on the following result which can be found in Ethier and Kurtz [20]:

Lemma 2.4.11. *If X is a separable metric space, then the set of bounded, continuous functions from X to \mathbb{R} which are uniformly continuous is convergence determining in the space of probability measures on X .*

Here when we say that a collection $\{f : f \in V\}$ of bounded, continuous functions is convergence determining, we mean that $\mu_n \implies \mu$ if and only if $\langle \mu_n, f \rangle \rightarrow \langle f, \mu \rangle$ for all $f \in V$. We prove one more topological lemma before stating our general result.

Lemma 2.4.12. *Let X be a metric space with subspace $A \subset X$. Then any bounded, uniformly continuous function $f : A \rightarrow \mathbb{R}$ can be extended to a bounded, continuous function \tilde{f} on $cl(A)$, the closure of A in X .*

Proof. Suppose $x \in cl(A)$, so there exists some sequence $\{x_n\}$ in A such that $x_n \rightarrow x$ as $n \rightarrow \infty$. We set $\tilde{f}(x) = \lim_{n \rightarrow \infty} f(x_n)$ and show that this function is well-defined.

We begin by showing that this limit exists. Fix some $\epsilon > 0$. Then by the uniform continuity of f on A there is some $\delta > 0$ such that

$$d(a_1, a_2) < \delta \implies |f(a_1) - f(a_2)| < \epsilon \quad \forall a_1, a_2 \in A. \quad (1)$$

As $x_n \rightarrow x$ there is some n_0 such that for $n \geq n_0$, $d(x_n, x) < \delta/2$. Therefore

$$n_1, n_2 \geq n_0 \implies d(x_{n_1}, x_{n_2}) \leq d(x_{n_1}, x) + d(x, x_{n_2}) < \delta \implies |f(x_{n_1}) - f(x_{n_2})| < \epsilon.$$

So the sequence $\{f(x_n)\}$ forms a Cauchy sequence in \mathbb{R} and so must converge to a limit.

Now we show that this limit does not depend on which converging sequence $\{x_n\}$ is chosen. Let $\{y_n\}$ be a further sequence in A which converges to x . Clearly there is some n_1 such that

$$n \geq n_1 \implies d(x_n, x) < \delta/2 \text{ and } d(y_n, x) < \delta/2$$

with δ as in (1) above. In a similar way to before it then follows that for any $n \geq n_1$ we have $|f(x_n) - f(y_n)| < \epsilon$. Consequently $|f(x_n) - f(y_n)| \rightarrow 0$ as $n \rightarrow \infty$ and so $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n)$.

To show continuity we suppose that $x_n \rightarrow x$ in $cl(A)$ and fix $\epsilon > 0$. Again using the uniform continuity of f on A we choose $\delta > 0$ small enough so that

$$d(x', y') < \delta \implies |f(x') - f(y')| < \epsilon/3 \quad \forall x', y' \in A. \quad (2)$$

By the definition of \tilde{f} we can choose $x' \in A$ so that $d(x, x') < \delta/3$ and $|\tilde{f}(x) - f(x')| < \epsilon/3$. Now we choose $n_0 \in \mathbb{N}$ large enough so that $d(x, x_n) < \delta/3$ for all $n \geq n_0$. For such n we write $\tilde{f}(x_n) = \lim_{m \rightarrow \infty} f(x_n(m))$ for some sequence $\{x_n(m) : m \in \mathbb{N}\}$ in A which converges to x_n . For sufficiently large m we have $d(x_n(m), x_n) < \delta/3$ and so

$$d(x', x_n(m)) \leq d(x', x) + d(x, x_n) + d(x_n, x_n(m)) < \delta/3 + \delta/3 + \delta/3 = \delta.$$

Using (2) this then gives

$$|\tilde{f}(x) - f(x_n(m))| \leq |\tilde{f}(x) - f(x')| + |f(x') - f(x_n(m))| < \epsilon/3 + \epsilon/3 = 2\epsilon/3,$$

so that in particular

$$|\tilde{f}(x) - \tilde{f}(x_n)| = |\tilde{f}(x) - \lim_{m \rightarrow \infty} f(x_n(m))| < \epsilon.$$

Thus for all $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ so that $|\tilde{f}(x) - \tilde{f}(x_n)| < \epsilon$ for all $n \geq n_0$. This shows that $\tilde{f}(x) = \lim_{n \rightarrow \infty} \tilde{f}(x_n)$ and gives the continuity required. The boundedness of \tilde{f} follows directly from the boundedness of f . \square

Proposition 2.4.13. *Let R be a Borel subset of the metric space S . Suppose the sequence μ_n converges to μ in $\mathcal{M}_1(S)$ and that none of these measures give any mass to the set $S \setminus R$. Then if R is separable in S it follows that μ_n converges to μ in $\mathcal{M}_1(R)$.*

Proof. As R is separable it follows from lemma 2.4.11 that to show convergence of μ_n to μ in $\mathcal{M}_1(R)$ it suffices to show

$$\int_R f(x) \mu_n(dx) \rightarrow \int_R f(x) \mu(dx)$$

for all bounded, uniformly continuous functions $f : R \rightarrow \mathbb{R}$. Let f be such a function. By lemma 2.4.12 above we can extend f to a bounded and continuous function \tilde{f} on $cl(R)$. Tietze's extension theorem, which can be found in [9] for example, states that any continuous function on a closed subset Y of a metric space X has a continuous extension on all of X . Further, if the original function is bounded then the extension can be bounded also. We let g be such a bounded continuous extension of \tilde{f} from $cl(R)$ onto the whole of S . Thus g is a bounded continuous extension of f from R to S . Now using the convergence of μ_n to μ in $\mathcal{M}_1(S)$ and the fact that these measures are supported on R we have

$$\int_R f(x) \mu_n(dx) = \int_S g(x) \mu_n(dx) \rightarrow \int_S g(x) \mu(dx) = \int_R f(x) \mu(dx).$$

This gives convergence in $\mathcal{M}_1(R)$ as desired. □

With this valuable abstract theory in place we return to our particle processes. Clearly we wish to apply the above work to our specific example and to do this we must show that the space F is separable.

Lemma 2.4.14. *F is separable as a subspace of $F \cup \{\infty_w\}$.*

Proof. As in the proof of lemma 2.4.10 we let F_n be the subset of F containing all those measures which consist of exactly n point masses. It is clear that F_1

is separable since we consider the set of measures of the form δ_x where x is a rational point on S (again identifying S with the interval $[0, 1]$). Any other point mass measure defined on S can be approximated by a sequence of measures of this form.

Now suppose it is known that F_n is separable for $n \leq k$, with D_n , $n = 1, \dots, k$ being the corresponding countable dense sets in F_n . Let x be some measure from the set F_{k+1} , so we can write

$$x = \sum_{i=0}^{k+1} \delta_{x_i} = \sum_{i=0}^k \delta_{x_i} + \delta_{x_{k+1}} = y + z$$

where $y \in F_k$ and $z \in F_1$. Let $\{y_n\} \subset D_k$ and $\{z_n\} \subset D_1$ be sequences converging weakly to y and z respectively, and set $x_n = y_n + z_n$. For any bounded continuous $f : S \rightarrow \mathbb{R}$ we have

$$\begin{aligned} \int_S f(s)x_n(ds) &= \int_S f(s)y_n(ds) + \int_S f(s)z_n(ds) \\ &\longrightarrow \int_S f(s)y(ds) + \int_S f(s)z(ds) = \int_S f(s)x(ds) \end{aligned}$$

so x_n converges weakly to x . It follows that the set $\{y + z : y \in D_k, z \in D_1\}$ is dense in F_{k+1} and is certainly countable, so that F_{k+1} is separable also.

In this way we have by induction that F_n is separable for each n . As F is the union of the sets F_n it is separable also. \square

We now conclude the proof of theorem 2.4.9 by showing that the limiting measure ν^x is stationary. The method used below, sometimes called the Krylov-Bogoliubov method, is a standard technique used to show the existence of stationary distributions.

Lemma 2.4.15. *The limiting measure ν^x which arises in 2.4.9 is stationary.*

Proof. For any $\mu \in \mathcal{M}_1(F)$ let $T_t\mu \in \mathcal{M}_1(F)$ denote the law at time t of the process started according to μ , so $T_t\mu(A) = P(\xi_t \in A | \xi_0 \sim \mu)$ for all measurable $A \subset F$. The measure μ is stationary if $T_t\mu = \mu$ for all $t \geq 0$, and so to show stationarity it suffices to show that

$$T_t\mu(f) = \mu(f) \quad \forall \text{ bounded, continuous } f : F \rightarrow \mathbb{R}.$$

It is already known that the measures ν_n^x converge weakly to ν^x in $\mathcal{M}_1(E)$, the space of probability measures on the set $E = F \cup \{\infty_w\}$. Since the process remains finite almost surely when started from any $x \in F$, the measures ν_n^x give no mass to ∞_w . Nor, by lemma 2.4.10, does the limiting measure ν^x . By lemma 2.4.15 above F is separable in E and thus we may now apply proposition 2.4.13 to see that

$$\nu_n^x \implies \nu^x \quad \text{in } \mathcal{M}_1(F). \quad (1)$$

It was shown in section 2.4.2 that the Feller property holds for this process on F . So if $f : F \rightarrow \mathbb{R}$ is bounded and continuous then so is the map $x \mapsto \mathbb{E}_x(f(\xi_t))$. Combining this with (1) above yields

$$\int_F \mathbb{E}_x(f(\xi_t)) \nu_n^x(dx) \longrightarrow \int_F \mathbb{E}_x(f(\xi_t)) \nu^x(dx) \quad (2)$$

for any such bounded and continuous f .

Now we write

$$T_t\mu(f) = \int_F f(x) T_t\mu(dx) = \int_F \mathbb{E}_x(f(\xi_t)) \mu(dx)$$

so that in the case of the limiting measure ν^x we use (2) to give

$$T_t\nu^x(f) = \int_F \mathbb{E}_x(f(\xi_t)) \nu^x(dx) \stackrel{(2)}{=} \lim_{n' \rightarrow \infty} \int_F \mathbb{E}_x(f(\xi_t)) \nu_{n'}^x(dx) = \lim_{n' \rightarrow \infty} T_t\nu_{n'}^x(f).$$

Additionally we have

$$T_t\nu_{n'}^x(f) = T_t\left(\frac{1}{n'} \int_0^{n'} \mu_s^x ds\right)(f) = \left(\frac{1}{n'} \int_0^{n'} T_t\mu_s^x ds\right)(f) = \left(\frac{1}{n'} \int_0^{n'} \mu_{t+s}^x ds\right)(f)$$

where the final equality follows from the Markov property of the processes. Putting these steps together we have

$$\begin{aligned}
 T_t \nu^x(f) &= \lim_{n' \rightarrow \infty} T_t \nu_{n'}^x(f) \\
 &= \lim_{n' \rightarrow \infty} \left(\frac{1}{n'} \int_0^{n'} \mu_{i+s}^x ds \right) (f) \\
 &= \lim_{n' \rightarrow \infty} \left(\frac{1}{n'} \int_0^{n'} \mu_s^x(f) ds + \frac{1}{n'} \int_{n'}^{n'+t} \mu_s^x(f) ds - \frac{1}{n'} \int_0^t \mu_s^x(f) ds \right) \\
 &= \lim_{n' \rightarrow \infty} \left(\frac{1}{n'} \int_0^{n'} \mu_s^x(f) ds \right) \\
 &= \lim_{n' \rightarrow \infty} \nu_{n'}^x(f) = \nu^x(f).
 \end{aligned}$$

Thus we have shown that the measure ν^x is stationary, which completes this lemma and so concludes the proof of theorem 2.4.9. \square

2.5 Stationary Distributions on S: Classification

The aim of this section is to identify and classify the stationary distributions for these processes. The form of these distributions, and whether or not they are unique, depends directly on the branching and interaction probabilities which define the processes. In the case of our main model, where $\beta > 1$ and $\mu < 2$ we can summarise the main results as follows:

- (i). If it is possible that the process can die out from any finite initial condition, then this will happen exponentially quickly. This gives a single, unique stationary measure, δ_0 .
- (ii). If there exists a set of finite initial conditions from which extinction is impossible then there is a single stationary measure supported on this set. All

other initial states lead to extinction. Convergence to stationarity will occur exponentially quickly in either case. This gives two extremal stationary measures, one of which is δ_0 .

2.5.1 The Recurrence of ‘Small’ Populations

The intention here is to show that the system will regularly return to configurations in which the population size is ‘low’. The result we prove is actually stronger than this. We show that there is some fixed $T > 0$ and some $n'_\epsilon \in \mathbb{N}$ so that, whatever the initial population $N_0 \in \mathbb{N}$, the expected population $\mathbb{E}(N_t)$ is less than n'_ϵ for some $t \leq T$ later. This fact, together with a simple coupling argument, will be used to show the convergence statements above.

We begin by proving a simple variation on theorem 2.3.4. This new result is not a strengthening of the original theorem, but rather a trade-off between two of its features. The time $t_n = k/n$ is replaced by the shorter time $t_n^\epsilon = k/n^{1+\epsilon}$, $\epsilon > 0$, but the reduction proportion, which was the constant α , is now α_n^ϵ which is dependent on n .

Proposition 2.5.1. *Suppose that n particles are somehow distributed on S at time $t = 0$ and fix $\epsilon \in (0, 1)$. Then there exists some $n_\epsilon \in \mathbb{N}$ and $\tilde{\alpha} \in (0, 1)$ so that, with $k > 0$ given in theorem 2.3.4, we have*

$$n \geq n_\epsilon \quad \implies \quad \mathbb{E}(N_{t_n^\epsilon}) \leq n \left(1 - \frac{\tilde{\alpha}}{n^\epsilon} \right) =: \alpha_n^\epsilon n$$

for $t_n^\epsilon = k/n^{1+\epsilon}$.

Proof. In the proof of theorem 2.3.4, the time interval $[0, t_n]$ was divided into n subintervals of length k/n^2 . Here the time interval $[0, t_n^\epsilon]$ is again divided into subintervals of this length, but can contain only $\lfloor n^{1-\epsilon} \rfloor$ such intervals. These

subintervals may not exactly cover the interval $[0, t_n^\epsilon]$, so there may be some remainder, a time interval of length $t_0 < k/n^2$. We allow the system to evolve for this small time t_0 first and then divide the remaining time into our $[n^{1-\epsilon}]$ subintervals. We stipulate that $n_\epsilon > n_0$, with n_0 given in theorem 2.3.4, so that if $\mathbb{E}(N_t) \leq 3n/4$ at the start of any of these intervals, then as in the proof of 2.3.4

$$\mathbb{E}(N_{t_n^\epsilon}) \leq 7n/8.$$

We now assume that this is not the case, so $\mathbb{E}(N_t) > 3n/4$ at the beginning of each subinterval. Adapting the expression marked (***) from our earlier proof to bound using exponential growth for the small time t_0 , we have

$$\begin{aligned} \mathbb{E}(N_{t_n^\epsilon}) &= \mathbb{E}(\mathbb{E}(N_{t_n^\epsilon} | N_{t_0})) = \mathbb{E}(N_{t_0} + \sum_{i=0}^{[n^{1-\epsilon}]-1} \mathbb{E}(\Delta_i^B + \Delta_i^I | N_{t_0})) \\ &\leq ne^{\lambda(\beta-1)k/n^2} + \sum_{i=0}^{[n^{1-\epsilon}]-1} \mathbb{E}(\Delta_i^B + \Delta_i^I) \end{aligned}$$

where Δ_i^B and Δ_i^I are the change in population due to branching and interaction over the i -th subinterval as before. As the subintervals are the same length as before, the upper bound for this population change per interval shown in our original proof still holds, so that

$$\mathbb{E}(N_{t_n^\epsilon}) \leq ne^{\lambda(\beta-1)k/n^2} - [n^{1-\epsilon}]\tilde{p} = n(e^{\lambda(\beta-1)k/n^2} - [n^{1-\epsilon}]\tilde{p}/n)$$

with \tilde{p} a fixed and strictly positive constant. Now assuming that n_ϵ , and hence n , is large enough so that

$$[n^{1-\epsilon}] \geq n^{1-\epsilon}/2 \quad \text{and} \quad e^{\lambda(\beta-1)k/n^2} \leq 1 + \tilde{p}/4n^\epsilon$$

then we have

$$\mathbb{E}(N_{t_n^\epsilon}) \leq n(e^{\lambda(\beta-1)k/n^2} - [n^{1-\epsilon}]\tilde{p}/n) \leq n(e^{\lambda(\beta-1)k/n^2} - \tilde{p}/2n^\epsilon) \leq n(1 - \tilde{p}/4n^\epsilon).$$

Letting $\tilde{\alpha} = \tilde{p}/4 > 0$ and combining the two cases gives

$$\mathbb{E}(N_{t_n^\epsilon}) \leq n \max\left\{7/8, \left(1 - \frac{\tilde{\alpha}}{n^\epsilon}\right)\right\}.$$

Choosing n_ϵ large enough so that $\tilde{\alpha}/n^\epsilon < 1/8$ for any $n \geq n_\epsilon$ gives the stated result. \square

Now we define a sequence of time points $\{t_n : n \in \mathbb{N}\}$ for the process, dependent on the initial configuration ξ_0 . This gives a corresponding sequence, $\{X_n : n \in \mathbb{N}\}$, of expected populations at these times. Recursively we have:

$$\begin{aligned} t_0 &= 0, & X_0 &= N_0 \\ t_1 &= \frac{k}{N_0^{1+\epsilon}}, & X_1 &= \mathbb{E}(N_{t_1}) \\ t_2 &= t_1 + \frac{k}{\mathbb{E}(N_{t_1})^{1+\epsilon}} = t_1 + \frac{k}{X_1^{1+\epsilon}}, & X_2 &= \mathbb{E}(N_{t_2}) \\ & & & \vdots \\ t_{n+1} &= t_n + \frac{k}{\mathbb{E}(N_{t_n})^{1+\epsilon}} = t_n + \frac{k}{X_n^{1+\epsilon}}, & X_{n+1} &= \mathbb{E}(N_{t_{n+1}}) \\ & & & \vdots \end{aligned}$$

Notice that these time points have been defined in keeping with the form of t_n^ϵ given in proposition 2.5.1 above, although the expected populations may not be integers. The following lemma is in the spirit of proposition 2.5.1, but is given in terms of this deterministic sequence $\{X_n : n \in \mathbb{N}\}$.

Lemma 2.5.2. *If $X_n \geq n_\epsilon$, then*

$$X_{n+1} \leq X_n \left(1 - \frac{\tilde{\alpha}}{X_n^\epsilon}\right)$$

with $\tilde{\alpha}$ and n_ϵ given in proposition 2.5.1.

Proof. We take the time interval $[t_n, t_{n+1}]$ of length $k/X_n^{1+\epsilon}$ and divide it into $[X_n^{1-\epsilon}]$ steps of length k/X_n^2 . These subintervals are placed consecutively at the end of the interval $[t_n, t_{n+1}]$, leaving perhaps a short length of time $< k/X_n^2$ at

the beginning. Let N_i be the population of the process at the start of the i -th subinterval, and suppose that $\mathbb{E}(N_i) < 3X_n/4$ for some i . Then since $X_n \geq n_\epsilon \geq n_0$ we have

$$X_{n+1} = \mathbb{E}(N_{t_{n+1}}) = \mathbb{E}(\mathbb{E}(N_{t_{n+1}}|N_i)) \leq \frac{3X_n}{4} e^{\lambda(\beta-1)k/X_n} \leq \frac{7X_n}{8}.$$

Now we assume that $\mathbb{E}(N_i) \geq 3X_n/4$ for each i . In the proof of our main boundedness theorem on S , theorem 2.3.4, it was shown that if $\mathbb{E}(N_i) \geq 3m/4$ with $m \geq n_0$, then

$$\mathbb{E}(\Delta_i^B + \Delta_i^I) \leq -\tilde{p} \quad \text{for some fixed } \tilde{p} > 0,$$

where $\Delta_i^B + \Delta_i^I$ represents the change in population in an interval of length k/m^2 . Using this fact, together with the methodology of the proof of proposition 2.5.1, allows us to write

$$\begin{aligned} X_{n+1} &= \mathbb{E}(\mathbb{E}(N_{t_{n+1}}|N_{t_n})) \leq \mathbb{E}\left(N_{t_n} e^{\lambda(\beta-1)k/X_n^2} + \sum_{i=0}^{[X_n^{1-\epsilon}] - 1} \mathbb{E}(\Delta_i^B + \Delta_i^I|N_{t_n})\right) \\ &= \mathbb{E}(N_{t_n}) e^{\lambda(\beta-1)k/X_n^2} + \sum_{i=0}^{[X_n^{1-\epsilon}] - 1} \mathbb{E}(\Delta_i^B + \Delta_i^I) \\ &\leq X_n e^{\lambda(\beta-1)k/X_n^2} - [X_n^{1-\epsilon}] \tilde{p}. \end{aligned}$$

Now since $X_n \geq n_\epsilon$ we know that

$$[X_n^{1-\epsilon}] \geq X_n^{1-\epsilon}/2 \quad \text{and} \quad e^{\lambda(\beta-1)k/X_n^2} \leq 1 + \tilde{p}/4X_n^\epsilon.$$

Hence it follows that

$$X_{n+1} \leq X_n e^{\lambda(\beta-1)k/X_n^2} - [X_n^{1-\epsilon}] \tilde{p} \leq X_n (e^{\lambda(\beta-1)k/X_n^2} - \tilde{p}/2X_n^\epsilon) \leq X_n (1 - \tilde{p}/4X_n^\epsilon)$$

so letting $\tilde{\alpha} = \tilde{p}/4$ as before gives

$$X_{n+1} \leq X_n \max\left\{7/8, \left(1 - \frac{\tilde{\alpha}}{X_n^\epsilon}\right)\right\} = X_n \left(1 - \frac{\tilde{\alpha}}{X_n^\epsilon}\right)$$

since $\tilde{\alpha}/X_n^\epsilon < 1/8$ for $X_n \geq n_\epsilon$. □

Corollary 2.5.3. *There exists $n'_\epsilon \in \mathbb{N}$ so that if $X_n > n'_\epsilon$ then $X_{n+1} \leq X_n - 1$.*

Proof. Choose $n'_\epsilon \geq n_\epsilon$ large enough so that $\tilde{\alpha}n^{1-\epsilon}$ is greater than 1 for any $n \geq n'_\epsilon$. Now applying lemma 2.5.2 for $X_n \geq n'_\epsilon$ we have

$$X_{n+1} \leq X_n \left(1 - \frac{\tilde{\alpha}}{X_n^\epsilon}\right) = X_n - \tilde{\alpha}X_n^{1-\epsilon} \leq X_n - 1.$$

□

Thus we have a sequence of numbers $\{X_n : n \in \mathbb{N}\}$, with $X_0 = N_0$, which decrease by at least 1 at each step until they fall below n'_ϵ . So in the case where $N_0 \geq n'_\epsilon$, we certainly have that the sequence drops below n'_ϵ before the $(N_0 - n'_\epsilon + 1)$ -th term.

Lemma 2.5.4. *Suppose that $X_0 = N_0 \geq n'_\epsilon$ and let t_m be the first time point at which the sequence $X_n = \mathbb{E}(N_{t_n})$ is less than n'_ϵ . Then we have*

$$t_m \leq \sum_{r=0}^{\infty} \frac{k}{r^{1+\epsilon}} = T.$$

Proof. Recalling the definition of the time steps for the sequence, we have

$$t_m = \frac{k}{N_0^{1+\epsilon}} + \frac{k}{\mathbb{E}(N_{t_1})^{1+\epsilon}} + \cdots + \frac{k}{\mathbb{E}(N_{t_{m-1}})^{1+\epsilon}}.$$

For $i \in \{0, \dots, m-1\}$ let r_i be the largest integer which is smaller than $\mathbb{E}(N_{t_i})$. By lemma 2.5.2 and the fact that m is the first integer for which $X_i < n'_\epsilon$, we have

$$|X_i - X_{i+1}| \geq 1 \quad \forall i \in \{0, \dots, m-1\}$$

and so it follows that $r_i \neq r_j$ for $i \neq j$. Thus we have

$$t_m \leq \frac{k}{r_0^{1+\epsilon}} + \frac{k}{r_1^{1+\epsilon}} + \cdots + \frac{k}{r_{m-1}^{1+\epsilon}} \leq \sum_{r=1}^{\infty} \frac{k}{r^{1+\epsilon}} = T.$$

□

To conclude this section we notice that

$$T = \sum_{r=1}^{\infty} \frac{k}{r^{1+\epsilon}} < \infty.$$

It is for this reason that the effort was made to prove proposition 2.5.1 and lemma 2.5.2. The time intervals in the original version of our boundedness theorem are of the form k/r and using such intervals would make the above sum non-finite. Weakening theorem 2.3.4 in one respect, so that the population reduction was no longer proportional to n , allowed us to use shorter time intervals so that the time T is finite. Thus whatever the value of $N_0 \in \mathbb{N}$, we have $\mathbb{E}(N_t) < n'_\epsilon$ for some $t \leq T$. We can now use this recurrence of small populations as our main tool in examining the stationary distributions of these processes. Before doing that we state a final corollary; this is a nice result which we get for free by combining the above with elements of our previous work.

Corollary 2.5.5. *There exist finite constants C and T so that for any finite N_0 , we have*

$$\mathbb{E}(N_t) \leq C \quad \text{for all } t \geq T.$$

Proof. From the above we know that the expected population will fall below n'_ϵ in the interval $[0, T]$. Now for all t after this time, we can use precisely the same logic as in the proof of proposition 2.3.10 to show that

$$\mathbb{E}(N_t) \leq \max \{ n'_\epsilon e^{\lambda(\beta-1)\tau}, M e^{2\lambda(\beta-1)\tau} \} =: C,$$

where τ and M are finite constants as defined in expression (1) of that proof. \square

2.5.2 The Stationary Distributions: Extinction

Before proceeding it is necessary to introduce an additional notion for these systems, namely that of *parity*. Quite simply this is whether the number of

particles in the system is odd or even at any given time. We say that our model has *fixed parity* if

$$q_n = 0 \quad \forall \text{ odd } n \quad \text{and} \quad p_n = 0 \quad \forall \text{ even } n$$

otherwise we say it has *changeable parity*. It is easy to see that in a *fixed parity* system, where a pairwise interaction cannot produce an odd number of offspring and a single particle branching cannot result in an even number of offspring, the parity will not change. If N_0 is odd, then N_t will be odd at all future times, and similarly if N_0 is even then N_t will be even also. This is not true in a *changeable parity* model. This notion will play a vital role in determining the forms of the stationary distributions for the processes, and the corresponding domains of attraction.

We remark that the following results refer to our main model. This means that the branching provides growth and the interactions are reductive, so that $\beta > 1$ and $\mu < 2$. Some observations regarding models lying in the $\beta \leq 1$ and $\mu < 2$ region of the parameter space will be made later.

Lemma 2.5.6. *In each of the following cases*

Case(i): $p_0 \neq 0$

Case(ii): $p_0 = 0$, *fixed parity*, N_0 *even*

Case(iii): $p_0 = 0$, *changeable parity*, $q_0 \neq 0$

there is some finite t_0 and some constant $p > 0$ so that, with n'_e given in 2.5.3, we have

$$P(N_{t_0} = 0 | N_0 < 2n'_e) \geq p.$$

Proof. In the first case, since $p_0 \neq 0$, we consider the probability that the original N_0 particles all branch into zero offspring before any interactions occur. If this happens before time t_0 then the system is extinct and $N_{t_0} = 0$. Let A_t be the event that the N_0 exponential waiting times governing the branching of

the original particles are all less than t , and that the first N_0 branching events produce zero offspring. Let B_t be the event that no interactions occur in the system before time t . Applying a logic similar to that used in the proof of lemma 2.4.7 we can show that

$$P(B_t|A_t, N_0 < 2n'_\epsilon) \geq 1 - 2n'_\epsilon(2n'_\epsilon - 1)P\left(\sup_{0 \leq s \leq t} X_s \geq e\right),$$

where X_s is a standard 1-dimensional Brownian motion and e is an exponential random variable of mean 1. As in 2.4.7 this holds regardless of the actual distribution of the N_0 particles. We can thus choose t_0 small enough so that $P(B_{t_0}|A_{t_0}, N_0 < 2n'_\epsilon) \geq 1/2$. Using this we have

$$\begin{aligned} P(N_{t_0} = 0|N_0 < 2n'_\epsilon) &\geq P(A_{t_0} \cap B_{t_0}|N_0 < 2n'_\epsilon) \\ &= P(B_{t_0}|A_{t_0}, N_0 < 2n'_\epsilon)P(A_{t_0}|N_0 < 2n'_\epsilon) \\ &\geq \frac{1}{2}(p_0(1 - e^{-\lambda t_0}))^{2n'_\epsilon} = p > 0. \end{aligned}$$

In the second case we note that fixed parity implies that $q_1 = 0$ and so consequently $q_0 \neq 0$ since $\mu < 2$. Let A_t be the event that the exponential waiting times governing the branching of the original particles are all greater than t , and that the first $N_0/2$ interaction events produce no offspring. Conditional on the event A_t , we see that up to time t the process is a pure annihilation process. So given that $N_0 < 2n'_\epsilon$ we can choose t_0 large enough so that

$$P(N_{t_0} = 0|A_{t_0}, N_0 < 2n'_\epsilon) \geq 1/2,$$

regardless of how the initial particles are distributed on S . From this we see that

$$\begin{aligned} P(N_{t_0} = 0|N_0 < 2n'_\epsilon) &\geq P(N_{t_0} = 0|A_{t_0}, N_0 < 2n'_\epsilon)P(A_{t_0}|N_0 < 2n'_\epsilon) \\ &\geq \frac{1}{2}q_0^{n'_\epsilon}(e^{-\lambda t_0})^{2n'_\epsilon} = p > 0. \end{aligned}$$

The third case is similar to the second case except that now we stipulate that the parity firstly becomes even if N_0 is odd. As $\beta > 1$ and the system

has changeable parity there is some event, or sequence of events, which causes some single particle to become an even number of particles. This could be a branching into an even number (≥ 2) of offspring, or a branching into an odd number (≥ 3) of offspring followed by a parity changing interaction between two of these offspring. We choose an event of this type which adds the least to the population. If N_0 is odd then there is some probability $p' > 0$ that this event is the first to happen and occurs by time $t = 1$. After this event the population is even and is less than $2n''_e$ for some fixed n''_e . As $q_0 \neq 0$ we use the method applied in case (ii) to give an additional time t_1 after which extinction has occurred with probability $p_1 > 0$. Letting $t_0 = 1 + t_1$, we have the stated result with $p = p'p_1 > 0$. \square

Now we prove that in each of the cases outlined in lemma 2.5.6 above the process becomes extinct almost surely, and that this extinction happens exponentially quickly. These facts are a consequence of the following proposition.

Proposition 2.5.7. *In each of the cases outlined in lemma 2.5.6 above we have*

$$P(N_{m(T+t_0)} > 0) \leq \left(1 - \frac{p}{2}\right)^m$$

with t_0 and p given by lemma 2.5.6, and T given by lemma 2.5.4.

Proof. Given the tools that are now in place, this proof is fairly straight-forward. The process is considered over consecutive time intervals of length $T + t_0$. We form a lower bound for the probability that a process which is alive at the start of such an interval becomes extinct before the end. This bound is not dependent on the population at the start of the interval.

Let $t = 0$ be the start of some interval $[0, T + t_0]$ which we want to consider. Lemma 2.5.4 tells us that, regardless of N_0 , there exists $\tau \in [0, T]$ such that

$\mathbb{E}(N_\tau | N_0 > 0) < n'_\epsilon$. By Chebychev's inequality

$$P(N_\tau \geq 2n'_\epsilon | N_0 > 0) \leq \mathbb{E}(N_\tau | N_0 > 0) / 2n'_\epsilon < 1/2,$$

so that $P(N_\tau < 2n'_\epsilon | N_0 > 0) > 1/2$. Now with a straight-forward application of lemma 2.5.6 and the Markov property we can write

$$\begin{aligned} P(N_{T+t_0} = 0 | N_0 > 0) &\geq P(N_{\tau+t_0} = 0 | N_0 > 0) \\ &\geq P(N_{\tau+t_0} = 0 | N_\tau < 2n'_\epsilon) P(N_\tau < 2n'_\epsilon | N_0 > 0) > \frac{p}{2}. \end{aligned}$$

Using this for $m \geq 0$ we can produce the following inductive step:

$$\begin{aligned} P(N_{(m+1)(T+t_0)} > 0) &= P(N_{(m+1)(T+t_0)} > 0 | N_{m(T+t_0)} > 0) P(N_{m(T+t_0)} > 0) \\ &\leq \left(1 - \frac{p}{2}\right) P(N_{m(T+t_0)} > 0). \end{aligned}$$

An inductive argument thus yields the stated result. \square

We have shown that in a number of cases the process will become extinct. The three cases outlined above are in fact the only cases in which extinction occurs, and this is precisely because they are the only cases in which extinction is possible. The aim now is to look at the remaining possibilities. As a concrete example consider a process in which particles branch into three and any pairwise interaction causes annihilation: from any odd initial number of particles this process cannot die out as it is of fixed parity.

2.5.3 The Stationary Distributions: Non-Extinction

Recalling that F is the natural state space for the process, we define $G \subset F$ to be those measures on S from which extinction is impossible. We require two results immediately: a categorisation of those models for which G is non-empty, and the fact that stationary distributions exist for the process on G . We begin by identifying the elements of G .

Lemma 2.5.8. *The set $G \subset F$ (those states from which extinction is impossible) is non-empty in the following cases:*

Case(iv): $p_0 = 0$, changeable parity, $q_0 = 0$

Case(v): $p_0 = 0$, fixed parity.

In all other cases G is empty. In the first of these two cases $G = F \setminus \{\emptyset\}$, whilst in the second

$$G = \left\{ \sum_{i=1}^n \delta_{x_i} : x_1, \dots, x_n \in S, n \text{ odd} \right\}.$$

Proof. Clearly G is empty in both cases (i) and (iii) as presented in the section above, since extinction occurs regardless of initial state $\xi_0 \in F$. In case (ii) however, in which $p_0 = 0$ and the system is of fixed parity, the extinction result above is stated only in those cases for which N_0 is even. From any initial state with odd parity, extinction is clearly impossible as parity is conserved. This gives case (v) above, along with the corresponding form of G . The remaining case follows simply since we have stipulated that neither pairwise annihilation nor single particle death occur, so clearly the process cannot die from any non-empty state. Considering cases (i)-(v) we see that they are exhaustive. All possible models have been covered and cases (iv) and (v) are the only ones in which G is non-empty. \square

Lemma 2.5.9. *In both the cases presented in lemma 2.5.8 above, there exists at least one stationary distribution for the process on G .*

Proof. This follows from theorem 2.4.9, although we require the additional fact that G is closed in F . Under the metric on F given by lemma 2.4.5 we see that measures of different masses in F are a distance at least $1/2$ apart. It is then easy to see that in both cases above the subspace G is closed in F . Now as in the proof of theorem 2.4.9 we assume some initial condition $\xi_0 = x \in G$, and

define the probability measures $\mu_t^x \in \mathcal{M}_1(F)$ by

$$\mu_t^x(A) = P(\xi_t \in A | \xi_0 = x), \quad \forall A \subset F.$$

These measures are clearly concentrated on G and hence so are

$$\nu_n^x = \frac{1}{n} \int_0^n \mu_s^x ds,$$

the Cesaro averages of the μ_s^x upto time $n \in \mathbb{N}$. By theorem 2.4.9 there is some stationary measure $\nu^x \in \mathcal{M}_1(F)$ which arises as the limit of some subsequence $\{\nu_{n'}^x\}$ of these measures. Now as G is closed we have

$$\nu^x(G) \geq \lim_{n' \rightarrow \infty} \nu_{n'}^x(G) = 1$$

so that ν^x is in fact a stationary measure on G . □

Now it remains to show that any such stationary measure on G is unique and that convergence to this occurs exponentially quickly from any initial state in G . The method by which this is done is similar to that used in the extinction proofs above, but now includes an additional coupling argument. We begin with a result which is analogous to lemma 2.5.6 except that rather than the population dying out entirely, it reduces to just a single particle.

Lemma 2.5.10. *In each of the cases presented in lemma 2.5.8, there is some finite t_0 and some constant $p > 0$ so that*

$$P(N_{t_0} = 1 | N_0 < 2n'_e) \geq p.$$

Proof. Here we apply similar ideas to those used to prove case (ii) in lemma 2.5.6.

In case (iv) we see that since $q_0 = 0$ and $\mu < 2$ we must have $q_1 > 0$. Let A_t be the event that the exponential waiting times governing the branching of the original particles are all greater than t , and that the first $N_0 - 1$ interaction

events produce just one offspring. Conditional on the event A_t , we see that up to time t the process is a pure coalescing process. So given that $N_0 < 2n'_\epsilon$ we can choose t_0 large enough so that

$$P(N_{t_0} = 1 | A_{t_0}, N_0 < 2n'_\epsilon) \geq 1/2,$$

regardless of how the initial particles are distributed on S . From this we see that

$$\begin{aligned} P(N_{t_0} = 1 | N_0 < 2n'_\epsilon) &= P(N_{t_0} = 1 | A_{t_0}, N_0 < 2n'_\epsilon) P(A_{t_0} | N_0 < 2n'_\epsilon) \\ &\geq \frac{1}{2} q_1^{2n'_\epsilon - 1} (e^{-\lambda t_0})^{2n'_\epsilon} = p > 0. \end{aligned}$$

In case (v) the fixed parity of the system implies that $q_1 = 0$ and so $q_0 > 0$. We now apply exactly the same method as for case (ii) in the proof of lemma 2.5.6, except that now the final annihilating interaction leaves a solitary particle rather than no particles at all. this gives

$$P(N_{t_0} = 1 | N_0 < 2n'_\epsilon) \geq \frac{1}{2} q_0^{n'_\epsilon} (e^{-\lambda t_0})^{2n'_\epsilon} = p > 0.$$

□

We now consider two copies of our process, ξ_t and η_t , started from different initial conditions (or initial distributions) in G . These two versions of the process evolve independently until such time as they couple, after which they evolve identically. Coupling is defined to occur if both processes contain just a single particle, and the two particles occupy the same position on S at the same time.

Proposition 2.5.11. *Let C_t be the event that the two versions ξ_t and η_t of the process are coupled by time t . Then*

$$P(C_{m(T+t_0+c/8)}) \geq 1 - \left(1 - \frac{p^2 e^{-2\lambda(T+c/8)}}{8}\right)^m$$

where t_0 and p are given in lemma 2.5.10, T is given in lemma 2.5.4 and c in corollary 2.3.3.

Proof. The proof of this result is similar in spirit to that of proposition 2.5.7. We consider the behaviour of the two processes over consecutive time intervals, here of length $T + t_0 + c/8$. Let $t = 0$ be the beginning of some such interval. We are interested in the probability that the two processes become coupled in the interval $[0, T + t_0 + c/8]$, conditional on them not being coupled at the start of the interval. For either of the two processes ξ_t and η_t , we know from lemma 2.5.4 that whatever the population at time 0, the expected population will fall below n'_e before time T , say at time $\tau \in [0, T]$. As in the proof of proposition 2.5.7, but using lemma 2.5.10, we write

$$P(N_{\tau+t_0} = 1) \geq P(N_{\tau+t_0} = 1 | N_\tau < 2n'_e) P(N_\tau < 2n'_e) > p/2.$$

To ensure the particle remains solitary until time T we stipulate that no further branching occurs given this event, so that certainly

$$P(N_{T+t_0} = 1) \geq \frac{p}{2} e^{-\lambda T}.$$

We see that regardless of the populations at time $t = 0$, the probability that both processes contain only a single particle at time $T + t_0$ is greater than $p^2 e^{-2\lambda T}/4$. Now these two particles are at most a distant $1/2$ apart on their respective versions of S . So applying corollary 2.3.3 tells us that these particles have probability at least $1/2$ of meeting in a further time $c/8$. If this happens before any further branching occurs then the processes are coupled. Hence we see that

$$P(C_{T+t_0+c/8}^c | C_0^c) \geq \frac{p^2}{4} e^{-2\lambda T} \frac{1}{2} e^{-2\lambda(c/8)} = \frac{p^2}{8} e^{-2\lambda(T+c/8)} =: \hat{p}.$$

Thus for any $m \geq 0$ we have

$$\begin{aligned} P(C_{(m+1)(T+t_0+c/8)}^c) &= P(C_{(m+1)(T+t_0+c/8)}^c | C_{m(T+t_0+c/8)}^c) P(C_{m(T+t_0+c/8)}^c) \\ &\leq (1 - \hat{p}) P(C_{m(T+t_0+c/8)}^c), \end{aligned}$$

so that inductively $P(C_{m(T+t_0+c/8)}^c) \leq (1 - \hat{p})^m$. Taking complements and writing \hat{p} out in full gives the result. \square

Corollary 2.5.12. *In the cases outlined in lemma 2.5.8 there exists a unique (and clearly non-trivial) stationary distribution for the process on G . From any initial configuration $\xi_0 \in G$ the system converges in total variation to this stationary distribution exponentially quickly.*

Proof. From lemma 2.5.9 there exists at least one stationary distribution supported by G . Consider two versions of the process, ξ_t and η_t , with ξ_0 distributed according to some such stationary distribution. By proposition 2.5.11 above, these two versions of the process will couple almost surely, and in fact exponentially quickly, after which their laws are identical. Consequently such a stationary distribution is unique and the process converges in total variation to this distribution exponentially quickly from any initial condition in G . \square

Remark 2.5.1. The work above relates to those models in which $\beta > 1$ and $\mu < 2$. We give a brief consideration now to those models in which the interactions remain reductive but $\beta \leq 1$. These models can be analysed using the techniques above, and exhibit exponentially quick extinction in all but the degenerate case with no branching. To see this we notice that if $\beta < 1$ then $p_0 > 0$, and so extinction follows as for case (i) in lemma 2.5.6. Similarly, if $\beta = 1$ with $p_1 < 1$, then again we must have $p_0 > 0$ and extinction follows. This leaves only the degenerate case $p_1 = 1$ which corresponds to no branching.

In this case it is clear that if the process reduces to just one particle then it becomes trapped in this state; it will remain as a solitary particle for all future times. Thus, unlike in any of the cases above, we have initial configurations from which the process could die out or survive forever. As illustration consider a model started from just two particles, but in which annihilation and coalescence are both possible. If the two particles annihilate then the process becomes extinct, whilst a coalescence leads to survival, albeit trivially. Obviously in several cases we can be certain of the behaviour: a pure coalescing model will

reduce to one particle, as will a fixed parity system started from an odd number of particles. A fixed parity system with an even number of initial particles will die out. In all cases the model will eventually end up in one of the trap states, either extinct or as a solitary particle.

2.6 Summary

This completes our analysis of these spatially interacting branching processes on S where the interaction mechanism is reductive and the branching mechanism provides growth. Our intuition presented in section 2.1 has been borne out by the mathematics: the reductive pairwise interactions do indeed dominate the branching when the population becomes large. The system returns swiftly to configurations with relatively low populations, and extinction, when possible, occurs exponentially quickly.

To summarise: in the following cases

Case(a): $p_0 \neq 0$

Case(b): $p_0 = 0$, changeable parity, $q_0 \neq 0$

there is a single unique stationary distribution, δ_0 , a unit mass on the empty set. From any other state in F the process will reach this state, so become extinct, exponentially quickly.

In the remaining cases

Case(c): $p_0 = 0$, changeable parity, $q_0 = 0$

Case(d): $p_0 = 0$, fixed parity

there is a unique other stationary state ν concentrated on $G \subset F$. If $\xi_0 \in G$ then the process converges to ν whilst if $\xi_0 \in F \setminus G$ the process becomes extinct. Both happen exponentially quickly.

Case(c) is the simpler of these two since G is all of F except the empty set. We have a family of stationary distributions of the form

$$\{(1 - \theta)\nu + \theta\delta_0 : \theta \in [0, 1]\}.$$

and there are no further stationary measures.

Case(d) is the most interesting since neither G nor $F \setminus G$ are trivial. Rather G is the set of all measures in F with odd total mass, and $F \setminus G$ is those measures with even mass. Again this yields a family of stationary measures on F of the form above, and this family is exhaustive.

The work we have done studying these processes on S is valid in its own right. However our intention was always that this should serve as a step towards studying the processes on \mathbb{R} . In what ways have our efforts helped towards this goal and what ideas and intuition should we take away with us? In what ways do we expect the analysis on \mathbb{R} to differ from that on S ?

To begin with we have, in section 2.2, a concrete proof that the process can be constructed on \mathbb{R} for arbitrary time, provided the number of initial particles is finite (see remark 2.2.1). Hence we can begin studying the finite processes on \mathbb{R} immediately, safe in the knowledge that such processes are non-explosive.

Secondly we are more confident in our intuitive understanding of the dynamics of these processes. High particle densities will be reduced as the interactions dominate the branching. Here however we must be cautious. On S a high population implies a high particle density, allowing the *close pair* arguments used to prove the stochastic boundedness of the total population. On \mathbb{R} we must consider the possibility that the population ‘spreads out’ from a finite initial condition, so that the total population grows but the particle density never becomes large enough for the interaction mechanism to dominate.

The final reason why this work is useful as a preliminary to further analysis on \mathbb{R} is that it gives an indication as to which methods and tools are most helpful. The *close pair* arguments developed here helped to bypass some of the difficulties inherent in the non-attractiveness of these processes. This notion will certainly be of use to us. It should also be noted that numerous results have been proved by controlling expected values related to these models. These include the existence of the processes in the first place (2.2.4), various steps in proving the existence of stationary distributions (2.3.10) and the majority of the work in characterising those stationary distributions (2.5.4). This is again a tool that we shall return to in our future work.

Chapter 3

The Process on \mathbb{R} : the Finite Case

It will be shown later that in several important cases we can construct the infinite process directly in a path-wise manner. However, as we do not have such a construction in every case, and wish to retain the generality of our work, we will show how to construct an infinite process as the limit of finite ones. Here the phrase infinite/finite process is shorthand used to refer to a version of our model started from an infinite/finite number of particles.

As has been mentioned, we already have a proof that the finite processes on \mathbb{R} can be constructed up to arbitrary time t without explosions (see section 2.2). We begin then by turning our attention to the development of a 'boundedness theorem' analogous to theorem 2.3.4. The additional problem here is that the population may spread out from a finite initial condition, so that whilst the population grows, this does not lead to a corresponding increase in the particle density. In this case our intuitive reasoning would break down - the population growing large would *not* lead to a domination by pairwise interactions. To

combat this we introduce a new mechanism: a test function which weights the particles according to their position on \mathbb{R} .

3.1 The Boundedness Theorem on \mathbb{R}

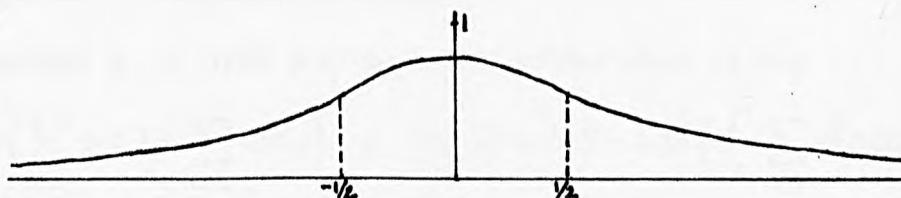
3.1.1 Preliminaries

Where possible we will carry over the notation developed in the previous chapters. Thus the process at time t is denoted ξ_t where

$$\xi_t = \{x_i : i \in I_t\},$$

with I_t being an index set labelling the particles alive at time t . Again $l_t^{i,j}$ will be the local time between particles i and j at time t , and $\beta < \infty$ and $\mu < 2$ will be the expected values of the single-particle and pairwise interaction offspring distributions respectively.

We wish to introduce a weighting function ϕ on \mathbb{R} . We are interested in a class of such test functions which have helpful properties. Outside the interval $[-\frac{1}{2}, \frac{1}{2}]$ we wish ϕ to have exponentially decreasing tails. So there is some $\gamma > 0$ such that $\phi(r) = e^{-\gamma|r|}$ for $r \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$. In the region $[-\frac{1}{2}, \frac{1}{2}]$ we complete ϕ in some continuous manner so that it has continuous second derivative, and so that $\gamma^2\phi \geq \phi''$ for all $x \in \mathbb{R}$. We have in mind a function of the form illustrated below:



Lemma 3.1.1. *For each $\gamma > 0$ there exists at least one function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ with the properties required.*

Proof. Clearly we define $\phi(r) = e^{-\gamma|r|}$ for $r \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$. Inside the interval $[-\frac{1}{2}, \frac{1}{2}]$ we complete ϕ with a quartic polynomial of the form $y(r) = ar^4 + br^2 + c$. In order for ϕ to have continuous second derivative and be of the desired form, we have the boundary conditions

$$y(\pm 1/2) = e^{-\frac{\gamma}{2}}, \quad y'(-1/2) = \gamma e^{-\frac{\gamma}{2}}, \quad y'(1/2) = -\gamma e^{-\frac{\gamma}{2}}, \quad y''(\pm 1/2) = \gamma^2 e^{-\frac{\gamma}{2}}.$$

Using these conditions to determine a, b and c gives

$$y(r) = \frac{1}{2}\gamma e^{-\frac{\gamma}{2}}(\gamma + 2)r^4 - \frac{1}{4}\gamma e^{-\frac{\gamma}{2}}(\gamma + 6)r^2 + \frac{1}{32}e^{-\frac{\gamma}{2}}(\gamma^2 + 10\gamma + 32).$$

Finally we notice that, after simplification, we have

$$\gamma^2 y(r) - y''(r) = \gamma e^{-\frac{\gamma}{2}} \left(\frac{1}{4} - r^2 \right) \left[12 + 6\gamma + \gamma^2 + \left(\frac{1}{4} - r^2 \right) \left(\gamma^2 + \frac{\gamma^3}{2} \right) \right]$$

which is non-negative for $r \in [-\frac{1}{2}, \frac{1}{2}]$ so that $\gamma^2 y \geq y''$. Thus letting

$$\phi(r) = \begin{cases} e^{-\gamma|r|} & r \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}] \\ y(r) & r \in [-\frac{1}{2}, \frac{1}{2}] \end{cases}$$

gives a function with the desired properties. \square

In the case of these processes on the ring S we spent much time investigating the behaviour of the total population process N_t . Now that we have progressed to the real line we wish to use similar results and methods, but this time for the *weighted* population process under ϕ ,

$$\Phi_t := \sum_{i \in I_t} \phi(x_i) = \sum_{i \in I} \phi(x_i(t)) 1_{\{i \in I_t\}}.$$

We begin with an important proposition.

Proposition 3.1.2. *With ϕ of the form described above we have*

$$\begin{aligned} \mathbb{E} \left(\sum_{i \in I_t} \phi(x_i) - \sum_{i \in I_0} \phi(x_i) \right) &\leq [(\gamma^2/2) + \lambda(\beta - 1)] \mathbb{E} \left(\int_0^t \sum_{i \in I_s} \phi(x_i) ds \right) \\ &\quad + \frac{1}{2} [\mu - 2] \mathbb{E} \left(\int_0^t \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j} \right). \end{aligned}$$

Remark 3.1.1. Before commencing with the proof of this result we make an additional remark concerning our notation for these processes. Recalling the note on the labelling of particles from the first chapter, when writing expressions of the form

$$\int_0^t \sum_{i \in I_s} f(x_i) dx_s^i$$

we understand this to be shorthand for the expression

$$\sum_{i \in I} \int_0^t f(x_i(s)) 1_{A_i}(s) dx_s^i.$$

where A_i is the event that there is a particle labelled i alive. This avoids the problem of having an integral over a random sum of integrators. In the finite particle process, the sums above are almost surely finite.

Proof. Applying Itô's formula for a process with jumps to the process $\sum_{i \in I} \phi(x_i(s)) 1_{\{i \in I_s\}}$ immediately gives

$$\begin{aligned} \sum_{i \in I} \phi(x_i(t)) 1_{\{i \in I_t\}} &= \sum_{i \in I} \phi(x_i(0)) 1_{\{i \in I_0\}} \\ &+ \sum_{i \in I} \int_0^t \phi'(x_i(s)) 1_{\{i \in I_s\}} dx_i(s) + \frac{1}{2} \sum_{i \in I} \int_0^t \phi''(x_i(s)) 1_{\{i \in I_s\}} ds \\ &+ \sum_{s \leq t} \left(\sum_{i \in I} \phi(x_i(s)) 1_{\{i \in I_s\}} - \sum_{i \in I} \phi(x_i(s-)) 1_{\{i \in I_{s-}\}} \right). \end{aligned}$$

The $dx_i(s)$ in the second term on the right-hand side above are the Brownian increments of the path followed by the i -th particle. The final term is the sum of the jumps made by the process $\sum_{i \in I} \phi(x_i(s)) 1_{\{i \in I_s\}}$ due to branching and interactions. Although the sum appears uncountable at first glance, it follows from our construction of the finite process that only finitely many terms are non-zero almost surely. We can write

$$\sum_{s \leq t} \left(\sum_{i \in I} \phi(x_i(s)) 1_{\{i \in I_s\}} - \sum_{i \in I} \phi(x_i(s-)) 1_{\{i \in I_{s-}\}} \right) = \Delta_t^B + \Delta_t^I,$$

where Δ_t^B and Δ_t^I are the changes by time t in the process $\sum_{i \in \mathcal{I}} \phi(x_i(s))1_{\{i \in I_s\}}$ due to branching and interacting respectively.

Now let τ_n be the random time at which the n -th branching or interaction occurs. Running the process upto time $t \wedge \tau_n$ and taking expectations yields

$$\begin{aligned} \mathbb{E}\left(\sum_{i \in \mathcal{I}} \phi(x_i(t \wedge \tau_n))1_{\{i \in I_{t \wedge \tau_n}\}}\right) &= \\ \mathbb{E}\left(\sum_{i \in \mathcal{I}} \phi(x_i(0))1_{\{i \in I_0\}}\right) &+ \mathbb{E}\left(\sum_{i \in \mathcal{I}} \int_0^{t \wedge \tau_n} \phi'(x_i(s))1_{\{i \in I_s\}} dx_i(s)\right) \\ + \frac{1}{2} \mathbb{E}\left(\sum_{i \in \mathcal{I}} \int_0^{t \wedge \tau_n} \phi''(x_i(s))1_{\{i \in I_s\}} ds\right) &+ \mathbb{E}\left(\Delta_{t \wedge \tau_n}^B + \Delta_{t \wedge \tau_n}^I\right). \end{aligned} \quad (1)$$

Given the times τ_k and τ_{k+1} of the k -th and $k+1$ -th branching or interaction, we know that the population is fixed in between. Thus letting $\tau_0 = 0$ we have

$$\begin{aligned} &\mathbb{E}\left(\sum_{i \in \mathcal{I}} \int_0^{t \wedge \tau_n} \phi'(x_i(s))1_{\{i \in I_s\}} dx_i(s)\right) \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left(\sum_{i \in \mathcal{I}} \int_{t \wedge \tau_k}^{t \wedge \tau_{k+1}} \phi'(x_i(s))1_{\{i \in I_{\tau_k}\}} dx_i(s)\right) \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left(\mathbb{E}\left(\sum_{i \in \mathcal{I}} \int_{t \wedge \tau_k}^{t \wedge \tau_{k+1}} \phi'(x_i(s))1_{\{i \in I_{\tau_k}\}} dx_i(s) \middle| \mathcal{F}_{\tau_k}\right)\right) \\ &= \sum_{k=0}^{n-1} \mathbb{E}\left(\sum_{i \in \mathcal{I}} 1_{\{i \in I_{\tau_k}\}} \mathbb{E}\left(\int_{t \wedge \tau_k}^{t \wedge \tau_{k+1}} \phi'(x_i(s)) dx_i(s) \middle| \mathcal{F}_{\tau_k}\right)\right) = 0. \end{aligned}$$

The equality to zero follows since the Brownian stochastic integrals in the final line are martingales. This calculation eradicates one of the terms in expression (1).

Now we turn our attention to those terms in (1) arising from jumps in the process. For $i \in \mathcal{I}$, let $\Delta_t^B(i)$ be the change in the process $\sum_{i \in \mathcal{I}} \phi(x_i(s))1_{\{i \in I_s\}}$ in the interval $[0, t]$ arising from the branching of the particle labelled i . Obviously

we have

$$\mathbb{E}(\Delta_{t \wedge \tau_n}^B) = \mathbb{E}\left(\sum_{i \in \mathcal{I}} \Delta_{t \wedge \tau_n}^B(i)\right).$$

As the process is being stopped after the n -th branching or interaction, the sum on the right-hand side has at most n non-zero terms. The value of $\Delta_{t \wedge \tau_n}^B(i)$, given the branching of particle i in the interval $[0, t \wedge \tau_n]$, is distributed according to $\phi(r)(B(i) - 1)$, where r is the position of the particle when it branches and $B(i)$ is a copy of the distribution governing branching offspring. As $0 \leq \phi \leq 1$ it is easy to see that $|\Delta_{t \wedge \tau_n}^B(i)|$ is dominated above by $B(i) + 1$, so that

$$\mathbb{E}\left(\sum_{i \in \mathcal{I}} |\Delta_{t \wedge \tau_n}^B(i)|\right) \leq \mathbb{E}\left(\sum_{r=1}^n B(r) + 1\right) = n(\beta + 1) < \infty.$$

From this we can thus apply Fubini's theorem to give

$$\mathbb{E}\left(\sum_{i \in \mathcal{I}} \Delta_{t \wedge \tau_n}^B(i)\right) = \sum_{i \in \mathcal{I}} \mathbb{E}(\Delta_{t \wedge \tau_n}^B(i)).$$

Now $\Delta_t^B(i)$ is a single-step jump process which jumps from 0 to $\phi(x_i(t))(B(i) - 1)$ at rate $\lambda 1_{\{i \in I_t\}} dt$, so applying lemma 2.2.1 gives

$$\mathbb{E}(\Delta_{t \wedge \tau_n}^B(i)) = \lambda(\beta - 1) \mathbb{E}\left(\int_0^{t \wedge \tau_n} \phi(x_i(s)) 1_{\{i \in I_s\}} ds\right).$$

We treat the $\Delta_{t \wedge \tau_n}^I$ in a very similar way. Letting $\Delta_{t \wedge \tau_n}^I(i, j)$ be the contribution to the total change due to interaction of the particles i and j , we have

$$\mathbb{E}(\Delta_{t \wedge \tau_n}^I) = \frac{1}{2} \mathbb{E}\left(\sum_{i \in \mathcal{I}} \sum_{\substack{j \in \mathcal{I} \\ j \neq i}} \Delta_{t \wedge \tau_n}^I(i, j)\right).$$

The factor $1/2$ arises since $\Delta_{t \wedge \tau_n}^I(i, j) = \Delta_{t \wedge \tau_n}^I(j, i)$ and we do not want to count the contribution from a single interaction twice. Dominating $|\Delta_{t \wedge \tau_n}^I(i, j)|$ above by $M(i, j) + 2$ gives

$$\mathbb{E}\left(\sum_{i \in \mathcal{I}} \sum_{\substack{j \in \mathcal{I} \\ j \neq i}} |\Delta_{t \wedge \tau_n}^I(i, j)|\right) \leq n(\mu + 2) < \infty.$$

This in turn justifies application of Fubini's theorem to get

$$\mathbb{E}\left(\sum_{i \in \mathcal{I}} \sum_{\substack{j \in \mathcal{I} \\ j \neq i}} \Delta_{t \wedge \tau_n}^I(i, j)\right) = \sum_{i \in \mathcal{I}} \sum_{\substack{j \in \mathcal{I} \\ j \neq i}} \mathbb{E}(\Delta_{t \wedge \tau_n}^I(i, j)).$$

As $\Delta_t^I(i, j)$ is a single-step jump process which jumps from 0 to $\phi(x_i(t))(M(i, j) - 2)$ at rate $1_{\{i \in I_s\}} 1_{\{j \in I_s\}} dl_s^{i, j}$, we again use lemma 2.2.1 to write

$$\mathbb{E}(\Delta_{t \wedge \tau_n}^I(i, j)) = (\mu - 2) \mathbb{E}\left(\int_0^{t \wedge \tau_n} \phi(x_i(s)) 1_{\{i \in I_s\}} 1_{\{j \in I_s\}} dl_s^{i, j}\right).$$

Using the work done so far, together with the fact that $\phi'' \leq \gamma^2 \phi$, we reformulate (1) as

$$\begin{aligned} \mathbb{E}(\Phi_{t \wedge \tau_n}) &\leq \mathbb{E}(\Phi_0) + \frac{\gamma^2}{2} \mathbb{E}\left(\sum_{i \in \mathcal{I}} \int_0^{t \wedge \tau_n} \phi(x_i(s)) 1_{\{i \in I_s\}} ds\right) \\ &\quad + \lambda(\beta - 1) \sum_{i \in \mathcal{I}} \mathbb{E}\left(\int_0^{t \wedge \tau_n} \phi(x_i(s)) 1_{\{i \in I_s\}} ds\right) \\ &\quad + \frac{1}{2}(\mu - 2) \sum_{i \in \mathcal{I}} \sum_{\substack{j \in \mathcal{I} \\ j \neq i}} \mathbb{E}\left(\int_0^{t \wedge \tau_n} \phi(x_i(s)) 1_{\{i \in I_s\}} 1_{\{j \in I_s\}} dl_s^{i, j}\right), \end{aligned}$$

where Φ_t is the obvious shorthand. The integrands on the right-hand side of this inequality are all non-negative, so repeated use of Fubini yields

$$\begin{aligned} \mathbb{E}(\Phi_{t \wedge \tau_n}) &\leq \mathbb{E}(\Phi_0) + [\gamma^2/2 + \lambda(\beta - 1)] \mathbb{E}\left(\int_0^{t \wedge \tau_n} \Phi_s ds\right) \\ &\quad + \frac{1}{2}(\mu - 2) \mathbb{E}\left(\int_0^{t \wedge \tau_n} \sum_{i \in \mathcal{I}} \phi(x_i(s)) 1_{\{i \in I_s\}} \sum_{\substack{j \in \mathcal{I} \\ j \neq i}} 1_{\{j \in I_s\}} dl_s^{i, j}\right). \end{aligned}$$

The work done in constructing the process, in particular lemma 2.2.4, tell us that $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$ almost surely. We take the limit as $n \rightarrow \infty$ of the above inequality, using Fatou's lemma on the left-hand side to give a bound on $\mathbb{E}(\Phi_t)$. Re-writing the resulting expression in our short-hand notation gives the proposition as stated. \square

The above proposition gives a bound on the expected change in the weighted population $\sum_{i \in I_t} \phi(x_i)$ over a time t . The first term on the right-hand side can be interpreted as the change due to movement and births; it is certainly positive when $\beta > 1$ as would be expected. The second term represents the effect of the pairwise interactions. For reductive interactions ($\mu < 2$) this contribution is negative as the local time processes are non-decreasing.

The next lemma in this section is a simple result which says that regardless of the form (or even presence) of the reductive pairwise interactions, the expected value of $\sum_{i \in I_t} \phi(x_i)$ grows at most exponentially.

Lemma 3.1.3. *Provided $\mu \leq 2$, including the degenerate case with no interactions, then*

$$\mathbb{E}\left(\sum_{i \in I_t} \phi(x_i)\right) \leq \mathbb{E}\left(\sum_{i \in I_0} \phi(x_i)\right) e^{[(\gamma^2/2) + \lambda(\beta-1)]t}.$$

Proof. Applying Fubini's theorem allows us to write

$$\mathbb{E}\left(\int_0^t \sum_{i \in I_s} \phi(x_i) ds\right) = \int_0^t \mathbb{E}\left(\sum_{i \in I_s} \phi(x_i)\right) ds.$$

Letting Φ_s denote $\sum_{i \in I_s} \phi(x_i)$, proposition 3.1.2 certainly gives

$$\mathbb{E}(\Phi_t) \leq \mathbb{E}(\Phi_0) + [(\gamma^2/2) + \lambda(\beta - 1)] \int_0^t \mathbb{E}(\Phi_s) ds. \quad (1)$$

Now we have

$$\mathbb{E}(\Phi_s) = \mathbb{E}\left(\sum_{i \in I_s} \phi(x_i)\right) \leq \mathbb{E}(|I_s| \sup_{x \in \mathbb{R}} \phi(x)) \leq c\mathbb{E}(N_s)$$

for some constant $c < \infty$. Now from lemma 2.2.5, which did not depend on the topology of S , we know that $\mathbb{E}(N_s) < \infty$. Thus $\mathbb{E}(\Phi_s) < \infty$ also, and we may apply Gronwall's inequality to expression (1) to yield the result. \square

We now prove an analogous result to the bound given in lemma 2.2.5 which was used to prove the main boundedness theorem.

Corollary 3.1.4. *With the notation as above we have the following bound on $\mathbb{E}(\Phi_t)$,*

$$\mathbb{E}(\Phi_t) \leq \mathbb{E}(\Phi_0)e^{[(\gamma^2/2)+\lambda(\beta-1)]t} + \frac{1}{2}(\mu - 2)\mathbb{E}\left(\int_0^t \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right).$$

Proof. From proposition 3.1.2 we can write

$$\begin{aligned} \mathbb{E}\left(\sum_{i \in I_t} \phi(x_i) - \sum_{i \in I_0} \phi(x_i)\right) &\leq [(\gamma^2/2) + \lambda(\beta - 1)] \int_0^t \mathbb{E}\left(\sum_{i \in I_s} \phi(x_i)\right) ds \\ &\quad + \frac{1}{2}(\mu - 2)\mathbb{E}\left(\int_0^t \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \quad (1) \end{aligned}$$

using Fubini's theorem to exchange the expectation and the integral in the first term on the right-hand side. Now bounding the integrand of this term above by the exponential growth given in lemma 3.1.3 gives

$$\begin{aligned} [(\gamma^2/2) + \lambda(\beta - 1)] \int_0^t \mathbb{E}(\Phi_s) ds &\leq [(\gamma^2/2) + \lambda(\beta - 1)]\mathbb{E}(\Phi_0) \int_0^t e^{[(\gamma^2/2)+\lambda(\beta-1)]s} ds \\ &= \mathbb{E}(\Phi_0)(e^{[(\gamma^2/2)+\lambda(\beta-1)]t} - 1). \end{aligned}$$

Substituting this back into (1) gives the result. \square

3.1.2 The Main First Moment Bound

Our intention now is to prove a result about our processes on \mathbb{R} analogous to the boundedness theorem 2.3.4 on S . We begin with a statement of the desired theorem and then devote the remainder of the section to its proof. It is here that our earlier work on the unit ring will pay off as we will use many of the ideas developed during our work on that simpler case.

Theorem 3.1.5. *There exists $R_0 \geq 0$, $\alpha \in (0, 1)$ and $C > 0$ so that if the initial distribution of particles on \mathbb{R} is such that $\Phi_0 \geq R_0$, then*

$$\mathbb{E}(\Phi_{t(\Phi_0)}) \leq \alpha\Phi_0,$$

where $t(\Phi_0) = C/\Phi_0$.

Thus, the theorem states that if the weighted density of the initial particles is sufficiently large then a short time later the expected weighted density will be less. The expected decrease will be at least proportional to the initial weighted density, whilst the required time interval is inversely proportional to it. It is clear that the result above is almost exactly like our earlier result (theorem 2.3.4) on the ring, but with the weighted particle density replacing the actual number of particles. The proof is correspondingly similar, using the idea of 'close pairs' and breaking the time interval $[0, C/\Phi_0]$ into smaller sub-intervals of length C/Φ_0^2 . The main differences arise due to the presence of the test function ϕ .

One of the key elements of the proof is again to compare our system to a system of independent pairs of particles. The following series of results are required for this comparison. We begin by considering a single pair of standard Brownian particles, B_t^1 and B_t^2 , started from the origin. We are particularly interested in the event E_t^x , that at least one of these two particles has travelled no further than a distance $x > 0$ from its initial position during the time interval $[0, t]$.

Lemma 3.1.6. *Let e be an exponential random variable with mean 1 and let $l_t^{1,2}$ be the local time by time t between the two particles. Then for any $x > 0$,*

$$P(E_t^x | l_t^{1,2} \geq e) \rightarrow 1 \quad \text{as } t \rightarrow 0.$$

Proof. Let $E_t^x(i) = \{|B_s^i| \leq x \ \forall s \leq t\}$ be the event that B_t^i has moved no further than a distance x from its initial position in the interval $[0, t]$. Hence $E_t^x = E_t^x(1) \cup E_t^x(2)$ and considering complements gives $(E_t^x)^c = E_t^x(1)^c \cap E_t^x(2)^c$. Now as

$$P(E_t^x | l_t^{1,2} \geq e) = 1 - P((E_t^x)^c | l_t^{1,2} \geq e)$$

it suffices to show that

$$P((E_t^x)^c | l_t^{1,2} \geq e) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

We can immediately write

$$P((E_t^x)^c | l_t^{1,2} \geq e) = \frac{P((E_t^x)^c \cap \{l_t^{1,2} \geq e\})}{P(l_t^{1,2} \geq e)} \leq \frac{P((E_t^x)^c)}{P(l_t^{1,2} \geq e)}$$

and using the independence of B_t^1 and B_t^2 this yields

$$P((E_t^x)^c | l_t^{1,2} \geq e) \leq \frac{P(E_t^x(1)^c)P(E_t^x(2)^c)}{P(l_t^{1,2} \geq e)}. \quad (1)$$

Letting B_t be another standard Brownian motion we use the fact that $l_t^{1,2}$ is equal in distribution to $\sup_{0 \leq s \leq t} B_s$ to write

$$\begin{aligned} P(l_t^{1,2} \geq e) &= P(\sup_{0 \leq s \leq t} B_s \geq e) \geq P(\{\sup_{0 \leq s \leq t} B_s \geq x\} \cap \{e \leq x\}) \\ &= P(\sup_{0 \leq s \leq t} B_s \geq x)P(e \leq x). \end{aligned}$$

Also we have

$$\begin{aligned} P(E_t^x(1)^c) &= P(E_t^x(2)^c) = P(\{\sup_{0 \leq s \leq t} B_s^1 \geq x\} \cup \{\inf_{0 \leq s \leq t} B_s^1 \leq -x\}) \\ &\leq P(\sup_{0 \leq s \leq t} B_s^1 \geq x) + P(\inf_{0 \leq s \leq t} B_s^1 \leq -x) \\ &= 2P(\sup_{0 \leq s \leq t} B_s^1 \geq x). \end{aligned}$$

Using these in the expression labelled (1) we have

$$P((E_t^x)^c | l_t^{1,2} \geq e) \leq \frac{4 \left(P(\sup_{0 \leq s \leq t} B_s^1 \geq x) \right)^2}{P(e \leq x)P(\sup_{0 \leq s \leq t} B_s \geq e)} = \frac{4}{P(e \leq x)} P(\sup_{0 \leq s \leq t} B_s^1 \geq x)$$

and the right-hand side of this expression clearly tends to zero as t tends to zero so we are done. \square

This result does not really seem surprising – certainly without the conditioning this simply says that as the available time interval gets smaller, the

probability that one of two Brownian particles makes a large excursion gets smaller also. All we have done is shown that this remains true if we condition on the local time between the particles being 'large'. The reason we require this result is so that we can prove the extension to corollary 2.3.8 which follows. In this we replace π_t , the number of interactions by time t , with $\tilde{\pi}_t$, the number of interactions by t in at which at least one of the pair has moved no further than $1/2$ from its initial point before the interaction.

Corollary 3.1.7. *Consider m independent pairs of Brownian particles subject to local-time annihilating interactions, with the two particles in each pair no more than a distance d apart at some initial time. There exist constants $t' > 0$ and $d' > 0$ so that if $d \leq d'$ then for any $\theta \in (0, 1)$ we have*

$$\mathbb{E}(\tilde{\pi}_t) \geq \frac{m}{8}(1 - \theta) \left\{ 1 - e^{-\frac{1}{2}\theta\sqrt{\pi t}} \right\}$$

provided $d^2c \leq t \leq t'$, with $c > 0$ given in lemma 2.3.1.

Proof. Label the pairs $z = 1, \dots, m$. Let \tilde{A}_t^z be the event that the z -th pair interact at some time $\tau \leq t$, with at least one particle having remained within a distance $1/2$ of its initial position during $[0, \tau]$. Clearly

$$\mathbb{E}(\tilde{\pi}_t) = \sum_{z=1}^m \mathbb{E}(1_{\tilde{A}_t^z}) = \sum_{z=1}^m P(\tilde{A}_t^z). \quad (1)$$

Now consider a single pair. By assumption $d^2c \leq t$, so corollary 2.3.3 states that the probability this pair meet before time $t/2$ is at least $1/2$. We can choose d sufficiently small to ensure that this first meeting has probability greater than $1/2$ of occurring before either particle has moved further than $1/4$ from its initial position. Thus with probability at least $1/4$ the z -th pair meet at some time $\tau \leq t/2$, with each particle still within $1/4$ of its initial point. We let this be the event B_z and we write

$$P(\tilde{A}_t^z) \geq P(\tilde{A}_t^z | B_z) P(B_z) \geq P(\tilde{A}_t^z | B_z) / 4. \quad (2)$$

Using lemma 3.1.6 we choose $t' > 0$ small enough so that, in the notation of that lemma,

$$t \leq t' \implies P(E_{t/2}^{1/4} | l_{t/2}^{1,2} \geq e) \geq 1/2. \quad (3)$$

Conditional on the event B_z , suppose the z -th pair undergo an interaction in the interval $[\tau, \tau + t/2]$, with $t \leq t'$. Then (3) implies that there is a probability at least $1/2$ that the interaction occurred whilst one of the particles was still within a distance $1/4$ of its position at τ . By B_z this position at time τ is no more than $1/4$ from each particle's start position, so giving event \tilde{A}_t^z . Thus

$$P(\tilde{A}_t^z | B_z, \{l_t^z \geq e_z\}) \geq 1/2$$

where e_z is the mean 1 exponential outcome governing the interaction of the z -th pair, and l_t^z is the local time of the pair. Now we have

$$P(\tilde{A}_t^z | B_z) \geq P(\tilde{A}_t^z | B_z, \{l_t^z \geq e_z\}) P(\{l_t^z \geq e_z\} | B_z) \geq \frac{1}{2} P(\{l_t^z \geq e_z\} | B_z).$$

This final term involves the probability that a pair of particles interact by time t given that they meet before time $t/2$. Using the same bound for this as in lemma 2.3.7 gives

$$P(\tilde{A}_t^z | B_z) \geq \frac{1}{2} (1 - \theta) \left\{ 1 - e^{-\frac{1}{2}\theta\sqrt{\pi t}} \right\}.$$

Substituting this into (2) and then (1) gives the stated result. \square

The extra spatial aspect of the above result is very important. We again intend to show that enough pairwise interactions occur to control the weighted particle density, but here we must show that the interactions occur between particles which have not moved too far under the weighting function ϕ so that they retain enough weight to be significant.

Proving Theorem 3.1.5

We now wish to move on from these technical preliminaries to the actual task of proving our main theorem. To do this we require some additional notation. For any $k \in \mathbb{Z}$ we let A_k be the interval $[k - \frac{1}{2}, k + \frac{1}{2}) \subset \mathbb{R}$ and a_k^+ and a_k^- be points in the closure of A_k such that

$$\phi(a_k^+) = \sup_{x \in A_k} \phi(x) \quad \text{and} \quad \phi(a_k^-) = \inf_{x \in A_k} \phi(x).$$

Using these points we can define the quantities

$$\Phi^+ = \sum_{k \in \mathbb{Z}} \phi(a_k^+) \quad \text{and} \quad \Phi^- = \sum_{k \in \mathbb{Z}} \phi(a_k^-)$$

noting that these are finite since ϕ has exponentially decreasing tails. Now let ξ_t^k be the restriction of the process at time t to A_k , so $\xi_t^k = \xi_t \cap A_k$, and hence $|\xi_t^k|$ is the number of particles of the process in the interval A_k at time t . Similarly let $I_t^k = \{i \in I_t : x_i \in A_k\}$.

The result that we actually prove is as follows: there exists $r' > 0$, $\alpha \in (0, 1)$ and $C' > 0$ such that if $\Phi_0 = r\Phi^+$ with $r \geq r'$, then

$$\mathbb{E}(\Phi_{t_r}) \leq \alpha \Phi_0$$

where $t_r = C'/r$. Having proved this result we set $C = C'\Phi^+$ and $R_0 = r'\Phi^+$ to give the theorem stated in 3.1.5.

As before we let $\tau = t_r/r = C'/r^2$ so that $[r]$ sub-intervals of length τ can be placed consecutively in $[0, t_r]$, possibly with some small remainder. In the result below we are assuming that r and thus τ are fixed, and that we are re-starting the process from some state with weighted particle density Φ (not necessarily Φ_0). A lower bound, varying in Φ , is then given on a certain local time integral over the fixed interval $[0, \tau]$.

Proposition 3.1.8. *Suppose the weighted particle density is Φ at some initial time $t = 0$. There exists a constant $C_1 > 0$, so that for r sufficiently large*

$$\mathbb{E}\left(\int_0^r \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \geq 2C_1 \left(\frac{2\Phi}{r} - \Phi^+\right).$$

Proof. As the local-time processes can only increase when both particles are alive, we can write

$$\begin{aligned} \mathbb{E}\left(\int_0^r \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) &\geq \mathbb{E}\left(\int_0^r \sum_{i \in I_0} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \\ &= \sum_{k \in \mathbb{Z}} \mathbb{E}\left(\int_0^r \sum_{i \in I_0^k} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right). \end{aligned}$$

In the first inequality we have simply restricted the sum to those particles which are in existence originally, discarding those terms arising from particles created later.

Now we consider the initial distribution ξ_0^k of particles on just one interval A_k . Letting $m = \lceil |\xi_0^k| / \lceil r/2 \rceil \rceil$, we can divide the particles $\{x_i : i \in I_0^k\}$ into m disjoint groups of $\lceil r/2 \rceil$ consecutive particles, which we label g_1, \dots, g_m . This holds true even if $|\xi_0^k| < \lceil r/2 \rceil$ because then $m = 0$. Throwing away the contribution from any extra particles which do not form a group we write

$$\mathbb{E}\left(\int_0^r \sum_{i \in I_0^k} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \geq \sum_{n=1}^m \mathbb{E}\left(\int_0^r \sum_{i \in g_n} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right). \quad (1)$$

We now wish to bound from below just one of the terms in the sum on the right-hand side of the above.

For $i \in I_0$ let T_i be the random time at which particle x_i undergoes either a branching or an interaction. Further let $E_i(s)$ be the event that the particle x_i remains within a distance $1/2$ of its initial position in the interval $[0, s \wedge T_i]$.

Then

$$\mathbb{E}\left(\int_0^\tau \sum_{i \in g} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \geq \mathbb{E}\left(\int_0^\tau \sum_{i \in g} \phi(x_i) 1_{E_i(s)} \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right). \quad (2)$$

Now for any particle x_i which begins in A_k we have

$$\phi(x_i(t)) 1_{E_i} \geq \inf\{\phi(x) : |x - x_i(0)| \leq 1/2\} 1_{E_i} \geq \phi(a_k^+) e^{-3\gamma/2} 1_{E_i}$$

throughout its lifetime. To see this consider that the smallest value of the function ϕ which lies within a distance $1/2$ of the region A_k is greater than the smallest value of ϕ which lies within $3/2$ of $\phi(a_k^+)$. The actual calculation of this value as a multiple of $\phi(a_k^+)$ follows from the exponential behaviour of ϕ , with an extra consideration for the region A_0 . Using this in equation (2) we have

$$\mathbb{E}\left(\int_0^\tau \sum_{i \in g} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \geq \phi(a_k^+) e^{-3\gamma/2} \sum_{i \in g} \mathbb{E}\left(\int_0^\tau 1_{E_i(s)} \sum_{\substack{j \in I \\ j \neq i}} dl_s^{i,j}\right). \quad (3)$$

To continue let \tilde{B}_i be the event that particle x_i undergoes an interaction in the interval $[0, s]$ before moving a distance greater than $1/2$ from its start position. Then $1_{\tilde{B}_i}$ is a single-step jump process which jumps from 0 to 1 at rate $1_{E_i(s)} \sum_{\substack{j \in I \\ j \neq i}} dl_s^{i,j}$, so from lemma 2.2.1 we have

$$\mathbb{E}\left(\int_0^\tau 1_{E_i(s)} \sum_{\substack{j \in I \\ j \neq i}} dl_s^{i,j}\right) = \mathbb{E}(1_{\tilde{B}_i}).$$

Using this we now write

$$\sum_{i \in g} \mathbb{E}\left(\int_0^\tau 1_{E_i(s)} \sum_{\substack{j \in I \\ j \neq i}} dl_s^{i,j}\right) = \sum_{i \in g} \mathbb{E}(1_{\tilde{B}_i}) = E\left(\sum_{i \in g} 1_{\tilde{B}_i}\right) \geq \frac{1}{2} E(\tilde{\pi}_\tau(g)), \quad (4)$$

with $\tilde{\pi}_\tau(g)$ being the number of interactions occurring in $[0, \tau]$ in which at least one of the particles is in g and has moved no further than $1/2$ from its initial position. The factor $1/2$ arises for the same reason as in the proof of the boundedness theorem 2.3.4 on the ring; it stops us counting one interaction twice.

As there are $[r/2]$ particles in any group g , our work on *close pairs* given in lemma 2.3.5 allows us to choose $[r/12]$ disjoint pairs of neighbouring particles which are a distance at most $2/[r/2] \leq 8/r$ apart. Let $\tilde{\pi}_\tau^{cl}(g)$ be analogous to $\tilde{\pi}_\tau(g)$, but with each interaction involving a *close pair* particle. Using the same logic as in step 1 of the proof of theorem 2.3.4, we have

$$\mathbb{E}(\tilde{\pi}_\tau(g)) \geq \mathbb{E}(\tilde{\pi}_\tau^{cl}(g)) \geq \frac{1}{2}e^{-2\lambda[r/12]\tau} \mathbb{E}(\tilde{\pi}'_\tau). \quad (5)$$

Here $\tilde{\pi}'_\tau$ is the object considered in corollary 3.1.7 and is defined on a separate probability space. Specifically $\tilde{\pi}'_\tau$ is defined on a system of $[r/12]$ independent pairs of particles, with the pairs starting a distance no more than $8/r$ apart. It is the number of interactions in this system by time τ in which at least one of the particles has remained within a distance $1/2$ of its initial position. In order to apply corollary 3.1.7 we now choose C' larger than $64c$ and assume that r is large enough that $8/r \leq d'$ and $C'/r^2 \leq t'$. Then

$$\mathbb{E}(\tilde{\pi}'_\tau) \geq \frac{1}{8}[r/12](1 - \theta) \left\{ 1 - e^{-\frac{1}{2r}\theta\sqrt{C'}\tau} \right\}. \quad (6)$$

Again using the methods given in step 1 of our proof of theorem 2.3.4, we use this to show that for large enough r we have

$$\mathbb{E}(\tilde{\pi}'_\tau) \geq p$$

with $p > 0$ a constant independent of r . Combining this with (5) gives

$$\mathbb{E}(\tilde{\pi}_\tau(g)) \geq \frac{p}{2}e^{-2\lambda[r/12]\tau} \geq \frac{p}{4}$$

provided we assume r large enough so that $e^{-2\lambda[r/12]\tau} = e^{-2\lambda C'[r/12]/r^2} \geq 1/2$.

Using this in (4) and then (3) gives

$$\mathbb{E}\left(\int_0^\tau \sum_{i \in g} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \geq \frac{p}{8}\phi(a_k^+)e^{-3\gamma/2}$$

and we set $2C_1 = (p/8)e^{-3\gamma/2} > 0$. With this (1) becomes

$$\mathbb{E}\left(\int_0^\tau \sum_{i \in I_0^k} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \geq 2C_1\phi(a_k^+) \left[\frac{|\xi_0^k|}{[r/2]} \right] \geq 2C_1\phi(a_k^+) \left(\frac{2|\xi_0^k|}{r} - 1 \right).$$

Finally, returning to the expression found at the beginning of this proof:

$$\begin{aligned}
\mathbb{E}\left(\int_0^\tau \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) &\geq \sum_{k \in \mathbb{Z}} \mathbb{E}\left(\int_0^\tau \sum_{i \in I_0^k} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j}\right) \\
&\geq \frac{4C_1}{r} \sum_{k \in \mathbb{Z}} \phi(a_k^+) |\zeta_0^k| - 2C_1 \sum_{k \in \mathbb{Z}} \phi(a_k^+) \\
&\geq \frac{4C_1}{r} \sum_{k \in \mathbb{Z}} \sum_{i \in I_0^k} \phi(x_i) - 2C_1 \Phi^+ \\
&= 2C_1 \left(\frac{2\Phi}{r} - \Phi^+\right).
\end{aligned}$$

□

We now possess all the ingredients required to prove theorem 3.1.5.

Proof. This proof follows similarly to step 3 in our original proof on the ring. The interval $[0, t_r]$ can certainly contain $[r]$ sub-intervals of length $\tau = C'/r^2$, which we label $J_1, \dots, J_{[r]}$. These are placed consecutively at the end of the larger interval, perhaps leaving some small time $t_0 < C'/r^2$ at the beginning which is in no subinterval. Using lemma 3.1.3 to bound the process by exponential growth over the interval $[0, t_0]$, we have

$$\mathbb{E}(\Phi_{t_r}) \leq \Phi_0 e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} + \sum_{z=1}^{[r]} \mathbb{E}(\Delta_z), \quad (1)$$

where Δ_z is the change in Φ_s in the interval J_z . Letting Φ_z denote the value of Φ_s at the beginning of the interval J_z , apply corollary 3.1.4 to give

$$\begin{aligned}
&\mathbb{E}(\Delta_z) = \mathbb{E}(\mathbb{E}(\Delta_z | \Phi_z)) \\
&\leq \mathbb{E}\left(\Phi_z (e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} - 1) + \frac{1}{2}(\mu - 2) \mathbb{E}\left(\int_{J_z} \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j} | \Phi_z\right)\right) \\
&= \mathbb{E}(\Phi_z) (e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} - 1) + \frac{1}{2}(\mu - 2) \mathbb{E}\left(\mathbb{E}\left(\int_{J_z} \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j} | \Phi_z\right)\right).
\end{aligned}$$

Now assuming r is large enough for proposition 3.1.8 to hold we have

$$\mathbb{E}\left(\int_{J_z} \sum_{i \in I_s} \phi(x_i) \sum_{\substack{j \in I_s \\ j \neq i}} dl_s^{i,j} | \Phi_z\right) \geq 2C_1\left(\frac{2\Phi_z}{r} - \Phi^+\right),$$

so that

$$\begin{aligned} \mathbb{E}(\Delta_z) &\leq \mathbb{E}(\Phi_z) \left(e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} - 1 \right) + C_1(\mu - 2) \mathbb{E}\left(\frac{2\Phi_z}{r} - \Phi^+\right) \\ &= \mathbb{E}(\Phi_z) \left(e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} - 1 + 2C_1(\mu - 2)/r \right) - C_1(\mu - 2)\Phi^+. \end{aligned}$$

We assume that r is sufficiently large so that $e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r} \leq 7/6$. Now if $\mathbb{E}(\Phi_z) \leq 3r\Phi^+/4$ for any z then

$$\mathbb{E}(\Phi_{t_r}) = \mathbb{E}(\mathbb{E}(\Phi_{t_r} | \Phi_z)) \leq \mathbb{E}(\Phi_z) e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r} \leq 7r\Phi^+/8 = 7\Phi_0/8.$$

From this point on we assume that this is not the case, so there is no z for which $\mathbb{E}(\Phi_z) \leq 3r\Phi^+/4$.

Choosing r large enough so that $e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} \leq 1 - C_1(\mu - 2)/2r$ and returning to the work above, we have

$$\mathbb{E}(\Delta_z) \leq \mathbb{E}(\Phi_z) 3C_1(\mu - 2)/2r - C_1(\mu - 2)\Phi^+.$$

Now with $\mathbb{E}(\Phi_z) \geq 3r\Phi^+/4$ and recalling that $(\mu - 2) < 0$ this yields

$$\mathbb{E}(\Delta_z) \leq \frac{3r\Phi^+}{4} \frac{3C_1(\mu - 2)}{2r} - C_1(\mu - 2)\Phi^+ = C_1(\mu - 2)\Phi^+/8 = -\tilde{p}\Phi^+$$

with $\tilde{p} > 0$. Using this in equation (1) from the beginning of this proof gives

$$\mathbb{E}(\Phi_{t_r}) \leq \Phi_0 e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} - [r]\tilde{p}\Phi^+.$$

Now ensuring that r is large enough so that

$$[r] \geq r/2 \quad \text{and} \quad e^{[(\gamma^2/2) + \lambda(\beta-1)]C'/r^2} \leq 1 + \tilde{p}/4$$

we have

$$\mathbb{E}(\Phi_{t_r}) \leq \Phi_0 \left(1 + \frac{\tilde{p}}{4}\right) - r\Phi + \frac{\tilde{p}}{2} = \Phi_0 \left(1 - \frac{\tilde{p}}{4}\right).$$

Consequently the statement of theorem 3.1.5 holds with

$$\alpha = \max\left\{\frac{7}{8}, 1 - \frac{\tilde{p}}{4}\right\} \in (0, 1).$$

□

3.1.3 Some Additional Results

This section contains a number of corollaries to the theorem proved above. The first result is directly analogous to corollary 2.3.9 for the process on the ring. In it we replace the variable time $t(\Phi_0)$ from theorem 3.1.5 with a fixed time $\tau > 0$. This is then used to show the second result; that the expected value of Φ_t is bounded by some constant for all future times. This second lemma is in the spirit of proposition 2.3.10, although somewhat disguised.

Lemma 3.1.9. *There exists $R_1 \geq 0$, $\alpha' \in (0, 1)$ and $\tau > 0$ so that if the initial distribution of particles on \mathbb{R} is such that $\Phi_0 \geq R_1$, then*

$$\mathbb{E}(\Phi_\tau) \leq \alpha' \Phi_0.$$

Proof. With R_0 , α and C given in theorem 3.1.5, choose R_1 such that

$$R_1 \geq R_0 \quad \text{and} \quad e^{[(\gamma^2/2) + \lambda(\beta-1)]C/R_1} < 1/\sqrt{\alpha}.$$

Now set $\tau = C/R_1$ and notice that if $\Phi_0 \geq R_1$, then $t(\Phi_0) \leq \tau$. Consequently

$$\mathbb{E}(\Phi_\tau) = \mathbb{E}(\mathbb{E}(\Phi_\tau | \Phi_{t(\Phi_0)})) \leq \mathbb{E}(\Phi_{t(\Phi_0)}) e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau} \leq \frac{1}{\sqrt{\alpha}} \alpha \Phi_0 = \sqrt{\alpha} \Phi_0,$$

where the final inequality follows from theorem 3.1.5 and the conditions on R_1 above. Thus the stated result holds if we let $\alpha' = \sqrt{\alpha}$. □

Lemma 3.1.10. *There is some constant, dependent on Φ_0 , so that $\mathbb{E}(\Phi_t)$ is bounded by this constant for all future time $t \geq 0$.*

Proof. Let R_1 , α' and τ be as in lemma 3.1.9 above and fix some M large enough so that

$$(1 - \alpha')M > R_1 e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau}. \quad (1)$$

Now we have

$$\begin{aligned} \mathbb{E}(\Phi_{t+\tau}) &\leq \mathbb{E}(\mathbb{E}(\Phi_{t+\tau} | \Phi_t, \Phi_t \geq R_1)) + \mathbb{E}(\mathbb{E}(\Phi_{t+\tau} | \Phi_t, \Phi_t < R_1)) \\ &\leq \alpha' \mathbb{E}(\Phi_t) + R_1 e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau}, \end{aligned}$$

where the first and last terms on the right-hand side are given by applications of lemmas 3.1.9 and 3.1.3 respectively. Now it follows that if $\mathbb{E}(\Phi_t) > M$ then

$$\begin{aligned} \mathbb{E}(\Phi_{t+\tau}) &\leq \alpha' \mathbb{E}(\Phi_t) + R_1 e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau} \\ &\stackrel{\text{using (1)}}{<} \alpha' \mathbb{E}(\Phi_t) + (1 - \alpha')M \\ &< \alpha' \mathbb{E}(\Phi_t) + (1 - \alpha')\mathbb{E}(\Phi_t) = \mathbb{E}(\Phi_t). \end{aligned}$$

So when $\mathbb{E}(\Phi_t) > M$ the value of $\mathbb{E}(\Phi_{t+\tau})$ is strictly smaller than $\mathbb{E}(\Phi_t)$. Examining $\mathbb{E}(\Phi_t)$ at each of the time points $t \in \{n\tau : n \in \mathbb{N}\}$ and bounding by exponential growth in between, we see that $\mathbb{E}(\Phi_t)$ is bounded by

$$\max\{\Phi_0 e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau}, M e^{2[(\gamma^2/2) + \lambda(\beta-1)]\tau}\}. \quad (2)$$

□

To state and prove the final results in this chapter we introduce a new notation. Suppose that we have a countable rather than finite initial number of particles, still fulfilling the condition $\sum_{i \in I_0} \phi(x_i) < \infty$. We label these initial particles $0, 1, 2, \dots$ according to increasing distance from the origin, with some arbitrary choice made in the case of a tie. Then let $I_0(n)$ be the indexing of the

first n particles in this list. We refer to these as the ' n particles closest to the origin', although that is not strictly true in the case of ties. Let $\xi_t(n)$ be the process at time t constructed from these n initial particles, and let $I_t(n)$ be the indexing of the particles in $\xi_t(n)$. Notice that although $\xi_t(n)$ is almost surely finite, it *does not* necessarily contain n particles; it is the process at time t when *started* from the initial state $\xi_0(n)$.

We now have a simple corollary to the lemma above in terms of these restricted processes:

Corollary 3.1.11. *Suppose I_0 is countable with $\sum_{i \in I_0} \phi(x_i) < \infty$. Then there exists some constant $M_1 < \infty$ such that*

$$\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) \leq M_1$$

for all $n \in \mathbb{N}$ and all $t \geq 0$.

Proof. Applying the bound labelled (2) from the proof of lemma 3.1.10 to give the first inequality and recalling that ϕ is positive to give the second we can write

$$\begin{aligned} \mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) &\leq \max\left\{\sum_{i \in I_0(n)} \phi(x_i) e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau}, \quad M e^{2[(\gamma^2/2) + \lambda(\beta-1)]\tau}\right\} \\ &\leq \max\left\{\sum_{i \in I_0} \phi(x_i) e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau}, \quad M e^{2[(\gamma^2/2) + \lambda(\beta-1)]\tau}\right\} =: M_1. \end{aligned}$$

Noticing that this final constant is independent of both n and t and is certainly finite we are done. \square

Proposition 3.1.12. *In the situation described in the above corollary*

$$\sup_{n \in \mathbb{N}} \sup_{t \geq 0} P\left(\sum_{i \in I_t(n)} \phi(x_i) \geq k\right) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof. By Markov's inequality

$$\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) \geq kP\left(\sum_{i \in I_t(n)} \phi(x_i) \geq k\right)$$

so using corollary 3.1.11 above to give the last inequality we have

$$\sup_{n \in \mathbb{N}} \sup_{t \geq 0} P\left(\sum_{i \in I_t(n)} \phi(x_i) \geq k\right) \leq \frac{1}{k} \sup_{n \in \mathbb{N}} \sup_{t \geq 0} \mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) \leq \frac{1}{k} \sup_{n \in \mathbb{N}} \sup_{t \geq 0} M_1 = \frac{M_1}{k}.$$

Consequently

$$\sup_{n \in \mathbb{N}} \sup_{t \geq 0} P\left(\sum_{i \in I_t(n)} \phi(x_i) \geq k\right) \leq \frac{M_1}{k} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

□

Chapter 4

Constructing the Infinite Process

We are finally in a position to construct our pairwise interacting branching processes on \mathbb{R} started from infinitely many particles. This will be the sole aim of this chapter.

We would perhaps hope for a direct path-wise construction for these processes, similar to that presented for the finite processes in section 1.4. It shall be seen at the end of this chapter that in several important cases we can provide such graphical constructions, relying on percolation-type arguments. However, when dealing with a general interacting branching process this percolation argument breaks down. Numerous additional difficulties arise and path-wise construction methods are no longer readily apparent. Still, although they do not cover the entire generality of models we wish to discuss, these graphical constructions are highly intuitive and can provide a helpful ‘picture’ in later work.

We concentrate instead on developing a ‘soft’ construction of our infinite processes as the limits of finite ones. This approach has the advantage of being valid for our entire class interacting branching processes, as well as meaning that

several important results pass over easily from the finite to the infinite setting. We begin by presenting the duality relation between our (finite) processes and certain stochastic PDEs. This result is due to Athreya and Tribe [3]. As discussed in the introduction this duality is not only one of the most powerful tools we have available to us, but is also one of the prime motivations for studying these particle systems. Athreya and Tribe used this duality to show uniqueness in law for solutions of a class of non-Lipschitz stochastic PDEs. We will use the duality in reverse as a tool to construct the law of our infinite processes as the limiting law of finite processes.

The next step is to embed our models in the framework of measure-valued processes. As in the case of our earlier work on the unit ring, S , this representation is both intuitive and useful. It yields a natural metric on the realisations of our processes. It is well known (see for example Dawson [11]) that the law of a random measure on \mathbb{R} can be determined by its Laplace functionals. We specify the Laplace functionals for our infinite process ξ_t as the limit of the Laplace functionals of the restricted processes $\xi_t(n)$, and use a tightness argument to give a corresponding law. The limits themselves are shown to exist via the duality formula.

4.1 The Duality Relation

Firstly we fix some pairwise interacting branching diffusion process of the type we have been discussing. Recall that the single particle births and the pairwise interaction mechanism have probability generating functions

$$\sum_{k=0}^{\infty} p_k s^k \quad \text{and} \quad \sum_{k=0}^{\infty} q_k s^k$$

respectively. Now we consider bounded solutions of the one-dimensional SPDE

$$\partial_t u = \frac{1}{2} \Delta u + b(u) + \sqrt{\sigma(u)} \dot{W}_{t,x}, \quad u_0 \in [a, 1] \quad (1)$$

where $\dot{W}_{t,x}$ is space-time white noise. The functions $b(u)$ and $\sigma(u)$ are analytic and are given in terms of the particle system offspring probabilities p_k and q_k , namely

$$b(u) = \lambda \left(\sum_{k=0}^{\infty} p_k u^k - u \right) \quad \text{and} \quad \sigma(u) = \sum_{k=0}^{\infty} q_k u^k - u^2. \quad (2)$$

As discussed in the introduction, this SPDE is well-defined and solutions bounded in $[a, 1]$, with $-1 \leq a \leq 0$, exist provided

$$\sum_{k=0}^{\infty} k q_k - 2 \leq 0.$$

This is exactly the condition $\mu \leq 2$, which equates to a particle system in which the interactions are either reductive or critical. It was noted that in the case $\mu > 2$ when the interactions provide growth, the SPDE is not well-defined.

The duality relation between solutions to the stochastic PDE and our particle process started from finitely many initial particles is as follows:

Proposition 4.1.1. *Suppose that u is a solution to the above SPDE satisfying $a \leq u_t(x) \leq 1$ for all $t \geq 0$, $x \in \mathbb{R}$ and that ξ_t is the corresponding particle system started from a finite set of points $\{x_i : i \in I_0\}$. Then the following duality relation holds*

$$\mathbb{E} \left(\prod_{i \in I_0} u_t(x_i) \right) = \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right).$$

No proof will be provided for this duality relation since the result is taken almost directly from Athreya and Tribe [3], but we do provide several remarks.

Firstly we notice that the form of the duality presented here is simpler than that found in [3], containing neither the exponential term nor the power of -1 . The reason for this is simple: whereas Athreya and Tribe began with a given SPDE and constructed the corresponding dual particle system, we have worked in the other direction. Given *general* analytic functions $b(u)$ and $\sigma(u)$ and writing them in the form set out in (2) above, it is highly unlikely that the p_k 's and q_k 's would give sensible probability values. Several of the coefficients could be negative and their sum is unlikely to be 1! In order to study a larger class of SPDEs Athreya and Tribe needed to normalise the p_k 's and q_k 's yielded from b and σ , and also introduced a counting process which ran along-side the particle system and was triggered every time a birth or interaction occurred which corresponded to a negative coefficient of b or σ . It is these ingredients which lead to the extra factors in the duality relation. Beginning with the particle system as we have and constructing the SPDE yields this simpler duality.

Secondly our result contains neither of the two hypotheses stated in the Athreya and Tribe theorem. Further, in the situation in which the branching is a growth mechanism ($\beta > 1$) our function $b(u)$ fails to fulfil either of these two hypotheses (the other function $\sigma(u)$ fulfils both provided $\mu < 2$). The arguments used in [3] to prove proposition 4.1.1 continue to hold, it is simply that the facts which follow from these hypotheses have either been covered already or are redundant. The first of the two hypotheses is designed to ensure that the dual particle system is non-explosive, and we have shown this to be true for our particle systems independently (see section 2.2 and in particular remark 2.2.1). The second hypothesis ensures integrability of the exponential terms which occur in the error bounds used when proving the duality. However as has been remarked earlier, there are no corresponding exponential terms in our simpler case and this hypothesis can be discarded. Thus it follows that whilst there are many SPDEs which do not occur as duals to our particle systems via

this simple duality, but which are covered in [3], so too there are certain classes of SPDEs which are excluded by the Athreya and Tribe hypotheses but which do occur as the associated SPDEs to our processes in this simple duality.

4.2 The Construction

4.2.1 ξ_t as a Measure-valued Process

As in the case of the models on the unit ring we can easily convert our processes into measures by placing a unit point mass at the position of each particle. Thus the natural state space for our processes when considered as measures is

$$M_f^\phi(\mathbb{R}) = \left\{ \sum_{i \in I} \delta_{x_i} : I \text{ countable, } \sum_{i \in I} \phi(x_i) < \infty \right\}.$$

As we wish to allow for the construction of infinite processes we see these measures will not always be finite. However we can use the test function ϕ as a push-forward map from $M_f^\phi(\mathbb{R})$ into $M_f(\mathbb{R})$, the space of finite measures on \mathbb{R} , namely

$$\phi : M_f^\phi(\mathbb{R}) \rightarrow M_f(\mathbb{R}), \quad \phi\left(\sum_{i \in I} \delta_{x_i}\right) = \sum_{i \in I} \phi(x_i) \delta_{x_i}.$$

It is easy to see that this map is well-defined since if $\sum_{i \in I} \delta_{x_i} \in M_f^\phi(\mathbb{R})$, then

$$\phi\left(\sum_{i \in I} \delta_{x_i}\right)(\mathbb{R}) = \left(\sum_{i \in I} \phi(x_i) \delta_{x_i}\right)(\mathbb{R}) = \sum_{i \in I} \phi(x_i) < \infty$$

by the definition of elements of $M_f^\phi(\mathbb{R})$. Further, since ϕ is non-zero on \mathbb{R} , the map is injective and has an obvious inverse:

$$\phi^{-1}\left(\phi\left(\sum_{i \in I} \delta_{x_i}\right)\right) = \phi^{-1}\left(\sum_{i \in I} \phi(x_i) \delta_{x_i}\right) = \sum_{i \in I} \frac{1}{\phi(x_i)} \phi(x_i) \delta_{x_i} = \sum_{i \in I} \delta_{x_i}.$$

We remark that this map $\phi : M_f^\phi(\mathbb{R}) \rightarrow M_f(\mathbb{R})$ is just a restriction of the more general map ϕ defined on all measures on \mathbb{R} via $(\phi\mu)(dx) = \phi(x)\mu(dx)$.

Similarly the inverse map ϕ^{-1} on $\phi(M_f^\phi(\mathbb{R}))$ is the restriction of the general map ϕ^{-1} defined by $(\phi^{-1}\mu)(dx) = \phi^{-1}(x)\mu(dx)$.

The topology of weak convergence on $M_f(\mathbb{R})$ can also be pushed back onto $M_f^\phi(\mathbb{R})$ in a similar way. So

$$\mu_n \xrightarrow{\phi} \mu \text{ in } M_f^\phi(\mathbb{R}) \quad \text{iff} \quad \langle \phi\mu_n, f \rangle \rightarrow \langle \phi\mu, f \rangle \quad \forall f \in bC(\mathbb{R})$$

where $bC(\mathbb{R})$ is the set of all bounded continuous functions on \mathbb{R} and $\langle \mu, f \rangle$ denotes $\int_{\mathbb{R}} f(x)\mu(dx)$.

Given any countable set of functions $\{f_n\} \subset bC(\mathbb{R})$ which is convergence determining, we can define a metric on $M_f(\mathbb{R})$ associated with the topology of weak convergence of the form

$$d(\mu, \nu) = \sum_{m=0}^{\infty} \frac{1}{2^m} (1 \wedge |\langle \mu, f_m \rangle - \langle \nu, f_m \rangle|) \quad \text{for } \mu, \nu \in M_f(\mathbb{R}).$$

In fact we can, and do, choose such a set of functions $V = \{f_n\}$ so that $f_0 \equiv 1$, the functions are continuous, non-negative and bounded by 1, and all except f_0 have tails reducing to zero as $|x|$ gets large.

Lemma 4.2.1. *Suppose μ is some measure on \mathbb{R} such that $\langle \mu, \phi \rangle < \infty$, and let $At(\mu)$ be the points of \mathbb{R} at which μ is atomic. Then*

$$\mu \in M_f^\phi(\mathbb{R}) \iff \mu([a, b]) \in \mathbb{N} \text{ for any interval } [a, b] \text{ on } \mathbb{R} \text{ with } a, b \notin At(\mu).$$

Proof. This result is similar to that shown in lemma 2.4.2 earlier. The implication from left to right is trivial, so we concentrate on the inverse. By assumption $\phi\mu$ is a finite measure, so as in 2.4.2 it follows that $At(\phi\mu)$ is countable and hence contains no interval. Trivially $At(\phi\mu) = At(\mu)$ so the same holds for $At(\mu)$. Let $r \in \mathbb{R}$ be some point which is not in $At(\mu)$ and define the functions f and g as follows:

$$f(s) = \mu([r, r + s]), \quad g(s) = \mu([r - s, r]) \quad s \in \mathbb{R}_{\geq 0}.$$

The domain of each function is restricted so that f is defined on those s for which $r + s \notin \text{At}(\mu)$, and g is defined on those s for which $r - s \notin \text{At}(\mu)$. Each function is \mathbb{N} -valued and increasing in s , with a series of jump discontinuities of integer size. Between these points the measure μ has no mass, whilst at each jump point it has a positive integer mass which can be considered as a number of unit point masses. Thus μ is of the form required to be in $M_f^\phi(\mathbb{R})$ and by assumption fulfils $\langle \mu, \phi \rangle < \infty$. \square

Lemma 4.2.2. *The image of the natural state space $M_f^\phi(\mathbb{R})$ under the map ϕ is closed in $M_f(\mathbb{R})$.*

Proof. Suppose that some sequence $\{\nu_n\} \subset \phi(M_f^\phi(\mathbb{R}))$ converges to ν in $M_f(\mathbb{R})$. We need to show that $\nu = \phi\mu$ for some element $\mu \in M_f^\phi(\mathbb{R})$. Firstly notice that, as mentioned earlier, we can define the measure $\phi^{-1}\nu$ regardless of whether ν is an element of $\phi(M_f^\phi(\mathbb{R}))$. We do this via $(\phi^{-1}\nu)(dx) = \phi^{-1}(x)\nu(dx)$, so that

$$\phi^{-1}\nu(A) = \int_A \frac{1}{\phi(x)} \nu(dx)$$

for any Borel set $A \subset \mathbb{R}$. Thus we are required to show that $\phi^{-1}\nu$ is an element of $M_f^\phi(\mathbb{R})$.

We certainly have $\langle \phi^{-1}\nu, \phi \rangle < \infty$ since

$$\langle \phi^{-1}\nu, \phi \rangle = \int_{\mathbb{R}} \phi(x) \phi^{-1}\nu(dx) = \int_{\mathbb{R}} \phi(x) \frac{1}{\phi(x)} \nu(dx) = \nu(\mathbb{R}) < \infty$$

as ν is a finite measure on \mathbb{R} . We can now apply lemma 4.2.1 above to show that $\phi^{-1}\nu$ is an element of $M_f^\phi(\mathbb{R})$. To yield a contradiction we suppose otherwise: that there exists some interval $[a, b] \subset \mathbb{R}$, with $a, b \notin \text{At}(\phi^{-1}\nu)$, such that

$$\phi^{-1}\nu([a, b]) = \gamma \notin \mathbb{N}. \quad (1)$$

Therefore there exists some $m \in \mathbb{N}$ so that $m < \gamma < m + 1$, and we define

$$\Delta = \min\{\gamma - m, m + 1 - \gamma\}.$$

Now as in the proof of lemma 2.4.3 we define continuous approximating super- and sub-functions f_ϵ and g_ϵ to the indicator on $[a, b]$ by

$$f_\epsilon(s) = \begin{cases} 1 & s \in [a, b] \\ 0 & s \notin [a - \epsilon, b + \epsilon] \\ \text{linear in between} & \end{cases} \quad g_\epsilon(s) = \begin{cases} 1 & s \in [a + \epsilon, b - \epsilon] \\ 0 & s \notin [a, b] \\ \text{linear in between} & \end{cases}$$

noticing that g_ϵ is well-defined for sufficiently small $\epsilon < 0$. Thus as ϵ tends to zero we have

$$\int_{\mathbb{R}} f_\epsilon(s) \phi^{-1} \nu(ds) \searrow \phi^{-1} \nu([a, b]) = \gamma$$

$$\int_{\mathbb{R}} g_\epsilon(s) \phi^{-1} \nu(ds) \nearrow \phi^{-1} \nu((a, b)) = \phi^{-1} \nu([a, b]) = \gamma.$$

Some sufficiently small $\delta > 0$ may now be fixed so that

$$\gamma - \frac{\Delta}{2} < \int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu(ds) \leq \gamma \leq \int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu(ds) < \gamma + \frac{\Delta}{2}.$$

As we have $\nu_n \Rightarrow \nu$ in $M_f(\mathbb{R})$ and since the functions f_δ/ϕ and g_δ/ϕ are continuous and bounded we can write

$$\int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu_n(ds) = \int_{\mathbb{R}} \frac{f_\delta(s)}{\phi(s)} \nu_n(ds) \longrightarrow \int_{\mathbb{R}} \frac{f_\delta(s)}{\phi(s)} \nu(ds) = \int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu(ds)$$

$$\int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu_n(ds) = \int_{\mathbb{R}} \frac{g_\delta(s)}{\phi(s)} \nu_n(ds) \longrightarrow \int_{\mathbb{R}} \frac{g_\delta(s)}{\phi(s)} \nu(ds) = \int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu(ds).$$

Now we may choose N sufficiently large so that both

$$\left| \int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu_N(ds) - \int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu(ds) \right|, \quad \left| \int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu_N(ds) - \int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu(ds) \right|$$

are less than $\Delta/2$.

Consequently

$$\phi^{-1} \nu_N([a, b]) \leq \int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu_N(ds) < \int_{\mathbb{R}} f_\delta(s) \phi^{-1} \nu(ds) + \frac{\Delta}{2} < \gamma + \Delta \leq m + 1$$

$$\phi^{-1} \nu_N([a, b]) \geq \int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu_N(ds) > \int_{\mathbb{R}} g_\delta(s) \phi^{-1} \nu(ds) - \frac{\Delta}{2} > \gamma + \Delta \geq m$$

which gives

$$m < \phi^{-1}\nu_N([a, b]) < m + 1.$$

By lemma 4.2.1 this implies that $\phi^{-1}\nu_N \notin M_f^\phi(\mathbb{R})$ which in turn contradicts $\nu_N \in \phi(M_f^\phi(\mathbb{R}))$. Thus there can be no such interval $[a, b]$ for which (1) holds and so $\phi^{-1}\nu \in M_f^\phi(\mathbb{R})$. From this we have $\nu \in \phi(M_f^\phi(\mathbb{R}))$ and so this image set is closed. \square

4.2.2 Duality and Laplace Functionals

Our intention is to construct the law of the process at time t started from some infinite initial condition, I_0 , as the limit as n tends to infinity of the laws of the restricted finite processes started from $I_0(n)$. Recall that $I_0(n)$ is the indexing of the n particles closest to the origin, with a more precise definition preceding corollary 3.1.11 in the previous chapter. In this limiting argument we make use of the finite duality relation discussed earlier, but we begin by re-formulating the result into a more helpful form.

Lemma 4.2.3. *For any finite I_0 and function $u_0 \in C(\mathbb{R}, (0, 1])$ we have:*

$$(i). \quad \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) = \int_{M_f^\phi(\mathbb{R})} e^{-\langle \mu, -\ln u_0(\cdot) \rangle} P_t^{\mu_0}(d\mu),$$

where $P_t^{\mu_0}$ is the law of the process on $M_f^\phi(\mathbb{R})$ at time t if started from an initial measure μ_0 corresponding to I_0 . So $P_t^{\mu_0}(A) = P(\xi_t \in A | \xi_0 = \mu_0)$ for $A \subset M_f^\phi(\mathbb{R})$ where $\mu_0 = \sum_{i \in I_0} \delta_{x_i}$.

$$(ii). \quad \int_{M_f^\phi(\mathbb{R})} e^{-\langle \mu, -\ln u_0(\cdot) \rangle} P_t^{\mu_0}(d\mu) = \int_{M_f(\mathbb{R})} e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle} \tilde{P}_t^{\mu_0}(d\nu)$$

where $\tilde{P}_t^{\mu_0}$ is the push-forward measure of $P_t^{\mu_0}$ onto $M_f(\mathbb{R})$ using ϕ , so that $\tilde{P}_t^{\mu_0}(A) = P_t^{\mu_0}(\phi^{-1}A)$ for $A \subset M_f(\mathbb{R})$.

Proof. These are simple re-arrangements:

$$\begin{aligned} \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) &= \mathbb{E}\left(e^{\sum_{i \in I_t} \ln u_0(x_i)}\right) = \mathbb{E}\left(e^{-\langle \xi_t, -\ln u_0(\cdot) \rangle}\right) = \int_{M_f^\phi(\mathbb{R})} e^{-\langle \mu, -\ln u_0(\cdot) \rangle} P_t^{\mu_0}(d\mu). \\ &= \int_{M_f^\phi(\mathbb{R})} e^{-\langle \mu, -\ln u_0(\cdot) \rangle} P_t^{\mu_0}(d\mu) = \int_{M_f(\mathbb{R})} e^{-\langle \phi^{-1}\nu, -\ln u_0(\cdot) \rangle} P_t^{\mu_0}(\phi^{-1}(d\nu)) \\ &= \int_{M_f(\mathbb{R})} e^{-\int -\ln u_0(s) \frac{1}{\phi(s)} \nu(ds)} \tilde{P}_t^{\mu_0}(d\nu) = \int_{M_f(\mathbb{R})} e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle} \tilde{P}_t^{\mu_0}(d\nu). \end{aligned}$$

□

This now allows us to write one side of our duality formula in terms of Laplace functionals:

Corollary 4.2.4. *For $u_0 \in C(\mathbb{R}, (0, 1])$, the finite duality relation given in proposition 4.1.1 can be written as*

$$\int_{M_f(\mathbb{R})} e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle} \tilde{P}_t^{\mu_0}(d\nu) = \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right).$$

Proof. We simply use (i) followed by (ii) from lemma 4.2.3 above to rewrite the term $\mathbb{E}(\prod_{i \in I_t} u_0(x_i))$ in the form given above. □

In order to construct our infinite process we wish to determine the laws $P_t^{\mu_0}$ for all $t \geq 0$ and all $\mu_0 \in M_f^\phi(\mathbb{R})$, not just those which are finite. Any such law $P_t^{\mu_0}$ is in turn determined by the corresponding push-forward measure $\tilde{P}_t^{\mu_0}$ since

$$P_t^{\mu_0}(A) = \tilde{P}_t^{\mu_0}(\phi(A))$$

for any Borel set $A \subset M_f^\phi(\mathbb{R})$. Thus we may consider our process as living in the space $M_f^\phi(\mathbb{R})$ with law $P_t^{\mu_0}$ at time t , or equivalently we may consider our weighted process living in the space $\phi(M_f^\phi(\mathbb{R}))$ with law $\tilde{P}_t^{\mu_0}$ at time t . The advantage of this second setting is that $\phi(M_f^\phi(\mathbb{R})) \subset M_f(\mathbb{R})$ so that the measures

defined by our process are finite. Our aim now is to specify $\tilde{P}_t^{\mu_0}$ for any t and μ_0 .

If $\mu_0 \in M_f^\phi(\mathbb{R})$ corresponds to some initial state with labelling I_0 , then we let $\mu_0(n)$ correspond to $I_0(n)$, the restriction of I_0 to the first n particles. One of the main steps required in showing the existence of the law $\tilde{P}_t^{\mu_0}$ is to show that the sequence of Laplace functionals

$$\int_{M_f(\mathbb{R})} e^{-\langle \nu, f \rangle} \tilde{P}_t^{\mu_0(n)}(d\nu)$$

converges as $n \rightarrow \infty$ for any $f \in V$, a set of convergence determining bounded and continuous functions on \mathbb{R} such as those given in section 4.2.1. This, together with tightness of the laws $\{\tilde{P}_t^{\mu_0(n)}\}$ and some careful topological arguments, will give a limiting $\tilde{P}_t^{\mu_0}$ supported on the space $\phi(M_f^\phi(\mathbb{R}))$, so that the limit is a probability on point mass type measures as required.

To show this convergence of Laplace functionals we can equivalently prove that

$$\int_{M_f(\mathbb{R})} e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle} \tilde{P}_t^{\mu_0(n)}(d\nu)$$

converges, where u_0 takes the form $u_0(\cdot) = e^{-\phi(\cdot)f(\cdot)}$ for $f \in V$. We notice that for any $f \in V$ the corresponding u_0 is continuous and bounded in $(0, 1]$, with tails which become arbitrarily close to 1. Thus each such u_0 is a suitable initial condition for the stochastic PDE used in the duality formula, and that formula can be written in the form given in corollary 4.2.4. Using this we have

$$\int_{M_f(\mathbb{R})} e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle} \tilde{P}_t^{\mu_0(n)}(d\nu) = \mathbb{E} \left(\prod_{i \in I_0(n)} u_t(x_i) \right).$$

Our aim now is to show that the sequence of expectations given on the right-hand side of the above converges. This will be done in the next section, which follows this small lemma.

Lemma 4.2.5. *Let μ be some element of $M_f^\phi(\mathbb{R})$ and again define $\mu(n)$ to be the restriction of μ to the n particles closest to the origin. Thus if μ corresponds to I_0 , then the restrictions $\mu(n)$ correspond to $I_0(n)$. Then under the topology of weak convergence we have*

$$\phi\mu(n) \implies \phi\mu \text{ in } M_f(\mathbb{R}) \text{ as } n \rightarrow \infty.$$

Proof. We must show that

$$d(\phi\mu, \phi\mu(n)) = \sum_{m=0}^{\infty} \frac{1}{2^m} (1 \wedge |\langle \phi\mu, f_m \rangle - \langle \phi\mu(n), f_m \rangle|)$$

becomes arbitrarily small as n gets large. Begin by fixing some $\epsilon > 0$.

Note that as $\mu \in M_f^\phi(\mathbb{R})$ then by definition we have $\langle \phi\mu, 1 \rangle = \sum_{i \in I_0} \phi(x_i) < \infty$.

Thus there exists some n_0 such that for $n \geq n_0$,

$$\sum_{i \in I_0 \setminus I_0(n)} \phi(x_i) < \frac{\epsilon}{2}.$$

Now since each f_m is non-negative and bounded by 1, we have

$$\left| \langle \phi\mu, f_m \rangle - \langle \phi\mu(n), f_m \rangle \right| = \left| \sum_{i \in I_0} f_m(x_i) \phi(x_i) - \sum_{i \in I_0(n)} f_m(x_i) \phi(x_i) \right| \leq \sum_{i \in I_0 \setminus I_0(n)} \phi(x_i).$$

Putting these together we have, for $n \geq n_0$,

$$\begin{aligned} d(\phi\mu, \phi\mu(n)) &= \sum_{m=0}^{\infty} \frac{1}{2^m} (1 \wedge |\langle \phi\mu, f_m \rangle - \langle \phi\mu(n), f_m \rangle|) \\ &\leq \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{i \in I_0 \setminus I_0(n)} \phi(x_i) < \sum_{m=0}^{\infty} \frac{1}{2^m} \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

4.2.3 Existence of the Limit

To begin we state and prove a simple analysis lemma. This result will prove very useful here and also in later work.

Lemma 4.2.6. Suppose $a_n(k) \in [-1, 1]$ for $n, k \in \mathbb{N}$. Further assume that $a_n(k) \searrow a(k)$ as $n \rightarrow \infty$. Then if the products exist we have $\prod_k a_n(k) \rightarrow \prod_k a(k)$ as $n \rightarrow \infty$.

Proof. Case(i): $\prod_k a(k) = 0$.

Fix $\epsilon > 0$ and choose m large enough so that $\prod_{k=0}^m a(k) < \epsilon/2$. Now as m is finite we can choose n large enough such that

$$\left| \prod_{k=0}^m a_n(k) \right| \leq \left| \prod_{k=0}^m a(k) \right| + \left| \prod_{k=0}^m a_n(k) - \prod_{k=0}^m a(k) \right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Now clearly for large enough n we have

$$\left| \prod_k a_n(k) \right| = \left| \prod_{k=0}^m a_n(k) \right| \left| \prod_{k=m+1}^{\infty} a_n(k) \right| \leq \left| \prod_{k=0}^m a_n(k) \right| < \epsilon.$$

As $\epsilon > 0$ was arbitrary this shows that $\prod_k a_n(k) \rightarrow 0$ as $n \rightarrow \infty$.

Case(ii): $\left| \prod_k a(k) \right| > 0$.

If the terms $a(k)$ are negative for infinitely many k then the partial products $\prod_{k=0}^m a(k)$ will change sign infinitely often as $m \rightarrow \infty$. In this case the only possible value of the limit $\prod_k a(k)$ of these partial products is 0. But here we have assumed that $\left| \prod_k a(k) \right| > 0$, so there is some $K \in \mathbb{N}$ such that $a(k) \geq 0 \forall k \geq K$. Consequently $a_n(k) > 0 \forall k \geq K$ for each n also. Further, $\left| \prod_k a(k) \right| > 0$ implies that $\prod_{k=m}^{\infty} a(k) \rightarrow 1$ as $m \rightarrow \infty$. Thus for any $\epsilon > 0$ we can choose $m \geq K$ large enough so that

$$\left| 1 - \prod_{k=m+1}^{\infty} a_n(k) \right| \leq \left| 1 - \prod_{k=m+1}^{\infty} a(k) \right| < \epsilon/3.$$

Now choosing n large enough so that $\left| \prod_{k=0}^m a(k) - \prod_{k=0}^m a_n(k) \right| < \epsilon/3$ we can

write

$$\begin{aligned}
 & \left| \prod_k a(k) - \prod_k a_n(k) \right| \\
 &= \left| \prod_k a(k) - \prod_{k=0}^m a(k) \right| + \left| \prod_{k=0}^m a(k) - \prod_{k=0}^m a_n(k) \right| + \left| \prod_{k=0}^m a_n(k) - \prod_k a_n(k) \right| \\
 &< \left| \prod_{k=0}^m a(k) \right| \left| 1 - \prod_{k=m+1}^{\infty} a(k) \right| + \epsilon/3 + \left| \prod_{k=0}^m a_n(k) \right| \left| 1 - \prod_{k=m+1}^{\infty} a_n(k) \right| \\
 &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
 \end{aligned}$$

This completes the second case and gives the convergence of products as required. \square

Our next result is a technical lemma concerning the SPDE started from an initial condition u_0 of the form we have discussed. As a corollary to this we see that only finitely many of the terms $\{u_t(x_i) : i \in I_0\}$ are negative with probability 1.

Lemma 4.2.7. *Let u_t be a solution to the SPDE started from an initial condition of the form $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$ for some non-negative, bounded and continuous f . Let $\{x_i : i \in I_0\}$ be a suitable initial state for the particle system, so that $\sum_{i \in I_0} \phi(x_i) < \infty$. Then*

$$\sum_{i \in I_0} \mathbb{E}(1 - u_t(x_i)) < \infty.$$

Proof. Letting $v_t(x) = 1 - u_t(x)$ and using the Green's representation for the solution u_t (see [38]) we can write

$$\begin{aligned}
 \mathbb{E}(v_t(x_i)) &= \mathbb{E}(1 - u_t(x_i)) \\
 &= 1 - \int_{\mathbb{R}} u_0(x_i) G_t(x, x_i) dx - \mathbb{E} \left(\int_{\mathbb{R}} \int_0^t b(u_s(x)) G_{t-s}(x, x_i) ds dx \right) \quad (1)
 \end{aligned}$$

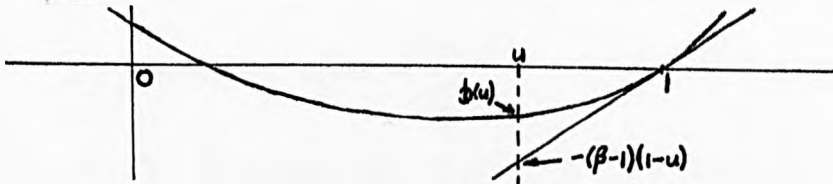
where $G_t(x, x_i) = \frac{1}{\sqrt{2\pi t}} e^{-(x-x_i)^2/2t}$. Firstly we notice that as $\int_{\mathbb{R}} G_t(x, x_i) dx = 1$ we have

$$1 - \int_{\mathbb{R}} u_0(x_i) G_t(x, x_i) dx = \int_{\mathbb{R}} v_0(x_i) G_t(x, x_i) dx.$$

Secondly, for $|u| \leq 1$ we have

$$b'(u) = \sum_{k=0}^{\infty} k p_k u^{k-1} - 1 \leq \sum_{k=0}^{\infty} k p_k - 1 = b'(1) = \beta - 1$$

and so $b(u) \geq -(\beta - 1)(1 - u)$ as illustrated below.



Using this together with Fubini in the final term on the right-hand side of (1) yields

$$-\mathbb{E}\left(\int_{\mathbb{R}} \int_0^t b(v_s(x)) G_{t-s}(x, x_i) ds dx\right) \leq (\beta - 1) \int_{\mathbb{R}} \int_0^t \mathbb{E}(v_s(x)) G_{t-s}(x, x_i) ds dx.$$

Thus we reformulate (1) as

$$\mathbb{E}(v_t(x_i)) \leq \int_{\mathbb{R}} v_0(x_i) G_t(x, x_i) dx + (\beta - 1) \int_{\mathbb{R}} \int_0^t \mathbb{E}(v_s(x)) G_{t-s}(x, x_i) ds dx. \quad (1')$$

Now we define a map $\Lambda_i : [0, t] \rightarrow \mathbb{R}$ by

$$\Lambda_i(s) = \int_{\mathbb{R}} \mathbb{E}(v_s(x)) G_{t-s}(x, x_i) dx.$$

This map Λ_i can be continuously extended to the closed interval $[0, t]$ provided we set

$$\Lambda_i(t) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \mathbb{E}(v_{t-\epsilon}(x)) G_{\epsilon}(x, x_i) dx = \mathbb{E}(v_t(x_i)).$$

Using Fubini to exchange the integrals in the final term of (1') we can write

$$\Lambda_i(t) \leq \Lambda_i(0) + (\beta - 1) \int_0^t \Lambda_i(s) ds.$$

Applying Gronwall's inequality here yields

$$\Lambda_i(t) = \mathbb{E}(v_t(x_i)) \leq e^{(\beta-1)t} \Lambda_i(0) = e^{(\beta-1)t} \int_{\mathbb{R}} v_0(x) G_t(x, x_i) dx$$

so to prove this lemma it suffices to show that $\sum_{i \in I_0} \int_{\mathbb{R}} v_0(x) G_t(x, x_i) dx < \infty$.

We know that $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$ for some non-negative, bounded and continuous f . Suppose therefore that $f(x) \leq M, \forall x \in \mathbb{R}$. Now using the fact that $1 - e^{-x} < x$ we have

$$\begin{aligned} \sum_{i \in I_0} \int_{\mathbb{R}} v_0(x) G_t(x, x_i) dx &= \sum_{i \in I_0} \int_{\mathbb{R}} (1 - e^{-f(x)\phi(x)}) G_t(x, x_i) dx \\ &\leq \sum_{i \in I_0} \int_{\mathbb{R}} f(x)\phi(x) G_t(x, x_i) dx \leq M \sum_{i \in I_0} \int_{\mathbb{R}} \phi(x) G_t(x, x_i) dx \end{aligned}$$

so it suffices to show that the sum in the final term in the above equation converges. Further, since ϕ is bounded and the number of points of I_0 in the interval $[-1, 1]$ is finite, it suffices to show that

$$\sum_{\substack{i \in I_0 \\ x_i \in [-1, 1]}} \int_{\mathbb{R}} \phi(x) G_t(x, x_i) dx < \infty. \quad (2)$$

Now $\phi(x)$ is bounded above by $e^{-\gamma|x|}$ so we have

$$\begin{aligned} \int_{\mathbb{R}} \phi(x) G_t(x, x_i) dx &\leq \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\gamma|x|} e^{-(x-x_i)^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(x^2+x_i^2-2x_i x+2t\gamma|x|)} dx \\ &\leq \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(x^2+x_i^2-2|x_i||x|+2t\gamma|x|)} dx \\ &= e^{t\gamma^2/2} e^{-\gamma|x_i|} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(|x|-(|x_i|-t\gamma))^2} dx. \end{aligned}$$

Now noticing that

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(|x|-(|x_i|-t\gamma))^2} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{1}{2t}(|x|-(|x_i|-t\gamma))^2} dx + \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^0 e^{-\frac{1}{2t}(|x|-(|x_i|-t\gamma))^2} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{1}{2t}(x-(|x_i|-t\gamma))^2} dx + \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^0 e^{-\frac{1}{2t}(x+(|x_i|-t\gamma))^2} dx \\
&\leq \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(x-(|x_i|-t\gamma))^2} dx + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(x+(|x_i|-t\gamma))^2} dx \\
&= 1 + 1 = 2
\end{aligned}$$

we can conclude that

$$\int_{\mathbb{R}} \phi(x) G_t(x, x_i) dx \leq k e^{-\gamma|x_i|}$$

where $k = 2e^{t\gamma^2/2}$. Finally since $\phi(x)$ is identically equal to $e^{-\gamma|x|}$ for any x outside the interval $[-\frac{1}{2}, \frac{1}{2}]$ we have

$$\sum_{\substack{i \in I_0 \\ x_i \notin [-1, 1]}} \int_{\mathbb{R}} \phi(x) G_t(x, x_i) dx \leq \sum_{\substack{i \in I_0 \\ x_i \notin [-1, 1]}} k e^{-\gamma|x_i|} = \sum_{\substack{i \in I_0 \\ x_i \notin [-1, 1]}} k \phi(x_i) \leq k \sum_{i \in I_0} \phi(x_i)$$

and we know the final term is finite since I_0 is an element of $M_f^\phi(\mathbb{R})$. This proves the criterion labelled (2) which is in turn enough to give the required result. \square

Corollary 4.2.8. *Let $a \in [-1, 1)$. In the situation described in lemma 4.2.7 above we have $P(u_t(x_i) < a \text{ for infinitely many } i \in I_0) = 0$.*

Consequently $\prod_{i \in I_0} u_t(x_i)$ exists almost surely.

Proof. Under the event $u_t(x_i) < a$ infinitely often the sum $\sum_{i \in I_0} 1 - u_t(x_i)$ is infinite. However from lemma 4.2.7 above we know that $\mathbb{E}(\sum_{i \in I_0} 1 - u_t(x_i)) < \infty$, so this event must have probability 0.

To prove the final line of the statement we consider $a = 0$. For almost every

realisation of u_t there is some $n \in \mathbb{N}$ such that $u_t(x_i) \geq 0 \forall i > n$. We then prove the existence of the required product by writing

$$\prod_{i \in I_0} u_t(x_i) = \prod_{i \in I_0(n)} u_t(x_i) \times \prod_{\substack{i \in I_0 \\ i \notin I_0(n)}} u_t(x_i).$$

The two products on the right-hand side exist since the first is finite and the second is a product of terms in $[0, 1]$, so we are done. \square

We are now in a position to prove the convergence of the sequence of expectations which we require to define our process. Whilst we only require the result for $u_0(\cdot) = e^{-\phi(\cdot)f(\cdot)}$ with $f \in V$, we have actually shown this for u_0 corresponding to any non-negative, bounded and continuous f .

Proposition 4.2.9. *The sequence $\mathbb{E}(\prod_{i \in I_0(n)} u_t(x_i))$ converges to a limit as $n \rightarrow \infty$, and this limit is $\mathbb{E}(\prod_{i \in I_0} u_t(x_i))$.*

Proof. Firstly we remark that when writing $\mathbb{E}(\prod_{i \in I_0} u_t(x_i))$ we actually consider $\mathbb{E}(X(u_t))$, where X is the random variable

$$X(u) = \begin{cases} \prod_{i \in I_0} u_t(x_i) & \text{if this product exists} \\ 0 & \text{otherwise.} \end{cases}$$

We know from the last part of corollary 4.2.8 that $X(u_t)$ will be identical to the product everywhere except on a set of measure zero, so this is a sensible definition of the expectation.

Now we define sequences $a_n(k)$ and $a(k)$. We let $a(k) = u_t(x_k)$, and for each $n \in \mathbb{N}$ we set

$$a_n(k) = \begin{cases} u_t(x_k) & \text{if } k \leq n \\ 1 & \text{if } k > n. \end{cases}$$

Clearly for any k we have $a_n(k) \searrow a(k)$ as $n \rightarrow \infty$. Using lemma 4.2.6, and again considering corollary 4.2.8, shows that almost everywhere we have

$$\prod_{k \in I_0(n)} u_t(x_k) = \prod_k a_n(k) \longrightarrow \prod_k a(k) = \prod_{k \in I_0} u_t(x_k) \quad \text{as } n \rightarrow \infty.$$

As everything is bounded in $[-1, 1]$ we can now apply the dominated convergence theorem to give

$$\mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) \longrightarrow \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) \quad \text{as } n \rightarrow \infty.$$

□

We conclude this section with a lemma stating that these limits are non-zero.

Lemma 4.2.10. *For u_0 corresponding to some non-negative, bounded and continuous f , the value*

$$L = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right)$$

is strictly positive.

Proof. By proposition 4.2.9 above we know that this limit exists and we denote this by L . Using the duality relation given in proposition 4.1.1

$$L = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t(n)} u_0(x_i)\right) \quad (1)$$

where again $I_t(n)$ denotes an indexing of the particles at time t in the finite process started from $I_0(n)$. Since for any non-negative f we correspondingly have

$$u_0(\cdot) = e^{-\phi(\cdot)f(\cdot)} \in [0, 1],$$

it is clear that the terms in the limit on the right-hand side are all non-negative. Therefore the corresponding limit L will be non-negative and it remains to eliminate the case $L = 0$.

We begin by recalling from 3.1.11 that there exists some $M_1 \in \mathbb{R}$ such that

$$\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) \leq M_1 \quad (2)$$

for any $t \geq 0$ and $n \in \mathbb{N}$. Now we assume $L = 0$ and show that this leads to a contradiction. Again using duality we write

$$\mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t(n)} u_0(x_i)\right) = \mathbb{E}\left(e^{-\sum_{i \in I_t(n)} \phi(x_i) f(x_i)}\right) \geq e^{-\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i) f(x_i)\right)}$$

where the inequality follows from Jensen's inequality. Rearranging the above yields

$$\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i) f(x_i)\right) \geq -\ln \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right). \quad (3)$$

Now f is bounded and non-negative so we may suppose that $0 \leq f(s) \leq F$, $\forall s \in \mathbb{R}$, and so consequently $0 \leq f(s)/F \leq 1$. Therefore

$$\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) \geq \mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i) \frac{f(x_i)}{F}\right) \stackrel{(3)}{\geq} -\frac{1}{F} \ln \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right). \quad (4)$$

So if $L = 0$ then $\mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) \rightarrow 0$ as n gets large, and so

$$-\frac{1}{F} \ln \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

and correspondingly

$$\mathbb{E}\left(\sum_{i \in I_t(n)} \phi(x_i)\right) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

by (4). This is a direct contradiction of the fact stated in (2) and so we may conclude that $L \neq 0$ and so must be strictly positive. \square

4.2.4 Topological Arguments

This section comprises of a collection of topological facts regarding the state space of these interacting branching processes on \mathbb{R} . Several of these results will

be required for us to complete the construction of the infinite processes. Some, however, will show their worth later when we prove that stationary distributions exist for these models.

Ideally we wish to consider the weighted state space $\phi(M_f^\phi(\mathbb{R}))$, denoted F from now on, as this is the natural space on which to consider these models. By construction we have $F \subset M_f(\mathbb{R})$, the set of all finite measures on \mathbb{R} . From lemma 4.2.2 we know that F is closed in $M_f(\mathbb{R})$.

Begin now by compactifying \mathbb{R} to give the space $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ in the usual way, so that here $x_n \rightarrow \infty$ if and only if x_n is eventually outside K for any compact set $K \subset \mathbb{R}$. This allows us to define the space $M_f(\overline{\mathbb{R}})$ analogously to $M_f(\mathbb{R})$; it is the space of finite measures on $\overline{\mathbb{R}}$ with the topology of weak convergence. We can embed $M_f(\mathbb{R})$ as a set in this new space via the identification

$$M_f(\mathbb{R}) \cong \{\mu \in M_f(\overline{\mathbb{R}}) : \mu(\infty) = 0\} \subset M_f(\overline{\mathbb{R}}).$$

Now as outlined in Dawson [11] we can compactify $M_f(E)$ for any compact space E to give the Watanabe compactification of $M_f(E)$, denoted $\overline{M}_f(E)$, by adding an extra point. The space $\overline{M}_f(E)$ is then itself both compact and metrizable. In our case, now that we have compactified \mathbb{R} we can form

$$\overline{M}_f(\overline{\mathbb{R}}) = M_f(\overline{\mathbb{R}}) \cup \{\infty_w\}.$$

This has the following topology derived from the weak topology on $M_f(\overline{\mathbb{R}})$

$$\begin{aligned} \mu_n \implies \mu \in M_f(\overline{\mathbb{R}}) &\iff \langle \mu_n, f \rangle \rightarrow \langle \mu, f \rangle, \forall f \in bC(\overline{\mathbb{R}}) \\ \mu_n \rightarrow \infty_w &\iff \langle \mu_n, 1 \rangle \rightarrow \infty, \end{aligned}$$

where $bC(\overline{\mathbb{R}})$ is the space of bounded, continuous functions on $\overline{\mathbb{R}}$.

We now define the space $\overline{F} \subset M_f(\overline{\mathbb{R}})$ to be

$$\overline{F} = \{\mu + \alpha\delta_\infty : \mu \in F, \alpha \geq 0\}.$$

Clearly F can be embedded in \overline{F} as the set of elements with $\alpha = 0$. For the remainder of this work we keep in mind the following embedding diagram:

$$\begin{array}{ccccc} M_f(\mathbb{R}) & \subset & M_f(\overline{\mathbb{R}}) & \subset & \overline{M}_f(\overline{\mathbb{R}}) \\ \cup & & \cup & & \cup \\ F & \subset & \overline{F} & \subset & \overline{F} \cup \{\infty_w\} \end{array} \quad (*)$$

Lemma 4.2.11. *The space \overline{F} is closed in $M_f(\overline{\mathbb{R}})$.*

Proof. Suppose that we have some sequence $\{\overline{x}_n = x_n + \alpha_n \delta_\infty : n \in \mathbb{N}\}$ in \overline{F} , such that $\overline{x}_n \implies \overline{x}$ in $M_f(\overline{\mathbb{R}})$ as $n \rightarrow \infty$. We must show that this implies that \overline{x} is in \overline{F} . To prove this result we make extensive use of the method used to prove 4.2.2.

We begin by writing $\overline{x} = x + \alpha \delta_\infty$ with $\alpha = \overline{x}(\infty)$, so that x is a finite measure on \mathbb{R} . We now wish to show that $x \in F$, or equivalently that $\phi^{-1}x$ is in $M_f^\phi(\mathbb{R})$. As in the proof of 4.2.2 we assume that there exists some interval $[a, b] \subset \mathbb{R}$, with $\phi^{-1}x$ not atomic at a or b , such that

$$\phi^{-1}x([a, b]) = \gamma \notin \mathbb{N}. \quad (1)$$

Now we remark that the functions f_ϵ , f_ϵ/ϕ , g_ϵ and g_ϵ/ϕ from that earlier proof can be extended continuously onto $\overline{\mathbb{R}}$ by setting them equal to 0 at ∞ . Then as before we can find some N so that $\phi^{-1}x_N([a, b]) \notin \mathbb{N}$, and consequently this contradicts $x_N \in F$. Thus there can be no such interval $[a, b]$ for which (1) holds. Hence from lemma 4.2.1 we see that $\phi^{-1}x \in M_f^\phi(\mathbb{R})$, so that $x \in F$ and $\overline{x} = x + \alpha \delta_\infty$ is in \overline{F} . This concludes the proof that \overline{F} is closed. \square

Corollary 4.2.12. *The space \overline{F} described above is the closure of F in $M_f(\overline{\mathbb{R}})$.*

Proof. From the lemma above \overline{F} is closed in $M_f(\overline{\mathbb{R}})$. As it also contains F it follows that $cl(F) \subset \overline{F}$. Now fix some arbitrary element $\overline{x} = x + \alpha \delta_\infty$ in \overline{F} , with

$x \in F$ and $\alpha \in \mathbb{R}$.

Any function $f \in bC(\overline{\mathbb{R}})$ must have tails which tend to some finite limit as the absolute value of $s \in \mathbb{R}$ gets large. Otherwise we could find some sequence of points $s_n \in \mathbb{R}$ which tend to ∞ in $\overline{\mathbb{R}}$ but for which the corresponding sequence $f(s_n)$ has no limit. But this contradicts the fact that by continuity $\lim_{n \rightarrow \infty} f(s_n) = f(\infty) < \infty$.

Now we choose an increasing sequence of points $\{s_n : n \in \mathbb{N}\}$ in \mathbb{R} so that $s_n \rightarrow \infty$ and for which $\alpha/\phi(s_n)$ is an integer. This is possible by the form of ϕ . We now define the sequence

$$x_n = x + \alpha \delta_{s_n} = x + \frac{\alpha}{\phi(s_n)} \phi(s_n) \delta_{s_n}$$

and note that by construction each of these terms is in F . For any $f \in bC(\overline{\mathbb{R}})$ it follows that

$$\int_{\overline{\mathbb{R}}} f(s) x_n(ds) = \int_{\overline{\mathbb{R}}} f(s) x(ds) + \alpha f(s_n) \rightarrow \int_{\overline{\mathbb{R}}} f(s) x(ds) + \alpha f(\infty) = \int_{\overline{\mathbb{R}}} f(s) \bar{x}(ds)$$

and so $x_n \Rightarrow \bar{x}$ in $M_f(\overline{\mathbb{R}})$. Consequently $\bar{x} \in cl(F)$ and since \bar{x} was arbitrary this gives $\overline{F} \subset cl(F)$. \square

Corollary 4.2.13. *The space $\overline{F} \cup \{\infty_w\} \subset \overline{M}_f(\overline{\mathbb{R}})$ is compact and metrizable.*

Proof. Metrizability is inherited from the larger space $\overline{M}_f(\overline{\mathbb{R}})$. Now suppose that we have some sequence $\{x_n : n \in \mathbb{N}\}$ in $\overline{F} \cup \{\infty_w\}$ which converges to x in $\overline{M}_f(\overline{\mathbb{R}})$. We wish to show that x is in $\overline{F} \cup \{\infty_w\}$.

If $x = \infty_w$ then we are done. Otherwise, by definition, $\langle x_n, f \rangle \rightarrow \langle x, f \rangle$ for all $f \in bC(\overline{\mathbb{R}})$. Since the unit function 1 is continuous, we have

$$x_n(\overline{\mathbb{R}}) = \int_{\overline{\mathbb{R}}} 1(s) x_n(ds) \rightarrow \int_{\overline{\mathbb{R}}} 1(s) x(ds) = x(\overline{\mathbb{R}}) < \infty$$

and it follows from this that the sequence is eventually in $\overline{F} \subset M_f(\overline{\mathbb{R}})$. As this set is closed from lemma 4.2.11, it follows that the limit x is in \overline{F} and so certainly

in $\overline{F} \cup \{\infty_w\}$. Thus $\overline{F} \cup \{\infty_w\}$ is closed in the compact space $\overline{M}_f(\overline{\mathbb{R}})$ and so is itself compact. \square

We conclude these topological notes with one final result:

Lemma 4.2.14. *F is separable as a subspace of $M_f(\overline{\mathbb{R}})$.*

Proof. We define a subset F_0 of F as

$$F_0 = \left\{ x \in F : x = \sum_{i \in I} \phi(x_i) \delta_{x_i}, I \text{ countable}, x_i \in \mathbb{Q} \right\}.$$

This set is countable as there is a natural mapping from F_0 into $\mathbb{Q}^{\mathbb{N}}$ which is injective.

Now fix some point $x = \sum_{i \in I} \phi(x_i) \delta_{x_i} \in F$. We choose a sequence $\{x_n : n \in \mathbb{N}\}$ in F_0 so that for each x_n there is a one-to-one correspondence between the particles of x_n and x , with corresponding particles a distance $< 1/n$ apart. Thus each x_n is of the form $\sum_{i \in I} \phi((x_n)_i) \delta_{(x_n)_i}$ with $|x_i - (x_n)_i| < 1/n$. This is of course possible since the rationals are dense in \mathbb{R} .

Now let $f : \overline{\mathbb{R}} \rightarrow \mathbb{R}$ be a bounded and continuous function, say with $|F(s)| \leq M \forall s \in \overline{\mathbb{R}}$. Fix some $\epsilon > 0$. As $\sum_{i \in I} \phi(x_i) < \infty$, we can choose $m \in \mathbb{N}$ large enough so that

$$\sum_{\substack{i \in I \\ i \notin I(m)}} \phi(x_i) < \epsilon/6M. \quad (1)$$

Here $I(m)$ is the indexing of the m atoms of x which are closest to the origin. Further to this we let $x(m)$ denote the measure consisting of only these m atoms, and $x \setminus x(m)$ be the measure consisting of all atoms of x except these m . Now choose n large enough so that $e^{\gamma/n} < 2$. It follows from the form of ϕ that for any such n , $\phi((x_n)_i) \leq e^{\gamma/n} \phi(x_i) < 2\phi(x_i)$. Consequently using (1)

$$\sum_{\substack{i \in I \\ i \notin I(m)}} \phi((x_n)_i) < 2 \sum_{\substack{i \in I \\ i \notin I(m)}} \phi(x_i) < \epsilon/3M. \quad (2)$$

Finally we choose n large enough so that

$$|f(x_i)\phi(x_i) - f((x_n)_i)\phi((x_n)_i)| < \epsilon/3m \quad \text{for all } i = 1, \dots, m. \quad (3)$$

This is possible since f and ϕ are continuous and $(x_n)_i \rightarrow x_i$ in $\overline{\mathbb{R}}$.

Now with n fulfilling all conditions above we have

$$\begin{aligned} & |\langle f, x \rangle - \langle f, x_n \rangle| \\ & \leq |\langle f, x \rangle - \langle f, x(m) \rangle| + |\langle f, x(m) \rangle - \langle f, x_n(m) \rangle| + |\langle f, x_n(m) \rangle - \langle f, x_n \rangle| \\ & \leq \left| \sum_{\substack{i \in I \\ i \notin I(m)}} \phi(x_i)f(x_i) \right| + \sum_{i=1}^m |f(x_i)\phi(x_i) - f((x_n)_i)\phi((x_n)_i)| \\ & \quad + \left| \sum_{\substack{i \in I \\ i \notin I(m)}} \phi((x_n)_i)f((x_n)_i) \right| \leq M \frac{\epsilon}{6M} + m \frac{\epsilon}{3m} + M \frac{\epsilon}{3M} < \epsilon \end{aligned}$$

where the final line comes from applying (1), (2) and (3). It follows that $\langle f, x_n \rangle \rightarrow \langle f, x \rangle$ for any $f \in bC(\overline{\mathbb{R}})$, so x_n converges weakly to x in $M_f(\overline{\mathbb{R}})$ and F is separable. \square

4.2.5 Constructing the Process

With all the tools in place we can complete this construction. We begin by showing that the laws of the restricted processes converge to some limiting law on F for any fixed $t \geq 0$. These will specify the finite-dimensional distributions of our process. General Markov process theory will tell us that there does exist a process with these finite-dimensional distributions.

Keeping in mind the embedding diagram (*) from the previous section, consider $\tilde{P}_t^{\mu_0(n)}$ as a probability measure on $M_f(\overline{\mathbb{R}})$ rather than the smaller space F .

Lemma 4.2.15. *The sequence $\{\tilde{P}_t^{\mu_0(n)} : n \in \mathbb{N}\}$ is tight in $\mathcal{M}_1(M_f(\overline{\mathbb{R}}))$.*

Proof. We must show that for any $\epsilon > 0$ there is some compact $K_\epsilon \subset M_f(\overline{\mathbb{R}})$ such that $\sup_{n \in \mathbb{N}} \tilde{P}_t^{\mu_0(n)}(K_\epsilon^c) < \epsilon$.

Define the sets $G_m \subset M_f(\overline{\mathbb{R}})$ to be $G_m = \{\mu \in M_f(\overline{\mathbb{R}}) : \mu(\overline{\mathbb{R}}) \leq m\}$.

These sets are closed as follows: suppose first that $\mu_n \implies \mu$ in $M_f(\overline{\mathbb{R}})$, with each μ_n in G_m . Then as the unit function on $\overline{\mathbb{R}}$ is bounded and continuous we have

$$\mu(\overline{\mathbb{R}}) = \lim_{n \rightarrow \infty} \mu_n(\overline{\mathbb{R}}) \leq m,$$

so μ must also be an element of G_m .

These sets remain closed when embedded in the larger space $M_f(\overline{\mathbb{R}}) \cup \{\infty_w\}$, and as this space is compact then so are the sets G_m . Now consider some general sequence $\{\mu_n : n \in \mathbb{N}\}$ in G_m . By compactness there exists $\mu \in G_m$ and some subsequence $\{\mu_{n'}\}$ of this sequence so that $\mu_{n'} \implies \mu$ in the topology of $M_f(\overline{\mathbb{R}}) \cup \{\infty_w\}$. However, since $\mu \neq \infty_w$ because $\infty_w \notin G_m$, it follows that this is the same as convergence in the topology of $M_f(\overline{\mathbb{R}})$. This shows that the sets G_m are compact in $M_f(\overline{\mathbb{R}})$.

To conclude, we show that $\sup_{n \in \mathbb{N}} \tilde{P}_t^{\mu_0(n)}(G_m^c) \rightarrow 0$ as m gets large. This proves tightness. Firstly, as each measure $\tilde{P}_t^{\mu_0(n)}$ is the law of a restricted particle process at time t , we have

$$\tilde{P}_t^{\mu_0(n)}(G_m^c) = P\left(\sum_{i \in I_t(n)} \phi(x_i) > m\right).$$

Consequently we have

$$\sup_{n \in \mathbb{N}} \tilde{P}_t^{\mu_0(n)}(G_m^c) \leq \sup_{n \in \mathbb{N}} \sup_{t \geq 0} P\left(\sum_{i \in I_t(n)} \phi(x_i) > m\right),$$

and the right-hand side of this tends to 0 as $m \rightarrow \infty$ by proposition 3.1.12. \square

Corollary 4.2.16. *There exists some probability measure $\tilde{P}_t^{\mu_0} \in \mathcal{M}_1(M_f(\overline{\mathbb{R}}))$ such that $\tilde{P}_t^{\mu_0(n)} \implies \tilde{P}_t^{\mu_0}$.*

Proof. From corollary 3.2.7 of Dawson [11] it follows that to prove this it suffices to show that the measures $\{\tilde{P}_t^{\mu_0(n)}\}$ are tight, and that the sequence

$$\int_{M_f(\overline{\mathbb{R}})} e^{-\langle \nu, f \rangle} \tilde{P}_t^{\mu_0(n)}(d\nu), \quad n \in \mathbb{N} \quad \text{converges for any non-negative } f \in bC(\overline{\mathbb{R}}).$$

The tightness of the measures was shown in lemma 4.2.15 above. To prove the second part, recall that each $\tilde{P}_t^{\mu_0(n)}$ is supported on $F \subset M_f(\mathbb{R})$. In fact, as we have seen in corollary 4.2.4,

$$\int_{M_f(\overline{\mathbb{R}})} e^{-\langle \nu, f \rangle} \tilde{P}_t^{\mu_0(n)}(d\nu) = \int_{M_f(\mathbb{R})} e^{-\langle \nu, f \rangle} \tilde{P}_t^{\mu_0(n)}(d\nu) = \mathbb{E} \left(\prod_{i \in I_0(n)} u_t(x_i) \right)$$

where $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$. We can of course restrict f to \mathbb{R} and it remains non-negative, continuous and bounded. Now applying proposition 4.2.9 we can see that the sequence of integrals converges for any such f and this gives us $\tilde{P}_t^{\mu_0}$ as desired. \square

This result gives us the limiting probability measure we require, but not on the space we want. The measure $\tilde{P}_t^{\mu_0}$ should represent the law at time t of a weighted *particle process* on \mathbb{R} , as each of the approximating measures $\tilde{P}_t^{\mu_0(n)}$ does. The result we really desire is that $\tilde{P}_t^{\mu_0(n)} \Rightarrow \tilde{P}_t^{\mu_0}$ in $\mathcal{M}_1(F)$, the space of probability measures on F . The next few results are geared to showing precisely that fact:

Lemma 4.2.17. $\tilde{P}_t^{\mu_0(n)} \Rightarrow \tilde{P}_t^{\mu_0}$ in $\mathcal{M}_1(\overline{F})$.

Proof. From lemma 4.2.11 it is known that \overline{F} is closed in $M_f(\overline{\mathbb{R}})$. Each of the probabilities $\tilde{P}_t^{\mu_0(n)}$ is supported on \overline{F} so consequently

$$\tilde{P}_t^{\mu_0}(\overline{F}) \geq \limsup_{n \rightarrow \infty} \tilde{P}_t^{\mu_0(n)}(\overline{F}) = 1$$

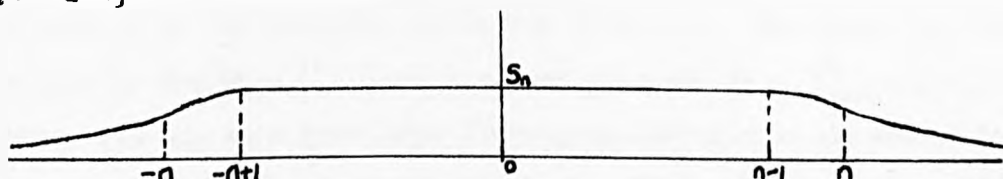
and so $\tilde{P}_t^{\mu_0}$ is supported on \overline{F} also.

Now let f be some arbitrary bounded and continuous function on \overline{F} . Tietze's extension theorem, for which the reader is again referred to [9], states that any continuous function on a closed subset A of a metric space X has a continuous extension on all of X . Further, if the original function is bounded then the extension can be bounded also. Let \tilde{f} denote such an extension of the function f to the whole of $M_f(\overline{\mathbb{R}})$. Then using the convergence laid out in corollary 4.2.16 above

$$\int_{\overline{F}} f(\nu) \tilde{P}_t^{\mu_0(n)} = \int_{M_f(\overline{\mathbb{R}})} \tilde{f}(\nu) \tilde{P}_t^{\mu_0(n)} \longrightarrow \int_{M_f(\overline{\mathbb{R}})} \tilde{f}(\nu) \tilde{P}_t^{\mu_0} = \int_{\overline{F}} f(\nu) \tilde{P}_t^{\mu_0}.$$

As this holds for any $f \in bC(\overline{F})$ we have the convergence required. \square

The final step is to prove that $\tilde{P}_t^{\mu_0(n)} \implies \tilde{P}_t^{\mu_0}$ in $\mathcal{M}_1(F)$. This requires a little extra work, and we begin by defining some new weighting functions $\{\phi_n : n \in \mathbb{N}\}$ on \mathbb{R} of the form



As in lemma 3.1.1 we can construct such functions in a piecewise manner from exponential tails, sections of quartic polynomials and a constant function. This can again be done in such a way that ϕ_n'' is continuous with $\phi_n'' \leq \gamma^2 \phi_n$, and such that $\phi_n = e^{-\gamma|x|}$ outside the interval $[-n, n]$. Additionally we remark that if s_n is the value of the function ϕ_n in the plateau region, then we have $s_n \leq \phi(n-1) = e^{-\gamma(n-1)}$.

The important thing about these new weighting functions is that they share with ϕ precisely the properties we have made use of in our work. Consequently nearly all the results proved so far can be shown for each ϕ_n in the same way. We need only check that we have not made use of any property unique to ϕ . In particular we have the following analogy and extension to corollary 3.1.11:

Lemma 4.2.18. *Suppose I_0 is countable with $\sum_{i \in I_0} \phi(x_i) < \infty$ (and so clearly $\sum_{i \in I_0} \phi_m(x_i) < \infty$ for all $m \in \mathbb{N}$). Then there exist constants $M_1(m)$ such that for all $n \in \mathbb{N}$ and all $t \geq 0$ we have*

$$\mathbb{E} \left(\sum_{i \in I_t(n)} \phi_m(x_i) \right) \leq M_1(m).$$

These constants are such that $M_1(m) \rightarrow 0$ as $m \rightarrow \infty$.

Proof. With the exception of the final statement concerning the limits, this result follows for each ϕ_m in exactly the way that corollary 3.1.11 did for the original ϕ . To prove the final statement we simply have to trace constants. It follows as in the original proof of 3.1.11 that

$$M_1(m) = \max \left\{ \sum_{i \in I_0} \phi_m(x_i) e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau(m)}, \quad M(m) e^{2[(\gamma^2/2) + \lambda(\beta-1)]\tau(m)} \right\}. \quad (1)$$

For each m we can certainly ensure that $\tau(m) \leq 1$. Hence the first term on the right-hand side of (1) above is a constant multiple of $\sum_{i \in I_0} \phi_m(x_i)$ which converges to zero as m gets large. Turning our attention to the second term in (1) we see that this is less than a constant multiple of $M(m)$, where $M(m)$ is any constant such that

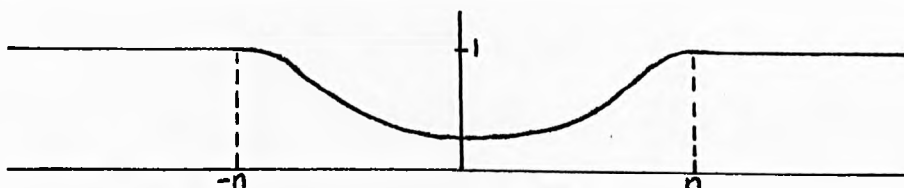
$$M(m) > \frac{R_1(m)}{(1 - \sqrt{\alpha(m)})} e^{[(\gamma^2/2) + \lambda(\beta-1)]\tau(m)}.$$

This is slightly more complicated: careful consideration of our methods in chapter 3 reveal that $\alpha(m)$ is not dependent on the test function, being derived from properties of Brownian particles. Further $R_1(m)$ is a constant multiple of $r'(m)\Phi^+(m)$, where $r'(m) = r'$ is not test function dependent, and $\Phi^+(m) = \sum_{k \in \mathbb{Z}} \phi_m(a_k^+)$. Consequently $M(m)$ is any constant which is larger than a constant multiple of $\Phi^+(m)$. But $\Phi^+(m)$ becomes arbitrarily small as m gets large, so we may choose $M(m)$ so that this sequence also tends to 0 as m tends to infinity.

Thus both terms on the right-hand side of (1) tend to 0 as $m \rightarrow \infty$ and hence so does $M_1(m)$. \square

Lemma 4.2.19. *The probability measure $\tilde{P}_t^{\mu_0}$ is supported on $F \subset \bar{F}$.*

Proof. Firstly we define the maps $\psi_m : \mathbb{R} \rightarrow \mathbb{R}$ to be $\psi_m(\cdot) = \phi_n(\cdot)/\phi(\cdot)$. These maps are bounded and continuous, and are of the form:



Each such map ψ_m has a trivial extension to $\bar{\mathbb{R}}$ by letting $\psi_m(\infty) = 1$. For any $\epsilon > 0$ we define corresponding maps $L_m^\epsilon : \bar{F} \rightarrow \mathbb{R}$ by

$$L_m^\epsilon(\mu) = \begin{cases} 1 & \text{if } \langle \psi_m, \mu \rangle > \epsilon \\ 0 & \text{if } \langle \psi_m, \mu \rangle \leq \epsilon/2 \\ \text{linear in between.} & \end{cases}$$

Notice that if $\mu_n \Rightarrow \mu$ in \bar{F} , then $\langle \psi_m, \mu_n \rangle \rightarrow \langle \psi_m, \mu \rangle$, and so $L_m^\epsilon(\mu_n) \rightarrow L_m^\epsilon(\mu)$. Consequently L_m^ϵ is bounded and continuous on \bar{F} . From lemma 4.2.17 we have $\tilde{P}_t^{\mu_0(n)} \Rightarrow \tilde{P}_t^{\mu_0}$ in $\mathcal{M}_1(\bar{F})$, so we write

$$\tilde{P}_t^{\mu_0}(\{\mu \in \bar{F} : \langle \psi_m, \mu \rangle > \epsilon\}) \leq \int_{\bar{F}} L_m^\epsilon(\mu) \tilde{P}_t^{\mu_0}(d\mu) = \lim_{n \rightarrow \infty} \int_{\bar{F}} L_m^\epsilon(\mu) \tilde{P}_t^{\mu_0(n)}(d\mu).$$

But now

$$\int_{\bar{F}} L_m^\epsilon(\mu) \tilde{P}_t^{\mu_0(n)}(d\mu) \leq \tilde{P}_t^{\mu_0(n)}(\{\mu \in \bar{F} : \langle \psi_m, \mu \rangle > \epsilon/2\}) = P\left(\sum_{i \in I_t(n)} \phi_m(x_i) > \frac{\epsilon}{2}\right).$$

Using the results outlined in lemma 4.2.18 together with Chebyshev's inequality yields

$$P\left(\sum_{i \in I_t(n)} \phi_m(x_i) > \epsilon/2\right) \leq \frac{2}{\epsilon} \mathbb{E}\left(\sum_{i \in I_t(n)} \phi_m(x_i)\right) \leq \frac{2}{\epsilon} M_1(m),$$

which is independent of n . Consequently

$$\tilde{P}_t^{\mu_0}(\{\mu \in \bar{F} : \langle \psi_m, \mu \rangle > \epsilon\}) \leq \frac{2}{\epsilon} M_1(m). \quad (1)$$

Now let I_m denote the indicator function of the set $[-m, m]^c \subset \overline{\mathbb{R}}$, noting that this set always contains ∞ . Define subsets $\{G_\epsilon : \epsilon > 0\}$ of \overline{F} as

$$G_\epsilon := \{\mu \in \overline{F} : \mu(\infty) > \epsilon\} \subseteq \bigcap_{m=0}^{\infty} \{\mu \in \overline{F} : \langle I_m, \mu \rangle > \epsilon\}.$$

Now as $I_m \leq \psi_m$ we see that $\langle I_m, \mu \rangle > \epsilon$ implies $\langle \psi_m, \mu \rangle > \epsilon$, so

$$\begin{aligned} \tilde{P}_t^{\mu_0}(G_\epsilon) &\leq \lim_{m \rightarrow \infty} \tilde{P}_t^{\mu_0}(\{\mu \in \overline{F} : \langle I_m, \mu \rangle > \epsilon\}) \\ &\leq \lim_{m \rightarrow \infty} \tilde{P}_t^{\mu_0}(\{\mu \in \overline{F} : \langle \psi_m, \mu \rangle > \epsilon\}) \leq \lim_{m \rightarrow \infty} \frac{2}{\epsilon} M_1(m), \end{aligned}$$

where the final inequality comes from (1) above. However, from lemma 4.2.18 we know that the right-hand limit above is in fact zero for any $\epsilon > 0$, so we have

$$\tilde{P}_t^{\mu_0}(\{\mu \in \overline{F} : \mu(\infty) > 0\}) = \lim_{\epsilon \rightarrow 0} \tilde{P}_t^{\mu_0}(G_\epsilon) = 0.$$

Consequently $\tilde{P}_t^{\mu_0}$ is supported on $F \subset \overline{F}$ as stated. \square

Corollary 4.2.20. $\tilde{P}_t^{\mu_0(n)} \implies \tilde{P}_t^{\mu_0}$ in $\mathcal{M}_1(F)$.

Proof. The probability measures $\tilde{P}_t^{\mu_0(n)}$ are supported on $F \subset \overline{F}$ and now from 4.2.19 above we know that the same is true for the limit $\tilde{P}_t^{\mu_0}$. Combining lemma 4.2.14 with corollary 4.2.12 we see that F is separable in \overline{F} . Now with $R = F$ and $S = \overline{F}$, apply proposition 2.4.13 from chapter 2 to give the result. \square

Thus, finally, we are able to construct a law on F of the infinite interacting process at time t started from initial measure μ_0 . In turn these transition functions specify the finite-dimensional distributions of the model. This will be sufficient for our needs, but ideally we would wish to know that there is some unique underlying F -valued Markov process with such finite-dimensional distributions. This fact follows from general Markov process theory such as may be found in [20]. For completeness we include the following result:

Proposition 4.2.21. *Let $\mu_0 \in M_f^\phi(\mathbb{R})$ represent some initial distribution of particles. Then there exists an F -valued Markov process ξ with initial distribution $\phi\mu_0$, so that the finite-dimensional distributions of ξ correspond to those given by the laws $\tilde{P}_t^{\mu_0}$ above. The sample paths $\xi(\cdot)(\omega) : [0, \infty) \rightarrow F$ of this process ξ are cadlag.*

Proof. We wish to directly apply theorem 2.7 from chapter 4 of [20]. The topological conditions of this result are that F be both locally compact and separable. Separability is known from lemma 4.2.14 and we show local compactness now. For each $m \in \mathbb{N}$ let $g_m : \mathbb{R} \rightarrow \mathbb{R}$ be the map

$$g_m(s) = \begin{cases} 0 & s \in [-m, m] \\ 1 & s \notin [-m-1, m+1] \\ \text{linear in between.} & \end{cases}$$

Now for any $\mu \in F$ we define the set $G_\mu \subset F$ to be

$$G_\mu = \{\nu \in F : \langle \nu, g_m \rangle \leq 2\langle \mu, g_m \rangle \quad \forall m\}.$$

This set G_μ certainly contains μ and we claim it is compact. Firstly we consider this set G_μ embedded in the larger space $\overline{F} \cup \{\infty_w\}$. Suppose that $\nu_n \implies \nu$ in $\overline{F} \cup \{\infty_w\}$ with each $\nu_n \in G_\mu$. Then since $g_0 \equiv 1$ we have

$$\langle \nu, 1 \rangle = \lim_{n \rightarrow \infty} \langle \nu_n, 1 \rangle \leq 2\langle \mu, 1 \rangle < \infty,$$

so consequently $\nu \neq \infty_w$. Further to this

$$\langle \nu, g_m \rangle = \lim_{n \rightarrow \infty} \langle \nu_n, g_m \rangle \leq 2\langle \mu, g_m \rangle$$

and so

$$\nu(\infty) = \lim_{m \rightarrow \infty} \langle \nu, g_m \rangle \leq 2 \lim_{m \rightarrow \infty} \langle \mu, g_m \rangle = 2\mu(\infty) = 0.$$

Combining all this we see that $\nu \in F$ and is an element of G_μ . G_μ is thus closed in compact $\overline{F} \cup \{\infty_w\}$ and must itself be compact.

Now let $\{\nu_n\}$ be some general sequence in G_μ . There must be some convergent subsequence $\{\nu_{n'}\}$ such that $\nu_{n'} \implies \nu$ in G_μ in the topology of $\overline{F} \cup \{\infty_w\}$. As $\infty_w \notin G_\mu$, then by the definition of the compactification topology this is equivalent to saying that $\nu_{n'} \implies \nu$ in the topology of \overline{F} .

Now let $f : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and continuous with $|f(s)| \leq M \quad \forall s$, and fix $\epsilon > 0$. We can certainly choose m large enough so that $\langle \mu, g_m \rangle < \epsilon/6M$. The function $(1 - g_m(\cdot))f(\cdot)$ can be extended continuously to all of $\overline{\mathbb{R}}$ by letting its value at ∞ be zero. Then, as $\nu_{n'} \implies \nu$ in the topology of \overline{F} , we can choose n_0 large enough so that for $n' \geq n_0$ we have

$$\left| \int_{\overline{\mathbb{R}}} (1 - g_m(s))f(s)\nu_{n'}(ds) - \int_{\overline{\mathbb{R}}} (1 - g_m(s))f(s)\nu(ds) \right| < \frac{\epsilon}{3}.$$

As the integrands are 0 at infinity, the above remains true if we only integrate over \mathbb{R} . Now for $n' \geq n_0$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}} f(s)\nu_{n'}(ds) - \int_{\mathbb{R}} f(s)\nu(ds) \right| &\leq \left| \int_{\mathbb{R}} f(s)\nu_{n'}(ds) - \int_{\mathbb{R}} (1 - g_m(s))f(s)\nu_{n'}(ds) \right| \\ &\quad + \left| \int_{\mathbb{R}} (1 - g_m(s))f(s)\nu_{n'}(ds) - \int_{\mathbb{R}} (1 - g_m(s))f(s)\nu(ds) \right| \\ &\quad + \left| \int_{\mathbb{R}} (1 - g_m(s))f(s)\nu(ds) - \int_{\mathbb{R}} f(s)\nu(ds) \right|. \end{aligned}$$

The second term on the right-hand side is less than $\epsilon/3$ by the choice of n_0 discussed above. For the first and third terms we notice that

$$\left| \int_{\mathbb{R}} g_m(s)f(s)\nu(ds) \right| \leq M\langle \nu, g_m \rangle \leq 2M\langle \mu, g_m \rangle < \epsilon/3,$$

where the final inequality comes from our choice of m . The same holds true if we replace ν with $\nu_{n'}$ for any n' , and so we see that the three terms on the right-hand side of the expression above sum to less than ϵ . From this we have

$$\int_{\mathbb{R}} f(s)\nu_{n'}(ds) \longrightarrow \int_{\mathbb{R}} f(s)\nu(ds) \quad \text{as } n' \rightarrow \infty \quad \text{for any } f \in bC(\mathbb{R}).$$

So we see that $\nu_{n'} \implies \nu$ in F and that G_μ is thus compact in F .

In the following chapter it will be shown that the laws $\tilde{P}_t^{\mu_0}$ fulfil the Feller property. This, together with the separability and local compactness of F shown here, allows us to use theorem 2.7 from chapter 4 of [20]. This guarantees the existence of an underlying Markov process with the properties stated in our proposition. \square

To summarise the construction of the process covered in this chapter we provide the following definition and theorem. Firstly we recall from lemma 3.1.1 that for each $\gamma > 0$ there exists a corresponding $\phi = \phi(\gamma)$ of the type used throughout this chapter. We let Θ be the set of all such functions ϕ , so $\Theta = \{\phi = \phi(\gamma) : \gamma > 0\}$. Now recalling the definition of the set of measures $M_f^\phi(\mathbb{R})$ from the start of section 4.2.1, we form the set

$$\mathcal{A}(\mathbb{R}) = \bigcup_{\phi \in \Theta} M_f^\phi(\mathbb{R}).$$

Thus $\mathcal{A}(\mathbb{R})$ is the set of all measures on \mathbb{R} which consist of countably many point masses and which are finite under *some* weight function ϕ . Now the construction of our process covered in this chapter can be summarised as:

Theorem 4.2.22. *Let $\xi_0 \in \mathcal{A}(\mathbb{R})$ represent some initial distribution of particles. Then there exists an $\mathcal{A}(\mathbb{R})$ -valued Markov process ξ with initial distribution ξ_0 , so that the finite-dimensional distributions of ξ correspond to the laws $P_t^{\xi_0}$ defined and constructed in the above work.*

4.3 Path-Wise Constructions: One Example

Thus far we have provided a ‘soft’ construction of any general interacting branching diffusion process with reductive interactions. We now turn our attention to one important sub-class of such processes for which a relatively simple path-wise

construction exists. It can be helpful to have a graphical model such as this in mind when considering problems relating to our general processes. This helps us to retain the intuitive picture which may have been lost during the abstract arguments above.

The processes we have studied so far have all incorporated *stochastically* reductive interaction mechanisms. In other words, the mean number of offspring produced by two interacting parents has been less than two. Now we further restrict our attention to those models which are *strictly* reductive, so that no interaction can produce more than two offspring. Thus the interaction offspring probabilities are such that $q_n = 0 \quad \forall n \geq 3$. This restriction still leaves us with an important class of processes: those representing systems in which inter-species interactions are of a competitive nature and reproduction is always asexual.

A final modification required for this construction relates to the initial distribution of particles. Previously we have only stipulated countable initial conditions with finite weight under the test function ϕ . Here however, as our path-wise construction relies on a continuum percolation argument, we restrict ourselves to initial conditions which have finite limiting density. By this we mean that we allow $\xi_0 = \sum_{i \in I_0} \delta_{x_i}$ such that

$$\limsup_{m \rightarrow \infty} \frac{\xi_0([-m, m])}{2m} = \theta,$$

for some $\theta < \infty$.

At this point we present a general continuum percolation result. The picture we have in mind is as follows: upon each point $x \in \xi_0$ is placed a random interval of the form $[x - R, x + R]$, where R is some random variable taking values in $[0, \infty)$. This is done independently for each of the points of ξ_0 .

Lemma 4.3.1. *In the situation described above suppose that $\mathbb{E}(R) < \infty$ and that $P(R \leq \epsilon) > 0$ for any $\epsilon > 0$. Then letting 0 be any arbitrary point on \mathbb{R}*

which is not in ξ_0 , we have

$$P(0 \text{ not covered by an interval}) > 0.$$

Proof. Firstly notice that as $\limsup_{m \rightarrow \infty} \xi_0([-m, m])/2m = \theta$, this sequence must also be bounded, say by K , so that

$$\xi_0([-m, m]) \leq 2mK \quad \forall m. \quad (*)$$

Letting E_x be the event that 0 is covered by a random interval $[x - R, x + R]$ centered on x , we use the above to write

$$\sum_{x \in \xi_0} P(E_x) = \sum_{x \in \xi_0} \mathbb{E}(1_{\{R \geq |x|\}}) = \mathbb{E}\left(\sum_{x \in \xi_0} 1_{\{R \geq |x|\}}\right) = \mathbb{E}(\xi_0([-R, R])) \leq 2K\mathbb{E}(R) < \infty.$$

Now by assumption $0 \notin \xi_0$ so that $|x - 0| > 0$ for any point x of ξ_0 . Hence, as R has some positive probability of being less than $|x - 0|$, we see that $P(E_x) < 1$. Consequently we see that

$$\sum_{x \in \xi_0} P(E_x) < \infty \implies \prod_{x \in \xi_0} (1 - P(E_x)) = \prod_{x \in \xi_0} P(E_x^c) > 0.$$

Using this, together with the independence of intervals on different points, we have

$$P(0 \text{ not covered by an interval}) = P\left(\bigcap_{x \in \xi_0} E_x^c\right) = \prod_{x \in \xi_0} P(E_x^c) > 0.$$

□

The above result encapsulates the idea we exploit to show that the union of random intervals on ξ_0 does not percolate. However to overcome some small technicalities, we are required to work a little bit harder. The following lemma and corollary yield a re-formulation of the result above which we can use directly.

Lemma 4.3.2. *With K given by (*) in the proof above, we can find arbitrarily large $y \in \mathbb{R}$ such that*

$$\min\{|y - x| : x \in \xi_0\} \wedge \min\{|(-y) - x| : x \in \xi_0\} \geq 1/8K.$$

Proof. Suppose that this is false. Then $\exists r \in [0, \infty)$ such that

$$y \geq r \implies \exists x \in \xi_0 \text{ such that } |y - x| < 1/8K \text{ or } |(-y) - x| < 1/8K.$$

Now consider the sequence of points $w_n = r + n(1/4K)$, $n = 0, 1, \dots$ in \mathbb{R} . We see that for each n , there is some point of ξ_0 within distance $1/8K$ of either w_n or $-w_n$, which we denote $x(w_n)$. If $n_1 \neq n_2$ then the distance between w_{n_1} and w_{n_2} (and also $-w_{n_1}$ and $-w_{n_2}$) is at least $1/4K$. Consequently we see that $x(w_{n_1}) \neq x(w_{n_2})$. Simply by counting only these special points of ξ_0 we have the bound

$$\xi_0([-r - n(1/4K), r + n(1/4K)]) \geq n.$$

But combining this with the upper bound denoted by (*) in the proof of lemma 4.3.1, we have

$$n \leq \xi_0([-r - n(1/4K), r + n(1/4K)]) \leq 2(r + n(1/4K))K,$$

which is impossible if $n > 4rK$. Thus there can be no such r and the result stands as stated in the lemma. \square

Corollary 4.3.3. *Under the conditions set out in lemma 4.3.1 above, let y be some arbitrary point such that*

$$\min\{|y - x| : x \in \xi_0\} \wedge \min\{|(-y) - x| : x \in \xi_0\} \geq 1/8K.$$

Then there is some $p_0 > 0$ independent of y , so that

$$P(y \text{ and } -y \text{ are not covered by an interval}) > p_0.$$

Proof. To prove this result we follow the outline of the proof of lemma 4.3.1 above. Let $E_x(y)$ and $E_x(-y)$ be the events that the random interval on $x \in \xi_0$ covers y or $-y$ respectively. Now using Boole's inequality we have

$$\sum_{x \in \xi_0} P(E_x(y) \cup E_x(-y)) \leq \sum_{x \in \xi_0} P(E_x(y)) + \sum_{x \in \xi_0} P(E_x(-y)) < 4K\mathbb{E}(R) < \infty.$$

Here the penultimate inequality follows by bounding each of the two sums by $2K\mathbb{E}(R)$ as in 4.3.1. Using the independence of the intervals on different points, we have

$$\begin{aligned} & P(y \text{ and } -y \text{ are not covered by an interval}) \\ &= P\left(\bigcap_{x \in \xi_0} (E_x^c(y) \cap E_x^c(-y))\right) = \prod_{x \in \xi_0} P(E_x^c(y) \cap E_x^c(-y)). \end{aligned}$$

It remains to show that the final term above is larger than some $p_0 > 0$ which does not depend on y . Letting $a_x = P(E_x(y) \cup E_x(-y))$ we see that $1 - a_x = P(E_x^c(y) \cap E_x^c(-y))$, so we are required to show that $\prod_{x \in \xi_0} (1 - a_x) > p_0$.

By assumption y and $-y$ are a distance at least $1/8K$ from any point of ξ_0 . By one of the properties specified for the width distribution R there is some $\alpha > 0$ such that $P(R < 1/8K) = \alpha$. We then see that $a_x < 1 - \alpha$ for any $x \in \xi_0$. Using this fact, along with basic power series expansions, we write

$$-\ln(1 - a_x) = a_x + a_x^2/2 + a_x^3/3 + \dots \leq a_x + a_x^2 + a_x^3 + \dots = \frac{a_x}{1 - a_x} < \frac{a_x}{\alpha}.$$

From this we have

$$\sum_{x \in \xi_0} -\ln(1 - a_x) \leq \sum_{x \in \xi_0} \frac{a_x}{\alpha} = \frac{1}{\alpha} \sum_{x \in \xi_0} P(E_x(y) \cup E_x(-y)) < \frac{4K}{\alpha} \mathbb{E}(R) < \infty,$$

using the expression appearing at the beginning of this proof. To finish we have

$$\prod_{x \in \xi_0} (1 - a_x) = e^{-\sum_{x \in \xi_0} -\ln(1 - a_x)} > e^{-4K\mathbb{E}(R)/\alpha} = p_0 > 0,$$

which completes the proof. □

The reason we require the above ‘two-point’ version of our original result will become clear in the following lemma.

Lemma 4.3.4. *In the situation described above, the union of the random intervals will not percolate almost surely, i.e. each connected component of this union of intervals is bounded.*

Proof. As ξ_0 is countable we label the particles $0, 1, 2, \dots$ in order of increasing distance from the origin, with some arbitrary choice made in the case of a tie as usual. Let R_i be the copy of the random variable R governing the width of the interval placed on the particle with label i . Now let l denote Lebesgue measure and note that for any finite n

$$\mathbb{E}\left(l\left(\bigcup_{i=0}^n [x_i - R_i, x_i + R_i]\right)\right) \leq \sum_{i=0}^n \mathbb{E}(l([x_i - R_i, x_i + R_i])) \leq \sum_{i=0}^n 2\mathbb{E}(R_i) < \infty.$$

So the union of the intervals on the first n particles is almost surely bounded. Removing these intervals will not affect the percolation of the whole system. By this we mean that the occurrence of an unbounded, connected component in the set $\bigcup_{i \in I_0} [x_i - R_i, x_i + R_i]$ will not be altered by removing finitely many of the intervals. Thus letting A be the event that the union of all intervals percolate, we see that A does not depend on the first n intervals. This shows that A is a tail-event of the independent random variables $R_i, i \in \mathbb{N}$, so applying Kolmogorov’s 0-1 law we have $P(A) \in \{0, 1\}$.

Now we assume that $P(A) = 1$ so that percolation occurs almost surely. We let $U \subset \mathbb{R}$ be the set given by the union of all the random intervals on the points $x \in \xi_0$. As percolation occurs with probability 1 there is almost surely some unbounded connected component of U . Thus we have

$$\lim_{y \rightarrow \infty} P(\{y \notin U\} \cap \{-y \notin U\}) = 0.$$

But from lemma 4.3.2 we know that we can find arbitrarily large $y \in \mathbb{R}$ such that y and $-y$ are at least $1/8K$ from any point of ξ_0 . Corollary 4.3.3 tells us that for such y , we have

$$P(\{y \notin U\} \cap \{-y \notin U\}) > p_0 > 0,$$

which contradicts the limit statement above. Thus we cannot have $P(A) = 1$ so we see that $P(A) = 0$. \square

We now consider a similar initial distribution of particles ξ_0 , but on each point we evolve a pure branching tree up to time t . This is done independently for each point, as in the super-position constructions discussed in the introduction. For each $x \in \xi_0$, let $T_x \subset \mathbb{R}$ be the set of all points on \mathbb{R} ever occupied by the particles of the branching tree on x in the time interval $[0, t]$. Then let $r_x = \max\{|x - y| : y \in T_x\}$, so clearly T_x is contained in the random interval $[x - r_x, x + r_x]$. The random variables $\{r_x : x \in \xi_0\}$ are independent and identically distributed, say with distribution R .

Lemma 4.3.5. *In this case $\mathbb{E}(R) < \infty$ and $P(R \leq \epsilon) > 0$ for any $\epsilon > 0$. Consequently the pure branching process run until time t does not percolate almost surely.*

Proof. Consider some arbitrary point $x \in \xi_0$ on which we evolve a branching tree up to time t . Let M_t be the total number of particles forming this tree, so M_t is the number of differently indexed particles which are ever alive in the tree. We remark that in this pure branching tree, M_t depends only on the exponential waiting times for births and the corresponding number of offspring. So M_t does not depend on the movement of the particles. Re-labelling these particles $i = 1, \dots, M_t$, let B_t^i be the Brownian motion which governs the motion of the i -th particle. As we have seen before, these Brownian paths are still defined even

after the particle has died. Letting B_t be a general Brownian path, we have

$$\begin{aligned} \mathbb{E}(R) &= \mathbb{E}(\mathbb{E}(R|M_t)) \leq \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{M_t} \max_{0 \leq r \leq s \leq t} |B_s^i - B_r^i| \middle| M_t\right)\right) \\ &= \mathbb{E}\left(\max_{0 \leq r \leq s \leq t} |B_s - B_r|\right)\mathbb{E}(M_t). \end{aligned}$$

As $\max_{0 \leq r \leq s \leq t} |B_s - B_r|$ has finite first moment, the fact that $\mathbb{E}(R)$ is finite follows if we show $\mathbb{E}(M_t) < \infty$.

Let $\Delta^M(i)$, $i \in \mathbb{N}$, be the increase in M_t caused by the branching of a particle labelled i . Clearly this is zero if no particle with label i branches in the interval $[0, t]$. Letting $I_s \subset \mathbb{N}$ be the set of indexes of the particles which are alive in this tree at time s , we see that $\Delta^M(i)$ jumps from 0 at rate $\lambda 1_{\{i \in I_s\}} ds$. In the event of a jump, $\Delta^M(i)$ jumps to some random value distributed according to the branching offspring distribution, B . Using lemma 2.2.1 we write

$$\mathbb{E}(M_t) = \sum_{i \in \mathbb{N}} \mathbb{E}(\Delta^M(i)) = \sum_{i \in \mathbb{N}} \lambda \beta \mathbb{E}\left(\int_0^t 1_{\{i \in I_s\}} ds\right) = \lambda \beta \mathbb{E}\left(\int_0^t N_s^x ds\right),$$

where N_s^x is the population of the tree on x at time s . As this is a pure branching tree starting from one particle we have

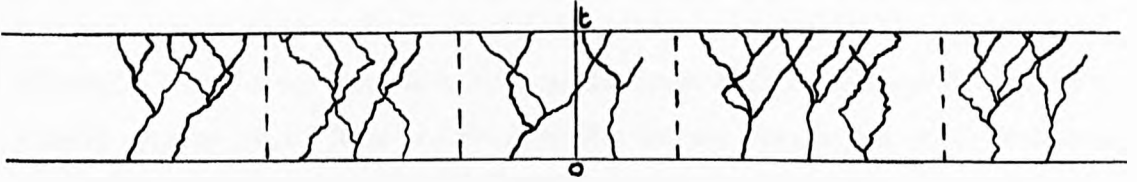
$$\mathbb{E}\left(\int_0^t N_s^x ds\right) = \int_0^t \mathbb{E}(N_s^x) ds \leq \int_0^t e^{\lambda(\beta-1)s} ds < \infty.$$

This shows that $\mathbb{E}(M_t) < \infty$, which in turn gives $\mathbb{E}(R) < \infty$.

To complete the proof we remark that there is some small but positive probability that the particle $x \in \xi_0$ does not branch in the interval $[0, t]$, and that its motion never takes it further than $\epsilon > 0$ from its initial position. In this case it is clear that $R \leq \epsilon$, so we see that $P(R \leq \epsilon) > 0$ for any positive ϵ . Now applying lemma 4.3.4 we see that the collection of pure branching trees on ξ_0 will not percolate almost surely. \square

The above lemma is the crux of this construction method. Given our initial distribution of points ξ_0 , we evolve the process as a pure branching model up to

time t . From the lemma above the ensuing graphical picture can almost surely be divided into disjoint bounded components. This idea is illustrated below:



Upon each bounded component we may now apply a thinning argument to give the interacting process required. We move up through some component of the graph until the first time an interaction occurs. As the component is bounded, there will almost surely be such a time, and a unique corresponding interaction event. We then 'trim' the branches forwards of the interaction point according to the offspring probabilities q_k . Thus with probability q_0 we remove both of the branches, with probability q_1 we remove one of the branches at random, and with probability q_2 we do nothing. The resulting graph is a subgraph of the original and so the component remains disjoint. We then repeat this process on this modified component, moving forward from the time of the interaction. This procedure will almost surely terminate after a finite number of steps, when there are no more interactions to consider. This is done for each component of the original tree, and what remains is a graphical representation of a path of our strictly reductive interacting branching process over the interval $[0, t]$.

Why does this method of construction fail for models with interactions which are stochastically rather than strictly reductive? The pure branching process will, as before, almost surely produce a realisation consisting of disjoint finite components. But now an interaction may add new particles to the graph whose motion leads them into neighbouring components, where they may branch or interact. Consequently we can no longer treat each component separately as we add the interactions; the evolution becomes dependent on the behaviour of the whole system. The interactions of the whole system have no sensible ordering in time if ξ_0 is infinite, so the construction breaks down.

The fact that such path-wise constructions are difficult to find, and often place restrictions on the offspring probabilities or initial condition, has led us to development of the soft construction method presented in this chapter. Additionally we will see that this soft construction makes it relatively simple to extend certain results from the finite to the infinite models, including a version of the Athreya and Tribe duality. Before turning our attention to such matters we sketch a proof that the process defined through the graphical method above corresponds to the equivalent process given by the soft construction. We can see this as some justification of the validity of our soft approach.

Lemma 4.3.6. *The state of the graphical construction above at time t defines a probability measure on the set of particle configurations. This corresponds to the probability measure given by the equivalent soft approach outlined in section 4.2.5.*

Proof. We only provide a sketch proof of this result:

Fix the Brownian paths, exponential random variables and outcomes of B and M which govern the movement, birth times and offspring numbers of the particles. The graphical method above then gives the corresponding realisations, firstly of the pure branching tree and then the thinned process. Let I_t^R and I_t^Q be an indexing of the particles existing at time t in the pure branching and thinned trees respectively. By construction we have $I_t^Q \subset I_t^R$. The state at time t of the pure branching process can be seen as the measure $\sum_{i \in I_t^R} \delta_{x_i}$, where x_i is the position at time t of the particle labelled i . Thus the state of the pure branching tree at time t started from ξ_0 defines a probability measure $R_t^{\xi_0}$ on the set of point mass type measures on \mathbb{R} . Similarly the thinned tree defines a probability measure $Q_t^{\xi_0}$ on the same space.

For each $x \in \xi_0$ we let $I_t^R(x)$ be the indexing at time t of the particles in the branching tree on x . We have seen above that there is some $K < \infty$ such

that $\xi_0([-m, m]) \leq 2mK$ for all $m > 0$, and we use this together with the exponential growth bound for pure branching trees (see lemma 3.1.3 in the case of no interactions), to write

$$\begin{aligned} \mathbb{E}\left(\sum_{i \in I_t^R} \phi(x_i)\right) &= \sum_{x \in \xi_0} \mathbb{E}\left(\sum_{i \in I_t^R(x)} \phi(x_i)\right) \leq \sum_{x \in \xi_0} \phi(x) e^{[(\gamma^2/2) + \lambda(\beta-1)]t} \\ &\leq C \sum_{m=1}^{\infty} \xi_0([-m, m]) \phi(m-1) \leq 2KC \sum_{m=1}^{\infty} m e^{-\lambda(m-1)} < \infty, \end{aligned}$$

where C is the constant exponential factor. From this we have

$$\mathbb{E}\left(\sum_{i \in I_t^Q} \phi(x_i)\right) \leq \mathbb{E}\left(\sum_{i \in I_t^R} \phi(x_i)\right) < \infty,$$

and thus both $Q_t^{\xi_0}$ and $R_t^{\xi_0}$ give no mass to measures which are not finite measures under the weight function ϕ . This allows us to define the push-forward probability measures $\tilde{Q}_t^{\xi_0}$ and $\tilde{R}_t^{\xi_0}$ on the space $F \subset M_f(\mathbb{R})$.

To complete the proof we must show that $\tilde{Q}_t^{\xi_0} \equiv \tilde{P}_t^{\xi_0}$, where $\tilde{P}_t^{\xi_0}$ is the law on F defined via the soft construction presented earlier. It is known from our earlier work that it suffices to show

$$\int_f e^{-\langle \mu, f \rangle} \tilde{Q}_t^{\xi_0}(d\mu) = \int_f e^{-\langle \mu, f \rangle} \tilde{P}_t^{\xi_0}(d\mu) \quad \text{for all } f \in V,$$

or equivalently

$$\mathbb{E}\left(\prod_{i \in I_t^Q} u_0(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t^P} u_0(x_i)\right) \quad \text{for } u_0(\cdot) = e^{-\phi(\cdot)f(\cdot)}, \quad f \in V.$$

By virtue of our soft construction we have

$$\mathbb{E}\left(\prod_{i \in I_t^P} u_0(x_i)\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t^P(n)} u_0(x_i)\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t^Q(n)} u_0(x_i)\right).$$

The second inequality requires some thought: $I_t^Q(n)$ is the indexing at time t of the particles in the graphical construction on the initial state $\xi_0(n)$. But

$I_t^P(n)$ is the same indexing for the finite process run from $\xi_0(n)$, and a 'nuts and bolts' construction of this process is given in the introduction. Thus if we fix the collection of Brownian paths, exponential outcomes and offspring numbers, then the particles indexed by $I_t^P(n)$ and $I_t^Q(n)$ are identical. Equality along every path gives $\mathbb{E}(\prod_{i \in I_t^P(n)} u_0(x_i)) = \mathbb{E}(\prod_{i \in I_t^Q(n)} u_0(x_i))$. Thus the final fact required is that

$$\mathbb{E}\left(\prod_{i \in I_t^Q} u_0(x_i)\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t^Q(n)} u_0(x_i)\right).$$

Take some realisation of the graphical construction, the pure branching stage of which almost surely consists of a collection of disjoint bounded components. For any $n \in \mathbb{N}$ let $C(n)$ be the union of those components which contain only branches stemming from particles in $\xi_0(n)$. We remark that the tree on ξ_0 and the tree on $\xi_0(n)$ agree exactly on $C(n)$, so that

$$\prod_{\substack{i \in I_t^Q \\ x_i \in C(n)}} u_0(x_i) = \prod_{\substack{i \in I_t^Q(n) \\ x_i \in C(n)}} u_0(x_i).$$

Using this we can write

$$\begin{aligned} \left| \prod_{i \in I_t^Q} u_0(x_i) - \prod_{i \in I_t^Q(n)} u_0(x_i) \right| &\leq \left| \prod_{i \in I_t^Q} u_0(x_i) - \prod_{\substack{i \in I_t^Q \\ x_i \in C(n)}} u_0(x_i) \right| + \left| \prod_{\substack{i \in I_t^Q(n) \\ x_i \in C(n)}} u_0(x_i) - \prod_{i \in I_t^Q(n)} u_0(x_i) \right| \\ &\leq \left| 1 - \prod_{\substack{i \in I_t^Q \\ x_i \notin C(n)}} u_0(x_i) \right| + \left| 1 - \prod_{\substack{i \in I_t^Q(n) \\ x_i \notin C(n)}} u_0(x_i) \right| \\ &\leq 2 \left(1 - \prod_{\substack{i \in I_t^R \\ x_i \notin C(n)}} u_0(x_i) \right). \end{aligned}$$

So the final fact required is that

$$\mathbb{E}\left(\prod_{\substack{i \in I_t^R \\ x_i \notin C(n)}} u_0(x_i)\right) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

To see this we use Jensen's inequality and the fact that each $f \in V$ is bounded by 1 to write

$$\mathbb{E}\left(\prod_{\substack{i \in I_t^R \\ x_i \notin C(n)}} u_0(x_i)\right) = \mathbb{E}\left(\exp\left\{-\sum_{\substack{i \in I_t^R \\ x_i \notin C(n)}} \phi(x_i) f(x_i)\right\}\right) \geq \exp\left\{-\mathbb{E}\left(\sum_{\substack{i \in I_t^R \\ x_i \notin C(n)}} \phi(x_i)\right)\right\}.$$

We know from above that $\mathbb{E}(\sum_{i \in I_t^R} \phi(x_i)) < \infty$, and that $C(n)$ almost surely grows arbitrarily large as $n \rightarrow \infty$, so

$$\mathbb{E}\left(\sum_{\substack{i \in I_t^R \\ x_i \notin C(n)}} \phi(x_i)\right) \rightarrow 0,$$

giving the convergence required. □

Chapter 5

Stationary Distributions, Duality Theory and an Example

We have worked hard over the previous chapters to construct our interacting branching processes on \mathbb{R} . With this construction behind us, we can begin formulating results about these general models. We begin with an extension of the much-used Athreya and Tribe duality relation to processes consisting of infinitely many particles. That the proof of this result is relatively simple is a consequence of the ‘soft’ construction method utilised earlier. The infinite duality relation allows us to show that the infinite processes fulfil the Feller property, which in turn is used to show the existence of stationary distributions for the models. Again we look to the work done on the unit ring for our methodology, and the framework of the proof is almost identical.

5.1 The Infinite Duality Relation I

The first general result presented for these infinite processes is a duality relation analogous to the Athreya and Tribe result discussed earlier. This result is not only interesting in itself, but will be used extensively throughout the remainder of this chapter. In the following result it is immaterial whether we choose to consider the particle process with one-dimensional distributions $P_t^{\mu_0}$ or the *weighted* particle process with distributions $\tilde{P}_t^{\mu_0}$. The important information is I_t , the indexing of the particles of the process at time t , and the positions $\{x_i : i \in I_t\}$ of these particles on \mathbb{R} . These remain the same regardless of whether the particles are weighted or not.

Proposition 5.1.1. *Suppose that ξ_t is a interacting branching process of the type discussed above, started from the infinite initial condition indexed by I_0 . Suppose that u_t is a solution of the corresponding SPDE where u_0 has the form $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$ for some non-negative, bounded and continuous $f : \mathbb{R} \rightarrow \mathbb{R}$. Then it follows that*

$$\mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right).$$

Before proving this result we remark that it is not a direct extension of the earlier finite duality relation. We now have an added condition requiring a certain form of the initial condition for the SPDE. Later, in the case of processes with no annihilating interactions, we will prove an additional general duality result which holds for all continuous u_0 . However, as shall be seen when discussing the Feller property, the result above can still be very useful.

Proof. Firstly, for any initial condition u_0 of this form we can apply proposition 4.2.9 to give

$$\mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right).$$

Secondly we have

$$\begin{aligned} \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) &= \mathbb{E}\left(e^{\sum_{i \in I_t} \ln u_0(x_i)}\right) = \mathbb{E}\left(e^{-\langle \xi_t, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle}\right) \\ &= \int_{M_f(\mathbb{R})} e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0(\cdot) \rangle} \tilde{P}_t^{\mu_0}(d\nu) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) \end{aligned}$$

where the final equality holds since this is precisely the way in which we defined our infinite process in section 4.2.

Thus we have

$$\mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) = \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right)$$

which completes our proof □

5.2 The Feller Property

As in the earlier case on the ring, S , the Feller property is an important tool in ensuring the existence of stationary distributions for these processes. Additionally the Feller property is the final ingredient in the construction of the underlying Markov process described in proposition 4.2.21. This section is devoted to showing that this property holds for the laws $\tilde{P}_t^{\mu_0}$ defining our infinite models. Thus we wish to establish that the map $x \mapsto \mathbb{E}_x(f(\xi_t))$ from F to \mathbb{R} is bounded and continuous whenever the function $f : F \rightarrow \mathbb{R}$ is. Here the expectation on the right-hand side is interpreted as

$$\mathbb{E}_x(f(\xi_t)) = \int_F f(\nu) \tilde{P}_t^{\phi^{-1}x}(d\nu),$$

since $\tilde{P}_t^{\phi^{-1}x}$ is the law on F of the process started from initial condition $\phi^{-1}x$.

In proving the Feller property for finite processes on the ring, we made use of a coupling argument. This told us that if two initial conditions were sufficiently similar, there was a high probability that the corresponding processes

would become coupled. We would like to use a similar argument here, but the coupling mechanism fails in the case of infinitely many particles. However, the construction of the law of the infinite process at time t as the limit of the laws of finite processes allows us to adapt that earlier work. We begin by adapting the coupling lemma 2.4.7 to the models on \mathbb{R} , at least in the case where the processes are finite.

Suppose that x and y are elements of F which correspond to initial conditions consisting of exactly n particles. Let ξ_t and η_t be copies of our F -valued interacting branching process started from x and y respectively. We consider a coupling mechanism exactly the same as that defined before lemma 2.4.7.

Lemma 5.2.1. *Fix any $T > 0$ and let ξ_t and η_t be as described above, with $x \in F$ fixed. Then for any $\alpha > 0$ there exists $\delta > 0$ so that*

$$d(x, y) < \delta \implies P(\xi_t \text{ and } \eta_t \text{ coupled by time } T) > 1 - \alpha.$$

Proof. The proof of this result follows almost directly from the methods used in the proofs of lemmas 2.4.6 and 2.4.7, so we do not repeat it here. \square

In order to show the continuity of the map $x \mapsto \mathbb{E}_x(f(\xi_t))$ we consider some convergent sequence, say $x_n \implies x$, in F . Let the corresponding indexing sets of the particles in x_n and x be I_0^n and I_0 respectively. We remark here that I_0^n and $I_0(n)$ are entirely different, the latter being the restriction of I_0 to the n particles nearest the origin as usual.

Let $x \in F$ correspond to an infinite collection of particles with indexing I_0 . We say that some integer m is a δ separation value for x (or for I_0) if the minimum distance between points indexed by $I_0(m)$ and those indexed by $I_0 \setminus I_0(m)$ is at least $\delta > 0$, so $\min\{|x_i - x_j| : i \in I_0(m), j \in I_0 \setminus I_0(m)\} \geq \delta$.

We say that m is a separation value if it is a δ separation value for some δ .

Lemma 5.2.2. *If $x \in F$, with the corresponding I_0 infinite, then we can find separation values for x which are arbitrarily large.*

Proof. Suppose that this result is false and that m is the last separation value for x . Let s_l and s_r be the positions of the left- and right-most particles of $I_0(m)$ respectively. As $m + 1$ is not a separation value it follows that the new particle introduced when moving from $I_0(m)$ to $I_0(m + 1)$ must lie at either s_l or s_r . The same follows for $m + 2$ and so on. Thus the remainder of the particles of I_0 lie at either s_l or s_r , and since I_0 is infinite this gives $\sum_{i \in I_0} \phi(x_i) = \infty$ which contradicts $x \in F$. \square

Remark 5.2.1. If $x \in F$ correspond to only finitely many particles, say $|I_0| = n$, then n is a separation value for x for any $\delta > 0$. It is important to note that the following work holds in this simpler case. In the proof of the Feller property where we use the above lemma to choose a ‘large enough’ separation value, we can in the finite case just use n .

We now have the following important lemma concerning these separation values. It is this lemma, together with coupling and the infinite duality relation proved in the previous section, which will provide the proof of the Feller property.

Lemma 5.2.3. *Suppose $x_n \implies x$ in F as described above and that m is a 3δ separation point for x . Suppose that the left- and right- hand points of $I_0(m)$ are at s_l and s_r respectively. Then there exists some n_0 such that for $n \geq n_0$*

- (i) there exists exactly m points of I_0^n in the interval $[s_l - \delta, s_r + \delta]$, and*
- (ii) there are no particles of I_0^n in the intervals $[s_l - 2\delta, s_l - \delta]$ and $[s_r + \delta, s_r + 2\delta]$.*

Proof. We can define a bounded and continuous function $f_{[a,b]}^\delta : \mathbb{R} \rightarrow \mathbb{R}$ so that

$f_{[a,b]}^\delta(\cdot)\phi(\cdot)$ has the form

$$f_{[a,b]}^\delta(x)\phi(x) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a - \delta, b + \delta] \\ \text{linear in between} & \end{cases}$$

Letting $|I_0^n[a, b]|$ denote the number of particles of I_0^n in the interval $[a, b]$ we have

$$\langle x_n, f_{[s_l, s_r]}^\delta \rangle \leq |I_0^n[s_l - \delta, s_r + \delta]| \leq \langle x_n, f_{[s_l - \delta, s_r + \delta]}^\delta \rangle.$$

Taking limits as $n \rightarrow \infty$ and using the fact that m is a 3δ separation point for x we have

$$m = \langle x, f_{[s_l, s_r]}^\delta \rangle \leq \lim_{n \rightarrow \infty} |I_0^n[s_l - \delta, s_r + \delta]| \leq \langle x, f_{[s_l - \delta, s_r + \delta]}^\delta \rangle = m$$

and so

$$\lim_{n \rightarrow \infty} |I_0^n[s_l - \delta, s_r + \delta]| = m.$$

As this is the limit of a sequence of integers we can conclude that the sequence eventually becomes m , so there exists $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies that $|I_0^n[s_l - \delta, s_r + \delta]| = m$.

We may now repeat the above argument beginning with the fact

$$\langle x_n, f_{[s_l - \delta, s_r + \delta]}^\delta \rangle \leq |I_0^n[s_l - 2\delta, s_r + 2\delta]| \leq \langle x_n, f_{[s_l - 2\delta, s_r + 2\delta]}^\delta \rangle$$

to show that there exists $n_2 \in \mathbb{N}$ such that $n \geq n_2$ implies $|I_0^n[s_l - 2\delta, s_r + 2\delta]| = m$. Taking $n_0 = \max\{n_1, n_2\}$ and combining these two properties gives the result. \square

To state the next result consider the following notation: if $y \in F$ is a measure corresponding to a collection of particles with index I , then $y(m) \in F$ denotes the measure formed by restricting y to the particles indexed $I(m)$.

Lemma 5.2.4. *If $x_n \Rightarrow x$ in F and m is a separation value for x , then $x_n(m) \Rightarrow x(m)$.*

Proof. As m is a separation value for x there exists some $\delta > 0$ so that m is a 3δ separation value for x . By lemma 5.2.3 above there are eventually exactly m points of each x_n in the interval $[s_l - \delta, s_r + \delta]$ and none in the two intervals of length δ either side of this. Thus for any bounded and continuous $g : \mathbb{R} \rightarrow \mathbb{R}$ we have

$$\lim_{n \rightarrow \infty} \langle x_n(m), g \rangle = \lim_{n \rightarrow \infty} \langle x_n, g \circ f_{[s_l - \delta, s_r + \delta]}^\delta \rangle = \langle x, g \circ f_{[s_l - \delta, s_r + \delta]}^\delta \rangle = \langle x(m), g \rangle$$

where $f_{[a,b]}^\delta$ is the bounded, continuous function defined in the proof of 5.2.3. Thus for any bounded and continuous g we have shown $\lim_{n \rightarrow \infty} \langle x_n(m), g \rangle = \langle x(m), g \rangle$ and thus that $x_n(m) \Rightarrow x(m)$ in F . \square

We are now in a position to prove the Feller property for our processes:

Proposition 5.2.5. *If $f : F \rightarrow \mathbb{R}$ is bounded and continuous on F , then so is the map $x \mapsto \mathbb{E}_x(f(\xi_t))$.*

Proof. Take some sequence $x_n \Rightarrow x$ in F . We want to show that for any bounded and continuous map $f : F \rightarrow \mathbb{R}$ we have

$$\int_F f(\nu) \tilde{P}_t^{\phi^{-1}x_n}(d\nu) \longrightarrow \int_F f(\nu) \tilde{P}_t^{\phi^{-1}x}(d\nu).$$

In other word we wish to show that $\tilde{P}_t^{\phi^{-1}x_n}$ converges weakly to $\tilde{P}_t^{\phi^{-1}x}$ in the space of probability measures on F . From Dawson [11] and Ethier and Kurtz [20] we know that it suffices to show that

$$\int_F e^{-\langle \nu, f \rangle} \tilde{P}_t^{\phi^{-1}x_n}(d\nu) \longrightarrow \int_F e^{-\langle \nu, f \rangle} \tilde{P}_t^{\phi^{-1}x}(d\nu)$$

for every non-negative, bounded and continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. This, as shown in the infinite construction chapter, is again equivalent to proving that

$$\mathbb{E}\left(\prod_{i \in I_t^n} u_0(x_i)\right) \longrightarrow \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right)$$

where u_0 has the form $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$.

Now fix some such $f : \mathbb{R} \rightarrow \mathbb{R}$ and some $\epsilon > 0$. As $\sum_{i \in I_0} \phi(x_i) < \infty$ we can choose m large enough so that

$$M \sum_{\substack{i \in I_0 \\ i \notin I_0(m)}} \phi(x_i) e^{[\gamma^2/2 + \lambda(\beta-1)]t} < \frac{\epsilon}{6} \quad (1)$$

where $M > 0$ is a constant with $f(x) \leq M$ for all $x \in \mathbb{R}$. If x corresponds to finite I_0 , say with $|I_0| = n$, then we can take $m = n$, so m is a separation value for x by remark 5.2.1. If I_0 is infinite then using lemma 5.2.2 we may also assume m is a separation value for x . In either case there some $\delta > 0$ for which m is a 3δ separation value for x .

For this fixed m we can write

$$\begin{aligned} \left| \mathbb{E} \left(\prod_{i \in I_t^n} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right) \right| &\leq \left| \mathbb{E} \left(\prod_{i \in I_t^n} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t^n(m)} u_0(x_i) \right) \right| \\ &+ \left| \mathbb{E} \left(\prod_{i \in I_t^n(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) \right| \\ &+ \left| \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right) \right|. \quad (*) \end{aligned}$$

We now assume that $n \geq n_0$ where n_0 is the value given in lemma 5.2.3. Thus, with s_l and s_r being the positions of the left- and right- hand points of $I_0(m)$, we have exactly m particles of x_n in the interval $[s_l - \delta, s_r + \delta]$ and no particles of x_n in the two intervals of length δ which lie either side of this.

Now we use the infinite duality relation set out in proposition 5.1.1, noting that the initial condition u_0 is of the form required for the result. Remarking once again that $I_t(m)$ is the indexing at time t of the particles of a process

started from $I_0(m)$ (not the m particles of I_t closest to the origin), we have

$$\begin{aligned} \left| \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right) \right| &= \left| \mathbb{E} \left(\prod_{i \in I_0(m)} u_t(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_0} u_t(x_i) \right) \right| \\ &= \left| \mathbb{E} \left(\prod_{i \in I_0(m)} u_t(x_i) \left(1 - \prod_{\substack{i \in I_0 \\ i \notin I_0(m)}} u_t(x_i) \right) \right) \right| \\ &\leq \mathbb{E} \left(\left| \prod_{i \in I_0(m)} u_t(x_i) \right| \left| 1 - \prod_{\substack{i \in I_0 \\ i \notin I_0(m)}} u_t(x_i) \right| \right) \\ &\leq 1 - \mathbb{E} \left(\prod_{\substack{i \in I_0 \\ i \notin I_0(m)}} u_t(x_i) \right). \end{aligned}$$

Now we let $I_0(m)_t^c$ denote the indexing at time t of the particles of the process started from the initial condition $I_0 \setminus I_0(m)$; in other words the process run from all except the m particles closest to the origin. We can define $I_0^n(m)_t^c$ in an analogous manner. Thus once again using the infinite duality, together with the fact that $1 - e^{-z} \leq z$ for non-negative z , we have

$$\begin{aligned} 1 - \mathbb{E} \left(\prod_{\substack{i \in I_0 \\ i \notin I_0(m)}} u_t(x_i) \right) &= 1 - \mathbb{E} \left(\prod_{i \in I_0(m)_t^c} u_0(x_i) \right) = \mathbb{E} \left(1 - e^{-\sum_i f(x_i) \phi(x_i)} \right) \\ &\leq \mathbb{E} \left(\sum_{i \in I_0(m)_t^c} f(x_i) \phi(x_i) \right) \leq M \mathbb{E} \left(\sum_{i \in I_0(m)_t^c} \phi(x_i) \right). \end{aligned}$$

Now combining the two calculations above and then using the exponential growth bound (lemma 3.1.3) on the sum, we have

$$\begin{aligned} \left| \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right) \right| &\leq M \mathbb{E} \left(\sum_{i \in I_0(m)_t^c} \phi(x_i) \right) \\ &\leq M \sum_{\substack{i \in I_0 \\ i \notin I_0(m)}} \phi(x_i) e^{[\gamma^2/2 + \lambda(\beta-1)]t}. \end{aligned}$$

In this case the choice of m gives condition (1) above and we can see that the right-hand side of the above inequality is less than $\epsilon/6$. Further, up to this point the same calculation for x_n rather than x yields

$$\left| \mathbb{E} \left(\prod_{i \in I_t^n(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t^n} u_0(x_i) \right) \right| \leq M \sum_{\substack{i \in I_0^n \\ i \notin I_0^n(m)}} \phi(x_i) e^{[\gamma^2/2 + \lambda(\beta-1)]t}.$$

We define the bounded and continuous map $h : \mathbb{R} \rightarrow \mathbb{R}$ to be

$$h(x) = \begin{cases} 0 & x \in [s_l - \delta, s_r + \delta] \\ 1 & x \notin [s_l - 2\delta, s_r + 2\delta] \\ \text{linear in between} & \end{cases}$$

As we have assumed that $n \geq n_0$, and since x_n converges to x , we can state immediately that

$$\sum_{\substack{i \in I_0^n \\ i \notin I_0^n(m)}} \phi(x_i) = \int_{\mathbb{R}} h(s) x_n(ds) = \langle x_n, h \rangle \longrightarrow \langle x, h \rangle = \sum_{\substack{i \in I_0 \\ i \notin I_0(m)}} \phi(x_i).$$

So we can choose $n_1 \geq n_0$ large enough, so that $n \geq n_1$ implies

$$M \sum_{\substack{i \in I_0^n \\ i \notin I_0^n(m)}} \phi(x_i) e^{[\gamma^2/2 + \lambda(\beta-1)]t} \leq M \sum_{\substack{i \in I_0 \\ i \notin I_0(m)}} \phi(x_i) e^{[\gamma^2/2 + \lambda(\beta-1)]t} + \frac{\epsilon}{6} \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.$$

Finally we consider the remaining term in (*), namely

$$\left| \mathbb{E} \left(\prod_{i \in I_t^n(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) \right|.$$

Again we wish to show that this term is small for large n and to do this we use a coupling argument. $I_t^n(m)$ and $I_t(m)$ are indexing sets for finite processes started from the same finite number of particles, namely m , so they are certainly candidates for the coupling referred to in lemma 5.2.1. If the two processes $\xi_s^n(m)$ and $\xi_s(m)$ have coupled by time t , then the difference above will be zero since we would have $\{x_i : i \in I_t^n(m)\} \equiv \{x_i : i \in I_t(m)\}$. Hence we have the trivial bound

$$\left| \mathbb{E} \left(\prod_{i \in I_t^n(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) \right| \leq 2P(\xi_s^n(m), \xi_s(m) \text{ not coupled by time } t).$$

Now by lemma 5.2.4 we know that $x_n(m) \Rightarrow x(m)$ in F , which can equivalently be written as $\xi_0^n(m) \Rightarrow \xi_0(m)$. Thus, by the coupling lemma 5.2.1, we can

choose n_2 large enough so that $n \geq n_2$ implies that the probability $\xi_s^n(m)$ and $\xi_s(m)$ are not coupled by time t is less than $\epsilon/6$. Consequently for $n \geq n_2$

$$\left| \mathbb{E} \left(\prod_{i \in I_t^n(m)} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t(m)} u_0(x_i) \right) \right| \leq \frac{\epsilon}{3}.$$

Combining all the work above into the expression (*) we see that for any fixed $\epsilon > 0$, we can choose $N = \max\{n_1, n_2\}$ and $m \in \mathbb{N}$ so that for any $n \geq N$ we have

$$\left| \mathbb{E} \left(\prod_{i \in I_t^n} u_0(x_i) \right) - \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right) \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{6} < \epsilon.$$

Thus we have shown that

$$\mathbb{E} \left(\prod_{i \in I_t^n} u_0(x_i) \right) \longrightarrow \mathbb{E} \left(\prod_{i \in I_t} u_0(x_i) \right)$$

for any u_0 of the form $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$ where f is non-negative, bounded and continuous. This in turn proves that the Feller property holds for these processes. \square

5.3 Existence of Stationary Measures

With the Feller property in place, we are now ready to show that stationary distributions exist for these processes. Following the work done for processes on the ring we again use the convergence result found in theorem 2.4.1 as our starting point.

Let E be the space $\overline{F} \cup \{\infty_w\}$, which we know from corollary 4.2.13 is both compact and metrizable. As usual $\mathcal{M}_1(E)$ denotes the space of probability measures on E with the topology of weak convergence. We fix some initial condition $x = \xi_0 \in F$ for our process and define the measures $\mu_t^x \in \mathcal{M}_1(E)$ by $\mu_t^x \equiv \tilde{P}_t^{\phi^{-1}x}$.

Now consider the sequence $\{\nu_n^x\}$ of Cesaro averages of these measures, namely

$$\nu_n^x = \frac{1}{n} \int_0^n \mu_s^x ds.$$

It is to this sequence of probability measures on E to which we apply theorem 2.4.1. Thus there is some convergent subsequence $\{\nu_{n'}^x\}$ such that $\nu_{n'}^x \implies \nu^x$, where ν^x is an element of $\mathcal{M}_1(E)$.

Our aim now is two-fold: firstly to show that in fact $\nu_{n'}^x \implies \nu^x$ in $\mathcal{M}_1(F)$, the space of probability measure on the natural state space F of our processes. Secondly, once we know that ν^x is a probability measure on F , we show that ν^x is a stationary distribution for our process.

Lemma 5.3.1. *The measure ν^x gives no mass to the set $\{\infty_w\}$.*

Proof. For each $k \in \mathbb{N}$ we define a map $J_k : \overline{F} \cup \{\infty_w\} \rightarrow \mathbb{R}$ by

$$J_k(x) = \begin{cases} 1 & \text{if } \langle 1, x \rangle \geq k \text{ or if } x = \infty_w \\ 0 & \text{if } \langle 1, x \rangle \leq k - 1 \\ \langle 1, x \rangle - (k - 1) & \text{if } \langle 1, x \rangle \in (k - 1, k). \end{cases}$$

This map is clearly bounded. To show it is also continuous suppose that $x_n \implies x$ in $\overline{F} \cup \{\infty_w\}$. If $x = \infty_w$ then by definition $\langle 1, x_n \rangle \rightarrow \infty$, so the sequence $J_k(x_n)$ is eventually identically 1. If $x \neq \infty_w$ then the situation is even easier since $\langle 1, x_n \rangle \rightarrow \langle 1, x \rangle$ by the definition of weak convergence. Thus J_k is a bounded, continuous function on $\overline{F} \cup \{\infty\}$.

Now we define the sets $A_k \subset \overline{F} \cup \{\infty_w\}$ to be

$$A_k = \{\infty_w\} \cup \{x : \langle 1, x \rangle \geq k\}.$$

It follows that $\nu^x(\{\infty_w\}) = \lim_{k \rightarrow \infty} \nu^x(A_k)$. But also we have

$$\nu^x(A_k) = \int_E 1_{A_k}(x) \nu^x(dx) \leq \int_E J_k(x) \nu^x(dx) = \langle J_k, \nu^x \rangle$$

so combining these gives

$$\nu^x(\{\infty_w\}) = \lim_{k \rightarrow \infty} \nu^x(A_k) \leq \lim_{k \rightarrow \infty} \langle J_k, \nu^x \rangle. \quad (1)$$

Now as J_k is bounded and continuous, and $\nu_{n'}^x \implies \nu^x$, we have $\langle J_k, \nu^x \rangle = \lim_{n' \rightarrow \infty} \langle J_k, \nu_{n'}^x \rangle$. Here though we may write

$$\langle J_k, \nu_{n'}^x \rangle \leq \nu_{n'}^x(A_{k-1}) = \frac{1}{n'} \int_0^{n'} \mu_s(A_{k-1}) ds \leq \sup_{s \geq 0} \mu_s(A_{k-1})$$

which no longer depends on n' so that we have

$$\langle J_k, \nu^x \rangle \leq \sup_{s \geq 0} \mu_s(A_{k-1}). \quad (2)$$

Recalling that our initial condition is $\xi_0 = x \in F$, we let $x(m)$ be the restriction of x to the m particles nearest the origin as usual. We have

$$\mu_s(A_{k-1}) \leq \langle J_{k-1}, \mu_s \rangle = \langle J_{k-1}, \tilde{P}_s^{\phi^{-1}x} \rangle.$$

But as $x(m) \implies x$ in F by lemma 4.2.5, and since J_{k-1} is bounded and continuous, we use the Feller property on F (proposition 5.2.5) to write

$$\begin{aligned} \langle J_{k-1}, \tilde{P}_s^{\phi^{-1}x} \rangle &= \lim_{m \rightarrow \infty} \langle J_{k-1}, \tilde{P}_s^{\phi^{-1}x(m)} \rangle \\ &\leq \lim_{m \rightarrow \infty} \tilde{P}_s^{\phi^{-1}x(m)}(A_{k-2}) \leq \sup_{m \in \mathbb{N}} P\left(\sum_{i \in I_s(m)} \phi(x_i) \geq k-2\right). \end{aligned}$$

Thus using this in (2) above we have

$$\langle J_k, \nu^x \rangle \leq \sup_{s \geq 0} \sup_{m \in \mathbb{N}} P\left(\sum_{i \in I_s(m)} \phi(x_i) \geq k-2\right)$$

which in turn can be used in (1) to give

$$\nu^x(\{\infty_w\}) \leq \lim_{k \rightarrow \infty} \sup_{s \geq 0} \sup_{m \in \mathbb{N}} P\left(\sum_{i \in I_s(m)} \phi(x_i) \geq k-2\right).$$

However we can now see directly from proposition 3.1.12 that this limit is zero, so that $\nu^x(\{\infty_w\}) = 0$ as required. \square

Corollary 5.3.2. *We have $\nu_{n'}^x \implies \nu^x$ in $\mathcal{M}_1(\overline{F})$.*

Proof. The measures $\{\nu_{n'}^x\}$ are certainly measures on \overline{F} and from lemma 5.3.1 above we know that so too is ν^x . Using similar methods to those in lemma 4.2.14 we can show that \overline{F} is separable. The result now follows from proposition 2.4.13 in chapter 2. \square

Using a slight adaptation of the method of lemma 4.2.19, is now easy to show that ν^x is supported on F . Then, as in corollary 4.2.20, it follows that:

Lemma 5.3.3. *We have $\nu_{n'}^x \implies \nu^x$ in $\mathcal{M}_1(F)$.*

We may now conclude this section with an existence theorem for stationary distributions of these processes. This result is analogous to theorem 2.4.9 for the processes on the ring.

Theorem 5.3.4. *For each $x \in F$ there is a convergent subsequence $\{\nu_{n'}^x\}$ of the Cesaro averages, such that $\nu_{n'}^x \implies \nu^x$ weakly in $\mathcal{M}_1(F)$ as $n' \rightarrow \infty$. The limit $\nu^x \in \mathcal{M}_1(F)$ is a stationary distribution for the process.*

Proof. We have already shown all parts of this result except the stationarity of the measure ν^x . This final fact follows as in the proof of lemma 2.4.15 in chapter 2. Beginning with the expression labelled (1) in that proof the work is identical, even using the same notation, and so we do not repeat it here. \square

5.4 The Infinite Duality Relation II

- Non-Annihilating Processes

In proposition 5.1.1 we already have an extension of the Athreya and Tribe duality to infinite particle systems. In this section we prove a second version of this result for non-annihilating processes, in which the restrictions on the initial condition u_0 of the SPDE are weakened. Although it is always preferable to have results which hold in the most general setting, there is a specific motivation for this work. It is hoped that the duality formula can be used to translate information about one system into information about the other. To do this requires careful choice of the initial condition of the SPDE, and it shall be seen that many of the most useful such u_0 fall outside the scope of proposition 5.1.1. Thus we desire to strengthen our result to a broader class of u_0 .

We have discussed solutions to the associated SPDE which are bounded in the interval $[a, 1]$, where $a \in [-1, 0]$ is the root of $\sigma(u)$ lying closest to 0. In the case of a non-annihilating process we must have $q_0 = 0$ which in turn implies that $\sigma(0) = 0$. This tells us that $a = 0$ and so the solutions of the SPDE under consideration lie in $[0, 1]$. This non-negativity of solutions will be important in the following work, and it is this aspect which breaks down when considering annihilating processes.

For any continuous initial condition $u_0 : \mathbb{R} \rightarrow [0, 1]$ of the SPDE, define the corresponding continuous maps u_0^m to be

$$u_0^m(s) = \begin{cases} u_0(s) \vee \frac{1}{m} & \text{if } s \in [-m, m] \\ 1 & \text{if } s \notin [-m-1, m+1] \\ (1-\theta)(u_0(s) \vee \frac{1}{m}) + \theta & \text{if } s = -m-\theta \text{ or } s = m+\theta \text{ for } \theta \in [0, 1]. \end{cases}$$

Each such map u_0^m is bounded in $[\frac{1}{m}, 1]$, has tails which are eventually 1 and is always greater than or equal to u_0 .

Lemma 5.4.1. *Let $u_0 : \mathbb{R} \rightarrow [0, 1]$ be continuous. For each corresponding u_0^m we define the function $f_m : F \rightarrow \mathbb{R}$ by*

$$f_m(\nu) = e^{-\langle \nu, -\frac{1}{\phi(\cdot)} \ln u_0^m(\cdot) \rangle}.$$

These maps are bounded and continuous on F .

Proof. From the properties of u_0^m discussed above, it is easy to see that the map $g_m : \mathbb{R} \rightarrow \mathbb{R}$ given by $g_m(\cdot) = -\frac{1}{\phi(\cdot)} \ln u_0^m(\cdot)$ is continuous, bounded and non-negative. It thus follows that f_m is bounded in $[0, 1]$ and is continuous by the definition of convergence in F . □

Corollary 5.4.2. *For u_0^m as defined above we have*

$$\mathbb{E} \left(\prod_{i \in I_t(n)} u_0^m(x_i) \right) \longrightarrow \mathbb{E} \left(\prod_{i \in I_t} u_0^m(x_i) \right) \quad \text{as } n \rightarrow \infty.$$

Proof. From corollary 4.2.20 we know that $\tilde{P}_t^{\mu_0(n)} \implies \tilde{P}_t^{\mu_0}$ in $\mathcal{M}_1(F)$. From lemma 5.4.1 above we know that the maps f_m are bounded and continuous on F , so that

$$\int_F f_m(\nu) \tilde{P}_t^{\mu_0(n)}(d\nu) \longrightarrow \int_F f_m(\nu) \tilde{P}_t^{\mu_0}(d\nu). \tag{1}$$

Now if $\nu \in F$ has I as the corresponding indexing of its atoms, we can see that $f_m(\nu) = \prod_{i \in I} u_0^m(x_i)$. Using this we may rewrite the expression (1) above as

$$\mathbb{E} \left(\prod_{i \in I_t(n)} u_0^m(x_i) \right) \longrightarrow \mathbb{E} \left(\prod_{i \in I_t} u_0^m(x_i) \right) \quad \text{as } n \rightarrow \infty.$$

□

Before we state and prove our duality result we quote a simple analysis lemma regarding sequences, together with its immediate corollary:

Lemma 5.4.3. *Suppose that $\{a(n, m) : n, m \in \mathbb{N}\}$ is decreasing in both n and m , and is bounded below. Then*

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a(n, m) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a(n, m).$$

Corollary 5.4.4. *With the maps u_0^m as defined above we have*

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{E} \left(\prod_{i \in I_t(n)} u_0^m(x_i) \right) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E} \left(\prod_{i \in I_t(n)} u_0^m(x_i) \right).$$

Proof. From lemma 5.4.3 above it suffices to show that $\mathbb{E} \left(\prod_{i \in I_t(n)} u_0^m(x_i) \right)$ is decreasing in both n and m and is bounded below. Each expectation is clearly bounded below by zero so the latter condition is simple. Also, by construction, we have $0 \leq u_0^{m+1}(x) \leq u_0^m(x)$ for all $x \in \mathbb{R}$, so clearly

$$\mathbb{E} \left(\prod_{i \in I_t(n)} u_0^{m+1}(x_i) \right) \leq \mathbb{E} \left(\prod_{i \in I_t(n)} u_0^m(x_i) \right).$$

It remains to show that this sequence is also decreasing in n . This is not easy to see with the expectations in their current form, since adding a particle to the system at time 0 may reduce the number of particles at time t . However applying the finite duality relation given in proposition 4.1.1 we see that it suffices to show that $\mathbb{E} \left(\prod_{i \in I_0(n)} u_t^m(x_i) \right)$ is decreasing in n . Here u_t^m denotes a solution at time t of the SPDE started from initial condition u_0^m . Now since u_t^m is bounded in $[0, 1]$ we write

$$\mathbb{E} \left(\prod_{i \in I_0(n+1)} u_t^m(x_i) \right) = \mathbb{E} \left(\left(\prod_{i \in I_0(n)} u_t^m(x_i) \right) u_t^m(x_{n+1}) \right) \leq \mathbb{E} \left(\prod_{i \in I_0(n)} u_t^m(x_i) \right).$$

This shows that the sequence is decreasing in n and so we may interchange the limits as stated. \square

We can now combine these ingredients to give the following duality result:

Proposition 5.4.5. *Let ξ_t be a non-annihilating particle system started from a possibly infinite set of points $\{x_i : i \in I_0\}$. Let u_t be a solution to the corresponding SPDE with the continuous initial condition $u_0 : \mathbb{R} \rightarrow [0, 1]$. Then the following duality relation holds*

$$\mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right).$$

Proof. Throughout this proof we will make repeated use of lemma 4.2.6, together with the dominated convergence theorem, to show the convergence in expectation of products of decreasing terms. A detailed example of such a calculation can be found in the proof of proposition 4.2.9. Here we can guarantee that all the infinite products exist since the terms lie in $[0, 1]$. This makes applying lemma 4.2.6 straight-forward.

Firstly we write

$$\begin{aligned} \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) &= \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0(n)} u_t(x_i)\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t(n)} u_0(x_i)\right) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t(n)} u_0^m(x_i)\right). \end{aligned} \quad (1)$$

The first and final equalities arise from applying lemma 4.2.6 as discussed, whilst the central equality follows from the original *finite* duality relation, which held for any continuous u_0 .

Beginning from the other side of the proposed relation, we write

$$\mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) = \lim_{m \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t} u_0^m(x_i)\right) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_t(n)} u_0^m(x_i)\right). \quad (2)$$

The first equality follows yet again from lemma 4.2.6, whilst the second is due to corollary 5.4.2 above. To complete the proof we notice that, by corollary 5.4.4, the right-hand sides of both (1) and (2) are equal. Thus the left-hand sides must be equal too and this gives the duality result stated. \square

5.5 Exploiting the Duality

Throughout this work much emphasis has been placed on the duality which exists between our particle systems and the solutions to certain SPDEs. This duality has proved to be an invaluable tool in helping us to construct and study these particle processes, from defining the finite dimensional distributions of the infinite process to proving the Feller property. The numerous duality results that have been presented allow information about one system to yield information about the corresponding dual system and vice-versa. In this section we give some general results which arise from exploiting the duality relation in both directions. We end the section with the study of a specific example in which previous work on an SPDE yields information about the survival of one of our interacting branching processes.

5.5.1 From SPDE to Particle System

This paper has been concerned with the construction and study of a certain class of interacting branching process. Thus we naturally begin by asking how the duality relation can be used to give information about our particle processes. However, before we investigate this we make a few observations concerning the associated SDPE.

We consider solutions to the associated SPDE which are bounded in the interval $[a, 1]$ for some $a \in [-1, 0]$. As $b(1) = \sigma(1) = 0$ it is clear that the state $u(x) \equiv 1$ is a trap, or 'extinct state', for the SPDE. In many ways it is more natural to consider $1 - u_t(x)$ which lies in the region $[0, 1 + |a|]$ and has an extinct state at 0. However, such a transformation would make the coefficients of b and σ less readily comparable with the offspring probabilities

of the corresponding particle system, and would lead to a slightly less elegant form for the duality relation. So we will retain the ‘upside down’ solutions and introduce some additional terminology. We say that an initial condition u_0 for the SPDE is *compactly supported at 1* if $1 - u_0$ is compactly supported. Also we say that a solution to the SPDE ‘dies out’ if it converges in distribution to the coffin state $u \equiv 1$. Here we consider our solutions in the space $C(\mathbb{R}, [-1, 1])$ of continuous maps from \mathbb{R} to $[-1, 1]$ with the topology of uniform convergence on compact sets.

We may now state our first general result: this tells us that certain behaviour of solutions to the SPDE ensures the local extinction of the associated particle process. By local extinction we mean that for any compact set $K \subset \mathbb{R}$, the probability that K contains particles of the process tends to 0 as t gets large.

Proposition 5.5.1. *Suppose that solutions to the SPDE die out from any initial condition which is compactly supported at 1. Then the associated particle system exhibits local extinction from any initial condition ξ_0 (provided $\sum_{i \in I_0} \phi(x_i) < \infty$ as usual).*

Proof. For any interval $[a, b] \subset \mathbb{R}$ and any arbitrary $\epsilon > 0$ we define the initial condition $u_0^{a,b}$ to be

$$u_0^{a,b}(s) = \begin{cases} 1/2 & \text{if } s \in [a, b] \\ 1 & \text{if } s \notin [a - \epsilon, b + \epsilon] \\ \text{linear in between.} & \end{cases}$$

Clearly there exists a non-negative, bounded and continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ so that we can write $u_0^{a,b}(\cdot) = e^{-f(\cdot)\phi(\cdot)}$. Consequently we can apply the duality relation as stated in proposition 5.1.1. This duality result holds for the entire class of particle systems we have discussed, requiring only that the offspring probabilities fulfil $\mu < 2$, $\beta < \infty$, and that $\sum_{i \in I_0} \phi(x_i) < \infty$.

Now by assumption the SPDE dies out from $u_0^{a,b}$, so we have

$$\mathbb{E}\left(\prod_{i \in I_0} u_t^{a,b}(x_i)\right) \longrightarrow 1 \quad \text{as } t \rightarrow \infty. \quad (1)$$

Now we notice that

$$\begin{aligned} \mathbb{E}\left(\prod_{i \in I_t} u_0^{a,b}(x_i)\right) &\leq \mathbb{E}\left(\prod_{\substack{i \in I_t \\ x_i \in [a,b]}} u_0^{a,b}(x_i)\right) = \mathbb{E}\left((1/2)^{|\xi_t[a,b]|}\right) \\ &\leq P(|\xi_t[a,b]| = 0) + \frac{1}{2}P(|\xi_t[a,b]| > 0) = \frac{1}{2} + \frac{1}{2}P(|\xi_t[a,b]| = 0), \end{aligned}$$

which can be re-arranged to give

$$P(|\xi_t[a,b]| = 0) \geq 2\mathbb{E}\left(\prod_{i \in I_t} u_0^{a,b}(x_i)\right) - 1.$$

Applying the duality relation and using (1) above we have

$$P(|\xi_t[a,b]| = 0) \geq 2\mathbb{E}\left(\prod_{i \in I_0} u_t^{a,b}(x_i)\right) - 1 \longrightarrow 1 \quad \text{as } t \rightarrow \infty.$$

Thus, for any interval $[a, b]$, the probability that there are no particles living in the interval tends to 1 as t gets large, giving local extinction. \square

This result is of particular interest to us since it not only holds for all the interacting branching processes we have discussed, but also gives information about systems started from an infinite number of particles. In the following section we prove the equivalent result in the other direction; that local extinction of the particle system gives convergence in distribution of solutions of the SPDE to $u \equiv 1$. We shall also see an example in which the behaviour of the associated SPDE is used to show non-extinction of the particle system, and the existence of non-trivial stationary distributions.

Before leaving this section we briefly consider how different choices of u_0 can give information about the particle system in terms of the finite dimensional

distributions of the SPDE. We remark that each of the following forms of u_0 take the value 0 at some point, so cannot be written in the form $u_0(\cdot) = e^{-f(\cdot)\phi(\cdot)}$ for any non-negative, bounded and continuous $f : \mathbb{R} \rightarrow \mathbb{R}$. Thus we can no longer apply proposition 5.1.1 and these observations hold only in those cases for which we have duality for all continuous u_0 , namely for any finite particle process (proposition 4.1.1) or where there is no annihilation (proposition 5.4.5).

Firstly if we modify the initial conditions $u_0^{a,b}$ defined above so that they take the value 0 on $[a, b]$, and also take $\epsilon > 0$ to be small, then we have

$$\mathbb{E}\left(\prod_{i \in I_0} u_t^{a,b}(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t} u_0^{a,b}(x_i)\right) \simeq P(|\xi_t[a, b]| = 0).$$

Thus the expected value on the left, which relies on knowledge of the SPDE, can be used to estimate the probability that the particle system is extinct in $[a, b]$ at time t . In particular, if the SPDE dies out from such initial conditions, then the rate at which this occurs gives the rate of local extinction of the particle system.

To be precise we remark that the approximate equality in the expression above is really a 'less than or equal to' relationship. Specifically we can write

$$P(|\xi_t[a - \epsilon, b + \epsilon]| = 0) \leq \mathbb{E}\left(\prod_{i \in I_t} u_0^{a,b}(x_i)\right) \leq P(|\xi_t[a, b]| = 0).$$

From this we see why the above approximate statement seems reasonable when $\epsilon > 0$ is small.

In a similar manner we can consider an initial condition u_0 which is identically 0. This can then be used to give the probability of global extinction of the particle system in terms of the SPDE. Clearly this is only helpful when considering systems started from finitely many particles, but we do note the result is exact rather than approximate. To see this we write

$$\mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) = P(\xi \text{ is totally extinct at time } t),$$

recalling that the product over an empty set is defined to be 1. Conversely knowing that the particle system becomes extinct from finite initial states tells us that the expected value of $u_t(x_i)$ must tend to 1 for finite sets of points $\{x_i : i \in I_0\}$. We shall expand upon this idea in the next section.

Finally, using initial conditions of the form

$$u_0^{a,b}(s) = \begin{cases} 1 & \text{if } s \in [a, b] \\ 0 & \text{if } s \notin [a - \epsilon, b + \epsilon] \\ \text{linear in between} & \end{cases}$$

with $\epsilon > 0$ small, we can write

$$\mathbb{E}\left(\prod_{i \in I_0} u_t^{a,b}(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t} u_0^{a,b}(x_i)\right) \simeq P(\{\xi_t \subset [a, b]\} \cup \{\xi_t = \emptyset\}).$$

Thus $1 - \mathbb{E}(\prod_{i \in I_0} u_t^{a,b}(x_i))$ approximates the probability that there are particles of the process living outside $[a, b]$ at time t . Choosing $\{x_i : i \in I_0\} \subset [a, b]$ we can hence estimate the probability that ξ has 'escaped' from $[a, b]$ by time t .

Of course the difficulty in applying the duality relation in such ways is knowing sufficient information about one of the systems. In the examples above the required expectations of the SPDE are not easy to find. Similarly the behaviour of the particle system yields information about the SPDE only if that behaviour is known! We present a specific example later, but firstly we present some more general theory.

5.5.2 From Particle System to SDPE

We have seen that particle systems are often used as tools to study the behaviour of solutions to SPDEs. The results presented in this section are of this type, categorising properties of the SPDE in terms of the behaviour of the corresponding

particle process. Before stating our results we require some general theory of these SPDEs.

Lemma 5.5.2. *Let $u_0 : \mathbb{R} \rightarrow [a, 1]$ be a continuous map. Let u_t be a solution to an SPDE of the type discussed above, started from the initial condition u_0 . If μ_t denotes the law on $C(\mathbb{R}, [a, 1])$ of u_t , then the sequence $\{\mu_t : t \geq 1\}$ is tight.*

Proof. We show that for each $L \geq 0$, $\exists C_L > 0$ such that

$$\sup_{t \geq 1} \mathbb{E} \left(|u_t(x) - u_t(y)|^r \right) \leq C_L |x - y|^{1+\alpha} \quad \forall x, y \in [-L, L], \quad (1)$$

where r and α are fixed strictly positive constants. As u_t is bounded for all $t \geq 0$ this is sufficient to show the required tightness, see for example [30] and [19]. Further, since C_L does not depend on u_0 and using the fact that these solutions are Markov and remain continuous and bounded in $[a, 1]$, we need only show that

$$\sup_{1 \leq t \leq T} \mathbb{E} \left(|u_t(x) - u_t(y)|^r \right) \leq C_L |x - y|^{1+\alpha} \quad \forall x, y \in [-L, L], \quad (1')$$

for some $T > 1$. To see this, note that for any $s \geq 0$ we can consider u_{t+s} to be a solution of the SPDE at time t started from the initial condition u_s . We know that u_s is a continuous map from \mathbb{R} to $[a, 1]$, so if (1') holds we have

$$\sup_{1 \leq t \leq T} \mathbb{E} \left(|u_{t+s}(x) - u_{t+s}(y)|^r \right) \leq C_L |x - y|^{1+\alpha} \quad \forall x, y \in [-L, L],$$

which in turn gives (1) as $s \geq 0$ was arbitrary.

Now for $1 \leq t \leq T$ we begin by using the Green's function representation of u_t (see [38]) to write

$$\begin{aligned} u_t(x+h) - u_t(x) &= \int_{\mathbb{R}} [G_t(x+h, y) - G_t(x, y)] u_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] b(u_s(y)) dy ds \\ &+ \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] \sqrt{\sigma(u_s(y))} \dot{W}(s, y) dy ds, \end{aligned}$$

where $G_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}$. Then clearly we have

$$\begin{aligned} & \mathbb{E}\left(|u_t(x+h) - u_t(x)|^r\right) \\ & \leq 3^r \left| \int_{\mathbb{R}} [G_t(x+h, y) - G_t(x, y)] u_0(y) dy \right|^r \\ & + 3^r \mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] b(u_s(y)) dy ds \right|^r\right) \quad (*) \\ & + 3^r \mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] \sqrt{\sigma(u_s(y))} \dot{W}(s, y) dy ds \right|^r\right). \end{aligned}$$

We treat each term on the right-hand side of (*) in turn, beginning with the last one. Applying the Burkholder-Davis-Grundy inequality (again see [30]) we write

$$\begin{aligned} & 3^r \mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] \sqrt{\sigma(u_s(y))} \dot{W}(s, y) dy ds \right|^r\right) \\ & \leq 3^r k(r) \mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)]^2 \sigma(u_s(y)) dy ds \right|^{r/2}\right) \\ & \leq 3^r k(r) M_\sigma^{r/2} \left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)]^2 dy ds \right|^{r/2}, \end{aligned}$$

where $k(r)$ is a positive constant dependent only on r , and $M_\sigma = \max_{a \leq u \leq 1} \sigma(u)$. At this stage we wish to apply the following result which can be found in [33] and which is adapted from earlier work by Sowers [39]. It states that for any $\kappa \in (0, 1/4)$ there exists $c_\kappa > 0$ such that for any $x \in \mathbb{R}$ and $t \geq 0$

$$\int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)]^2 dy ds \leq c_\kappa |h|^{2\kappa}.$$

Using this in the above we see that

$$\begin{aligned} & 3^r \mathbb{E}\left(\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] \sqrt{\sigma(u_s(y))} \dot{W}(s, y) dy ds \right|^r\right) \\ & \leq 3^r k(r) M_\sigma^{r/2} c_\kappa^{r/2} |h|^{r\kappa} = C_3(r, \kappa) |h|^{r\kappa}. \end{aligned}$$

To treat the remaining two terms we remark that since $G_s(x+h, y) \geq G_s(x, y)$ if and only if $y \geq h/2$, we have

$$\int_{\mathbb{R}} |G_s(x+h, y) - G_s(x, y)| dy \leq 2 \int_{-h/2}^{h/2} \frac{1}{\sqrt{2\pi s}} dy \leq \frac{2|h|}{\sqrt{2\pi s}}. \quad (2)$$

Recalling that $|u_0| \leq 1$, we use this immediately on the first term in (*) to write

$$\begin{aligned} \left| \int_{\mathbb{R}} [G_t(x+h, y) - G_t(x, y)] u_0(y) dy \right| &\leq \int_{\mathbb{R}} |G_t(x+h, y) - G_t(x, y)| dy \\ &\leq \frac{2|h|}{\sqrt{2\pi t}} \leq \sqrt{2/\pi} |h|. \end{aligned}$$

The final inequality follows since $t \geq 1$ and we see now why this is stipulated in the statement of the result. This shows that the map $f_1(x) = \int_{\mathbb{R}} G_t(x, y) u_0(y) dy$ is Lipschitz continuous, and therefore it follows that it is Hölder continuous for any exponent $\alpha \leq 1$. In particular f_1 is Hölder continuous with exponent $\kappa \in (0, 1/4)$, and using this we see that there is some constant $C_1(r, \kappa)$ such that

$$3^r \left| \int_{\mathbb{R}} [G_t(x+h, y) - G_t(x, y)] u_0(y) dy \right|^r \leq C_1(r, \kappa) |h|^{r\kappa}.$$

Again using expression (2) above we now write

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] b(u_s(y)) dy ds \right| \\ &\leq \left(\max_{a \leq u \leq 1} |b(u)| \right) \int_0^t \int_{\mathbb{R}} |G_s(x+h, y) - G_s(x, y)| dy ds \\ &\leq \left(\max_{a \leq u \leq 1} |b(u)| \right) \int_0^t \frac{2|h|}{\sqrt{2\pi s}} ds \\ &\leq \left(\max_{a \leq u \leq 1} |b(u)| \right) \frac{4|h|}{\sqrt{2\pi}} t^{1/2} \leq \left(\max_{a \leq u \leq 1} |b(u)| \right) \frac{4|h|}{\sqrt{2\pi}} T^{1/2}. \end{aligned}$$

This shows that the function $f_2(x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x, y) b(u_s(y)) dy ds$ is Lipschitz continuous for any u_s and so again is Hölder continuous with exponent $\kappa \in (0, 1/4)$. Thus there is some $C_2(r, \kappa)$ such that

$$3^r \mathbb{E} \left(\left| \int_0^t \int_{\mathbb{R}} [G_{t-s}(x+h, y) - G_{t-s}(x, y)] b(u_s(y)) dy ds \right|^r \right) \leq C_2(r, \kappa) |h|^{r\kappa}.$$

Letting $C(r, \kappa) = C_1(r, \kappa) + C_2(r, \kappa) + C_3(r, \kappa)$ we can now use (*) to write

$$\mathbb{E} \left(|u_t(x+h) - u_t(x)|^r \right) \leq C(r, \kappa) |h|^{r\kappa}.$$

Choosing $r > 1/\kappa$ shows (1') which is sufficient to give tightness. □

Using this tightness allows us to prove the following results. By a finite initial configuration or state, we mean an arrangement of finitely many particles on \mathbb{R} at time $t = 0$.

Proposition 5.5.3.

(i). *Suppose the particle system becomes extinct from any finite initial configuration almost surely. Then solutions to the corresponding SPDE converge in distribution to the dead state $u \equiv 1$ from any continuous initial condition u_0 .*

(ii). *Suppose the particle system becomes locally extinct from any finite initial configuration almost surely. Then solutions to the corresponding SPDE converge in distribution to the dead state $u \equiv 1$ from any continuous initial condition u_0 which is compactly supported at 1.*

Proof. Firstly we notice that in both cases we have

$$\mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) \longrightarrow 1 \quad \text{as } t \rightarrow \infty \quad (1)$$

whenever the particle process is started from finitely many initial particles. In case (i) this follows from the almost sure eventual extinction of the process. In case (ii) the map u_0 is identically 1 outside some closed interval, and so (1) follows from the local extinction of the particle process.

Now as we have already shown tightness for these solutions, we know that the convergence we require follows from showing convergence of the finite dimensional distributions (see [36]). Consequently to show convergence of solutions to the extinct state $u \equiv 1$, we must show that

$$\mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) \longrightarrow \mathbb{E}\left(\prod_{i \in I_0} u(x_i)\right) = 1 \quad \text{as } t \rightarrow \infty,$$

for any finite collection of points $\{x_i : i \in I_0\} \subset \mathbb{R}$. In the cases outlined in the proposition this follows from the duality relation and expression (1), since

$$\mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) = \mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) \longrightarrow 1 \quad \text{as } t \rightarrow \infty.$$

□

To illustrate the necessity of the second case in the above proposition, we remark that it may be impossible for the process to die out from some finite initial states. Consider a system with no single particle branching in which the pairwise interactions preserve the parity of the model. In other words the number of offspring produced in an interaction event is always even. Such a system cannot become extinct when started from an odd number of particles. Similarly consider a system with no single particle branching and with no annihilating interactions. This also can never die out. However if $\mu < 2$ then both of these two classes of system will exhibit local extinction. This allows us to conclude from case (ii) of proposition 5.5.3 that the SPDEs

$$\partial_t u = \frac{1}{2} \Delta u + \sqrt{\left(\sum_{k=0}^{\infty} q_{2k} u^{2k} - u^2\right)} \dot{W}_{t,x}, \quad \sum_{k=0}^{\infty} 2k q_{2k} < 2$$

and

$$\partial_t u = \frac{1}{2} \Delta u + \sqrt{\left(\sum_{k=1}^{\infty} q_k u^k - u^2\right)} \dot{W}_{t,x}, \quad \sum_{k=1}^{\infty} k q_k < 2$$

have solutions which converge in distribution to $u \equiv 1$ from any u_0 which is compactly supported at 1. These facts would not have followed from part (i).

We remark through the following lemma that the property ‘solutions to the SPDE converge in distribution to $u \equiv 1$ ’ can be interpreted as a local extinction property.

Lemma 5.5.4. *Suppose that u_t converges in distribution to $u \equiv 1$. Then for any $\epsilon > 0$ and any interval $[a, b] \subset \mathbb{R}$, we have*

$$P(\inf\{u_t(x) : x \in [a, b]\} < 1 - \epsilon) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Proof. As u_t converges in distribution to $u \equiv 1$, this means that the corresponding measures μ_t converge weakly to μ in the space $C(\mathbb{R}, [-1, 1])$ with the topology of uniform convergence on compacts. Noting that μ is the point-measure on

$u \equiv 1$ we have

$$\int_{C(\mathbb{R},[-1,1])} f(x)\mu_t(dx) \rightarrow f(u \equiv 1), \tag{1}$$

for any bounded, continuous map $f : C(\mathbb{R},[-1,1]) \rightarrow \mathbb{R}$. Now for the interval $[a, b] \subset \mathbb{R}$ define the map f by $f(x) = \inf\{x(s) : s \in [a, b]\}$. This map is clearly bounded. It is continuous since if x_n converges to x in $C(\mathbb{R},[-1,1])$, then x_n certainly converges uniformly to x on $[a, b]$, so that

$$f(x_n) = \inf\{x_n(s) : s \in [a, b]\} \rightarrow \inf\{x(s) : s \in [a, b]\} = f(x).$$

From (1) this gives

$$\int_{C(\mathbb{R},[-1,1])} f(x)\mu_t(dx) \rightarrow f(u \equiv 1) = 1 \quad \text{as } t \rightarrow \infty.$$

In particular, for any $\epsilon > 0$, $\mu_t(\{x : f(x) < 1 - \epsilon\}) \rightarrow 0$ as $t \rightarrow \infty$, which can be re-formulated as

$$P(\inf\{u_t(x) : x \in [a, b]\} < 1 - \epsilon) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

□

The information we now have, in particular the two propositions 5.5.1 and 5.5.3, leads to the following if and only if results for our particle processes and their corresponding SPDEs. Proposition 5.5.1 additionally gives local extinction conditions for the infinite particle process.

Corollary 5.5.5. *For an interacting branching process and its dual SPDE we have:*

<i>Particle system becomes extinct from any finite initial state almost surely</i>	\Leftrightarrow	<i>Solutions to SPDE converge in distribution to $u \equiv 1$ from any continuous u_0</i>
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<i>Particle system becomes locally extinct almost surely from any state with $\sum_{i \in I_0} \phi(x_i) < \infty$</i>	\Leftrightarrow	<i>Solutions to SPDE converge in $dist^n$ to $u \equiv 1$ from any continuous u_0 compactly supported at 1</i>
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Proof. Except for the one from right to left in the first statement, all implications follow from one of the two propositions. The final implication comes from considering the duality when $u_0 \equiv 0$ as at the end of section 5.5.1. \square

We have seen that the final implication from left to right actually follows from the local extinction of the particle system from *finite* initial states, rather than the stronger condition given here. These results are stated in terms of extinction of the particle system and SPDE but can of course be re-written in terms of survival. We shall use this in the following section to provide a concrete example of a reductive interacting branching process which exhibits non-trivial behaviour.

5.5.3 An Example

We conclude this thesis with a study of the particular example alluded to in the introduction. This particle system has single parent binary branching, so that each particle splits into two at rate λ , and pairwise local-time coalescence. We show that if λ is sufficiently large, then the particle system will not exhibit local extinction.

Our main tool in this analysis is the work of Mueller and Sowers [33] on interface solutions to the SPDE

$$\partial_t v = \Delta v + v - v^2 + \epsilon \sqrt{v(1-v)} \dot{W}. \quad (1)$$

We summarise their results here: suppose that the initial condition v_0 for the above equation is bounded in $[0, 1]$ and has an interface. By this we mean that $v_0(x) = 1$ for $x < a$ and $v_0(x) = 0$ for $x > b$, where $a < b$ are real constants. Then there exist solutions to (1) which remain bounded in $[0, 1]$ and continue to have this interface form for all $t \geq 0$. Let $a(t) = \inf\{x \in \mathbb{R} : v_t(x) < 1\}$ and $b(t) = \sup\{x \in \mathbb{R} : v_t(x) > 0\}$ be the extreme points of the interface. Then for

small $\epsilon > 0$ and with probability 1,

$$\alpha := \lim_{t \rightarrow \infty} \frac{b(t)}{t} \text{ exists and lies in } (0, \infty). \quad (2)$$

This limit α is dependent only on ϵ . Further the length $L(t)$ of the interface region $[a(t), b(t)]$ tends towards a stationary distribution under which $L(t)$ is almost surely finite. Thus the interface does not degenerate and has a positive limiting speed.

Clearly we wish to make use of these results via the duality relation, but the SPDE (1) is not currently in the form of one of our dual equations. To see how the work in [33] will be of use we make the transformation $\tilde{v}_t(x) = 1 - v_t(-x)$, as Mueller and Sowers themselves do, to give

$$\partial_t \tilde{v} = \Delta \tilde{v} + (\tilde{v}^2 - \tilde{v}) + \epsilon \sqrt{\tilde{v} - \tilde{v}^2} \dot{W}.$$

Finally we introduce a linear scaling of time and space by letting $u_t(x) = \tilde{v}_{t/\epsilon}(x\sqrt{2/\epsilon})$, which gives

$$\partial_t u = \frac{1}{2} \Delta u + \frac{1}{\epsilon} (u^2 - u) + \sqrt{u - u^2} \dot{W}. \quad (1')$$

Letting $\lambda = 1/\epsilon$ we now see that (1') is in the form of one of our dual equations. In particular it is the dual SPDE to a particle system in which $p_2 = 1$ and $q_1 = 1$, precisely the process discussed above. We now prove some properties of solutions to (1) which then tell us about the transformed solutions in (1'). This in turn gives information about the particle process.

Lemma 5.5.6. *For $\epsilon > 0$ small enough so that (2) holds we also have*

$$\lim_{t \rightarrow \infty} \frac{a(t)}{t} = \alpha \quad \text{a.s.},$$

with $\alpha \in (0, \infty)$ as given in (2).

Proof. By definition we have $a(t) = b(t) - L(t)$ so that we can write

$$\begin{aligned} P(\lim_{t \rightarrow \infty} a(t)/t = \alpha) &\geq P(\{\lim_{t \rightarrow \infty} b(t)/t = \alpha\} \cap \{\lim_{t \rightarrow \infty} L(t)/t = 0\}) \\ &= 1 - P(\{\lim_{t \rightarrow \infty} b(t)/t \neq \alpha\} \cup \{\lim_{t \rightarrow \infty} L(t)/t > 0\}) \\ &\geq 1 - P(\lim_{t \rightarrow \infty} b(t)/t \neq \alpha) - P(\lim_{t \rightarrow \infty} L(t)/t > 0). \end{aligned}$$

In the final line we already know from (2) that $P(\lim_{t \rightarrow \infty} b(t)/t \neq \alpha) = 0$. Thus it remains to show that $P(\lim_{t \rightarrow \infty} L(t)/t > 0) = 0$ also. To do this we use the following result which is lemma 4.1 in [33]:

Fix $\gamma \in (0, 1)$. There is a finite constant $L_0(\gamma) > 0$ which depends only on γ , $L(0)$ and ϵ such that

$$P(L(t) > L_0(\gamma)) < \gamma \quad \forall t \geq 0. \quad (*)$$

To make use of this fact we recall that $L(t)$ converges in distribution to some stationary distribution, denoted L , as $t \rightarrow \infty$. For any $\delta > 0$ and $N > 0$ the set $[0, \delta N]$ is closed, so we can write

$$P(\lim_{t \rightarrow \infty} L(t)/N \leq \delta) = P(\lim_{t \rightarrow \infty} L(t) \leq \delta N) = P(L \leq \delta N) \geq \limsup_{t \rightarrow \infty} P(L(t) \leq \delta N).$$

Now for any $\gamma \in (0, 1)$, and regardless of how small we choose $\delta > 0$, if N is large enough so that $\delta N > L_0(\gamma)$, then from (*) we have

$$P(\lim_{t \rightarrow \infty} L(t)/N \leq \delta) \geq \limsup_{t \rightarrow \infty} P(L(t) \leq \delta N) \geq \limsup_{t \rightarrow \infty} P(L(t) \leq L_0(\gamma)) > 1 - \gamma.$$

For such sufficiently large N we may now write

$$P(\lim_{t \rightarrow \infty} L(t)/t \leq \delta) \geq P(\lim_{t \rightarrow \infty} L(t)/N \leq \delta) \geq 1 - \gamma.$$

This is true for arbitrary positive γ , and so $P(\lim_{t \rightarrow \infty} L(t)/t > \delta) = 0$. Similarly as this holds for arbitrary $\delta > 0$ we have $P(\lim_{t \rightarrow \infty} L(t)/t = 0) = 1$ which completes our proof. \square

This lemma tells us that the limiting speed of the left-hand side of the interface is the same as that of the right-hand side of the interface. This is not really surprising as we know that the length of the interval converges to a stationary distribution. The reason why we need this knowledge of $a(t)$ rather than $b(t)$ will now become clear. Consider an initial condition v_0 for the SPDE (1) of the form

$$v_0(s) = \begin{cases} 1 & \text{if } s \in [-m, m] \\ 0 & \text{if } s \notin [m - \delta, m + \delta] \\ \text{linear in between,} & \end{cases}$$

with $\delta > 0$. Define $L^1(t)$ to be the length of the interval around the origin for which $v_t(x) = 1$, so that $L^1(0) = 2m$. Our next result tells us that there is a positive chance that $L^1(t)$ grows arbitrarily large and never takes the value 0.

Lemma 5.5.7. *For small $\epsilon > 0$ and sufficiently large m we can guarantee that the event $\{L^1(t) \rightarrow \infty \text{ and never takes the value } 0\}$ occurs with probability at least $1/4$.*

Proof. This proof is adapted from an oriented percolation proof which can be found in section 3 of [17]. The first important step is to notice that until the two interfaces meet, so until the first time at which $L^1(t) = 0$, they evolve independently of each other. Thus to consider the right-hand interface we can consider the SPDE started from the initial condition

$$v_0^r(s) = \begin{cases} 1 & \text{if } s \leq m \\ 0 & \text{if } s \geq m + \delta \\ \text{linear in between.} & \end{cases}$$

This is exactly the situation discussed in [33] and in the work above, so with $a(t)$ defined as before, we know from lemma 5.5.6 that for small ϵ

$$\lim_{t \rightarrow \infty} \frac{a(t)}{t} = \alpha \in (0, \infty) \quad \text{a.s.}$$

Consequently we may choose m large enough so that if $a(0) = m$ as it does here, then $P(\inf_{t \geq 0} a(t) > 0) \geq 1/2$. Thus with probability at least $1/2$, the right-hand interface will tend to $+\infty$ and never move below the origin.

Using the spatial reflection $\tilde{v}_t(x) = v_t(-x)$ we can see that the same is true for the left-hand interface but in the opposite direction. So, again with probability at least $1/2$, the left-hand interface will tend to $-\infty$ and never move above the origin.

If both of these events occur, which happens with probability at least $1/4$, then $L^1(t) \rightarrow \infty$ and never takes the value 0. □

This result concerns solutions to the SPDE (1) rather than our dual SPDE (1'). However these SPDEs are related through a simple inversion and scaling presented earlier. Thus a solution u_t of (1') started from an initial condition

$$u_0(x) = \begin{cases} 0 & \text{if } x \in [-M, M] \\ 1 & \text{if } x \notin [-M - \delta_1, M + \delta_1] \\ \text{linear in between} & \end{cases}$$

corresponds to a solution v_t of (1) started from

$$v_0(x) = \begin{cases} 1 & \text{if } x \in [-m, m] \\ 0 & \text{if } x \notin [-m - \delta_2, m + \delta_2] \\ \text{linear in between.} & \end{cases}$$

Here m and δ_2 are scalar multiples of M and δ_1 respectively. From lemma 5.5.7 above we know that if m is sufficiently large then, with probability at least $1/4$, the region of v_t which is 1 will grow to be infinite. Consequently the same is true for the region of u_t which is 0 if M is large enough. Through the duality this leads us to the following result.

Proposition 5.5.8. *Consider the model described above in which particles split into two at rate λ and coalesce according to pairwise local-time. Let $\xi_0 = x$ be a*

non-empty initial state indexed by I_0 . Then for sufficiently large λ there exists a positive probability that the particle process does not become locally extinct from x . Additionally there exists at least one non-trivial stationary distribution for this process.

Proof. In re-writing (1) in our dual SPDE form (1') we noted that $\lambda = 1/\epsilon$. Thus small ϵ corresponds to large λ and vice-versa. We choose ϵ small enough, and hence λ large enough, so that lemma 5.5.7 holds. Let u_0 be as defined above, so that it is 0 on $[-M, M]$ and 1 outside the slightly larger interval $[-M-\delta_1, M+\delta_1]$. Here M is chosen to be large enough so that the remarks above hold, and we let G be the event that the region of u_t which is zero grows to infinity. Then from the above remarks we have

$$\begin{aligned} \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) &= \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i) | G\right) P(G) + \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i) | G^c\right) P(G^c) \\ &\leq \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i) | G\right) P(G) + \frac{3}{4} \longrightarrow \frac{3}{4} \text{ as } t \rightarrow \infty. \end{aligned}$$

The convergence in the last line holds because under G there must be a point $x_i, i \in I_0$ at which u_t is eventually always zero, making the product zero also. Recalling the general theory presented in section 5.5.1, we have

$$\limsup_{t \rightarrow \infty} P(|\xi_t[-M-\delta_1, M+\delta_1]| = 0) \leq \limsup_{t \rightarrow \infty} \mathbb{E}\left(\prod_{i \in I_0} u_t(x_i)\right) \leq 3/4.$$

Thus there is some t_0 such that for all $t \geq t_0$, $P(|\xi_t[-M-\delta_1, M+\delta_1]| > 0) > 1/8$. This shows that the process does not always exhibit local extinction.

To prove the stated result concerning stationary distributions we first need to show that the set $A = \{x \in F : x([a, b]) > 0\}$ is a closed subset of the state space F . We suppose that $x_n \implies x$ in F with each $x_n \in A$. For a contradiction we assume that $x \notin A$, so that $x([a, b]) = 0$. As there are only finitely many atoms of x in the intervals $[a-1, a)$ and $(b, b+1]$, there is some $\epsilon > 0$ such that

no atoms of x lie in $[a - \epsilon, b + \epsilon]$. As usual we define f_ϵ to be 1 on $[a, b]$, 0 outside $[a - \epsilon, b + \epsilon]$ and linear in between. This map is bounded and continuous so

$$x_n([a, b]) \leq \int_{\mathbb{R}} f_\epsilon(s)x_n(ds) \longrightarrow \int_{\mathbb{R}} f_\epsilon(s)x(ds) = 0.$$

But the left-hand side of the above is a sequence of positive integers, so the convergence above is impossible. This contradiction shows that A is closed.

From our earlier work on stationary distributions, we know from theorem 5.3.4 that there is a subsequence $\{\nu_{n'}^x\}$ of the Cesaro averages, such that $\nu_{n'}^x \implies \nu^x$ weakly in $\mathcal{M}_1(F)$ as $n' \rightarrow \infty$. This limit ν^x is stationary for the process. Letting $[a, b] = [-M - \delta_1, M + \delta_1]$ in the definition of the set A above, we know since A is closed that

$$\nu^x(A) \geq \limsup_{n' \rightarrow \infty} \nu_{n'}^x(A). \quad (1)$$

But then for $n' \geq t_0$, with t_0 given above, we have

$$\begin{aligned} \nu_{n'}^x(A) &= \frac{1}{n'} \int_0^{n'} \tilde{P}_s^{\phi^{-1}x}(A) ds = \frac{1}{n'} \int_0^{n'} P(|\xi_s[-M - \delta_1, M + \delta_1]| > 0) ds \\ &\geq \frac{1}{n'} \int_{t_0}^{n'} P(|\xi_s[-M - \delta_1, M + \delta_1]| > 0) ds \geq \frac{n' - t_0}{8n'}. \end{aligned}$$

As n' gets large we have $(n' - t_0)/8n' \rightarrow 1/8$, so that using (1),

$$\nu^x(A) \geq \limsup_{n' \rightarrow \infty} \nu_{n'}^x(A) \geq 1/8.$$

This shows that the stationary distribution $\nu^x \in \mathcal{M}_1(F)$ is non-trivial. \square

We have shown that for sufficiently large $\lambda \in [0, \infty)$ the particle system does not necessarily exhibit local extinction. At the opposite end of the scale when $\lambda = 0$ there is no branching and this coalescing system will almost surely exhibit local extinction. We can thus consider this example as a class of systems parameterised by λ . Now we use a coupling argument for the associated SPDE to prove that the property of almost sure local extinction of the particle system

is monotone in λ . This then tells us that there is some $\lambda_c \in [0, \infty)$ such that local extinction is guaranteed for $\lambda < \lambda_c$ and not for $\lambda > \lambda_c$.

Consider two coalescing branching processes ξ_t^1 and ξ_t^2 of the type above, with branching rates λ_1 and λ_2 respectively. Let I_t^1 and I_t^2 denote the indexing of the particles alive in each process at time t . Suppose that both processes begin from the same initial state ξ_0 . Let u_t^1 and u_t^2 denote solutions to the associated SPDEs of ξ_t^1 and ξ_t^2 respectively, both started from the initial condition u_0 . So u_t^1 is a solution to the associated SPDE with parameter λ_1 , and u_t^2 a solution to a similar SPDE with parameter λ_2 . Using this notation we have the following lemma and corollary:

Lemma 5.5.9. *If $u_0 : \mathbb{R} \rightarrow [0, 1]$ is continuous and $\lambda_1 \geq \lambda_2$, then*

$$\mathbb{E}\left(\prod_{i \in I_t^1} u_0(x_i)\right) \rightarrow 1 \text{ as } t \rightarrow \infty \implies \mathbb{E}\left(\prod_{i \in I_t^2} u_0(x_i)\right) \rightarrow 1 \text{ as } t \rightarrow \infty.$$

Proof. It is easy to see that $u^2 - u$ is negative in the interval $[0, 1]$, so that

$$\lambda_1(u^2 - u) \leq \lambda_2(u^2 - u) \quad \forall u \in [0, 1].$$

The method of Shiga [38] allows us to construct solutions u_t^1 and u_t^2 as described above such that, with probability 1, $u_t^1(x) \leq u_t^2(x)$ for all $x \in \mathbb{R}$. Examples of this technique can be found in [33] and [26]. As these solutions are bounded in $[0, 1]$ we have

$$\mathbb{E}\left(\prod_{i \in I_0} u_t^1(x_i)\right) \leq \mathbb{E}\left(\prod_{i \in I_0} u_t^2(x_i)\right),$$

so that

$$\mathbb{E}\left(\prod_{i \in I_0} u_t^1(x_i)\right) \rightarrow 1 \text{ as } t \rightarrow \infty \implies \mathbb{E}\left(\prod_{i \in I_0} u_t^2(x_i)\right) \rightarrow 1 \text{ as } t \rightarrow \infty.$$

Applying the duality laid out in proposition 5.4.5 gives the stated result. \square

Corollary 5.5.10. *Let ξ_t^2 and ξ_t^1 be as above and again suppose that $\lambda_1 \geq \lambda_2$. Then if ξ_t^1 becomes locally extinct almost surely, then so does ξ_t^2 .*

Proof. We have seen that a necessary and sufficient condition for the almost sure local extinction of a process ξ_t is that

$$\mathbb{E}\left(\prod_{i \in I_t} u_0(x_i)\right) \rightarrow 1 \quad \text{as } t \rightarrow \infty,$$

for any continuous u_0 which is compactly supported at 1. Now applying lemma 5.5.9 from above, we see that if this holds for the process ξ_t^1 then it also holds for the process ξ_t^2 . \square

Corollary 5.5.11. *In the binary branching and pairwise coalescing model described above, there is some $\lambda_c \in [0, \infty)$ such that local extinction of the process is guaranteed for $\lambda < \lambda_c$, but not for $\lambda > \lambda_c$.*

Proof. This follows from the monotonicity outlined in corollary 5.5.10 above and the comments on the previous page. \square

We conclude this work with a further illustration of this coupling technique by considering a simple generalisation of the model above. The pairwise interaction mechanism is again coalescence and single particle branching occurs at rate λ as usual. Here though the branching distribution depends on a parameter $p \in [0, 1]$, so that $p_0 = 1 - p$ and $p_2 = p$. Thus if a particle branches, it splits into two with probability p or dies with no offspring with probability $1 - p$.

We fix the branching rate λ large enough so that proposition 5.5.8 holds, so now these models depend only on the parameter p . Setting $p = 1$ we recover the model discussed above and hence know that in this case there is some positive probability that local extinction does not occur. Conversely, if $p < 1/2$ then

$\beta < 1$ and so local extinction occurs almost surely. Using a similar method to above we show the following:

Lemma 5.5.12. *The property of almost sure local extinction is monotone in p , so there is some $p_c \in [1/2, 1]$ below which local extinction is guaranteed and above which it is not.*

Proof. The associated SPDE for these models has the form

$$\partial_t u = \frac{1}{2} \Delta u + \lambda(pu^2 + (1-p) - u) + \sqrt{u - u^2} \dot{W}.$$

To use the method above consider that

$$p_1 u^2 + (1 - p_1) - u - (p_2 u^2 + (1 - p_2) - u) = (p_2 - p_1)(1 - u^2),$$

so that if $p_1 \geq p_2$, then

$$p_1 u^2 + (1 - p_1) - u \leq p_2 u^2 + (1 - p_2) - u \quad \text{for all } u \in [0, 1].$$

This then allows us to use a coupling of SPDEs to prove a lemma analogous to 5.5.9 above. As in corollary 5.5.10 we then show that if the model with parameter p_1 exhibits almost sure local extinction, then so will any model with parameter $p_2 \leq p_1$. This gives the monotonicity desired and the statement of the lemma follows. \square

Bibliography

- [1] Arratia, R. (1981), *Limiting point processes for Rescalings of Coalescing and Annihilating Random Walks on \mathbb{Z}^d* . Ann. of Prob. 9 No.6 909-936.
- [2] Athreya, K.B. and Ney, P.E. (1972), *Branching Processes*. Springer.
- [3] Athreya, S. and Tribe, R. (2000), *Uniqueness for a Class of One-Dimensional Stochastic PDEs via Moment Duality*. To appear in Ann fo Prob.
- [4] Badalbaev, I.S. and Dryakhlov, A.V. (1995), *The Asymptotic Behaviour of Nonextinction Probability of a Branching Process with Pairwise Interaction of Particles*. Theory Prob. and Appl. 41 (4) 617-631.
- [5] Blount, D. (1996), *Diffusion Limits for a Nonlinear Density Dependent Space-Time Population Model*. Ann. of Prob. 24 No.2 639-659.
- [6] Bramson, M. Cox, J.T. and Greven, A. (1993), *Ergodicity of Critical Spatial Branching Processes in Low Dimensions*. Ann. of Prob. 21 No.4 1946-1957.
- [7] Bramson, M. Cox, J.T. and Greven, A. (1997), *Invariant Measures of Critical Spatial Branching Processes in High Dimensions*. Ann. of Prob. 25 No.1 56-70.
- [8] Bramson, M. and Gray, L. (1985), *The Survival of Branching Annihilating Random Walks*. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 68 447-460.

- [9] Cohn, D.L. (1980), *Measure Theory*. Birkhäuser.
- [10] Dawson, D.A. (1977), *The Critical Measure Diffusion Process*. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 40 125-145.
- [11] Dawson, D.A. (1991), *Measure-Valued Markov Processes*. Lect. Notes in Math. 1541 (Saint-Flour), 1-262 Springer.
- [12] Dawson, D.A. and Ivanoff, B.G. (1978), *Branching Diffusions and Random Measures*. Adv. in Prob. : Branching Processes, 5 61-109, Dekker, New York.
- [13] Dawson, D.A. Iscoe, I. and Perkins E.A. (1989), *Super-Brownian Motion - Path Properties and Hitting Probabilities*. Probab. Theory Rel. Fields, 83 1-2 135-205.
- [14] Dawson, D.A. and Perkins, E.A. (1998), *Long-time behaviour and Coexistence in a Mutually-Catalytic Branching Model*. Ann. of Prob. 23 3 1088-1138.
- [15] Dawson, D.A. and Perkins, E.A. (1999), *Measure-Valued Processes and Renormalisation of Branching Particle Systems*. Stochastic Partial Differential Equations: Six Perspectives, edited by Carmona, Rozovskii, Amer. Math. Soc.
- [16] Durrett, R. (1979), *An Infinite Particle System with Additive Interactions*. Adv. in Appl. Prob., 11 355-383.
- [17] Durrett, R. (1984), *Oriented Percolation in Two Dimensions*. Ann. of Prob. 12 4 999-1040.
- [18] Durrett, R. (1995), *Ten Lectures on Particle Systems*. Lect. Notes in Math. 1608 (Saint-Flour), 97-201 Springer.

- [19] Durrett, R. (1996), *Stochastic Calculus: A Practical Introduction*. CRC Press.
- [20] Ethier, S. and Kurtz, T. (1986), *Markov Processes: Characterization and Convergence*. Wiley, New York.
- [21] Evans, S.N. and Perkins, E.A. (1994), *Measure-Valued Branching Diffusions with Singular Interactions*. Can. J. Math. 46 1 120-168.
- [22] Feller, W. (1970), *An Introduction to Probability Theory and its Applications*. Wiley, 3rd Ed.
- [23] Fleischman, J. (1978), *Limiting Distributions for Branching Random Fields*. Trans. Amer. Math. Soc. 239 353-389.
- [24] Griffeath, D. (1979), *Additive and Cancellative Interacting Particle Systems*. Lect. Notes in Math. 724 Springer.
- [25] Harris, T.E. (1963), *The Theory of Branching Processes*. Springer.
- [26] Horridge, P. (2001), *PhD Thesis*. University of Warwick.
- [27] Kalinkin, A.V. (1983), *Extinction Probability of a Branching Process with Interaction of Particles*. Theory Prob. and Appl. 27 (1) 201-205.
- [28] Kalinkin, A.V. (1999), *Final Probabilities for a Branching Process with Interaction of Particles and an Epidemic Process*. Theory Prob. and Appl. 43 (4) 633-639.
- [29] Kallenberg, O. (1983), *Random Measures*. Akademie-Verlag, Berlin.
- [30] Karatzas, I. and Shreve, S.E. (1991), *Brownian Motion and Stochastic Calculus*. 2nd Ed., Springer.
- [31] Liggett, T.M. (1985), *Interacting Particle Systems*. Springer.

- [32] Liggett, T.M. (1999), *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Springer.
- [33] Mueller, C. and Sowers, R.B. (1995), *Random Travelling Waves for the KPP Equation with Noise*. Jour. Funct. Analysis 128 439-498.
- [34] Mueller, C. and Tribe, R. (1995), *Stochastic PDEs Arising from the Long Range Contact and Long Range Voter Processes*. Probab. Theory Rel. Fields, 102 519-545.
- [35] Perkins, E.A. (1992), *Measure-Valued Branching Diffusions with Spatial Interactions*. Probab. Theory Rel. Fields, 94 189-245.
- [36] Rogers, L.C.G. and Williams, D. (1994), *Diffusions, Markov Processes, and Martingales*. 2nd Ed., Wiley.
- [37] Shiga, T. (1988), *Stepping Stone Models in Population Genetics and Population Dynamics*. Stochastic Processes in Physics and Engineering, Albevario, S. et al (editors).
- [38] Shiga, T. (1993), *Two Contrasting Properties of Solutions for One-Dimensional Stochastic Partial Differential Equations*. Can. J. Math. 46 No.2 415-437.
- [39] Sowers, R. (1992), *Large Deviations for a Reaction-Diffusion Equation with Non-Gaussian Perturbations*. Ann. of Prob. 20 504-537.
- [40] Tribe, R. (1995), *Large Time Behaviour of Interface Solutions to the Heat Equation with Fisher-Wright White Noise*. Probab. Theory Rel. Fields, 102 289-311.
- [41] Watanabe, S. (1968), *A Limit Theorem of Branching Processes and Continuous State Branching*. J. Math. Kyoto Univ. 8 141-167.