Three Problems on Multiplicative and Probabilistic Number Theory

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

I declare that the work in this thesis was carried out in accordance with the regulations of the University of Warwick.

Parts of this thesis have been published or submitted for publication by the author:

- "A lower bound for the variance of generalized divisor functions in arithmetic progressions", published in *The Ramanujan Journal* [53];
- "A lower bound for the variance in arithmetic progressions of some multiplicative functions close to 1", *Submitted* [52];
- "On maximal product sets of random sets", published in *The Journal of Number Theory* [54];
- "An almost sure upper bound for random multiplicative functions on integers with a large prime factor", *Submitted* [51].

Abstract

We will present three different problems of a multiplicative nature.

The first one concerns proving a lower bound for the variance of generalized divisor functions in arithmetic progressions. Amongst them we recognize the α-fold divisor functions $d_\alpha(n)$, for complex parameters $\alpha \in \mathbb{C} \setminus -\mathbb{N}$, for which our result will extend that of Harper and Soundararajan which only dealt with integers $\alpha \geq 2$, and for complex sequences α_N approaching 1, which is a limit case for the variance behaviour. In the first case, our approach consists in replacing the use of Perron's formula, to produce asymptotics for partial sums of a specific class of multiplicative functions as in the work of Harper and Soundararajan, with the recent extension of the Landau–Selberg–Delange's method, given by Granville and Koukoulopoulos. In the second case, through Taylor expanding $d_{\alpha_N}(n)$ roughly as $1+\omega(n)|\alpha_N - 1|$ we will reduce the problem to lower bounding the variance in arithmetic progressions of the simpler additive function $\omega(n)$.

The second problem treated examines product sets of finite random integer sets. Improving on work of Cilleruelo, Ramana and Ramaré, we will completely characterize those sets whose selfproduct is almost surely maximal. Achieving this requires a technical study of a function that looks like the 2-fold divisor function.

In the third problem we will obtain close to optimal almost sure upper bounds for the sums of a Rademacher or a Steinhaus random multiplicative function over positive integers up to x with the largest prime factor $> \sqrt{x}$. A corresponding Omega result has been very recently discovered by Harper. A main ingredient in our work is the implementation of Harper's recent low moments estimates for the full partial sums of the aforementioned random multiplicative functions.

Chapter 1

Notations

In this chapter I will insert all the basic notations I will need for this thesis. All the other ones will be introduced where necessary.

1.1 Landau notations

For real valued functions $f(x)$ and $g(x)$, with $g(x) > 0$, we write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ when there exists an absolute constant $C > 0$ such that $|f(x)| \leq Cg(x)$, for x sufficiently large. When the implicit constant C depends on a parameter α we instead write $f(x) \ll_\alpha g(x)$ or equivalently $f(x) = O_\alpha(g(x))$. Similarly, for a positive function $f(x)$ we say $f(x) \gg g(x)$ when instead there exists an absolute constant $c > 0$ such that $|g(x)| \leq cf(x)$, for x sufficiently large. When these conditions both simultaneously hold we write $f(x) \asymp g(x)$. We instead write $f(x) \sim g(x)$ to indicate asymptotic equality, which is equivalent to ask that $\lim_{x\longrightarrow\infty} f(x)/g(x) = 1$.

1.2 Some well-known arithmetic functions

A multiplicative function f is an arithmetic function satisfying $f(mn) = f(m)f(n)$, for any $m, n \in \mathbb{N}$ with $(m, n) = 1$. For two multiplicative arithmetic functions f and g, we indicate with $f * g$ the Dirichlet convolution between f and g defined by $f * g(n) := \sum_{d|n} f(d)g(n/d)$.

As usual, φ denotes the Euler totient function, which is the multiplicative function defined on prime powers by $\varphi(p^k) = p^{k-1}(p-1)$, for any $k \in \mathbb{N}$, and μ the Möbius function, which is the multiplicative function supported on squarefree numbers and given on the prime numbers by $\mu(p) = -1$. As common, we indicate with $\Omega(n)$ and $\omega(n)$ the counting functions of the number of prime factors of a positive integer n, with or without multiplicity; we will indicate with $\varpi(n)$ instead the function $\omega(n)$ or $\Omega(n)$, when a statement holds for both.

For any $\alpha \in \mathbb{C}$ we let $d_\alpha(n)$, the *α–fold divisor function*, be defined as the n-th coefficient in the Dirichlet series of $\zeta(s)^\alpha$ on the half plane $\Re(s) > 1$, so that

$$\sum_{n\geq 1} \frac{d_\alpha(n)}{n^s} = \Big(\sum_{n\geq 1} \frac{1}{n^s}\Big)^\alpha.$$

In particular, when $\alpha = k \in \mathbb{N}$, $d_\alpha(n) = \sum_{d_1\cdots d_k = n} 1$. Otherwise, we may define $d_\alpha(n)$ as the multiplicative function defined on the prime powers p^ν, with $\nu \geq 0$, by:

$$d_\alpha(p^\nu) := \binom{\alpha+\nu-1}{\nu} := \frac{1}{\nu!}\prod_{0\leq j<\nu}(\alpha+\nu-1-j),$$

where throughout in this thesis the letter p is reserved for a prime number, unless differently specified.

We write (a, b) and $[a, b]$ to denote the greatest common divisor and the least common multiple of integers a, b.

As usual, we denote with $\lfloor w \rfloor$ the integer part of a real number w.

We indicate with $P(n)$ the greatest prime factor of a positive integer n. We say that n is a y-smooth number if $P(n) \leq y$.

We let $\Gamma(\alpha)$ be the Gamma function, defined for any complex α as the meromorphic extension to the complex plane, with simple poles at $-\mathbb{N} \cup \{0\}$, of the function given on the half-plane $\Re(\alpha) > 0$ by:

$$\Gamma(\alpha) := \int_0^{+\infty} x^{\alpha-1}e^{-x}dx.$$

Chapter 2

Preliminaries

In this chapter I will collect some background results on the partial sums of non-negative multiplicative functions and of the ϖ-function.

2.1 Some results on the average of multiplicative functions

For any non-negative multiplicative function, with a nice behaviour on average over prime powers, we can estimate its partial sums in terms of truncated Euler products.

Lemma 2.1.1. *Let f be a non-negative multiplicative function. Suppose that C is a constant such that*

$$\sum_{p \le x} f(p) \log p \le Cx \tag{2.1}$$

for all $x \ge 1$ and that

$$\sum_{\substack{p^k, \\ k \ge 2.}} \frac{f(p^k) k \log p}{p^k} \le C. \tag{2.2}$$

Then for $x \geq 2$ we have

$$\sum_{n\leq x} f(n) \ll (C+1)\frac{x}{\log x}\sum_{n\leq x}\frac{f(n)}{n}.$$

Moreover, for any positive multiplicative function $f(n)$ we have

$$\sum_{n\leq x}\frac{f(n)}{n} \leq \prod_{p\leq x}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right)$$

and if f is uniformly bounded on the prime numbers by C, then we also have

$$\sum_{n\leq x}\frac{f(n)}{n} \gg_C \prod_{p\leq x}\left(1+\frac{f(p)}{p}\right).$$

The last estimate in Lemma 2.1.1 will be vital in Chapter "The variance of multiplicative functions in arithmetic progressions" to lower bound certain logarithmic averages of some non-negative multiplicative functions, so doing producing a lower bound for their corresponding variance.

Proof. The first conclusion is [69, Ch. III, Theorem 3.5] and the second one is a special case of Tenenbaum's result [70, Théorème 1.1] (see also Tenenbaum [71, Théorème 1.1] and corrig. [72] or Elliott and Kish [7, Lemma 20]). □

In particular, the following corollary will be useful for instance in Chapter "Random product sets", to estimate certain divisor sums resulting after an application of Rankin's trick.

Corollary 2.1.2. *Let $\Omega(n)$ be the number of prime factors of n counted with multiplicity. For any fixed $0.1 < y < 1.9$ we have the uniform bound*

$$\sum_{n\leq x} y^{\Omega(n)} \ll x(\log x)^{y-1} \quad (x\geq 2),$$

with a uniformly bounded implied constant.

Furthermore, if $\Omega_2(n)$ is the function which counts the number of prime factors of n different from 2 and counted with multiplicity, we have

$$\sum_{n\leq x} 2^{\Omega_2(n)} \ll x\log x \quad (x\geq 2).$$

Proof. Define $f(n) := y^{\Omega(n)}$. Clearly, f is a non-negative multiplicative function, satisfying (2.1) and (2.2). Then by Lemma 2.1.1, we have

$$\begin{aligned}\sum_{n\le x} y^{\Omega(n)} &\ll \frac{x}{\log x}\prod_{2\le p\le x}\left(1+\frac{y}{p}+\frac{y^2}{p^2}+\cdots\right)\\ &= \frac{x}{\log x}\prod_{2\le p\le x}\left(1-\frac{y}{p}\right)^{-1}\\ &\ll x(\log x)^{y-1},\end{aligned}$$

by Mertens' theorem. This shows the first part of the Corollary. For the second part, let $f(n) := 2^{\Omega_2(n)}$. Since $f(2^k) = 1$, for any $k \ge 0$, again (2.1) and (2.2) are satisfied. Then by Lemma 2.1.1, we have

$$\begin{aligned}\sum_{n\le x} 2^{\Omega_2(n)} &\ll \frac{x}{\log x}\left(1+\frac{1}{2}+\frac{1}{4^2}+\cdots\right)\prod_{3\le p\le x}\left(1+\frac{2}{p}+\frac{4}{p^2}+\cdots\right)\\ &= \frac{x}{\log x}\left(1-\frac{1}{2}\right)^{-1}\prod_{3\le p\le x}\left(1-\frac{2}{p}\right)^{-1}\\ &\ll x\log x,\end{aligned}$$

by Mertens' theorem, which concludes the second part of the Corollary. □

2.2 The distribution of the $\varpi(n)$ function

It is a classical result going back to Hardy and Ramanujan (see also Diaconis' paper [5]) that the partial sum of the ϖ–function satisfies the following asymptotic expansion:

$$\sum_{n\le x}\varpi(n) = x\log\log x + B_\varpi x + O\left(\frac{x}{\log x}\right) \quad (x\ge 2), \tag{2.3}$$

where B_ϖ is a constant depending on the function ϖ. In particular, we deduce that the mean value of $\varpi(n)$, over the integers $n \le x$, is roughly $\log\log x$. Regarding its variance, we can appeal to the Turán-Kubilius' inequality (see e.g. [69, Ch. III, Theorem 3.1]), which states that

$$\sum_{n\le x}(\varpi(n)-\log\log n)^2 \ll x\log\log x \quad (x\ge 2). \tag{2.4}$$

In particular, (2.3) and (2.4) together give

$$\sum_{n\le x}\varpi(n)^2 \ll x(\log\log x)^2 \quad (x\ge 2). \tag{2.5}$$

The above estimates will be employed in Chapter "The variance in arithmetic progressions of divisor functions and other sequences close to 1" to study the variance of the ϖ-function, which constitutes an intermediate step towards understanding that of divisor functions close to 1.

We define the probability space $([N], \mathcal{P}([N]), \mathbb{P}_N)$, where $\mathcal{P}([N])$ is the power set of $[N]$ and $\mathbb{P}_N$ denotes the discrete uniform measure on $[N]$. A classical consequence of the Turán–Kubilius inequality (see e.g. [69, Ch. III, Theorem 3.4]) is the following result.

Proposition 2.2.1. *Given any function $t(N)\ge 1$, we have*

$$\mathbb{P}_N(|\varpi(n)-\log\log N|> t(N)\sqrt{\log\log N}) \ll \frac{1}{t(N)^2}.$$

In particular, if $t(N)\longrightarrow +\infty$, as $N\longrightarrow +\infty$, then "almost all" numbers $n\le N^2$ (in the sense of asymptotic density) satisfy:

$$|\varpi(n)-\log\log N|\le t(N)\sqrt{\log\log N}.$$

An application of the method of moments leads to the following celebrated special case of the Erdős–Kac's theorem (see e.g. [69, Ch. III, Theorem 4.15]), which will be essential in Chapter "Random product sets" to estimate certain sums where the constraint is on the size of $\Omega(n)$ in intervals.

Proposition 2.2.2. *Under the probability measure $\mathbb{P}_N$, we have*

$$\frac{\varpi(n)-\log\log N}{\sqrt{\log\log N}} \xrightarrow{d} N(0,1) \text{ as } N\longrightarrow +\infty,$$

where $N(0,1)$ indicates a random variable of standard normal distribution.

Therefore, for any fixed $t \geq 1$ we have

$$\mathbb{P}\left(\frac{|\varpi(n) - \log\log N|}{\sqrt{\log\log N}} > t\right) \longrightarrow \frac{2}{\sqrt{2\pi}} \int_t^{+\infty} e^{-s^2/2} ds \leq \frac{2e^{-t^2/2}}{t\sqrt{2\pi}}, \tag{2.6}$$

where the last inequality follows from the fact that for any $s \geq t > 0$ we have

$$\int_t^{+\infty} 1 \cdot e^{-s^2/2} ds \leq \int_t^{+\infty} \frac{s}{t} e^{-s^2/2} ds = \frac{e^{-t^2/2}}{t}.$$

Finally, we remind of the following bound on the maximal size of $\varpi(n)$ (see e.g. [69, Ch. I, Eq. 5.9]):

$$\varpi(n) \leq (\log x)/(\log 2) \quad (1 \leq n \leq x). \tag{2.7}$$

Chapter 3

Introduction

3.1 The variance of multiplicative functions in arithmetic progressions

For many complex multiplicative functions $f(n)$ we expect uniform distribution in arithmetic progressions, namely that

$$\frac{1}{N}\sum_{\substack{n\le N\\ n\equiv a \pmod q}} f(n) \approx \frac{1}{\varphi(q)N}\sum_{\substack{n\le N\\ (n,q)=1}} f(n),$$

for $(a,q)=1$ and N a large positive integer.

It is often difficult to turn this prediction into a quantitative statement for individual a and q. To study the extent to which the above approximation holds, we introduce a second centred moment, averaged over all the arithmetic progressions for moduli $q\le Q\le N$.

Definition 3.1.1. We define the variance of f in arithmetic progressions by

$$V(N,Q;f)=\sum_{q\le Q}\sum_{h|q}\sum_{\substack{a \mod q\\ (a,q)=h}}\Bigg|\sum_{\substack{n\le N\\ n\equiv a \mod q}} f(n)-\frac{1}{\varphi(q/h)}\sum_{\substack{n\le N\\ (n,q)=h}} f(n)\Bigg|^2. \tag{3.1}$$

An asymptotic equality for $V(N,Q;d_2)$ has been given by Motohashi [56] for any $Q\le N$, whereas for $V(N,Q;d_k)$, for any $k\ge 2$, by Rodgers and Soundararajan [63], but only in the range $N^{1/(1+2/k-\delta)}\le Q\le N^{1/\delta}$, for any

$\delta > 0$ (see Sect. Notations for the definition of a divisor function d_k). On the other hand, Harper and Soundararajan [28, Theorem 2] found a lower bound for the variance of d_k,

$$V(N, Q; d_k) \gg_{k,\delta} QN(\log N)^{k^2-1},$$

for any integer $k \geq 2$, holding in the range $N^{1/2+\delta} \leq Q \leq N$, for any small $\delta > 0$ and N large enough with respect to δ.

Our aim is to generalize their result to a wider class of multiplicative functions.

Definition 3.1.2. A *generalized divisor function* is a multiplicative function for which there exist a complex number α and positive real numbers β, A_1, A_2, with α and β possibly depending on N, such that the following statistics hold

$$\sum_{p \leq x} f(p) \log p = \alpha x + O\left(\frac{x}{(\log x)^{A_1}}\right) \quad (2 \leq x \leq N), \tag{3.2}$$

$$\sum_{p \leq x} |f(p) - 1|^2 \log p = \beta x + O\left(\frac{x}{(\log x)^{A_2}}\right) \quad (2 \leq x \leq N) \tag{3.3}$$

and such that $|f(n)| \leq d_\kappa(n)$, for a constant $\kappa > 0$ and every N-smooth positive integer n.

By the Prime Number Theorem, the equations (3.2) and (3.3) are trivially satisfied with $\beta = |\alpha - 1|^2$, for $\alpha \neq 1$, $\kappa = |\alpha| + 2$ and any $A_1, A_2 > 0$, when $f(n) = d_\alpha(n)$, meaning that each α-fold divisor function, with $\alpha \neq 1$, is, in particular, a generalized divisor function. For them we are able to prove the following result.

Proposition 3.1.1. *Let $\delta > 0$ sufficiently small and consider $N^{1/2+\delta} \leq Q \leq N$. For any complex number $\alpha \notin -\mathbb{N} \cup \{1\}$, we have*

$$V(N, Q; d_\alpha) \gg_{\alpha,\delta} Q \sum_{n \leq N} |d_\alpha(n)|^2, \tag{3.4}$$

if N large enough with respect to α and δ.

The values of α excluded in Proposition 3.1.1 reflect the different size of the variance of d_1, which we will prove to be Q^2, and the behaviour of the asymptotic expansion of the partial sum of $d_\alpha(n)$. Indeed, when α is a pole of the Gamma function, we lose control on it and we are not able to deduce a lower bound for the corresponding variance, even though (3.4) might still be true. An exception is given by $\alpha = 0$, in which case the variance can be easily computed and roughly equals Q.

Another example of generalized divisor function is the indicator of the the set S of all integer sums of two squares, for which we will add to the existing literature (see for instance Fiorilli [11], Iwaniec [38], Lin and Zhan [44] and Rieger [61, 62], just to name a few) by proving the following result.

Proposition 3.1.2. *Let $\delta > 0$ sufficiently small and consider $N^{1/2+\delta} \leq Q \leq N$. Then we have*

$$V(N, Q; \boldsymbol{1}_S) \gg_\delta \frac{QN}{\sqrt{\log N}}, \tag{3.5}$$

if N is large enough with respect to δ.

Propositions 3.1.1 and 3.1.2 are a corollary of our new main contribution.

Theorem 3.1.3. *Let δ be a sufficiently small positive real number and N be a large positive integer. Suppose that $N^{1/2+\delta} \leq Q \leq N$. Let $f(n)$ be a generalized divisor function as in Definition 3.1.2 with $\alpha \notin -\mathbb{N} \cup \{0\}$ and let $\kappa(\alpha, \beta) := (\kappa + 1)^2 + \kappa - \Re(\alpha) - \beta + 4$. Furthermore, assume that*

$$\begin{aligned} &A_1 > \max\{\kappa(\alpha, \beta), \kappa + 2\}; \\ &A_2 > A_1 - \kappa(\alpha, \beta) + 1; \\ &\beta \geq (\log N)^{\kappa(\alpha,\beta)-A_1}; \\ &|\Gamma(\alpha)| \leq \log N, \end{aligned} \tag{3.6}$$

where the Gamma function has been defined in Ch. Notations. Finally, let

$$c_0 = \prod_{p \leq N} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right)\left(1 - \frac{1}{p}\right)^\alpha$$

and suppose that

$$(\log N)^{1-\delta}|c_0|\geq 1. \tag{3.7}$$

Then we have

$$V(N,Q;f)\gg\left|\frac{c_0\beta}{\Gamma(\alpha)}\right|^2 Q\sum_{n\leq N}|f(n)|^2. \tag{3.8}$$

The implicit constant above may depend on δ,κ,A_1,A_2 and the implicit constants in (3.2)–(3.3)*, but not on α, and we take N large enough depending on all of these parameters.*

To show Theorem 3.1.3 we employ Harper and Soundararajan [28, Proposition 1], which roughly gives:

$$V(N,Q;f)\gg Q\int_{\mathfrak{m}}\left|\sum_{n\leq N}f(n)e(n\theta)\right|^2 d\theta, \tag{3.9}$$

with $e(t)=e^{2\pi it}$, for any $t\in\mathbb{R}$, and $\mathfrak{m}$ the set of minor arcs, which is made of small arcs around rational fractions with large denominator. For functions that fluctuate like random, we usually expect the above integral to be well approximated by that over the whole circle. This leads, by Parseval's identity, to the following guess:

$$V(N,Q;f)\gg Q\sum_{n\leq N}|f(n)|^2, \tag{3.10}$$

in the full range $N^{1/2+\delta}\leq Q\leq N$, with $\delta>0$ small.

Proposition 3.1.1 shows that (3.10) is verified for all divisor functions d_α, with $\alpha\notin-\mathbb{N}\cup\{1\}$, and Theorem 3.1.3 extends this to all generalized divisor functions, except for the possible loss coming from the factor $|c_0\beta/\Gamma(\alpha)|^2$. From this point of view, we believe that (3.4) and (3.5) are presumably sharp.

We lower bound the L^2-integral in (3.9) in terms of an L^1-integral by means of Cauchy–Schwarz's inequality:

$$\geq \frac{(\int_{\mathfrak{m}} |\sum_{n\leq N} f(n)e(n\theta)\sum_{n\leq N} h(n)e(n\theta)|d\theta)^2}{\int_{\mathfrak{m}} |\sum_{n\leq N} h(n)e(n\theta)|^2 d\theta}, \tag{3.11}$$

where $h(n)$ represents a truncation of the Dirichlet convolution $(f * \mu) * \mathbf{1}(n)$.

We extend the integral in the denominator of (3.11) to the whole circle to then, by Parseval's identity, bound it with the partial sum of $|h(n)|^2$, which can be fairly easy estimated with $\ll N(\log N)^{\beta+2|\Re(\alpha)-1|}$.

To produce a lower bound for the L^1-integral in the numerator of (3.11) instead, we employ Harper and Soundararajan [28, Proposition 3], which roughly gives

$$\begin{aligned} \int_{\mathfrak{m}} \Big|\sum_{n\leq N} f(n)e(n\theta)\sum_{n\leq N} h(n)e(n\theta)\Big|d\theta \\ \gg \sum_{q\leq R}\Big|\sum_{\substack{r\leq R\\ q|r}} \frac{f*\mu(r)}{r}\Big|\Big|\sum_{n\leq N} f(n)c_q(n)\Big|, \end{aligned} \tag{3.12}$$

where $c_q(n)$ stands for a Ramanujan sum, defined as

$$c_q(n) = \sum_{\substack{a=1,\dots,q\\ (a,q)=1}} e(an/q).$$

Our strategy is to exploit in depth the asymptotic expansion of the partial sums of $f(n)$ to analyze that of $f(n)c_q(n)$. To this aim, we note that we roughly have

$$\sum_{n\leq N} f(n) \approx \frac{c_0 N(\log N)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.13}$$

by the Selberg–Delange's theorem, explaining the presence of the factors c_0 and $\Gamma(\alpha)$ in (3.8). In particular, (3.7) and the last condition in (3.6) have been inserted to maintain control on the average of $f(n)$ over integers $n \leq N$, which otherwise would preclude us from producing a lower bound for $V(N,Q;f)$.

We replace Harper and Soundararajan's strategy to produce the lower bound [28, Theorem 2] for $V(N,Q;d_k)$, which consisted in rephrasing the partial sums of $d_k(n)c_q(n)$, by means of Perron's formula, in terms of an integral of the associated Dirichlet series, with the use of Selberg–Delange's method. Indeed, in the first case the corresponding Dirichlet series could be meromorphically extended to the complex plane with just one pole at 1, whereas for generalized divisor functions a similar extension does not subsist. By properties of the Ramanujan sums, we can rewrite

$$\sum_{n\leq N} f(n)c_q(n) = \sum_{\substack{b\leq N\\ p|b\Rightarrow p|q}} f(b)c_q(b) \sum_{\substack{a\leq N/b\\ (a,q)=1}} f(a), \tag{3.14}$$

using the unique substitution $n = ab$, with $(a,q) = 1$ and $b = n/a$. Next, we develop the innermost sum as roughly (3.13). The application of the Selberg–Delange's method here requires a specific control on the size of the derivatives of the Dirichlet series of $f(n)\mathbf{1}_{(n,q)=1}$; to achieve that, we ask that q avoids having lots of small prime factors. Whence, we are left with analyzing the truncated Dirichlet series

$$\sum_{\substack{b\leq N:\\ p|b\Rightarrow p|q}} \frac{f(b)c_q(b)}{b}(\log(N/b))^{\alpha-1}. \tag{3.15}$$

By splitting the integers $q = rs \leq N^{1/2}$, with s the part of q supported only on prime numbers smaller than a logarithmic power, we may rewrite the sum in (3.15) as roughly

$$\sum_{\substack{b_1\leq\sqrt{N}\\ p|b_1\Rightarrow p|r}} \frac{f(b_1)c_r(b_1)}{b_1} \sum_{\substack{b_2\leq N/b_1\\ p|b_2\Rightarrow p|s}} \frac{f(b_2)c_s(b_2)}{b_2}(\log(N/b_1b_2))^{\alpha-1}, \tag{3.16}$$

since by multiplicativity of $c_q(n)$ as function of q we have

$$c_q(b) = c_r(b)c_s(b) = c_r(b_1)c_s(b_2).$$

Next, we expand the $(\alpha-1)$-power of the logarithm $\log(N/b_1b_2)$, using the generalized binomial theorem. In this way, we obtain a sum of successive derivatives of the Dirichlet series of $f(b)c_s(b)\mathbf{1}_{p|b\Rightarrow p|s}$, which can be handled by means of several applications of the Faà di Bruno's formula. To avoid such derivatives blowing up, we impose that the value of f at prime numbers is suitably away from 1.

To manage the contribution of such derivatives into (3.16), we insert a key hypothesis on the structure of q, i.e. to be divisible by a prime number $t \approx \sqrt{N}$. Indeed, it seems crucial to avoid the situation in which $r = ts'$ possesses several large divisors, thus gaining more control on the factor $\log(N/b_1 b_2)$. This is another main difference with the approach taken in [28], where q was restricted to just being an N^{ε}–smooth number, for a carefully chosen small $\varepsilon > 0$.

Under our assumption, we can conclude the computation of the sum in (3.16) by using the multinomial coefficient formula: to bound the Euler product derivatives arising from this procedure, we require f to be never too close to 1 on several prime factors of s'.

Overall, we end up showing that

$$\left| \sum_{n \le N} f(n) c_q(n) \right| \gg |c_0| \frac{N(\log N)^{\Re(\alpha)-1}}{|\Gamma(\alpha)|} |(f * \mu)(q)|,$$

which, together with the estimate

$$\left| \sum_{\substack{r \le R \\ q|r}} \frac{f * \mu(r)}{r} \right| \gg \frac{|f * \mu(q)|}{q} (\log N)^{|\Re(\alpha)-1|},$$

gives a lower bound for the numerator in (3.12) of

$$\int_{\mathfrak{m}} \left| \sum_{n \le N} f(n) e(n\theta) \sum_{n \le N} h(n) e(n\theta) \right| d\theta$$
$$\gg |c_0| \frac{N(\log N)^{\Re(\alpha)-1+|\Re(\alpha)-1|}}{|\Gamma(\alpha)|} \sum_{q \le R} \frac{|(f * \mu)(q)|^2}{q},$$

which in turn, by (3.9), leads to

$$V(N, Q; f) \gg |c_0|^2 \frac{QN(\log N)^{-\beta+2(\Re(\alpha)-1)}}{|\Gamma(\alpha)|^2} \left(\sum_{q \le N} \frac{|f * \mu(q)|^2}{q} \right)^2.$$

Here the sum over $q = tss'$ is subject to some restrictions and can be estimated as

$$\sum_{q\le N}\frac{|f*\mu(q)|^2}{q}\gg\beta(\log N)^{\beta},$$

with the help of (3.3). We see that the assumption on β in (3.6) is necessary, because a smaller value of β corresponds to a function f closer to d_1, which variance we will prove in Ch. "The variance in arithmetic progressions of divisor functions and other sequences close to 1" to have size Q^2.

The proof now ends after showing that

$$\sum_{n\le N}|f(n)|^2\ll N(\log N)^{\beta+2(\Re(\alpha)-1)},$$

so as to deduce

$$V(N,Q;f)\gg\left|\frac{c_0\beta}{\Gamma(\alpha)}\right|^2 QN(\log N)^{\beta+2(\Re(\alpha)-1)}\gg\left|\frac{c_0\beta}{\Gamma(\alpha)}\right|^2 Q\sum_{n\le N}|f(n)|^2.$$

3.2 The variance in arithmetic progressions of divisor functions close to 1

In the previous section, we saw that the variance of a divisor function d_α, for a parameter $\alpha>1$, cannot be any smaller than QN, which followed from Proposition 3.1.1. On the other hand, we also pointed out that $V(N,Q;d_1)$ has size Q^2. To better understand this change in size, we study lower bounds for the variance in arithmetic progressions of a sequence of divisor functions $d_{\alpha_N}^{\varpi}(n)=\alpha_N^{\varpi(n)}$ (where ϖ stands both for ω and Ω), with a parameter α_N increasingly approaching 1.

Theorem 3.2.1. *Let $\alpha_N=1+1/R(N)$, where $R(N)$ is a non-zero real function. Assume $N^{1/2+\delta}\le Q\le N$, with $\delta>0$ sufficiently small. Then there exists a constant $B>0$ such that if $B\le|R(N)|\le A\log\log N$, with $A>0$, we have*

$$V(N,Q;d_{\alpha_N}^{\varpi})\gg_{\delta,A}\frac{QN}{R(N)^4}\exp\left(\left(2+\frac{1}{R(N)}\right)\frac{\log\log N}{R(N)}\right),$$

if N is large enough with respect to δ and A. Furthermore, there exists a constant $C = C(\delta) > 0$ such that if $C \log\log N \leq |R(N)| \leq N^{\delta/12}$ and N is large in terms of δ, we have

$$V(N, Q; d^{\varpi}_{\alpha_N}) \gg_\delta \frac{QN}{R(N)^2} \log\left(\frac{\log N}{\log(2N/Q)}\right) + Q^2. \tag{3.17}$$

The first part follows almost immediately from Theorem 3.1.3. The proof of the second part instead has a different flavour: as in the outline of the demonstration of Theorem 3.1.3 given in the previous section, we start again from Harper and Soundararajan [28, Proposition 1] to roughly get

$$V(N, Q; d^{\varpi}_{\alpha_N}) \gg Q \int_{\mathfrak{m}} \Big| \sum_{n \leq N} d^{\varpi}_{\alpha_N}(n) e(n\theta) \Big|^2 d\theta, \tag{3.18}$$

but now we use Cauchy–Schwarz's inequality to express the lower bound for the above L^2-integral in the following form:

$$\geq \frac{|\int_{\mathfrak{m}} \sum_{n \leq N} d^{\varpi}_{\alpha_N}(n) e(n\theta) \sum_{n \leq N} \overline{h(n)} \phi(n/N) e(-n\theta) d\theta|^2}{\int_{\mathfrak{m}} |\sum_{n \leq N} h(n) \phi(n/N) e(n\theta)|^2 d\theta}, \tag{3.19}$$

where we choose $h(n) = \sum_{p \leq R,\ p|n} 1$, with $R \approx \sqrt{N}$, and $\phi(t)$ an opportune smooth function. Here $h(n)$ is a truncated version of $\varpi(n)$ (whereas in Theorem 3.1.3, $h(n)$ was a truncation of a generalized divisor function) since we will simplify the study of (3.19) by replacing $d^{\varpi}_{\alpha_N}(n)$ with $\varpi(n)$.

Previously, we extended the integral in the denominator to the whole circle and used Parseval's identity to compute it. Here, this procedure would be inefficient and we instead need to work out carefully the exponential sum with coefficients $h(n)$ over the minor arcs. In this way, we obtain that

$$\int_{\mathfrak{m}} \Big| \sum_{n \leq N} h(n) \phi(n/N) e(n\theta) \Big|^2 d\theta \ll N \log\left(\frac{\log N}{\log(2N/Q)}\right).$$

In contrast to Theorem 3.1.3, we cannot directly lower bound the integral in the numerator, by means of [28, Proposition 3]. In fact, this is only possible when the main contribution comes from minor arcs centred on fractions with denominator smaller than $\sqrt{N}$, which is not the case for divisor functions approaching 1. Therefore, we rewrite the numerator in (3.19) as $\int_{\mathfrak{m}} = \int_0^1 - \int_{\mathfrak{M}}$ and proceed by asymptotically estimating both integrals: to this aim we rely on Harper and Soundararajan [28, Proposition 2] which restates this difference roughly as

$$\begin{aligned} (3.20)\quad & \sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)\overline{h(n)}\phi\left(\frac{n}{N}\right) \\ & - N\sum_{q\leq Q_0}\int_{-1/qQ}^{1/qQ}\left(\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_q(n)e(n\beta)\right)\left(\sum_{\substack{r\leq R\\ q\mid r}}\frac{\overline{h*\mu}(r)}{r}\right)\hat{\phi}(\beta N)d\beta, \end{aligned}$$

for a suitable choice of $Q_0 < Q$, where $\hat{\phi}$ stands by the Fourier transform of ϕ.

To compute (3.20), we Taylor expand the divisor functions $d_{\alpha_N}^{\varpi}(n) = (1+1/R(N))^{\varpi(n)}$ as $1+\varpi(n)/R(N)+O(\varpi(n)^2/R(N)^2)$ to reduce the problem to analyze similar quantities, but with the additive and simpler function $\varpi(n)$ in place of $d_{\alpha_N}^{\varpi}(n)$. Since such function is, for the majority of positive integers $n \leq N$, of size roughly $\log\log N$ (see e.g. Eq. (2.3)), this justifies the condition $|R(N)| \geq C\log\log N$ in the hypotheses of Theorem 3.2.1.

Since we might think of $\varpi(n)$ as made of a deterministic part $\log\log N$ and a more random one $\varpi(n)-\log\log N$, we might see $d_{\alpha_N}^{\varpi}(n)$ as well roughly as $1+\log\log N/R(N) \approx 1$, when $|R(N)| \geq C\log\log N$ and C is large enough, plus $(\varpi(n)-\log\log N)/R(N)$. Considering their contribution to (3.18) individually, we will get that the former contributes an amount of Q^2, whereas the latter one of $QN\log(\frac{\log N}{\log(2N/Q)})/R(N)^2$. This explains the structure of the lower bound in Theorem 3.2.1.

The sequence of functions $d_{\alpha_N}^{\varpi}(n)$ is only one instance of a wide class of multiplicative functions 'close' to 1. Another interesting representative of such class is the characteristic function of the y–smooth numbers (see Ch. Notations for a definition thereof), for parameters y near to N.

Theorem 3.2.2. *Let $N^{1/2+\delta} \leq Q \leq N$, with $\delta > 0$ sufficiently small. There exists a large constant $C > 0$ such that the following holds. If*

$$\sqrt{N} \leq y \leq N/C$$

and N *is large enough in terms of* δ*, we have*

$$V(N,Q;\boldsymbol{1}_{y-smooth}) \gg_\delta QN\log\left(\frac{\log N}{\log y}\right)+Q^2.$$

We observe that Harper's result [22, Theorem 2] gives a tight corresponding upper bound for the variance above, when $Q = N/(\log N)^A$, with $A > 0$, and $\sqrt{N} \le y \le N^{1-\delta}$, say.

3.3 On maximal random product sets

The origin of the Multiplication Table Problem traces back to 1955, when Erdős [8] asked about the number of *distinct* products in a multiplication table of integers. More precisely, for a set $A \subset [N] := \{1,\dots,N\}$, he pondered about the size of the product set

$$AA := \{ab : a \in A, b \in A\}.$$

Erdős [9], Hall and Tenenbaum [20] and lastly Ford [14], in a series of increasingly preciser results, determined the exact order of magnitude of $|[N][N]|$, showing that

$$|[N][N]| \asymp \frac{N^2}{(\log N)^\delta(\log\log N)^{3/2}}, \tag{3.21}$$

with

$$\delta := 1-\frac{1+\log\log 2}{\log 2} = 0.086071....$$

Therefore, the size of $[N][N]$ is much smaller than $|[N]|^2$. The same happens for instance for the set $\mathcal{Q}_N$ of sums of two squares [50, Theorem 1.4]. These two examples suggest that both the relative density in $[N]$ and the multiplicative structure (if we extend both sets to infinite sets they become closed under multiplication) of a set reduce the size of its selfproduct. In particular, if $|A| = N\alpha$ with $|AA| \sim (|A|^2+|A|)/2$, i.e. of maximal size possible, by seeing

$AA \subset [N][N]$, we deduce from (3.21) that

$$\alpha \ll \frac{1}{(\log N)^{\delta/2}(\log\log N)^{3/4}}.$$

When instead the elements of a relatively small set A look like more randomly distributed (compared to $[N]$), we might expect to have a corresponding product set AA of maximal size possible. In fact, we might believe that the number of pairs of elements in A with identical product should be much less than the total, which is roughly $|A|^2$. This should happen because the likelihood of having equal product is directly proportional to that of sharing same prime factors, which should be small, if they behave quite randomly. This is indeed the situation when A is the set of primes $\mathcal{P}_N$, shifted primes $\mathcal{P}_N - 1$ or shifted sums of two squares $\mathcal{Q}_N - 1$, as shown by Cilleruelo, Ramana and Ramaré:

$$\begin{aligned}
|\mathcal{P}_N\mathcal{P}_N| &\sim \frac{|\mathcal{P}_N|^2}{2}, \\
|(\mathcal{P}_N - 1)(\mathcal{P}_N - 1)| &\sim \frac{|\mathcal{P}_N - 1|^2}{2}, \\
|(\mathcal{Q}_N - 1)(\mathcal{Q}_N - 1)| &\asymp |\mathcal{Q}_N - 1|^2,
\end{aligned}$$

In order to better investigate this situation, the following random model has been introduced:

For every $\alpha \in [0, 1)$, let $B(N, \alpha)$ denote the probabilistic set up in which a random set $A \subset [N]$ is constructed by choosing independently every element of $[N]$ with probability α.

We can then interpret the random variable $|A| = \sum_{1\le i\le N} \mathbf{1}_{i\in A}$ as a random variable with binomial distribution $\mathrm{Bi}(N, \alpha)$. In particular, A has expected size $N\alpha$; thus, it can be viewed as a random model for a subset of the positive integers smaller than N with natural relative density approximately α and whose elements look like independently randomly distributed.

A sufficient condition to guarantee *asymptotic* maximality for the size of a product set has been given in the following result (see [3, Theorem 1.2]).

Theorem 3.3.1 (Cilleruelo, Ramana and Ramaré)**.** *Let A be a random set in $B(N, \alpha)$. If $\alpha = o((\log N)^{-1/2})$, then $|AA| \sim |A|^2/2$ with probability $1 - o(1)$ (or, in other words, the quotient $2|AA|/|A|^2$ converges in probability to 1).*

From the proof of Theorem 3.3.1 it is not clear whether the value $\alpha = o((\log N)^{-1/2})$ is sharp, and, if not, there would not be any obvious way to extend the proof to improve it. Our new next result instead completely determines all the values of α corresponding to maximal random product sets.

Theorem 3.3.2. *Let A be a random set in $B(N, \alpha)$, with $\alpha \in [0, 1)$. Then, we have $|AA| \sim |A|^2/2$ with probability $1 - o(1)$, as $N \longrightarrow +\infty$, if and only if*

$$\frac{\log(\alpha^2 (\log N)^{\log 4 - 1})}{\sqrt{\log\log N}} \longrightarrow -\infty.$$

In other words, all the random sets under the model $B(N, \alpha)$, with α approximately $o((\log N)^{-\log 2 + 1/2})$, with $-\log 2 + 1/2 = -0.19314718... > -0.5$, have a product set of asymptotically maximal size, whereas larger ones do not, thus substantially improving on Theorem 3.3.1.

We remark that both Theorems 3.3.1 and 3.3.2 are not concerned with random sets A for which $|AA| \sim |A|^2/2$ holds with an intermediate probability, and the maximality comes from both the size of the product set and the likelihood of the relation. It would be nice though to figure out what are the choices of α for which such relation happens with half probability, for example.

We now sketch the proof of the Theorem 3.3.2. For the sufficient part, one realizes that it is enough to show a version of the result in expectation, i.e. that $\mathbb{E}[|AA|] \sim \mathbb{E}[|A|^2]/2$, as $N \longrightarrow +\infty$. Then, by using an explicit expression for the mean value of $|AA|$, as given in [3], and properties of the binomial distribution, the problem is reduced to checking that

$$\sum_{1 \le n \le N^2} \left(\frac{\alpha^2 \tau_N(n)}{2} - 1 + (1 - \alpha^2)^{\tau_N(n)/2} \right) = o(\alpha^2 N^2), \tag{3.22}$$

where

$$\tau_N(n) := \#\{(j, k) \in [N] \times [N] : n = jk\}$$

is the number of restricted representations of a positive integer n as product $n = jk$, with $1 \le j, k \le N$.

Cilleruelo, Ramana and Ramaré, too, started from (3.22) and Taylor expanded the binomial up to the second order, to then compute the average of τ_N^2 and produce a saving in the range of $\alpha = o((\log N)^{-1/2})$. Unfortunately, this approach cannot be extended any further, because the higher moments of τ_N grow too fast to be controlled in this way; to make progress we need to better study the summand in (3.22). In order to do that, we need a deep understanding of the distribution of the function τ_N. Some heuristic considerations that we introduce reveal that τ_N is mostly concentrated, at least on an average sense, on integers where the additive function $\omega(n)$, which follows a normal distribution (see Proposition 2.2.2), is close to double its mean value $\log\log N$ for at most a factor of its standard deviation $\sqrt{\log\log N}$. Equivalently, they are positive integers n such that:

$$|\omega(n) - 2\log\log N| \leq M\sqrt{\log\log N}, \tag{3.23}$$

with $M > 0$. Let us indicate with $\mathcal{S}_1$ the set of those $n \leq N^2$. In other words, if we let $\mathcal{S}_2$ be the complementary of $\mathcal{S}_1$ in $[N^2]$, then

$$\sum_{n\in\mathcal{S}_2} \tau_N(n) = o\bigg(\sum_{n\leq N^2} \tau_N(n)\bigg) \text{ if } M \longrightarrow +\infty,$$

and therefore restricting the summation in (3.22) on numbers $n \in \mathcal{S}_2$ already produces an acceptable contribution, since

$$\sum_{1\leq n\leq N^2} \tau_N(n) = N^2, \tag{3.24}$$

as it can be easily deduced from the definition of τ_N.

On the other hand, crucially, the numbers in (3.23) are outside the set of integers where, on average, τ_N^2 is mostly concentrated on, meaning that

$$\sum_{n\in\mathcal{S}_1} \tau_N^2(n) = o\bigg(\sum_{n\leq N^2} \tau_N^2(n)\bigg) \text{ if } M \longrightarrow +\infty.$$

Indeed, the main contribution to the average of τ_N^2 comes from those numbers n such that

$$|\omega(n) - 4\log\log N| \leq M\sqrt{\log\log N}.$$

Consequently, by Taylor expanding the binomial $(1-\alpha^2)^{\tau_N(n)/2}$ to the second order, we obtain a certain amount of saving in (3.22) from averaging τ_N^2 over the numbers $n \in \mathcal{S}_1$. Quantifying this amount requires the use of the Erdős–Kac's theorem (Proposition 2.2.2) and information on the distribution of the divisor function (like Corollary 2.1.2). We are then just left with checking that all the values of α satisfying the condition stated in Theorem 3.3.2 make such contribution acceptable.

Regarding the necessary part, we suppose that the limit in Theorem 3.3.2 either does not exist or it gives a value different from $-\infty$. We also assume that even in these cases the associated choices of α lead to random product sets of maximal size, seeking for a contradiction.

We then realize that when a random set A in $B(N, \alpha)$ is such that $|AA| \sim (|A|^2+|A|)/2$ with probability $1 - o(1)$, we necessarily have $\mathbb{E}[|AA|] \sim \mathbb{E}[(|A|^2+|A|)/2]$, as $N \longrightarrow +\infty$. This can be restated as in (3.22), which we would like now to contradict.

To this aim, we reduce to proving that

$$\sum_{n\in\mathcal{S}'}\left(\frac{\alpha^2\tau_N(n)}{2}-1\right)\geq\varepsilon\alpha^2N^2, \tag{3.25}$$

for a certain possibly small constant $\varepsilon > 0$, where we let

$$\mathcal{S}' := \{1 \leq n \leq N^2 : M\sqrt{\log\log N} < \omega(n) - 2\log\log N \leq 2M\sqrt{\log\log N}\}$$

and $M > 0$. The above set has been chosen so that to increase in (3.25) the contribution of τ_N (in relation to (3.24) and previous considerations) and diminish that of the constant function 1, on average. This is because, by the Hardy–Ramanujan's theorem (see Eq. (2.3)), most numbers $n \leq N^2$ have $\omega(n)$ roughly equal to $\log\log N$. It is then not surprising that, to compute the sum in (3.25) and get the desired contradiction, we need to invoke another application of the Erdős–Kac's theorem.

3.4 On random multiplicative functions

A main problem in the theory of numbers concerns demonstrating squareroot cancellation for the partial sums of the Möbius function $\mu(n)$. More precisely, one ponders the validity of the following statement:

$$\sum_{n\le x}\mu(n) \ll_\varepsilon x^{1/2+\varepsilon} \tag{3.26}$$

for all $\varepsilon>0$ and x large with respect to ε, which is equivalent to the Riemann hypothesis (see Soundararajan [67] for a refinement of such relation).

Being an extremely difficult task, one focuses instead on producing good estimates for the partial sums of certain random models for $\mu(n)$. An interesting one is the following:

a Rademacher random multiplicative function f is a multiplicative function supported on the squarefree integers and defined on the prime numbers p by letting the $f(p)$ be independent random variables taking values ±1 with probability $1/2$ each.

This model has been introduced by Wintner [76], in 1944, who was also able to prove that, for any fixed $\varepsilon>0$, one almost surely has

$$\begin{aligned}\sum_{n\le x} f(n) &= O(x^{1/2+\varepsilon})\\ \sum_{n\le x} f(n) &\ne O(x^{1/2-\varepsilon}).\end{aligned} \tag{3.27}$$

The first estimate above was later improved by Erdős [10], Halász [19] and finally Basquin [1] and independently Lau, Tenenbaum and Wu [43], who gave, for any $\varepsilon>0$, the following almost sure upper bound:

$$\Big|\sum_{n\le x} f(n)\Big| \le \sqrt{x}(\log\log x)^{2+\varepsilon} \text{ as } x\longrightarrow +\infty. \tag{3.28}$$

On the opposite side, Harper [21] found that, for any function $V(x)$ tending to infinity with x, there almost surely exist arbitrarily large values of x for which

$$\Big|\sum_{n\le x} f(n)\Big| \ge \frac{\sqrt{x}(\log\log x)^{1/4}}{V(x)}, \tag{3.29}$$

which he deduced by proving, as an intermediate step, that there almost surely exist arbitrarily large values of x for which

$$\Big|\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} f(n)\Big| \ge \sqrt{x}(\log\log x)^{1/4+o(1)}, \tag{3.30}$$

where $P(n)$ indicates the largest prime factor of n.

The lower bound (3.30), and hence (3.29), also holds for Steinhaus random multiplicative functions f, where $\{f(p)\}_{p \text{ prime}}$ is a sequence of independent Steinhaus random variables (i.e. distributed uniformly on the unit circle $\{|z|=1\}$) and the function f is taken to be completely multiplicative.

The bounds (3.28) and (3.29) together give the feeling of the existence of a *Law of the Iterated Logarithm* for the partial sums of $f(n)$, which, due to the lack of independency among the values of f, cannot be accessed from classical probability results. It has been conjectured though (see Harper [21]) that the almost sure size of the largest fluctuations of f should roughly be $\sqrt{x}(\log\log x)^{1/4+o(1)}$.

Considering (3.29), the following new theorem, which is our main result here, may be seen as a partial progress in this direction.

Theorem 3.4.1. *Let f be a Rademacher or a Steinhaus random multiplicative function. Let $\varepsilon > 0$ small. As $x \longrightarrow +\infty$, we almost surely have*

$$\Big|\sum_{\substack{n\le x\\ P(n)>\sqrt{x}}} f(n)\Big| \le \sqrt{x}(\log\log x)^{1/4+\varepsilon}.$$

Considering (3.30), the bound in Theorem 3.4.1 is close to be sharp. Moreover, the set of numbers $n \le x$ with $P(n) > \sqrt{x}$ consists in a positive proportion of all the positive integers up to x. Hence, the partial sums in Theorem 3.4.1 might make a big contribution to the full partial sums of f. However, we cannot directly use Theorem 3.4.1 to deduce an almost sure upper bound for the full partial sums of f, which is until today not known. Indeed, we should anyway deal with the complementary portion on integers $n \le x$ with $P(n) \le \sqrt{x}$, which requires exploiting more the intricate dependence structure of the values of $f(n)$.

We now sketch the proof of Theorem 3.4.1:

- similarly to Basquin [1] and Lau–Tenenbaum–Wu [43], we reduce our analysis to what happens on a suitable subsequence of '*test points*' x_i. They are such that it is possible to study the tail of the distribution of the partial sums of f and easily control the increments of f between any two of them. We will globally recollect at the end the information by means of the first Borel–Cantelli's lemma;

- by writing, for any $n \leq x$ with $P(n) > \sqrt{x}$, $n = pm$, with $\sqrt{x} < p \leq x$ a prime and $m \leq x/p$ a positive integer, we deduce

$$\sum_{\substack{n \leq x \\ P(n) > \sqrt{x}}} f(n) = \sum_{\sqrt{x} < p \leq x} f(p) \sum_{m \leq x/p} f(m); \tag{3.31}$$

- by *conditioning* on the value of $f(q)$, for prime numbers $q \leq \sqrt{x}$, we can interpret the above as a sum of many independent random variables $f(p)$ times some coefficients. Its conditional probability distribution possesses a conditional *Gaussian* tail, thanks to Hoeffding's inequality. This allows us to sharply estimate the sum in (3.31);

- we express the aforementioned conditioning in terms of the size of a certain smooth weighted version of the conditional *variance* $V(x)$ of the partial sums in (3.31). Arguing as in Harper [25], we recast this in terms of an L^2-integral of a truncated Euler product corresponding to f. This gives rise to a *submartingale* sequence in x, which roughly speaking is a sequence of integrable random variables non-decreasing on average;

- finally, we input *low moments* estimates for the partial sums of f, to show that, with high probability, $V(x)$ has *uniformly* in x inside a wide interval size close to $x/\sqrt{\log\log x}$. To this aim, we need to drastically *increase* the number of test points x_i we simultaneously consider, in contrast with [1] and [43]. This forces us to introducing a suitable *normalized* version of the aforementioned submartingale sequence. To uniformly control its size we make use of Doob's maximal inequality, which in turn bounds the probability that all the elements of such sequence lie in an interval in terms of the biggest one.

Chapter 4

The variance of multiplicative functions in arithmetic progressions

Summary

The first section in this chapter is introductory to the problem of lower bounding the variance of complex arithmetic functions in arithmetic progressions: the first part covers some history on this problem, including past results related to divisor functions; the second part introduces the class of generalized divisor functions and presents a new theorem on a lower bound for their variance in arithmetic progressions; the third part covers the basics of the circle method approach, which will be used to derive our new contribution; finally, the last part contains the sketch of the proof of our new main discovery.

The first part in the second section of this chapter analyses some relations among the main parameters that define a generalized divisor function and sets all the conditions we ask the moduli q of the arithmetic progressions determining the variance of generalized divisor functions to be subject to in order to deduce a lower bound for it. The second part instead studies an asymptotic equality for the partial sums of generalized divisor functions under a coprimality condition, which will be involved in the proof of the corresponding variance lower bound.

The third section of this chapter is devoted to the proof of some corollaries of our main new contribution: we prove a lower bound for the variance in arithmetic progressions of divisor functions d_α, for a fixed parameter $\alpha \in \mathbb{C} \setminus \{-\mathbb{N} \cup \{1\}\}$, and of the indicator of the set of sums of two squares.

From section four to section twelve, we will develop the proof of the lower bound for the variance of generalized divisor functions in arithmetic progressions: we first reduce the problem to understanding L^2-integrals over subarcs of the circle of the exponential sum with coefficients our functions, estimating in sections four to six all the errors and minor terms arising from this procedure; in section seven we then focus on producing an asymptotic equality for the partial sums of a generalized divisor function twisted with Ramanujan sums $c_q(n)$, which is crucial to lower bound the aforementioned integrals; sections eight and nine analyse the contribution of respectively small and large prime factors of q to the study of such partial sums, by carefully estimating the derivatives of the Euler products associated to the functions in question; finally, sections ten to twelve collect all the various results obtained in the previous ones to conclude the proof of our new theorem.

4.1 Introduction

4.1.1 Some history on the variance in arithmetic progressions

Let $f(n)$ be a complex arithmetic function. In many arithmetic problems one needs information on the average of f over a fixed arithmetic progression $a \pmod q$ with $(a, q) = 1$:

$$\frac{1}{N} \sum_{\substack{n \le N \\ n \equiv a \pmod q}} f(n).$$

If f is roughly uniformly distributed in arithmetic progressions, one expects the above average to be close to

$$\frac{1}{\varphi(q) N} \sum_{\substack{n \le N \\ (n,q)=1}} f(n).$$

In fact, there is a vast literature around the topic of estimating the pointwise discrepancy $\Delta(f;a,q)$, defined by

$$\Delta(f;a,q) := \Bigg| \sum_{\substack{n\leq N\\ n\equiv a \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\leq N\\ (n,q)=1}} f(n)\Bigg|.$$

If f happens to satisfy the common bound $|f(n)|\leq d_2(n)^B(\log N)^B$, for a certain constant $B>0$, then a trivial estimate for the discrepancy is $\ll N(\log N)^{B'}$, for another constant $B'>0$. At this point, one usually seeks for logarithmic power savings, i.e. bounds of order $N/(\log N)^A$, with $A>0$, or even more for polynomial power savings, and hence for bounds of size $N^{1-\varepsilon}$, for a certain $0<\varepsilon<1$. For instance, the former can be achieved for the Von Mangoldt function $\Lambda(n)$, uniformly for moduli $q\leq(\log N)^C$, with $C>0$, being a consequence of the classical Siegel–Walfisz's theorem; the latter instead is known for the k-fold divisor functions $d_k(n)$ (we refer to [58, Table 1]). However, this task remains extremely difficult to undertake for a general function f and modulus q.

In order to investigate the distribution of a function f in arithmetic progressions, where the direct analysis of $\Delta(f;a,q)$ turns out to be unsuccessful, one looks at a sort of averaged second centred moment of the partial sums of f, with the aim of finding their 'typical' value.

Definition 4.1.1. We define the variance of f in arithmetic progressions by

$$V(N,Q;f) = \sum_{q\leq Q}\sum_{h|q}\sum_{\substack{a \mod q\\ (a,q)=h}} \Bigg| \sum_{\substack{n\leq N\\ n\equiv a \mod q}} f(n) - \frac{1}{\varphi(q/h)} \sum_{\substack{n\leq N\\ (n,q)=h}} f(n)\Bigg|^2. \tag{4.1}$$

The effect of averaging over all the arithmetic progressions for moduli $q\leq Q\leq N$ is to remove possible issues arising from single 'bad' moduli. The works of Hooley [32, 33, 34, 36, 35] and Vaughan [74, 75] are pioneering for the application of the circle method to produce asymptotic formulae for the variance in arithmetic progressions of some general sequences. Motohashi [56] instead succeded in establishing an asymptotic equality for $V(N,Q;d_2)$, for any $Q\leq N$, which was extended to (a smooth version of) d_k, for any $k\geq 2$, by Rodgers and Soundararajan [63] (see also de la Bretèche and Fiorilli [4] for a similar result). It is important to notice though that the last article only deals with values of Q lying in a limited range: for any $\delta > 0$, it is roughly required that $N^{1/(1+2/k-\delta)} \leq Q \leq N^{1/\delta}$.

On the other hand, Harper and Soundararajan [28] found a lower bound for the variance of $\Lambda(n)$ and of d_k, for any integer $k \geq 2$, holding in the range $N^{1/2+\delta} \leq Q \leq N$, for any small $\delta > 0$. In the first case [28, Theorem 1], they showed that

$$V(N, Q; \Lambda) \gg_\delta QN \log N,$$

if N is large enough with respect to δ. This admits a corresponding upper bound in the range $N/(\log N)^A \leq Q \leq N$, for any fixed $A > 0$, known as the Barban–Davenport–Halberstam's theorem, and matches a conditional, on the Generalized Riemann Hypothesis, asymptotic equality in the full range $N^{1/2+\delta} \leq Q \leq N$ (see Hooley [31]). In the second case [28, Theorem 2], they proved that

$$V(N, Q; d_k) \gg_{k,\delta} QN(\log N)^{k^2-1},$$

if N is large enough in terms of δ, for which a matching upper bound has been given by Nguyen [58, Theorem 3], but only in the range $N^{1-1/6(k+2)} \leq Q \leq N$.

Harper and Soundararajan [28] set up the bases for the study of lower bounds of variances (4.1) of complex sequences in arithmetic progressions. They showed that for a wide class of functions we can lower bound the variance (4.1) with the L^2-norm of the exponential sum with coefficients $f(n)$ over small subarcs of the circle around rational fractions a/q with large denominator. This point of view was already widespread in the literature (see for example in the works of Liu [46, 47] and Perelli [59]). However, the previous arguments relied on the connection between character sums and exponential sums, which can only be made to work for some particular sequences. In contrast, as pointed out in [28], Harper and Soundararajan avoided the use of Dirichlet characters in favour of Hooley's approach, connecting the variance of $f(n)$ in arithmetic progressions with the variance of the exponential sums $\sum_{n\leq N} f(n)e(na/q)$, where $e(t) = e^{2\pi it}$ for any $t \in \mathbb{R}$.

4.1.2 The variance of generalized divisor functions

Our aim is to generalize the result of Harper and Soundararajan [28, Theorem 2] to a wider class of multiplicative functions that contains the simple divisor functions d_k, for $k \geq 2$, as a particular instance.

Definition 4.1.2. A *generalized divisor function* is a multiplicative function for which there exist a complex number α and positive real numbers β, A_1, A_2, with α and β possibly depending on N, such that the following statistics hold

$$\sum_{p\leq x} f(p)\log p = \alpha x + O\left(\frac{x}{(\log x)^{A_1}}\right) \quad (2\leq x\leq N), \tag{4.2}$$

$$\sum_{p\leq x} |f(p)-1|^2 \log p = \beta x + O\left(\frac{x}{(\log x)^{A_2}}\right) \quad (2\leq x\leq N) \tag{4.3}$$

and such that $|f(n)|\leq d_\kappa(n)$, for a constant $\kappa > 0$ and every N-smooth positive integer n (see Ch. Notations for the definition of smooth numbers).

In other words, we require that, on average over prime numbers, f looks like α and is far from 1, since $\beta > 0$. The bigger A_1, A_2 are, the more precise the approximations (4.2)-(4.3) are, the easier it is to produce a lower bound for the variance of f in arithmetic progressions. Moreover, to allow for uniformity in N in our results, we only require to know a limited number of values of $f(n)$, i.e. those corresponding to $n \leq N$.

By the Prime Number Theorem, the equations (4.2) and (4.3) are trivially satisfied with $\beta = |\alpha - 1|^2$, for $\alpha \neq 1$, $\kappa = |\alpha|+2$ and any $A_1, A_2 > 0$, when $f(n) = d_\alpha(n)$. Whence, each α-fold divisor function (see Ch. Notations for a definition thereof), with $\alpha \neq 1$, is, in particular, a generalized divisor function. When α is a pole of the Gamma function, it is conjectured that the partial sums of d_α exhibit squareroot cancellation, as for $d_{-1}(n) = \mu(n)$ (see Ch. Notations for the definition of the Möbius function). Losing control on the exact size of their partial sums makes the study of their variance hard. This is not true though for the special case of d_0, whose variance can be trivially estimated by direct inspection, and its value is of size Q. Indeed, it is clear that the contribution to (4.1) is not zero only if $h = 1$ and in such case the term inside the square reduces to $\mathbf{1}_{a\equiv 1 \pmod q}(a) - 1/\varphi(q)$ (see Ch. Notations for the definition of the Euler totient function $\varphi(n)$). Hence, we get

$$\begin{aligned} V(N,Q;d_0) &= \sum_{q\le Q}\left(1-\frac{1}{\varphi(q)}\right)^2 + \sum_{q\le Q}\sum_{\substack{a \pmod q\\ (a,q)=1\\ a\not\equiv 1 \pmod q}}\frac{1}{\varphi(q)^2} \\ &= \sum_{q\le Q}\left(1-\frac{1}{\varphi(q)}\right) \\ &= Q+O(\log Q), \end{aligned}$$

by Landau's result [40, p. 184].

The previous considerations explain the hypothesis in the following extension of Harper and Soundararajan's result [28, Theorem 2].

Proposition 4.1.1. *Let $\delta>0$ sufficiently small and consider $N^{1/2+\delta}\le Q\le N$. For any complex number $\alpha\notin -\mathbb{N}\cup\{1\}$, we have*

$$V(N,Q;d_\alpha) \gg_{\alpha,\delta} Q\sum_{n\le N}|d_\alpha(n)|^2, \tag{4.4}$$

if N is large enough with respect to α and δ.

Even though for $\alpha\in-\mathbb{N}$ we do not know how to deduce the lower bound for the corresponding variance, we believe that (4.4) might still be true.

4.1.3 The main new result and some corollaries

Proposition 4.1.1 is a corollary of our new main contribution.

Theorem 4.1.2. *Let δ be a sufficiently small positive real number and N be a large positive integer. Suppose that $N^{1/2+\delta}\le Q\le N$. Let $f(n)$ be a generalized divisor function as in Definition 4.1.2 with $\alpha\notin-\mathbb{N}\cup\{0\}$ and let $\kappa(\alpha,\beta):=(\kappa+1)^2+\kappa-\Re(\alpha)-\beta+4$. Furthermore, assume that*

$$\begin{aligned} &A_1>\max\{\kappa(\alpha,\beta),\kappa+2\}; \\ &A_2>A_1-\kappa(\alpha,\beta)+1; \\ &\beta\ge(\log N)^{\kappa(\alpha,\beta)-A_1}; \\ &|\Gamma(\alpha)|\le\log N, \end{aligned} \tag{4.5}$$

where the Gamma function has been defined in Ch. Notations. Finally, let

$$c_0 = \prod_{p \le N} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots \right) \left(1 - \frac{1}{p}\right)^{\alpha}$$

and suppose that

$$(\log N)^{1-\delta} |c_0| \ge 1. \tag{4.6}$$

Then we have

$$V(N,Q;f) \gg \left| \frac{c_0 \beta}{\Gamma(\alpha)} \right|^2 Q \sum_{n \le N} |f(n)|^2. \tag{4.7}$$

The implicit constant above may depend on δ, κ, A_1, A_2 *and the implicit constants in* (4.2)–(4.3)*, but not on* α*, and we take* N *large enough depending on all of these parameters.*

Other consequences of Theorem 4.1.2

Theorem 4.1.2 is quite technical, but it does not merely represent an improvement upon the result of Proposition 4.1.1. Indeed, it allows us to lower bound the variance of multiplicative functions that arise from divisor functions as well, such as for instance positive integer powers of $d_2(n)$ or products of divisor functions as $d_2(n)d_3(n)$, but also, and most importantly, of those that behave very differently from the simple divisor functions. As a concrete example of this last case we will deduce from Theorem 4.1.2 the next lower bound on the variance in arithmetic progressions of the indicator of the integers sums of two squares. The distribution of sums of two squares in arithmetic progressions has been studied by a number of authors, amongst them Fiorilli [11], Iwaniec [38], Lin and Zhan [44] and Rieger [61, 62].

Corollary 4.1.3. *Let* $\delta > 0$ *sufficiently small and consider* $N^{1/2+\delta} \le Q \le N$*. Let* S *be the set of all integer sums of two squares. Then we have*

$$V(N,Q;\mathbf{1}_S) \gg_\delta \frac{QN}{\sqrt{\log N}}, \tag{4.8}$$

if N is large enough with respect to δ.

Furthermore, Theorem 4.1.2 also allows us to investigate the variance of sequences of divisor functions $d_{\alpha_N}(n)$ depending uniformly with N. Amongst them, an interesting case is when α_N approaches a pole of the Gamma function, with α_N subject to the last condition of (4.5), or when α_N is close to 1, with $|\alpha_N - 1| \geq (\log N)^{\frac{\kappa(\alpha,\beta)-A_1}{2}}$. This last case will be covered in the next chapter, where we will extend the range of $|\alpha_N - 1|$ where we are able to produce a lower bound for $V(N, Q; d_{\alpha_N})$ in the full range $N^{1/2+\delta} \leq Q \leq N$, obtaining a presumably sharp one as soon as $|\alpha_N - 1| \leq 1/\log\log N$.

Some considerations about the statement of Theorem 4.1.2

To deduce Theorem 4.1.2 we follow Harper and Soundararajan's seminal work [28], where, roughly speaking, they discovered that, under suitable conditions on f, one has

$$V(N, Q; f) \gg Q \int_{\mathfrak{m}} \Big| \sum_{n \leq N} f(n) e(n\theta) \Big|^2 d\theta, \tag{4.9}$$

with $e(t) = e^{2\pi i t}$, for any $t \in \mathbb{R}$, and $\mathfrak{m}$ the set of minor arcs (see the next subsection for a definition).

For functions that fluctuate like random, we usually expect the above integral to be well approximated by that over the whole circle. This leads, by Parseval's identity, to the following guess:

$$V(N, Q; f) \gg Q \sum_{n \leq N} |f(n)|^2, \tag{4.10}$$

in the full range $N^{1/2+\delta} \leq Q \leq N$.

For Q in such range, the lower bound (4.10) happens to hold when $f(n) = \Lambda(n)$ and $f(n) = d_k(n)$, for $k \geq 2$ a positive integer, by Theorem 1 and Theorem 2 in [28]. More generally, it is satisfied by $f(n) = d_\alpha(n)$, for any $\alpha \notin -\mathbb{N} \cup \{1\}$, by Proposition 4.1.1. By Theorem 4.1.2, the same conclusion holds for other generalized divisor functions, like for instance the characteristic function of the sums of two squares, as given in Corollary 4.1.3, except for the possible loss coming from the factor $|c_0\beta/\Gamma(\alpha)|^2$. From this point of view, we believe that (4.4) and (4.8) are presumably sharp.

To produce an explicit lower bound for the variance in arithmetic progressions, Harper and Soundararajan reduced the estimate of the right-hand side of (4.9) to that of the partial sums of $f(n)$ twisted with Ramanujan sums $c_q(n)$. These sums are simply defined as

$$c_q(n) = \sum_{\substack{a=1,\dots,q \\ (a,q)=1}} e(an/q)$$

and characterized by their main property that we will make use of several times in the future

$$c_q(n) = \sum_{k|(n,q)} k\mu(q/k). \tag{4.11}$$

Our strategy is to exploit in depth the asymptotic expansion of the partial sums of $f(n)$ to analyze that of $f(n)c_q(n)$. To this aim, note that we roughly have

$$\sum_{n\leq N} f(n) \approx \frac{c_0 N(\log N)^{\alpha-1}}{\Gamma(\alpha)},$$

by the Selberg–Delange's theorem. Here c_0 is the truncated Euler product of $f*d_{-\alpha}$, which naturally arises from the Selberg–Delange's method by comparing the partial sums of f with those of d_α; similar reasons explain the presence of the factor $\Gamma(\alpha)$, here and in (4.7). In particular, the last condition in (4.5) has been inserted to avoid the scenario in which α is too close to a pole of the Gamma function, which would make us lose control on the average of $f(n)$ over integers $n \leq N$, thus precluding us from producing a lower bound for $V(N,Q;f)$. However, it may be possible to relax such restriction to $|\Gamma(\alpha)|$ being smaller than a suitable large power of $\log N$, but we are not going to work this out here. The condition (4.6) instead makes sure that f looks nice on the primes $p \leq \kappa$, thus excluding certain patterns of $f(p)$ where c_0 is too close to 0. However, one might still be able to replace (4.7) with a lower bound of a different shape by carefully understanding some non-trivial derivatives of the Euler product of f, but we are not going to pursue such extension here.

To compute an average over the moduli $q \leq Q$ of the partial sums of $f(n)c_q(n)$, we need to restrict the set of positive integers q to those satisfying some special properties. One of them is to possess a very large prime factor t, of size approximately $\sqrt{N}$. This leads to expressing the lower bound of $V(N,Q;f)$ in terms of the sum $\sum_t |f(t)-1|^2/t$. Using (4.3) to asymptotically estimate it produces the factor β in the expression (4.7). We see that the assumption on β in (4.5) is necessary, because a smaller value of β corresponds to a function f closer to d_1, which variance we will see in the next chapter has size Q^2.

4.1.4 The circle method approach

For suitably chosen parameters K, Q and Q_0, we are going to define the so called set of major arcs $\mathfrak{M} = \mathfrak{M}(Q_0, Q; K)$, consisting of those $\theta \in \mathbb{R}/\mathbb{Z}$ having an approximation $|\theta - a/q| \leq K/(qQ)$, with $q \leq KQ_0$ and $(a,q) = 1$. Let $\mathfrak{m}$, the minor arcs, denote the complement of the major arcs in $\mathbb{R}/\mathbb{Z}$. Clearly, this last set occupies almost the totality of the circle, depending on K, Q and Q_0, and it consists of real numbers well approximated by rational fractions with large denominator, as large as depending on K, Q_0 and Q.

The first step in Harper and Soundararajan work [28, Proposition 1] is to reduce the problem of estimating the variance in arithmetic progressions to that of lower bounding L^2-integrals of exponential sums over minor arcs.

Proposition 4.1.4. *Let $f(n)$ be any complex sequence. Let N be a large positive integer, $K \geq 5$ be a parameter and K, Q_0 and Q be such that*

$$K\sqrt{N \log N} \leq Q \leq N \ \text{and} \ \frac{N \log N}{Q} \leq Q_0 \leq \frac{Q}{K^2}. \tag{4.12}$$

Then we have

$$V(N,Q;f) \geq Q\Big(1 + O\Big(\frac{\log K}{K}\Big)\Big)\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)|^2 d\theta + O\Big(\frac{NK}{Q_0}\sum_{n\leq N}|f(n)|^2\Big) \tag{4.13}$$
$$+ O\Big(\sum_{q \leq Q}\frac{1}{q}\sum_{\substack{d|q \\ d > Q_0}}\frac{1}{\varphi(d)}\Big|\sum_{n\leq N} f(n)c_d(n)\Big|^2\Big),$$

where $\mathcal{S}_f(\theta) := \sum_{n\leq N} f(n)e(n\theta)$.

To handle the L^2-integrals over minor arcs as in (4.13) for the generalized divisor functions, we will appeal to [28, Proposition 3], which we next report adapted to our context.

Proposition 4.1.5. *Under the usual notations, we assume* $KQ_0 < R := N^{1/2-\delta/2}$. *Then we have*

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)|^2 d\theta \geq \left(\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta \right)^2 \left(\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \right)^{-1}, \tag{4.14}$$

where

$$\mathcal{G}(\theta) = \sum_{n \leq N} \Big(\sum_{\substack{r | n \\ r \leq R}} g(r) \Big) e(n\theta),$$

for any complex arithmetic function $g(r)$.

If moreover there exists a constant $\kappa > 1$ *for which* $|g(n)| \leq d_\kappa(n)$, *for any* $n \leq N$, *we also have*

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta \geq \sum_{KQ_0 < q \leq R} \Big| \sum_{\substack{r \leq R \\ q | r}} \frac{g(r)}{r} \Big| \Big| \sum_{n \leq N} f(n) c_q(n) \Big| + O_{\delta,\kappa}(N^{1-\delta/11}), \tag{4.15}$$

if N *is large enough in terms of* δ *and* κ.

This is a slight and simplified variation of [28, Proposition 3] for functions bounded by a divisor function, in which a smooth weight in the average of f has been removed by introducing a small error term. We will prove it in Subsect. 4.2.5.

4.1.5 Sketch of the proof of Theorem 4.1.2

To begin with, we make use of Harper and Soundararajan [28, Proposition 1] (see Proposition 4.1.4 above) to roughly get

$$V(N, Q; f) \gg Q \int_{\mathfrak{m}} \Big| \sum_{n \leq N} f(n) e(n\theta) \Big|^2 d\theta. \tag{4.16}$$

We lower bound the L^2-integral in (4.16) in terms of an L^1-integral by means of Cauchy–Schwarz's inequality:

$$\geq \frac{(\int_{\mathfrak{m}} |\sum_{n\leq N} f(n)e(n\theta) \sum_{n\leq N} h(n)e(n\theta)| d\theta)^2}{\int_{\mathfrak{m}} |\sum_{n\leq N} h(n)e(n\theta)|^2 d\theta}, \tag{4.17}$$

where $h(n) := \sum_{r|n,\ r\leq R} g(r)\Phi(\frac{n}{N})$. Here $R := N^{1/2-\delta/2}$ and $g(n)$ is a suitable arithmetic function with $\Phi(t)$ a suitable smooth function compactly supported in $[0,1]$ with $0 \leq \Phi(t) \leq 1$ for all $0 \leq t \leq 1$. The choice of g is fundamental for succeeding in the proof of Theorem 4.1.2: we consider a multiplicative function supported on the squarefree integers and zero on all the prime numbers smaller than C, where $C > 0$ is a suitable constant to be chosen later; on the prime numbers $C < p \leq N$ we put $g(p) = f(p) - 1$, if $\Re(\alpha) \geq 1$, and $g(p) = 1 - f(p)$, otherwise.

We extend the integral in the denominator of (4.17) to the whole circle to then, by Parseval's identity, bound it with the partial sum of $|h(n)|^2$.

Proposition 4.1.6. *Let $R = N^{1/2-\delta/2}$ as before. Then we have*

$$\sum_{n\leq N} \Bigg| \sum_{\substack{r|n \\ r\leq R}} g(r) \Bigg|^2 \ll N(\log N)^{\beta+2|\Re(\alpha)-1|}, \tag{4.18}$$

where the implicit constant may depend on δ, κ, A_1, A_2 and that in (4.2)–(4.3)*.*

To produce a lower bound for the L^1-integral in the numerator of (4.17), we employ Harper and Soundararajan [28, Proposition 3] (see Proposition 4.1.5 above), which gives for it a value

$$\gg \sum_{q\leq R} \Bigg| \sum_{\substack{r\leq R \\ q|r}} \frac{g(r)}{r} \Bigg| \Bigg| \sum_{n\leq N} f(n)c_q(n) \Bigg|. \tag{4.19}$$

Applying this strategy to the sequence of divisor functions $d_k(n)$ for a positive integer $k \geq 2$, Harper and Soundararajan produced a lower bound for $\sum_{n\leq N} d_k(n)c_q(n)$, by suitably restricting the range in which q varies, thus deducing a lower bound for $V(N,Q;d_k)$ [28, Theorem 2]. However, their technique, which consists in rewriting the sum in question as an integral of the associated Dirichlet series, by means of Perron's formula, does not extend to the case of any generalized divisor function. In fact, in the former case the corresponding Dirichlet series can be extended to a meromorphic function on the whole complex plane with just one pole at 1, whereas in the latter it may only be defined on the half plane of complex numbers with real part greater then or equal to 1.

One possible way to handle these differences, and to work on full generality at the same time, is to apply the Selberg–Delange's method to asymptotically compute the sum $\sum_{n\leq N} f(n)c_q(n)$. Since the product $f(n)c_q(n)$ is not a multiplicative function, its Dirichlet series is hard to analyze. To overcome this, we break the above sum down into smaller chunks that are easier to understand so reducing ourselves to apply the Selberg–Delange's method to the much more manageable average of f over a coprimality condition. More precisely, we notice that

$$\sum_{n\leq N} f(n)c_q(n) = \sum_{\substack{b\leq N\\ p|b\Rightarrow p|q}} f(b)c_q(b) \sum_{\substack{a\leq N/b\\ (a,q)=1}} f(a), \tag{4.20}$$

using the substitution $n = ab$, with $(a, q) = 1$ and $b = n/a$, which is unique, and properties of the Ramanujan sums. By the Selberg–Delange's method we can develop the innermost sum as in the following result.

Theorem 4.1.7. *Let $f(n)$ be a multiplicative function with complex values such that there exists $\kappa > 1$ with $|f(n)|\leq d_\kappa(n)$, for any N-smooth positive integer n, and satisfying* (4.2) *with $\alpha \in \mathbb{C} \setminus \{\{0\} \cup -\mathbb{N}\}$. Moreover, suppose that q is a positive squarefree number smaller than N such that for any prime $p|q$ we have $p > C$, where $C > \kappa^2$ will be chosen later on in terms of δ, κ, A_1 and the implicit constant in* (4.2)*. Then for any $4 \leq x \leq N$ we have*

$$\begin{aligned}\sum_{\substack{n\leq x\\ (n,q)=1}} f(n) &= x(\log x)^{\alpha-1} \sum_{j=0}^{J} \frac{\lambda_j}{(\log x)^j} \\ &\quad + O(|\tilde{G}_q^{(2\lfloor A_1\rfloor+2)}(1)|x(\log x)^{\kappa-A_1-1}(\log\log x)) \\ &\quad + O\biggl(x^{3/4}\sum_{d|q}\frac{d_\kappa(d)}{d^{3/4}}\biggr),\end{aligned} \tag{4.21}$$

where J is the largest integer $< A_1$, $\lfloor A_1 \rfloor$ is the integer part of A_1 and we define

$$\lambda_j = \lambda_j(f,\alpha,q) = \frac{1}{\Gamma(\alpha-j)} \sum_{l+h=j} \frac{(H_q^{-1})^{(h)}(1) c_l}{h!},$$

with

$$H_q(z) = \prod_{p|q} \left(1 + \frac{f(p)}{p^z} + \frac{f(p^2)}{p^{2z}} + \cdots \right) \text{ and } \tilde{G}_q(z) = \prod_{p|q} \left(1 + \frac{|f(p)|}{p^z}\right)$$

on $\Re(z) \geq 1$, and

$$c_l = \frac{1}{l!} \frac{d^l}{dz^l} \left(\zeta_N(z)^{-\alpha} F(z) \frac{((z-1)\zeta(z))^{\alpha}}{z} \right)_{z=1}, \text{ for any } l \leq J,$$

with

$$F(z) = \sum_{\substack{n: \\ p|n \Rightarrow p \leq N}} \frac{f(n)}{n^z}, \text{ and } \zeta_N(z) = \sum_{\substack{n: \\ p|n \Rightarrow p \leq N}} \frac{1}{n^z}.$$

Here the big-Oh constant depends on κ, A_1 and the implicit constant in (4.2).

Plugging the asymptotic expansion for $\sum_{n \leq N/b,\ (a,q)=1} f(n)$, given by Theorem 4.1.7, into (4.20), we are basically left to evaluate the truncated Dirichlet series

$$\sum_{\substack{b \leq N: \\ p|b \Rightarrow p|q}} \frac{f(b) c_q(b)}{b} (\log(N/b))^{\tilde{\alpha}}, \tag{4.22}$$

with $\tilde{\alpha} \in \{\alpha-1, \alpha-2, \ldots, \alpha-J-1\}$. Naively, we might expect them to behave like

$$(\log N)^{\tilde{\alpha}} \sum_{b|q} f(b)\mu(q/b). \tag{4.23}$$

Technically speaking, there are several details to take into account which make the evaluation of those sums quite complicated, as for instance the presence of possibly very large divisors of q, that make $(\log(N/b))^{\tilde{\alpha}}$ hard to analyze. Therefore, we decide to exploit more the structure of the Ramanujan sums, leading to a useful decomposition of (4.22) given by splitting the integers

$q = rs \leq N^{1/2}$, with s the part of q supported only on prime numbers smaller than a logarithmic power. In view of this factorization we may rewrite the sum in (4.22) as roughly

$$(4.24) \qquad \sum_{\substack{b_1 \leq \sqrt{N} \\ p|b_1 \Rightarrow p|r}} \frac{f(b_1)c_r(b_1)}{b_1} \sum_{\substack{b_2 \leq N/b_1 \\ p|b_2 \Rightarrow p|s}} \frac{f(b_2)c_s(b_2)}{b_2} (\log(N/b_1 b_2))^{\tilde{\alpha}},$$

since by multiplicativity of $c_q(n)$ as function of q we have

$$c_q(b) = c_r(b)c_s(b) = c_r(b_1)c_s(b_2).$$

To estimate the innermost sum, we expand the $\tilde{\alpha}$-power of the logarithm $\log(N/b_1b_2)$, using the generalized binomial theorem. In this way we obtain a sum of successive derivatives of the Dirichlet series of $f(b)c_s(b)\mathbf{1}_{p|b\Rightarrow p|s}$, which can be handled by means of several applications of the Faà di Bruno's formula. This last one is a combinatorial expression for the derivative of the composition of two functions (see for instance Roman's paper [64]). To avoid such derivatives blowing up we input further restrictions on the value of f at prime numbers, as to be suitably away from 1.

It remains to estimate the outermost sum twisted again with a fractional power of $\log(N/b_1b_2)$. In order to compute it, we insert a key hypothesis on the structure of q, i.e. to be divisible by an extremely large prime number $t \approx \sqrt{N}$. Indeed, it seems crucial to avoid the situation in which $r = ts'$ has several large divisors, thus gaining more control on the factor $\log(N/b_1b_2)$. This is another main difference with the approach employed in [28], where q was restricted to just being an N^{ε}–smooth number, for a carefully chosen small $\varepsilon > 0$. Under our assumption, the aforementioned sum can be handled by using the multinomial coefficient formula, which supplies the expansion for a positive integer power of a multinomial sum (see for example Netto [57]). To control the Euler product derivatives arising from this procedure, we require f to be *never* too close to 1 on several prime factors of s'.

Overall, we end up showing that

$$\Big|\sum_{n\leq N} f(n)c_q(n)\Big| \gg |c_0| \frac{N(\log N)^{\Re(\alpha)-1}}{|\Gamma(\alpha)|} |g(s')g(s)\theta_{N,\alpha}(t)|,$$

where c_0 is as in the statement of Theorem 4.1.2 and for any prime $p \le N$,

$$\theta_{N,\alpha}(p) := 1 - f(p)\left(1 - \frac{\log p}{\log N}\right)^{\alpha-1}.$$

We may naively think of $\theta_{N,\alpha}(p)$ as roughly $1 - f(p) = g(p)$, on average over prime numbers $p \le N$; therefore, we might look at $g(s')g(s)\theta_{N,\alpha}(t)$ as $g(q)$.

The above lower bound will be used into (4.19) in combination with the following estimate.

Proposition 4.1.8. *Let q be a positive integer with $KQ_0 \le q \le N^{1/2-3\delta/4}$ such that any prime divisor of q is larger than C and $q \in \mathcal{A}$, where*

$$\mathcal{A} := \left\{q : \sum_{p|q} \frac{(\log p)^{A_1+1}}{p^{3/4}} \le D\right\},$$

for a certain constant $D > 0$ to be determined later on. Then we have

$$\Big|\sum_{\substack{r\le R\\ q|r}} \frac{g(r)}{r}\Big| \gg \frac{|g(q)|}{q}(\log N)^{|\Re(\alpha)-1|}, \tag{4.25}$$

where the implicit constant may depend on δ, κ, A_1, D and the implicit constant in (4.2)*. Moreover, we are assuming N and C to be sufficiently large with respect to all of these parameters.*

From (4.19), we then now deduce that the integral in the numerator of (4.17) is roughly

$$\gg |c_0| \frac{N(\log N)^{\Re(\alpha)-1+|\Re(\alpha)-1|}}{|\Gamma(\alpha)|} \sum_{KQ_0\le q\le RN^{-\delta/4}} \frac{|g(q)g(s')g(s)\theta_{N,\alpha}(t)|}{q},$$

where the sum over q is subject to some restrictions. This, together with Proposition 4.1.6 and (4.16), by means of (4.17), gives a lower bound for $V(N,Q;f)$ of

$$|c_0|^2 \frac{QN(\log N)^{-\beta+2(\Re(\alpha)-1)}}{|\Gamma(\alpha)|^2} \left(\sum_{KQ_0\le q\le RN^{-\delta/4}} \frac{|g(q)g(s')g(s)\theta_{N,\alpha}(t)|}{q}\right)^2.$$

The computation of the above sum is quite delicate and require an accurate analysis of the contribution coming from the large prime factors t.

Lemma 4.1.9. *Let $f(n) : \mathbb{N} \longrightarrow \mathbb{C}$ be a generalized divisor function as in Definition 4.1.2, for parameters $\alpha, \beta, \kappa, A_1, A_2$ satisfying* (4.5)*. Then there exists a small $\delta_0 = \delta_0(\kappa)$, such that either for $\delta \leq \delta_0$ or for $\delta/2$, we have*

$$\sum_{\substack{t \text{ prime}: \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|\theta_{N,\alpha}(t)(f(t)-1)|}{t} \geq \eta\beta\frac{\delta}{V}, \tag{4.26}$$

for a certain $\eta = \eta(\delta, \kappa) > 0$, if V is large enough with respect to δ, κ, A_1, A_2 and the implicit constants in (4.2)–(4.3) *and N is sufficiently large in terms of all these parameters.*

Using Lemma 4.1.9, and other calculations, we can overall find that

$$\sum_{KQ_0 \leq q \leq RN^{-\delta/4}} \frac{|g(q)g(s')g(s)\theta_{N,\alpha}(t)|}{q} \gg \beta(\log N)^{\beta}.$$

The proof of Theorem 4.1.2 now ends after showing that $\sum_{n\leq N}|f(n)|^2 \ll N(\log N)^{\beta+2(\Re(\alpha)-1)}$, so as to deduce

$$V(N,Q;f) \gg \left|\frac{c_0\beta}{\Gamma(\alpha)}\right|^2 QN(\log N)^{\beta+2(\Re(\alpha)-1)} \gg \left|\frac{c_0\beta}{\Gamma(\alpha)}\right|^2 Q\sum_{n\leq N}|f(n)|^2.$$

4.2 Preliminary notions and results

In this section we first investigate some relations between the parameters that determine a generalized divisor function as in Definition 4.1.2. Then we set all the conditions the moduli q need to adhere to in order to produce a lower bound for the variance of a generalized divisor function in arithmetic progressions as in Theorem 4.1.2. Next, for such functions we study their behaviour on average over integers n coprime with q, whence showing Theorem 4.1.7. Finally, we deduce Proposition 4.1.5 from the original [28, Proposition 3].

4.2.1 Main parameters

We consider K as a large constant so that the term $(\log K)/K$ in Proposition 4.1.4 is small enough. Since we are assuming $N^{1/2+\delta} \leq Q \leq N$, with $\delta > 0$ small, we let $R := N^{1/2-\delta/2}$.

We note that from (4.2) it follows that when $\alpha \neq 0$

$$\begin{aligned}|\alpha| N\left(1+O\left(\frac{1}{(\log N)^{A_1}}\right)\right) &= |\sum_{p\leq N} f(p)\log p| \\ &\leq \sum_{p\leq N} |f(p)| \log p \\ &\leq \sum_{p\leq N} \kappa \log p \\ &= \kappa N\left(1+O\left(\frac{1}{(\log N)^{A_1}}\right)\right),\end{aligned}$$

by the Prime Number Theorem (see e.g. [55, Theorem 6.9]). We conclude that for any α we have $|\Re(\alpha)| \leq |\alpha| \leq \kappa(1+O(1/(\log N)^{A_1}))$. Similarly, but using (4.3), we get $\beta \leq (\kappa+1)^2(1+O(1/(\log N)^{A_2}))$. In particular, we deduce that $|\alpha| \leq \kappa+1$ and $\beta \leq (\kappa+2)^2$, if N is large enough in terms of κ, A_1, A_2 and the implicit constants (4.2)–(4.3). By the monotonicity of $d_\kappa(n)$ as function of $\kappa > 0$ and by replacing κ with $\kappa+1$, we may thus assume that $\kappa > 1$ and $|\alpha| \leq \kappa$ and $\beta \leq (\kappa+1)^2$. Therefore, from the previous considerations we get $\kappa(\alpha,\beta) \geq 4$. In particular, we deduce that, in the statement of Theorem 4.1.2, $A_1, A_2 > 1$. We collect these information together in the next remark.

Remark 4.2.1. *With notations as in Theorem 4.1.2, we can assume that*

- $\kappa > 1$*;*
- $|\alpha| \leq \kappa$ *and* $\beta \leq (\kappa+1)^2$*;*
- $\kappa(\alpha,\beta) \geq 4$*;*
- $A_1, A_2 > 1$*.*

The following lemma studies the relation, produced by equations (4.2)-(4.3), between the parameters α and β

Lemma 4.2.2. *Under the usual notation, we have*

$$|\alpha - 1|^2 \leq \beta + O_\kappa((\log N)^{-\min\{A_1, A_2\}}) \ll \beta,$$

if N is large enough with respect to A_1, A_2, κ and the implicit constants (4.2)–(4.3).

Proof. If $\alpha = 1$, the result is trivial. Assume then $\alpha \neq 1$. By an application of the Cauchy–Schwarz's inequality, we have

$$\begin{aligned} \Big|\sum_{p\leq N}(f(p)-1)\log p\Big|^2 &\leq \sum_{p\leq N}|f(p)-1|^2 \log p \sum_{p\leq N}\log p \\ &= (\beta N + O(N(\log N)^{-A_2}))(N + O(Ne^{-c\sqrt{\log N}})), \end{aligned}$$

for a suitable $c > 0$, by the prime number theorem and equation (4.3). The left hand side of the above inequality instead is $|\alpha - 1|^2 N^2 + O_\kappa(N^2(\log N)^{-A_1})$, by (4.2). This implies the thesis if we assume N as in the statement of the lemma and thanks to conditions (4.5). □

4.2.2 The conditions on q

In the proof of Theorem 4.1.2, as explained in the introduction to this chapter, we will need to restrict the set of moduli q we will work with, which we can do since we are seeking for a lower bound for the variance. In the following, we summarize for the sake of readability and for future reference all the conditions we ask q to be subject to. Let ε be a small positive real number to be chosen at the end in terms of δ, κ, A_1, A_2 and the implicit constants in (4.2)–(4.3). Moreover, let A, B, C and D positive real constants to be chosen in due course. Then we ask that:

1. $q \leq N^{1/2-3\delta/4}$ and squarefree;
2. $\omega(q) \leq A \log\log N$. Equivalently, we are asking that the number of prime factors of q is bounded by the expected one;
3. $q := tss'$, with
   (a) $p|s \Rightarrow p \leq (\log N)^B$, i.e. s is $(\log N)^B$–smooth;

(b) $s' \le N^{\varepsilon}$, with $p|s' \Rightarrow p > (\log N)^B$, i.e. s' smaller than a suitably small power of N and $(\log N)^B$–rough;

(c) $s' \in \mathcal{A}'$, where

$$\mathcal{A}' := \left\{ s' : \sum_{p|s'} \frac{\log p}{\min\{|f(p)-1|, 1\}} \le \frac{\varepsilon \log N}{\kappa} \right\}.$$

It is equivalent to ask that f is *never* too close to 1 on several prime factors of s';

(d) t a prime in $[N^{1/2-3\delta/4-\varepsilon}, N^{1/2-3\delta/4-\varepsilon/2}]$, supposing the existence of a *unique large prime factor* in the prime factorization of q;

4. for any prime $p|q$ we have $p > C$, i.e. q does not have any very small prime factor;

5. (a) $|f(p)-1| > 1/\sqrt{\log\log N}$, if $p|ss'$, i.e. on those primes f is never too close to 1;

   (b) $p > C/|f(p)-1|$, for any $p|ss'$;

   (c) $f(t) \neq 1$;

6. To avoid the scenario in which q has lots of small prime factors, we require $q \in \mathcal{A}$, where

$$\mathcal{A} := \left\{ q : \sum_{p|q} \frac{(\log p)^{A_1+1}}{p^{3/4}} \le D \right\}.$$

4.2.3 Mean value of multiplicative functions under a coprimality condition

We will now show how to handle averages of multiplicative functions satisfying (4.2) under a coprimality condition. Since we are going to use the full strength of [17, Theorem 1], we report it here for the sake of readiness and in a form more suitable for our purposes.

Theorem 4.2.3. *Let f be a multiplicative function satisfying* (4.2) *and such that there exists $\kappa > 1$ with $|f(n)| \le d_\kappa(n)$, for any N-smooth positive integer* *n. Let J be the largest integer $< A_1$ and the coefficients $c_j = c_j(f,\alpha)$ defined by*

$$c_j := \frac{1}{j!}\frac{d^j}{dz^j}\left(\zeta_N(z)^{-\alpha}F(z)\frac{((z-1)\zeta(z))^\alpha}{z}\right)_{z=1}, \text{ for any } j \le J,$$

with

$$F(z) := \sum_{\substack{n:\\ p|n\Rightarrow p\le N}} \frac{f(n)}{n^z}, \quad \zeta_N(z) := \sum_{\substack{n:\\ p|n\Rightarrow p\le N}} \frac{1}{n^z}.$$

Then we have

$$\sum_{n\le x} f(n) = x\sum_{j=0}^{J} c_j \frac{(\log x)^{\alpha-j-1}}{\Gamma(\alpha-j)} + O(x(\log x)^{\kappa-1-A_1}(\log\log x)), \quad (2\le x\le N). \tag{4.27}$$

The big-Oh constant depends at most on κ, A_1 and the implicit constant in (4.2). The dependence on A_1 comes from both its size and its distance from the nearest integer. Moreover, the condition $|f(n)|\le d_\kappa(n)$ can be relaxed to the following two ones on average over prime powers

$$\sum_{p\le x}\frac{|f(p)|\log p}{p} \le \kappa\log x + O(1) \quad (2\le x\le N); \tag{4.28}$$
$$\sum_{\substack{p\le x\\ j\ge 1}}\frac{|f(p^j)|^2}{p^j} \le \kappa^2\log\log x + O(1) \quad (2\le x\le N),$$

where the big-Oh terms here depend only on κ.

Proof. We first note that [17, Theorem 1] gives an asymptotic for the mean value of multiplicative functions for which we know their behaviour on average over all the prime numbers, including those much larger than N. Here instead, we are interested only in the value of $f(p^k)$, for prime powers p^k with $p\le N$. However, we can freely replace f with the function equal to f itself on such prime powers and such that

$$f(p^k) := d_\alpha(p^k), \text{ for any } p > N \text{ and } k\ge 1.$$

Then Theorem 4.2.3 readily follows from [17, Theorem 1]. Indeed, it is clear that the statistic [17, Eq. 1.2] corresponds to (4.2) here. Moreover, the condition $|f(n)|\le d_\kappa(n)$, for every $n \le N$, trivially translates to our condition only on N-smooth numbers, since it is equivalent to the corresponding one on prime powers. Same considerations for the statistics (4.28), which are slightly weaker than the corresponding conditions [17, Eq. (7.1)–(7.2)].

The only main difference is in the representation of the coefficients c_j. Indeed, in [17] such coefficients are defined as

$$c_j := \frac{1}{j!}\frac{d^j}{dz^j}\frac{(z-1)^\alpha \tilde{F}(z)}{z}\bigg|_{z=1},$$

with $\tilde{F}(z)$ the Dirichlet series of $f(n)$. Here we multiply and divide the above expression by $\zeta(z)^{-\alpha}$ and notice that

$$\zeta(z)^{-\alpha}\tilde{F}(z) = \zeta_N(z)^{-\alpha}F(z),$$

where $\zeta_N(z)$ and $F(z)$ are defined as in the statement of the theorem. Since the function $((z-1)\zeta(z))^\alpha$ is a holomorphic function on $\Re(z) \ge 1$, for any $\alpha \in \mathbb{C}$, we see that each coefficient c_j is basically the j-th derivative of an Euler product. In particular, we have

$$c_0 = \prod_{p\le N}\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right)\left(1-\frac{1}{p}\right)^\alpha.$$

Potentially, the coefficients c_j could grow together with N and α. However, the next lemma shows that under our hypotheses on f they are indeed uniformly bounded.

Lemma 4.2.4. *Let f be a multiplicative function satisfying* (4.2) *for some $\alpha \in \mathbb{C}$ and such that $|f(n)|\le d_\kappa(n)$, for some $\kappa > 1$ and every N-smooth number n. Then*

$$c_j \ll 1, \text{ for any } 0 \le j \le J,$$

where the implicit constant may depend on κ, A_1 and the implicit constant in (4.2) *and we take N large enough with respect to these parameters.*

Proof. It is clear that c_0 is uniformly bounded in N and α. Indeed, since by hypothesis $f(p^k) \le d_\kappa(p^k)$, for any prime $p \le N$ and integer $k \ge 0$, either

$c_0 = 0$ or we can write

$$c_0 = \exp\left(\sum_{p\le N}\left(\frac{f(p)-\alpha}{p} + O_\kappa\left(\frac{1}{p^2}\right)\right)\right) \asymp 1,$$

by partial summation from (4.2), where the implicit constant may depend on κ, A_1 and the implicit constant in (4.2) and we take N large enough with respect to these parameters.

It is not that straightforward though to show that each c_j, for $j \ge 1$, is uniformly bounded in N and α. To this aim we employ the following procedure borrowing some ideas from the discussion in [17, Sect. 2]. Since c_j is the j-th derivative at $z = 1$ of the product between $H(z) := \zeta_N(z)^{-\alpha}F(z)$ and $Z_\alpha(z) := ((z-1)\zeta(z))^\alpha/z$, we only need to show that all the l-derivatives of $H(z)$ at $z = 1$ are uniformly bounded, for any $l \le J$. Indeed, this is certainly true for all the m-derivatives of $Z_\alpha(z)$ at $z = 1$, for any $m \le J$, and we have

$$(H(z)Z_\alpha(z))^{(j)}(1) = \sum_{l+m=j}\binom{j}{l}H^{(l)}(1)Z_\alpha^{(m)}(1).$$

We have $F(z) := F_1(z)F_2(z)$, where

$$F_1(z) := \sum_{\substack{n\ge1:\\ p|n\Rightarrow p\le N}}\frac{d_f(n)}{n^z} = \prod_{p\le N}\left(1-\frac{1}{p^z}\right)^{-f(p)},$$

where $d_f(n)$ is the multiplicative function satisfying

$$d_f(p^k) := \binom{f(p)+k-1}{k}$$

over all the prime powers p^k, with $p \le N$, and

$$F_2(z) := \sum_{\substack{n\ge1:\\ p|n\Rightarrow p\le N}}\frac{R_f(n)}{n^z},$$

with $f(n) =: d_f * R_f(n)$. Since R_f is supported only on square-full integers,

$|R_f(n)| \le d_{2\kappa}(n)$ and for every $l \ge 0$ we have

$$F_2^{(l)}(1) = \sum_{\substack{n\ge 1:\\ p|n \Rightarrow p\le N}} \frac{R_f(n)(-\log n)^l}{n},$$

it is clear that all the derivatives of F_2 at $z=1$ are uniformly bounded.

Arguing similarly as before, we are left with showing that all the derivatives of

$$H_1(z) := \zeta_N(z)^{-\alpha} F_1(z) = \prod_{p\le N} \left(1 - \frac{1}{p^z}\right)^{-f(p)+\alpha}$$

at $z=1$ are uniformly bounded.

To this aim, for any $1 \le l \le J$ we use the Faà di Bruno's formula [64, p. 807, Theorem 2] to find

$$H_1^{(l)}(1) = H_1(1) l! \sum_{m_1+2m_2+\ldots+lm_l=l} \frac{\prod_{i=1}^{l} (h^{(i-1)}(1))^{m_i}}{1!^{m_1}\, m_1!\, 2!^{m_2}\, m_2! \cdots l!^{m_l}\, m_l!}, \tag{4.29}$$

where

$$h(z) := \frac{H_1'}{H_1}(z) = \sum_{p\le N} (\alpha - f(p)) \log p \sum_{k=0}^{\infty} \frac{d_{\alpha-f}(p^k)}{p^{(k+1)z}},$$

where as before $d_{\alpha-f}(n)$ is the multiplicative function satisfying

$$d_{\alpha-f}(p^k) := \binom{\alpha - f(p) + k - 1}{k}$$

over all the prime powers p^k, with $p \le N$. From this we deduce that

$$\begin{aligned} h^{(i-1)}(1) &= \sum_{p\le N}(\alpha - f(p))(\log p)^i \sum_{k=0}^{\infty} \frac{d_{\alpha-f}(p^k)(-k-1)^{i-1}}{p^{(k+1)}} \\ &= (-1)^{i-1}\sum_{p\le N}\frac{(\alpha - f(p))(\log p)^i}{p} \\ &+ \sum_{p\le N}(\alpha - f(p))(\log p)^i \sum_{k=1}^{\infty} \frac{d_{\alpha-f}(p^k)(-k-1)^{i-1}}{p^{(k+1)}} \\ &= (-1)^{i-1}\sum_{p\le N}\frac{(\alpha - f(p))(\log p)^i}{p} + O_{i,\kappa}\Bigg(\sum_{p\le N}\frac{(\log p)^i}{p^2}\Bigg) \\ &= (-1)^{i-1}\sum_{p\le N}\frac{(\alpha - f(p))(\log p)^i}{p} + O_{i,\kappa}(1), \end{aligned}$$

where we used that $|d_{\alpha-f}(n)| \le d_{2\kappa}(n)$. The last sum above can be estimated with a partial summation argument from (4.2), for any $1 \le i \le J$. We thus find $|h^{(i-1)}(1)| \ll 1$, with an implicit constant depending on κ, A_1, i and the implicit constant in (4.2). Inserting this into (4.29) also gives $H^{(l)}(1) \ll 1$, with now an implicit constant depending on κ, A_1, l and the implicit constant in (4.2), since we can prove that $H(1)$ is uniformly bounded in much the same way as we did for c_0. Together with previous considerations this concludes the proof of the lemma. □

The previous lemma shows that the coefficients in the asymptotic expansion (4.27) are well defined and indeed uniformly bounded independently of N and α, for given A_1. This will also turn out to be useful in several future applications of Theorem 4.2.3, in which in order to make sure that the first term in the asymptotic expansion (4.27) dominates, we will need a careful control on the other terms. Together with previous observations, it proves this version of [17, Theorem 1]. □

We are going to apply the above theorem to prove its slight variation about sums restricted to those integers up to x coprime with a parameter q that satisfies certain nice properties. This is the content of Theorem 4.1.7, where in the statement we take C as in condition (4) on q.

Proof of Theorem 4.1.7. To begin with, let us define an auxiliary multiplicative function $\tilde{f}$ such that

$$\tilde{f}(p^j) := \begin{cases} f(p^j) & \text{if } p \nmid q; \\ f(p)^j & \text{otherwise.} \end{cases}$$

Then we may rewrite the sum in question as

$$\begin{aligned} \sum_{\substack{n\le x \\ (n,q)=1}} \tilde{f}(n) = \sum_{n\le x} \tilde{f}(n) \sum_{d|n,d|q} \mu(d) &= \sum_{d|q} \mu(d) \sum_{\substack{n\le x \\ d|n}} \tilde{f}(n) \\ &= \sum_{d|q} \mu(d) \sum_{k\le x/d} \tilde{f}(dk) \\ &= \sum_{\substack{d|q \\ d\le x/2}} \mu(d) \sum_{k\le x/d} \tilde{f}(dk) + \sum_{\substack{d|q \\ x/2<d\le x}} \mu(d)\tilde{f}(d). \end{aligned} \tag{4.30}$$

The completely multiplicative structure of $\tilde{f}$ on the numbers divisible only by prime factors of q allows us to rewrite the first double sum in (4.30) as

$$\sum_{\substack{d|q \\ d\le x/2}} \mu(d)\tilde{f}(d) \sum_{k\le x/d} \tilde{f}(k). \tag{4.31}$$

Moreover, since $\tilde{f}$ equals f on the primes, we have

$$\sum_{p\le x} \tilde{f}(p)\log p = \sum_{p\le x} f(p)\log p$$

and it is not difficult to show that the two conditions (4.28) hold for $\tilde{f}$ as well, if $C > \kappa^2$. Thus, an application of Theorem 4.2.3 leads to an evaluation of (4.31) as

$$\begin{aligned} &= x\sum_{l=0}^{J} \frac{\tilde{c}_l}{\Gamma(\alpha-l)} \sum_{\substack{d|q \\ d\le x/2}} \frac{\mu(d)\tilde{f}(d)}{d}(\log(x/d))^{\alpha-l-1} \\ &+ O\Bigg(x \sum_{\substack{d|q \\ d\le x/2}} \frac{|\tilde{f}(d)|}{d}(\log(x/d))^{\kappa-A_1-1}(\log\log x)\Bigg), \end{aligned} \tag{4.32}$$

where analogously to the definition of c_l we define

$$\tilde{c}_l := \frac{1}{l!}\frac{d^l}{dz^l}\left(\zeta_N(z)^{-\alpha}\tilde{F}(z)\frac{((z-1)\zeta(z))^\alpha}{z}\right)_{z=1},$$

with

$$\tilde{F}(z) := G_q(z)^{-1}\prod_{\substack{p\le N\\ p\nmid q:}}\sum_{k=0}^{\infty}\frac{f(p^k)}{p^{kz}} \text{ and } G_q(z) := \prod_{p|q}\left(1-\frac{f(p)}{p^z}\right).$$

The second double sum in (4.30) instead is upper bounded by

$$\ll x^{3/4}\sum_{d|q}\frac{|\tilde{f}(d)|}{d^{3/4}} \le x^{3/4}\sum_{d|q}\frac{d_\kappa(d)}{d^{3/4}}, \tag{4.33}$$

since q is squarefree. We see that we may rewrite $\tilde{F}(z)$ as $F(z)G_q(z)^{-1}H_q(z)^{-1}$.

Hence, $\tilde{c}_l$ will be

$$\tilde{c}_l = \sum_{k=0}^{l}\frac{\frac{d^k}{dz^k}(G_q(z)^{-1}H_q(z)^{-1})|_{z=1}}{k!}c_{l-k}.$$

By Lemma 4.2.4 each $c_{l-k}(\alpha)$ is uniformly bounded by a constant depending on κ, A_1 and the implicit constant in (4.2). The coefficients $\tilde{c}_l$ may potentially depend on q. However, we have

$$\begin{aligned} G_q(z)H_q(z) &= \prod_{p|q}\left(1-\frac{f(p)}{p^z}\right)\left(1+\frac{f(p)}{p^z}+\frac{f(p^2)}{p^{2z}}+\cdots\right)\\ &= \prod_{p|q}\left(1+O_\kappa\left(\frac{1}{p^{2\Re(z)}}\right)\right), \end{aligned}$$

as we can see from: $|f(p^j)-f(p^{j-1})f(p)|\le(\kappa+1)(d_\kappa(p^j)+d_\kappa(p^{j-1}))$, for any $j\ge 2$.

We deduce that $G_q(z)H_q(z)$ defines a non-vanishing analytic function on $\Re(z)\ge 1$ and so does its inverse. This shows the possibility to estimate the coefficient $\tilde{c}_l$ with a bound free on the dependence of q. Another way to show this could be to argue as in the proof of Lemma 4.2.4, because $(G_q(z)H_q(z))^{-1}$ coincides with the Dirichlet series of a function with a controlled growth and supported only on square-full integers.

Let us now focus on studying the sums over d in the main term of (4.32). By the generalized binomial expansion (see e.g. the first paragraph in Ch. II.5 of [69]), we find for any $0 \le l \le J$

$$\sum_{\substack{d|q\\ d\le x/2}} \frac{\mu(d)\tilde{f}(d)}{d}(\log(x/d))^{\alpha-l-1} \tag{4.34}$$
$$= (\log x)^{\alpha-l-1}\sum_{h=0}^{J-l}\frac{\binom{\alpha-l-1}{h}(-1)^h}{\log^h x}\sum_{\substack{d|q\\ d\le x/2}}\frac{\mu(d)\tilde{f}(d)}{d}\log^h d$$
$$+ O_{\kappa,J}\biggl((\log x)^{\Re(\alpha)-J-2}\sum_{d|q}\frac{|\tilde{f}(d)|}{d}(\log d)^{J-l+1}\biggr).$$

Completing the above sums to all the divisors of q gives an error in (4.34) of at most

$$\ll_{\kappa,J} 2^E(\log x)^{\Re(\alpha)-l-1-E}\sum_{d|q}\frac{|\tilde{f}(d)|}{d}(\log d)^{J-l+E},$$

for any $E>0$, since $x/2 \ge \sqrt{x}$ on $x \ge 4$. Similarly, the error term in (4.32) can be estimated with

$$\ll_{\kappa,A_1} (\log x)^{\kappa-A_1-1}(\log\log x)\biggl(\sum_{d|q}\frac{|\tilde{f}(d)|}{d}+\sum_{d|q}\frac{|\tilde{f}(d)|}{d}\frac{\log d}{\log x}\biggr). \tag{4.35}$$

Next, since q is squarefree we have

$$\sum_{d|q}\frac{|\tilde{f}(d)|}{d^z}=\prod_{p|q}\left(1+\frac{|f(p)|}{p^z}\right)=\tilde{G}_q(z)$$

and we can rewrite (4.34) as

$$(4.36) \qquad = (\log x)^{\alpha-l-1} \sum_{h=0}^{J-l} \frac{\binom{\alpha-l-1}{h}}{\log^h x} G_q^{(h)}(1) + O_{\kappa,J}\Big((\log x)^{\Re(\alpha)-J-2} |\tilde{G}_q^{(J-l+1)}(1)| \Big) + O_{\kappa,J}\Big(2^E (\log x)^{\Re(\alpha)-l-1-E} |\tilde{G}_q^{(J-l+E)}(1)| \Big)$$

and (4.35) as

$$(4.37) \qquad \ll_{\kappa,A_1} (\log x)^{\kappa-A_1-1} (\log\log x) \bigg(|\tilde{G}_q(1)| + \frac{|\tilde{G}_q^{(1)}(1)|}{\log x} \bigg).$$

Inserting (4.36) and (4.37) into (4.32) and rearranging, we have overall found

$$(4.38) \qquad \sum_{\substack{n\leq x\\(n,q)=1}} f(n) = x \sum_{j=0}^{J} \frac{(\log x)^{\alpha-j-1}}{\Gamma(\alpha-j)} \sum_{h+l=j} \frac{G_q^{(h)}(1)\tilde{c}_l}{h!} + O\bigg(x(\log x)^{\Re(\alpha)-J-2} \sum_{l=0}^{J} \frac{|\tilde{c}_l \tilde{G}_q^{(J-l+1)}(1)|}{|\Gamma(\alpha-l)|} \bigg) + O\bigg(2^E x (\log x)^{\Re(\alpha)-1-E} \sum_{l=0}^{J} \frac{|\tilde{c}_l \tilde{G}_q^{(J-l+E)}(1)|}{|\Gamma(\alpha-l)|(\log x)^l} \bigg) + O\bigg(x(\log x)^{\kappa-A_1-1}(\log\log x) \bigg(|\tilde{G}_q(1)| + \frac{|\tilde{G}_q^{(1)}(1)|}{\log x} \bigg) \bigg) + O\bigg(x^{3/4} \sum_{d|q} \frac{d_\kappa(d)}{d^{3/4}} \bigg).$$

By definition of $\tilde{c}_l$, the j-th coefficient in the sum in the main term in the displayed equation above can be rewritten as

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha-j)}\sum_{h=0}^{j}\frac{G_q^{(h)}(1)}{h!}\sum_{k=0}^{j-h}\frac{(G_q(z)^{-1}H_q(z)^{-1})^{(k)}(1)}{k!}c_{j-h-k}\\
&=\frac{1}{\Gamma(\alpha-j)}\sum_{l=0}^{j}c_{j-l}\sum_{k+h=l}\frac{G_q^{(h)}(1)}{h!}\frac{(G_q(z)^{-1}H_q(z)^{-1})^{(k)}(1)}{k!}\\
&=\frac{1}{\Gamma(\alpha-j)}\sum_{l=0}^{j}\frac{c_{j-l}}{l!}(H_q(z)^{-1})^{(l)}(1)\\
&=\frac{1}{\Gamma(\alpha-j)}\sum_{l+h=j}\frac{(H_q^{-1})^{(l)}(1)c_h}{l!}
\end{aligned}$$

and in this way is presented as in the statement of the theorem.

Regarding the error term instead, by Lemma 4.2.4 and previous considerations, we can prove that all the coefficients $\tilde{c}_l$ are uniformly bounded, thus finding an upper bound of

$$\begin{aligned}
&\ll_{\kappa,J}|\tilde{G}_q^{(J+1)}(1)|x(\log x)^{\Re(\alpha)-J-2}+2^E|\tilde{G}_q^{(J+E)}(1)|x(\log x)^{\Re(\alpha)-1-E}\\
&+(|\tilde{G}_q(1)|+|\tilde{G}_q^{(1)}(1)|)x(\log x)^{\kappa-A_1-1}(\log\log x)+x^{3/4}\sum_{d|q}\frac{d_\kappa(d)}{d^{3/4}}.
\end{aligned}$$

Here we also used the continuity of $\Gamma(\alpha-l)^{-1}$ over the compact set $|\alpha|\leq\kappa$. Moreover, since we have

$$|\tilde{G}_q^{(h)}(1)|=\sum_{d|q}\frac{|\tilde{f}(d)|\log^h d}{d},$$

for any $h\geq 0$, it is clear that $|\tilde{G}_q^{(a)}(1)|\leq|\tilde{G}_q^{(b)}(1)|$, for any $0\leq a\leq b$. Thanks to this last inequality we may simplify the error term in (4.38) further to

$$\ll_{\kappa,A_1}|\tilde{G}_q^{(2\lfloor A_1\rfloor+2)}(1)|x(\log x)^{\kappa-A_1-1}(\log\log x)+x^{3/4}\sum_{d|q}\frac{d_\kappa(d)}{d^{3/4}},$$

if we let $E:=\lfloor A_1\rfloor+2\geq A_1+1+|\Re(\alpha)|-\kappa$. Therefore, also the error term above is in the form contained in the statement of the theorem, thus concluding its proof. $\square$

4.2.4 Some partial sums over integers sharing all their prime factors with q

To deal with certain error terms arising from our computation of the partial sums of a generalized divisor function twisted with Ramanujan sums, we will make several times use of the next lemma.

Lemma 4.2.5. *For all positive integers $q \in \mathcal{A}$, as in condition* (6)*, we have*

$$\sum_{p|b \Rightarrow p|q} \frac{d_\kappa(b)(q,b)}{b^{3/4}} \ll_{\kappa,D} q^{1/4} d_{\kappa+1}(q), \quad \sum_{p|b \Rightarrow p|q} \frac{d_\kappa(b)(q,b)}{b} \ll_{\kappa,D} d_{\kappa+1}(q).$$

Proof. The first sum in the statement is upper bounded by

$$\sum_{e|q} e \sum_{\substack{e|b \\ p|b \Rightarrow p|q}} \frac{d_\kappa(b)}{b^{3/4}}. \tag{4.39}$$

Since $\kappa > 1$, it is

$$\leq \sum_{e|q} e^{1/4} d_\kappa(e) \sum_{\substack{f \\ p|f \Rightarrow p|q}} \frac{d_\kappa(f)}{f^{3/4}} \leq q^{1/4} d_{\kappa+1}(q) \prod_{p|q} \left(1 - \frac{1}{p^{3/4}}\right)^{-\kappa}$$
$$\ll_{\kappa,D} q^{1/4} d_{\kappa+1}(q).$$

Similarly the second sum in the statement is bounded by

$$\sum_{e|q} e \sum_{\substack{e|b \\ p|b \Rightarrow p|q}} \frac{d_\kappa(b)}{b}, \tag{4.40}$$

which is

$$\leq \sum_{e|q} d_\kappa(e) \sum_{\substack{f \\ p|f \Rightarrow p|q}} \frac{d_\kappa(f)}{f} \leq d_{\kappa+1}(q) \prod_{p|q} \left(1 - \frac{1}{p}\right)^{-\kappa} \ll_{\kappa,D} d_{\kappa+1}(q). \qquad \square$$

4.2.5 Proof of Proposition 4.1.5

Let $\Phi(t)$ indicate a smooth function, compactly supported in $[0,1]$, with $0 \leq \Phi(t) \leq 1$ for all $0 \leq t \leq 1$. Let g be a multiplicative function supported on the squarefree integers and zero on all the prime numbers smaller than C, where C is as in condition (4) on q; on the prime numbers $C < p \leq N$ we put instead $g(p) = f(p) - 1$, if $\Re(\alpha) \geq 1$, and $g(p) = 1 - f(p)$, otherwise.

The aim here is to prove Proposition 4.1.5, which is a variation of [28, Proposition 3], which we next report.

Proposition 4.2.6. *Keep notations as above and assume that $KQ_0 \leq R \leq \sqrt{N}$ and $|f(n)| \ll_\epsilon N^\epsilon$ for any $\epsilon > 0$ when $n \leq N$. Moreover, let*

$$\mathcal{S}_f(\theta) := \sum_{n \leq N} f(n) e(n\theta) \text{ and } \mathcal{G}(\theta) := \sum_{n \leq N} \Bigg(\sum_{\substack{r|n \\ r \leq R}} g(r) \Bigg) e(n\theta).$$

Then

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta \geq \sum_{KQ_0 < q \leq R} \Bigg| \sum_{\substack{r \leq R \\ q|r}} \frac{g(r)}{r} \Bigg| \Bigg| \sum_{n \leq N} f(n) c_q(n) \Phi(n/N) \Bigg| + O_{\epsilon,\Phi}(\Delta R N^{\frac{1}{2}+\epsilon}),$$

where $\Delta = \max_{r \leq R} |g(r)|$ and $\Phi(t)$ is as in [28, Proposition 3].

Next, we show that there exists a smooth function $\Phi(t)$, satisfying the hypotheses in [28, Proposition 3], for which we can remove in our set up the smooth cut-off in the sum above.

Proof of Proposition 4.1.5. Let $\Psi(t) : \mathbb{R} \to [0,1]$ be a smooth function compactly supported on $[-1,1]$ with

$$\int_{\mathbb{R}} \Psi(t) dt := 1.$$

Then consider the following convolution

$$\Phi(t) := T \mathbf{1}_{[1/T, 1-1/T]}(t) * \Psi(Tt) = T \int_{1/T}^{1-1/T} \Psi(T(s-t)) ds,$$

for any real number $T \geq 4$. A quick analysis of this integral reveals that $\Phi(t)$ is a smooth function such that

$$\Phi(t) = \begin{cases} 1 & \text{if } 2/T \leq t \leq 1 - 2/T; \\ \in [0,1] & \text{if } 1 - 2/T \leq t \leq 1 \text{ or } 0 \leq t \leq 2/T; \\ 0 & \text{if } t \geq 1 \text{ or } t \leq 0. \end{cases}$$

In particular, $\Phi(t)$ is a smooth function, compactly supported in $[0, 1]$, with $0 \leq \Phi(t) \leq 1$ for all $0 \leq t \leq 1$, and with

$$\int_0^1 \Phi(t)dt \geq 1 - \frac{4}{T}.$$

It is easy to see that

$$\Phi^{(k)}(t) \ll T^k ||\Psi^{(k)}||_{L^1},$$

for every $k \geq 0$. Let

$$F(\xi) := \int_{\mathbb{R}} \Phi(t) e^{-2\pi i \xi t} dt$$

be the Fourier transform of $\Phi(t)$. Then F is continuous and $F(0) = \int_{\mathbb{R}} \Phi(t)dt < \infty$. Moreover, by using k times integration by parts and the definition of $\Phi(t)$ we immediately deduce that

$$\begin{aligned} F(\xi) &= \frac{1}{(2\pi i \xi)^k} \int_{1-2/T}^1 \Phi^{(k)}(t) e^{-2\pi i \xi t} dt + \frac{1}{(2\pi i \xi)^k} \int_0^{2/T} \Phi^{(k)}(t) e^{-2\pi i \xi t} dt \\ &\ll \frac{T^{k-1} ||\Psi^{(k)}||_{L^1}}{(2\pi |\xi|)^k}, \end{aligned}$$

where the implicit constant is absolute. In particular, we get for all $\xi \in \mathbb{R}$ that $F(\xi) \ll T(1 + |\xi|)^{-2}$. This is equivalent to say that Φ as defined satisfies the conditions of the smooth weight introduced in [28]. Moreover, observe that in the proof of [28, Proposition 3], which corresponds to Proposition 4.2.6 here, it was only used the bound $F(\xi) \ll (1 + |\xi|)^{-2}$, where the implicit constant here is directly proportional to that in the error term of Proposition 4.2.6. Therefore, we may conclude that this last one is indeed

$$\ll_\epsilon T \Delta R N^{1/2+\epsilon}.$$

For any $4 \le T \le \sqrt{N}$, we may write

$$\sum_{n\le N} f(n)c_q(n)\Phi\left(\frac{n}{N}\right) = \sum_{n\le N} f(n)c_q(n) + O\Bigg(\sum_{N(1-2/T)<n\le N} d_\kappa(n)(q,n) + \sum_{n\le 2N/T} d_\kappa(n)(q,n)\Bigg).$$

We can estimate the first sum in the big-Oh term with

$$\ll \sum_{e|q} ed_\kappa(e) \sum_{N(1-2/T)/e<l\le N/e} d_\kappa(l) \ll_\delta d_{\kappa+1}(q)\frac{N}{T\log N}\exp\left(\sum_{p\le N}\frac{\kappa}{p}\right) \ll_{\delta,\kappa} d_{\kappa+1}(q)\frac{N(\log N)^{\kappa-1}}{T},$$

for every $4 \le T \le \sqrt{N}$, using Shiu's theorem [66, Theorem 1], Mertens' theorem and considering δ small enough. On average over q in Proposition 4.1.5, by upper bounding $|g(r)|\le d_{\kappa+1}(r)$, it will contribute

$$\ll_{\delta,\kappa} \frac{N(\log N)^{\kappa-1}}{T}\sum_{q\le R}\frac{d_{\kappa+1}(q)d_{\kappa+1}(q)}{q}\sum_{r\le R}\frac{d_{\kappa+1}(r)}{r} \ll_\kappa \frac{N(\log N)^{(\kappa+1)^2+2\kappa}}{T} = \frac{N(\log N)^{\kappa^2+4\kappa+1}}{T},$$

say, for any $4 \le T \le \sqrt{N}$. The second sum in the big-Oh term above can be estimated similarly, but replacing the application of Shiu's theorem with an application of Lemma 2.1.1, and gives the same contribution.

Finally, observe that Δ satisfies $\Delta \ll R^{O_\kappa(1/\log\log R)}$ (see e.g. [69, Ch. I, Theorem 5.4] for the case of d_2, which can be easily generalized to a general d_κ). Then, it is easy to see that letting $\epsilon := \delta/4$, say, the error term in Proposition 4.2.6 becomes

$$\ll_\delta TN^{1-\delta/4+O_\kappa(1/\log\log N)} \le TN^{1-\delta/5} \le N^{1-\delta/10},$$

if N is large enough in terms of δ and κ, by letting $T := N^{\delta/10}$. Putting the above considerations together we can now deduce Proposition 4.1.5 from Proposition 4.2.6. $\square$

4.3 Proof of corollaries of Theorem 4.1.2

This section is devoted to the proof of some applications of our main theorem that have already been mentioned in the introduction to this chapter.

Proof of Proposition 4.1.1. We have already noticed in the introduction to this chapter that $V(N, Q; d_0)$ has size Q. For a divisor function d_α, with $\alpha \notin -\mathbb{N} \cup \{0, 1\}$, Theorem 4.1.2 can be applied.

Notice than that $c_0 = 1$ as we can see from the following identities:

$$\sum_{k\geq 0} \frac{d_\alpha(p^k)}{p^k} = \left(1 - \frac{1}{p}\right)^{-\alpha},$$

for any prime p and any complex number α. Thus, assumption (4.6) is satisfied. Moreover, since $\alpha \notin -\mathbb{N} \cup \{0, 1\}$ and $\beta = |\alpha - 1|^2 > 0$ are constant and equations (4.2)–(4.3) are satisfied with any $A_1, A_2 > 0$, also the relations (4.5) hold. Theorem 4.1.2 now gives the thesis. □

Proof of Corollary 4.1.3. The function $\mathbf{1}_S$ is multiplicative and satisfies (4.2) and (4.3) with $\alpha = \beta = 1/2$ and any $A_1, A_2 > 0$, by the Prime Number Theorem for the arithmetic progression 1 (mod 4) (see e.g. [55, Corollary 11.20]). Thus, the inequalities (4.5) hold. By Mertens' theorem for the arithmetic progressions 1 (mod 4) and 3 (mod 4) (see e.g. [55, Corollary 4.12]) we have $c_0 \gg 1$, thus implying assumption (4.6). We again conclude by using (4.7). □

4.4 The error terms in Proposition 4.1.4

Proposition 4.1.4 is our first step to deduce the lower bound for the variance of a generalized divisor function in arithmetic progressions as in Theorem 4.1.2. To this aim, we need to work out the size of the error terms there. We start here with the first one. An application of [55, Theorem 2.14] leads to

$$\begin{aligned}\sum_{n\leq N} |f(n)|^2 &\ll_\kappa \frac{N}{\log N} \sum_{n\leq N} \frac{|f(n)|^2}{n} \\ &\leq \frac{N}{\log N} \prod_{p\leq N} \left(1 + \frac{|f(p)|^2}{p} + \frac{|f(p^2)|^2}{p^2} + \cdots\right).\end{aligned}$$

By Mertens' theorem we deduce

$$\sum_{n\le N}|f(n)|^2 \ll_\kappa c_0(|f|^2,\beta+2\Re(\alpha)-1)N(\log N)^{\beta+2\Re(\alpha)-2}, \tag{4.41}$$

where

$$c_0(|f|^2,\beta+2\Re(\alpha)-1)=\prod_{p\le N}\left(1+\frac{|f(p)|^2}{p}+\frac{|f(p^2)|^2}{p^2}+\cdots\right)\left(1-\frac{1}{p}\right)^{\beta+2\Re(\alpha)-1}$$

is a positive constant. In particular, it is uniformly bounded in terms of κ, A_1, A_2 and the implicit constants in (4.2)–(4.3), as we may see by applying partial summation from (4.2)–(4.3) and considering the relations (4.5). In conclusion, the first error term in (4.13) is

$$\ll \frac{KN^2(\log N)^{\beta+2(\Re(\alpha)-1)}}{Q_0}, \tag{4.42}$$

with the implicit constant depending on all the aforestated parameters.

We now turn to the estimate of the second error term in (4.13).

Proposition 4.4.1. *We have*

$$\sum_{q\le Q}\frac{1}{q}\sum_{\substack{d|q\\ d>Q_0}}\frac{1}{\varphi(d)}\Big|\sum_n f(n)c_d(n)\Big|^2 \ll_\kappa \frac{N^2(\log N)^{\kappa^2+4\kappa+2}}{Q_0}. \tag{4.43}$$

Proof. We initially observe that

$$\Big|\sum_{n\le N} f(n)c_d(n)\Big| \le \sum_{n\le N} d_\kappa(n)\sum_{e|(n,d)} e \le \sum_{e|d} e\sum_{\substack{n\le N\\ e|n}} d_\kappa(n), \tag{4.44}$$

by (4.11). Since $\kappa > 1$

$$\sum_{\substack{n\le N\\ e|n}} d_\kappa(n)=\sum_{k\le N/e} d_\kappa(ek)\le \frac{d_\kappa(e)}{e}N\sum_{k\le N/e}\frac{d_\kappa(k)}{k} \ll_\kappa \frac{N}{e}d_\kappa(e)(\log N)^\kappa,$$

by Lemma 2.1.1, which inserted in (4.44) gives

$$\Big|\sum_{n\le N} f(n)c_d(n)\Big|^2 \ll_\kappa N^2(\log N)^{2\kappa} d_{\kappa+1}^2(d).$$

From this we deduce that the left hand side in (4.43) is

$$\begin{aligned}\ll_\kappa N^2(\log N)^{2\kappa} \sum_{Q_0<d\le Q} \frac{d_{\kappa+1}^2(d)}{\varphi(d)} \sum_{\substack{q\le Q\\ d|q}} \frac{1}{q} &\le N^2(\log N)^{2\kappa} \sum_{Q_0<d\le Q} \frac{d_{\kappa+1}^2(d)}{d\varphi(d)} \sum_{q\le Q/d} \frac{1}{q} \\ &\ll N^2(\log N)^{2\kappa+1} \sum_{Q_0<d\le Q} \frac{d_{\kappa+1}^2(d)}{d\varphi(d)} \\ &\ll_\kappa \frac{N^2(\log N)^{\kappa^2+4\kappa+2}}{Q_0},\end{aligned}$$

again by Lemma 2.1.1. □

4.5 The partial sum of a truncated generalized divisor function

To lower bound the integral in (4.13) we use Proposition 4.1.5. After the application of (4.14) we are left with two main task: lower bounding the integral $(\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta)^2$ and upper bounding $\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta$. In this section we deal with the second one. By Parseval's identity and the definition of $\mathcal{G}(\theta)$, we have

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \le \sum_{n\le N} \Big|\sum_{\substack{r|n\\ r\le R}} g(r)\Big|^2. \tag{4.45}$$

We now find an upper bound for the above sum, but before we state the next lemma which will be useful later.

Lemma 4.5.1. *Let $g(n)$ be a multiplicative function supported on squarefree integers such that $|g(n)| \le d_{\kappa+1}(n)$ and*

$$\sum_{p\le x} |g(p)|^2 \log p = \beta x + O\Big(\frac{x}{(\log x)^{A_2}}\Big) \quad (2\le x\le R), \tag{4.46}$$

with κ, β, A_2 *and* R *as usual. Then we have*

$$\sum_{q\le R} \frac{|g(q)|^2}{q}\Big(\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\Big)^2 \ll (\log N)^{\beta}, \tag{4.47}$$

with an implicit constant depending on κ, A_2 *and that in* (4.46).

Proof. Expanding the square out and swapping summation we find that the sum in (4.47) is

$$\begin{aligned}&\sum_{d_1,d_2\le R \text{ squarefree}} \frac{d_{\kappa+1}(d_1)d_{\kappa+1}(d_2)}{d_1^{3/4}d_2^{3/4}} \sum_{\substack{q\le R\\ q\equiv 0 \pmod{[d_1,d_2]}}} \frac{|g(q)|^2}{q}\\ &\le \sum_{d_1,d_2\le R}\frac{d_{\kappa+1}(d_1)d_{\kappa+1}(d_2)|g([d_1,d_2])|^2(d_1,d_2)}{d_1^{7/4}d_2^{7/4}}\sum_{k\le R}\frac{|g(k)|^2}{k},\end{aligned}$$

where $[a,b]$ stands for the least common multiple of integers a and b. The innermost sum is $\ll (\log N)^{\beta}$, by Lemma 2.1.1 and partial summation from (4.46), with an implicit constant depending on κ, A_2 and that of (4.46). On the other hand, the double sum over d_1, d_2 is

$$\le \sum_{d_1,d_2\le R}\frac{d_{\kappa+1}(d_1)^3 d_{\kappa+1}(d_2)^3(d_1,d_2)}{d_1^{7/4}d_2^{7/4}} \le \sum_{e\le R}\frac{d_{\kappa+1}(e)^6}{e^{5/2}}\Big(\sum_k \frac{d_{\kappa+1}(k)^3}{k^{7/4}}\Big)^2.$$

Since

$$\sum_k \frac{d_{\kappa+1}(k)^3}{k^{7/4}} \ll_\kappa 1,$$

by using e.g. $d_{\kappa+1}(k) \ll_\kappa k^{3/24}$, we obtain that the final double sum above is

$$\ll_\kappa \sum_{e\le R}\frac{d_{\kappa+1}(e)^6}{e^{5/2}} \ll_\kappa 1.$$

Collecting the above estimate together we get (4.47). □

We are now ready to bound (4.45), whence proving Proposition 4.1.6.

Remark 4.5.2. *It is crucial to have a* sharp *upper bound for the sum in* (4.18) *to guarantee a sharp lower bound for the variance in arithmetic progressions. Indeed,* (4.18) *provides an upper bound for the integral in* (4.45) *which coincides* *with the denominator in* (4.14)*. Finding a sharp lower bound for the* L^2*–integral on the left-hand side of* (4.14) *is a key step towards proving Theorem 4.1.2.*

Proof. To begin with, we expand the square in (4.45) out and swap summations to find that the sum in (4.45) is

$$\leq N \sum_{r,r' \leq R} \frac{g(r)\bar{g}(r')}{[r,r']} + O\Big(\sum_{r \leq R} |g(r)|\Big)^2. \tag{4.48}$$

Regarding the error term in (4.48), we notice the sum may be upper bounded by

$$R \sum_{r \leq R} \frac{|g(r)|}{r} \ll_\kappa R(\log R)^{\kappa+1} \leq N^{1/2-\delta/2}(\log N)^{\kappa+1}, \tag{4.49}$$

by an application of Lemma 2.1.1.

Using a manipulation which traces back to the work of Dress, Iwaniec and Tenenbaum (see e.g. [6, Eq. 1]), we may rearrange the sum in the main term of (4.48) as

$$\begin{aligned} \sum_{r,r' \leq R} \frac{g(r)\bar{g}(r')}{rr'} \sum_{q \mid r, q \mid r'} \varphi(q) &= \sum_{q \leq R} \varphi(q) \Big| \sum_{\substack{r \leq R \\ q \mid r}} \frac{g(r)}{r} \Big|^2 \\ &= \sum_{q \leq R} \frac{\varphi(q)|g(q)|^2}{q^2} \Big| \sum_{\substack{k \leq R/q \\ (q,k)=1}} \frac{g(k)}{k} \Big|^2. \end{aligned} \tag{4.50}$$

We now need a careful estimate for the innermost sum in the second line of (4.50). We restrict first to the case $\Re(\alpha) \geq 1$. If $\alpha = 1$, we define the auxiliary multiplicative function $\tilde{g}$ such that

$$\tilde{g}(p^j) := \begin{cases} g(p^j) & \text{if } p \nmid q; \\ g(p)^j & \text{otherwise.} \end{cases}$$

In this way the innermost sum above may be rewritten as

$$\sum_{\substack{k\le R/q\\(q,k)=1}}\frac{\tilde g(k)}{k}=\sum_{d|q}\frac{g(d)\mu(d)}{d}\sum_{l\le R/dq}\frac{\tilde g(l)}{l},$$

since q is squarefree and arguing as at the start of the proof of Theorem 4.1.7. We notice that since $\alpha=1$, we have

$$\sum_{p\le x}\tilde g(p)\log p=\sum_{p\le x}g(p)\log p\ll x/\log^{A_1}x\quad(2\le x\le N).$$

By Theorem 4.2.3, we deduce that

$$\sum_{n\le x}\tilde g(l)\ll x(\log x)^{\kappa-A_1-1}\log\log x\quad(2\le x\le N).$$

By partial summation, remembering $A_1>\kappa+2$ from the hypothesis of Theorem 4.1.2, we get that

$$\sum_{l\le R/dq}\frac{\tilde g(l)}{l}\ll 1,\ \text{for any } d|q \text{ and } q\le R.$$

We then conclude that

$$\sum_{q\le R}\frac{\varphi(q)|g(q)|^2}{q^2}\Big|\sum_{\substack{k\le R/q\\(q,k)=1}}\frac{g(k)}{k}\Big|^2\ll\sum_{q\le R}\frac{|g(q)|^2}{q}\Big(\sum_{d|q}\frac{|g(d)|}{d}\Big)^2\ll(\log N)^{\beta},$$

for any $\beta>0$, with an implicit constant depending on κ,A_1,A_2 and that in (4.2)–(4.3). The last estimate follows from Lemma 4.5.1.

From now on we will work under the hypothesis $\alpha\neq 1$ and $\Re(\alpha)\ge 1$.

We first note that (4.50) is bounded by

$$\ll_\kappa \sum_{q\le R}\frac{|g(q)|^2}{q}+\sum_{q\le R/4}\frac{|g(q)|^2}{q}\Bigg|\sum_{\substack{4\le k\le R/q\\(q,k)=1}}\frac{g(k)}{k}\Bigg|^2 \tag{4.51}$$

$$= O_\kappa((\log N)^\beta)+\sum_{q\le R/4}\frac{|g(q)|^2}{q}\Bigg|\sum_{\substack{4\le k\le R/q\\(q,k)=1}}\frac{g(k)}{k}\Bigg|^2.$$

By Theorem 4.1.7, we have

$$\sum_{\substack{4\le k\le x\\(k,q)=1}} g(k) = x(\log x)^{\alpha-2}\sum_{j=0}^{J}\frac{\lambda_j}{(\log x)^j} \tag{4.52}$$

$$+\,O(|\tilde{G}_q^{(2\lfloor A_1\rfloor+2)}(1)|x(\log x)^{\kappa-A_1}(\log\log x)),$$

$$+\,O\Bigg(x^{3/4}\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\Bigg),$$

where

$$\lambda_j=\lambda_j(g,\alpha,q):=\frac{1}{\Gamma(\alpha-1-j)}\sum_{l+h=j}\frac{(H_q^{-1})^{(h)}(1)c_l}{h!}=:\frac{\lambda_j'}{\Gamma(\alpha-1-j)},$$

with

$$H_q(z):=\prod_{p|q}\left(1+\frac{g(p)}{p^z}\right),\quad c_l:=\frac{1}{l!}\frac{d^l}{dz^l}\left(\zeta_N(z)^{-(\alpha-1)}G(z)\frac{((z-1)\zeta(z))^{\alpha-1}}{z}\right)_{z=1}$$

and

$$G(z):=\sum_{\substack{n:\\p|n\Rightarrow p\le N}}\frac{g(n)}{n^z},\quad \tilde{G}_q(z):=\sum_{d|q}\frac{|g(d)|}{d^z}=\prod_{p|q}\left(1+\frac{|g(p)|}{p^z}\right)$$

on $\Re(z)\ge 1$. Here each $c_l=c_l(g,\alpha)$ is uniformly bounded on $|\alpha|\le\kappa$, thanks to an application of Lemma 4.2.4 with f replaced by g here and α by $\alpha-1$.

Using partial summation, we get

$$
\begin{aligned}
(4.53) \qquad & \Big| \sum_{\substack{4\le k\le R/q\\(k,q)=1}} \frac{g(k)}{k} \Big| \\
& \ll \frac{(\log(R/q))^{\Re(\alpha)-1}}{|\Gamma(\alpha)|}\bigg(\sum_{j=0}^{J}|\lambda_j'|+|\tilde{G}_q^{(2\lfloor A_1\rfloor+2)}(1)|+\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\bigg),
\end{aligned}
$$

where the implicit constant depends on κ, A_1 and the implicit constant in (4.2). Here we used that $\Gamma(\alpha)^{-1}$ is an entire function on the whole complex plane satisfying two main properties:

$$
|\Gamma(\alpha)|\le \Gamma(\Re(\alpha)) \text{ and } \Gamma(\alpha-l)=\frac{\Gamma(\alpha)}{(\alpha-l)\cdots(\alpha-1)},
$$

for any $l\ge 1$ and $\alpha\in\mathbb{C}$ such that $\Re(\alpha)\ge 1$ and $|\alpha|\le \kappa$. We can pretty easily deduce that $|\lambda_j'|\ll_{\kappa,j}\sum_{h=0}^{j}|(H_q^{-1})^{(h)}(1)|$. Likewise as in the proof of Theorem 4.1.7, we can write

$$
H_q^{-1}(z)=\prod_{p|q}\left(1-\frac{g(p)}{p^z}\right)\prod_{p|q}\left(1+\frac{g(p)}{p^z}\right)^{-1}\left(1-\frac{g(p)}{p^z}\right)^{-1}:=\tilde{H}_q(z)\tilde{\tilde{H}}_q(z)
$$

and show that we can bound all the derivatives of $\tilde{\tilde{H}}_q(z)$ with a constant independent of q. By linearity, all the derivatives of H_q^{-1} will be a linear combination with complex coefficients of those of $\tilde{H}_q$, which are given by

$$
(\tilde{H}_q)^{(h)}(1)=(-1)^h\sum_{d|q}\frac{\mu(d)g(d)}{d}(\log d)^h\ll\sum_{d|q}\frac{|g(d)|}{d}(\log d)^h,
$$

for any $0\le h\le J$. Hence

$$
\begin{aligned}
\sum_{j=0}^{J}|\lambda_j'|\ll_{\kappa,J}\sum_{j=0}^{J}\sum_{h=0}^{j}\sum_{d|q}\frac{|g(d)|}{d}(\log d)^h &\ll_{\kappa,J}\sum_{j=0}^{J}\sum_{d|q}\frac{|g(d)|}{d}((\log d)^j+1)\\
&\ll_{\kappa,J}\sum_{d|q}\frac{|g(d)|}{d}((\log d)^J+1).
\end{aligned}
$$

Thus, we deduce that (4.53) is

$$\ll \frac{(\log(R/q))^{\Re(\alpha)-1}}{|\Gamma(\alpha)|}\bigg(\sum_{d|q}\frac{|g(d)|}{d}((\log d)^{2A_1+2}+1)+\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\bigg),$$

where the implicit constant depends on κ, A_1 and that in (4.2). We conclude that (4.51) is

$$\begin{aligned}
&\ll (\log N)^{\beta} \\
&+ \frac{(\log N)^{2(\Re(\alpha)-1)}}{|\Gamma(\alpha)|^2}\sum_{q\le R}\frac{|g(q)|^2}{q}\bigg(\sum_{d|q}\frac{|g(d)|}{d}((\log d)^{2A_1+2}+1)+\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\bigg)^2 \\
&\ll \frac{(\log N)^{\beta+2(\Re(\alpha)-1)}}{|\Gamma(\alpha)|^2} \ll_{\kappa} (\log N)^{\beta+2(\Re(\alpha)-1)},
\end{aligned}$$

by Lemma 4.5.1, with an implicit constant depending on δ, κ, A_1, A_2 and that in (4.2)–(4.3). This concludes the proof when $\Re(\alpha) \ge 1$, since the error (4.48) will be negligible, thanks to (4.49).

When instead $\Re(\alpha) < 1$, by definition of g, we now get from Theorem 4.1.7 that

$$\begin{aligned}
\sum_{\substack{4\le k\le x\\(k,q)=1}} g(k) &= x(\log x)^{-\alpha}\sum_{j=0}^{J}\frac{\lambda_j}{(\log x)^j} \\
&+ O(|\tilde{G}_q^{(2\lfloor A_1\rfloor+2)}(1)|x(\log x)^{\kappa-A_1}(\log\log x)) \\
&+ O\bigg(x^{3/4}\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\bigg),
\end{aligned}$$

where

$$\lambda_j = \lambda_j(g,\alpha,q) := \frac{1}{\Gamma(1-\alpha-j)}\sum_{l+h=j}\frac{(H_q^{-1})^{(h)}(1)c_l}{h!} =: \frac{\lambda_j'}{\Gamma(1-\alpha-j)},$$

with

$$c_l := \frac{1}{l!}\frac{d^l}{dz^l}\bigg(\zeta_N(z)^{-(1-\alpha)}G(z)\frac{((z-1)\zeta(z))^{1-\alpha}}{z}\bigg)_{z=1}$$

and $G(z), \tilde{G}_q(z)$ and $H_q(z)$ defined as before. Again by partial summation we get

$$\Big|\sum_{\substack{4\le k\le R/q\\(k,q)=1}}\frac{g(k)}{k}\Big|\ll\frac{(\log(R/q))^{1-\Re(\alpha)}}{|\Gamma(2-\alpha)|}\bigg(\sum_{j=0}^{J}|\lambda_j'|+|\tilde{G}_q^{(2\lfloor A_1\rfloor+2)}(1)|+\sum_{d|q}\frac{d_{\kappa+1}(d)}{d^{3/4}}\bigg),$$

from which we can conclude as before, since all the other considerations and computations carry over exactly the same. □

4.6 The summation over r

The next step in the application of (4.15) consists in finding a sharp lower bound for the summation over r there, the content of which is Proposition 4.1.8. Before starting with the proof we insert here a lemma which will be useful later.

Lemma 4.6.1. *Let g be a multiplicative function supported on the squarefree numbers and such that $|g(n)|\le d_{\kappa+1}(n)$, for a certain real positive constant $\kappa>1$ and any N-smooth integer n. Assume moreover that $g(p)=0$, for any prime $p\le C$, and define the following Euler products*

$$H_q(z):=\prod_{p|q}\bigg(1+\frac{g(p)}{p^z}\bigg),\ \ \tilde{G}_q(z):=\prod_{p|q}\bigg(1+\frac{|g(p)|}{p^z}\bigg)\quad(\Re(z)\ge 1),$$

where q is a squarefree positive integer smaller than N satisfying conditions (4) and (6). Then for every positive integer h we have

$$\max\{|(H_q^{-1})^{(h)}(1)|,|\tilde{G}_q^{(h)}(1)|\}\ll_{h,\kappa,D}C^{-1/5},$$

if $C=C(\kappa,h)>\kappa+1$ is sufficiently large. Moreover, under our assumptions on q we also have

$$\max\{|H_q^{-1}(1)|,|\tilde{G}_q(1)|\}\asymp_{\kappa,D}1.$$

Proof. Let us focus on $\tilde{G}_q$, since similar computations also hold for H_q^{-1}. For values of $h\ge 1$ we use the Faà di Bruno's formula [64, p. 807, Theorem 2] to find

$$\tilde{G}_q^{(h)}(1) = \tilde{G}_q(1)h! \sum_{m_1+2m_2+\ldots+hm_h=h} \frac{\prod_{i=1}^{h}(\gamma_q^{(i-1)}(1))^{m_i}}{1!^{m_1}\, m_1!\, 2!^{m_2}\, m_2! \cdots h!^{m_h}\, m_h!},$$

where

$$\gamma_q(z) := \frac{\tilde{G}'_q}{\tilde{G}_q}(z) =: -\sum_{n\geq 1} \frac{\Lambda_{\tilde{g}}(n)}{n^z} = -\sum_{p|q}\sum_{k=1}^{\infty} \frac{\Lambda_{\tilde{g}}(p^k)}{p^{kz}}.$$

Here, we indicated with $\tilde{g}(n) := |g(n)|\mathbf{1}_{n|q}$ and defined $\Lambda_{\tilde{g}}(n)$ exactly as the n-th coefficient in the Dirichlet series corresponding to minus the logarithmic derivative of $\tilde{G}_q(z) = \sum_n \frac{\tilde{g}(n)}{n^z}$. Analysing the values of the $\Lambda_{\tilde{g}}$ function, we see that it is supported only on prime powers for primes dividing q. More precisely, on those powers we have the following relation

$$\Lambda_{\tilde{g}}(p^k) = (-1)^{k+1}|g(p)|^k \log p,$$

which in turn follows from

$$\tilde{g}(n)\log(n) = \Lambda_{\tilde{g}} * \tilde{g}(n), \text{ for any } n. \tag{4.54}$$

The above also shows that $\Lambda_{\tilde{g}}(p^k) = 0$ whenever $p \leq C$, by the support of g, and choosing $C = C(\kappa) > \kappa + 1$ large enough makes the series over k on $\Re(z) \geq 1$ convergent. We clearly obtain

$$\gamma_q^{(i)}(1) = -\sum_{p|q}\sum_{k=1}^{\infty} \frac{(-1)^{k+i+1}|g(p)|^k(k\log p)^{i+1}}{kp^k} \ll_{i,\kappa} \sum_{\substack{p|q\\ p>C}} \frac{(\log p)^{i+1}}{p}$$
$$\leq \frac{1}{C^{1/5}} \sum_{p|q} \frac{(\log p)^{i+1}}{p^{4/5}},$$

since $|g(p)|$ is uniformly bounded by $\kappa + 1 > 0$ and supported only on large primes. Remembering that $q \in \mathcal{A}$ by condition (6), we immediately deduce that the last sum is bounded for $i \leq h - 1$, implying that $\tilde{G}_q^{(h)}(1) \ll_{h,\kappa,D}$

$\tilde{G}_q(1)/C^{1/5}$, for any $h \geq 1$. However, $\tilde{G}_q(1)$ is itself bounded, because

$$\tilde{G}_q(1) = \exp\Bigg(\sum_{p|q} \frac{|g(p)|}{p} + O(1)\Bigg) = \exp(O_{\kappa,D}(1)) \asymp_{\kappa,D} 1,$$

by condition (6) and $C > \kappa + 1$. Similarly, we can show the same for all the derivatives of H_q^{-1}, by first showing that the bound for those of its logarithmic derivative coincides with the bound for the derivative of γ_q. Indeed, observe that we have

$$\frac{d}{dz}\log(H_q^{-1}(z)) = \sum_{p|q} \frac{g(p)\log p}{p^z + g(p)} = \sum_{p|q} g(p)\log p \sum_{k=0}^{\infty} \frac{(-g(p))^k}{p^{(k+1)z}},$$

where the series converges since $\Re(z) \geq 1$ and $p > C > \kappa + 1$. Therefore, its corresponding j-th derivative is

$$\sum_{p|q} g(p)(\log p)^{j+1} \sum_{k=0}^{\infty} \frac{(-g(p))^k(-k-1)^j}{p^{(k+1)z}},$$

from which by taking the absolute value we recover the analogous bound for $\gamma_q^{(j)}$. Finally, note that since g vanishes on the primes smaller than a large constant C, the product $H_q(1)$ is not zero. Moreover, we see that

$$H_q^{-1}(1) = \exp\Bigg(-\sum_{p|q} \frac{g(p)}{p} + O(1)\Bigg) = \exp(O_{\kappa,D}(1)) \asymp_{\kappa,D} 1,$$

again by condition (6) on q and since $C > \kappa + 1$. □

Proof of Proposition 4.1.8. First of all, note that

$$\sum_{\substack{r \leq R \\ q|r}} \frac{g(r)}{r} = \frac{g(q)}{q} \sum_{\substack{1 \leq r \leq R/q \\ (q,r)=1}} \frac{g(r)}{r} = \frac{g(q)}{q}\Bigg(1 + \sum_{\substack{C \leq r \leq R/q \\ (q,r)=1}} \frac{g(r)}{r}\Bigg), \tag{4.55}$$

since $g(1) = 1$ and g is supported on squarefree numbers larger than C.

In order to evaluate the last sum on the right hand side of (4.55) we again apply Theorem 4.1.7, as it was done in Proposition 4.1.6, to conclude with a partial summation argument. In this case our task is facilitated by restricting q to lie in the subset $\mathcal{A} \subset [KQ_0, RN^{-\delta/4}]$, as in condition (6). In particular, since $q \leq RN^{-\delta/4}$ we notice that $\log(R/q) \asymp_\delta \log N$ and the condition $q \in \mathcal{A}$ allows us to simplify the asymptotic expansion of the average of $g(n)$. However, since here we are looking for a lower bound, some difficulties arise when $\Re(\alpha)$ is near 1, for which we will need to invoke condition (4) on q and divide the argument into two different cases according to the size of $|\Re(\alpha) - 1|$.

We first restrict our attention to the case $\Re(\alpha) \geq 1$ in which case we can compute the average of g over the coprimality condition using Theorem 4.1.7. Assuming $C > 4$ sufficiently large, we obtain

$$\sum_{\substack{C \leq k \leq x \\ (k,q)=1}} g(k) = \sum_{j=0}^{J} \frac{\lambda'_j}{\Gamma(\alpha - 1 - j)} (\log x)^{\alpha-2-j} + O\Big(x(\log x)^{\kappa - A_1}(\log\log x)\Big),$$

where the big-Oh term depends on κ, A_1, D and the implicit constant in (4.2) and the λ'_j are as in (4.52). Here we simplified the expression in the error term by using Lemma 4.6.1 and noticing that

$$\sum_{d|q} \frac{d_\kappa(d)}{d^{3/4}} = \prod_{p|q}\left(1 + \frac{\kappa}{p^{3/4}}\right) = \exp\left(\sum_{p|q} \frac{\kappa}{p^{3/4}} + O_\kappa(1)\right) \ll_{\kappa,D} 1,$$

by the conditions (4) and (6) on q, if $C = C(\kappa, A_1) > \kappa^{4/3}$. Moreover, note that

$$\lambda'_0 = \prod_{p|q}\left(1 + \frac{g(p)}{p}\right)^{-1} \prod_{p \leq N}\left(1 + \frac{g(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{\alpha-1} \asymp 1, \tag{4.56}$$

with an implicit constant depending on κ, A_1, C, D and that in (4.2), since $g(p) = 0$ when $p \leq C$ with $C > \kappa + 1$ and using Lemma 4.6.1 and partial summation from (4.2), as at the start of the proof of Lemma 4.2.4. By partial summation and similar considerations to those employed in the proof of Proposition 4.1.6, remembering the hypothesis (4.5) on A_1, we deduce that

$$\Bigg|\sum_{\substack{C \leq r \leq R/q \\ (q,r)=1}} \frac{g(r)}{r}\Bigg| \geq E\left|\frac{\lambda'_0}{\Gamma(\alpha)}\right|(\log(R/q))^{\Re(\alpha)-1}, \tag{4.57}$$

where E depends on δ, κ, A_1, D and the implicit constant in (4.2), if we think of C as large enough in terms of δ, κ, A_1, D and take N sufficiently large to all of these parameters.

Now, write $\alpha = 1 + L/\log\log N + i\tau$, with $L \geq 0$, and suppose that $L > L_0$, where

$$L_0 := \min\left\{ l \in \mathbb{R}^+ : e^{l-1} \geq \frac{2}{E}\left|\frac{\Gamma(\kappa)}{\lambda_0'}\right| \right\}$$

depends on δ, κ, A_1, D and the implicit constant in (4.2). Then (4.57) is clearly

$$= E\left|\frac{\lambda_0'}{\Gamma(\alpha)}\right| e^{L+O(|\log\delta|/\log\log N)} \geq 2,$$

if we take N large enough. This, together with (4.56), concludes the proof in this subcase.

Suppose now $0 \leq L \leq L_0$. We remark that when τ is either 0 or a possibly small function of N and $\Re(\alpha)$ is suitably close to 1, the above partial summation argument could lose its efficiency. For this reason, a direct argument is needed, one where only the value of the $\Re(\alpha)$ counts. Hence, we start again from (4.55) and note that

$$\sum_{\substack{1\leq r\leq R/q \\ (q,r)=1}} \frac{g(r)}{r} = \sum_{(q,r)=1} \frac{g(r)}{r} - \sum_{\substack{r>R/q \\ (q,r)=1}} \frac{g(r)}{r}. \tag{4.58}$$

Moreover, the complete series in (4.58) converges, since it is equal to

$$\prod_{\substack{p\leq N \\ p\nmid q}} \left(1 + \frac{g(p)}{p}\right) = H_q^{-1}(1) \prod_{p\leq N} \left(1 + \frac{g(p)}{p}\right) \asymp 1,$$

with an implicit constant depending on κ, A_1, D and that in (4.2). Indeed, Lemma 4.6.1 gives

$$H_q^{-1}(1) \asymp_{\kappa,D} 1$$

and we have

$$\left|\prod_{p\le N}\left(1+\frac{g(p)}{p}\right)\right| = \exp\left(\Re\Big(\sum_{C<p\le N}\left(\frac{g(p)}{p}+O_\kappa\left(\frac{1}{p^2}\right)\right)\Big)\right)$$
$$\asymp_\kappa \exp\left(\sum_{C<p\le N}\frac{\Re(g(p))}{p}\right)\asymp 1,$$

with an implicit constant depending on δ, κ, A_1, D and that in (4.2), if C and N are sufficiently large in terms of those parameters. The last estimate follows through partial summation from (4.2).

Finally, since $R/q \ge N^{\delta/4}$, the tail of the series in (4.58) can be made arbitrary small if we choose N large enough. Therefore, we simply have

$$\sum_{\substack{1\le r\le R\\ q|r}}\frac{g(r)}{r}\gg\frac{|g(q)|}{q},$$

with an implicit constant depending on δ, κ, A_1, D and that in (4.2), if C and N are large enough in terms of those parameters. This matches the expression in (4.25), since

$$|\Gamma(1+L/\log\log N+i\tau)|\le\Gamma(1+L/\log\log N)\ll 1,$$

choosing N sufficiently large, by the continuity of $\Gamma(\alpha)$. This concludes the proof in the case $\Re(\alpha)\ge 1$.

In the complementary case, i.e. $\Re(\alpha) < 1$, we just note that g has average $1-\alpha\ne 0$ over the primes. All the above computations then carry over, with the opportune modifications already explained at the end of the proof of Proposition 4.1.6, and the overall result may be written as in (4.25). □

4.7 Twisting with Ramanujan's sums

By inserting the conclusion of Proposition 4.4.1 and estimate (4.41) in Proposition 4.1.4, so far we have found

$$\begin{aligned} V(N,Q;f) \geq Q\Big(1+O\Big(\frac{\log K}{K}\Big)\Big)\int_{\mathfrak{m}}|\mathcal{S}_f(\theta)|^2 d\theta \\ + O_{\kappa,K}\bigg(\frac{N^2(\log N)^{\kappa^2+4\kappa+2}}{Q_0}+\frac{N^2(\log N)^{\beta+2\Re(\alpha)-2}}{Q_0}\bigg), \end{aligned}$$

where $\int_{\mathfrak{m}}|\mathcal{S}_f(\theta)|^2 d\theta$ may be lower bounded using the results of Proposition 4.1.5, Proposition 4.1.6 and Proposition 4.1.8, with K a large constant. Hence, we have proved that $V(N,Q;f)$ is

$$\begin{aligned} &\gg \frac{Q}{N(\log N)^{\beta}} \\ &\times\bigg(\sideset{}{'}\sum_{KQ_0\leq q\leq RN^{-\delta/4}}\frac{|g(q)|}{q}\Big|\sum_{n\leq N}f(n)c_q(n)\Big|+O_{\delta,\kappa}(N^{1-\delta/11})\bigg)^2 \\ &+O_{\kappa}\bigg(\frac{N^2(\log N)^{\kappa^2+4\kappa+2}}{Q_0}+\frac{N^2(\log N)^{\beta+2\Re(\alpha)-2}}{Q_0}\bigg). \end{aligned} \tag{4.59}$$

Here, Σ' indicates a sum over all the squarefree $KQ_0 \leq q \leq RN^{-\delta/4}$ under the restrictions (4) and (6) and the $\gg$ constant may depend on $\delta, \kappa, A_1, A_2, D$ and the implicit constants in (4.2)–(4.3). Moreover, we are assuming N sufficiently large with respect to all of them as well as to C.

In this section we explain how to deal with the average of $f(n)$ twisted with Ramanujan's sums, which is indeed the heart of the proof of Theorem 4.1.2. We begin with the following observation:

$$\sum_{n\leq N}f(n)c_q(n)=\sum_{\substack{b\leq N\\ p|b\Rightarrow p|q}}f(b)c_q(b)\sum_{\substack{a\leq N/b\\ (a,q)=1}}f(a), \tag{4.60}$$

using the substitution $n=ab$, with $(a,q)=1$ and $b=n/a$, which is unique, and noticing that

$$c_q(n)=\frac{\mu(q/(n,q))\varphi(q)}{\varphi(q/(n,q))}=\frac{\mu(q/(b,q))\varphi(q)}{\varphi(q/(b,q))}=c_q(b),$$

which can be deduced from (4.11). For any $b \le N/4$ we can apply Theorem 4.1.7 to find

$$\sum_{\substack{a\le N/b\\(a,q)=1}} f(a) = \frac{N}{b}(\log(N/b))^{\alpha-1}\sum_{j=0}^{J}\frac{\lambda_j}{(\log(N/b))^j} + O\left(\frac{N}{b}(\log(N/b))^{\kappa-1-A_1}(\log\log N)\right). \tag{4.61}$$

Here, we simplified the expression in the error term by using Lemma 4.6.1 and noticing that

$$\sum_{d|q}\frac{d_\kappa(d)}{d^{3/4}} = \prod_{p|q}\left(1+\frac{\kappa}{p^{3/4}}\right) = \exp\left(\sum_{p|q}\frac{\kappa}{p^{3/4}} + O_\kappa(1)\right) \ll_{\kappa,D} 1,$$

by the conditions (4) and (6) on q, if $C = C(\kappa, A_1) > \kappa^{4/3}$.

In (4.61), we indicated with J the largest integer smaller than A_1 and the coefficients λ_j are as in the statement of Theorem 4.1.7.

Since the asymptotic holds only when $N/b \ge 4$ we need to estimate

$$\sum_{\substack{N/4<b\le N\\p|b\Rightarrow p|q}} d_\kappa(b)|c_q(b)| \le N^{3/4}\sum_{p|b\Rightarrow p|q}\frac{d_\kappa(b)(q,b)}{b^{3/4}}. \tag{4.62}$$

By Lemma 4.2.5, we conclude that (4.62) is

$$\ll N^{3/4}q^{1/4}d_{\kappa+1}(q) \ll_{\kappa,D} N^{7/8-3\delta/16}d_{\kappa+1}(q).$$

Plugging (4.61) into (4.60) we get that the sum $\sum_{n\le N} f(n)c_q(n)$ is

$$\begin{aligned}&= N\sum_{\substack{b\le N/4\\p|b\Rightarrow p|q}}\frac{f(b)c_q(b)}{b}(\log(N/b))^{\alpha-1}\left(\sum_{j=0}^{J}\frac{\lambda_j}{(\log(N/b))^j}\right)\\&+ O\left(N\sum_{\substack{b\le N/4\\p|b\Rightarrow p|q}}\frac{d_\kappa(b)|c_q(b)|}{b}(\log(N/b))^{\kappa-1-A_1}(\log\log N)\right)\\&+ O(N^{7/8-3\delta/16}d_{\kappa+1}(q)).\end{aligned} \tag{4.63}$$

To estimate the sum in the error term here we use same considerations employed in the case of (4.62). Since in the hypothesis of Theorem 4.1.2 we assumed $A_1 > \kappa + 2$, the function $(\log(N/b))^{\kappa-1-A_1}$ is an increasing function of b. Therefore, by an application of Lemma 4.2.5 we immediately deduce that the sum in the error term corresponding to values of $b \leq \sqrt{N}$ is

$$\begin{aligned}
&\ll (\log N)^{\kappa-1-A_1}(\log\log N) \sum_{p|b \Rightarrow p|q} \frac{d_\kappa(b)(q,b)}{b} \\
&\ll_{\kappa,D} (\log N)^{\kappa-1-A_1}(\log\log N) d_{\kappa+1}(q).
\end{aligned}$$

On the other hand, the one corresponding to $b > \sqrt{N}$ is simply

$$\ll N^{-1/8}(\log\log N) \sum_{p|b \Rightarrow p|q} \frac{d_\kappa(b)(q,b)}{b^{3/4}} \ll_{\kappa,D} N^{-3\delta/16}(\log\log N) d_{\kappa+1}(q),$$

again by an application of Lemma 4.2.5. We conclude that

$$\begin{aligned}
\sum_{n \leq N} f(n)c_q(n) = N &\sum_{\substack{b \leq N/4 \\ p|b \Rightarrow p|q}} \frac{f(b)c_q(b)}{b} (\log(N/b))^{\alpha-1} \left(\sum_{j=0}^{J} \frac{\lambda_j}{(\log(N/b))^j} \right) \\
&+ O\left(N(\log N)^{\kappa-1-A_1}(\log\log N) d_{\kappa+1}(q) \right),
\end{aligned} \tag{4.64}$$

where the constant in the big-Oh term may depend on δ, κ, A_1, D and the implicit constant in (4.2) and we take N large enough with respect to these parameters.

The principal aim from now on is to evaluate the following family of sums

$$\sum_{\substack{b \leq N/4 \\ p|b \Rightarrow p|q}} \frac{f(b)c_q(b)}{b} \log^{\tilde{\alpha}}(N/b), \tag{4.65}$$

with $\tilde{\alpha} \in \{\alpha-1, \alpha-2, \ldots, \alpha-J-1\}$. In order to do that, we employ condition (3.a) and write $q = rs$, with $p|r \Rightarrow p > (\log N)^B$ and $p|s \Rightarrow p \leq (\log N)^B$, for a large constant $B > 0$ to be chosen later. In view of this factorization we have the following identity for the quantity in (4.65):

$$= \sum_{\substack{b_1 \le \sqrt{N} \\ p|b_1 \Rightarrow p|r}} \frac{f(b_1)c_r(b_1)}{b_1} \sum_{\substack{b_2 \le N/4b_1 \\ p|b_2 \Rightarrow p|s}} \frac{f(b_2)c_s(b_2)}{b_2} \log^{\tilde{\alpha}}(N/b_1b_2) + \sum_{\substack{b_2 \le \sqrt{N} \\ p|b_2 \Rightarrow p|s}} \frac{f(b_2)c_s(b_2)}{b_2} \sum_{\substack{N^{1/2} < b_1 \le N/4b_2 \\ p|b_1 \Rightarrow p|r}} \frac{f(b_1)c_r(b_1)}{b_1} \log^{\tilde{\alpha}}(N/b_1b_2), \tag{4.66}$$

since by multiplicativity of $c_q(n)$ as function of q and definition of r, s we have

$$c_q(b) = c_r(b)c_s(b) = c_r(b_1)c_s(b_2).$$

We inserted the above structural information on q to reduce the estimate of (4.65) to that of a sum over smooth integers (see Ch. Notations for a definition thereof), which is easier to handle, and one over integers divisible only by large primes which will turn out to be basically over squarefree integers, notably simplifying its computation. Let us focus our attention on the second double sum on the right-hand side of (4.66). By Lemma 4.2.5, the innermost sum there is

$$\ll \frac{(\log N)^{\max\{\Re(\alpha)-1,0\}}}{N^{1/8}} \sum_{p|b_1 \Rightarrow p|r} \frac{d_\kappa(b_1)|c_r(b_1)|}{b_1^{3/4}} \ll_{\kappa,D} \frac{(\log N)^{\max\{\Re(\alpha)-1,0\}}}{N^{1/8}} r^{1/4} d_{\kappa+1}(r) \ll \frac{(\log N)^{\max\{\Re(\alpha)-1,0\}}}{N^{3\delta/16}} d_{\kappa+1}(r). \tag{4.67}$$

Since this bound is independent of b_2 we only need to consider

$$\sum_{\substack{b_2 \le \sqrt{N} \\ p|b_2 \Rightarrow p|s}} \frac{|f(b_2)c_s(b_2)|}{b_2} \le \sum_{p|b_2 \Rightarrow p|s} \frac{d_\kappa(b_2)|c_s(b_2)|}{b_2} \ll_{\kappa,D} d_{\kappa+1}(s),$$

again by Lemma 4.2.5. In conclusion, the contribution from the second double sum in (4.66) is

$$\ll_{\kappa,D} \frac{d_{\kappa+1}(q)(\log N)^{\max\{\Re(\alpha)-1,0\}}}{N^{3\delta/16}}. \tag{4.68}$$

4.8 The contribution from small prime factors

We are left with the estimate of the first double sum in (4.66). In this section we explain how to work with the innermost sum there, which is on integers b_2 supported only on small prime factors, i.e. on those dividing s.

For brevity, let us write $M = N/b_1$. We need to consider first

$$\begin{aligned}
&\sum_{\substack{b_2 \le M/4 \\ p|b_2 \Rightarrow p|s}} \frac{f(b_2)c_s(b_2)}{b_2} \log^{\tilde{\alpha}}(M/b_2) \\
&= \sum_{k=0}^{K} \binom{-\tilde{\alpha}+k-1}{k} (\log M)^{\tilde{\alpha}-k} \sum_{\substack{b_2 \le M/4 \\ p|b_2 \Rightarrow p|s}} \frac{f(b_2)c_s(b_2)}{b_2} \log^k b_2 \\
&+ O_{\kappa,K}\Bigg((\log M)^{\Re(\tilde{\alpha})-K-1} \sum_{\substack{b_2 \le M/4 \\ p|b_2 \Rightarrow p|s}} \frac{|f(b_2)c_s(b_2)|}{b_2} (\log b_2)^{K+1} \Bigg),
\end{aligned}$$

for a constant K that will be chosen in terms of A_1 later on. Let us move on now to estimate the sums

$$\sum_{b \le M/4,\ p|b \Rightarrow p|s} \frac{f(b)c_s(b)}{b} \log^k b, \quad (\forall\ 0 \le k \le K).$$

First, we observe that we can remove the condition $b \le M/4$, because using the monotonicity of $b^{1/4}/(\log b)^k$, for fixed k and b large, and applying Lemma 4.2.5, we may deduce that

$$\begin{aligned}
\sum_{\substack{b > M/4 \\ p|b \Rightarrow p|s}} \frac{f(b)c_s(b)}{b} \log^k b &\ll_k \frac{\log^k M}{M^{1/4}} \sum_{p|b \Rightarrow p|s} \frac{d_\kappa(b)|c_s(b)|}{b^{3/4}} \\
&\ll_{k,\kappa,D} \frac{\log^k M}{M^{1/4}} s^{1/4} d_{\kappa+1}(s).
\end{aligned}$$

Therefore, the error in replacing the finite sums above with the complete series is

$$\ll_{k,\kappa,D} \frac{\log^{\Re(\tilde{\alpha})} M}{M^{1/4}} s^{1/4} d_{\kappa+1}(s) \ll_{\delta,\kappa,A_1} \frac{\log^{\Re(\tilde{\alpha})} N}{N^{3\delta/16}} d_{\kappa+1}(s),$$

for any $\tilde{\alpha}$, using that $M \ge N^{1/2}$ and $s \le q \le N^{1/2-3\delta/4}$.

We have obtained so far

$$\begin{aligned}(4.69)\qquad &\sum_{\substack{b_2\le M/4\\ p|b_2\Rightarrow p|s}}\frac{f(b_2)c_s(b_2)}{b_2}\log^{\tilde{\alpha}}(M/b_2)\\ &=\sum_{k=0}^{K}\binom{-\tilde{\alpha}+k-1}{k}(\log M)^{\tilde{\alpha}-k}\sum_{p|b_2\Rightarrow p|s}\frac{f(b_2)c_s(b_2)}{b_2}\log^k b_2\\ &+O_{\kappa,K}\left((\log M)^{\Re(\tilde{\alpha})-K-1}\sum_{p|b_2\Rightarrow p|s}\frac{d_\kappa(b_2)(b_2,s)}{b_2}(\log b_2)^{K+1}\right)\\ &+O_{\delta,\kappa,A_1,D,K}\left(\frac{\log^{\Re(\tilde{\alpha})}N}{N^{3\delta/16}}d_{\kappa+1}(s)\right).\end{aligned}$$

Let us define the following Dirichlet series

$$\Theta(\sigma):=\sum_{p|b\Rightarrow p|s}\frac{f(b)c_s(b)}{b^\sigma},\quad \tilde{\Theta}(\sigma):=\sum_{p|b\Rightarrow p|s}\frac{d_\kappa(b)(b,s)}{b^\sigma},\quad (\sigma\ge 1).$$

In order to find a better and manageable form for them, we will prove the following lemma.

Lemma 4.8.1. *For squarefree values of s, we have*

$$\begin{aligned}\Theta(\sigma)&=\prod_{p|s}\left(-p+(p-1)\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^{\nu\sigma}}\right),\\ \tilde{\Theta}(\sigma)&=\prod_{p|s}\left(1-p+p(1-1/p^\sigma)^{-\kappa}\right).\end{aligned}$$

Proof. For a general multiplicative function $f(n)$ we have

$$\sum_n\frac{f(n)c_s(n)}{n^\sigma}=\sum_n\frac{f(n)}{n^\sigma}\sum_{d|n,d|s}\mu(s/d)d=\sum_{d|s}\mu(s/d)d^{1-\sigma}\sum_k\frac{f(dk)}{k^\sigma},$$

by (4.11). Let $F(\sigma)$ indicate the Dirichlet series of f. We denote with $v_p(n)$

the p-adic valuation of n. Then we get

$$\sum_k \frac{f(dk)}{k^\sigma} = \prod_p \sum_{\nu\geq 0} \frac{f(p^{\nu+v_p(d)})}{p^{\nu\sigma}}$$
$$= \prod_{p\nmid d}\left(1+\frac{f(p)}{p^\sigma}+\frac{f(p^2)}{p^{2\sigma}}+\cdots\right)\prod_{\substack{p^a||d\\ a\geq 1}}\sum_{\nu\geq 0}\frac{f(p^{\nu+a})}{p^{\nu\sigma}}.$$

Therefore, we can write

$$\sum_n \frac{f(n)c_s(n)}{n^\sigma} = F(\sigma)\sum_{d|s}\mu(s/d)d^{1-\sigma}\ell(d),$$

where $\ell(d)$ is the multiplicative function given by

$$\ell(d) = \prod_{\substack{p^a||d\\ a\geq 1}}\left(1+\frac{f(p)}{p^\sigma}+\frac{f(p^2)}{p^{2\sigma}}+\cdots\right)^{-1}\sum_{\nu\geq 0}\frac{f(p^{\nu+a})}{p^{\nu\sigma}}.$$

From this, we immediately find that

$$\sum_n \frac{f(n)c_s(n)}{n^\sigma} = F(\sigma)F_s(\sigma),$$

with $F_s(\sigma)$ equal to

$$\prod_{p|s}\left(1+\frac{f(p)}{p^\sigma}+\frac{f(p^2)}{p^{2\sigma}}+\cdots\right)^{-1}\left(-1+(p-1)\sum_{\nu\geq 1}\frac{f(p^\nu)}{p^{\nu\sigma}}\right),$$

since s is square-free. Therefore, it follows that

$$\Theta(\sigma) = \sum_{p|b\Rightarrow p|s}\frac{f(b)c_s(b)}{b^\sigma} = \prod_{p|s}\left(-p+(p-1)\sum_{\nu\geq 0}\frac{f(p^\nu)}{p^{\nu\sigma}}\right).$$

This concludes the search for the Euler product form of $\Theta(\sigma)$. Regarding $\tilde{\Theta}(\sigma)$ instead, if we indicate with $G(\sigma)$ the Dirichlet series of $d_\kappa(n)\mathbf{1}_{p|n\Rightarrow p|s}$, by using the identity $(n,s)=\sum_{d|n,d|s}\varphi(d)$ we get

$$\begin{aligned}
&\sum_{\substack{n:\\ p|n\Rightarrow p|s}}\frac{d_\kappa(n)(n,s)}{n^\sigma}\\
&=G(\sigma)\prod_{p|s}\left(1+\frac{(p-1)}{p^\sigma}\left(1+\frac{d_\kappa(p)}{p^\sigma}+\frac{d_\kappa(p^2)}{p^{2\sigma}}+\cdots\right)^{-1}\sum_{\nu\geq 0}\frac{d_\kappa(p^{\nu+1})}{p^{\nu\sigma}}\right)\\
&=G(\sigma)\prod_{p|s}\left(p-(p-1)\left(1-\frac{1}{p^\sigma}\right)^{\kappa}\right)\\
&=\prod_{p|s}\left(1-p+p\left(1-\frac{1}{p^\sigma}\right)^{-\kappa}\right),
\end{aligned}$$

since s is squarefree. The proof of the lemma is completed. □

We now show that each term in the sum on the right-hand side of (4.69) corresponding to a $k\geq 1$ gives a smaller contribution compared to the $k=0$ term. Let us start by noticing that

$$\Theta^{(k)}(\sigma)=(-1)^k\sum_{p|b\Rightarrow p|s}\frac{f(b)c_s(b)}{b^\sigma}\log^k b.$$

We let

$$\theta(\sigma):=\frac{\Theta'}{\Theta}(\sigma)=\sum_{p|s}\frac{\gamma_p'(\sigma)}{\gamma_p(\sigma)}, \tag{4.70}$$

where

$$\gamma_p(\sigma):=-p+(p-1)\sum_{\nu\geq 0}\frac{f(p^\nu)}{p^{\nu\sigma}}. \tag{4.71}$$

Using the Faà di Bruno's formula [64, p. 807, Theorem 2] we see that

$$\begin{aligned}
\Theta^{(k)}(1)&=(e^{\log\Theta(\sigma)})^{(k)}|_{\sigma=1}\\
&=\Theta(1)k!\sum_{m_1+2m_2+\ldots+km_k=k}\frac{\prod_{j=1}^k(\theta^{(j-1)}(1))^{m_j}}{1!^{m_1}\,m_1!\cdots k!^{m_k}\,m_k!}.
\end{aligned} \tag{4.72}$$

Consequently, we need an estimate for the logarithmic derivative of Θ and its derivatives. To this aim we first note that $(\gamma_p'/\gamma_p)^{(h)}(1) = (\log \gamma_p(\sigma))^{(h+1)}|_{\sigma=1}$, which again by the Faà di Bruno's formula is

$$(4.73) \qquad = (h+1)! \sum_{m_1+2m_2+...+(h+1)m_{h+1}=h+1} \frac{(-1+m_1+m_2+...+m_{h+1})!}{1!^{m_1}\, m_1! \cdots (h+1)!^{m_{h+1}}\, m_{h+1}!} \times \prod_{j=1}^{h+1} \left(\frac{-\gamma_p^{(j)}(1)}{\gamma_p(1)} \right)^{m_j}.$$

We observe that

$$\gamma_p(1) = -1 + f(p) - \frac{f(p)}{p} + \frac{f(p^2)}{p} - \cdots = \sum_{\nu \geq 0} \frac{g(p^{\nu+1})}{p^\nu},$$

where $g(n)$ is the multiplicative function defined by $f(n) =: g * \mathbf{1}(n)$. Hence,

$$|\gamma_p(1)| \geq |g(p)| + O_\kappa(1/p),$$

since for any $j \geq 2$ we have $g(p^j) = f(p^j) - f(p^{j-1})$, from which $|g(p^j)| \leq d_\kappa(p^j) + d_\kappa(p^{j-1})$. We note that $|g(p)|$ coincides exactly with the absolute value of the previously defined function g at p, without notational issues. Moreover, thanks to restriction $(5.b)$ we get $|\gamma_p(1)| > 0$, if we choose $C = C(\kappa)$ large enough, thus making (4.73) well defined.

On the other hand, we can rewrite $\gamma_p'(\sigma)$ as

$$\gamma_p'(\sigma) = (p-1) \sum_{\nu \geq 1} \frac{f(p^\nu)}{p^{\nu\sigma}} (-\nu \log p)$$

from which we immediately deduce that

$$\gamma_p^{(j)}(1) = (p-1) \sum_{\nu \geq 1} \frac{f(p^\nu)}{p^\nu} (-\nu \log p)^j \quad (j \geq 1).$$

Clearly, $|\gamma_p^{(j)}(1)| \leq C_j (\log p)^j$, for fixed values of $j \geq 1$ and a certain constant $C_j = C_j(\kappa) > 0$.

Inserting the above estimates in (4.73) we obtain

$$\left|\left(\frac{\gamma_p'}{\gamma_p}\right)^{(h)}(1)\right| \leq \tilde{C}_h\left(\frac{\log p}{\min\{|\gamma_p(1)|,1\}}\right)^{h+1},$$

for fixed values of h and suitable constants $\tilde{C}_h = \tilde{C}_h(\kappa) > 0$. Plugging this into (4.70) gives

$$\begin{aligned}|\theta^{(j-1)}(1)| \leq \tilde{C}_j \sum_{p|s}\left(\frac{\log p}{\min\{|\gamma_p(1)|,1\}}\right)^j &\leq \tilde{C}_j \tilde{\gamma}_s^j \max_{p|s}\{(\log p)^j\}\omega(s)\\ &\leq \tilde{C}_j B^j \tilde{\gamma}_s^j \omega(s)(\log\log N)^j,\end{aligned}$$

where we defined $\tilde{\gamma}_s := \max_{p|s}\min\{|\gamma_p(1)|,1\}^{-1}$. Finally, by restriction (2) on q, we deduce

$$|\theta^{(j-1)}(1)| \leq A\tilde{C}_j B^j \tilde{\gamma}_s^j (\log\log N)^{j+1}.$$

Inserting this into (4.72) we obtain

$$\Theta^{(k)}(1) \ll_{k,\kappa} |\Theta(1)|\xi^k \tilde{\gamma}_s^k (\log\log N)^{2k}, \tag{4.74}$$

for fixed values of k and a constant $\xi = \xi(A, B, \tilde{C}_1(\kappa), \ldots, \tilde{C}_h(\kappa))$.

For future reference we observe that the explicit multiplicative form of $\Theta(1)$ is given by

$$\begin{aligned}\Theta(1) = \prod_{p|s}\left(-p + (p-1)\sum_{\nu\geq 0}\frac{f(p^\nu)}{p^\nu}\right) &= \prod_{p|s}\left(\sum_{\nu\geq 0}\frac{g(p^{\nu+1})}{p^\nu}\right)\\ &= \prod_{p|s}\left(g(p) + O_\kappa\left(\frac{1}{p}\right)\right).\end{aligned} \tag{4.75}$$

We conclude this section by estimating the series in the first error term in (4.69). First, since $q \in \mathcal{A}$, we also have

$$\tilde{\Theta}(1) = \prod_{p|s}\left(1 + \kappa + O_\kappa\left(\frac{1}{p}\right)\right) \ll_{\kappa,D} d_{\kappa+1}(s).$$

Second, by Lemma 4.8.1 and arguing as above, we find

$$\tilde{\Theta}^{(k)}(1) \ll_{k,\kappa} |\tilde{\Theta}(1)|\tilde{\xi}^k(\log\log N)^{2k} \ll_{\kappa,D} \tilde{\xi}^k d_{\kappa+1}(s)(\log\log N)^{2k}, \tag{4.76}$$

for a suitable $\tilde{\xi} = \tilde{\xi}(A, B, \kappa) > 0$. Plugging the bound for $\tilde{\Theta}^{(K+1)}(1)$ inside the first error term in (4.69), we obtain that this last one is

$$\ll_{\kappa,D,K} \tilde{\xi}^{K+1} d_{\kappa+1}(s)(\log N)^{\Re(\tilde{\alpha})-K-1}(\log\log N)^{2K+2}, \tag{4.77}$$

using that $\sqrt{N} \ll M \ll N$. This exceeds the second error term in (4.69), if N is large enough in terms of $\delta, \kappa, A, A_1, B, D, K$. Collecting together (4.68), (4.69) and (4.77), we conclude that

$$\begin{aligned}
&\sum_{\substack{b\le N/4\\ p|b\Rightarrow p|q}} \frac{f(b)c_q(b)}{b}\log^{\tilde{\alpha}}(N/b)\\
&= \sum_{k=0}^{K}(-1)^k\binom{-\tilde{\alpha}+k-1}{k}\Theta^{(k)}(1)\sum_{\substack{b_1\le\sqrt{N}\\ p|b_1\Rightarrow p|r}} \frac{f(b_1)c_r(b_1)}{b_1}(\log(N/b_1))^{\tilde{\alpha}-k}\\
&+ O\Bigg(\tilde{\xi}^{K+1}\sum_{\substack{b_1\le\sqrt{N}\\ p|b_1\Rightarrow p|r}} \frac{d_\kappa(b_1)|c_r(b_1)|}{b_1} d_{\kappa+1}(s)(\log N)^{\Re(\tilde{\alpha})-K-1}(\log\log N)^{2K+2}\Bigg)\\
&+ O_{\kappa,D}\bigg(\frac{(\log N)^{\kappa-1}}{N^{3\delta/16}}d_{\kappa+1}(q)\bigg).
\end{aligned}$$

By Lemma 4.2.5, it can be rewritten as

$$\begin{aligned}
&\sum_{k=0}^{K}(-1)^k\binom{-\tilde{\alpha}+k-1}{k}\Theta^{(k)}(1)\sum_{\substack{b_1\le\sqrt{N}\\ p|b_1\Rightarrow p|r}} \frac{f(b_1)c_r(b_1)}{b_1}(\log(N/b_1))^{\tilde{\alpha}-k}\\
&+ O_{\delta,\kappa,D,K}\Big(\tilde{\xi}^{K+1}d_{\kappa+1}(q)(\log N)^{\Re(\tilde{\alpha})-K-1}(\log\log N)^{2K+2}\Big),
\end{aligned} \tag{4.78}$$

if we choose N large enough compared to $\delta, \kappa, A, A_1, B, D$ and K.

4.9 The contribution from large prime factors

In this section we compute the innermost sums in (4.78), which are over integers b_1 supported only on large prime factors, i.e. on those of r.

In order to simplify the calculations we observe that the main contribution comes only from squarefree values. Indeed, since $\kappa > 1$, we have

$$\begin{aligned}
&\sum_{\substack{b_1 \leq \sqrt{N} \\ p|b_1 \Rightarrow p|r \\ b_1 \text{ not-squarefree}}} \frac{f(b_1)c_r(b_1)}{b_1}(\log(N/b_1))^{\tilde{\alpha}-k} \\
&\ll \sum_{e|r} e \sum_{\substack{b_1 \leq \sqrt{N} \\ p|b_1 \Rightarrow p|r \\ b_1 \text{ not-squarefree} \\ e|b_1}} \frac{d_\kappa(b_1)}{b_1}(\log(N/b_1))^{\Re(\tilde{\alpha})-k} \\
&\ll (\log N)^{\Re(\tilde{\alpha})-k} \sum_{e|r} d_\kappa(e) \sum_{\substack{t \leq \sqrt{N}/e \\ p|t \Rightarrow p|r \\ t \neq 1}} \frac{d_\kappa(t)}{t} \\
&\leq (\log N)^{\Re(\tilde{\alpha})-k-B/4} d_{\kappa+1}(r) \sum_{p|t \Rightarrow p|r} \frac{d_\kappa(t)}{t^{3/4}} \\
&\ll_{\kappa,D} (\log N)^{\Re(\tilde{\alpha})-k-B/4} d_{\kappa+1}(r),
\end{aligned}$$

by condition (6) on q. Using (4.74) we find an overall contribution to (4.78) of at most

$$(4.79) \qquad \begin{aligned} &(\log N)^{\Re(\tilde{\alpha})-B/4} |\Theta(1)| d_{\kappa+1}(r) \\ &\times \sum_{k=0}^{K} \left| \binom{-\tilde{\alpha}+k-1}{k} \right| \xi^k \tilde{\gamma}_s^k \left(\frac{(\log\log N)^2}{\log N} \right)^k. \end{aligned}$$

Now, by conditions (4) and (5.a) on s, we have $\tilde{\gamma}_s \ll \sqrt{\log\log N}$. Moreover

$$|\Theta(1)| \leq \prod_{p|s} \left(\kappa + 1 + O_\kappa\left(\frac{1}{p}\right) \right) \ll_{\kappa,D} d_{\kappa+1}(s),$$

by condition (6) on q. Therefore, taking e.g. $B = 4(K+2)$ and remembering that $\xi = \xi(A, B, \kappa)$, where we will be taking A as a function of only κ and A_1, we may conclude that (4.79) will contribute $\ll_{\kappa,A_1,D,K} d_{\kappa+1}(q)(\log N)^{\Re(\tilde{\alpha})-K-2}$. This will be absorbed into the error term of (4.78), if we choose N sufficiently large with respect to δ, κ, A_1, D and K.

We are left with the estimate of

$$
\begin{aligned}
(4.80)\qquad &\sum_{\substack{b_1\le\sqrt{N}\\ p|b_1\Rightarrow p|r\\ b_1 \text{ squarefree}}} \frac{f(b_1)c_r(b_1)}{b_1}(\log(N/b_1))^{\tilde{\alpha}-k}\\
&= \sum_{b_1|r}\frac{f(b_1)c_r(b_1)}{b_1}(\log(N/b_1))^{\tilde{\alpha}-k}\\
&= \mu(r)\sum_{b_1|r}\frac{f(b_1)\varphi(b_1)\mu(b_1)}{b_1}(\log(N/b_1))^{\tilde{\alpha}-k}
\end{aligned}
$$

since r is square-free and $r \le N^{1/2-3\delta/4}$. Note that we can replace the last sum with

$$
(4.81)\qquad \mu(r)\sum_{b|r} f(b)\mu(b)(\log(N/b))^{\tilde{\alpha}-k}
$$

at the cost of a small error term. Indeed

$$
\begin{aligned}
\left|\frac{\varphi(b)}{b}-1\right| = \left|\exp\left(\sum_{p|b}\log\left(1-\frac{1}{p}\right)\right)-1\right| &= \left|\exp\left(O\Big(\sum_{p|b}\frac{1}{p}\Big)\right)-1\right|\\
&\ll \sum_{p|b}\frac{1}{p}\\
&\ll \frac{1}{(\log N)^{B/4}}\sum_{p|b}\frac{1}{p^{3/4}}\\
&\ll_D \frac{1}{(\log N)^{B/4}}.
\end{aligned}
$$

Arguing as before, its overall contribution to (4.78) will be absorbed in the big-Oh term there.

Now, assuming that r is of the form $r = ts'$, with t and s' as in restrictions $(3.b) - (3.d)$ on q, we can rewrite (4.81) as

$$
(4.82)\quad \mu(r)\sum_{b|s'} f(b)\mu(b)(\log(N/b))^{\tilde{\alpha}-k} - f(t)\mu(r)\sum_{b|s'} f(b)\mu(b)(\log(N/tb))^{\tilde{\alpha}-k}.
$$

For $M \in \{N, N/t\}$, we write

$$\sum_{b|s'} f(b)\mu(b)(\log(M/b))^{\tilde{\alpha}-k} = \sum_{h=0}^{\infty} \binom{-\tilde{\alpha}+k+h-1}{h} (\log M)^{\tilde{\alpha}-k-h} \sum_{b|s'} f(b)\mu(b)\log^h b. \tag{4.83}$$

In the next section we will need an estimate for $\sum_{b|s'} f(b)\mu(b)\log^h b$, when $h \geq 1$. This is what we achieve next.

Lemma 4.9.1. *For any $h \geq 1$ and s' as before, satisfying in particular condition (3.c) and (5.a), we have*

$$\left|\sum_{b|s'} f(b)\mu(b)\log^h b\right| \leq |g(s')|(\varepsilon \log N)^h.$$

Proof. With the same spirit of what was previously done in Sect. 4.8, we can write

$$\begin{aligned}
&\sum_{b|s'} f(b)\mu(b)\log^h b \\
&= (-1)^h \frac{d^h}{d\sigma^h}\left(\prod_{p|s'}\left(1-\frac{f(p)}{p^\sigma}\right)\right)\bigg|_{\sigma=0} \\
&= (-1)^h \sum_{j_1+j_2+\cdots+j_{\omega(s')}=h} \binom{h}{j_1, j_2, \ldots, j_{\omega(s')}} \prod_{i=1}^{\omega(s')}\left(1-\frac{f(p_i)}{p_i^\sigma}\right)^{(j_i)}\bigg|_{\sigma=0}.
\end{aligned}$$

We have

$$\left(1-\frac{f(p_i)}{p_i^\sigma}\right)^{(j_i)}\bigg|_{\sigma=0} = \begin{cases} 1-f(p_i) & \text{if } j_i = 0; \\ -f(p_i)(-\log p_i)^{j_i} & \text{if } j_i \neq 0. \end{cases}$$

Hence, we can rewrite the above expression as

$$(-1)^h \sum_{j_1+j_2+\cdots+j_{\omega(s')}=h} \binom{h}{j_1, j_2, \ldots, j_{\omega(s')}}$$
$$\times \prod_{\substack{i=1,\ldots,\omega(s') \\ j_i \neq 0}} (-f(p_i)) \prod_{\substack{i=1,\ldots,\omega(s') \\ j_i = 0}} (1-f(p_i)) \prod_{i=1}^{\omega(s')} (-\log p_i)^{j_i}.$$

Since s' satisfies condition $(5.a)$ and in particular for any prime $p|s'$ we have $f(p) \neq 1$, we may further restate the above as

$$\prod_{p|s'}(1-f(p)) \sum_{j_1+j_2+\cdots+j_{\omega(s')}=h} \binom{h}{j_1, j_2, \ldots, j_{\omega(s')}}$$
$$\times \prod_{\substack{i=1,\ldots,\omega(s') \\ j_i \neq 0}} \left(\frac{-f(p_i)}{1-f(p_i)} \right) \prod_{i=1}^{\omega(s')} (\log p_i)^{j_i}.$$

Now, observe the above expression is upper bounded in absolute value by

$$\begin{aligned} &\le |g(s')| \sum_{j_1+j_2+\cdots+j_{\omega(s')}=h} \binom{h}{j_1, j_2, \ldots, j_{\omega(s')}} \\ &\times \prod_{i=1}^{\omega(s')} \left(\max\left\{ \left| \frac{f(p_i)}{g(p_i)} \right|, 1 \right\} \log p_i \right)^{j_i} \\ &= |g(s')| \left(\sum_{p|s'} \max\left\{ \left| \frac{f(p)}{g(p)} \right|, 1 \right\} \log p \right)^h, \end{aligned} \tag{4.84}$$

by the multinomial theorem [57]. Finally, note that

$$\max\left\{ \left| \frac{f(p)}{g(p)} \right|, 1 \right\} \le \max\left\{ \frac{\kappa}{|g(p)|}, 1 \right\} \le \frac{\kappa}{\min\{|g(p)|, 1\}}.$$

Since s' satisfies restriction $(3.c)$, the second line of (4.84) is

$$\le |g(s')|(\varepsilon \log N)^h,$$

which proves the lemma. $\square$

4.10 Combining the different pieces

Collecting the results (4.64), (4.78) and (4.82)–(4.83) together, we see that $\sum_{n\leq N} f(n)c_q(n)$ equals to a main term of

$$N(\log N)^{\alpha-1}\sum_{j=0}^{J}\frac{-\lambda_j}{(\log N)^j}\sum_{k=0}^{K}(-1)^k\binom{-\alpha+j+k}{k}\frac{\Theta^{(k)}(1)}{(\log N)^k} \times\sum_{h=0}^{\infty}\binom{-\alpha+j+k+h}{h}\sum_{b|s'}f(b)\mu(s'/b)\frac{(\log b)^h}{(\log N)^h} \times\left(1-f(t)\left(1-\frac{\log t}{\log N}\right)^{\alpha-1-j-k-h}\right), \tag{4.85}$$

since $\mu(r)=\mu(t)\mu(s')=-\mu(s')$, plus an error term of

$$O\left(d_{\kappa+1}(q)N\sum_{j=0}^{J}|\lambda_j|(\log N)^{\Re(\alpha)-j-K-2}(\log\log N)^{2(K+1)}\right) + O\left(d_{\kappa+1}(q)N(\log N)^{\kappa-1-A_1}(\log\log N)\right), \tag{4.86}$$

where the big-Oh terms may depend on δ,κ,A_1,D,K and the implicit constant in (4.2) and the λ_j are as in Theorem 4.1.7. We remind that t indicates a prime number in the interval

$$[N^{1/2-3\delta/4-\varepsilon},N^{1/2-3\delta/4-\varepsilon/2}].$$

In order to estimate the contribution of the sum of the λ_j's we are going to prove the following lemma.

Lemma 4.10.1. *Let f be a multiplicative function such that $|f(n)|\leq d_\kappa(n)$, for a certain real positive constant $\kappa>1$ and any N-smooth integer n. Let q be a squarefree positive integer smaller than N satisfying condition* (4)*, with a large $C=C(\kappa,A_1)>\kappa+1$, and* (6)*.*

Then the coefficients $\lambda'_j=\Gamma(\alpha-j)\lambda_j$, where λ_j are as defined in the statement of Theorem 4.1.7, satisfy $\lambda'_j\ll 1$, for $j=0,\ldots,J$, with an implicit constant depending on κ,A_1,D and that one in (4.2)*.*

Proof. We remind that the coefficients λ'_j are defined as

$$\lambda'_j = \lambda'_j(f, \alpha, q) = \sum_{l+h=j} \frac{(H_q^{-1})^{(h)}(1)c_l}{h!},$$

where

$$H_q(z) := \prod_{p|q} \left(1 + \frac{f(p)}{p^z} + \frac{f(p^2)}{p^{2z}} + \cdots\right) \quad (\Re(z) \geq 1)$$

and for any $0 \leq l \leq J$ the coefficients c_l are as in the statement of Theorem 4.2.3. By Lemma 4.2.4 each c_l is uniformly bounded by a constant possibly depending on κ, A_1, l and that in (4.2).

Therefore, to conclude the proof of the lemma we only need to show that each derivative $(H_q^{-1})^{(h)}(1)$ is bounded. However, we can write

$$\begin{aligned} H_q(z) &= \prod_{p|q} \left(1 + \frac{f(p)}{p^z}\right) \prod_{p|q} \left(1 + \frac{f(p)}{p^z} + \frac{f(p^2)}{p^{2z}} + \cdots\right)\left(1 + \frac{f(p)}{p^z}\right)^{-1} \\ &=: \tilde{H}_q(z)\tilde{\tilde{H}}_q(z). \end{aligned}$$

Now it is not difficult to show that all the derivatives of $\tilde{\tilde{H}}_q(z)^{-1}$ at $z = 1$ are uniformly bounded in q and by Lemma 4.6.1 the same is true for those of $\tilde{H}_q(z)^{-1}$ at $z = 1$. Finally, since we have

$$\frac{d^h}{dz^h} H_q^{-1}(z)|_{z=1} = \sum_{l+k=h} \binom{h}{l} \frac{d^l}{dz^l} \tilde{H}_q^{-1}(z)|_{z=1} \frac{d^k}{dz^k} \tilde{\tilde{H}}_q^{-1}(z)|_{z=1}$$

we obtain the desired conclusion. □

By Lemma 4.10.1, choosing $K := A_1$ and taking N large enough in terms of δ, κ, A_1, D and the implicit constant in (4.2), we see that the error term (4.86) reduces to

$$\ll d_{\kappa+1}(q) N (\log N)^{\kappa-1-A_1} (\log\log N).$$

Let us now focus on the main term (4.85). In the following we will make use several times of the following trivial estimates:

- for any $t \in [N^{1/2-3\delta/4-\varepsilon}, N^{1/2-3\delta/4-\varepsilon/2}]$, we have

$$\left|1 - f(t)\left(1 - \frac{\log t}{\log N}\right)^{\alpha-1-j-k-h}\right| \leq 1 + \kappa(1/2 + 3\delta/4 + \varepsilon/2)^{-\kappa-1-j-k-h}$$
$$\leq 1 + \kappa 2^{\kappa+1+j+k+h},$$

  if δ, ε small;

- the binomial coefficient $|\binom{-\alpha+j+k+h}{h}|$ equals

$$\frac{|(-\alpha+j+k+h)(-\alpha+j+k+h-1)\cdots(-\alpha+j+k+1)|}{h!}$$
$$\leq \frac{(|\alpha|+j+k+h)(|\alpha|+j+k+h-1)\cdots(|\alpha|+j+k+1)}{h!}$$
$$= \binom{|\alpha|+j+k+h}{h} \leq \binom{\kappa+J+k+h}{h};$$

- similarly as in the previous bullet point, we have

$$\left|\binom{-\alpha+j+k}{k}\right| \leq \binom{\kappa+J+k}{k}.$$

The contribution of the sum over $j \geq 1$ in (4.85) is

$$\ll N(\log N)^{\Re(\alpha)-1}|g(s')\Theta(1)|\sum_{j=1}^{J}\frac{2^j|\lambda_j|}{(\log N)^j}$$
$$\ll N(\log N)^{\Re(\alpha)-2}\frac{|g(s')\Theta(1)(\alpha-1)|}{|\Gamma(\alpha)|},$$

by using in sequence Lemma 4.9.1, the upper bound (4.74), conditions (4) and (5.a) on s to estimate $\tilde{\gamma}_s$ and Lemma 4.10.1. Thus, the main term in (4.85) reduces to

$$-N(\log N)^{\alpha-1}\lambda_0 \sum_{k=0}^{K}(-1)^k \binom{-\alpha+k}{k} \frac{\Theta^{(k)}(1)}{(\log N)^k} \tag{4.87}$$
$$\times \sum_{h=0}^{\infty} \binom{-\alpha+k+h}{h} \sum_{b|s'} f(b)\mu(s'/b) \frac{(\log b)^h}{(\log N)^h}$$
$$\times \left(1 - f(t)\left(1 - \frac{\log t}{\log N}\right)^{\alpha-1-k-h}\right).$$

Working in a similar way as before, the contribution of the sum over $k \geq 1$ in (4.87) is

$$\ll N(\log N)^{\Re(\alpha)-2}(\log\log N)^{5/2} \frac{|c_0 g(s')\Theta(1)(\alpha-1)|}{|\Gamma(\alpha)|}.$$

Here, we noticed that

$$\binom{-\alpha+k}{k} = \frac{(-\alpha+1)}{k}\binom{-\alpha+1+k-1}{k-1},$$

for any $k \geq 1$, we replaced λ_0 with

$$\lambda_0 = \frac{c_0 H_q^{-1}(1)}{\Gamma(\alpha)},$$

as in Theorem 4.1.7, and used Lemma 4.6.1 to estimate $|H_q^{-1}(1)|$.

Thus, (4.87) reduces to

$$-N(\log N)^{\alpha-1}\lambda_0\Theta(1) \sum_{h=0}^{\infty} \binom{-\alpha+h}{h} \sum_{b|s'} f(b)\mu(s'/b) \frac{(\log b)^h}{(\log N)^h}$$
$$\times \left(1 - f(t)\left(1 - \frac{\log t}{\log N}\right)^{\alpha-1-h}\right).$$

Again, similar considerations lead to the following estimate for the contribution of the sum over $h \geq 1$ above:

$$\ll N(\log N)^{\Re(\alpha)-1} \frac{\varepsilon|c_0 g(s')\Theta(1)(\alpha-1)|}{|\Gamma(\alpha)|}.$$

Assume now that ε is of the form $\varepsilon := \delta/V$, with $V \geq 5$ sufficiently large in terms of κ in order to make the above binomial series convergent and to be determined later on in terms of the other parameters. Collecting the above error terms together, thanks to (4.6) and taking $N \geq N_0$, with $N_0(\delta, \kappa, A_1, V)$ sufficiently large, we have obtained the following expression for $\sum_{n \leq N} f(n)c_q(n)$

$$
\begin{aligned}
(4.88) \qquad & -\frac{c_0}{\Gamma(\alpha)}(f * \mu)(s')H_q^{-1}(1)\Theta(1)\theta_{N,\alpha}(t)N(\log N)^{\alpha-1} \\
& + O\left(N(\log N)^{\Re(\alpha)-1}\frac{\varepsilon|c_0 g(s')\Theta(1)(\alpha-1)|}{|\Gamma(\alpha)|}\right) \\
& + O\left(d_{\kappa+1}(q)N(\log N)^{\kappa-1-A_1}(\log\log N)\right).
\end{aligned}
$$

Here, the big-Oh terms may depend on δ, κ, A_1, D and the implicit constant in (4.2) and we define

$$\theta_{N,\alpha}(p) := 1 - f(p)\left(1 - \frac{\log p}{\log N}\right)^{\alpha-1},$$

for any prime $p \leq N$.

4.11 A Mertens' type estimate with $\theta_{N,\alpha}$

In this section we prove Lemma 4.1.9, which will play a fundamental role in the next one, where we will produce a lower bound for (4.88) on average over q.

Proof of Lemma 4.1.9. We split the proof into three main cases. First of all, if $\alpha = 1$ then

$$\theta_{N,1}(t) = 1 - f(t).$$

Therefore, the sum (4.26) reduces to

$$\begin{aligned}
&\sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|f(t)-1|^2}{t} \\
&= \int_{N^{1/2-3\delta/4-\delta/V}}^{N^{1/2-3\delta/4-\delta/2V}} \frac{d(\beta t + R(t))}{t\log t} \\
&= \beta \log\left(1+\frac{\delta/2V}{1/2-3\delta/4-\delta/V}\right) + \int_{N^{1/2-3\delta/4-\delta/V}}^{N^{1/2-3\delta/4-\delta/2V}} \frac{d(R(t))}{t\log t} \\
&= \beta \log\left(1+\frac{\delta/2V}{1/2-3\delta/4-\delta/V}\right) + O((\log N)^{-A_2}),
\end{aligned}$$

by partial summation from (4.3), where $R(t) = O(t(\log t)^{-A_2})$ and the implicit constant in the big-Oh error term may depend on κ, A_2 and that of (4.3).

By taking N, V sufficiently large, δ small enough and thanks to (4.5), the above reduces to

$$\beta\left(\frac{\delta}{V}(1+O(\delta)) + O\left(\frac{\delta}{V}\right)^2\right) + O\left(\frac{\beta\delta^2}{V}\right) = \frac{\beta\delta}{V} + O\left(\frac{\beta\delta^2}{V}\right),$$

where now the implicit constant in the big-Oh error term is absolute. It is clear that now (4.26) follows with $\eta = 1/2$, say, by choosing δ suitably small.

In particular, we have proved that

$$\sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|f(t)-1|^2}{t} = \frac{\beta\delta}{V} + O\left(\frac{\beta\delta^2}{V}\right), \tag{4.89}$$

$$\sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{1}{t} = \frac{\delta}{V} + O\left(\frac{\delta^2}{V}\right), \tag{4.90}$$

where (4.90) follows in as much as the same way of (4.89).

Let us now assume $\alpha \neq 1$ and $|\alpha - 1| \leq \delta$. Then we can Taylor expand

$\theta_{N,\alpha}(t)$ as

$$\begin{aligned}
&= 1 - f(t)(1 + (\alpha - 1)\log(1/2 + 3\delta/4 + \theta(t)) + O(|\alpha - 1|^2)) \\
&= 1 - f(t) - f(t)(\alpha - 1)\log(1/2 + 3\delta/4 + \theta(t)) + O_\kappa(|\alpha - 1|^2) \\
&= (1 - f(t))(1 + (\alpha - 1)\log(1/2 + O(\delta))) - (\alpha - 1)\log(1/2 + O(\delta)) \\
&+ O_\kappa(|\alpha - 1|^2),
\end{aligned}$$

where $\theta(t) \in [\delta/V, \delta/2V]$ is defined by $t =: N^{1/2-3\delta/4-\theta(t)}$ and we take $V \geq 5$ and δ small.

Inserting this into (4.26) and using the triangle inequality, we get a lower bound for (4.26) of

$$\begin{aligned}
&\geq |1 + (\alpha - 1)\log(1/2 + O(\delta))| \sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|f(t) - 1|^2}{t} \\
&- |(\alpha - 1)\log(1/2 + O(\delta))| \sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|f(t) - 1|}{t} \\
&+ O_\kappa\Bigg(|\alpha - 1|^2 \sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|f(t) - 1|}{t}\Bigg).
\end{aligned}$$

By Cauchy–Schwarz's inequality and equations (4.89)–(4.90), we immediately deduce

$$\sum_{\substack{t \text{ prime:} \\ N^{1/2-3\delta/4-\delta/V} \leq t \leq N^{1/2-3\delta/4-\delta/2V}}} \frac{|f(t) - 1|}{t} \leq \frac{\sqrt{\beta}\delta}{V} + O\bigg(\frac{\sqrt{\beta}\delta^2}{V}\bigg). \tag{4.91}$$

Moreover, by Lemma 4.2.2 and taking N sufficiently large with respect to δ, κ, A_1, A_2 and the implicit constants (4.2)–(4.3), we also have

$$|\alpha - 1| \leq \sqrt{\beta} + O(\sqrt{\beta}\delta), \tag{4.92}$$

thanks to (4.5). Hence, using (4.89) and (4.91) we can further lower bound

(4.26) with

$$
\begin{aligned}
&\geq |1+(\alpha-1)\log(1/2+O(\delta))|\bigg(\frac{\beta\delta}{V}+O\bigg(\frac{\beta\delta^2}{V}\bigg)\bigg)\\
&-|(\alpha-1)\log(1/2+O(\delta))|\bigg(\frac{\sqrt{\beta}\delta}{V}+O\bigg(\frac{\sqrt{\beta}\delta^2}{V}\bigg)\bigg)\\
&+O_\kappa\bigg(|\alpha-1|^2\bigg(\frac{\sqrt{\beta}\delta}{V}+O\bigg(\frac{\sqrt{\beta}\delta^2}{V}\bigg)\bigg)\bigg).
\end{aligned}
$$

Thanks to (4.92) and the hypothesis $|\alpha-1|\leq\delta$, the above becomes

$$
\begin{aligned}
&\geq \frac{\beta\delta}{V}(1+O(\delta))-\log(2+O(\delta))\frac{\beta\delta}{V}(1+O(\delta))+O_\kappa\bigg(\frac{\beta\delta^2}{V}(1+O(\delta))\bigg)\\
&=\frac{\beta\delta}{V}(1-\log(2+O(\delta))+O(\delta)),
\end{aligned}
$$

which proves the lemma with $\eta=1/10$, say, if we take δ small enough.

Finally, we are left with the case $|\alpha-1|>\delta$. To handle it, we split the set of prime numbers into three sets:

$$
\begin{aligned}
\mathcal{A}_1 &:= \{p : |\theta_{N,\alpha}(p)|\leq\delta^5\}\\
\mathcal{A}_2 &:= \{p : |f(p)-1|\leq\delta^5\}\\
\mathcal{A}_3 &:= \{p : |\theta_{N,\alpha}(p)|>\delta^5, |f(p)-1|>\delta^5\}.
\end{aligned}
$$

Remark 4.11.1. *We expect the set $\mathcal{A}_3$, i.e. the set of primes where $\theta_{N,\alpha}$ and f are respectively bounded away from 0 and 1, to contain a positive proportion of primes, at least on a small scale. Indeed, their complementary conditions should force α to be either very close to 1 (which situation we dealt with before) or very close to 2, in which case we will succeed by adjusting the value of δ.*

We cover the interval $I:=[N^{1/2-3\delta/4-\delta/V}, N^{1/2-3\delta/4-\delta/2V}]$ with dyadic subintervals

$$
I=:I'\cup\bigcup_{k=0}^{\lfloor\frac{\delta\log N}{2V\log 2}\rfloor-1}[N^{1/2-3\delta/4-\delta/V}2^k, N^{1/2-3\delta/4-\delta/V}2^{k+1}),
$$

with I' the possible rest of the above dyadic dissection. However, since we are looking for just a lower bound for (4.26), we can forget about I'.

Let us first suppose that for any $[x, 2x)$ in the above union we have

$$|\mathcal{A}_3 \cap [x, 2x)| \geq \delta^5 \frac{x}{\log x}.$$

Hence, in accordance with the prime number theorem, we are asking for a proportion of at least δ^5 primes in the intersection $\mathcal{A}_3 \cap [x, 2x)$, for *any* such x. From here it is easy to conclude, since (4.26) will follow with a constant η proportional to $\delta^{15}/(\kappa+1)^2$, since $\beta \leq (\kappa+1)^2$.

Suppose now that there exists an interval

$$[x, 2x) := [N^{1/2-3\delta/4-\delta/V} 2^k, N^{1/2-3\delta/4-\delta/V} 2^{k+1}), \tag{4.93}$$

for a certain $k \in \{0, \ldots, \lfloor (\delta \log N)/(2V \log 2) \rfloor - 1\}$, for which

$$|\mathcal{A}_3 \cap [x, 2x)| < \delta^5 \frac{x}{\log x}.$$

This clearly implies that

$$|(\mathcal{A}_1 \cup \mathcal{A}_2) \cap [x, 2x)| \geq (1 - \delta^5) \frac{x}{\log x}.$$

Now, let

$$|\mathcal{A}_1 \cap [x, 2x)| = d_1 \frac{x}{\log x},$$

for a certain $d_1 \in [0, 1]$.

Remark 4.11.2. *One specific dyadic interval does not in general supply us with enough information on a function f verifying* (4.2)–(4.3) *for single fixed values of x. However, we ask statistics* (4.2)–(4.3) *to hold uniformly on $2 \leq x \leq N$. This imposes a* rigidity *on the distribution of f along the prime numbers, from which the "local" behaviour of f is determined by the "global" one. In particular, the information that f on average over all the primes smaller than* any *$x \leq N$ is roughly α, which we are now supposing to be bounded away from* 1*, forces f to be on* any *dyadic interval $[x, 2x)$, for large x, not too close to* 1*, apart for a small proportion of primes. This, together with some structural information on f over the primes that will be deducted from the definition of the sets $\mathcal{A}_1$ and $\mathcal{A}_2$, will negate the assumption that the almost totality of* *primes lies now in the union* $\mathcal{A}_1 \cup \mathcal{A}_2$.

We note that for primes in $\mathcal{A}_2$ we have

$$f(p) = 1 + O(\delta^5)$$

and for those in $\mathcal{A}_1$ we have instead

$$f(p) = \left(\frac{1}{2} + \frac{3\delta}{4} + \theta(p)\right)^{1-\alpha} + O(\delta^5) = \left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} + O_\kappa(\delta^5 + \delta/V).$$

Therefore, from (4.2) and choosing $V = V(\delta, \kappa)$ sufficiently large, we get

$$\begin{aligned}
&\alpha x + O\left(\frac{x}{(\log x)^{A_1}}\right)\\
&= \sum_{p \in (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3) \cap [x, 2x)} f(p) \log p\\
&= \sum_{p \in \mathcal{A}_1 \cap [x,2x)} \left(\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} + O_\kappa(\delta^5)\right) \log p + \sum_{p \in (\mathcal{A}_2 \backslash \mathcal{A}_1) \cap [x,2x)} f(p) \log p\\
&+ \sum_{p \in \mathcal{A}_3 \cap [x,2x)} f(p) \log p\\
&= \sum_{p \in \mathcal{A}_1 \cap [x,2x)} \left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} \log p + \sum_{p \in (\mathcal{A}_2 \backslash \mathcal{A}_1) \cap [x,2x)} \log p + O(\delta^5 x).
\end{aligned}$$

From this, considering that $|(\mathcal{A}_1 \cup \mathcal{A}_2) \cap [x, 2x)| = (1 + O(\delta^5))x/\log x$, we deduce that

$$\begin{aligned}
&\alpha x + O\left(\frac{x}{(\log x)^{A_1}}\right)\\
&= \sum_{p \in \mathcal{A}_1 \cap [x,2x)} \left(\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} - 1\right) \log p + \sum_{p \in (\mathcal{A}_1 \cup \mathcal{A}_2) \cap [x,2x)} \log p + O(\delta^5 x)\\
&= \left(\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} - 1\right) \log(x + O(1)) d_1 \frac{x}{\log x}\\
&+ \log(x + O(1))(1 + O(\delta^5)) \frac{x}{\log x} + O(\delta^5 x)\\
&= \left(\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} - 1\right) d_1 x + x + O_\kappa(\delta^5 x + 1/\log x).
\end{aligned}$$

Finally, since $x \in [N^{1/2-3\delta/4-\delta/V}, N^{1/2-3\delta/4-\delta/2V})$, by choosing N sufficiently large with respect to δ, κ, A_1 and the implicit constant in (4.2), and dividing through by x, we conclude that

$$\alpha - 1 + O_\kappa(\delta^5) = \left(\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} - 1\right) d_1. \tag{4.94}$$

Similar computations, but working with (4.3) instead, lead to

$$\beta + O_\kappa(\delta^5) = \left|\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} - 1\right|^2 d_1, \tag{4.95}$$

if now N is also sufficiently large with respect to A_2 and the implicit constant in (4.3).

By substituting the value of d_1 from (4.94) into (4.95), we find

$$\beta + O_\kappa(\delta^5) = (\alpha - 1)\left(\overline{\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha}} - 1\right). \tag{4.96}$$

By dividing through by $\alpha - 1$, remembering that $\delta < |\alpha - 1| \leq \kappa + 1$, and taking the conjugate, we can rewrite the above as

$$\frac{\beta}{\bar{\alpha} - 1} + 1 = \left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} + O_\kappa(\delta^4).$$

If the left-hand side in the above equation vanishes, we have

$$\left(\frac{1}{2} + \frac{3\delta}{4}\right)^{1-\alpha} = O_\kappa(\delta^4),$$

which already leads to a contradiction, since δ can be chosen sufficiently small with respect to κ. Otherwise, we can pass to the logarithm on both sides and deduce that

$$\frac{1}{1-\alpha} \log\left(\frac{\beta}{\bar{\alpha} - 1} + 1\right) = \log\left(\frac{1}{2} + \frac{3\delta}{4}\right) + O_\kappa(\delta^3).$$

By Taylor expanding the logarithmic factor on the right-hand side above as

$$\log\left(\frac{1}{2}+\frac{3\delta}{4}\right) = -\log 2 + \frac{3\delta}{2} + O(\delta^2),$$

and considering δ small enough in terms of κ, we finally get

$$\frac{1}{1-\alpha}\log\left(\frac{\beta}{\bar{\alpha}-1}+1\right) + \log 2 = \frac{3\delta}{2} + O(\delta^2).$$

A consequence of this is that, by shrinking δ if necessary, we should have:

$$R(\alpha,\beta) := \left|\frac{1}{1-\alpha}\log\left(\frac{\beta}{\bar{\alpha}-1}+1\right) + \log 2\right| \in \left[\frac{7\delta}{5}, \frac{8\delta}{5}\right]. \tag{4.97}$$

However, (4.97) fails either for any $\delta > 0$, when $R(\alpha,\beta) = 0$, or else by possibly replacing δ with $\delta/2$. In both cases we reach a contradiction, thus concluding the proof of the lemma. □

4.12 The lower bound for the variance

In this section we will conclude the proof of Theorem 4.1.2, by, roughly speaking, finding a lower bound for the logarithmic average of $|g(q)|^2$ restricted on integers q satisfying all the conditions as in Subsect. 4.2.2.

4.12.1 Collecting the main results

Plugging (4.88) into (4.59), we find for $V(N,Q;f)$ a lower bound of:

$$\begin{aligned} &\gg QN(\log N)^{-\beta+2(\Re(\alpha)-1)} \\ &\times \Bigg(\sideset{}{'}\sum_{KQ_0\le q\le RN^{-\delta/4}} \left(\left|\frac{c_0}{\Gamma(\alpha)}\right| h_1(q) + R_\alpha(N,q)\right) + E(N)\Bigg)^2 \\ &+ O_\kappa\left(\frac{N^2(\log N)^{\kappa^2+4\kappa+2}}{Q_0} + \frac{N^2(\log N)^{\beta+2\Re(\alpha)-2}}{Q_0}\right), \end{aligned} \tag{4.98}$$

where we let

$$E(N) := O_{\delta,\kappa}\left(\frac{(\log N)^{-\Re(\alpha)+1}}{N^{\delta/11}}\right)$$

$$R_\alpha(N,q) := O_{\delta,\kappa,A_1,A_2,D}\left(\frac{\varepsilon|c_0(\alpha-1)|h_2(q)}{|\Gamma(\alpha)|}+\frac{h_3(q)(\log\log N)}{(\log N)^{-\kappa+\Re(\alpha)+A_1}}\right),$$

with

$$h_1(q) := \frac{|g(q)g(s')H_q^{-1}(1)\Theta(1)\theta_{N,\alpha}(t)|}{q}; \tag{4.99}$$

$$h_2(q) := \frac{|g(q)g(s')\Theta(1)|}{q};$$

$$h_3(q) := \frac{|g(q)|d_{\kappa+1}(q)}{q}.$$

Here $\sum'$ indicates a sum over all numbers q satisfying restrictions (1) to (6) and the $\gg$ constant may depend on $\delta, \kappa, A_1, A_2, D$ and the implicit constants in (4.2)–(4.3). Moreover, c_0 is as in the statement of Theorem 4.1.2 and $\Gamma(\alpha)$ has been defined in Ch. Notations.

4.12.2 The sum of $R_\alpha(N,q)$

We can easily estimate the sum of $R_\alpha(N,q)$ by using Lemma 2.1.1. For the sum involving h_3 the contribution will be

$$\ll_\kappa (\log N)^{(\kappa+1)^2+\kappa-\Re(\alpha)-A_1}(\log\log N).$$

Regarding the sum involving h_2 instead, it may be bounded by

$$\ll |c_0|\frac{\varepsilon|\alpha-1|}{|\Gamma(\alpha)|} \times \sum_{\substack{s'\leq N^\varepsilon\\ p|s'\Rightarrow p>(\log N)^B}}\frac{|g(s')|^2}{s'}\sum_{\substack{t\text{ prime:}\\ t\in I_\delta(N)}}\frac{|g(t)|}{t}\sum_{\substack{s\leq N^{1/2}\\ p|s\Rightarrow p\leq(\log N)^B}}\frac{|\Theta(1)g(s)|}{s}, \tag{4.100}$$

where $I_\delta(N) := [N^{1/2-3\delta/4-\delta/V}, N^{1/2-3\delta/4-\delta/2V}]$.

Now, observe from (4.75) that $|\Theta(1)|\leq \prod_{p|s}(|g(p)|+O_\kappa(1/p))$ and trivially $|g(s)|=\prod_{p|s}|g(p)|$. Hence, by Rankin's trick the innermost sum in (4.100) is

$$\leq \prod_{p \leq (\log N)^B} \left(1 + \frac{|g(p)|^2}{p} + O_\kappa\left(\frac{1}{p^2}\right)\right) \asymp_\kappa \prod_{p \leq (\log N)^B} \left(1 + \frac{|g(p)|^2}{p}\right).$$

Regarding the sum over s', arguing similarly and since $(\log N)^B \leq N^\varepsilon$, if we take N large enough with respect to ε and A_1, we see it is

$$\ll_\kappa \prod_{(\log N)^B < p \leq N^\varepsilon} \left(1 + \frac{|g(p)|^2}{p}\right) = \frac{\prod_{p \leq N^\varepsilon}\left(1 + \frac{|g(p)|^2}{p}\right)}{\prod_{p \leq (\log N)^B}\left(1 + \frac{|g(p)|^2}{p}\right)}$$
$$\ll \frac{\varepsilon^\beta (\log N)^\beta}{\prod_{p \leq (\log N)^B}\left(1 + \frac{|g(p)|^2}{p}\right)},$$

by partial summation from (4.3), which is made possible thanks to the hypothesis (4.5) on A_2. Here the implicit constant depends on κ, A_1, A_2 and the implicit constants in (4.2)–(4.3) and we take N large enough with respect to these parameters.

Finally, we come to the sum over the primes t. By Cauchy–Schwarz and equations (4.89)–(4.90) it is $\ll \sqrt{\beta}\varepsilon$.

Hence, overall we get a bound for (4.100) of

$$\ll \varepsilon^{2+\beta} \frac{|c_0(\alpha - 1)|}{|\Gamma(\alpha)|} \sqrt{\beta} (\log N)^\beta \ll |c_0| \frac{\beta \varepsilon^{2+\beta}}{|\Gamma(\alpha)|} (\log N)^\beta,$$

where we used $|\alpha - 1| \ll \sqrt{\beta}$ from Lemma 4.2.2 and where again the implicit constant depends on κ, A_1, A_2 and those in (4.2)–(4.3) and we take N sufficiently large with respect to these parameters.

Remark 4.12.1. *It is essential here to relate $\alpha - 1$ to $\sqrt{\beta}$ by means of the tight bound supplied by Lemma 4.2.2. Otherwise, the above error coming from the sum involving h_2 could potentially overcome the main term coming from the sum of h_1.*

4.12.3 The main term

By expanding the definition of $h_1(q)$ and all the conditions q is subject to, we see that the precise shape of $\Sigma'_q h_1(q)$ is

$$\sum_{\substack{s'\le N^{\varepsilon}\\ p|s'\Rightarrow p>(\log N)^B,\\ p|s'\Rightarrow |g(p)|>(\log\log N)^{-1/2}\\ s'\in\mathcal{A}'\\ s' \text{ squarefree}}} \frac{|g(s')|^2|H_{s'}^{-1}(1)|}{s'} \times \sum_{\substack{t \text{ prime:}\\ N^{1/2-3\delta/4-\varepsilon}\le t\le N^{1/2-3\delta/4-\varepsilon/2}\\ f(t)\neq 1}} \frac{|\theta_{N,\alpha}(t)g(t)H_t^{-1}(1)|}{t} \times \sum_{\substack{KQ_0/ts'\le s\le RN^{-\delta/4}/ts'\\ p|s\Rightarrow C<p\le(\log N)^B,\ p>C/|g(p)|,\ |g(p)|>(\log\log N)^{-1/2}\\ \omega(tss')\le A\log\log N\\ tss'\in\mathcal{A}\\ s \text{ squarefree}}} \frac{|\Theta(1)g(s)H_s^{-1}(1)|}{s}. \tag{4.101}$$

We now insert a series of observations to simplify its estimate.

4.12.4 Removal of some extra conditions

To begin with, by Lemma 4.6.1 we have

$$|H_q^{-1}(1)| \gg_{\kappa,D} 1.$$

In the following we then replace $h_1(q)$ with the value in (4.99) without the factor $H_q^{-1}(1)$.

Let us now focus on the condition (2). To this aim, we note that

$$\sideset{}{'}\sum_{\substack{KQ_0\le q\le RN^{-\delta/4}\\ \omega(q)\le A\log\log N}} h_1(q) = \sideset{}{'}\sum_{KQ_0\le q\le RN^{-\delta/4}} h_1(q) - \sideset{}{'}\sum_{\substack{KQ_0\le q\le RN^{-\delta/4}\\ \omega(q)>A\log\log N}} h_1(q),$$

where now $\sum'$ indicates the sum over all of the other remaining restrictions on q. The last sum on the right-hand side above can be upper bounded by

$$\frac{1}{(\log N)^A}\sum_{q\leq N}h_1(q)e^{\omega(q)}.$$

Using again

$$|\Theta(1)|\leq\prod_{p|s}\left(|g(p)|+O_\kappa\left(\frac{1}{p}\right)\right)$$

and

$$|g(s')|=\prod_{p|s'}|g(p)|,$$

as well as the trivial bound

$$|\theta_{N,\alpha}(t)|\ll_\kappa 1,$$

we are left with estimating

$$\frac{1}{(\log N)^A}\sum_{\substack{t\leq\sqrt{N}\\ t\text{ prime}}}\frac{1}{t}\sum_{q'\leq N}\frac{\prod_{p|q'}(e(\kappa+1)^2+O(1/p))}{q'}.$$

This can be done by means of Lemma 2.1.1 and Mertens' theorem, so getting

$$\ll_\kappa(\log\log N)(\log N)^{e(\kappa+1)^2-A}.$$

So far, if we collect together all the error terms inside the parenthesis in (4.98), we have got an overall error of

$$\begin{aligned}(4.102)\qquad &\ll(\log\log N)(\log N)^{e(\kappa+1)^2-A}+(\log N)^{(\kappa+1)^2+\kappa-\Re(\alpha)-A_1}(\log\log N)\\ &+|c_0|\frac{\beta\varepsilon^{2+\beta}}{|\Gamma(\alpha)|}(\log N)^\beta+\frac{(\log N)^{-\Re(\alpha)+1}}{N^{\delta/11}}\\ &\ll(\log N)^{(\kappa+1)^2+\kappa-\Re(\alpha)-A_1}(\log\log N)+|c_0|\frac{\beta\varepsilon^{2+\beta}}{|\Gamma(\alpha)|}(\log N)^\beta.\end{aligned}$$

Here, we chose

$$A:=A_1+e(\kappa+1)^2+1$$

and took N sufficiently large in terms of δ and κ, where the implicit constant above depends on $\delta, \kappa, A_1, A_2, D$ and those in (4.2)–(4.3).

We now concentrate on the condition (6). It is certainly equivalent to

$$\frac{(\log t)^{A_1+1}}{t^{3/4}} + \sum_{p|s} \frac{(\log p)^{A_1+1}}{p^{3/4}} + \sum_{p|s'} \frac{(\log p)^{A_1+1}}{p^{3/4}} \leq D.$$

Since t is extremely large and all the primes dividing s' are at least $(\log N)^B$, with $B = 4(K+2) = 4(A_1+2)$, it is actually equivalent to the fact that the corresponding sum over the prime factors of s must be slightly smaller than D. So we can lower bound (4.101) with the same expression, but having the innermost sum switched with that over those numbers s satisfying:

$$\sum_{p|s} \frac{(\log p)^{A_1+1}}{p^{3/4}} \leq D - 1.$$

Now, it becomes the complete sum minus that under the condition complementary to the above one. This last sum is upper bounded by

$$\begin{aligned}
&\leq \sum_{\substack{KQ_0/ts' \leq s \leq RN^{-\delta/4}/ts' \\ p|s \Rightarrow C<p\leq(\log N)^B \\ \sum_{p|s} \frac{(\log p)^{A_1+1}}{p^{3/4}} > D-1}} \frac{\prod_{p|s}(|g(p)|^2+O(1/p))}{s} \\
&\leq \sum_{\substack{C<r\leq(\log N)^B \\ r \text{ prime}}} \frac{(\log r)^{A_1+1}}{(D-1)r^{3/4}} \sum_{\substack{s\leq RN^{-\delta/4} \\ p|s\Rightarrow C<p\leq(\log N)^B \\ r|s}} \frac{\prod_{p|s}(|g(p)|^2+O(1/p))}{s} \\
&\ll_\kappa \sum_{r\leq(\log N)^B} \frac{(\log r)^{A_1+1}}{(D-1)r^{7/4}} \sum_{\substack{s\leq RN^{-\delta/4} \\ p|s\Rightarrow C<p\leq(\log N)^B}} \frac{\prod_{p|s}(|g(p)|^2+O(1/p))}{s} \\
&\ll_\kappa \sum_{r\leq(\log N)^B} \frac{(\log r)^{A_1+1}}{(D-1)r^{7/4}} \prod_{C<p\leq(\log N)^B} \left(1 + \frac{|g(p)|^2+O_\kappa(1/p)}{p}\right) \\
&\ll_{A_1} \frac{1}{D-1} \prod_{C<p\leq(\log N)^B} \left(1 + \frac{|g(p)|^2+O_\kappa(1/p)}{p}\right),
\end{aligned}$$

by Rankin's trick. By the arbitrariness of $D = D(\kappa, A_1)$, this term will be negligible. Indeed, we will now show that the complete sum over s contributes

$$\gg \prod_{C<p\leq(\log N)^B} \left(1+\frac{|g(p)|^2+O_\kappa(1/p)}{p}\right).$$

4.12.5 The estimate of the sum over s

We start by setting the value of Q_0 as

$$Q_0 := \frac{N(\log N)^{\eta_0}}{Q\beta^2},$$

where

$$\eta_0 := (\kappa+2)^2 - \beta - 2(\Re(\alpha)-1) + 3 = (\kappa+1)^2 - \beta + 2(\kappa - \Re(\alpha)) + 8 \geq 8$$

and β is as in the statement of Theorem 4.1.2. Note that this choice satisfies the conditions in (4.12) and Proposition 4.1.5, if N is large enough in terms of δ, κ and A_1. By condition (4.5), we deduce that

$$\frac{KQ_0}{ts'} \ll \frac{(\log N)^{\eta_0} N^{-\delta/4+\varepsilon}}{\beta^2} \leq N^{-\delta/4+\varepsilon}(\log N)^{\eta_0+2(A_1-\kappa(\alpha,\beta))},$$

with

$$\kappa(\alpha,\beta) = (\kappa+1)^2 + \kappa - \Re(\alpha) - \beta + 4.$$

Thus, recalling that $\varepsilon = \delta/V$, with $V \geq 5$, and taking N large enough in terms of δ, κ and A_1, we have $KQ_0/ts' < 1$. Thanks to this, the sum over s becomes a sum over a *long* interval, which heavily simplifies its computation. In particular, it coincides with

$$\sum_{\substack{s\leq RN^{-\delta/4}/ts' \\ p|s\Rightarrow C<p\leq(\log N)^B,\ p>C/|g(p)|,\ |g(p)|>(\log\log N)^{-1/2}}} \frac{|g(s)\prod_{p|s}(g(p)+O(1/p))|}{s}.$$

Applying Lemma 2.1.1, we find it is

$$\gg_\kappa \prod_{\substack{C<p\leq\min\{RN^{-\delta/4}/ts',(\log N)^B\} \\ p>C/|g(p)|,\ |g(p)|>(\log\log N)^{-1/2}}} \left(1+\frac{|g(p)|^2+O_\kappa(1/p)}{p}\right).$$

We restrict now the sum over s' to those numbers $\leq N^{\varepsilon/W}$, for a certain $W \geq 3$ to determine later. In this way, it is immediate to check that

$$\frac{RN^{-\delta/4}}{ts'} = \frac{N^{1/2-3\delta/4}}{ts'} \geq \frac{N^{1/2-3\delta/4}}{N^{1/2-3\delta/4-\varepsilon/2+\varepsilon/W}} = \exp\left(\varepsilon\left(\frac{1}{2}-\frac{1}{W}\right)\log N\right)$$
$$\geq (\log N)^B,$$

for N large enough with respect to ε and A_1. Thus, the product above is indeed only over the prime numbers $C < p \leq (\log N)^B$ and it equals P_1/P_2, where

$$P_1 := \prod_{\substack{C<p\leq(\log N)^B \\ p>C/|g(p)|}} \left(1 + \frac{|g(p)|^2 + O_\kappa(1/p)}{p}\right),$$
$$P_2 := \prod_{\substack{C<p\leq(\log N)^B \\ p>C/|g(p)|, \\ |g(p)|\leq(\log\log N)^{-1/2}}} \left(1 + \frac{|g(p)|^2 + O_\kappa(1/p)}{p}\right).$$

However P_2 is of bounded order, since

$$\sum_{\substack{C<p\leq(\log N)^B \\ p>C/|g(p)|,\ |g(p)|\leq(\log\log N)^{-1/2}}} \frac{|g(p)|^2 + O_\kappa(1/p)}{p}$$
$$\leq \sum_{\substack{p\leq(\log N)^B \\ |g(p)|\leq(\log\log N)^{-1/2}}} \left(\frac{|g(p)|^2}{p} + O_\kappa\left(\frac{1}{p^2}\right)\right) \ll_\kappa 1,$$

by Mertens' theorem, if N is large compared to κ and A_1. Regarding P_1 instead, it coincides with P_3/P_4, where

$$P_3 := \prod_{C<p\leq(\log N)^B} \left(1 + \frac{|g(p)|^2 + O_\kappa(1/p)}{p}\right),$$
$$P_4 := \prod_{\substack{C<p\leq(\log N)^B \\ p\leq C/|g(p)|}} \left(1 + \frac{|g(p)|^2 + O_\kappa(1/p)}{p}\right).$$

As before, one can show that P_4 is bounded, which makes the sum over s at least of order

$$\gg_\kappa \prod_{C<p\leq(\log N)^B}\left(1+\frac{|g(p)|^2+O_\kappa(1/p)}{p}\right) \asymp_{\kappa,C} \prod_{C<p\leq(\log N)^B}\left(1+\frac{|g(p)|^2}{p}\right).$$

4.12.6 The estimate of the sum over t

We remind that $\varepsilon=\delta/V$ and we assume δ, N and V to be as in Lemma 4.1.9. We then make use of (4.26) to lower bound the sum over t in (4.101).

4.12.7 The estimate of the sum over s'

By previous considerations, the sum over s' is

$$\sum_{\substack{s'\leq N^{\varepsilon/W},\\ p|s'\Rightarrow p>(\log N)^B\\ |g(p)|>(\log\log N)^{-1/2}\\ s'\in\mathcal{A}'}} \frac{|g(s')|^2}{s'} = \sum_{\substack{s'\leq N^{\varepsilon/W},\\ p|s'\Rightarrow p>(\log N)^B\\ |g(p)|>(\log\log N)^{-1/2}}} \frac{|g(s')|^2}{s'} - \sum_{\substack{s'\leq N^{\varepsilon/W},\\ p|s'\Rightarrow p>(\log N)^B\\ |g(p)|>(\log\log N)^{-1/2}\\ s'\notin\mathcal{A}'}} \frac{|g(s')|^2}{s'}. \tag{4.103}$$

We may deal with the second sum on the right-hand side of (4.103) using the definition of the set $\mathcal{A}'$ in condition (3.c) in the following way:

$$\begin{aligned}
&\sum_{\substack{s'\le N^{\varepsilon/W}, \\ p|s'\Rightarrow p>(\log N)^B \\ |g(p)|>(\log\log N)^{-1/2} \\ s'\notin\mathcal{A}'}} \frac{|g(s')|^2}{s'} \\
&\ll_\kappa \frac{1}{\varepsilon\log N} \sum_{\substack{s'\le N^{\varepsilon/W}, \\ p|s'\Rightarrow p>(\log N)^B \\ |g(p)|>(\log\log N)^{-1/2}}} \frac{|g(s')|^2}{s'} \sum_{\substack{r|s': \\ r \text{ prime}}} \frac{\log r}{\min\{|f(r)-1|,1\}} \\
&= \frac{1}{\varepsilon\log N} \sum_{\substack{(\log N)^B<r\le N^{\varepsilon/W} \\ |g(r)|>(\log\log N)^{-1/2} \\ r \text{ prime}}} \frac{\log r}{\min\{|f(r)-1|,1\}} \sum_{\substack{s'\le N^{\varepsilon/W} \\ r|s' \\ p|s'\Rightarrow p>(\log N)^B \\ |g(p)|>(\log\log N)^{-1/2}}} \frac{|g(s')|^2}{s'} \\
&\le \frac{1}{\varepsilon\log N} \sum_{\substack{(\log N)^B<r\le N^{\varepsilon/W} \\ |g(r)|>(\log\log N)^{-1/2} \\ r \text{ prime}}} \frac{|f(r)-1|^2\log r}{\min\{|f(r)-1|,1\}r} \sum_{\substack{s'\le N^{\varepsilon/W} \\ p|s'\Rightarrow p>(\log N)^B \\ |g(p)|>(\log\log N)^{-1/2}}} \frac{|g(s')|^2}{s'} \\
&\ll_\kappa \frac{1}{W} \sum_{\substack{s'\le N^{\varepsilon/W} \\ p|s'\Rightarrow p>(\log N)^B \\ |g(p)|>(\log\log N)^{-1/2}}} \frac{|g(s')|^2}{s'}.
\end{aligned}$$

In the above, the fraction $|f(r)-1|^2/\min\{|f(r)-1|,1\}$ is easily seen to be bounded and we used Mertens' theorem to compute the sum over the primes.

Thus, choosing a value of $W = W(\kappa) \ge 3$ large enough we deduce that (4.103) is

$$\begin{aligned}
\gg_\kappa \prod_{\substack{(\log N)^B<p\le N^{\varepsilon/W} \\ |g(p)|>(\log\log N)^{-1/2}}} \left(1+\frac{|g(p)|^2}{p}\right) &= \frac{\prod_{(\log N)^B<p\le N^{\varepsilon/W}}\left(1+\frac{|g(p)|^2}{p}\right)}{\prod_{\substack{(\log N)^B<p\le N^{\varepsilon/W} \\ |g(p)|\le(\log\log N)^{-1/2}}}\left(1+\frac{|g(p)|^2}{p}\right)} \\
&\gg \prod_{(\log N)^B<p\le N^{\varepsilon/W}} \left(1+\frac{|g(p)|^2}{p}\right),
\end{aligned}$$

by Lemma 2.1.1 and since the product in the denominator above is bounded.

4.12.8 Completion of the proof of Theorem 4.1.2

Collecting the above estimates together, we have found an overall lower bound for the sum involving $h_1(q)$ in (4.98) of

$$\gg \left|\frac{\eta c_0}{\Gamma(\alpha)}\right| \frac{\beta\delta}{V} \prod_{C<p\leq N^{\varepsilon/W}} \left(1+\frac{|g(p)|^2}{p}\right),$$

with c_0 as in the statement of Theorem 4.1.2 and η as in Lemma 4.1.9.

The above product can be estimated through partial summation, giving a contribution of

$$\gg \left(\frac{\varepsilon}{W}\right)^{\beta} (\log N)^{\beta} \gg \varepsilon^{\beta} (\log N)^{\beta},$$

where the $\gg$ constant depends on κ, A_2, C and that in (4.3).

Recalling that $\varepsilon = \delta/V$, that C depends on δ, κ, A_1 and the implicit constant in (4.2) and collecting the previous two estimates together, we have proved that the sum of $h_1(q)$ in (4.98) is

$$\gg \left|\frac{\eta c_0}{\Gamma(\alpha)}\right| \frac{\delta^{1+\beta}}{V^{1+\beta}} \beta (\log N)^{\beta}.$$

The above implicit constant may depend on δ, κ, A_1, A_2 and the implicit constant in (4.2)–(4.3) and we consider N as sufficiently large with respect to all these parameters.

We deduce a lower bound for the term inside parenthesis in (4.98) of

$$\begin{aligned} &\gg \left|\frac{\eta c_0}{\Gamma(\alpha)}\right| \frac{\delta^{1+\beta}}{V^{1+\beta}} \beta (\log N)^{\beta} \\ &+ O\biggl((\log N)^{(\kappa+1)^2+\kappa-\Re(\alpha)-A_1} (\log\log N) + |c_0| \frac{\beta \varepsilon^{2+\beta}}{|\Gamma(\alpha)|} (\log N)^{\beta} \biggr) \\ &\gg \left|\frac{\eta c_0}{\Gamma(\alpha)}\right| \frac{\delta^{1+\beta}}{V^{1+\beta}} \beta (\log N)^{\beta}. \end{aligned}$$

Here, we used the conditions (4.5)–(4.6) and took V large enough in terms of δ, κ, A_1, A_2 and the implicit constants in (4.2)–(4.3), and took N sufficiently large in terms of all these parameters.

Remembering that

$$Q_0 = \frac{N(\log N)^{\eta_0}}{Q\beta^2},$$

where

$$\eta_0 = (\kappa + 2)^2 - \beta - 2(\Re(\alpha) - 1) + 3,$$

as well as the relations (4.5), we have overall found that

$$V(N, Q; f) \gg \left| \frac{c_0 \beta}{\Gamma(\alpha)} \right|^2 \left(\frac{\delta}{V} \right)^{2(1+\beta)} QN(\log N)^{\beta + 2(\Re(\alpha) - 1)}. \tag{4.104}$$

The implicit constant above may depend on δ, κ, A_1, A_2 and those in (4.2)–(4.3) and $N \geq N_0$, with N_0 large enough depending on all these parameters. Since the term $(\delta/V)^{2(1+\beta)}$ is uniformly bounded in terms of the aforementioned parameters, it may be absorbed in the implicit constant in (4.104). Finally, recalling the estimate (4.41), we notice that equation (4.104) is actually in the form stated in Theorem 4.1.2, thus concluding its proof.

Chapter 5

The variance in arithmetic progressions of divisor functions and other sequences close to 1

Summary

This chapter extends the previous one, by proving presumably sharp lower bounds for the variance of divisor functions d_{α_N}, for a sequence of parameters α_N close to 1, and of the indicator of $y-$smooth numbers, for parameters y close to N; we sketch their proof in the Introduction. To do this, we again reduce the problem to understanding L^2-integrals over subarcs of the circle of the exponential sum with coefficients such functions: in section two we list the results we obtained about their size and in section four we proceed to the proof of their upper bound part; regarding the corresponding lower bounds, we first produce some estimates for the partial sums of our functions twisted with Ramanujan sums, which is done in section five, to implement them inside a circle method approach in sections six to nine. Finally, sections ten and eleven are devoted to the proof of our new results related to the variance of respectively the constant function 1 and the above multiplicative functions close to 1.

5.1 Introduction

In the previous chapter we have proved a lower bound for the variance of a generalized divisor function f in arithmetic progressions, where f needed to be suitably away from the constant function 1. The measure of the distance between f and 1 was given by the parameter $\beta > 0$, that in Theorem 4.1.2 was required to satisfy $\beta \geq (\log N)^{\kappa(\alpha,\beta)-A_1}$ as in (4.5). When $f = d_\alpha$ is just a fixed α-fold divisor function, the condition turns into $\alpha \neq 1$.

The first result of this chapter is the estimate of the variance of the constant function 1, which satisfies (4.2)–(4.3) with $\alpha = 1$ and $\beta = 0$.

Proposition 5.1.1. *For any $Q \geq 1$, we have*

$$V(N, Q; d_1) \ll Q^2.$$

On the other hand, there exists an absolute constant $c > 0$ such that for any $cN^{2/3} \leq Q \leq N$ and N large enough, we have

$$V(N, Q; d_1) \gg Q^2.$$

We will prove Proposition 5.1.1 by employing a circle method approach, even though it is possible to use more elementary methods, which also extend the result to all $N^\delta \leq Q = o(N)$, for any $0 < \delta < 1$, as pointed out in a previous referee report of [52].

Proposition 5.1.1 tells us that $V(N, Q; d_1)$ is of a different shape than $V(N, Q; d_\alpha)$, for any parameter $\alpha \neq 1$. This change motivated us to look at intermediate cases, like those corresponding to sequences of divisor functions $d^{\varpi}_{\alpha_N}(n) = \alpha_N^{\varpi(n)}$ (see Ch. Notations for the definition of $\varpi(n)$), for parameters α_N increasingly approaching 1. If the rate of convergence of α_N to 1 is not too high, then Theorem 4.1.2 gives the following non-trivial lower bound.

Corollary 5.1.2. *Let $A > 0$ be a real number and $\alpha_N := 1 + 1/R(N)$, where $R(N)$ is a real non-vanishing function such that $|R(N)| \leq (\log N)^A$. Let $\delta > 0$ small enough and $N^{1/2+\delta} \leq Q \leq N$. Then there exists a constant $B > 0$ such* *that if* $|R(N)|\geq B$ *we have*

$$V(N,Q;d^{\varpi}_{\alpha_N}) \gg_{\delta,A} \frac{Q}{R(N)^4}\sum_{n\leq N} d^{\varpi}_{\alpha_N}(n)^2 \tag{5.1}$$
$$\gg \frac{QN}{R(N)^4}\exp\left(\left(2+\frac{1}{R(N)}\right)\frac{\log\log N}{R(N)}\right),$$

if N *is large enough with respect to* δ *and* A*.*

The lower bound of Corollary 5.1.2 lacks in three aspects:

- when $|R(N)|> \log\log N$ it is always of size $QN/R(N)^4$, but it turns out that the dependence on $R(N)$ is not optimal;
- it holds only for somewhat small values of $R(N)$ (bounded by a power of a logarithm);
- considering Proposition 5.1.1, extra terms of different shape should occur.

The next result, which is one of the main new contributions of this chapter, improves on Corollary 5.1.2 in all of the above three points.

Theorem 5.1.3. *Let* $\alpha_N = 1+1/R(N)$*, where* $R(N)$ *is a non-zero real function. Assume* $N^{1/2+\delta}\leq Q\leq N$*, with* $\delta>0$ *sufficiently small. Then there exists a constant* $C=C(\delta)>0$ *such that if* $C\log\log N\leq |R(N)|\leq N^{\delta/12}$ *and* N *is large in terms of* δ*, we have*

$$V(N,Q;d^{\varpi}_{\alpha_N}) \gg_{\delta} \frac{QN}{R(N)^2}\log\left(\frac{\log N}{\log(2N/Q)}\right)+Q^2. \tag{5.2}$$

Compared to (5.1), the lower bound (5.2) improves the exponent of $R(N)$, shows the presence of the extra factor Q^2, which dominates on certain ranges of $R(N)$, and $|R(N)|$ is allowed to grow much bigger than an arbitrarily large power of $\log N$.

The proof of Theorem 5.1.3 has a different flavour to that of Corollary 5.1.2. The latter is a consequence of the fine study of the asymptotic expansion of partial sums of the divisor functions $d^{\varpi}_{\alpha_N}(n)$. The former instead makes use of the Taylor expansion $d^{\varpi}_{\alpha_N}(n) = (1+1/R(N))^{\varpi(n)} =$

$1 + \varpi(n)/R(N) + O(\varpi(n)^2/R(N)^2)$, thus reducing the problem on understanding the variance of the additive function $\varpi(n)$ in arithmetic progressions. Since such function is, for the majority of positive integers $n \leq N$, of size roughly $\log\log N$ (see Proposition 2.2.1), this justifies the condition $|R(N)|\geq C\log\log N$ in the hypotheses of Theorem 5.1.3.

Theorem 5.1.4. *Assume $N^{1/2+\delta} \leq Q \leq N$, with $\delta > 0$ sufficiently small. Then we have*

$$V(N, Q; \varpi) \gg_\delta Q^2(\log\log N)^2 + QN\log\left(\frac{\log N}{\log(2N/Q)}\right),$$

if N is large enough in terms of δ.

We might think of $\varpi(n)$ as made of a deterministic part $\log\log N$ and a more random one $\varpi(n) - \log\log N$. We will show that the L^2-integral over minor arcs of the exponential sum with coefficients the former function has size $Q(\log\log N)^2$ and that of the exponential sum with coefficients the latter has size $N\log(\frac{\log N}{\log(2N/Q)})$. Since we saw in the previous chapter (see (4.9)) that we might heuristically lower bound the variance of a random looking function f with the L^2-integral over minor arcs of the exponential sum with coefficients f, the above claimed estimates explain the structure of the lower bound in Theorem 5.1.4. Similarly, they explain also that in Theorem 5.1.3, if we look at $d_{\alpha_N}^{\varpi}(n)$ as roughly $1 + \log\log N/R(N) \approx 1$, when $|R(N)|\geq C\log\log N$ and C is large enough, plus $(\varpi(n) - \log\log N)/R(N)$. In this sense, Theorem 5.1.3 might be seen as a sort of interpolation result between those of Proposition 4.1.1 and of Proposition 5.1.1.

The sequence of functions $d_{\alpha_N}^{\varpi}(n)$ is only one instance of a wide class of multiplicative functions 'close' to 1. Another interesting representative of such class is the characteristic function of the y–smooth numbers (see Ch. Notations for a definition thereof), for parameters y near to N.

Theorem 5.1.5. *Let $N^{1/2+\delta} \leq Q \leq N$, with $\delta > 0$ sufficiently small. There exists a large constant $C > 0$ such that the following holds. If*

$$\sqrt{N} \leq y \leq N/C$$

and N is large enough in terms of δ, we have

$$V(N,Q;\mathbf{1}_{y-smooth}) \gg_\delta QN\log\left(\frac{\log N}{\log y}\right)+Q^2.$$

We observe that Harper's result [22, Theorem 2] gives a tight corresponding upper bound for the variance above, when $Q = N/(\log N)^A$, with $A > 0$, and $\sqrt{N} \le y \le N^{1-\delta}$, say.

For the characteristic function of the y-smooth numbers, we recognize the contribution from the constant part 1 inside $\mathbf{1}_{y-\text{smooth}}(n)$ and that from the more random part $\mathbf{1}_{y-\text{smooth}}(n) - 1$. Moreover, we will prove that the L^2-integral over minor arcs of the exponential sum with coefficients 1 has size Q and that of the exponential sum with coefficients $\mathbf{1}_{y-\text{smooth}}(n) - 1$ has size $N\log(\frac{\log N}{\log y})$. Therefore, considering (4.9), we might then interpret Theorem 5.1.5 as a sort of interpolation result between that of Proposition 5.1.1 and the Parseval bound (4.10).

5.1.1 Sketch of the proof of Theorem 5.1.1

The upper bound can be easily deduced by elementarily estimating the number of integers up to N inside an arithmetic progression $a \pmod q$ and those coprime with q, to then study their difference on average over $q \le Q$.

Regarding the lower bound instead, we use Proposition 4.1.4 to reduce to work with the L^2-integral of the exponential sum with coefficients 1. If we restrict such integral only over suitable subarcs of the circle, we can then read this sum as a geometric progression, which we know how to compute. We end up with an expression roughly like $2 - 2\Re(e(N/q))$ that we need to sum up over all the arcs considered. An application of the van der Corput's inequality takes care of the phase term $e(N/q)$, showing some saving on average.

5.1.2 Sketch of the proof of Theorem 5.1.5

The proof of Theorem 5.1.5 follows the same basic strategy of that of Theorem 4.1.2, with three main differences:

1. we are also required to find a sharp upper bound for the L^2-integral $\int_{\mathfrak{m}}|\sum_{n\le N,\ p|n\Rightarrow p\le y} e(n\theta)|^2 d\theta$ to then, through the use of triangle inequal-

ity, produce the term Q^2 in the lower bound of the variance of the $y-$smooth numbers in arithmetic progressions;

2. we restrict the set of moduli q to lie in different ranges, according to the size of y:

   (a) if y is somewhat small, say $\sqrt{N} \le y \le N^{1-\delta/8}$, with $\delta > 0$ small, then we confine q in the set of prime numbers in the range $(\log N, \sqrt{N}]$;

   (b) if y is somewhat large, say $N^{1-\delta/8} < y \le N/C$, where C is a suitably large constant, we consider values of $q > 1$ that are squarefree integers smaller than $\sqrt{N}$ with all their prime factors larger than N/y.

3. we need to choose an approximating function $h(n)$ conceptually completely different from that used in Theorem 4.1.2. Indeed, we take $h(n) = \sum_{r|n,\ r\le R} g(r)$, with $R := N^{1/2-\delta/2}$, where:

   (a) when y is as in 2.(a), we roughly take $g(r)$ as the indicator of the prime numbers $r \le R$;

   (b) when y is as in 2.(b), we let $g(r)$ be the multiplicative function supported on the squarefree numbers r and given on the primes by

$$g(p) = \begin{cases} 1 & \text{if } N/y < p \le R; \\ 0 & \text{otherwise.} \end{cases}$$

As in Theorem 4.1.2, a key point here consists in evaluating the partial sums of $\mathbf{1}_{y-\text{smooth}}(n)$ twisted with Ramanujan sums $c_q(n)$ (defined in (4.11)), with q as in point 2, for which we have the following result.

Lemma 5.1.6. *Let C be a sufficiently large positive constant and consider $\sqrt{N} \le y \le N/C$. Then for any prime number $\log N < q \le \sqrt{N}$ and N large enough, we have*

$$\Bigg| \sum_{\substack{n\le N \\ p|n \Rightarrow p\le y}} c_q(n) \Bigg| \gg N \log\left(\frac{\log N}{\log(\max\{N/q, y\})}\right)$$

and for any squarefree integer $1 < q \le \sqrt{N}$ *with all the prime factors larger than* N/y*, we have*

$$\Bigg|\sum_{\substack{n\le N\\ p|n\Rightarrow p\le y}} c_q(n)\Bigg| \gg N\log\left(\frac{\log N}{\log y}\right).$$

5.1.3 Sketch of the proof of Theorem 5.1.3

As in the demonstration of Theorem 4.1.2 given in the previous chapter, we start again from Harper and Soundararajan [28, Proposition 1] to roughly get

$$V(N,Q;d_{\alpha_N}^{\varpi}) \gg Q\int_{\mathfrak{m}}\Bigg|\sum_{n\le N} d_{\alpha_N}^{\varpi}(n)e(n\theta)\Bigg|^2 d\theta, \tag{5.3}$$

but now we use Cauchy–Schwarz's inequality to express the lower bound for the above L^2-integral in the following form:

$$\ge \frac{|\int_{\mathfrak{m}}\sum_{n\le N} d_{\alpha_N}^{\varpi}(n)e(n\theta)\sum_{n\le N}\overline{h(n)}\phi(n/N)e(-n\theta)d\theta|^2}{\int_{\mathfrak{m}}|\sum_{n\le N} h(n)\phi(n/N)e(n\theta)|^2 d\theta}, \tag{5.4}$$

where we choose $h(n) := \sum_{p\le R,\ p|n} 1$, with $R := N^{1/2-\delta/2}$. Here $h(n)$ is a truncated version of $\varpi(n)$ (whereas in Theorem 4.1.2, $h(n)$ was a truncation of a generalized divisor function) since we will simplify the study of (5.4) by replacing $d_{\alpha_N}^{\varpi}(n)$ with $\varpi(n)$. Moreover, we take $\phi(t)$ to be a real smooth function belonging to the "Fourier class" $\mathcal{F}$ of functions $\Psi(t)$ satisfying the following properties:

- $\Psi(t)$ is compactly supported in $[0,1]$;
- $0\le\Psi(t)\le 1$, for all $0\le t\le 1$;
- $\int_0^1\Psi(t)dt \ge 1/2$;
- $|\hat{\Psi}(\xi)| \ll_A (1+|\xi|)^{-A}$, for any $A>0$, where $\hat{\Psi}(\xi) := \int_{-\infty}^{+\infty}\Psi(t)e(-\xi t)dt$ denotes the Fourier transform of $\Psi(t)$.

In the proof of Theorem 4.1.2, we extended the integral in the denominator to the whole circle and used Parseval's identity to compute it. Here, this procedure would be inefficient and we instead need to work out carefully the exponential sum with coefficients $h(n)$ over the minor arcs. In this way, we obtain that

$$\int_{\mathfrak{m}} \Big| \sum_{n\le N} h(n)\phi(n/N)e(n\theta)\Big|^2 d\theta \ll N \log\left(\frac{\log N}{\log(2N/Q)}\right).$$

In contrast to Theorem 4.1.2, we cannot directly lower bound the integral in the numerator, by means of [28, Proposition 3]. In fact, this is only possible when the main contribution comes from minor arcs centred on fractions with denominator smaller than R, which is not the case for divisor functions approaching 1. Therefore, we rewrite the numerator in (5.4) as $\int_{\mathfrak{m}} = \int_0^1 - \int_{\mathfrak{M}}$ and proceed by asymptotically estimating both integrals. To this aim we rely on Harper and Soundararajan [28, Proposition 2], which we next report in a more compact form.

Proposition 5.1.7. *Let $f(n)$ be any complex sequence. Let N be a large positive integer, $K \ge 5$ be a parameter and K, Q_0 and Q be such that*

$$K\sqrt{N\log N} \le Q \le N \text{ and } \frac{N\log N}{Q} \le Q_0 \le \frac{Q}{K^2}.$$

Assume moreover that $KQ_0 < R \le Q/2K$. Then we have

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)|^2 d\theta \ge \Big| \int_{\mathfrak{m}} \mathcal{S}_f(\theta)\overline{\mathcal{G}(\theta)} d\theta\Big|^2 \left(\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta\right)^{-1}, \tag{5.5}$$

where

$$\mathcal{S}_f(\theta) := \sum_{n\le N} f(n)e(n\theta) \text{ and } \mathcal{G}(\theta) := \sum_{n\le N}\Big(\sum_{\substack{r|n\\ r\le R}} g(r)\Big)\phi\left(\frac{n}{N}\right)e(n\theta),$$

for any complex arithmetic function $g(r)$ and real function $\phi(t)$.

Let $M := \max_{r \leq R}|g(r)|$. If $\phi(t) \in \mathcal{F}$, then we also have

$$
\begin{aligned}
(5.6)\qquad & \int_{\mathfrak{m}} \mathcal{S}_f(\theta)\overline{\mathcal{G}(\theta)}d\theta \\
&= \sum_{n\leq N} f(n)\bigg(\sum_{\substack{r|n\\ r\leq R}} \overline{g(r)}\bigg)\phi\bigg(\frac{n}{N}\bigg) \\
&- N\sum_{q\leq KQ_0}\int_{-K/qQ}^{K/qQ}\bigg(\sum_{n\leq N} f(n)c_q(n)e(n\beta)\bigg)\bigg(\sum_{\substack{r\leq R\\ q|r}}\frac{\overline{g(r)}}{r}\bigg)\hat{\phi}(\beta N)d\beta \\
&+ O\bigg(\frac{MKR\sqrt{Q_0}\log N}{\sqrt{Q}}\sqrt{\sum_{n\leq N} f(n)^2}\bigg).
\end{aligned}
$$

To compute (5.6) when $f(n) = d_{\alpha_N}^{\varpi}(n)$, $g(n) = \mathbf{1}_{\text{prime}}(n)$ and $\phi(t) \in \mathcal{F}$, we split the analysis into two cases: when $|R(N)| \leq (\log\log N)^3$ or not. In the first case, we make use of the following estimate.

Lemma 5.1.8. *Let $\alpha_N = 1 + 1/R(N)$, where $R(N)$ is a non-zero real function, and $R := N^{1/2-\delta/2}$, for $\delta > 0$ small. Assume that $N^{1/2+\delta} \leq Q < cN(\log\log N)/R(N)^2$, for a certain absolute constant $c > 0$. There exists a sufficiently large constant $C = C(\delta) > 0$ such that if $C\log\log N \leq |R(N)| \leq (\log\log N)^3$ and N is large enough with respect to δ, we have*

$$\bigg|\sum_{2N/Q<p\leq R}\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_p(n)\phi\left(\frac{n}{N}\right)\bigg| \gg \frac{N}{|R(N)|}\log\left(\frac{\log R}{\log(2N/Q)}\right).$$

When $|R(N)| > (\log\log N)^3$ instead, we Taylor expand the divisor functions $d_{\alpha_N}^{\varpi}(n) = (1+1/R(N))^{\varpi(n)}$ as $1+\varpi(n)/R(N)+O(\varpi(n)^2/R(N)^2)$. This is justified in our range of $R(N)$ since $\varpi(n)$ is, for the majority of positive integers $n \leq N$, of size roughly $\log\log N$ (see Proposition 2.2.1). The contribution of the constant function 1 into (5.6) will be evaluated with the help of the following lemma, which shows a huge amount of cancellation in the partial sums of a Ramanujan sum weighted with an exponential phase.

Lemma 5.1.9. *Let $R := N^{1/2-\delta/2}$, for $\delta > 0$ small, and $q < R$ be a prime* *number. Then we have*

$$\sum_{n\le N} c_q(n) e\left(\frac{nu}{N}\right) \ll q(1+|u|),$$

uniformly for all real numbers u.

On the other hand, the contribution of the ϖ-function into (5.6) will be handled with the aid of the next result.

Lemma 5.1.10. *Let $R := N^{1/2-\delta/2}$, for $\delta > 0$ small, and suppose that $N^{1/2+\delta} \le Q \le cN/\log\log N$, for a certain absolute constant $c > 0$. Then for any N large enough with respect to δ, we have*

$$\Bigg| \sum_{2N/Q<p\le R} \frac{1}{p} \sum_{n\le N} \varpi(n) c_p(n) \phi\left(\frac{n}{N}\right) \Bigg| \gg N \log\left(\frac{\log R}{\log(2N/Q)}\right).$$

We might think of $\varpi(n)$ as made of a deterministic part $\log\log N$ and a more random one $\varpi(n) - \log\log N$. Whence, we might see $d^{\varpi}_{\alpha_N}(n)$ as well roughly as $1 + \log\log N/R(N) \approx 1$, when $|R(N)| \ge C \log\log N$ and C is large enough, plus $(\varpi(n) - \log\log N)/R(N)$. Considering their contribution to (5.3) individually, we will get that the former contributes an amount of Q^2, whereas the latter one of $QN\log(\frac{\log N}{\log(2N/Q)})/R(N)^2$. This explains the structure of the lower bound in Theorem 5.1.3.

5.2 Proof of Corollary 5.1.2

In this section we are going to prove the lower bound for the variance in arithmetic progressions of the sequence of divisor functions $d^{\varpi}_{\alpha_N}(n)$ in the form that easily follows as a consequence of Theorem 4.1.2.

Proof of Corollary 5.1.2. We let $f(n) = d^{\varpi}_{\alpha_N}(n)$, with $\alpha_N = 1 + 1/R(N)$ as in the statement. By choosing B large enough, we may assume $1/2 \le \alpha_N^2 \le 3/2$. Using [17, Theorem 1] with $A_1 = 4$ we see that

$$(5.7) \qquad \sum_{n\le N} f^2(n) = \sum_{j=0}^{3} \frac{c_j(\alpha_N^2)}{\Gamma(\alpha_N^2 - j)} N(\log N)^{\alpha_N^2 - j - 1} + O\left(\frac{N\log\log N}{\log N}\right),$$

where the Gamma function has been defined in Ch. Notations and the coefficients $c_j(\alpha_N^2)$ are given by

$$c_j(\alpha_N^2) := \frac{d^j}{dz^j} \frac{(z-1)^{\alpha_N^2} F(z)}{z}\bigg|_{z=1},$$

where $F(z)$ is the Dirichlet series of $f^2(n)$. By adapting the proof of the C^4-continuation of $F(z)(z-1)^{\alpha_N^2}$ to the half-plane $\Re(z) \geq 1$ at the start of [17, Sect. 2], we can easily check that every $c_j(\alpha_N^2)$ is uniformly bounded, for every $1/2 \leq \alpha_N^2 \leq 3/2$. Moreover, for any $j \geq 1$

$$\Gamma(\alpha_N^2 - j) = \frac{\Gamma(\alpha_N^2)}{(\alpha_N^2 - j)(\alpha_N^2 - j + 1) \cdots (\alpha_N^2 - 1)},$$

from which we deduce that

$$|\Gamma(\alpha_N^2)| \asymp 1 \text{ and } |\Gamma(\alpha_N^2 - j)| \gg 1$$

thanks to the continuity of $\Gamma(z)$ and our hypothesis on α_N. Hence, we conclude that

$$\sum_{n \leq N} f^2(n) \gg N(\log N)^{\alpha_N^2 - 1} = N \exp\left(\left(2 + \frac{1}{R(N)}\right) \frac{\log\log N}{R(N)}\right), \tag{5.8}$$

if N is large enough, where we also used that

$$c_0(\alpha_N^2) = \prod_p \left(1 + \frac{f(p)^2}{p} + \frac{f(p^2)^2}{p^2} + \cdots\right)\left(1 - \frac{1}{p}\right)^{\alpha_N^2} \gg 1.$$

Similarly, we have

$$c_0 = \prod_{p \leq N} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right)\left(1 - \frac{1}{p}\right)^{\alpha_N} \gg 1.$$

We notice that the relations (4.5) are trivially satisfied, with $\beta = 1/R(N)^2$, since $R(N)$ is allowed to grow at most as a large power of $\log N$ and we can take A_1, A_2 there arbitrarily large.

An application of Theorem 4.1.2, together with equation (5.8), leads to

(5.1), since again by continuity we have $|\Gamma(\alpha_N)|\ll 1$. $\square$

5.3 The L^2-integral of some exponential sums over minor arcs

Our main aim from now until the end of this chapter is to prove Theorems 5.1.1, 5.1.3, 5.1.4 and 5.1.5, so getting a presumably sharp lower bound for the variance in arithmetic progressions of respectively the functions $f(n) = 1$, $f(n) = \varpi(n)$, $f(n) = d_{\alpha_N}^{\varpi}(n)$ and $f(n) = \mathbf{1}_{y-\text{smooth}}(n)$. In particular, showing Theorem 5.1.3 we will be able to improve on the result of Corollary 5.1.2, as discussed in the introduction to this chapter.

To do this, as already done in the previous chapter, we will invoke Proposition 4.1.4 to switch our attention to L^2-integrals $\int_{\mathfrak{m}}|\mathcal{S}_f(\theta)|^2 d\theta$ over unions of minor arcs $\mathfrak{m}$ of exponential sums $\mathcal{S}_f(\theta) := \sum_{n\leq N} f(n)e(n\theta)$ with coefficients such functions $f(n)$. Our plan is to employ this strategy with the choice of minor arcs $\mathfrak{m} = \mathfrak{m}(K, Q, Q_0)$ given by K a large positive constant, $N^{1/2+\delta} \leq Q \leq N$, for any suitably small $\delta > 0$, and Q_0 satisfying (4.12). This will indeed be the underlying choice of minor arcs in the next propositions.

Regarding the constant function 1, we have the following result.

Proposition 5.3.1. *For any N large enough with respect to δ, we have*

$$\int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta \gg Q. \tag{5.9}$$

Regarding the additive function $\varpi(n)$, we will prove the next proposition.

Proposition 5.3.2. *Suppose $KQ_0 < N^{1/2-\delta/2}$. If N is sufficiently large in terms of δ, we have*

$$\int_{\mathfrak{m}} |\mathcal{S}_{\varpi}(\theta)|^2 d\theta \gg_{\delta} Q(\log\log N)^2 + N \log\left(\frac{\log N}{\log(2N/Q)}\right). \tag{5.10}$$

Regarding the multiplicative function $d_{\alpha_N}^{\varpi}(n)$, the result is the following.

Proposition 5.3.3. *Suppose $KQ_0 < N^{1/2-\delta/2}$. There exists a large constant $C = C(\delta) > 0$ such that if $C \log\log N < |R(N)| \leq N^{\delta/12}$ and N is large enough in terms of δ, we have*

$$\int_{\mathfrak{m}} |\mathcal{S}_{d_{\overline{\alpha}_N}}(\theta)|^2 d\theta \gg_\delta \frac{N}{R(N)^2} \log\left(\frac{\log N}{\log(2N/Q)}\right) + Q. \tag{5.11}$$

Remark 5.3.4. *From the proof of Theorem 4.1.2 it can be easily evinced that*

$$\int_{\mathfrak{m}} |\mathcal{S}_{d_{\overline{\alpha}_N}}(\theta)|^2 d\theta \gg_\delta \frac{N}{R(N)^4} \exp\left(\left(2 + \frac{1}{R(N)}\right)\frac{\log\log N}{R(N)}\right),$$

whenever $B < |R(N)| \leq \log\log N$, for a suitable large constant $B \geq 3$.

Regarding the indicator of y–smooth numbers, we will show the following lower bound.

Proposition 5.3.5. *Assume that $KQ_0 \leq N^{1/2-\delta}(\log N)^{17}$. Let*

$$u := (\log N)/(\log y).$$

There exists a large constant $C > 0$ such that the following holds. If

$$1 + \frac{\log C}{\log N} \leq u \leq 2$$

and N is large enough in terms of δ, we have

$$\int_{\mathfrak{m}} |\mathcal{S}_{\mathbf{1}_{y-smooth}}(\theta)|^2 d\theta \gg_\delta N \log u + Q. \tag{5.12}$$

In order to show that Q-times our lower bounds (5.9), (5.10), (5.11) and (5.12) provides us with the expected best possible approximation for the related variances, we will produce corresponding sharp upper bounds for them. In some cases, they will also turn out to be useful to deduce the aforementioned lower bounds.

Proposition 5.3.6. *With notations as in Propositions 5.3.1, 5.3.2, 5.3.3 and 5.3.5, we have that*

a) (5.9) *is sharp;*

b) (5.10) *is sharp;*

c) the estimate (5.11) *is sharp when* $|R(N)| > (\log\log N)^{3/2}$*;*

d) (5.12) *is sharp.*

Remark 5.3.7. *It should be possible to produce a sharp upper bound for the integral in* (5.11) *in the whole range* $|R(N)| > C\log\log N$ *(see Remark 5.4.1 below).*

To work out the size of the L^2-integral over minor arcs of the exponential sum with coefficients $f(n) = d^{\varpi}_{\alpha_N}(n)$, we will split f into a sum $f = f_d + f_r$ of a deterministic part f_d, constant, and a 'pseudorandom' one f_r. By triangle inequality we will separate their contribution to the integrals to then analyse them individually. To deal with $\int_{\mathfrak{m}}|\mathcal{S}_{f_d}(\theta)|^2 d\theta$ we will unfold the definition of minor arcs and insert classical estimates for the size of a complete exponential sum. Regarding $\int_{\mathfrak{m}}|\mathcal{S}_{f_r}(\theta)|^2 d\theta$ instead, when $|R(N)| > (\log\log N)^{3/2}$, we will reduce the problem to estimate the L^2-integral over minor arcs of the exponential sum with coefficients $\varpi(n)$. To this aim, we will write $\varpi(n) = \Sigma_1 + \Sigma_2$, where Σ_1 is a sum over prime numbers smaller than a power of $2N/Q$ and Σ_2 the remaining part, and again use triangle inequality. To estimate $\int_{\mathfrak{m}}|\mathcal{S}_{\Sigma_2}(\theta)|^2 d\theta$ we will use Parseval's identity and an application of Turán–Kubilius' inequality. Regarding $\int_{\mathfrak{m}}|\mathcal{S}_{\Sigma_1}(\theta)|^2 d\theta$ instead we will expand out the square inside the integral and unfold the definition of minor arcs to then conclude by counting the number of primes which are solution to certain systems of congruences.

5.4 Proof of Proposition 5.3.6

We set the parameter K to be a large constant, $N^{1/2+\delta} \leq Q \leq N$, with N sufficiently large in terms of δ, and Q_0 satisfying (4.12). We keep these notations throughout the rest of this section, where our aim is to prove each case of Proposition 5.3.6, thus showing that the lower bounds (5.9), (5.10), (5.11) and (5.12) are sharp, at least in some ranges of the parameters considered.

5.4.1 The case of the constant function 1

We use the well-known bound

$$|\mathcal{S}_1(\theta)| \ll \min\left\{N, \frac{1}{||\theta||}\right\}, \tag{5.13}$$

where $||\theta||$ indicates the distance of θ from the nearest integer. Since $\theta = a/q + \delta$, with $|\delta| \leq K/qQ$ and $q > KQ_0$, we have that either $||\theta|| = |\theta|$ or $||\theta|| = 1 - |\theta|$. Hence, by symmetry, we find that

$$\begin{aligned}\int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta &\ll \sum_{KQ_0 < q \leq Q} \sum_{\substack{1 \leq a < q/2 \\ (a,q)=1}} \int_{a/q - \frac{K}{qQ}}^{a/q + \frac{K}{qQ}} \frac{1}{\theta^2} d\theta \\ &= \frac{2K}{Q} \sum_{KQ_0 < q \leq Q} q \sum_{\substack{1 \leq a < q/2 \\ (a,q)=1}} \frac{1}{a^2 - (K/Q)^2} \ll Q.\end{aligned}$$

Here, we used that $a^2 - (K/Q)^2 \geq a^2/2$, for any $a \geq 1$, if N is large enough. This shows Proposition 5.3.6 a).

5.4.2 The case of smooth numbers

We first observe that for any two complex numbers w, z we have

$$|w + z|^2 \leq 2(|w|^2 + |z|^2). \tag{5.14}$$

By writing $\mathbf{1}_{y-\text{smooth}}(n) = 1 - \mathbf{1}_{\exists p | n: p > y}(n)$ and using (5.14) to separate their contribution to the integral, we get

$$\int_{\mathfrak{m}} |\mathcal{S}_{\mathbf{1}_{y-\text{smooth}}}(\theta)|^2 d\theta \ll Q + \sum_{\substack{n \leq N \\ \exists p | n: p > y}} 1 \leq Q + N \sum_{y < p \leq N} \frac{1}{p} \ll Q + N \log u,$$

by Proposition 5.3.6 a), Parseval's identity and Mertens' theorem, where $u := (\log N)/(\log y) \in [1 + 1/\log N, 2]$. This shows Proposition 5.3.6 d).

5.4.3 The case of divisor functions close to 1

Let $\alpha_N = 1 + 1/R(N)$, where $R(N)$ is a non-vanishing real function with $|R(N)|> C\log\log N$, for a constant $C>0$ to determine later on. By (5.14), one has

$$\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}}(\theta)|^2 d\theta \le 2\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}-1}(\theta)|^2 d\theta + 2\int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta. \tag{5.15}$$

Now, we split the exponential sum with coefficients $d^{\varpi}_{\alpha_N}(n)-1$ according to whether $\varpi(n)\le A\log\log N$ or $\varpi(n) > A\log\log N$, with $A>0$ large to be chosen later. We do this only when $|R(N)|\le (\log N)/(\log 2)$. Then, we separate their contribution to the integral by (5.14). The second one is bounded by Parseval's identity by

$$\begin{aligned}\sum_{\substack{n\le N\\ \varpi(n)>A\log\log N}} (\alpha_N^{\varpi(n)}-1)^2 &\le \sum_{\substack{n\le N\\ \varpi(n)>A\log\log N}} (\alpha_N^{2\varpi(n)}+1)\\ &\le \frac{1}{(\log N)^{A\log(5/4)}}\sum_{n\le N}\left(\left(\frac{3}{2}\right)^{\varpi(n)}+1\right)\left(\frac{5}{4}\right)^{\varpi(n)}\\ &\ll \frac{N}{(\log N)^3},\end{aligned}$$

say, by Corollary 2.1.2 and choosing A large enough.

Let

$$\mathrm{Err}(N) := \begin{cases} \frac{N}{(\log N)^3} & \text{if } |R(N)|\le (\log N)/(\log 2);\\ 0 & \text{otherwise.}\end{cases} \tag{5.16}$$

From the above considerations and Proposition 5.3.6 a), we deduce that

$$\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}}(\theta)|^2 d\theta \ll \int_{\mathfrak{m}} \Bigg|\sum_{\substack{n\le N\\ \varpi(n)\le A\log\log N}} (\alpha_N^{\varpi(n)}-1)e(n\theta)\Bigg|^2 d\theta + Q + \mathrm{Err}(N),$$

where the restriction on the sum is there only when $|R(N)|\le (\log N)/(\log 2)$.

The integral on the right-hand side by (5.14) is

$$\ll \frac{1}{R(N)^2} \int_{\mathfrak{m}} \Bigg| \sum_{\substack{n \le N \\ \varpi(n) \le A \log\log N}} \varpi(n) e(n\theta) \Bigg|^2 d\theta + \int_{\mathfrak{m}} |\mathcal{S}_{T_N}(\theta)|^2 d\theta,$$

where we let

$$T_N(n) := \left(\alpha_N^{\varpi(n)} - 1 - \frac{\varpi(n)}{R(N)} \right) \mathbf{1}_{\varpi(n) \le A \log\log N}.$$

The second integral above, again by (5.14), is

$$\ll M_N^2 \int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta + \int_{\mathfrak{m}} |\mathcal{S}_{T_N - M_N}(\theta)|^2 d\theta,$$

where

$$M_N := \alpha_N^{\log\log N} - 1 - \frac{\log\log N}{R(N)}.$$

By Proposition 5.3.6 a), the first term above is $\ll Q(\log\log N)^4/R(N)^4 \le Q$, if C is large enough. On the other hand, the second one can be estimated with

$$\ll \frac{(\log\log N)^2}{R(N)^4} \sum_{n \le N} (\varpi(n) \mathbf{1}_{\varpi(n) \le A \log\log N} - \log\log N)^2 \ll \frac{N (\log\log N)^3}{R(N)^4},$$

if $A, C(A)$ and N are sufficiently large. To deduce it, we used in sequence Parseval's identity, the Taylor expansion of $\alpha_N^{\varpi(n)}$ and $\alpha_N^{\log\log N}$ (which is possible thanks to the restriction in the sum and reminding of the maximal size (2.7) of $\varpi(n)$) and the well-known identity $a^k - b^k = (a-b) \sum_{j=0}^{k-1} a^j b^{k-1-j}$, which holds for a couple of positive real numbers a, b and any positive integer k. Moreover, to simplify its computation, we inserted and after removed the condition $\varpi(n) \le A \log\log N$ on the sum, at a cost of an acceptable error term, and performed the mean square estimate using (2.4).

Overall, by gathering all of the above considerations, we have showed that

$$\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}}(\theta)|^2 d\theta \ll \frac{1}{R(N)^2}\int_{\mathfrak{m}} |\mathcal{S}_{\varpi}(\theta)|^2 d\theta + Q + \frac{N(\log\log N)^3}{R(N)^4} + \frac{N}{R(N)^2\log N}, \tag{5.17}$$

say, whenever $|R(N)|> C\log\log N$ and C and N are sufficiently large.

It is then clear that assuming the upper bound in Proposition 5.3.6 b) for $\int_{\mathfrak{m}}|\mathcal{S}_{\varpi}(\theta)|^2 d\theta$ and $|R(N)|> (\log\log N)^{3/2}$ we get Proposition 5.3.6 c).

Remark 5.4.1. *If we had $T_N(n) = \varpi(n)^2/(2R(N)^2)$, we believe that we would roughly find*

$$\int_{\mathfrak{m}} |\mathcal{S}_{T_N}(\theta)|^2 d\theta \approx \frac{N(\log\log N)^2}{R(N)^4}\log\left(\frac{\log N}{\log(2N/Q)}\right).$$

This would imply that the lower bound (5.11) *for the integral $\int_{\mathfrak{m}}|\mathcal{S}_{d^{\varpi}_{\alpha_N}}(\theta)|^2 d\theta$ is sharp in the whole range $|R(N)|> C\log\log N$, with C large. In practice, by writing $T_N(n)$ as a truncated Taylor series up to order k, plus a remainder term, we believe we would get to prove that* (5.11) *is sharp in the range $|R(N)|>$ $(\log\log N)^{1+1/k}$, for any fixed positive integer k, by inspecting the structure of the minor arcs. Even though this would constitute an improvement on the result of Proposition 5.3.6 c), we will not commit ourselves to formally proving this here.*

5.4.4 The case of the ϖ function

To begin with, we write

$$\sum_{n\leq N}\omega(n)e(n\theta) = \sum_{n\leq N}\omega_1(n)e(n\theta) + \sum_{n\leq N}\omega_2(n)e(n\theta),$$

where $\omega_1(n)$ is the number of prime factors of n smaller than or equal to $\sqrt[4]{2N/Q}$ and $\omega_2(n)$ instead that of prime divisors contained in the interval $(\sqrt[4]{2N/Q}, N]$. By (5.14), one has

$$\int_{\mathfrak{m}} |S_{\omega}(\theta)|^2 d\theta \ll \int_{\mathfrak{m}} |S_{\omega_1}(\theta)|^2 d\theta + \int_{\mathfrak{m}} |S_{\omega_2}(\theta)|^2 d\theta. \tag{5.18}$$

A simple calculation shows that $\omega_2(n)$ has a mean value of size

$$\log\left(\frac{4\log N}{\log(2N/Q)}\right).$$

Hence, isolating this term inside the corresponding integral gives

$$\begin{aligned}\int_{\mathfrak{m}}|S_{\omega_2}(\theta)|^2 d\theta \ll Q\biggl(\log\left(\frac{4\log N}{\log(2N/Q)}\right)\biggr)^2 \\ + \int_{\mathfrak{m}}\left|\sum_{n\leq N}\left(\omega_2(n)-\log\left(\frac{4\log N}{\log(2N/Q)}\right)\right)e(n\theta)\right|^2 d\theta \\ \ll Q\biggl(\log\left(\frac{4\log N}{\log(2N/Q)}\right)\biggr)^2 + N\log\left(\frac{4\log N}{\log(2N/Q)}\right),\end{aligned}$$

by Proposition 5.3.6 a), Parseval's identity and an application of the general form of the Turán–Kubilius' inequality, which gives an analogue for $\omega_2(n)$ of (2.4) (see e.g. [69, Ch. III, Theorem 3.1]).

Moreover, from

$$\sum_{n\leq N}\Omega(n)e(n\theta) = \sum_{n\leq N}\omega(n)e(n\theta) + \sum_{n\leq N}\left(\sum_{\substack{p^k|n\\ k\geq 2}}1\right)e(n\theta)$$

we immediately get

$$\int_{\mathfrak{m}}|\mathcal{S}_{\Omega}(\theta)|^2 d\theta \ll \int_{\mathfrak{m}}|\mathcal{S}_{\omega}(\theta)|^2 d\theta + \sum_{n\leq N}\left(\sum_{\substack{p^k|n\\ k\geq 2}}1\right)^2.$$

By expanding the square out and swapping summations, we see that the above sum is

$$\sum_{n\le N}\sum_{\substack{p_1^k|n\\ k\ge 2}}\sum_{\substack{p_2^j|n\\ j\ge 2}} 1 = \sum_{p_1\le\sqrt{N}}\sum_{k=2}^{\left\lfloor\frac{\log N}{\log p_1}\right\rfloor}\sum_{p_2\le\sqrt{N}}\sum_{j=2}^{\left\lfloor\frac{\log N}{\log p_2}\right\rfloor}\sum_{\substack{n\le N\\ n\equiv 0 \pmod{[p_1^k,p_2^j]}}} 1$$

$$\le N\sum_{p_1\le\sqrt{N}}\sum_{k=2}^{\left\lfloor\frac{\log N}{\log p_1}\right\rfloor}\sum_{j=2}^{\left\lfloor\frac{\log N}{\log p_1}\right\rfloor}\frac{1}{p_1^{\max\{k,j\}}}$$

$$+ N\sum_{p_1\le\sqrt{N}}\sum_{k=2}^{\left\lfloor\frac{\log N}{\log p_1}\right\rfloor}\frac{1}{p_1^k}\sum_{\substack{p_2\le\sqrt{N}\\ p_2\ne p_1}}\sum_{j=2}^{\left\lfloor\frac{\log N}{\log p_2}\right\rfloor}\frac{1}{p_2^j} \ll N.$$

For the rest of this subsection, we will focus on showing the following statement.

Claim 5.4.2. *Let K be a large constant, $N^{1/2+\delta}\le Q\le N$, with N sufficiently large in terms of δ, and Q_0 satisfying* (4.12)*. Then we have*

$$\int_{\mathfrak{m}} |\mathcal{S}_{\omega_1}(\theta)|^2 d\theta \ll N.$$

Assuming the validity of Claim 5.4.2, and collecting the above observations together, it is immediate to deduce Proposition 5.3.6 b).

We now then move to the proof of Claim 5.4.2. By expanding the integral, we find

$$\begin{aligned}(5.19)\qquad &\int_{\mathfrak{m}} |\mathcal{S}_{\omega_1}(\theta)|^2 d\theta \\ &= \sum_{KQ_0<q\le Q}\sum_{\substack{a=1,\dots,q\\ (a,q)=1}}\int_{a/q-K/qQ}^{a/q+K/qQ}\Bigg|\sum_{p\le\sqrt[4]{2N/Q}}\sum_{k\le N/p} e(kp\theta)\Bigg|^2 d\theta.\end{aligned}$$

We notice that each innermost exponential sum is quite 'long', since for any $p\le\sqrt[4]{2N/Q}$ it always runs over at least Q numbers. We thus expect to individually observe cancellation. Hence, we should not lose much by trivially upper bounding the double sum using the triangle inequality followed by

(5.13). Since $p\theta = pa/q + p\beta$ and by (4.12)

$$|p\beta| \ll \frac{N}{qQ^2} \le \frac{1}{q} \le \frac{1}{Q_0} \le \frac{1}{\log N},$$

we deduce that

$$||p\theta|| = ||\overline{pa}/q + p\beta|| = \min\{|\overline{pa}/q + p\beta|, |1 - \overline{pa}/q - p\beta|\},$$

where $\overline{pa}$ stands for the residue class of pa modulo q. We will only focus on the case $\overline{pa} \le q/2$, so that the above minimum always coincides with $|\overline{pa}/q + p\beta|$, since the complementary one can be similarly dealt with. We notice that $\overline{pa} > 0$. For, if $\overline{pa} = 0$ then $q|p$ and $p \le 2N/Q$, which cannot happen since $q > KQ_0$. Hence, $|\overline{pa}/q + p\beta| \ge \overline{pa}/2q$. Indeed, for any N large enough compared to δ, we have

$$p|\beta| \le \frac{KN}{qQ^2} \le \frac{1}{2q} \le \frac{\overline{pa}}{2q}.$$

Putting together the above information, we see that (5.19) is

$$\ll \frac{1}{Q} \sum_{KQ_0 < q \le Q} \frac{1}{q} \sum_{\substack{a=1,\ldots,q \\ (a,q)=1}} \left(\sum_{p \le \sqrt[4]{2N/Q}} \min\left\{\frac{N}{p}, \frac{q}{\overline{pa}}\right\}\right)^2. \tag{5.20}$$

Note that the above minimum is always of size $q/\overline{pa}$. So, the above reduces to be

$$\begin{aligned} &= \frac{1}{Q} \sum_{KQ_0 < q \le Q} q \sum_{\substack{a=1,\ldots,q \\ (a,q)=1}} \left(\sum_{p \le \sqrt[4]{2N/Q}} \frac{1}{\overline{pa}}\right)^2 \\ &= \frac{1}{Q} \sum_{KQ_0 < q \le Q} q \sum_{\substack{a=1,\ldots,q \\ (a,q)=1}} \sum_{p_1,p_2 \le \sqrt[4]{2N/Q}} \frac{1}{\overline{p_1 a}} \frac{1}{\overline{p_2 a}} \\ &\le \frac{1}{Q} \sum_{KQ_0 < q \le Q} q \sum_{p_1,p_2 \le \sqrt[4]{2N/Q}} \sum_{b_1,b_2 \le q} \frac{1}{b_1 b_2} \sum_{\substack{a=1,\ldots,q \\ (a,q)=1 \\ p_1 a \equiv b_1 \pmod q \\ p_2 a \equiv b_2 \pmod q}} 1. \end{aligned} \tag{5.21}$$

The system of congruences

$$\begin{cases} p_1 a \equiv b_1 \pmod q \\ p_2 a \equiv b_2 \pmod q \end{cases}$$

has always at most $\min\{p_1, p_2\}$ solutions. By multiplying through the first equation by b_2 and the second one by b_1, we need to have

$$p_1 b_2 a \equiv p_2 b_1 a \pmod q \Leftrightarrow p_1 b_2 \equiv p_2 b_1 \pmod q.$$

Therefore, we may upper bound the quantity in the last line of (5.21) with

$$\frac{1}{Q} \sum_{KQ_0 < q \le Q} q \sum_{p_1, p_2 \le \sqrt[4]{2N/Q}} \min\{p_1, p_2\} \sum_{\substack{b_1, b_2 \le q \\ p_1 b_2 \equiv p_2 b_1 \pmod q}} \frac{1}{b_1 b_2}. \tag{5.22}$$

It is easy to verify that we have at most p_1 solutions $b_2 \pmod q$ of the congruence relation $p_1 b_2 \equiv p_2 b_1 \pmod q$, with $b_2 \ge p_2 b_1 / p_1$. Hence, (5.22) may be upper bounded with

$$\begin{aligned} &\frac{1}{Q} \sum_{KQ_0 < q \le Q} q \sum_{p_1, p_2 \le \sqrt[4]{2N/Q}} \frac{p_1^2 \min\{p_1, p_2\}}{p_2} \sum_{b_1 \le q} \frac{1}{b_1^2} \\ &\ll \frac{1}{Q} \sum_{KQ_0 < q \le Q} q \sum_{p_1, p_2 \le \sqrt[4]{2N/Q}} p_1^2 \ll N, \end{aligned}$$

thus concluding the proof of Claim 5.4.2.

Remark 5.4.3. *Note that we have been able to facilitate the estimate of* (5.21) *thanks to our choice of parameter* $\sqrt[4]{2N/Q}$ *in* (5.18).

5.5 The partial sum of some arithmetic functions twisted with Ramanujan sums

To deduce Propositions 5.3.2, 5.3.3 and 5.3.5 we will use a circle method approach, where, as explained in the introduction to this chapter, a key step consists in asymptotically estimating the partial sum of respectively the functions $f(n) = \varpi(n)$, $f(n) = d^{\varpi}_{\alpha_N}(n)$ and $f(n) = \mathbf{1}_{y-\mathrm{smooth}}(n)$ twisted with the Ramanujan sums $c_q(n)$. In this section we are going to deal with this problem by starting with a lower bound for the partial sums of Ramanujan sums $c_q(n)$ over y-smooth numbers and under suitable assumptions on q. This is the content of Lemma 5.1.6, which will be employed in an application of Proposition 4.1.5.

Proof of Lemma 5.1.6. By [69, Ch. III, Theorem 5.8] we know that

$$\Psi\left(\frac{N}{d}, y\right) := \sum_{\substack{n \leq N/d \\ p|n \Rightarrow p \leq y}} 1 = \begin{cases} \lfloor \frac{N}{d} \rfloor & \text{if } d > N/y; \\ \frac{N}{d}(1 - \log(\frac{\log(N/d)}{\log y})) + O(\frac{N}{d \log y}) & \text{if } d \leq N/y. \end{cases}$$

For any prime number q the identity (4.11) reduces to $c_q(n) = -1 + q\mathbf{1}_{q|n}$. It is then immediate to verify the following equality:

$$\sum_{\substack{n \leq N \\ p|n \Rightarrow p \leq y}} c_q(n) = -\Psi(N, y) + q\Psi\left(\frac{N}{q}, y\right),$$

from which it is straightforward to deduce the first estimate of the lemma.

By (4.11), and letting $\sigma(q) := \sum_{d|q} d$, we can always rewrite the sum in the statement as

$$\begin{aligned} \sum_{d|q} d\mu\left(\frac{q}{d}\right) \Psi\left(\frac{N}{d}, y\right) = N &\sum_{\substack{d|q \\ d > N/y}} \mu\left(\frac{q}{d}\right) \\ &+ N \sum_{\substack{d|q \\ d \leq N/y}} \mu\left(\frac{q}{d}\right) \left(1 - \log\left(\frac{\log(N/d)}{\log y}\right)\right) \\ &+ O\biggl(\frac{N}{\log N} \sum_{\substack{d|q \\ d \leq N/y}} 1 + \sigma(q)\biggr). \end{aligned}$$

In the hypothesis that $q > 1$ has all the prime factors larger than N/y, the sums over the divisors of q smaller than or equal to N/y reduce only to the single term corresponding to $d = 1$. Hence, we actually have

$$\sum_{\substack{n\le N\\ p|n\Rightarrow p\le y}} c_q(n) = -N\mu(q) + N\mu(q)(1-\log u) + O\biggl(\frac{N}{\log N}\biggr)$$
$$= -N\mu(q)\log u + O\biggl(\frac{N}{\log N}\biggr),$$

since $\sigma(q) \ll q\log\log q \ll \sqrt{N}\log\log N \le N/\log N$ (see [69, Ch. I, Theorem 5.7]), if N is large, which immediately leads to deduce the second estimate of the lemma. □

To prove Proposition 5.3.2 we will instead need to invoke Proposition 5.1.7, where again a crucial step consists in estimating the partial sum of $\varpi(n)$ twisted with Ramanujan sums and weighted by a smooth weight $\phi(n/N)$, with $\phi(t)$ belonging to the Fourier class $\mathcal{F}$ as in Proposition 5.1.7. The precise shape of the result we will need is contained in Lemma 5.1.10, which we now prove.

Proof of Lemma 5.1.10. To begin with, we note that for prime numbers p the identity (4.11) reduces to $c_p(n) = -1 + p\mathbf{1}_{p|n}$. Hence, the sum over n in the statement is

$$= -\sum_{n\le N}\varpi(n)\phi\left(\frac{n}{N}\right) + p\sum_{\substack{n\le N\\ p|n}}\varpi(n)\phi\left(\frac{n}{N}\right)$$
$$= -\sum_{n\le N}\varpi(n)\phi\left(\frac{n}{N}\right) + p\sum_{k\le N/p}(\varpi(k)+1)\phi\left(\frac{kp}{N}\right) + O\biggl(p\sum_{k\le N/p^2}(\varpi(k)+2)\biggr),$$

where we used that $\varpi(pk) \le \varpi(k) + \varpi(p) = \varpi(k) + 1$. By (2.3) the above big-Oh error term contributes at most $\ll N(\log\log N)/p$.

By partial summation from (2.3), it is easy to show that

$$\sum_{n\le N}\varpi(n)\phi\left(\frac{n}{N}\right) = JN\log\log N + JNB_\varpi + O\biggl(\frac{N\log\log N}{\log N}\biggr),$$

for any N large enough, where $J := \int_0^1 \phi(t)dt \in [1/2, 1]$. This, applied once with N and once with N/p, together with the previous observations, gives

$$\sum_{n\le N}\varpi(n)c_p(n)\phi\left(\frac{n}{N}\right)=JN\left(1+\log\left(1-\frac{\log p}{\log N}\right)\right)$$
$$+O\left(\frac{N\log\log N}{\log N}+\frac{N\log\log N}{p}\right).$$

Therefore, we see that the double sum in the statement is

$$=JN\sum_{2N/Q<p\le R}\frac{1+\log(1-\frac{\log p}{\log N})}{p}$$
$$+O\left(\frac{N(\log\log N)^2}{\log N}+N\log\log N\sum_{2N/Q<p\le R}\frac{1}{p^2}\right)$$
$$\gg N\log\left(\frac{\log R}{\log(2N/Q)}\right)+O\left(\frac{N(\log\log N)^2}{\log N}+\frac{Q\log\log N}{\log(2N/Q)}\right),$$

by Mertens' theorem, from which the thesis follows on our range of Q, if N is large enough with respect to δ. □

Similarly as before, to get Proposition 5.3.3 we will again make use of Proposition 5.1.7 and therefore we will need as well to estimate an averaged partial sum of the divisor functions $d^{\varpi}_{\alpha_N}(n)$ twisted with Ramanujan sums and weighted by a smooth weight $\phi(n/N)$, with $\phi(t)$ belonging to the Fourier class $\mathcal{F}$ as in Proposition 5.1.7. The precise shape of the result we will need is the content of Lemma 5.1.8, which we now prove.

Proof of Lemma 5.1.8. By adapting the proof of Corollary 5.1.2, it is not difficult to show that

$$\sum_{n\le t}d^{\varpi}_{\alpha_N}(n)=\frac{c_0(\alpha_N,\varpi)}{\Gamma(\alpha_N)}t(\log N)^{\alpha_N-1}\left(1+O\left(\frac{\log\log N}{|R(N)|\log N}\right)\right)$$
$$+O\left(\frac{N\log\log N}{\log N}\right),\tag{5.23}$$

for any $t\in[N/\log N,N]$, if N is large enough. Here, $\Gamma(z)$ stands for the

Gamma function (see Ch. Notations) and

$$c_0(\alpha_N, \varpi) := \begin{cases} \prod_p \left(1 - \frac{1}{p}\right)^{\alpha_N} \left(1 + \frac{\alpha_N}{p-1}\right) & \text{if } \varpi(n) = \omega(n); \\ \prod_p \left(1 - \frac{1}{p}\right)^{\alpha_N} \left(1 - \frac{\alpha_N}{p}\right)^{-1} & \text{if } \varpi(n) = \Omega(n). \end{cases}$$

It is easy to verify that

$$c_0(\alpha_N, \varpi) = 1 + O\left(\frac{1}{|R(N)|}\right) = \Gamma(\alpha_N), \tag{5.24}$$

if N is large enough (see [55, Appendix C] for basic results on the Gamma function).

By Corollary 2.1.2, we certainly have

$$\begin{aligned} \sum_{n \le N/\log N} d^{\varpi}_{\alpha_N}(n) \phi\left(\frac{n}{N}\right) &\ll \sum_{n \le N/\log N} \left(1 + \frac{1}{|R(N)|}\right)^{\varpi(n)} \\ &\ll \frac{N}{\log N} (\log N)^{1/|R(N)|} \ll \frac{N}{\log N}. \end{aligned}$$

This, together with partial summation from (5.23) applied to the remaining part of the sum, leads to

$$\sum_{n \le N} d^{\varpi}_{\alpha_N}(n) \phi\left(\frac{n}{N}\right) = \frac{c_0(\alpha_N, \varpi)}{\Gamma(\alpha_N)} J N e^{\frac{\log\log N}{R(N)}} + O\left(\frac{N \log\log N}{\log N}\right),$$

where $J := \int_0^1 \phi(t) dt \in [1/2, 1]$ and we made use of (5.24) to simplify the error term.

Applying this asymptotic estimate with length of the sum N/p in place of N, we find

$$\begin{aligned} \sum_{\substack{n \le N \\ p | n}} d^{\varpi}_{\alpha_N}(n) \phi\left(\frac{n}{N}\right) &= \alpha_N \sum_{\substack{k \le N/p \\ p \nmid k}} d^{\varpi}_{\alpha_N}(k) \phi\left(\frac{pk}{N}\right) + \sum_{k \le N/p^2} d^{\varpi}_{\alpha_N}(kp^2) \phi\left(\frac{kp^2}{N}\right) \\ &= \alpha_N \sum_{k \le N/p} d^{\varpi}_{\alpha_N}(k) \phi\left(\frac{pk}{N}\right) + O\left(\sum_{k \le N/p^2} d^{\varpi}_{1+1/|R(N)|}(k)\right) \\ &= \frac{c_0(\alpha_N, \varpi)}{\Gamma(\alpha_N)} \frac{J N \alpha_N}{p} e^{\frac{\log\log(N/p)}{R(N)}} + O\left(\frac{N \log\log N}{p \log N} + \frac{N}{p^2}\right), \end{aligned}$$

where we used $\varpi(pk) \le \varpi(k) + 1$ and Corollary 2.1.2 to handle the error term contribution.

The collection of the above estimates, taking into account of the identity (4.11) for the Ramanujan sums, makes the sum over n in the statement equal to

$$\frac{c_0(\alpha_N, \varpi)}{\Gamma(\alpha_N)} JNe^{\frac{\log\log N}{R(N)}} (\alpha_N e^{\frac{\log(1-\frac{\log p}{\log N})}{R(N)}} - 1) + O\Big(\frac{N \log\log N}{\log N} + \frac{N}{p}\Big). \tag{5.25}$$

By Taylor expansion and thanks to (5.24), one has

$$\begin{aligned} \alpha_N e^{\frac{\log(1-\frac{\log p}{\log N})}{R(N)}} - 1 &= \Big(1 + \frac{1}{R(N)}\Big)\Big(1 + \frac{\log(1-\frac{\log p}{\log N})}{R(N)} + O\Big(\frac{1}{R(N)^2}\Big)\Big) - 1 \\ &= \frac{1 + \log(1-\frac{\log p}{\log N})}{R(N)} + O\Big(\frac{1}{R(N)^2}\Big) \end{aligned}$$

and

$$\begin{aligned} \frac{c_0(\alpha_N, \varpi)}{\Gamma(\alpha_N)} e^{\frac{\log\log N}{R(N)}} &= \Big(1 + O\Big(\frac{1}{|R(N)|}\Big)\Big)\Big(1 + O\Big(\frac{\log\log N}{|R(N)|}\Big)\Big) \\ &= 1 + O\Big(\frac{\log\log N}{|R(N)|}\Big). \end{aligned}$$

Inserting the above estimates into (5.25), we see that the double sum in the statement is

$$\begin{aligned} &= \Big(\frac{JN}{R(N)} \sum_{2N/Q<p\le R} \frac{1+\log(1-\frac{\log p}{\log N})}{p}\Big)\Big(1 + O\Big(\frac{\log\log N}{|R(N)|}\Big)\Big) \\ &+ O\Big(\frac{N\log\log N}{R(N)^2} + \frac{N(\log\log N)^2}{\log N} + N \sum_{2N/Q<p\le R} \frac{1}{p^2}\Big) \\ &\gg \frac{N}{|R(N)|} \log\Big(\frac{\log R}{\log(2N/Q)}\Big) + O\Big(\frac{Q}{\log(2N/Q)}\Big), \end{aligned}$$

by Mertens' theorem, by taking C and N large enough with respect to δ and thanks to our assumption on $|R(N)|$, from which we get the thesis on our range of Q. □

Notice that Lemma 5.1.8 requires values of $|R(N)|$ smaller than or equal to $(\log\log N)^3$: this is indeed the range of $R(N)$ where we are able to directly estimate the twisted partial sums of $d^{\varpi}_{\alpha_N}(n)$ as in the statement; in the remaining range, we Taylor expand $d^{\varpi}_{\alpha_N}(n) = (1+1/R(N))^{\varpi(n)}$ as $1+\varpi(n)/R(N)+O(\varpi(n)^2/R(N)^2)$, and use Lemma 5.1.10 to tackle the contribution of the ϖ-function whereas Lemma 5.1.9 to handle that of the constant function 1.

Proof of Lemma 5.1.9. To begin with, we notice that for any prime number q, the following estimate holds:

$$S(t) := \sum_{n\leq t} c_q(n) = \sum_{\substack{n\leq t\\ q|n}} q - \sum_{n\leq t} 1 \ll q, \tag{5.26}$$

by (4.11), for any $t \geq 1$. Hence, by partial summation we find

$$\begin{aligned}\sum_{n\leq N} c_q(n) e\left(\frac{nu}{N}\right) &= \int_1^N e\left(\frac{tu}{N}\right) dS(t)\\ &= S(N)e(u) - S(1)e\left(\frac{u}{N}\right) - \frac{u}{N}\int_1^N S(t)e\left(\frac{tu}{N}\right) dt,\end{aligned}$$

from which, by using (5.26), the thesis follows. □

5.6 Proof of Proposition 5.3.1

By restricting the integral in the statement over minor arcs of the form $(1/q - 1/KqQ, 1/q + 1/KqQ)$, for positive integers q in the range $Q/(2M^2) < q \leq Q/M^2$, where M is a large positive constant to be chosen later, we can lower bound it with

$$\sum_{Q/(2M^2)<q\leq Q/M^2} \int_{-1/KqQ}^{1/KqQ} \left|\sum_{n\leq N} e(n/q)e(n\theta)\right|^2 d\theta.$$

Since, by definition of minor arcs, $q > KQ_0$ and by (4.12) $Q_0 \leq Q/K^2$, we require $K > 2M^2$, say. Moreover, we remind that K, and thus M, are absolute constants here. By partial summation it is easy to verify that

$$\Big|\sum_{1\leq n\leq N} e(n/q)e(n\theta)\Big| = \Big|\frac{e^{2\pi i(N+1)/q} - e^{2\pi i/q}}{e^{2\pi i/q}-1}\Big| + O\Big(\frac{N}{Q}\Big).$$

We deduce that

$$\begin{aligned}\int_{\mathfrak{m}(K,Q_0,Q)} |\mathcal{S}_1(\theta)|^2 d\theta \geq & \sum_{Q/(2M^2)<q\leq Q/M^2} \frac{2}{KqQ}\Big|\frac{e^{2\pi i(N+1)/q} - e^{2\pi i/q}}{e^{2\pi i/q}-1}\Big|^2 \\ & + O\Big(\frac{N^2}{Q^3} + \frac{N}{Q}\sum_{Q/(2M^2)<q\leq Q/M^2} \frac{1}{qQ}\Big|\frac{e^{2\pi i(N+1)/q} - e^{2\pi i/q}}{e^{2\pi i/q}-1}\Big|\Big) \\ \gg & \sum_{Q/(2M^2)<q\leq Q/M^2} \frac{q}{Q}|e^{2\pi i(N+1)/q} - e^{2\pi i/q}|^2 \\ & + O\Big(\frac{N^2}{Q^3} + \frac{N}{Q}\Big),\end{aligned}$$

by expanding

$$e^{2\pi i/q} - 1 = \frac{2\pi i}{q} + O\Big(\frac{1}{q^2}\Big) \asymp \frac{1}{q}.$$

Notice that

$$|e^{2\pi i(N+1)/q} - e^{2\pi i/q}|^2 = 2 - 2\Re(e^{2\pi i N/q}).$$

Therefore, to conclude, we only have to produce some saving on the size of the partial sum of $\Re(e^{2\pi i N/q})$ over the interval $I := [Q/(2M^2) < q \leq Q/M^2]$ compared to its length. Once done that, we immediately deduce that

$$\int_{\mathfrak{m}(K,Q_0,Q)} |\mathcal{S}_1(\theta)|^2 d\theta \gg Q + O\Big(\frac{N^2}{Q^3} + \frac{N}{Q}\Big),$$

where the term Q dominates whenever $Q \geq c\sqrt{N}$, for a suitable absolute constant $c > 0$. To this aim, we apply the van der Corput's inequality (see e.g. [69, Ch. I, Theorem 6.5]) to the function $f_N(t) := N/t$, for which $f_N(t) \in C^2(I)$ with $f_N''(t) \asymp NM^6/Q^3$, for $t \in I$. We thus get

$$\Big|\sum_{q\in I} \Re(e^{2\pi i f_N(q)})\Big| \ll \frac{Q}{M^3},$$

for any $M^{8/3}N^{1/3} \leq Q \leq N$, if we take N sufficiently large, from which the thesis follows, by taking M large enough.

5.7 Proof of Proposition 5.3.2

Let K be a large constant, Q_0 and Q be real numbers satisfying (4.12).

5.7.1 Large values of Q

By isolating the constant term $Z := \log\log N$ and expanding the square out, we have

$$\begin{aligned}\int_{\mathfrak{m}} |\mathcal{S}_{\varpi}(\theta)|^2 d\theta &\geq \int_{\mathfrak{m}} |\mathcal{S}_{\varpi-Z}(\theta)|^2 d\theta + \int_{\mathfrak{m}} |\mathcal{S}_Z(\theta)|^2 d\theta - 2\int_{\mathfrak{m}} |\mathcal{S}_{\varpi-Z}(\theta)\mathcal{S}_Z(\theta)| d\theta \\ &\geq \int_{\mathfrak{m}} |\mathcal{S}_Z(\theta)|^2 d\theta - 2\sqrt{\int_{\mathfrak{m}} |\mathcal{S}_{\varpi-Z}(\theta)|^2 d\theta \int_{\mathfrak{m}} |\mathcal{S}_Z(\theta)|^2 d\theta},\end{aligned}$$

by an application of Cauchy–Schwarz's inequality.

By completing the integral $\int_{\mathfrak{m}} |\mathcal{S}_{\varpi-Z}(\theta)|^2 d\theta$ to the whole circle and using Parseval's identity followed by an application of the upper bound (2.4) on the second centred moment of $\varpi(n)$, we find it is $\ll N\log\log N$.

Since from Propositions 5.9 and 5.3.6 a) we know that $\int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta \asymp Q$, on a wide range of Q, we also in particular have

$$\int_{\mathfrak{m}} |\mathcal{S}_Z(\theta)|^2 d\theta \asymp Q(\log\log N)^2,$$

whenever e.g. $Q \geq cN/\log\log N$, for any fixed constant $c > 0$. By choosing c suitably large, we then get the lower bound (5.10) on such range of Q.

5.7.2 Small values of Q

Assume now $N^{1/2+\delta} \leq Q < cN/\log\log N$, with c as in the previous subsection, and $KQ_0 < R$, where $R := N^{1/2-\delta/2}$, for a small $\delta > 0$. Let $g(r)$ be the characteristic function of the set of prime numbers smaller than R. We apply Proposition 5.1.7 with such sets of minor arcs and functions $g(r)$ and $f(n) = \varpi(n)$.

Remark 5.7.1. *In order to successfully apply Proposition 5.1.7, as a rule of thumb, we might think of $g(r)$ as an approximation of the Dirichlet convolution $f * \mu(r)$, where $\mu(r)$ is the Möbius function (see Ch. Notations for a definition thereof). This motivates our choice of g, since for any $n \le N$ we either have $g * 1(n) = \omega(n)$ or $g * 1(n) = \omega(n) - 1$, with $\omega(n) \approx \log\log N \approx \Omega(n)$, for most of the integers $n \le N$, by* (2.3).

With the notations introduced in Proposition 5.1.7, we have

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \ll N \log\left(\frac{\log N}{\log(2N/Q)}\right), \tag{5.27}$$

which follows from Proposition 5.3.6 b), on our range of Q.

Next, by (5.6), with $f(n) = \varpi(n)$, the integral $\int_{\mathfrak{m}} \mathcal{S}_f(\theta)\overline{\mathcal{G}(\theta)} d\theta$ is

$$\begin{aligned} &= \sum_{n \le N} \varpi(n) \Big(\sum_{\substack{p \mid n \\ p \le R}} 1 \Big) \phi\left(\frac{n}{N}\right) \\ &- N \sum_{q \le KQ_0} \frac{\mathbf{1}_{q>2,\ \text{prime}}}{q} \int_{-K/qQ}^{K/qQ} \Big(\sum_{n \le N} \varpi(n) c_q(n) e(n\beta) \Big) \hat{\phi}(\beta N) d\beta \\ &- N \sum_{p \le R} \frac{1}{p} \sum_{q \le KQ_0} \mathbf{1}_{q=1} \int_{-K/qQ}^{K/qQ} \Big(\sum_{n \le N} \varpi(n) c_q(n) e(n\beta) \Big) \hat{\phi}(\beta N) d\beta \\ &+ O(N^{1-\delta}), \end{aligned} \tag{5.28}$$

if N is large enough with respect to δ. Here, the error term has been trivially estimated by using the bound (2.7) on the maximal size of $\varpi(n)$ and our hypotheses on Q_0, Q and R. The second and third expressions in (5.28) equal

$$- N \sum_{n \le N} \varpi(n) \sum_{p \le R} \frac{1}{p} \int_{-K/Q}^{K/Q} e(n\beta) \hat{\phi}(\beta N) d\beta \tag{5.29}$$

$$- N \sum_{\substack{2 \le q \le KQ_0 \\ q \text{ prime}}} \frac{1}{q} \sum_{n \le N} \varpi(n) c_q(n) \int_{-K/qQ}^{K/qQ} e(n\beta) \hat{\phi}(\beta N) d\beta. \tag{5.30}$$

By changing variable and since $\phi(t)$ belongs to the Fourier class $\mathcal{F}$ as in Proposition 5.1.7, one has

$$N\int_{-K/Q}^{K/Q} e(n\beta)\hat{\phi}(\beta N)d\beta \tag{5.31}$$
$$= \phi\left(\frac{n}{N}\right) + O\bigg(\int_{KN/Q}^{+\infty} \hat{\phi}(u)du + \int_{-\infty}^{-KN/Q} \hat{\phi}(u)du\bigg)$$
$$= \phi\left(\frac{n}{N}\right) + O\bigg(\frac{Q^4}{N^4}\bigg),$$

where we remind that $Q < cN/\log\log N$. Thus, by the asymptotic expansion (2.3) for the partial sum of $\varpi(n)$ and Mertens' theorem, (5.29) equals

$$-\sum_{p\le R}\frac{1}{p}\sum_{n\le N}\varpi(n)\phi\left(\frac{n}{N}\right) + O\bigg(\frac{Q}{\log\log N}\bigg). \tag{5.32}$$

We now split the sum over q in (5.30) into two parts according to whether $q \le 2N/Q$ or $q > 2N/Q$. In the second case, since $\hat{\phi}(\xi)$ is bounded, we find

$$N\int_{-K/qQ}^{K/qQ} e(n\beta)\hat{\phi}(\beta N)d\beta = \int_{-KN/qQ}^{KN/qQ} e\left(\frac{nu}{N}\right)\hat{\phi}(u)du \ll \frac{N}{qQ}. \tag{5.33}$$

We deduce that the contribution in (5.30) from the primes $q > 2N/Q$ is

$$\ll \frac{N}{Q}\sum_{\substack{q>2N/Q\\ q\ \text{prime}}}\frac{1}{q^2}\sum_{n\le N}\varpi(n)|c_q(n)| \ll \frac{N^2\log\log N}{Q}\sum_{\substack{q>2N/Q\\ q\ \text{prime}}}\frac{1}{q^2} \tag{5.34}$$
$$\ll \frac{N\log\log N}{\log(2N/Q)}.$$

On the other hand, for values of $q \le 2N/Q$, by changing variable and by definition of $\phi(t)$, we can rewrite the integral $\int_{-KN/qQ}^{KN/qQ} e(nu/N)\hat{\phi}(u)du$ as

$$\phi\left(\frac{n}{N}\right) + \int_{KN/qQ}^{+\infty} e\left(\frac{nu}{N}\right)\hat{\phi}(u)du + \int_{-\infty}^{-KN/qQ} e\left(\frac{nu}{N}\right)\hat{\phi}(u)du \tag{5.35}$$
$$= \phi\left(\frac{n}{N}\right) + O\bigg(\frac{qQ}{N}\bigg).$$

We may then deduce that the contribution in (5.30) coming from those primes is

$$(5.36)\qquad -\sum_{\substack{2\le q\le 2N/Q\\ q\ \text{prime}}}\frac{1}{q}\sum_{n\le N}\varpi(n)c_q(n)\phi\left(\frac{n}{N}\right)+O\bigg(\frac{N\log\log N}{\log(2N/Q)}\bigg).$$

Collecting together (5.32), (5.36) and previous observations and thanks to the identity (4.11) for the Ramanujan sums, we see that (5.28) equals to

$$\sum_{2N/Q<p\le R}\frac{1}{p}\sum_{n\le N}\varpi(n)c_p(n)\phi\left(\frac{n}{N}\right)+O\bigg(\frac{N\log\log N}{\log(2N/Q)}\bigg),$$

if N is large enough with respect to δ. Also, note that a lower bound for the size of the above sum has already been given in Lemma 5.1.10. Overall, we have thus found that

$$\int_{\mathfrak{m}}\mathcal{S}_f(\theta)\overline{\mathcal{G}(\theta)}d\theta\gg N\log\left(\frac{\log R}{\log(2N/Q)}\right),$$

in the range $N^{1/2+\delta}\le Q\le cN/\log\log N$. This, together with the upper bound (5.27) for the integral $\int_{\mathfrak{m}}|\mathcal{G}(\theta)|^2d\theta$, concludes the proof of the lower bound (5.10) for the integral $\int_{\mathfrak{m}}|\mathcal{S}_{\varpi}(\theta)|^2d\theta$, via an application of Proposition 5.1.7, whenever N is suitably large with respect to δ. Indeed, to rewrite the result as in the statement of Proposition 5.3.2 we appeal to the following lemma.

Lemma 5.7.2. *For any δ small enough and N sufficiently large with respect to δ, we have*

$$\log\left(\frac{\log R}{\log(2N/Q)}\right)\ge\delta\log\left(\frac{\log N}{\log(2N/Q)}\right).$$

Proof. The aimed inequality is equivalent to

$$\left(\frac{1}{2}-\frac{\delta}{2}\right)\left(\frac{\log N}{\log(2N/Q)}\right)^{1-\delta}\ge 1,$$

which is satisfied when in particular

$$\left(\frac{1}{2}-\frac{\delta}{2}\right)\ge\left(\frac{1}{2}-\delta+O(\delta^2)\right)^{1-\delta}$$

and N is sufficiently large with respect to δ. The above in turn is equivalent to

$$\frac{1+\frac{\delta}{\log 2}+O(\delta^2)}{1+\frac{2\delta}{\log 2}+O(\delta^2)} \leq 1-\delta.$$

Since the left-hand side above equals to $1-\delta/\log 2+O(\delta^2)$, the thesis immediately follows if δ is taken small enough. □

5.8 Proof of Proposition 5.3.3

Let K be a large constant, Q_0 and Q be real numbers satisfying (4.12). Moreover, let $C \log\log N \leq |R(N)| \leq N^{\delta/12}$, with C as in Lemma 5.1.8.

5.8.1 Large values of Q

By isolating the constant term 1 and expanding the square out, we have

$$\begin{aligned}\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}}(\theta)|^2 d\theta &\geq \int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}-1}(\theta)|^2 d\theta + \int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta - 2\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}-1}(\theta)\mathcal{S}_1(\theta)| d\theta \\ &\geq \int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta - 2\sqrt{\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}-1}(\theta)|^2 d\theta \int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta},\end{aligned}$$

by an application of Cauchy–Schwarz's inequality. The estimate of the integral $\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}-1}(\theta)|^2 d\theta$ has already been performed in Subsect. 5.4.3, where we found (see Eq. (5.17)):

$$\int_{\mathfrak{m}} |\mathcal{S}_{d^{\varpi}_{\alpha_N}-1}(\theta)|^2 d\theta \ll \frac{1}{R(N)^2}\int_{\mathfrak{m}} |\mathcal{S}_{\varpi}(\theta)|^2 d\theta + \frac{N(\log\log N)^3}{R(N)^4} + \frac{N}{R(N)^2 \log N}.$$

By Propositions 5.3.1 and 5.3.6 a), which together give $\int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta \asymp Q$, by Proposition 5.3.6 b), which shows that

$$\int_{\mathfrak{m}} |\mathcal{S}_{\varpi}(\theta)|^2 d\theta \ll Q(\log\log N)^2 + N\log\left(\frac{\log N}{\log(2N/Q)}\right),$$

and by the above considerations, we may deduce the lower bound (5.11), at least when $Q \geq cN(\log\log N)/R(N)^2$, for c a suitable positive constant, by taking N large enough and possibly replacing C with a larger value.

5.8.2 Small values of Q

Let us now assume $N^{1/2+\delta} \leq Q < cN(\log\log N)/R(N)^2$ and $KQ_0 < R$, where $R := N^{1/2-\delta/2}$, for a small $\delta > 0$. Let $g(r)$ be the characteristic function of the set of prime numbers smaller than R. We apply Proposition 5.1.7 with such sets of minor arcs and functions $g(r)$ and $f(n) = d_{\alpha_N}^{\varpi}(n)$. With the notations introduced there, we again have

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \ll N \log\left(\frac{\log N}{\log(2N/Q)}\right), \tag{5.37}$$

which follows from Proposition 5.3.6 b), since by assumption on $|R(N)|$ we always at least have $Q \ll N/\log\log N$.

Next, by (5.6), with $f(n) = d_{\alpha_N}^{\varpi}(n)$, the integral $\int_{\mathfrak{m}} \mathcal{S}_f(\theta)\overline{\mathcal{G}(\theta)} d\theta$ is

$$\begin{aligned} &= \sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)\Bigg(\sum_{\substack{p|n\\ p\leq R}} 1\Bigg)\phi\left(\frac{n}{N}\right) \\ &- N\sum_{p\leq R}\frac{1}{p}\sum_{q\leq KQ_0} \mathbf{1}_{q=1}\int_{-K/qQ}^{K/qQ}\Bigg(\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_q(n)e(n\beta)\Bigg)\hat{\phi}(\beta N)d\beta \\ &- N\sum_{q\leq KQ_0}\frac{\mathbf{1}_{q>2,\ \text{prime}}}{q}\int_{-K/qQ}^{K/qQ}\Bigg(\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_q(n)e(n\beta)\Bigg)\hat{\phi}(\beta N)d\beta \\ &+ O(N^{1-\delta}), \end{aligned} \tag{5.38}$$

if N is large enough with respect to δ. Here, the error term has been trivially estimated by using Corollary 2.1.2 and our hypotheses on Q_0, Q and R.

The second and third expressions in the above displayed equation equal

$$-N\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)\sum_{p\leq R}\frac{1}{p}\int_{-K/Q}^{K/Q} e(n\beta)\hat{\phi}(\beta N)d\beta \tag{5.39}$$

$$-N\sum_{\substack{2\leq q\leq KQ_0\\ q\ \text{prime}}}\frac{1}{q}\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_q(n)\int_{-K/qQ}^{K/qQ} e(n\beta)\hat{\phi}(\beta N)d\beta. \tag{5.40}$$

By the second identity in (5.31) for $N\int_{-K/Q}^{K/Q} e(n\beta)\hat{\phi}(\beta N)d\beta$, we see that (5.39) is

$$= -\sum_{p\le R}\frac{1}{p}\sum_{n\le N} d_{\alpha_N}^{\varpi}(n)\phi(n/N) + O\left(\frac{Q}{R(N)^2}\right), \tag{5.41}$$

where we used Corollary 2.1.2 and our hypothesis on Q to estimate the error term.

We now split the sum over q in (5.40) into two parts according to whether $q \le 2N/Q$ or $q > 2N/Q$. The term corresponding to the second set of primes equals to

$$\begin{aligned}(5.42)\qquad & -\frac{N}{R(N)}\sum_{\substack{2N/Q<q\le KQ_0\\ q \text{ prime}}}\frac{1}{q}\sum_{n\le N}\varpi(n)c_q(n)\int_{-K/qQ}^{K/qQ} e(n\beta)\hat{\phi}(\beta N)d\beta\\ & - N\sum_{\substack{2N/Q<q\le KQ_0\\ q \text{ prime}}}\frac{1}{q}\sum_{n\le N} c_q(n)\int_{-K/qQ}^{K/qQ} e(n\beta)\hat{\phi}(\beta N)d\beta\\ & - N\sum_{\substack{2N/Q<q\le KQ_0\\ q \text{ prime}}}\frac{1}{q}\sum_{n\le N} E(n)c_q(n)\int_{-K/qQ}^{K/qQ} e(n\beta)\hat{\phi}(\beta N)d\beta,\end{aligned}$$

where for the sake of readiness we defined $E(n) := d_{\alpha_N}^{\varpi}(n) - 1 - \varpi(n)/R(N)$. The sum in the first term above has already been estimated before, with the result given in (5.34). Whence, the first expression in (5.42) is

$$\ll \frac{N\log\log N}{|R(N)|\log(2N/Q)}.$$

Regarding the second term in (5.42), by changing variable inside the integral and swapping integral and summation, it is

$$\begin{aligned}-\sum_{\substack{2N/Q<q\le KQ_0\\ q \text{ prime}}}\frac{1}{q}\int_{-KN/qQ}^{KN/qQ}\sum_{n\le N} c_q(n)e\left(\frac{nu}{N}\right)\hat{\phi}(u)du &\ll \frac{N}{Q}\sum_{\substack{2N/Q<q\le KQ_0\\ q \text{ prime}}}\frac{1}{q}\\ &\ll \frac{N\log\log N}{Q} \le \sqrt{N},\end{aligned}$$

by Lemma 5.1.9, Mertens' theorem and taking N large enough with respect to δ.

Finally, regarding the third term in (5.42), we employ the estimate (5.33) for the integral $N\int_{-K/qQ}^{K/qQ} e(n\beta)\hat{\phi}(\beta N)d\beta$, the identity (4.11) for the Ramanujan sums and the bound (2.5) on the second moment of $\varpi(n)$. Thanks to them, it is easily seen to be

$$\ll \frac{N^2(\log\log N)^2}{QR(N)^2} \sum_{\substack{2N/Q<q\leq KQ_0 \\ q \text{ prime}}} \frac{1}{q^2} \ll \frac{N(\log\log N)^2}{R(N)^2\log(2N/Q)}.$$

Here, to estimate the sum over n in (5.42) we argued as in Subsect. 5.4.3, by dividing the argument according to whether $|R(N)|\leq (\log N)/(\log 2)$ or not; in the first case, we further split the sum over those integers n such that $\varpi(n) \leq C(\log\log N)$ or the opposite holds.

Regarding the part of (5.40) corresponding to primes $q \leq 2N/Q$, we first rewrite the integral $\int_{-KN/qQ}^{KN/qQ} e(nu/N)\hat{\phi}(u)du$ as in (5.35) and write $d_{\alpha_N}^{\varpi}(n) =: 1 + \varpi(n)/R(N) + E(n)$. Then, we use Lemma 5.1.9 to handle the contribution coming from the constant function 1 and argue similarly as before to compute the contribution from $\varpi(n)$ and $E(n)$. So that, we readily see that such part equals to

$$-\sum_{\substack{2\leq q\leq 2N/Q \\ q \text{ prime}}} \frac{1}{q}\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_q(n)\phi\left(\frac{n}{N}\right) + O\left(\frac{N\log\log N}{|R(N)|\log(2N/Q)}\right).$$

Overall, we have found that (5.38) is

$$\begin{aligned}(5.43)\qquad &\sum_{\substack{2N/Q<q\leq R \\ q \text{ prime}}} \frac{1}{q}\sum_{n\leq N} d_{\alpha_N}^{\varpi}(n)c_q(n)\phi\left(\frac{n}{N}\right) \\ &+ O\left(\frac{N\log\log N}{|R(N)|\log(2N/Q)} + \frac{Q}{R(N)^2}\right),\end{aligned}$$

if N is sufficiently large with respect to δ.

We now split the argument into two parts, according to whether we have $|R(N)|\leq (\log\log N)^3$ or not. In the first case, we remind that the size of the above sum has already been estimated in Lemma 5.1.8. From this, from the upper bound (5.37) for the integral $\int_{\mathfrak{m}}|\mathcal{G}(\theta)|^2d\theta$ and taking into account of Lemma 5.7.2, we may deduce the lower bound (5.11) for the integral $\int_{\mathfrak{m}}|\mathcal{S}_{d_{\alpha_N}^{\varpi}}(\theta)|^2 d\theta$ in such range of $|R(N)|$, via an application of Proposition 5.1.7, if N is suitably large with respect to δ.

On the other hand, when $|R(N)| > (\log\log N)^3$, we replace $d_{\alpha_N}^{\varpi}(n)$ inside (5.43) with $1 + \varpi(n)/R(N) + E(n)$. Afterwards, we estimate the error contribution coming from the constant function 1 using partial summation from the bound (5.26) on the partial sum of $c_q(n)$. Moreover, we trivially estimate the error contribution coming from $E(N)$ thanks to our current assumption on $|R(N)|$ and arguing as before. Finally, the main contribution coming from $\varpi(n)/R(N)$ can be immediately handled by Lemma 5.1.10. Combining the estimate we get, by proceeding in this way, for (5.38) together with the bound (5.37) via an application of Proposition 5.1.7, we may deduce the lower bound (5.11) also on this range of $|R(N)|$, thus concluding the proof of Proposition 5.3.3.

5.9 Proof of Proposition 5.3.5

5.9.1 Large values of Q

We always have that the integral $\int_{\mathfrak{m}}|\mathcal{S}_{\mathbf{1}_{y-\text{smooth}}}(\theta)|^2 d\theta$ is

$$\geq \int_{\mathfrak{m}}|\mathcal{S}_1(\theta)|^2 d\theta + \int_{\mathfrak{m}}\Bigg|\sum_{\substack{n\leq N\\ \exists p|n:p>y}} e(n\theta)\Bigg|^2 d\theta - 2\int_{\mathfrak{m}}\Bigg|\mathcal{S}_1(\theta)\sum_{\substack{n\leq N\\ \exists p|n:p>y}} e(n\theta)\Bigg| d\theta.$$

By Parseval's identity and Mertens' theorem, the second integral on the right-hand side above is $\ll N\log u$, where $u := (\log N)/(\log y)$. This, together with the upper bound for $\int_{\mathfrak{m}}|\mathcal{S}_1(\theta)|^2 d\theta$ given in Proposition 5.3.6 a) and Cauchy–Schwarz's inequality, makes the third integral instead of size $\ll \sqrt{QN\log u}$. By using the lower bound (5.9) for the integral $\int_{\mathfrak{m}}|\mathcal{S}_1(\theta)|^2 d\theta$, for values $DN\log u \leq Q \leq N$, with $D > 0$ a large constant, we may deduce the lower bound (5.12) on such range of Q.

5.9.2 Small values of Q

Let $\delta > 0$ small. Let K be a large constant, Q_0 and Q be real numbers satisfying (4.12) and such that $N^{1/2+\delta} \leq Q < DN \log u$, with D as in the previous subsection, and $\log N < Q_0 \leq Q_0^{\max} := N^{1/2-\delta}(\log N)^{17}/K$. Let $R := N^{1/2-\delta/2}$. We keep these notations throughout the rest of this section.

Remark 5.9.1. *The choice of the maximal possible size of Q_0 only reflects the fact that, to deduce the lower bound on the variance of the y–smooth numbers in arithmetic progressions as in Theorem 5.1.5, we will take $Q_0 = N(\log N)^{17}/Q$ in Proposition 4.1.4.*

Case y small

Let $\sqrt{N} \leq y \leq N^{1-\delta/8}$. Let $g(r)$ be the indicator of the prime numbers $r \in [Q_0^{\max}, R]$. We apply Proposition 4.1.5 with functions $f(n) = \mathbf{1}_{y-\text{smooth}}(n)$ and $g(r)$ as above.

Remark 5.9.2. *The choice of g here has been inspired by the fact that the Dirichlet convolution $\mathbf{1}_{y-smooth} * \mu(n)$, with μ the Möbius function (see Ch. Notations), equals $\mathbf{1}_{primes\ \in(y,N]}(n)$.*

With notations as in Proposition 4.1.5, by Parseval's identity, we have

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \leq \sum_{n\leq N} \Bigg(\sum_{\substack{p|n \\ Q_0^{\max}<p\leq R}} 1 \Bigg)^2 \leq \sum_{Q_0^{\max}<p\leq R} \frac{N}{p} + \sum_{\substack{Q_0^{\max}<p_1,p_2\leq R \\ p_1\neq p_2}} \frac{N}{p_1 p_2} \ll_\delta N,$$

by expanding the square out and swapping summations.

Let $W := \min\{N/y, R\}$ and $Z := \max\{KQ_0^{\max}, N/y\}$. By (4.15), with $f(n) = \mathbf{1}_{y-\text{smooth}}(n)$, and employing the first part of Lemma 5.1.6, we get

$$\begin{aligned}\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta &\gg \frac{N}{\log N} \sum_{\substack{KQ_0^{\max}<q\leq W \\ q \text{ prime}}} \frac{\log q}{q} + N \log u \sum_{\substack{Z<q\leq R \\ q \text{ prime}}} \frac{1}{q} + O_\delta(N^{1-\delta/11}) \\ &\gg_\delta N,\end{aligned}$$

by Mertens' theorem, if N is large with respect to δ. This concludes the proof of Proposition 5.3.5 when $\sqrt{N} \leq y \leq N^{1-\delta/8}$ via the application of Proposition

4.1.5 and the results just proved.

Case y large

Let us now consider $N^{1-\delta/8} < y \leq N/C$, where C is as in Lemma 5.1.6. Let g be a multiplicative function supported on the squarefree numbers and given on the primes by

$$g(p) = \begin{cases} 1 & \text{if } N/y < p \leq R; \\ 0 & \text{otherwise.} \end{cases}$$

We again apply Proposition 4.1.5 with functions $f(n) = \mathbf{1}_{y-\text{smooth}}(n)$ and $g(r)$ as above.

Remark 5.9.3. *From the work in the y small case, it is clear that we cannot make use of the same type of g even when y is very close to N. Indeed, we would always have*

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \ll N \max\left\{ \sum_{p \in Supp(g) \cap [KQ_0, R]} \frac{1}{p}, \left(\sum_{p \in Supp(g) \cap [KQ_0, R]} \frac{1}{p} \right)^2 \right\},$$

where $Supp(g) := \{n : g(n) \neq 0\}$.

On the other hand, by (4.15) *and Lemma 5.1.6 we would always also have*

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta \gg N \log u \sum_{p \in Supp(g) \cap [KQ_0, R]} \frac{1}{p},$$

which are not of comparable size, whenever u is close to 1. *For such values of y, we then opted for a multiplicative function g with the right logarithmic density, suggested to us from the second part of Lemma 5.1.6 and the following computations.*

By Parseval's identity, we have

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \le \sum_{n\le N}\Bigg(\sum_{\substack{r|n\\ r\le R\\ p|r\Rightarrow N/y<p\le R}} 1\Bigg)^2 \le \sum_{\substack{r_1,r_2\le R\\ p|r_1,r_2\Rightarrow N/y<p\le R}} \sum_{\substack{n\le N\\ [r_1,r_2]|n}} 1$$
$$\le N \sum_{\substack{r_1,r_2\le R\\ p|r_1,r_2\Rightarrow N/y<p\le R}} \frac{1}{[r_1,r_2]},$$

by expanding the square and swapping summations. By using a manipulation employed in a work of Dress, Iwaniec and Tenenbaum (see [6, Eq. 1]) we can rewrite the last sum above as

$$\sum_{\substack{r_1,r_2\le R\\ p|r_1,r_2\Rightarrow N/y<p\le R}} \frac{1}{r_1r_2}\sum_{d|r_1,r_2}\varphi(d) \le \sum_{\substack{d\le R\\ p|d\Rightarrow N/y<p\le R}} \frac{\varphi(d)}{d^2}\Bigg(\sum_{\substack{k\le R\\ p|k\Rightarrow N/y<p\le R}} \frac{1}{k}\Bigg)^2$$
$$\le \Bigg(\sum_{\substack{k\le R\\ p|k\Rightarrow N/y<p\le R}} \frac{1}{k}\Bigg)^3.$$

The last sum in the above displayed equation is

$$\ll \prod_{N/y<p\le R}\left(1+\frac{1}{p}\right) \ll \exp\Bigg(\sum_{N/y<p\le R}\frac{1}{p}\Bigg) \ll \frac{\log R}{\log(N/y)} \ll \frac{1}{u-1},$$

thanks to Lemma 2.1.1 and Mertens' theorem. We deduce that

$$\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta \ll \frac{N}{(u-1)^3}. \tag{5.44}$$

We note that

$$\sum_{\substack{r\le R\\ q|r\\ \mu^2(r)=1}} \frac{g(r)}{r} = \frac{g(q)}{q}\sum_{\substack{k\le R/q\\ (q,k)=1\\ \mu^2(k)=1}} \frac{g(k)}{k} \ge \frac{g(q)}{q}\prod_{p|q}\left(1+\frac{g(p)}{p}\right)^{-1}\sum_{\substack{k\le R/q\\ \mu^2(k)=1}}\frac{g(k)}{k}$$
$$=: \frac{h(q)}{q}\sum_{\substack{k\le R/q\\ \mu^2(k)=1}}\frac{g(k)}{k},$$

where we observe that $h(q)$ is a positive multiplicative function. Supposing $q \le N^{1/2-3\delta/4}$, using the last part of Lemma 2.1.1 and Mertens' theorem, we have

$$\sum_{\substack{k\le R/q\\ \mu^2(k)=1}} \frac{g(k)}{k} \gg \exp\Bigg(\sum_{N/y<p\le N^{\delta/4}} \frac{1}{p}\Bigg) \gg \frac{\log N^{\delta/4}}{\log(N/y)} \gg_\delta \frac{1}{u-1}.$$

By (4.15), with $f(n)=\mathbf{1}_{y-\mathrm{smooth}}(n)$, we find

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)|d\theta \gg_\delta \frac{N\log u}{u-1} \sum_{N^{1/2-5\delta/6}<q\le N^{1/2-3\delta/4}} \frac{h(q)}{q},$$

where we restricted the summation over q on those integers $N^{1/2-5\delta/6} < q \le N^{1/2-3\delta/4}$, employed the second part of Lemma 5.1.6 and took N large enough also with respect to δ.

Let $\mathcal{P} := \prod_{p\le N/y} p$. For any integer $k\ge 0$, we let

$$S_1(k) := \Bigg(\sum_{\substack{2^kN^{1/2-5\delta/6}<q\le 2^{k+1}N^{1/2-5\delta/6}\\ (q,\mathcal{P})=1\\ \mu^2(q)=1}} 1\Bigg)^2$$

$$S_2(k) := \sum_{\substack{2^kN^{1/2-5\delta/6}<q\le 2^{k+1}N^{1/2-5\delta/6}\\ (q,\mathcal{P})=1\\ \mu^2(q)=1}} \frac{q}{h(q)}.$$

By dyadic subdivision, one has

$$\begin{aligned}\sum_{N^{1/2-5\delta/6}<q\le N^{1/2-3\delta/4}} \frac{h(q)}{q} &\ge \sum_{k=0}^{\frac{\delta\log N}{12\log 2}-1} \sum_{2^kN^{1/2-5\delta/6}<q\le 2^{k+1}N^{1/2-5\delta/6}} \frac{h(q)}{q}\\ &\ge \sum_{k=0}^{\frac{\delta\log N}{12\log 2}-1} \frac{S_1(k)}{S_2(k)},\end{aligned}$$

by Cauchy–Schwarz's inequality, where we have restated the condition on the support of q, implicit in $h(q)$, as $\mu^2(q)=1$ and $(q,\mathcal{P})=1$. By the fundamental lemma of sieve theory (see e.g. [69, Ch. I, Theorem 4.4]), applied with a choice of δ small enough, and Mertens' theorem, we have

$$S_1(k) \gg_\delta \left(2^k N^{1/2-5\delta/6} \frac{\varphi(\mathcal{P})}{\mathcal{P}}\right)^2 \gg \left(\frac{2^k N^{1/2-5\delta/6}}{\log(N/y)}\right)^2.$$

On the other hand, by Lemma 2.1.1 and Mertens' theorem, we get that $S_2(k)$ is

$$\leq \sum_{\substack{q \leq 2^{k+1} N^{1/2-5\delta/6} \\ (q,\mathcal{P})=1 \\ \mu^2(q)=1}} \frac{2^{k+1} N^{1/2-5\delta/6}}{h(q)} \ll \frac{(2^k N^{1/2-5\delta/6})^2}{\log N} \prod_{N/y<p\leq N^{1/2-3\delta/4}} \left(1+\frac{1}{p}\right)$$
$$\ll \frac{(2^k N^{1/2-5\delta/6})^2}{\log(N/y)}.$$

Putting things together, we have proved that

$$\sum_{N^{1/2-5\delta/6}<q\leq N^{1/2-3\delta/4}} \frac{h(q)}{q} \gg_\delta \sum_{k=0}^{\frac{\delta \log N}{12\log 2}-1} \frac{1}{\log(N/y)} \gg_\delta \frac{\log N}{\log(N/y)} \geq \frac{1}{u-1}.$$

Consequently, we conclude that

$$\int_{\mathfrak{m}} |\mathcal{S}_f(\theta)\mathcal{G}(\theta)| d\theta \gg_\delta \frac{N \log u}{(u-1)^2}.$$

This, in combination with the upper bound (5.44) for the integral $\int_{\mathfrak{m}} |\mathcal{G}(\theta)|^2 d\theta$ and the trivial inequality $\log u \gg u-1$, if δ is small, finishes the proof of Proposition 5.3.5 via the application of Proposition 4.1.5.

5.10 Deduction of Theorem 5.1.1

By Proposition 4.1.4, we have

$$V(N,Q;d_1) \gg Q \int_{\mathfrak{m}} |\mathcal{S}_1(\theta)|^2 d\theta + O\Big(\frac{N^2}{Q_0} + \sum_{q\leq Q} \frac{1}{q} \sum_{\substack{d|q \\ d>Q_0}} \frac{1}{\varphi(d)} \Big| \sum_{n\leq N} c_d(n)\Big|^2\Big), \tag{5.45}$$

by choosing K large and where Q and Q_0 need to satisfy (4.12).

The sum in the big-Oh error term has already been estimated in Proposition 4.4.1, but here we are going to produce a better bound when we specify $f(n)$ to be the function $d_1(n)$.

First of all, by (4.11), we notice that

$$\sum_{n\le N} c_d(n) = \sum_{n\le N}\sum_{k|(n,d)} k\mu\left(\frac{d}{k}\right) = \sum_{k|d} k\mu\left(\frac{d}{k}\right)\sum_{\substack{n\le N\\ k|n}} 1$$
$$= \sum_{k|d} k\mu\left(\frac{d}{k}\right)\left\lfloor\frac{N}{k}\right\rfloor = O(\sigma(d)),$$

where we let $\sigma(d) := \sum_{k|d} k$ and where we used the well-known identity $\sum_{k|d}\mu(k) = 0$, for any $d > 1$. Therefore, we need to study the following sum:

$$\sum_{q\le Q}\frac{1}{q}\sum_{\substack{d|q\\ d>Q_0}}\frac{\sigma(d)^2}{\varphi(d)} = \sum_{Q_0<d\le Q}\frac{\sigma(d)^2}{\varphi(d)}\sum_{\substack{q\le Q\\ d|q}}\frac{1}{q} \tag{5.46}$$
$$\ll \sum_{Q_0<d\le Q}\frac{\sigma(d)^2}{d\varphi(d)}\left(\log\left(\frac{Q}{d}\right)+1\right).$$

Now, let

$$S(t) := \sum_{d\le t}\frac{\sigma(d)^2}{d\varphi(d)} \quad (t\ge 1).$$

It is not difficult to verify that the summand satisfies the hypotheses of Lemma 2.1.1, from which we easily deduce that $S(t) \ll t$, for any $t \ge 1$. By partial summation, we find that the last sum in (5.46) is $\ll Q$, on our range of parameters K, Q_0 and Q satisfying (4.12).

We employ Proposition 5.3.1 to lower bound the integral in (5.45). Choosing $Q_0 := CN^2/Q^2$, with $C > 0$ a large constant, we get the thesis for any Q in the range $C^{1/3}K^{2/3}N^{2/3} \le Q \le CN/\log N$ (remember that Q_0 has to satisfy (4.12)). By taking instead $Q_0 := N^2(\log N)/Q^2$, we get the thesis for any Q in the range $K^{2/3}N^{2/3}(\log N)^{1/3} \le Q \le N$. Together, they give Theorem 5.1.1, whenever N is sufficiently large.

5.11 Deduction of Theorem 5.1.3

In this final section of this chapter we prove the lower bound for the variance of $d^{\varpi}_{\alpha_N}(n)$ in arithmetic progressions as presented in Theorem 5.1.3. The proofs of Theorems 5.1.4 and 5.1.5 are similar, so they will be omitted.

By plugging the lower bound (5.11) for the integral $\int_{\mathfrak{m}}|\mathcal{S}_{d^{\varpi}_{\alpha_N}}(\theta)|^2 d\theta$ into the lower bound expression (4.13) for the variance of $f(n) = d^{\varpi}_{\alpha_N}(n)$ in arithmetic progressions, and choosing K large enough, we find

$$V(N,Q;d^{\varpi}_{\alpha_N}) \gg_{\delta} \frac{QN}{R(N)^2}\log\left(\frac{\log N}{\log(2N/Q)}\right) + Q^2 + O\left(\frac{N^2(\log N)^{14}}{Q_0}\right). \tag{5.47}$$

Here, to estimate the error term, we used Proposition 4.4.1 with $\kappa = 2$, say, and Corollary 2.1.2. Taking $Q_0 := NR(N)^2(\log N)^{15}/Q$, which satisfies the hypotheses of Proposition 5.3.3, we get the thesis, if N is large enough with respect to δ.

Chapter 6

Random product sets

Summary

The first section introduces the problem of estimating the size of deterministic and random product sets, and we present a new theorem. More precisely, we show that if, and only if, α is roughly $o((\log N)^{-\log 2+1/2})$, a random product set under the model $B(N, \alpha)$ is asymptotically almost surely maximal.

The second section contains the definition of the almost sure asymptotic and some basic information on the distribution of a random set in $B(N, \alpha)$.

The third section deals with the proof of a new minor result, which gives equality between the size of the product set AA and the maximal cardinality $(|A|^2+|A|)/2$, when $\alpha = o(1/\sqrt{N})$. Moreover, it contains an innovative proof of Cilleruelo, Ramana and Ramaré's result [3, Theorem 1.2], which gives a sufficient condition on α to have $|AA| \sim |A|^2/2$, with probability $1 - o(1)$ under $B(N, \alpha)$, and that was an inspiration for the problem considered here.

Finally, the fourth and fifth sections are devoted to the demonstration of the sufficient and necessary, respectively, part of our new theorem, which characterizes the random product sets of largest size possible.

6.1 Introduction

6.1.1 General background on product sets

The multiplication table problem

For every positive integer N denote by $[N] := \{1, \ldots, N\}$ the set of all positive integers between 1 and N.

In 1955, Erdős [8] asked about the number of *distinct* products in a multiplication table of integers. More specifically, he pondered about the size of the product set

$$[N][N] := \{mn : 1 \leq n \leq N, 1 \leq m \leq N\}.$$

Using estimates on the number of integers with a given number of prime factors, he deduced that $|[N][N]|= o(N^2)$, showing that there are only few distinct such products. Five years later, he established a more precise estimate (see [9])

$$|[N][N]|= \frac{N^2}{(\log N)^{\delta+o(1)}},$$

where

$$\delta := 1 - \frac{1 + \log\log 2}{\log 2} = 0.086071.... \tag{6.1}$$

Since Erdős' result, there was an explosion of activity aimed at nailing down the lower order term $(\log N)^{o(1)}$, which gave rise to new spectacular results on the closely related topic about the distribution of divisors into short intervals. As a consequence of such results, Hall and Tenenbaum [20] deduced that there exists a certain constant $c > 0$ such that

$$\frac{N^2}{(\log N)^{\delta} \exp(c\sqrt{\log N \log\log N})} \ll |[N][N]|\ll \frac{N^2}{(\log N)^{\delta}(\log\log N)^{1/2}}.$$

In 2008, Ford [14] established the exact order of magnitude of the number of distinct positive integers in a multiplication table (see also [15], where an easier proof of the same result is presented):

$$|[N][N]| \asymp \frac{N^2}{(\log N)^{\delta}(\log\log N)^{3/2}}, \tag{6.2}$$

where δ is as in (6.1). The parameter δ arises from the fact that most entries in an N-by-N multiplication table have about $(\log\log N)/(\log 2)$ prime factors (a heuristic for this is given in the introduction of [14]). For a k-dimensional generalization of the multiplication table problem, see the PhD thesis of Koukoulopoulos [39].

Uncompleted tables

One may ask whether there are multiplicative subsets of the first N positive integers with small selfproduct. In this regard, we consider the set of all bounded sums of two squares $Q_N := \{a^2 + b^2 : 1 \leq a \leq b \leq N\}$. Each of its elements follows a specific pattern: they factorize into a product of prime powers, each of which is a sum of two squares. We may then expect the product set $Q_N Q_N$ to contain many pairs of identical elements, as in the case of $[N][N]$. In fact, in his master's thesis, Mangerel [50], generalizing Ford's result on the multiplication table problem to sets of integral ideals of a rational number field (and even more to a certain class of arithmetical semigroups), proved, as a special case, that

$$|Q_N Q_N| \asymp \frac{|Q_N|^2}{(\log N)^{\delta}(\log\log N)^{3/2}},$$

with δ as in (6.1) (see [50, Theorem 1.4]).

The examples of $[N]$ and Q_N suggest that both the relative density in $[N]$ and the multiplicative structure (if we extend both sets to infinite sets they become closed under multiplication) of a set reduce the size of its selfproduct. In particular, if $|A| = N\alpha$ with $|AA| \sim (|A|^2 + |A|)/2$, i.e. of maximal size possible, by seeing $AA \subset [N][N]$, we deduce from (6.2) that

$$\alpha \ll \frac{1}{(\log N)^{\delta/2}(\log\log N)^{3/4}}. \tag{6.3}$$

On the other hand, when instead the elements of a relatively small set A

look like more randomly distributed (compared to $[N]$), we might expect to have a corresponding product set AA of maximal size possible. Let us for example consider the set of all the prime numbers up to N, say $\mathcal{P}_N$. Then, it is immediate to verify that $|\mathcal{P}_N\mathcal{P}_N| = (|\mathcal{P}_N|^2 + |\mathcal{P}_N|)/2$. Indeed, all the products pp', with $p, p' \in \mathcal{P}_N$, are distinct, apart from the order of the factors (sets with this property are usually called "multiplicative Sidon sets").

However, if we slightly tweak this set by shifting every element by 1, it no longer is so easy to determine the size of the corresponding product set. A main obstacle is that, for every prime number p, $p-1$ is coprime with p and we cannot predict its prime factorization.

Let $\mathcal{P}_N - 1 := \{p - 1 : p \in \mathcal{P}_N\}$. One may ask:

what is the cardinality of the product set $(\mathcal{P}_N - 1)(\mathcal{P}_N - 1)$*?*

Here, we have a different situation. It may be actually possible now to have two equal products $(p-1)(q-1) = (r-1)(s-1)$ inside $(\mathcal{P}_N - 1)(\mathcal{P}_N - 1)$, for certain primes p, q, r, s, but in such case the possibly many prime factors of the left-hand side product need to be shared with those of the right-hand side one. However, since the primes are considered to possess a random looking behaviour, we also expect the prime factorization of shifted primes to behave enough randomly, which makes the above identity quite unlikely. In a 2017 paper, Cilleruelo, Ramana and Ramaré [3] formally deduced this by showing the following asymptotic equality:

$$(\mathcal{P}_N - 1)(\mathcal{P}_N - 1) \sim |\mathcal{P}_N - 1|^2/2 \text{ as } N \longrightarrow +\infty. \tag{6.4}$$

As another example of the phenomena under which a relatively small random looking set should have a largest possible selfproduct, we pay attention to the set of shifted squares $Q_N - 1$. Compared to Q_N, it no longer retains a multiplicative structure and there is no obvious way to predict the prime factorization of its elements, making them look like more randomly distributed. Cilleruelo, Ramana and Ramaré [3, Theorem 1.5] indeed showed that

$$|(Q_N - 1)(Q_N - 1)| \asymp |Q_N - 1|^2, \tag{6.5}$$

but it is important to notice that they were unable to determine whether (6.5) might be replaced with an asymptotic relation. One thing though is certain:

the set of shifted primes and of shifted squares have a very different cardinality, since $|\mathcal{P}_N - 1| \sim N/\log N$, by the Prime Number Theorem [55, Theorem 6.9], whereas $|\mathcal{Q}_N - 1| \sim cN/\sqrt{\log N}$, for a constant $c > 0$, by Landau's theorem [41]. This is another indication of the fact that the relative density of a set might affect the relation between the exact size of its selfproduct and the maximal one. Our main theorem in this chapter demonstrates that this is indeed the case, at least when looking at suitably defined random sets. As a consequence, we will also provide heuristic evidence that the set of shifted sums of two squares should have asymptotically maximal product set.

6.1.2 Random product sets

The probabilistic model $B(N, \alpha)$

For every $\alpha \in [0, 1)$, let $B(N, \alpha)$ denote the probabilistic set up in which a random set $A \subset [N]$ is constructed by choosing independently every element of $[N]$ with probability α.

We can interpret the random variable $|A| = \sum_{1 \le i \le N} \mathbf{1}_{i \in A}$ as a random variable with binomial distribution $\mathrm{Bi}(N, \alpha)$, which motivates the choice of label for $B(N, \alpha)$. In particular, A has expected size $N\alpha$. Thus, it can be viewed as a random model for a subset of the positive integers smaller than N with natural density approximately α and whose elements look like independently randomly distributed.

Under the probabilistic model $B(N, \alpha)$, our focus will be on the study of the expected size of product sets.

A sufficient condition on α

When α is small enough, the next proposition, which proof is postponed to Sect. 6.3, guarantees equality between the size of an associated random product set and its maximal possible cardinality.

Proposition 6.1.1. *Let A be a random set in $B(N, \alpha)$ and assume that $\alpha = o(1/\sqrt{N}(\log N)^{1/4})$. Then $|AA| = (|A|^2 + |A|)/2$ with probability $1 - o(1)$.*

If we increase the value of α we might lose the above equality, but we could nevertheless still have an *asymptotic* equality. A sufficient condition for this to happen has been given in the following result (see [3, Theorem 1.2], and see instead Sanna [65, Theorem 1.2] for a generalization thereof to iterated product sets of random sets).

Theorem 6.1.2 (Cilleruelo, Ramana and Ramaré)**.** *Let A be a random set in $B(N,\alpha)$. If $\alpha = o((\log N)^{-1/2})$, then we have $|AA| \sim |A|^2/2$ with probability $1-o(1)$.*

The above asymptotic relation is equivalent to the convergence in probability of the quotient $2|AA|/|A|^2$ to 1.

The proof of Theorem 6.1.2 proceeds as follows. First, one realizes that it is enough to show a version of Theorem 6.1.2 in expectation, or equivalently that $\mathbb{E}[|AA|] \sim \mathbb{E}[|A|^2]/2$, as $N \longrightarrow +\infty$, through an application of Markov's inequality. To this aim, one establishes the following explicit expression for the expectation of $|AA|$:

$$\mathbb{E}[|AA|] = \sum_{1\le n\le N^2} (1-(1-\alpha^2)^{\tau_N(n)/2}) + O(N\alpha), \tag{6.6}$$

where

$$\tau_N(n) := \#\{(j,k) \in [N]\times[N] : n = jk\}$$

is the number of restricted representations of a positive integer n as product $n = jk$, with $1 \le j,k \le N$. Indeed, as shown in [3, Proposition 3.2]

$$\begin{aligned}\mathbb{P}(n \in AA) &= \mathbb{P}\Bigg(\bigcup_{m_1m_2=n,\ m_1<m_2\le N} \{m_1 \in A, m_2 \in A\}\Bigg)\\ &= 1 - \prod_{m_1m_2=n,\ m_1<m_2\le N} (1-\alpha^2) = 1-(1-\alpha^2)^{\tau_N(n)/2},\end{aligned}$$

if n is not square, otherwise $\mathbb{P}(n \in AA) = 1-(1-\alpha^2)^{\tau_N(n)/2} + O(\alpha)$.

Since

$$\mathbb{E}[|A|^2] = (N\alpha)^2 + N\alpha(1-\alpha),$$

as for a binomially distributed random variable, one reduces to prove that

$$\sum_{1\le n\le N^2}\left(\frac{\alpha^2\tau_N(n)}{2} - 1 + (1-\alpha^2)^{\tau_N(n)/2}\right) + O(N\alpha) = o(\alpha^2N^2). \tag{6.7}$$

Now, since

$$\sum_{1\leq n\leq N^2} \tau_N(n) = N^2, \tag{6.8}$$

which immediately follows from the definition of τ_N, it is clear that one seeks for some saving in the sum in (6.7). This is achieved, when $\alpha = o((\log N)^{-1/2})$, by Taylor expanding the binomial $(1-\alpha^2)^{\tau_N(n)/2}$ to the second order and making use of the fact that

$$\sum_{1\leq n\leq N^2} \tau_N(n)^2 \ll N^2 \log N, \tag{6.9}$$

which is the content of [3, Lemma 2.1].

Plainly, Theorem 6.1.2 leaves open the following questions:

is it true that the condition $\alpha = o((\log N)^{-1/2})$ is also necessary? Otherwise, can we improve it? and, in such case, what is the maximum value of α one can take?

Similarly, in the deterministic setting, Cilleruelo, Ramana and Ramaré raised the following question:

is it true that whenever $A \subset [N]$ is such that $|AA| \sim |A|^2/2$, as $N \longrightarrow +\infty$, then $|A| = o(N(\log N)^{-1/2})$?

This was answered negatively by Ford [16], by proving the following result.

Theorem 6.1.3 (Ford)**.** *Let $D > 7/2$. For each $N \geq 10$ there is a set $A \subset [N]$ of size*

$$|A| \geq \frac{N}{(\log N)^{\delta/2}(\log\log N)^D},$$

with δ as in (6.1)*, for which $|AA| \sim |A|^2/2$, as $N \longrightarrow +\infty$.*

Coming back to the approach taken by Cilleruelo, Ramana and Ramaré to prove Theorem 6.1.2, as sketched before, one could start again from (6.7) and Taylor expand the binomial a little more, making use this time of the higher moments of τ_N. Unfortunately, the Taylor expansion approach cannot be pursued any further here, because already the third moment of τ_N happens to be of size $N^2(\log N)^4$, as opposed to $N^2(\log N)^2$, in which case

$\alpha = o((\log N)^{-1/2})$ would have presumably been sharp. More specifically, the positive integers moments of τ_N, by analogy with those of the divisor function $\tau(n) := \sum_{d|n} 1$, should grow exponentially in the exponent of $\log N$.

The main new theorem

Our new next result is aimed at answering the previous questions in the random setting, by completely determining all the values of α corresponding to maximal random product sets.

Theorem 6.1.4. *Let A be a random set in $B(N, \alpha)$, with $\alpha \in [0, 1)$. Then we have $|AA| \sim |A|^2/2$ with probability $1 - o(1)$, as $N \longrightarrow +\infty$, if and only if*

$$\frac{\log(\alpha^2 (\log N)^{\log 4 - 1})}{\sqrt{\log \log N}} \longrightarrow -\infty.$$

In other words, all the random sets under the model $B(N, \alpha)$, with α approximately $o((\log N)^{-\log 2 + 1/2})$, with $-\log 2 + 1/2 = -0.19314718... > -0.5$, have a product set of asymptotically maximal size, whereas larger ones do not, thus substantially improving on Theorem 6.1.2. In particular, closing a gap present in [3], for sets $A \in B(N, \alpha)$ with $\alpha \asymp 1/\sqrt{\log N}$, such as random models of the set of shifted sums of two squares, it follows that $|AA| \sim |A|^2/2$, with probability $1 - o(1)$. We then conjecture that

$$|(Q_N - 1)(Q_N - 1)| \sim |Q_N - 1|^2/2 \text{ as } N \longrightarrow +\infty.$$

We also remark that both Theorems 6.1.2 and 6.1.4 are not concerned with random sets A for which $|AA| \sim |A|^2/2$ holds with an intermediate probability, and the maximality comes from both the size of the product set and the likelihood of the relation. It would be nice though to figure out what are the choices of α for which such relation happens with half probability, for example.

6.1.3 Sketch of the proof of Theorem 6.1.4

The sufficient part

We start again from (6.7). We saw before that exclusively Taylor expanding further the binomial $(1 - \alpha^2)^{\tau_N(n)/2}$ does not lead to any extension in the set of values of α corresponding to maximal random product sets, due to the fast blowing up of the moments of τ_N. We then need to incorporate new information on the distribution of τ_N in order to exploit more saving in the sum in (6.7).

Heuristics for τ_N

When studying the multiplication table problem, Hall and Tenenbaum [20] made the following assumption on the distribution of divisors of a positive integer, which seemed to match well with their results:

for most positive integers $n \leq N^2$, the set $\{(\log d)/(2\log N) : d|n\}$ is roughly uniformly distributed over the interval $[0, 1]$.

In light of this and of well-known properties of the distribution of the divisor function $\tau(n)$, we will deduce in Subsect. 6.4.1 that we may think of τ_N as

$$\tau_N(n) \approx \frac{\tau(n)}{\log N}, \tag{6.10}$$

at least on average over a 'large' set of integers. In particular, in analogy with $\tau(n)$, we will deduce that the numbers n that should contribute the most to (6.8) are those for which the additive function $\omega(n)$, which follows a normal distribution (see Proposition 2.2.2), is close to double its mean value $\log\log N$ for at most a factor of its standard deviation $\sqrt{\log\log N}$. Equivalently, they are positive integers n such that:

$$|\omega(n) - 2\log\log N| \leq M\sqrt{\log\log N}, \tag{6.11}$$

with $M > 0$. Let us indicate with $\mathcal{S}_1$ the set of those $n \leq N^2$. In other words, if we let $\mathcal{S}_2$ be the complement of $\mathcal{S}_1$ in $[N^2]$, then

$$\sum_{n\in\mathcal{S}_2} \tau_N(n) = o\bigg(\sum_{n\leq N^2} \tau_N(n)\bigg) \text{ if } M \longrightarrow +\infty. \tag{6.12}$$

On the other hand, crucially, again in analogy with $\tau(n)$, the numbers in (6.11) should be outside the set of integers where, on average, τ_N^2 is mostly concentrated on, meaning that we now expect

$$\sum_{n\in\mathcal{S}_1}\tau_N^2(n)=o\bigg(\sum_{n\le N^2}\tau_N^2(n)\bigg)\text{ if }M\longrightarrow+\infty. \tag{6.13}$$

In fact, the main contribution to the average of τ_N^2 should come from those numbers n such that

$$|\omega(n)-4\log\log N|\le M\sqrt{\log\log N}.$$

The main new idea

We split the sum in (6.7) into two parts: one on the integers in $\mathcal{S}_1$ and the other on their complementary set $\mathcal{S}_2$. By Taylor expanding over the first set the binomial $(1-\alpha^2)^{\tau_N(n)/2}$ to the second order, we bound (6.7) from the above with roughly

$$\alpha^2\sum_{n\in\mathcal{S}_2}\tau_N(n)+\alpha^4\sum_{n\in\mathcal{S}_1}\tau_N(n)^2.$$

In Subsect. 6.4.3, we will formally prove (6.12), which, together with (6.8), produces an acceptable contribution to (6.7). After, in Subsect. 6.4.4, we will formally deduce (6.13), but in a quantitative form, so as to capture the largest value of α that makes also the second sum above small enough to get (6.7). The main tools needed to achieve that consist in: a twisting of the sum over $\mathcal{S}_2$ with the divisor function, to recast in a more amenable form its characterizing condition on the number of prime factors; in the use of the Erdős–Kac's theorem (Proposition 2.2.2), about the normal distribution of $\Omega(n)$, to sharpen its estimate.

The necessary part

Here, we suppose that the limit in Theorem 6.1.4 either does not exist or it gives a value different from $-\infty$. We also assume that even in these cases the associated choice of α leads to a random product set of maximal size, seeking for a contradiction.

To this aim, we realize that when, for a random set A in $B(N,\alpha)$,

$|AA| \sim (|A|^2+|A|)/2$ with probability $1-o(1)$, we necessarily have $\mathbb{E}[|AA|] \sim \mathbb{E}[(|A|^2+|A|)/2]$, as $N \longrightarrow +\infty$. This can be restated as in (6.7), which we would like now to contradict.

Since the binomial $(1-\alpha^2)^{\tau_N(n)/2}$ is nonnegative, and since we are not seeking for any saving here, we can forget about it and reduce ourselves to prove that

$$\sum_{n\in\mathcal{S}'}\left(\frac{\alpha^2\tau_N(n)}{2}-1\right) \geq \varepsilon\alpha^2N^2, \tag{6.14}$$

for a certain possibly small constant $\varepsilon > 0$, over a suitable set $\mathcal{S}' \subset [N]$.

Taking into account of (6.8) and of (6.12), it is evident that $\mathcal{S}'$ needs to be contained in the set of numbers n where $\omega(n)$ is close to $2\log\log N$ for at most a factor times $\sqrt{\log\log N}$. On the other hand, we would also like the constant function -1 to contribute less than $\varepsilon\alpha^2N^2/2$, say, when averaged over $\mathcal{S}'$. For these reasons, the natural choice is to take

$$\mathcal{S}' := \{1 \leq n \leq N^2 : M\sqrt{\log\log N} < \omega(n) - 2\log\log N \leq 2M\sqrt{\log\log N}\},$$

for a certain constant $M > 0$, since by the Hardy–Ramanujan's theorem most numbers $n \leq N^2$ have $\omega(n)$ roughly equal to $\log\log N$ (see e.g. [55, Corollary 2.13] or Proposition 2.2.1 here). This last one means that the constant function -1 averaged over $\mathcal{S}'$ is certainly smaller than N^2. Then an application of the Erdős–Kac's theorem, and a sensible choice of M, make possible to precisely estimate the sums in (6.14) and get the desired conclusion, by finding that the assumed values of α are exactly those that compensate the loss in averaging -1 over $\mathcal{S}'$.

6.2 Preliminaries to the proof of the Theorem 6.1.4

6.2.1 Notations

For every $\alpha \in [0,1)$, let $B(N,\alpha)$ denote the probabilistic set up in which a random set $A \subset [N]$ is constructed by choosing independently every element of $[N]$ with probability α. This is the probabilistic setting we will operate on.

For two sequences of random variables $X_1^{(N)}, X_2^{(N)}$, we say $X_1^{(N)} \sim X_2^{(N)}$ if for any $\delta > 0$ and $\varepsilon > 0$ there exists $N_0 = N_0(\delta, \varepsilon) \geq 1$ such that

$$\mathbb{P}(|X_1^{(N)} - X_2^{(N)}| \geq \delta X_2^{(N)}) \leq \varepsilon, \text{ if } N \geq N_0.$$

In short, we may write that for any $\delta > 0$

$$\mathbb{P}(|X_1^{(N)} - X_2^{(N)}| \geq \delta X_2^{(N)}) = o_\delta(1) \text{ as } N \longrightarrow +\infty.$$

Furthermore, we will simply denote with X_1, X_2 two such sequences of random variables, thus omitting the explicit dependence on N, and say that $X_1 \sim X_2$ with probability $1 - o(1)$ or asymptotically almost surely. This is the asymptotic relation we will use between objects in $B(N, \alpha)$.

To any set $A \subset [N]$ we can associate a quantity called the *multiplicative energy* of A, defined as

$$E(A) := \{(a, b, c, d) \in A^4 : ab = cd\}.$$

In the definition of $E(A)$ we tacitly assume that each quadruple is taken once without accounting for the multiplicity coming from possible symmetries (e.g. from swapping a with b or c with d). The multiplicative energy thus counts the number of 'collisions' between elements in the product set. Note that we can always find inside $E(A)$ the set of quadruples (a, b, a, b) (without the multiplicity from swapping a with b), which we call the set of 'trivial solutions' (to the equation $ab = cd$), and the complementary set of 'non-trivial solutions'; the former has size $(|A|^2 + |A|)/2$. The multiplicative energy will be used in the new proof of [3, Theorem 1.2] that we will give in Sect. 6.3.

6.2.2 Basic results

We can interpret the random variable $|A| = \sum_{1 \leq i \leq N} \mathbf{1}_{i \in A}$ as a random variable with binomial distribution $\mathrm{Bi}(N, \alpha)$. From this it follows that

- $\mathbb{E}[|A|] = N\alpha$;
- $\mathrm{Var}(|A|) = N\alpha(1 - \alpha)$;

- $\mathbb{E}[|A|^2] = (N\alpha)^2 + N\alpha(1-\alpha)$;
- $\mathrm{Var}(|A|^2) = 4N^3\alpha^3(1-\alpha) + O(N^2\alpha^2)$;
- $\mathbb{E}[|A|^4] = N^4\alpha^4 + 6N^3\alpha^3(1-\alpha) + O(N^2\alpha^2)$.

For an easy direct proof of the above equalities see the paper of Cilleruelo, Ramana and Ramaré [3]. In particular, it follows that

$$\mathbb{E}[(|A|^2{+}|A|)/2] = \frac{N^2\alpha^2}{2} + N\alpha - \frac{N\alpha^2}{2} = \mathbb{E}[|A|^2/2] + O(N\alpha) \tag{6.15}$$

and when $N\alpha \longrightarrow +\infty$ that

$$|A| \sim N\alpha \text{ and } |A|^2 \sim (N\alpha)^2 \sim |A|^2{+}|A| \tag{6.16}$$

with probability $1 - o(1)$, which is the content of [3, Lemma 3.1].

The next lemma is about some basic inequalities between the exponential function and truncations of its Taylor series expansion, that will be useful to estimate, where needed, the binomial $(1-\alpha^2)^{\tau_N(n)/2}$.

Lemma 6.2.1. *Let*

$$T_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$$

be the Taylor series for $\exp(x)$ *at* 0 *truncated after* n *terms. Then for* $x > 0$ *we have*

$$\exp(x) > T_n(x).$$

On the other hand, for $x < 0$*, we have*

$$\begin{cases} \exp(x) > T_n(x) & \text{if } n \text{ odd;} \\ \exp(x) < T_n(x) & \text{if } n \text{ even.} \end{cases}$$

Proof. By the Taylor expansion of the exponential at 0 with the Lagrange remainder, we have:

$$\exp(x) = T_n(x) + \frac{\exp(\xi)}{(n+1)!}x^{n+1},$$

for a certain ξ between 0 and x. Since $\exp(\xi) \geq 0$, we immediately deduce the stated results. $\square$

We conclude this subsection by proving that when two sequences of positive random variables are asymptotic and if we have some control on the second moment of at least one of them, then their mean values will be asymptotic, too. We explain this in detail in the following lemma, in which the particular case of $|AA|$ and $(|A|^2+|A|)/2$ has been analysed. By arguing by contradiction, it will let us restate the necessary condition in Theorem 6.1.4 in a more tractable form for computations.

Lemma 6.2.2. *As $N\alpha \longrightarrow +\infty$, if $|AA| \sim (|A|^2+|A|)/2$ with probability $1 - o(1)$, then we have*

$$\mathbb{E}[|AA|] \sim \mathbb{E}[(|A|^2+|A|)/2] \text{ as } N \longrightarrow +\infty.$$

Proof. To simplify notations, let us put

$$X_1 = (|A|^2+|A|)/2$$
$$X_2 = |AA|.$$

We certainly have

$$\mathbb{E}[X_1] = \mathbb{E}[X_1 - X_2] + \mathbb{E}[X_2],$$

where the first mean value on the right-hand side above is, for any $\varepsilon > 0$,

$$\begin{aligned}
&= \mathbb{E}[(X_1 - X_2)\mathbf{1}_{(X_1-X_2)\geq\ \varepsilon X_2}] + \mathbb{E}[(X_1 - X_2)\mathbf{1}_{(X_1-X_2)\leq\ \varepsilon X_2}] \\
&\leq \mathbb{E}[(X_1 - X_2)\mathbf{1}_{(X_1-X_2)\geq\ \varepsilon X_2}] + \varepsilon\mathbb{E}[X_2] \\
&\leq \sqrt{\mathbb{E}[(X_1 - X_2)^2]\mathbb{P}(X_1 - X_2 \geq \varepsilon X_2)} + \varepsilon\mathbb{E}[X_2] \\
&\leq o_\varepsilon\left(\sqrt{\mathbb{E}[X_1^2 + X_2^2]}\right) + \varepsilon\mathbb{E}[X_2] \\
&\leq o_\varepsilon\left(\sqrt{\mathbb{E}[X_1^2]}\right) + \varepsilon\mathbb{E}[X_2].
\end{aligned}$$

The expression in the third line above is a consequence of the Cauchy–Schwarz inequality, that in the fourth one follows by hypothesis and in the last one we used the fact that $X_2 \le X_1$.

Using (6.16) and the asymptotics on the moments of $|A|$, it is immediately seen that $\mathbb{E}[X_1^2] \sim \mathbb{E}[X_1]^2$. Putting the above estimates together we deduce that

$$\mathbb{E}[X_1](1 - o_\varepsilon(1)) \le \mathbb{E}[X_2](1 + \varepsilon).$$

From this we can reach the required conclusion. Indeed, choose $N_0 = N_0(\varepsilon)$ such that $o_\varepsilon(1) \le \varepsilon$, for any $N \ge N_0$. Then

$$\left| \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_2]} - 1 \right| \le 2\varepsilon + O(\varepsilon^2),$$

for any $N \ge N_0$, from which the stated result easily follows. □

6.3 Proofs of the introductory results

In this section we are going to prove Proposition 6.1.1, which says that whenever $\alpha = o(1/\sqrt{N}(\log N)^{1/4})$, we have $|AA| = (|A|^2 + |A|)/2$, with probability $1 - o(1)$. Furthermore, we are going to present a new alternative proof of Proposition 6.1.2, which is the content of [3, Theorem 1.2].

Proof of Proposition 6.1.1. Every element in AA is by definition a product ab, with $a, b \in A$. The number of such products is, without accounting for the multiplicity coming from the symmetry $ab = ba$, at most $(|A|^2 + |A|)/2$. We will now show that the probability of having $|AA| = (|A|^2 + |A|)/2$ tends to 1. Equivalently, if we let

$$\Sigma(A) := \frac{|A|^2 + |A|}{2} - |AA|$$

we will show that

$$\mathbb{P}(\Sigma(A) \ge 1) = o(1).$$

To this end, remember from the introduction to this chapter that

$$\tau_N(n) := \#\{(j, k) \in [N] \times [N] : n = jk\}$$

and that it verifies

$$\sum_{1\leq n\leq N^2} \tau_N(n) = N^2 \tag{6.17}$$

and

$$\sum_{1\leq n\leq N^2} \tau_N(n)^2 \ll N^2 \log N. \tag{6.18}$$

Hence, we can infer that

$$\begin{aligned}
\mathbb{P}(\Sigma(A) \geq 1) &= \mathbb{P}(\exists (a,b,c,d) \in A^4 : ab = cd \text{ and } a \neq c, d) \\
&\leq \alpha^4 \sum_{ab\in[N][N]} \sum_{\substack{d|ab \\ d\neq a,b \\ d\leq N, ab/d\leq N}} 1 \\
&\leq \alpha^4 \sum_{ab\in[N][N]} \tau_N(ab) \\
&\leq \alpha^4 \sum_{n\leq N^2} \tau_N(n)^2 \\
&\ll \alpha^4 N^2 \log N,
\end{aligned}$$

by the union bound and (6.18). Since by hypothesis $\alpha = o(1/\sqrt{N}(\log N)^{1/4})$, we get $\mathbb{P}(\Sigma(A) \geq 1) = o(1)$, as required. □

When the product set AA has maximal cardinality it is intuitive to expect the set of trivial solutions inside the multiplicative energy $E(A)$ to be much larger than the complementary set of non-trivial ones. In other words, when the number of non-trivial solutions inside $E(A)$ is somewhat 'small' compared to $|A|^2$ we expect few collisions on average and thus a product set AA of size as large as possible. This is indeed the idea behind the new proof we next give of [3, Theorem 1.2].

Proof of Proposition 6.1.2. By an application of the Cauchy–Schwarz inequality, we have

$$\left(\frac{|A|^2+|A|}{2}\right)^2 = \left(\sum_{x\in AA} r_{AA}(x)\right)^2 \le |AA|\left(\sum_{x\in AA} r_{AA}(x)^2\right) = |AA|E(A), \tag{6.19}$$

where $r_{AA}(x)$ is the number of representations of x as a product of two elements in A, without accounting for possible symmetries. For an illustration of the use of inequality (6.19) to produce a lower bound for the size of product sets see Tao and Vu's textbook [68, Lemma 2.30].

Since $E(A) = (|A|^2+|A|)/2 + R(A)$, where $R(A)$ is the number of non-trivial solutions to $ab = cd$ in A, from (6.19) we get

$$\frac{((|A|^2+|A|)/2)^2}{(|A|^2+|A|)/2 + R(A)} \le |AA|. \tag{6.20}$$

Moreover, we have

$$\begin{aligned}\mathbb{E}[R(A)] &= \sum_{\substack{1\le a,b,c,d\le N\\ ab=cd\\ a\ne b,c,d}} \mathbb{P}(a,b,c,d\in A) + \sum_{\substack{1\le a,c,d\le N\\ a^2=cd\\ a\ne c,d}} \mathbb{P}(a,c,d\in A)\\ &\le \sum_{\substack{1\le a,b,c,d\le N\\ ab=cd}} \alpha^4 + \sum_{1\le a\le N}\sum_{\substack{1\le d\le N\\ d|a^2}} \alpha^3\\ &\le \alpha^4 E([N]) + \alpha^3 \sum_{1\le a\le N} \tau(a^2).\end{aligned}$$

It has been proven in [3, Lemma 2.1] that $E([N]) \ll N^2\log N$. Moreover, we have

$$\sum_{n\le x} \tau(n^2) \ll x(\log x)^3 \quad (x\ge 2),$$

which can be easily derived from Lemma 2.1.1 (the exact order of magnitude for the partial sum of $\tau(n^2)$ over the positive integers n up to x is $x(\log x)^2$, but we do not need this degree of precision here). We deduce that

$$\mathbb{E}[R(A)] \ll \alpha^4 N^2 \log N + \alpha^3 N (\log N)^3.$$

We conclude that taking $\alpha = o((\log N)^{-1/2})$ makes the above of size $o(\alpha^2 N^2)$. By Markov's inequality we then have, for any $\varepsilon > 0$, that

$$\mathbb{P}(R(A) > \varepsilon \alpha^2 N^2) \leq \frac{\mathbb{E}[R(A)]}{\varepsilon \alpha^2 N^2} = o_\varepsilon(1).$$

Combining this with (6.16) and (6.20), we deduce that

$$\frac{|A|^2 + |A|}{2}(1 + O(\varepsilon)) \leq |AA| \leq \frac{|A|^2 + |A|}{2}$$

with probability $1 - o_\varepsilon(1)$. Since $\varepsilon > 0$ is arbitrary, we get the result. □

6.4 The sufficient condition

In this section we are going to prove the sufficient condition of Theorem 6.1.4. To set up the argument, let us suppose that $N\alpha \longrightarrow +\infty$ and $\alpha \longrightarrow 0$ and consider a random set $A \in B(N, \alpha)$. We know that we can restrict α in this way thanks to Proposition 6.1.1 and the bound (6.3).

Let us then define

$$X_A := \frac{|A|^2 + |A|}{2} - |AA| \geq 0.$$

By (6.15) we have

$$\mathbb{E}[X_A] = \frac{\mathbb{E}[|A|^2]}{2} - \mathbb{E}[|AA|] + O(N\alpha).$$

Our aim is to find conditions on α for which the following holds:

for any δ and $\varepsilon > 0$ there exists an $N_0 = N_0(\delta, \varepsilon)$ such that

$$\mathbb{P}(X_A \geq \delta(|A|^2 + |A|)/2) \leq \varepsilon \text{ if } N \geq N_0.$$

However, since by (6.16), $|A|^2 + |A| \sim |A|^2 \sim (N\alpha)^2$ with probability $1 - o(1)$, we can replace inside the above probability the expression $(|A|^2+|A|)/2$ with just $(N\alpha)^2/2$, without changing the desired estimate.

By Markov's inequality we have

$$\mathbb{P}(X_A \geq \delta(N\alpha)^2/2) \leq \frac{2\mathbb{E}[X_A]}{\delta(N\alpha)^2}. \tag{6.21}$$

So, the aim is to show that $\mathbb{E}[X_A] \leq \delta\varepsilon\alpha^2 N^2/2$, say.

From the proof of [3, Proposition 3.2], and as explained in the introduction to this chapter, we have

$$\mathbb{E}[|AA|] = \sum_{1\leq n\leq N^2} \left(1 - (1-\alpha^2)^{\tau_N(n)/2}\right) + O(N\alpha)$$

and by (6.15) and (6.17) also that

$$\mathbb{E}[|A|^2] = \sum_{1\leq n\leq N^2} \alpha^2\tau_N(n) + O(N\alpha).$$

Putting the above two identities together we can rewrite the mean of X_A as

$$\mathbb{E}[X_A] = \sum_{1\leq n\leq N^2} \left(\frac{\alpha^2\tau_N(n)}{2} - 1 + (1-\alpha^2)^{\tau_N(n)/2}\right) + O(N\alpha). \tag{6.22}$$

The term inside the parenthesis is the difference between the binomial $(1-\alpha^2)^{\tau_N(n)/2}$ and its first order Taylor expansion. We then split the sum into two parts. The first one being on those integers $\mathcal{S}_1 \subset [N^2]$ where by Taylor expanding the above binomial to the second order we may obtain a certain amount of saving from the partial sum of τ_N^2. The second one being on the rest, denoted by $\mathcal{S}_2$, where the saving just comes from averaging τ_N. To this aim, we need to better understand the distribution of the function τ_N.

6.4.1 Heuristic behaviour of τ_N

We claim that roughly speaking we may think of $\tau_N(n)$ as

$$\tau_N(n) \approx 2\tau(n)\left(1 - \frac{\log n}{2\log N}\right) \quad \text{(for most } n \leq N^2\text{)},$$

at least when we consider τ_N on average over a 'large' set of integers.

Indeed, if we assume that for most positive integers $n \le N^2$ the set $\{\log d/\log N : d|n\}$ is roughly uniformly distributed over the interval $[0,1]$, we have

$$\begin{aligned}
\tau_N(n) = \#\{d|n : n/N \le d \le N\} &\approx \sum_{k=\lfloor \frac{\log(n/N)}{\log 2} \rfloor}^{\lfloor \frac{\log N}{\log 2} - 1 \rfloor} \sum_{\substack{d|n \\ 2^k < d \le 2^{k+1}}} 1 \\
&\approx \tau(n) \sum_{k=\lfloor \frac{\log(n/N)}{\log 2} \rfloor}^{\lfloor \frac{\log N}{\log 2} - 1 \rfloor} \frac{\log 2}{\log N} \\
&\approx \frac{\tau(n)\log 2}{\log N}\left(\frac{\log N - \log(n/N)}{\log 2}\right) \\
&= \frac{\tau(n)}{\log N}(2\log N - \log n) \\
&= 2\tau(n)\left(1 - \frac{\log n}{2\log N}\right).
\end{aligned}$$

We note that the mass of the average of $\tau(n)$ over the integers $n \le N^2$ is mainly concentrated around those integers close, but not too much, to N^2. Indeed, for the k-th moment of $\tau(n)$ we have

$$\sum_{n \le N^2} \tau(n)^k \sim c_k N^2 (\log N)^{2^k - 1} \text{ as } N \longrightarrow +\infty, \tag{6.23}$$

for a certain $c_k > 0$ (see e.g. Luca and Tóth's paper [48]). We deduce that, for any $B \ge 1$, the part of the sum over $n \le N^2/B$, say, contributes

$$\ll \frac{N^2(\log N)^{2^k-1}}{B},$$

thus making a negligible contribution to (6.23) when $B > 0$ is large enough.

On the other hand, for the part of the sum over $n > N^2(1 - 1/C)$, we again get a negligible contribution to (6.23), when $C > 0$ is large enough, by Shiu's theorem [66, Theorem 1].

In conclusion, the main contribution to the sum in (6.23) comes from those integers $n \asymp N^2$. Therefore, we can recast our heuristic as

$$\tau_N(n) \approx \frac{\tau(n)}{\log N}. \tag{6.24}$$

It is well-known that the average of $\tau(n)$ is small (compared to the whole average given by (6.23) for $k = 1$) on those integers $n \le N^2$ with a number of distinct prime factors $\omega(n)$ far from $2 \log\log N$. More precisely, we can prove the following lemma.

Lemma 6.4.1. *For any* $0 < \varepsilon < 1$ *we have*

$$\sum_{\substack{1 \le n \le N^2 \\ |\omega(n) - 2\log\log N| > \varepsilon \log\log N}} \tau(n) \ll N^2 (\log N)^{1-2\eta},$$

with

$$\eta := \left(1 + \frac{\varepsilon}{2}\right) \log\left(1 + \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2}$$

and a uniformly bounded implied constant.

Proof. We focus on estimating only the part of the sum corresponding to integers $n \le N^2$ for which

$$\omega(n) > (2 + \varepsilon) \log\log N,$$

since the estimate for the complementary part can be then similarly deduced. The sum we would like to handle can be interpreted as the mean value of the indicator function on the above condition weighted with $\tau(n)$. In analogy to the exponential moment method in probability theory, we let $y > 1$ be a parameter to determine later and bound from the above the aforementioned sum by:

$$\begin{aligned} y^{-(2+\varepsilon)\log\log N} \sum_{1 \le n \le N^2} \tau(n) y^{\omega(n)} &\ll N^2 (\log N)^{2y-1} y^{-(2+\varepsilon)\log\log N} \\ &= N^2 (\log N)^{2y-1-(2+\varepsilon)\log y}, \end{aligned}$$

by Lemma 2.1.1, with a uniformly bounded implicit constant. Indeed, conditions (2.1) and (2.2) are satisfied by

$$\sum_{p\leq x}\tau(p)y^{\omega(p)}\log p = 2y\sum_{p\leq x}\log p$$
$$\sum_{\substack{p^k,\\ k\geq 2.}}\frac{\tau(p^k)y^{\omega(p^k)}k\log p}{p^k} = y\sum_{\substack{p^k,\\ k\geq 2.}}\frac{k(k+1)\log p}{p^k}$$

and by Chebyshev's estimates [69, Ch. I, Corollary 2.12]. Moreover, for any $x\geq 2$ we have

$$\prod_{p\leq x}\left(\sum_{k\geq 0}\frac{\tau(p^k)y^{\omega(p^k)}}{p^k}\right) = \prod_{p\leq x}\left(1+y\sum_{k\geq 1}\frac{k+1}{p^k}\right) \ll \prod_{p\leq x}\left(1+\frac{2y}{p}\right) \ll (\log x)^{2y},$$

by Mertens' formula [69, Ch. I, Theorem 1.12], with an implicit constant independent of y.

We can now optimize in y: letting $y := 1+\varepsilon/2$ we reach our claim, since

$$2y-1-(2+\varepsilon)\log y = 1+\varepsilon-(2+\varepsilon)\log\left(1+\frac{\varepsilon}{2}\right) = 1-2\eta.$$

Note that $\eta > 0$, if ε small enough, so that the upper bound we found is non-trivial. □

Observe that on a positive proportion of integers $n\leq N^2$ (the squarefree numbers) we may identify $\tau(n)$ with $2^{\omega(n)}$. Moreover, on $\mathcal{S}_1$ we have $\omega(n)$ equal to $2\log\log N$ plus a smaller error term. Therefore, in view of our previous heuristic (6.24), we can expect

$$\log(\tau_N(n)) \approx (\log 4-1)\log\log N \quad \text{(for most } n\leq N^2).$$

This can be considered as the 'normal' order of $\log(\tau_N(n))$ (for a rigorous definition of the normal order of an arithmetical function, see e.g. [69, Ch. III, Eq. (3.1)]).

6.4.2 The sets $\mathcal{S}_1$ and $\mathcal{S}_2$

Coming back to the estimate of (6.22), we remind that we would like to split the sum there into two parts: one over the integers in a set $\mathcal{S}_1$, where τ_N^2 is far from being concentrated on; the other over the complementary set $\mathcal{S}_2 := [N^2] \setminus \mathcal{S}_1$, where τ_N has a small average.

Let M be a positive real number that will be chosen at the end as sufficiently large in terms of δ and ε. Following the previous heuristic considerations, and by working in analogy with the Turán–Kubilius inequality (see Proposition 2.2.1), we define the set $\mathcal{S}_1$ as:

$$\mathcal{S}_1 := \{n \leq N^2 : |\Omega(n) - 2\log\log N| \leq M\sqrt{\log\log N}\}.$$

We then write

$$\begin{aligned}(6.25)\quad \mathbb{E}[X_A] = &\sum_{\substack{1\leq n\leq N^2 \\ |\Omega(n)-2\log\log N|\leq M\sqrt{\log\log N}}} \left(\frac{\alpha^2\tau_N(n)}{2} - 1 + (1-\alpha^2)^{\tau_N(n)/2}\right) \\ &+ \sum_{\substack{1\leq n\leq N^2 \\ |\Omega(n)-2\log\log N|> M\sqrt{\log\log N}}} \left(\frac{\alpha^2\tau_N(n)}{2} - 1 + (1-\alpha^2)^{\tau_N(n)/2}\right) \\ &+ O(N\alpha).\end{aligned}$$

6.4.3 Computation of the sum over $\mathcal{S}_2$

Since $-1 + (1-\alpha^2)^{\tau_N(n)/2} \leq 0$, the second sum in (6.25) is plainly bounded above by

$$\alpha^2 \sum_{\substack{1\leq n\leq N^2 \\ |\Omega(n)-2\log\log N|> M\sqrt{\log\log N}}} \tau_N(n).$$

By plugging the definition of $\tau_N(n)$ into this sum we get

$$\begin{aligned}
\sum_{\substack{1\le n\le N^2\\ |\Omega(n)-2\log\log N|>M\sqrt{\log\log N}}} \tau_N(n) &= \sum_{\substack{1\le n\le N^2\\ |\Omega(n)-2\log\log N|>M\sqrt{\log\log N}}} \sum_{\substack{n/N\le d\le N\\ d|n}} 1\\
&= \sum_{d\le N} \sum_{\substack{n\le Nd\\ d|n\\ |\Omega(n)-2\log\log N|>M\sqrt{\log\log N}}} 1\\
&= \sum_{d\le N} \sum_{\substack{k\le N\\ |\Omega(d)+\Omega(k)-2\log\log N|>M\sqrt{\log\log N}}} 1\\
&\ll \sum_{\substack{d\le N\\ |\Omega(d)-\log\log N|>\frac{M}{2}\sqrt{\log\log N}}} \sum_{k\le N} 1\\
&\le N \sum_{\substack{d\le N\\ |\Omega(d)-\log\log N|>\frac{M}{2}\sqrt{\log\log N}}} 1.
\end{aligned}$$

To compute the last sum above we use the Erdős–Kac theorem, Proposition 2.2.2. We then conclude that the second line in (6.25) is bounded as:

$$\ll \frac{\alpha^2N^2}{M}\exp\left(-\frac{M^2}{8}\right) + O(N\alpha), \tag{6.26}$$

thanks to the bound (2.6). Clearly, we can make (6.26) $\le \delta\varepsilon\alpha^2N^2/4$, say, if $M = M(\delta,\varepsilon)$ is sufficiently large.

Overall, we have so far proved that

$$\begin{aligned}
\mathbb{E}[X_A] \le &\sum_{\substack{1\le n\le N^2\\ |\Omega(n)-2\log\log N|\le M\sqrt{\log\log N}}} \left(\frac{\alpha^2\tau_N(n)}{2} - 1 + (1-\alpha^2)^{\tau_N(n)/2}\right)\\
&+\frac{\delta\varepsilon}{4}\alpha^2N^2.
\end{aligned} \tag{6.27}$$

6.4.4 Computation of the sum over $\mathcal{S}_1$

By Lemma 6.2.1 we have

$$\begin{aligned}(1-\alpha^2)^{\tau_N(n)/2} &= \exp\left(\frac{\tau_N(n)}{2}\log(1-\alpha^2)\right)\\ &\le \exp\left(-\frac{\alpha^2\tau_N(n)}{2}\right)\\ &\le 1-\frac{\alpha^2\tau_N(n)}{2}+\frac{\alpha^4\tau_N(n)^2}{8},\end{aligned}$$

which used in the sum in (6.27) now gives

$$\mathbb{E}[X_A] \le \sum_{\substack{1\le n\le N^2\\ |\Omega(n)-2\log\log N|\le M\sqrt{\log\log N}}} \frac{\alpha^4\tau_N(n)^2}{8}+\frac{\delta\varepsilon}{4}\alpha^2N^2. \tag{6.28}$$

Note that the above sum is on the double condition

$$2\log\log N - M\sqrt{\log\log N} \le \Omega(n) \le 2\log\log N + M\sqrt{\log\log N}.$$

By exponentiating both members of the rightmost inequality with base 2 and letting $z := 1/2$, we may bound the sum in (6.28) from above by

$$\frac{\alpha^4}{8}2^{2\log\log N+M\sqrt{\log\log N}}\sum_{1\le n\le N^2}\tau_N(n)^2z^{\Omega(n)}. \tag{6.29}$$

Plugging the definition of $\tau_N(n)$ into this sum, we find

$$\sum_{1\le n\le N^2}\tau_N(n)^2z^{\Omega(n)} = \sum_{1\le n\le N^2} z^{\Omega(n)}\Bigg(\sum_{\substack{d|n\\ n/N\le d\le N}}1\Bigg)^2.$$

By expanding the square and swapping summations we get that the above is

$$(6.30)\qquad \begin{aligned} &= \sum_{1\le n\le N^2} z^{\Omega(n)} \sum_{\substack{d_1|n\\ n/N\le d_1\le N}} 1 \sum_{\substack{d_2|n\\ n/N\le d_2\le N}} 1 \\ &\ll \sum_{1\le d_1<d_2\le N} \sum_{\substack{1\le n\le Nd_1\\ n\equiv 0 \pmod{[d_1,d_2]}}} z^{\Omega(n)} + \sum_{1\le d\le N} \sum_{\substack{1\le n\le Nd\\ n\equiv 0 \pmod{d}}} z^{\Omega(n)}. \end{aligned}$$

In the second double sum in (6.30) we make the change of variables $n = dk$, with $k \le N$, and see that this sum is

$$(6.31)\qquad = \sum_{1\le d\le N} z^{\Omega(d)} \sum_{1\le k\le N} z^{\Omega(k)} \ll N^2(\log N)^{2z-2} = \frac{N^2}{\log N},$$

by two applications of Corollary 2.1.2.

Regarding the first double sum in (6.30) we use the following substitution: $d_1 = \ell t_1$, $d_2 = \ell t_2$ and $n = t_1 t_2 \ell k$. We can then bound it as

$$(6.32)\qquad \le \sum_{1\le \ell\le N} z^{\Omega(\ell)} \sum_{1\le t_2\le N/\ell} z^{\Omega(t_2)} \sum_{1\le t_1<t_2} z^{\Omega(t_1)} \sum_{k\le N/t_2} z^{\Omega(k)}.$$

Notice that the condition $t_1 < t_2$ forces $t_2 \ge 2$. Moreover, $1 \le N/t_2$ implies $2 \le 2N/t_2$. So, two applications of Corollary 2.1.2 make (6.32)

$$\begin{aligned} &\ll N \sum_{1\le \ell\le N} z^{\Omega(\ell)} \sum_{2\le t_2\le N/\ell} \frac{z^{\Omega(t_2)}}{t_2} (\log(2N/t_2))^{z-1} \sum_{1\le t_1<t_2} z^{\Omega(t_1)} \\ &\ll N \sum_{1\le \ell\le N} z^{\Omega(\ell)} \sum_{2\le t_2\le N/\ell} z^{\Omega(t_2)} (\log(2N/t_2))^{z-1} (\log t_2)^{z-1}. \end{aligned}$$

By swapping summations and by another application of Corollary 2.1.2 the above is

$$
\begin{aligned}
&= N \sum_{2 \le t_2 \le N} z^{\Omega(t_2)} (\log(2N/t_2))^{z-1} (\log t_2)^{z-1} \sum_{\ell \le N/t_2} z^{\Omega(\ell)} \\
&\ll N^2 \sum_{2 \le t_2 \le N} \frac{z^{\Omega(t_2)}}{t_2} (\log(2N/t_2))^{2(z-1)} (\log t_2)^{z-1} \\
&= N^2 \sum_{2 \le t \le N} \frac{1}{2^{\Omega(t)} t \sqrt{\log t} \log(2N/t)},
\end{aligned}
$$

on recalling $z = 1/2$.

We now pause a moment to understand the behaviour of the last sum above.

Lemma 6.4.2. *For any $N \ge 12$ we have*

$$
\sum_{2 \le t \le N/2} \frac{1}{2^{\Omega(t)} t \sqrt{\log t} \log(N/t)} \ll \frac{\log \log N}{\log N}.
$$

Proof. To begin with, we split the sum into dyadic intervals to find it is:

$$
\begin{aligned}
&\le \sum_{k=1}^{\left\lfloor \frac{\log N}{\log 2} \right\rfloor - 2} \sum_{\max\{2, N/2^{k+1}\} < t \le N/2^k} \frac{1}{2^{\Omega(t)} t \sqrt{\log t} \log(N/t)} \\
&\ll \frac{1}{N} \sum_{k=1}^{\left\lfloor \frac{\log N}{\log 2} \right\rfloor - 2} \frac{2^k}{k \sqrt{\log(N/2^{k+1})}} \sum_{\max\{2, N/2^{k+1}\} < t \le N/2^k} \frac{1}{2^{\Omega(t)}}.
\end{aligned}
$$

By Corollary 2.1.2 the innermost sum on the second line above is bounded by

$$
\ll \frac{N}{2^k} \frac{1}{\sqrt{\log(N/2^k)}}.
$$

Plugging this last estimate in, we find that the sum we wish to analyse is

$$
\begin{aligned}
&\ll \sum_{k=1}^{\lfloor \frac{\log N}{\log 2} \rfloor -2} \frac{1}{k \log(N/2^{k+1})} \\
&\leq \frac{1}{\log(N/4)} + \int_1^{\lfloor \frac{\log N}{\log 2} \rfloor -2} \frac{dt}{t \log(N/2^{t+1})} \\
&= \frac{1}{\log(N/4)} + \frac{\log t - \log\log(N/2^{t+1})}{\log(N/2^{t+1}) + t\log 2}\Bigg|_1^{\lfloor \frac{\log N}{\log 2} \rfloor -2} \\
&\leq \frac{1}{\log(N/4)} + \frac{\log\log N + O(1)}{\log N + O(1)} + \frac{\log\log(N/4)}{\log(N/4) + \log 2} \\
&\ll \frac{\log\log N}{\log N},
\end{aligned}
$$

using that

$$\left\lfloor \frac{\log N}{\log 2} \right\rfloor - 2 = \frac{\log N}{\log 2} + O(1),$$

which proves the lemma. □

With the help of Lemma 6.4.2 we can now conclude the estimation of the sum in (6.32), producing for it a bound of

$$\ll \frac{N^2 \log\log N}{\log N}. \tag{6.33}$$

Collecting together (6.29), (6.31) and (6.33), we have found an overall contribution for the sum in (6.28) of

$$\ll \alpha^4 N^2 (\log N)^{2\log 2 - 1} \exp((M \log 2 + o(1))\sqrt{\log\log N}).$$

6.4.5 Conclusion of the sufficient part of Theorem 6.1.4

Now suppose that α is such that the quantity

$$\frac{\log(\alpha^2 (\log N)^{\log 4 - 1})}{\sqrt{\log\log N}}$$

tends to $-\infty$ as $N \longrightarrow +\infty$. This is equivalent to saying that for any $K > 0$ there exists an $N_0 = N_0(K) \in \mathbb{N}$ such that for any $N \geq N_0$ we have

$$\alpha^2 \leq \frac{1}{(\log N)^{2\log 2-1}\exp(K\sqrt{\log\log N})}.$$

Now, take $K = 2M \log 2$ so that the sum in (6.28) becomes

$$\ll \alpha^2 N^2 \exp((-M\log 2 + o(1))\sqrt{\log\log N})$$

hence $\leq \delta\varepsilon\alpha^2 N^2/4$, say, if N is large enough in terms of δ and ε. From (6.27) it follows that there exists an $N_0 = N_0(\delta, \varepsilon)$ such that for any $N \geq N_0$ we have

$$\mathbb{E}[X_A] \leq \frac{\delta\varepsilon}{2}\alpha^2 N^2.$$

Plugging this into (6.21) we conclude that

$$\mathbb{P}(X_A \geq \delta(N\alpha)^2/2) \leq \varepsilon,$$

for any $N \geq N_0$, for a sufficiently large $N_0 = N_0(\delta, \varepsilon) > 0$. This shows the sufficient part in Theorem 6.1.4.

6.5 The necessary condition

In this section we are going to prove the necessary condition of Theorem 6.1.4.

Let $\alpha \in [0, 1)$. We have already noticed that we can confine ourselves with values of $\alpha \longrightarrow 0$ and $N\alpha \longrightarrow +\infty$, thanks to Proposition 6.1.1 and the bound (6.3).

Now suppose that we either have that the quantity

$$\frac{\log(\alpha^2(\log N)^{\log 4-1})}{\sqrt{\log\log N}}$$

does not converge as $N \longrightarrow +\infty$ or it does, but to a limit different from $-\infty$.

Then there exists a real number K and a sequence $\{N_k\}_{k\geq 1}$ such that for any $k \geq 1$ we have

$$\alpha^2 \geq \frac{\exp(K\sqrt{\log\log N_k})}{(\log N_k)^{\log 4 - 1}}.$$

In the following to shorten notations we will write N for a generic term of the sequence N_k.

Assume further that even for this choice of α we have a random product set of maximal size, i.e. that $|AA| \sim (|A|^2 + |A|)/2$ with probability $1 - o(1)$, for a random set A in $B(N, \alpha)$.

By Lemma 6.2.2 we deduce that $\mathbb{E}[|AA|] \sim \mathbb{E}[(|A|^2 + |A|)/2]$, as $N \longrightarrow +\infty$. Moreover, by the proof of [3, Proposition 3.2] and equations (6.15) and (6.17) we can restate this last asymptotic equality as:

$$\sum_{1 \leq n \leq N^2} \left(\frac{\alpha^2 \tau_N(n)}{2} - 1 + (1 - \alpha^2)^{\tau_N(n)/2} \right) = o(N^2\alpha^2). \tag{6.34}$$

The goal is to show that the above sum is larger than a small positive constant times $N^2\alpha^2$, thus contradicting (6.34).

From the heuristic considerations in the Subsect. 6.4.1, the summand can be roughly seen as

$$\frac{\alpha^2 t(N)}{2} - 1 + (1 - \alpha^2)^{t(N)/2}, \tag{6.35}$$

at least on the set $\mathcal{S}_1$, where we define

$$t(N) := (\log N)^{\log 4 - 1}.$$

This can be considered as an approximation to the normal order of the function $\tau_N(n)$ over the integers $n \leq N^2$.

When α is such that $\alpha^2 t(N) \longrightarrow 0$, as $N \longrightarrow +\infty$, we can clearly Taylor expand the binomial in (6.35); this has indeed been crucial before to producing some saving in the expression for the mean of X_A.

On the other hand, in the case when $\alpha^2 t(N)$ is bounded away from 0, it is clear that the binomial term in (6.35) can now be considered as 'smaller' than the other term $\alpha^2 t(N)/2 - 1$. In other words, in this range of α we no longer achieve a saving in (6.35) due to Taylor expansion, but instead the term $\alpha^2 t(N)/2 - 1$ dominates.

Moreover, by Lemma 6.2.1 we have

$$\begin{aligned}(1-\alpha^2)^{\tau_N(n)/2} &= \exp\left(\frac{\tau_N(n)}{2}\log(1-\alpha^2)\right)\\ &\geq 1 + \frac{\tau_N(n)}{2}\log(1-\alpha^2)\\ &= 1 - \frac{\tau_N(n)\alpha^2}{2} + O(\alpha^4 \tau_N(n)).\end{aligned}$$

Whence, by (6.17) and since $\alpha \longrightarrow 0$, the term inside parenthesis in (6.34) is positive apart from an overall error contribution of $o(\alpha^2 N^2)$. Hence, we can freely discard some unnecessary pieces from the sum to get a lower bound.

In particular, a first lower bound for the sum in (6.34) is given by

$$\sum_{\substack{1\leq n\leq N^2\\ h<\Omega_2(n)\leq m}} \left(\frac{\alpha^2\tau_N(n)}{2} - 1\right) = \frac{\alpha^2}{2}\sum_{\substack{1\leq n\leq N^2\\ h<\Omega_2(n)\leq m}} \tau_N(n) - \sum_{\substack{1\leq n\leq N^2\\ h<\Omega_2(n)\leq m}} 1, \tag{6.36}$$

where M is a sufficiently large positive real number that will be chosen later. Here, $\Omega_2(n)$ denotes the function which counts the number of all prime factors of n different form 2 and counted with multiplicity and we let

$$\begin{aligned}h &:= 2\log\log N + M\sqrt{\log\log N}\\ m &:= 2\log\log N + 2M\sqrt{\log\log N}.\end{aligned}$$

The choice of h and m has been inspired from our heuristics on the distribution of the function τ_N, as in Subsect. 6.4.1, and from the normal order of the function $\Omega(n)$, as given in Proposition 2.2.2.

The plan is to exhibit a lower bound for the first sum on the right-hand side of (6.36) and an upper bound for the second one there and compare them. Let us start with the former task. By expanding the definition of $\tau_N(n)$ it is immediate to see that

$$\sum_{\substack{1\leq n\leq N^2\\ h<\Omega_2(n)\leq m}} \tau_N(n) = \sum_{\substack{1\leq n\leq N^2\\ h<\Omega_2(n)\leq m}} \sum_{\substack{d\mid n\\ n/N\leq d\leq N}} 1 = \sum_{1\leq d\leq N} \sum_{\substack{1\leq k\leq N\\ h<\Omega_2(d)+\Omega_2(k)\leq m}} 1,$$

since clearly $\Omega_2(n)$ is still a completely additive function. Moreover, we can bound the above as:

$$\geq \sum_{\substack{1\leq d\leq N\\ h/2<\Omega_2(d)\leq m/2}} \sum_{\substack{1\leq k\leq N\\ h/2<\Omega_2(k)\leq m/2}} 1 = \Bigg(\sum_{\substack{1\leq j\leq N\\ h/2<\Omega_2(j)\leq m/2}} 1\Bigg)^2.$$

To compute the sum within parenthesis we use a variation of Proposition 2.2.2 for the function $\Omega_2(n)$ which follows from [69, Ch. III, Theorem 4.15]: we deduce that

$$\sum_{\substack{1\leq j\leq N\\ h/2<\Omega_2(j)\leq m/2}} 1 = \frac{N}{\sqrt{2\pi}} \int_{M/2}^{M} e^{-t^2/2}dt + O\bigg(\frac{N}{\sqrt{\log\log N}}\bigg),$$

with a big-Oh constant independent of M.

In conclusion, the first term on the right-hand side of (6.36) is

$$\begin{aligned} &\gg \alpha^2 N^2 \bigg(\int_{M/2}^{M} e^{-t^2/2}dt\bigg)^2 \\ &\gg_M \frac{N^2}{(\log N)^{\log 4-1}} \exp(K\sqrt{\log\log N}), \end{aligned} \tag{6.37}$$

if N is sufficiently large with respect to M and since M is positive.

On the other hand, we can rewrite the second sum on the right-hand side of (6.36) as

$$\sum_{h<k\leq m} \Pi(N^2,k), \tag{6.38}$$

where

$$\Pi(N^2,k) := \sum_{\substack{n\leq N^2\\ \Omega_2(n)=k}} 1.$$

Now, we can trivially bound $\Pi(N^2,k)$ from above by

$$\sum_{n\leq N^2} \frac{2^{\Omega_2(n)}}{2^k} \ll \frac{N^2\log N}{2^k},$$

thanks to the second part of Corollary 2.1.2. This, inserted into (6.38), gives an upper bound for (6.38) of

$$
\begin{aligned}
(6.39) \qquad & \ll N^2 \log N \sum_{h<k\leq m} \frac{1}{2^k} \\
& \ll \frac{N^2 \log N}{2^h} \\
& \ll \frac{N^2}{(\log N)^{\log 4 - 1}} \exp((-M \log 2)\sqrt{\log\log N}),
\end{aligned}
$$

by summing the geometric progression.

By choosing $M := 2|K|/\log 2 + 1$, and thanks to (6.37) and (6.39), we have overall showed that (6.36) is

$$\gg_K \alpha^2 N^2,$$

if N is large enough in terms of $|K|$. This contradicts the assertion (6.34) and concludes the proof of the necessary part of Theorem 6.1.4.

Chapter 7

Random multiplicative functions

Summary

The first section introduces Rademacher random multiplicative functions as a reasonable heuristic model for the Möbius function, with the aim of shedding more light on the size of its partial sums. A theorem concerning a close to optimal almost sure upper bound for the average of random multiplicative functions over integers with a large prime factor is presented.

The second section contains some preliminary results on the distribution of a random multiplicative function, like hypercontractive inequalities, bounds on the expectation of their Euler products and Parseval's identity for their Dirichlet series. It also contains classical results on the distribution of sequences of random variables, like Hoeffding's inequality for independent random variables and Doob's maximal inequality for submartingale sequences.

Finally, the third section deals with the proof of our new main result in this context and is divided into three parts: the first one concerns the basic and usual strategy to deduce almost sure bounds for sums of random variables; in the second one we estimate the tail probability of short increments of a random multiplicative function; in the third one we compute the tail probability of averages of random multiplicative functions over suitable well-spaced integers with a large prime factor.

7.1 Introduction

7.1.1 Motivations

Averages of the Möbius function

A fundamental and classical problem in Analytic Number Theory concerns demonstrating squareroot cancellation for the partial sums of the Möbius function $\mu(n)$. More precisely, one ponders the validity of the following statement:

$$\sum_{n\leq x}\mu(n) \ll_{\varepsilon} x^{1/2+\varepsilon} \tag{7.1}$$

for all $\varepsilon > 0$ and x large with respect to ε. Littlewood [45] was the first to realize that (7.1) is equivalent to the well-known Riemann hypothesis, which, in turn, is equivalent to the following asymptotic relation:

$$|\{p \leq x\}| = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x) \text{ as } x \longrightarrow +\infty.$$

Landau [42], Titchmarsh [73], Maier and Montgomery [49] and more recently Soundararajan [67] increasingly refined this relation, obtaining the equivalence between the Riemann hypothesis and the following improvement of (7.1):

$$\sum_{n\leq x}\mu(n) \ll \sqrt{x}\exp((\log x)^{1/2}(\log\log x)^{14}) \text{ as } x \longrightarrow +\infty. \tag{7.2}$$

This is pretty far from the best to date *unconditional* known bound on the partial sums of the Möbius function,

$$\sum_{n\leq x}\mu(n) \ll x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}),$$

with $c = 0.2098$, which is a consequence of Ford's result [13, Theorem 1], the work of Ford in [12] and Pintz's result [60, Theorem 8].

Since there are serious limitations to our knowledge of the distribution of prime numbers, a first attempt to investigate the validity of the Riemann hypothesis is through the study of the average behaviour of suitable probabilistic models for the Möbius function.

A first random approach

To model the Möbius function $(\mu(n))_{n=1}^{+\infty}$, a first possibility is to consider a sequence of *independent* random variables $(\varepsilon_n)_{n=1}^{+\infty}$ taking values ± 1 with probability $1/2$ each.

This is a well-studied sequence, which we know a lot about. For instance, the partial sums of ε_n also satisfy the analogous bound of (7.1):

$$\sum_{n \le x} \varepsilon_n \ll x^{1/2+\varepsilon}, \tag{7.3}$$

almost surely. This can be easily seen by noticing that the associated Dirichlet series $\sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n^s}$ has variance:

$$\mathbb{E}\left[\left|\sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n^s}\right|^2\right] = \mathbb{E}\left[\sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n^s} \sum_{m=1}^{+\infty} \overline{\frac{\varepsilon_m}{m^s}}\right] = \sum_{n=1}^{+\infty} \frac{1}{n^{2\Re(s)}},$$

since each ε_n has mean zero and variance 1, and they are orthogonal, where we switched expectation and summations thanks to the Fubini–Tonelli's theorem. Then, the Kolmogorov two-series theorem [18, Theorem 5.2] implies the almost sure convergence of the Dirichlet series $\sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n^s}$ on the half plane $\Re(s) > 1/2$. Finally, the Phragmén-Landau's oscillation theorem [69, Ch. I, Theorem 1.12] immediately leads to (7.3).

Actually, the estimate (7.3) can be improved: the sequence ε_n satisfies the so called Khintchine's Law of the Iterated Logarithm, which consists in the following almost sure statements (see for instance Gut [18, Ch. 8] for an extensive account of this result):

$$\limsup_{x \longrightarrow +\infty} \frac{\sum_{n \le x} \varepsilon_n}{\sqrt{2x \log\log x}} = 1 \quad \text{and} \quad \liminf_{x \longrightarrow +\infty} \frac{\sum_{n \le x} \varepsilon_n}{\sqrt{2x \log\log x}} = -1.$$

To compare, for large x, the sum $\sum_{n \le x} \varepsilon_n$ typically has size close to $\sqrt{x}$, by the Central Limit Theorem, from which its almost sure largest fluctuations are obtained by rescaling this size by a $\sqrt{\log\log x}$ factor, which describes the impact of the dependence amongst the sums $\sum_{n \le x} \varepsilon_n$ as x varies.

However, the sequence of independent random variables ε_n does not quite catch the multiplicative structure of the Möbius function, which inevitably leads to some dependences among its values. Moreover, it has been conjectured by Gonek (unpublished) that there should exist a finite number $B > 0$ such that

$$\limsup_{x\longrightarrow+\infty} \frac{|\sum_{n\leq x}\mu(n)|}{\sqrt{x}(\log\log\log x)^{5/4}} = B \quad \text{and} \quad \liminf_{x\longrightarrow+\infty} \frac{|\sum_{n\leq x}\mu(n)|}{\sqrt{x}(\log\log\log x)^{5/4}} = -B,$$

which defies a Law of the Iterated Logarithm for $\mu(n)$.

An improved random approach

To better investigate the Riemann hypothesis, Wintner [76], in 1944, introduced the following random model for $\mu(n)$:

a Rademacher random multiplicative function f is a multiplicative function supported on the squarefree integers and defined on the prime numbers p by letting the $f(p)$ be independent random variables taking values ±1 with probability $1/2$ each.

As explained in Lau, Tenenbaum and Wu's paper [43], the probability that $f = \mu$ is 0; however, $f(n) = \mu^2(n)\mu(d_n)$, on a random subsequence $(d_n)_{n=1}^{+\infty}$. Moreover, f and μ are both multiplicative, supported on the squarefree numbers and take values ±1. So, a Rademacher random multiplicative function might represent a reasonable heuristic model for the Möbius function. In fact, Wintner himself [76] was able to show that, for any fixed $\varepsilon > 0$, one almost surely has

$$\sum_{n\leq x} f(n) = O(x^{1/2+\varepsilon}) \qquad (7.4)$$
$$\sum_{n\leq x} f(n) \neq O(x^{1/2-\varepsilon}).$$

Indeed, to deduce the upper bound in (7.4), he observed that

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s}\right) = \exp\left(\sum_p \frac{f(p)}{p^s}\right) A(s),$$

with $A(s)$ holomorphic, on the half plane $\Re(s) > 1/2$. Whence, an application of the Kolmogorov two-series theorem to the above random sum over primes gives the almost sure convergence of the Euler product of f on $\Re(s) > 1/2$, from which the thesis follows by means of the Phragmén-Landau's theorem.

7.1.2 Some background on random multiplicative functions

The average distribution

A question that naturally arises when seeking to improve (7.4) is the following:

does a normal approximation hold for the partial sums of $f(n)$, as for the independent sequence ε_n?

In fact, in such case, the partial sums of f would typically have size $\sqrt{x}$ and one could hope to prove a Law of the Iterated Logarithm for them.

Restricting the attention to subsums over integers with a fixed number of prime factors seems to indicate a positive answer. In fact, Harper [24, Theorem 1], notably improving on Hough [37], showed that the sequence of random variables $M_{x,k}/\sqrt{\mathbb{E}[|M_{x,k}|^2]}$, where $M_{x,k} := \sum_{n\leq x,\ \omega(n)=k} f(n)$ and $k = o(\log\log x)$, converges in distribution to a standard normal random variable, as $x \longrightarrow +\infty$. Of the same spirit is the result of Chatterjee and Soundararajan [2, Theorem 1.1], who proved that the sequence of random variables $S_{x,y}/\sqrt{\mathbb{E}[|S_{x,y}|^2]}$, where $S_{x,y} := \sum_{x<n\leq x+y} f(n)$, converges in distribution to a standard normal random variable, as $x \longrightarrow +\infty$, as long as y verifies some technical conditions, among them $y = o(x/\log x)$. From these results, it seems plausible that the full partial sums of $f(n)$ were normally distributed. However, surprisingly, Harper also realized that this is actually not the case, by finding that the sequence $M_{x,k}/\sqrt{\mathbb{E}[|M_{x,k}|^2]}$ is not approximately Gaussian when $k = c\log\log x$, with $c > 0$ constant [24, Theorem 2]. Later, he made some advances on this direction by proving that $\frac{1}{\sqrt{x}}\sum_{n\leq x} f(n)$ converges in probability to 0 [25, Corollary 1]. Furthermore, he suggested that the distribution of the renormalization $\frac{\sum_{n\leq x} f(n)}{\sqrt{x}/(\log\log x)^{1/4}}$ might be linked to that of the total mass of a "critical multiplicative chaos" (see [25] for an explanation).

Understanding moments

The problem of determining the limit distribution of the partial sums of a Rademacher random multiplicative function $f(n)$ appears a real challenge, and until today stays unsolved. As an attempt, people have also looked at the moments of such partial sums.

Harper, Nikeghbali and Radziwiłł [27, Theorem 3], and also independently Heap and Lindqvist [29, Theorem 1], obtained an asymptotic for all the $2q$-th moment, with $q \in \mathbb{N}$. Later, Harper [25, 26] found the order of magnitude of all the $2q$-th moments, uniformly for $0 < q \leq c(\log x)/(\log\log x)$, with $c > 0$ constant. He proved that, for $q > 1$, very roughly speaking, the $2q$-th moment goes like $C_q x^q (\log x)^{O(q^2)}$, where C_q is a certain function of q (see [26, Theorem 1 and Theorem 2]). On the other hand, for $0 \leq q \leq 1$, he showed that (see [25, Theorem 1 and Theorem 2]):

$$\mathbb{E}\Big[\Big|\sum_{n\leq x} f(n)\Big|^{2q}\Big] \asymp \Big(\frac{x}{1+(1-q)\sqrt{\log\log x}}\Big)^{q}.$$

As a consequence [25, Corollary 2], Harper deduced that *typically*

$$\Big|\sum_{n\leq x} f(n)\Big| \asymp \frac{\sqrt{x}}{(\log\log x)^{1/4}}. \tag{7.5}$$

Incidentally, this constitutes another indication that the sequence of partial sums of f does not follow a normal distribution, since their typical value does not agree with their standard deviation, which is easily seen to be roughly $\sqrt{x}$.

7.1.3 The almost sure size of the partial sums of random multiplicative functions

Wintner's result (7.4) implies that the Riemann hypothesis is 'almost always' true. However, it is weaker than the conjectural bound (7.2) and one may wonder whether it might be improved and what might then be the exact size of the largest fluctuations of a Rademacher random multiplicative function.

In this regard, Erdős [10] was the first to improve (7.4) to

$$\sum_{n\leq x} f(n) = O(\sqrt{x}(\log x)^{A})$$
$$\sum_{n\leq x} f(n) \neq O(\sqrt{x}(\log x)^{-B}),$$

almost surely, for certain constants $A, B > 0$. This was later refined by Halász [19] to

$$\sum_{n\leq x} f(n) = O(\sqrt{x}e^{C\sqrt{\log\log x \log\log\log x}})$$
$$\sum_{n\leq x} f(n) \neq O(\sqrt{x}e^{-D\sqrt{\log\log x \log\log\log x}}),$$

almost surely, for certain constants $C, D > 0$. Notice that for any real number A, $(\log x)^{A} = \exp(A \log\log x)$; hence, we may interpret Halász's estimates as an exponentially squareroot-improvement of Erdős' bounds.

To deduce the upper bound, Halász was the first introducing a conditioning approach, based on the randomness coming from the 'small' primes. He then masterfully combined it with the use of hypercontractive inequalities (a fancy name commonly adopted in Harmonic Analysis to denote certain high moment bounds) to handle the randomness coming from 'large' primes.

Most recently, Basquin [1], and independently Lau, Tenenbaum and Wu [43], inserted a splitting device into Halász's argument, which notably increased the efficiency of the hypercontractive inequalities, yielding, for any $\varepsilon > 0$, to the following almost sure upper bound:

$$\sum_{n\leq x} f(n) \ll \sqrt{x}(\log\log x)^{2+\varepsilon}. \tag{7.6}$$

Their proof goes as follows:

- one reduces to studying the partial sums over a suitable subsequence of test points x_i. They are such that it is possible to study the tail of the distribution of the partial sums of f and easily control the increments of f between any two of them. One then globally recollects at the end the information through Borel-Cantelli's lemma;

- one splits the partial sums into several pieces, according to the size of the largest prime factor $P(n)$ in intervals, and bound them through the use of high moments inequalities;
- to control the size of such chunks of partial sums, one introduces some events, which assume a nice behaviour of f on small primes, and conditions on them;
- one shows that such events happen quite often, by rewriting them in terms of the size of a submartingale sequence in x_i, which roughly speaking is a sequence of integrable random variables non-decreasing on average. To uniformly control its size in intervals, one appeals to Doob's maximal inequality, which in turn bounds the probability that all the elements of such sequence lie in an interval in terms of the biggest one.

On the opposite side, Harper [21], improving on his own previous result [23], showed that, for any function $V(x)$ tending to infinity with x, there almost surely exist arbitrarily large values of x for which

$$\left|\sum_{n\leq x} f(n)\right| \geq \frac{\sqrt{x}(\log\log x)^{1/4}}{V(x)}. \tag{7.7}$$

This result also holds for Steinhaus random multiplicative functions f, where $\{f(p)\}_{p \text{ prime}}$ is a sequence of independent Steinhaus random variables (i.e. distributed uniformly on the unit circle $\{|z|=1\}$) and the function f is taken to be completely multiplicative.

To achieve (7.7), Harper reduced the problem to showing a similar statement, but where the sum runs only over integers with the largest prime factor $> \sqrt{x}$ (actually, he worked with a slightly different condition, but his argument may be adapted to this case). At the same time, he localised the problem by considering such statement only for a collection of values of x simultaneously. Then, he showed a multivariate Gaussian approximation for such sums, conditional on the behaviour of f at small primes. Finally, he controlled the interaction among these sums as x varies, showing conditional almost independency, through a careful and delicate study of the size of their covariances. In particular, as a consequence of the proof of (7.7), we may infer that there almost surely exist arbitrarily large values of x for which

$$\left|\sum_{\substack{n\leq x\\ P(n)>\sqrt{x}}} f(n)\right| \geq \sqrt{x}(\log\log x)^{1/4+o(1)}, \tag{7.8}$$

where $P(n)$ indicates the largest prime factor of n.

The main new contribution

The bounds (7.6) and (7.7) together give the feeling of the existence of a *Law of the Iterated Logarithm* for the partial sums of $f(n)$. However, Khintchine's theorem cannot be applied to study random multiplicative functions, because their values are clearly not all independent. Nevertheless, we might believe that a suitable version of the Law of the Iterated Logarithm might hold for them, in the hope that their multiplicative structure does not completely disrupt statistical cancellations. Having said that, the exact size of their almost sure largest fluctuations is not yet clear. Following Harper [21], in relation to Khintchine's law we might reason that, for a Rademacher or Steinhaus random multiplicative function f, it might be obtained by adjusting the typical size (7.5) of the partial sums of f (which we saw before does no more coincide with their standard deviation) with the usual Law of the Iterated Logarithm "correction factor" $\sqrt{\log\log x}$. This last one, in (7.7), is a result of the previously mentioned multivariate Gaussian approximation. So doing, we arrive at the following prediction:

we almost surely have

$$\left|\sum_{n\leq x} f(n)\right| \leq \sqrt{x}(\log\log x)^{1/4+o(1)} \quad \textit{as } x \longrightarrow +\infty$$

and the opposite inequality almost surely holds on a subsequence of points x.

Clearly, the lower bound already follows from (7.7); the validity of the upper bound is currently being investigated in a joint project with Adam J. Harper.

The next theorem, which is our main result of this chapter, may be seen as a partial progress in this direction.

Theorem 7.1.1. *Let f be a Rademacher or a Steinhaus random multiplicative function. Let $\varepsilon > 0$ small. As $x \longrightarrow +\infty$, we almost surely have*

$$\Bigg|\sum_{\substack{n\leq x\\ P(n)>\sqrt{x}}} f(n)\Bigg| \leq \sqrt{x}(\log\log x)^{1/4+\varepsilon}.$$

We observe that, considering (7.8), the bound in Theorem 7.1.1 is close to be optimal. Moreover, the set of numbers $n \leq x$ with $P(n) > \sqrt{x}$ consists in a positive proportion of all the positive integers up to x. Hence, the partial sums in Theorem 7.1.1 might make a big contribution to the full partial sums of f. However, we cannot directly use Theorem 7.1.1 to deduce an almost sure upper bound for the full partial sums of f, which remains so far unknown. Indeed, one should also be able to estimate the complementary portion over $\sqrt{x}$-smooth numbers (i.e., numbers n with $P(n) \leq \sqrt{x}$), which requires exploiting more the intricate dependence structure of the values of $f(n)$.

7.1.4 Sketch of the proof of Theorem 7.1.1

As usual when seeking to produce almost sure bounds for a sum of random variables, we will reduce our analysis to what happens on a sequence of 'test points' x_i, with the property of being sparse, but not too sparse so that we can easily control the increments of f between any pair x_{i-1}, x_i. We will then collect together the information we gather from each single point, by means of the first Borel–Cantelli's lemma. For random multiplicative functions this is indeed the approach that was taken by Basquin [1] and Lau–Tenenbaum–Wu [43] and others before, but the way we analyze the distribution of the partial sums of $f(n)$ on test points is a key difference with them. In fact, the study of their distribution is made available through the use of high moments inequalities. To avoid them blowing up, Basquin and Lau–Tenenbaum–Wu split the *full* partial sums of $f(n)$ up into several pieces where the constraints were on the size of the largest prime factor $P(n)$ in intervals. Here, we are in an easier setting, since we have to deal with just *one* such intervals where moreover there is a *unique* large prime factor, which leads to improving the efficiency of the high moments bounds. This allows us to save a $\log\log x$ factor in the estimate of Theorem 7.1.1 compared to [1] and [43]. More specifically, we note that any positive integer $n \leq x$ with $P(n) > \sqrt{x}$ can be uniquely written as $n = pm$, where $\sqrt{x} < p \leq x$ is a prime and $m \leq x/p$ is a positive integer. Consequently, by multiplicativity, we deduce that

$$\sum_{\substack{n \leq x \\ P(n) > \sqrt{x}}} f(n) = \sum_{\sqrt{x} < p \leq x} f(p) \sum_{m \leq x/p} f(m). \tag{7.9}$$

Conditional on the value of $f(q)$, for prime numbers $q \leq \sqrt{x}$, the above can be interpreted as a sum of many independent random variables $f(p)$ times some coefficients. Its conditional probability distribution possesses a conditional *Gaussian* tail, thanks to Hoeffding's inequality. This is exactly how we gain the $\log \log x$ factor mentioned above, by replacing the use of several high moments bounds (one for each of the roughly $\log \log x$ sums related to the size of the largest prime factor, as in [1] and [43]) with that of a single one.

The use of Hoeffding's inequality constitutes a difference also in relation to Harper's lower bound result (7.7), where instead, as explained before, it was necessary to establish a conditional jointly Gaussian approximation for the partial sums of f over integers with a large prime factor (which required in [21] a much greater effort).

Our second conditioning will be on the size of a certain smooth weighted version of the conditional variance $V(x)$ of the partial sums in (7.9), which is equal to $\sum_{\sqrt{x} < p \leq x} |\sum_{m \leq x/p} f(m)|^2$. Arguing as in Harper [25], we will recast this in terms of an L^2-integral of a truncated Euler product corresponding to f, which will give rise to a submartingale sequence in x.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with, additionally, a sequence $\{\mathcal{F}_n\}_{n \geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$, which is called a *filtration*, a *submartingale* is a sequence of random variables $(X_n)_{n \geq 0}$ which satisfies, for any time n, the following properties:

X_n is $\mathcal{F}_n -$ measurable	(X_n is adapted)
$\mathbb{E}[\lvert X_n \rvert] < +\infty$	(X_n is integrable)
$\mathbb{E}[X_{n+1} \lvert \mathcal{F}_n] \geq X_n$ almost surely	(X_n is non-decreasing on average).

Rewriting a smooth version of the partial sums of f in terms of a submartingale sequence is a common feature with [1] and [43]. However, differently from them, our sequence involves the Euler product of f and, most importantly, we will be able to input *low moments* estimates for the partial sums of f to better bound its size. This will allow us to gain a further $(\log\log x)^{1/4}$ factor compared to [1] and [43]. More precisely, the gain will come from showing that with high probability $V(x)$ has uniformly in x inside a wide interval size close to $x/\sqrt{\log\log x}$, which is what we can pointwise deduce from Harper's low moments estimates [25].

To implement such results successfully, we will need to drastically increase the number of test points x_i we simultaneosuly consider, in contrast with [1] and [43]. This will force us to introduce a suitable *normalized* version of the aforementioned submartingale sequence, with the renormalization given by the reciprocal of its expected value times a correction factor. By means of Doob's maximal inequality, to control such sequence uniformly on test points inside wide intervals, we will decrease the gap between the actual size of $V(x)$ and its uniform in x value $x/\sqrt{\log\log x}$ by a $\log\log x$ factor, compared to [1] and [43]. We recall that in previous works $V(x)$ was instead put in relation with its expected size x and the precision loss was indeed roughly $\log\log x$. This will lead to a last gain of roughly $\sqrt{\log\log x}$ in Theorem 7.1.1, thus overall reducing the upper bound from $(\log\log x)^{2+\varepsilon}$ to $(\log\log x)^{1/4+\varepsilon}$, in our case.

To recap, unlike the approach taken in [1] and [43], we are going to introduce three key tools, which, compared to the result obtained in [1] and [43], permit us to save:

- a $\log\log x$ factor, by improving the use of the high moments inequalities to study the distribution of the partial sums of f, having a single large prime factor to take out;
- a $(\log\log x)^{1/4}$ factor, by inputting low moments estimates for the full partial sums of f into our argument;
- a final $\sqrt{\log\log x}$ factor, by analysing the partial sums of f over a larger sample of points and simultaneously controlling them by associating a suitably normalized submartingale sequence.

7.2 Preliminaries to the proof of the Theorem 7.1.1

7.2.1 Probabilistic number theoretic results

As crucial in Lau–Tenenbaum–Wu's paper and previous works, we will need a control on the $2m$-th moment of weighted sums of random multiplicative functions. The following lemma allows us to do so by shifting the problem to computing the L_2-norm of a sum of such weights. Because m can be arbitrary, this explains the name of such a result (see e.g. Harper's paper [26, Proof of Probability Result 1]).

Lemma 7.2.1. *(Hypercontractive inequality). Let f be a Rademacher or Steinhaus random multiplicative function. For any sequence $(a_n)_{n=1}^{+\infty}$ of complex numbers and any positive integer $m \geq 1$, we have*

$$\mathbb{E}\Big[\Big|\sum_{n\geq 1} a_n f(n)\Big|^{2m}\Big] \leq \Big(\sum_{n\geq 1} |a_n|^2 d_{2m-1}(n)\Big)^m,$$

where for any $m \geq 1$, $d_m(n)$ is the m-fold divisor function defined as in the Chapter Preliminaries.

By proceeding similarly as in Harper [25, Sect. 2.5] and previously as in Harper, Nikeghbali and Radziwiłł [27, Sect. 2.2], we will smoothen certain partial sums of random multiplicative functions to replace them with an integral of a corresponding Dirichlet series. To this aim, we will need the following version of Parseval's identity for Dirichlet series.

Lemma 7.2.2 (Parseval's identity)**.** *Let $(a_n)_{n=1}^{+\infty}$ be any sequence of complex numbers and let $A(s) := \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$ denote the corresponding Dirichlet series and let also σ_c denote its abscissa of convergence. Then for any $\sigma > \max\{0, \sigma_c\}$, we have*

$$\int_0^{+\infty} \frac{|\sum_{n\leq x} a_n|^2}{x^{1+2\sigma}} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\frac{A(\sigma+it)}{\sigma+it}\right|^2 dt.$$

Proof. This is [55, Eq. (5.26)]. □

We will apply Lemma 7.2.2 to the sequence given by a random multiplicative function. By multiplicativity, its Dirichlet series can be recast in terms of an Euler product, for which we then need an L^2-estimate.

Lemma 7.2.3 (Euler product result)**.** *If f is a Rademacher random multiplicative function, then for any real numbers t and $2 \leq x \leq y$, we have*

$$\mathbb{E}\Big[\prod_{x<p\leq y}\Big|1+\frac{f(p)}{p^{1/2+it}}\Big|^2\Big] = \prod_{x<p\leq y}\Big(1+\frac{1}{p}\Big).$$

When f is a Steinhaus random multiplicative function, we instead have

$$\mathbb{E}\Big[\prod_{x<p\leq y}\Big|1-\frac{f(p)}{p^{1/2+it}}\Big|^{-2}\Big] = \prod_{x<p\leq y}\Big(1-\frac{1}{p}\Big)^{-1}.$$

Proof. Let f be a Rademacher random multiplicative function. By the independence of the $f(p)$'s, for different prime numbers p, we get

$$\mathbb{E}\Big[\prod_{x<p\leq y}\Big|1+\frac{f(p)}{p^{1/2+it}}\Big|^2\Big] = \prod_{x<p\leq y}\mathbb{E}\Big[\Big|1+\frac{f(p)}{p^{1/2+it}}\Big|^2\Big].$$

Expanding the square gives

$$\Big|1+\frac{f(p)}{p^{1/2+it}}\Big|^2 = 1+\frac{f(p)}{p^{1/2+it}}+\frac{f(p)}{p^{1/2-it}}+\frac{1}{p}.$$

Since $\mathbb{E}[f(p)] = 0$, for any prime p, it is immediate to reach the claim.

Let now f be a Steinhaus random multiplicative function and, for any prime number p, write

$$\Big(1-\frac{f(p)}{p^{1/2+it}}\Big)^{-1} = \sum_{k\geq 0}\frac{f(p^k)}{p^{k(1/2+it)}}.$$

Then, we clearly have

$$\mathbb{E}\Big[\Big|1-\frac{f(p)}{p^{1/2+it}}\Big|^{-2}\Big] = \mathbb{E}\Big[\sum_{k\geq 0}\frac{f(p^k)}{p^{k(1/2+it)}}\sum_{j\geq 0}\frac{\overline{f(p^j)}}{p^{j(1/2-it)}}\Big] = \sum_{k\geq 0}\frac{1}{p^k},$$

where we can exchange summations and expectation thanks to Tonelli–Fubini's theorem. By the independence of the $f(p)$'s, for different prime numbers p, we deduce the claim also in this case. □

7.2.2 Pure probabilistic results

Common tools to tackle Law of the Iterated Logarithm type results, both classically and related to random multiplicative functions, are the well-known Borel–Cantelli's lemmas. Since the values of a random multiplicative function are not all independent from one another, so do many of the events we will have to deal with. Hence, we are interested only in applications of the first Borel–Cantelli's lemma (see e.g. [18, Theorem 18.1]).

Lemma 7.2.4. *(The first Borel–Cantelli's lemma) Let $\{A_n\}_{n\geq 1}$ be any sequence of events. Then*

$$\sum_{n=1}^{+\infty} \mathbb{P}(A_n) < +\infty \Rightarrow \mathbb{P}(\limsup_{n\longrightarrow+\infty} A_n) = 0,$$

where

$$\limsup_{n\longrightarrow+\infty} A_n := \bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} A_m.$$

The next result is the celebrated Hoeffding's inequality, which gives Gaussian-type tails for the probability that a sum of many bounded *independent* random variables deviates from its mean value by more than a certain amount (see e.g. Hoeffding [30, Theorem 2]).

Lemma 7.2.5 (Hoeffding's inequality)**.** *Let $X_1, \ldots, X_n$ be independent random variables bounded by the intervals $[a_i, b_i]$. Let $S_n = X_1 + \cdots + X_n$. Then we have*

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]|\geq t) \leq 2\exp\bigg(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\bigg).$$

The next lemma (see e.g. [18, Theorem 9.1]) gives a strong uniform control on the supremum of a finite number of terms in a submartingale sequence (see Sect. Introduction for a definition thereof).

Lemma 7.2.6 (Doob's maximal inequality). *Let $\lambda > 0$. Suppose that the sequence of random variables and of σ-algebras $\{(X_n, \mathcal{F}_n)\}_{n\geq 0}$ is a nonnegative submartingale. Then*

$$\lambda \mathbb{P}(\max_{0\leq k\leq n} X_k > \lambda) \leq \mathbb{E}[X_n].$$

On the other hand, regarding the moments of such supremum, we have the following result (see e.g. [18, Theorem 9.4]).

Lemma 7.2.7 (Doob's L^p-inequality). *Let $p > 1$. Suppose that the sequence of random variables and of σ-algebras $\{(X_n, \mathcal{F}_n)\}_{n\geq 0}$ is a nonnegative submartingale bounded in L^p. Then*

$$\mathbb{E}[(\max_{0\leq k\leq n} X_k)^p] \leq \left(\frac{p}{p-1}\right)^p \max_{0\leq k\leq n} \mathbb{E}[X_k^p].$$

7.3 Proof of Theorem 7.1.1: setting up the argument

Let $\varepsilon > 0$ and define

$$M_f(x) := \sum_{\substack{n\leq x\\ P(n)>\sqrt{x}}} f(n).$$

We would like to show that the event

$$\mathcal{A} := \left\{|M_f(x)| > 6\sqrt{x}(\log\log x)^{1/4+\varepsilon},\ \text{for infinitely many } x\right\},$$

holds with null probability.

As in Basquin [1] and in Lau–Tenenbaum–Wu [43], we are going to check the condition of the event $\mathcal{A}$ on a suitable sequence of test points x_i, not too much sparse so that we can guarantee enough control on the size of $M_f(x)$ between two consecutive such points. As in the aforementioned works, we take $x_i := \lfloor e^{i^\varepsilon} \rfloor$.

Moreover, again following previous arguments, we are going to focus our analysis on the test points contained in very wide intervals $[X_{\ell-1}, X_\ell]$ so that $\mathcal{A} \subset \cup_{\ell \geq 1} \mathcal{A}_\ell$, where

$$\mathcal{A}_\ell := \left\{ \sup_{X_{\ell-1} < x_{i-1} \leq X_\ell} \sup_{x_{i-1} < x \leq x_i} \frac{|M_f(x)|}{\sqrt{x} R(x)} > 6 \right\}$$

and where, for the sake of readability, we let $R(x) := (\log \log x)^{1/4+\varepsilon}$.

We here choose $X_\ell := e^{2^{\ell^K}}$, with $K := 4/\varepsilon$. Unlike in [1] and [43], where K was equal to 1, we will work with an extremely sparser sequence X_ℓ. We then note that

$$\begin{aligned} 2^{(\ell-1)^K} < \log x_{i-1} \leq 2^{\ell^K} &\Rightarrow (\ell-1)^K \log 2 < \log \log x_{i-1} \leq \ell^K \log 2 \\ &\Rightarrow \log \log x_i \sim \ell^K \log 2, \text{ as } \ell \longrightarrow +\infty, \end{aligned}$$

for any $x_{i-1} \in [X_{\ell-1}, X_\ell]$.

For any $x \in [x_{i-1}, x_i]$, we may write

$$M_f(x) = M_f(x_{i-1}) + (M_f(x) - M_f(x_{i-1})).$$

Hence, we get

$$|M_f(x)| \leq |M_f(x_{i-1})| + \Bigg| \sum_{\substack{n \leq x_{i-1} \\ \sqrt{x_{i-1}} < P(n) \leq \sqrt{x}}} f(n) \Bigg| + \Bigg| \sum_{\substack{x_{i-1} < n \leq x \\ P(n) > \sqrt{x}}} f(n) \Bigg|.$$

Since the function $\sqrt{x} R(x)$ is an increasing function of x, we see that $\mathcal{A}_\ell \subset \mathcal{B}_\ell \cup \mathcal{C}_\ell \cup \mathcal{D}_\ell$, where

$$\begin{aligned} \mathcal{B}_\ell &:= \left\{ \sup_{X_{\ell-1} < x_i \leq X_\ell} \frac{|M_f(x_i)|}{\sqrt{x_i} R(x_i)} > 2 \right\} \\ \mathcal{C}_\ell &:= \left\{ \sup_{X_{\ell-1} < x_{i-1} \leq X_\ell} \frac{1}{\sqrt{x_{i-1}} R(x_{i-1})} \sup_{x_{i-1} < x \leq x_i} \Bigg| \sum_{\substack{n \leq x_{i-1} \\ \sqrt{x_{i-1}} < P(n) \leq \sqrt{x}}} f(n) \Bigg| > 2 \right\} \\ \mathcal{D}_\ell &:= \left\{ \sup_{X_{\ell-1} < x_{i-1} \leq X_\ell} \frac{1}{\sqrt{x_{i-1}} R(x_{i-1})} \sup_{x_{i-1} < x \leq x_i} \Bigg| \sum_{\substack{x_{i-1} < n \leq x \\ P(n) > \sqrt{x}}} f(n) \Bigg| > 2 \right\}. \end{aligned}$$

The event $\mathcal{B}_\ell$ encodes information about the size of $M_f(x)$ on test points. On the other hand, the events $\mathcal{C}_\ell$ and $\mathcal{D}_\ell$ together control the size of the increments of the partial sums of f between two consecutive test points.

Now, suppose that we are given upper bounds B_ℓ, C_ℓ and D_ℓ on the probabilities of $\mathcal{B}_\ell, \mathcal{C}_\ell$ and $\mathcal{D}_\ell$, respectively. If $\sum_{\ell\geq 1}(B_\ell + C_\ell + D_\ell) < +\infty$, so is $\sum_{\ell\geq 1}\mathbb{P}(\mathcal{A}_\ell)$. By the first Borel–Cantelli's lemma, Lemma 7.2.4, we would then deduce that $\mathbb{P}(\limsup_{\ell\geq 1}\mathcal{A}_\ell) = 0$. This, in turn, implies that, for any sufficiently large x, we would almost surely have $|M_f(x)| \leq 6\sqrt{x}(\log\log x)^{1/4+\varepsilon}$, which is the content of Theorem 7.1.1.

7.4 The sum between test points

The aim in this section is to find a bound summable on ℓ for the probability of the events $\mathcal{C}_\ell$ and $\mathcal{D}_\ell$. As explained in [43], this would show that, almost surely, the partial sum $M_f(x)$ fluctuates moderately in appropriate short intervals and that the problem of bounding $M_f(x)$ everywhere may be reduced to doing so at the suitable test points x_i.

7.4.1 The probability of $\mathcal{C}_\ell$

By the union bound, the probability of $\mathcal{C}_\ell$ is

$$\begin{aligned}
&\leq \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \mathbb{P}\Bigg(\sup_{x_{i-1}<x\leq x_i}\Bigg|\sum_{\substack{n\leq x_{i-1}\\ \sqrt{x_{i-1}}<P(n)\leq\sqrt{x}}} f(n)\Bigg| > 2\sqrt{x_{i-1}}R(x_{i-1})\Bigg)\\
&= \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \mathbb{P}\Bigg(\sup_{x_{i-1}<x\leq x_i}\Bigg|\sum_{\substack{n\leq x_{i-1}\\ \sqrt{x_{i-1}}<P(n)\leq\sqrt{x}}} f(n)\Bigg|^2 > 4x_{i-1}R(x_{i-1})^2\Bigg).
\end{aligned}$$

By Chebyshev's inequality, the above is

$$\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{1}{x_{i-1}^2 R(x_{i-1})^4}\mathbb{E}\Bigg[\Bigg(\sup_{x_{i-1}<x\leq x_i}\Bigg|\sum_{\substack{n\leq x_{i-1}\\ \sqrt{x_{i-1}}<P(n)\leq\sqrt{x}}} f(n)\Bigg|^2\Bigg)^2\Bigg].$$

Now, consider the sequence of random variables $(Z_k)_{k\geq 1}$ given by

$$Z_k := \Bigg| \sum_{\substack{n\leq x_{i-1} \\ \sqrt{x_{i-1}}<P(n)\leq\lfloor\sqrt{k}\rfloor}} f(n) \Bigg|^2$$

(see Ch. Introduction for the definition of the floor function).

To move from one element of the sequence Z_k to the next, we reveal at most one new prime at a time. This usually corresponds to having a submartingale structure (see Sect. Introduction for a definition thereof). In fact, $(Z_k)_{k\geq 1}$ does form a nonnegative submartingale with respect to the filtration $\mathcal{F}_k := \sigma(\{f(p) : p \leq \lfloor\sqrt{k}\rfloor\})$. Indeed, Z_k is clearly $\mathcal{F}_k$-adapted and L^1-bounded. Furthermore,

$$\begin{aligned}
\mathbb{E}[Z_{k+1}|\mathcal{F}_k] &= Z_k + \mathbb{E}\Bigg[\Bigg| \sum_{\substack{n\leq x_{i-1} \\ \lfloor\sqrt{k}\rfloor<P(n)\leq\lfloor\sqrt{k+1}\rfloor}} f(n)\Bigg|^2 \Bigg|\mathcal{F}_k\Bigg] \\
&\quad + 2\Re\Bigg(\sum_{\substack{n\leq x_{i-1} \\ \sqrt{x_{i-1}}<P(n)\leq\lfloor\sqrt{k}\rfloor}} \overline{f(n)}\mathbb{E}\Bigg[\sum_{\substack{n\leq x_{i-1} \\ \lfloor\sqrt{k}\rfloor<P(n)\leq\lfloor\sqrt{k+1}\rfloor}} f(n)\Bigg|\mathcal{F}_k\Bigg]\Bigg) \\
&\geq Z_k,
\end{aligned}$$

because for any n in the innermost sum on the second line above we have $f(n) = f(p)f(m)$, with $\lfloor\sqrt{k}\rfloor < p \leq \lfloor\sqrt{k+1}\rfloor$ and m divided only by primes smaller than $\lfloor\sqrt{k}\rfloor$, so that $\mathbb{E}[f(n)|\mathcal{F}_k] = f(m)\mathbb{E}[f(p)] = 0$.

Whence, an application of Doob's L^2-inequality, Lemma 7.2.7, leads to a bound for the probability of $\mathcal{C}_\ell$

$$\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{1}{x_{i-1}^2 R(x_{i-1})^4} \sup_{x_{i-1}<x\leq x_i} \mathbb{E}\Bigg[\Bigg| \sum_{\substack{n\leq x_{i-1} \\ \sqrt{x_{i-1}}<P(n)\leq\sqrt{x}}} f(n)\Bigg|^4\Bigg].$$

To compute the fourth moment we appeal to Lemma 7.2.1, which gives a bound

$$\leq \Bigg(\sum_{\substack{n\leq x_{i-1} \\ \sqrt{x_{i-1}}<P(n)\leq\sqrt{x_i}}} d_3(n) \Bigg)^2.$$

Finally, we write

$$\sum_{\substack{n\leq x_{i-1} \\ \sqrt{x_{i-1}}<P(n)\leq\sqrt{x_i}}} d_3(n) = 3 \sum_{\sqrt{x_{i-1}}<p\leq\sqrt{x_i}} \sum_{k\leq x_{i-1}/p} d_3(k)$$

and estimate the divisor sum on the right-hand side of the previous displayed equation by using Lemma 2.1.1. So doing, we get an overall bound for $\mathbb{P}(\mathcal{C}_\ell)$

$$\begin{aligned} &\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{1}{x_{i-1}^2 R(x_{i-1})^4} x_{i-1}^2 (\log x_{i-1})^4 \Bigg(\sum_{\sqrt{x_{i-1}}<p\leq\sqrt{x_i}} \frac{1}{p} \Bigg)^2 \\ &\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{(\log x_{i-1})^4}{i^2 R(x_{i-1})^4} \ll \sum_{i\geq 2^{\frac{(\ell-1)^K}{\varepsilon}}} \frac{1}{i^{2-4\varepsilon}} \ll 2^{-(\ell-1)^K(1-4\varepsilon)/\varepsilon}, \end{aligned}$$

by a strong form of Mertens' theorem (with error term given by the Prime Number Theorem), if ℓ is sufficiently large with respect to ε. This is certainly summable on ℓ, if $\varepsilon < 1/4$.

7.4.2 The probability of $\mathcal{D}_\ell$

By the union bound

$$\mathbb{P}(\mathcal{D}_l) \leq \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \mathbb{P}\Bigg(\sup_{x_{i-1}<x\leq x_i} \Bigg| \sum_{\substack{x_{i-1}<n\leq x \\ P(n)>\sqrt{x}}} f(n) \Bigg| > 2\sqrt{x_{i-1}} R(x_{i-1}) \Bigg).$$

The probability of the above event where instead the partial sum of $f(n)$ runs over the full short interval $[x_{i-1}, x]$ has already been studied by Basquin [1] and Lau–Tenenbaum–Wu [43]. Here, we have to deal with the extra condition on the largest prime factor, which can still be handled by adapting the proof in the aforementioned papers.

We split $[x_{i-1}, x]$ into a disjoint union of at most $2 \log x_i$ subintervals with limit points

$$u_k := x_{i-1} + \sum_{1\le j\le k} 2^{\nu_j} \quad (0 \le k \le h),$$

with $\nu_1 > \nu_2 > \cdots > \nu_h$ positive integers. In particular, $u_h = x$, with x the (random) point where the maximum is attained, and the sequence $\{u_k\}_k$ is in fact random, since it depends on f.

Then, by seeing

$$S_f(x) := \sum_{\substack{x_{i-1}<n\le x\\ P(n)>\sqrt{x}}} f(n) = S_f(x) - S_f(x_{i-1}),$$

we can bound the above probability with

$$\begin{aligned}
&\le \mathbb{P}\Bigg(\sum_{u_k} |S_f(u_{k+1}) - S_f(u_k)| > 2\sqrt{x_{i-1}}R(x_{i-1})\Bigg)\\
&\le \mathbb{P}\Bigg(\bigcup_{u_k}\bigg\{|S_f(u_{k+1}) - S_f(u_k)| > \frac{\sqrt{x_{i-1}}R(x_{i-1})}{\log x_i}\bigg\}\Bigg)\\
&= \mathbb{P}\Bigg(\sup_{u_k}|S_f(u_{k+1}) - S_f(u_k)| > \frac{\sqrt{x_{i-1}}R(x_{i-1})}{\log x_i}\Bigg).
\end{aligned}$$

Moreover, note that for any $x_{i-1} \le u \le v \le x$, we have

$$S_f(v) - S_f(u) = -\sum_{\substack{x_{i-1}<n\le u\\ \sqrt{u}<P(n)\le\sqrt{v}}} f(n) + \sum_{\substack{u<n\le v\\ P(n)>\sqrt{v}}} f(n).$$

Now, write $u = x_{i-1}+(l-1)2^m$ and $v = x_{i-1}+l2^m$, where $l := \sum_{1\le j\le k} 2^{\nu_j-\nu_k} \ge 1$ and $m := \nu_k \ge 0$ are such that $l2^m \le x_i - x_{i-1}$.

By the union bound, Markov's inequality for the fourth moment and the hypercontractive inequality as stated in Lemma 7.2.1, we have a bound for the probability of $\mathcal{D}_\ell$

$$\ll \sum_{X_{\ell-1}<x_{i-1}\le X_\ell} \frac{(\log x_i)^4}{x_{i-1}^2} \sum_{\substack{l\ge 1, m\ge 0\\ l2^m\le x_i-x_{i-1}}} \Bigg(\sum_{u<n\le v} d_3(n)\Bigg)^2 \tag{7.10}$$

$$+ \sum_{X_{\ell-1}<x_{i-1}\le X_\ell} \frac{(\log x_i)^4}{x_{i-1}^2} \sum_{\substack{l\ge 1, m\ge 0\\ l2^m\le x_i-x_{i-1}}} \Bigg(\sum_{\substack{x_{i-1}<n\le u\\ \sqrt{u}<P(n)\le\sqrt{v}}} d_3(n)\Bigg)^2. \tag{7.11}$$

By Hölder's inequality, we get

$$\begin{aligned}\Bigg(\sum_{u<n\le v} d_3(n)\Bigg)^2 &\le \Bigg(\sum_{n\le v} d_3(n)^3\Bigg)^{2/3}\Bigg(\sum_{u<n\le v} 1\Bigg)^{4/3}\\ &\ll x_i^{2/3}(\log x_i)^{52/3}(v-u)^{4/3},\end{aligned}$$

where the estimate for the partial sum of $d_3(n)^3$ easily follows from [69, Ch. III, Corollary 3.6] or from Lemma 2.1.1 here. Summing this up over all possible realizations of u and v, we get an overall bound for (7.10) of

$$\begin{aligned}&\ll \sum_{X_{\ell-1}<x_{i-1}\le X_\ell} (\log x_i)^{64/3}\left(\frac{x_i-x_{i-1}}{x_i}\right)^{4/3}\\ &\ll \sum_{X_{\ell-1}<x_{i-1}\le X_\ell} \frac{1}{i^{4/3-68\varepsilon/3}}\\ &\ll 2^{-(\ell-1)^K(1/3-68\varepsilon/3)/\varepsilon}.\end{aligned} \tag{7.12}$$

Here and subsequently we use that

$$\frac{x_i-x_{i-1}}{x_i} \asymp \frac{1}{i^{1-\varepsilon}}.$$

Regarding (7.11), we notice that

$$\sum_{\substack{x_{i-1}<n\le u\\ \sqrt{u}<P(n)\le\sqrt{v}}} d_3(n) = 3\sum_{\sqrt{u}<p\le\sqrt{v}} \sum_{\frac{x_{i-1}}{p}<k\le\frac{u}{p}} d_3(k). \tag{7.13}$$

If $\sqrt{v}-\sqrt{u}\ge 1$, we simply upper bound the innermost sum on the right-hand side above with $\ll u(\log u)^2/p$, by Lemma 2.1.1, and get a bound for (7.13) of

$$\ll u(\log u)^2 \sum_{\sqrt{u}<p\leq\sqrt{v}} \frac{1}{p} \ll \sqrt{u}(\log u)^2(\sqrt{v}-\sqrt{u})$$
$$\leq (v-u)(\log x_i)^2.$$

This contibutes to (7.11) an amount of

$$\begin{aligned}(7.14) \qquad &\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{(\log x_i)^8}{x_{i-1}^2} \sum_{\substack{l\geq 1, m\geq 0\\ l2^m\leq x_i-x_{i-1}}} (v-u)^2 \\ &\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{(\log x_i)^8(x_i-x_{i-1})^2}{x_{i-1}^2} \\ &\ll \sum_{X_{\ell-1}<x_{i-1}\leq X_\ell} \frac{1}{i^{2-10\varepsilon}} \\ &\ll 2^{-(\ell-1)^K(1-10\varepsilon)/\varepsilon}.\end{aligned}$$

On the other hand, if $\sqrt{v}-\sqrt{u}<1$, we extend the innermost sum on the right-hand side of (7.13) to all the integers in the interval $[x_{i-1}/p, x_i/p]$ to then, by Shiu's theorem [66, Theorem 1], upper bound it with

$$\frac{(x_i-x_{i-1})(\log x_i)^2}{p} \leq \frac{(x_i-x_{i-1})(\log x_i)^2}{\sqrt{x_{i-1}}}.$$

The application of Shiu's theorem is justified by the fact that

$$\frac{x_i}{p} - \frac{x_{i-1}}{p} > \sqrt[3]{\frac{x_i}{p}},$$

if x_i is sufficiently large with respect to ε, as it can be easily verified. This bound contributes to (7.11) an amount of

$$\begin{aligned}(7.15)\qquad &\ll \sum_{X_{\ell-1}<x_{i-1}\le X_\ell} \frac{(x_i - x_{i-1})^3(\log x_i)^8}{x_{i-1}^3} \\ &\ll \sum_{X_{\ell-1}<x_{i-1}\le X_\ell} \frac{1}{i^{3-11\varepsilon}} \\ &\ll 2^{-(\ell-1)^K(2-11\varepsilon)/\varepsilon}.\end{aligned}$$

Together, the estimates (7.12), (7.14) and (7.15) give a total bound for the probability of $\mathcal{D}_\ell$ that is summable on ℓ, if ε is small enough.

7.5 The sum on test points and conditional conclusion of the proof of Theorem 7.1.1

Thanks to the work done in the previous sections, to prove Theorem 7.1.1, we are left with understanding the size of the partial sums of f over test points. More specifically, we need to bound the probability of the following event

$$\mathcal{B}_\ell := \left\{ \sup_{X_{\ell-1}<x_i\le X_\ell} \frac{|M_f(x_i)|}{\sqrt{x_i}R(x_i)} > 2 \right\}.$$

Assume f to be a Rademacher random multiplicative function. To this aim, we first notice that we may rewrite the partial sums of f over integers with a large prime factor as a sum of many independent random variables, if we allow for conditioning on the smaller primes. In fact,

$$M_f(x_i) = \sum_{\sqrt{x_i}<p\le x_i} Y_p,$$

where, for any $p > \sqrt{x_i}$, we let

$$Y_p := f(p) \sum_{m\le x_i/p} f(m).$$

The random variables $(Y_p)_{\sqrt{x_i}<p\le x_i}$, conditioned on

$$\mathcal{F}(\sqrt{x_i}) := \sigma(\{f(p) : \ p \le \sqrt{x_i}\}),$$

are independent, with $\mathbb{E}[Y_p|\mathcal{F}(\sqrt{x_i})] = 0$.

We are then in position to apply Hoeffding's inequality, Lemma 7.2.5, to get

$$\mathbb{P}\Big(|M_f(x_i)| \ge 2\sqrt{x_i}R(x_i) \ |\mathcal{F}(\sqrt{x_i})\Big) \ll \exp\Big(-\frac{4x_iR(x_i)^2}{V(x_i)}\Big), \tag{7.16}$$

where

$$V(x_i) := \sum_{\sqrt{x_i}<p\le x_i}\Big|\sum_{m\le x_i/p} f(m)\Big|^2. \tag{7.17}$$

We arrive to the same bound (7.16), where the constants 2 and 4 are replaced by 1, if, for a Steinhaus random multiplicative function, we replace $M_f(x)$ with $\Re(M_f(x))$ and $\Im(M_f(x))$.

Clearly, the right-hand side of (7.16) is still a random variable. However, if we condition on the size of $V(x_i)$, it will lead to an estimate for the probability of $\mathcal{B}_\ell$. To this regard, we will show that with high probability (depending on ℓ)

$$V(x_i) \ll \frac{x_i}{\sqrt{\log\log x_i}}, \tag{7.18}$$

uniformly on $x_i \in [X_{\ell-1}, X_\ell]$. The scaling factor $\sqrt{\log\log x_i}$, compared to $\mathbb{E}[V(x_i)] \asymp x_i$, is characteristic of low moments of partial sums of Rademacher and Steinhaus random multiplicative functions (see the introduction to [25]) and we can already pointwise derive (7.18) using Harper's low moments results [25]. The uniformity in (7.18) will come from rewriting $V(x_i)$ in terms of a submartingale sequence and managing its size via Doob's inequality. These features, together with the Gaussian-type control (7.16) on the tail distribution of $M_f(x_i)$, are what determines the exponent 1/4 in Theorem 7.1.1.

Following previous considerations, we define

$$\mathcal{E}_\ell := \left\{ \sup_{X_{\ell-1} < x_i \le X_\ell} \frac{V(x_i)\sqrt{\log\log x_i}}{x_i} \le T \right\}.$$

Here $T \ge 1$ is a parameter that will be chosen later and that measures how much we can lose, compared to (7.18), to still be able to successfully estimate the probability of $\mathcal{B}_\ell$. We now show how to deduce Theorem 7.1.1 from the next lemma.

Lemma 7.5.1. *Let $\varepsilon > 0$ and $K = 4/\varepsilon$. Then, for any $T \ge 1$, we have*

$$\mathbb{P}(\bar{\mathcal{E}}_\ell) \ll_\varepsilon \left(\frac{\sqrt{\ell^K}}{T} \right)^{\lfloor \frac{4}{\varepsilon} \rfloor} 2^{-\frac{\ell^K}{\varepsilon}} + \frac{1}{T^{1/4}}.$$

Conclusion of the proof of Theorem 7.1.1, assuming Lemma 7.5.1. Let f be a Rademacher random multiplicative function. Plainly,

$$\begin{aligned}
\mathbb{P}(\mathcal{B}_\ell) &= \mathbb{P}\Big(\bigcup_{X_{\ell-1} < x_i \le X_\ell} \Big\{ \frac{|M_f(x_i)|}{\sqrt{x_i} R(x_i)} > 2 \Big\} \Big) \\
&\le \mathbb{P}\Big(\bigcup_{X_{\ell-1} < x_i \le X_\ell} \Big\{ \frac{|M_f(x_i)|}{\sqrt{x_i} R(x_i)} > 2 \Big\} \cap \Big\{ \frac{V(x_i)\sqrt{\log\log x_i}}{x_i} \le T \Big\} \Big) \\
&+ \mathbb{P}\Big(\bigcup_{X_{\ell-1} < x_i \le X_\ell} \Big\{ \frac{|M_f(x_i)|}{\sqrt{x_i} R(x_i)} > 2 \Big\} \cap \Big\{ \frac{V(x_i)\sqrt{\log\log x_i}}{x_i} > T \Big\} \Big).
\end{aligned}$$

By a repeated application of (7.16), where we condition on the event

$$\{V(x_i)\sqrt{\log\log x_i} \le T x_i\},$$

and the union bound, we get the above is

$$\ll \sum_{X_{\ell-1} < x_i \le X_\ell} \exp\left(- \frac{4R(x_i)^2\sqrt{\log\log x_i}}{T} \right) + \mathbb{P}(\bar{\mathcal{E}}_\ell).$$

We arrive to the same above bound, by replacing $M_f(x)$ with $\Re(M_f(x))$ and $\Im(M_f(x))$ and the constants 2 and 4 with 1, in the Steinhaus case.

Remind that $R(x_i) = (\log\log x_i)^{1/4+\varepsilon} \sim \ell^{K/4+K\varepsilon}(\log 2)^{1/4+\varepsilon}$, with $K\varepsilon =$

4. Whence, by letting $T = T(\ell) := \varepsilon^2\ell^8$, the above becomes

$$\ll \sum_{X_{\ell-1}<x_i\leq X_\ell} \exp\left(-\frac{c\ell^K}{\varepsilon^2}\right) + \mathbb{P}(\bar{\mathcal{E}}_\ell),$$

where $c > 0$ is a constant. By Lemma 7.5.1, we overall deduce that

$$\mathbb{P}(\mathcal{B}_\ell) \ll_\varepsilon 2^{\frac{\ell^K}{\varepsilon}} e^{-\frac{c\ell^K}{\varepsilon^2}} + 2^{-\frac{\ell^K}{\varepsilon}} \ell^{8/\varepsilon^2-32/\varepsilon} + \ell^{-2},$$

which is evidently a bound summable on ℓ, if ε is taken small enough. Since the same holds for the probabilities of $\mathcal{C}_\ell$ and $\mathcal{D}_\ell$, as proved in Subsect. 4.1 and 4.2, we overall get a summable bound for the probability of $\mathcal{A}_\ell$. This concludes the proof of Theorem 7.1.1, in the way it was described at the end of Sect. 7.3. □

Remark 7.5.2. *We would like to stress how important is the introduction of the exponent K in the definition of X_ℓ. Even though it makes our task harder, by drastically increasing the number of test points x_i contained between two consecutive elements of the sequence $\{X_\ell\}_{\ell\geq 1}$, it allows us to choose T much* smaller *compared to what is possible to do in Basquin [1] and Lau–Tenenbaum–Wu [43], leading to a superior bound in Theorem 7.1.1. Another source of saving in the choice of T comes from inputting the new information about the low moments of the partial sums of f.*

7.6 A smoothing argument

In this section we start the proof of Lemma 7.5.1. From now on, f will indicate both a Rademacher and a Steinhaus random multiplicative function. To begin with, we make $V(x_i)$ more amenable to perform the computation of the probability of $\bar{\mathcal{E}}_\ell$.

By taking inspiration from Harper's work [26] and previously from Harper, Nikeghbali and Radziwiłłs' work [27], we insert a logarithmic weight into the summation defining $V(x_i)$ to then smoothen the summation further by inserting an integral average. More specifically, we have

$$
\begin{aligned}
V(x_i) &\leq \frac{2}{\log x_i} \sum_{\sqrt{x_i}<p\leq x_i} \log p \Big| \sum_{m\leq x_i/p} f(m)\Big|^2 \\
&= \frac{2X}{\log x_i} \sum_{\sqrt{x_i}<p\leq x_i} \frac{\log p}{p} \int_p^{p(1+1/X)} \Big| \sum_{m\leq x_i/p} f(m)\Big|^2 dt \\
&\ll \frac{X}{\log x_i} \sum_{\sqrt{x_i}<p\leq x_i} \frac{\log p}{p} \int_p^{p(1+1/X)} \Big| \sum_{m\leq x_i/t} f(m)\Big|^2 dt \\
&+ \frac{X}{\log x_i} \sum_{\sqrt{x_i}<p\leq x_i} \frac{\log p}{p} \int_p^{p(1+1/X)} \Big| \sum_{x_i/t<m\leq x_i/p} f(m)\Big|^2 dt,
\end{aligned}
$$

by using $|a+b|^2 \ll |a|^2+|b|^2$, for any complex numbers a and b, and where $X \geq 1$ will be chosen later. Hence, the probability of $\bar{\mathcal{E}}_\ell$ may be bounded from above by $\mathbb{P}_1 + \mathbb{P}_2$, where

$$
\begin{aligned}
\mathbb{P}_1 := \mathbb{P}\Bigg(\Bigg\{ &\sup_{X_{\ell-1}<x_i\leq X_\ell} \frac{X\sqrt{\log\log x_i}}{x_i \log x_i} \\
&\times \sum_{\sqrt{x_i}<p\leq x_i} \frac{\log p}{p} \int_p^{p(1+1/X)} \Big| \sum_{m\leq x_i/t} f(m)\Big|^2 dt > \frac{T}{2}\Bigg\}\Bigg)
\end{aligned}
\tag{7.19}
$$

$$
\begin{aligned}
\mathbb{P}_2 := \mathbb{P}\Bigg(\Bigg\{ &\sup_{X_{\ell-1}<x_i\leq X_\ell} \frac{X\sqrt{\log\log x_i}}{x_i \log x_i} \\
&\times \sum_{\sqrt{x_i}<p\leq x_i} \frac{\log p}{p} \int_p^{p(1+1/X)} \Big| \sum_{x_i/t<m\leq x_i/p} f(m)\Big|^2 dt > \frac{T}{2}\Bigg\}\Bigg).
\end{aligned}
\tag{7.20}
$$

By the union bound and Markov's inequality for the power $q > 1$, we have that

$$
\begin{aligned}
\mathbb{P}_2 \ll_q \frac{1}{T^q} &\sum_{X_{\ell-1}<x_i\leq X_\ell} \left(\frac{\sqrt{\log\log x_i}}{x_i}\right)^q \\
&\times \mathbb{E}\Bigg[\Bigg(\sum_{\sqrt{x_i}<p\leq x_i} \frac{X}{p} \int_p^{p(1+1/X)} \Big| \sum_{x_i/t<m\leq x_i/p} f(m)\Big|^2 dt\Bigg)^q\Bigg].
\end{aligned}
\tag{7.21}
$$

We fix $q := \lfloor \frac{4}{\varepsilon} \rfloor$, because we would like to roughly have $(\log x_i)^q$ of size comparable to the number of test points x_i in $[X_{\ell-1}, X_\ell]$.

The expectation above can be seen as the qth power of the qth norm of a sum of random variables. Then, it is natural to swap norm and summation, by appealing to Minkowski's inequality. We can thus bound such expectation with

$$\leq \Bigg(\sum_{\sqrt{x_i} < p \leq x_i} \Bigg(\mathbb{E}\Bigg[\Bigg(\frac{X}{p} \int_p^{p(1+1/X)} \Bigg| \sum_{x_i/t < m \leq x_i/p} f(m) \Bigg|^2 dt \Bigg)^q \Bigg] \Bigg)^{\frac{1}{q}} \Bigg)^q .$$

The next step, arguing as in Harper [26], is to switch the expectation with the integral. This is achieved by an application of Hölder's inequality to the normalised integral $\frac{X}{p} \int_p^{p(1+1/X)} dt$ with parameters $1/q$ and $(q-1)/q$. We then estimate the above with

$$\leq \Bigg(\sum_{\sqrt{x_i} < p \leq x_i} \Bigg(\frac{X}{p} \int_p^{p(1+1/X)} \mathbb{E}\Bigg[\Bigg| \sum_{x_i/t < m \leq x_i/p} f(m) \Bigg|^{2q} \Bigg] dt \Bigg)^{\frac{1}{q}} \Bigg)^q . \tag{7.22}$$

The problem is then reduced to bound the $2q$th moment of a partial sum of f over short intervals. This is addressed by an application of the hypercontractive inequality. Indeed, arguing as in Harper [26], we notice that if $x_i/(X+1) < p \leq x_i$, then

$$\frac{X}{p} \int_p^{p(1+1/X)} \mathbb{E}\Bigg[\Bigg| \sum_{x_i/t < m \leq x_i/p} f(m) \Bigg|^{2q} \Bigg] dt \leq 1,$$

since the sum contains at most one element, having length $\frac{x_i(t-p)}{tp} < 1$. Otherwise, by again following Harper as in the proof of Proposition 2 in [26], we apply the Cauchy–Schwarz's inequality to bound the expectation in (7.22) with

$$\sqrt{\mathbb{E}\Bigg[\Bigg| \sum_{x_i/t < m \leq x_i/p} f(m) \Bigg|^{2} \Bigg] \mathbb{E}\Bigg[\Bigg| \sum_{x_i/t < m \leq x_i/p} f(m) \Bigg|^{2(2q-1)} \Bigg]} .$$

Now, since $t < p(1+1/X)$, we clearly have

$$\mathbb{E}\Bigg[\Big|\sum_{x_i/t<m\le x_i/p} f(m)\Big|^2\Bigg] \le \sum_{x_i/(p(1+1/X))<m\le x_i/p} 1 \ll \frac{x_i}{pX},$$

where we used that $p \le x_i/(X+1)$.

On the other hand, by Lemma 7.2.1, we find

$$\begin{aligned}\mathbb{E}\Bigg[\Big|\sum_{x_i/t<m\le x_i/p} f(m)\Big|^{2(2q-1)}\Bigg] &\le \Bigg(\sum_{m\le x_i/p} d_{4q-3}(m)\Bigg)^{2q-1} \\ &\ll_q \left(\frac{x_i}{p}(\log x_i)^{4q-4}\right)^{2q-1},\end{aligned}$$

by Lemma 2.1.1.

Collecting the previous computations together, we have found

$$\mathbb{E}\Bigg[\Big|\sum_{x_i/t<m\le x_i/p} f(m)\Big|^{2q}\Bigg] \ll_q \left(\frac{x_i}{p}\right)^q \frac{(\log x_i)^{4q^2-6q+2}}{\sqrt{X}}.$$

Hence, (7.22) is

$$\begin{aligned}&\ll_q \Bigg(\sum_{x_i/(X+1)<p\le x_i} 1 + \sum_{\sqrt{x_i}<p\le x_i/(X+1)} \frac{x_i}{pX^{1/2q}}(\log x_i)^{4q-6+2/q}\Bigg)^q \\ &\ll_q \frac{x_i^q}{(\log x_i)^q},\end{aligned}$$

by choosing e.g. $X := (\log x_i)^{8q^2-10q+4}$ and using estimates of Chebyshev and Mertens. Inserting this back into (7.21), we deduce:

$$\begin{aligned}\mathbb{P}_2 \ll_q \frac{1}{T^q}\sum_{X_{\ell-1}<x_i\le X_\ell}\left(\frac{\sqrt{\log\log x_i}}{\log x_i}\right)^q &\ll_q \frac{2^{\frac{\ell^K}{\varepsilon}}}{T^q}\left(\frac{\sqrt{\ell^K}}{2^{(\ell-1)^K}}\right)^q \\ &\le \left(\frac{\sqrt{\ell^K}}{T}\right)^{\lfloor\frac{4}{\varepsilon}\rfloor} 2^{-\frac{\ell^K}{\varepsilon}},\end{aligned}$$

reminding that $q = \lfloor\frac{4}{\varepsilon}\rfloor$ and taking ℓ large enough with respect to ε. This gives the first term in the upper bound of Lemma 7.5.1.

7.7 Inputting low moments estimates

In this section we continue the proof of Lemma 7.5.1, by now turning the attention to the study of the probability in (7.19).

7.7.1 Introducing a submartingale sequence

Swapping integral and summation, we have

$$\begin{aligned} &X \sum_{\sqrt{x_i}<p\leq x_i} \frac{\log p}{p} \int_{p}^{p(1+1/X)} \Big| \sum_{m\leq x_i/t} f(m)\Big|^2 dt \\ &\leq X \int_{\sqrt{x_i}}^{x_i(1+1/X)} \sum_{t/(1+1/X)<p\leq t} \frac{\log p}{p} \Big| \sum_{m\leq x_i/t} f(m)\Big|^2 dt. \end{aligned}$$

Since $\log t \asymp \log x_i$, and reminding that $X = (\log x_i)^{8q^2-10q+4}$, by a strong form of Mertens' theorem (with error term following from the reminder in the Prime Number Theorem) we find

$$\sum_{t/(1+1/X)<p\leq t} \frac{\log p}{p} \ll \log\left(1+\frac{1}{X}\right) \ll \frac{1}{X},$$

if x_i is sufficiently large with respect to ε.

Inserting the last estimate in the previous expression, and changing variables $x_i/t =: z$ inside the integral, we find it is

$$\ll x_i \int_{0}^{\sqrt{x_i}} \Big| \sum_{m\leq z} f(m)\Big|^2 \frac{dz}{z^2}. \tag{7.23}$$

It will be soon clear that the above random variable generates a nonnegative submartingale sequence. This observation will help us out later to deal with a supremum of such sequence over the test points x_i, via the use of Doob's maximal inequality. However, an immediate application of such result would only lead to a too weak bound for $\mathbb{P}_1$. This is due to the fact that Doob's maximal inequality relates the probability of a supremum of a submartingale sequence only to the expectations of its members, not instead to their low moments (which we need here, because of the presence of the factors $\sqrt{\log\log x_i}$

in (7.19), which are related to the size of the low moments of the random variables in (7.23)). For similar reasons, even an application of Doob's L^p inequality, Lemma 7.2.7, would be inefficient, considering that it only deals with high moments. To overcome this, we will first condition on the event that the contribution from the values of f on the small primes is dominated by the size of its low moments, and what follows goes in the direction of rewriting the integral in (7.23) in a way to make more accessible this kind of information.

By extending the integral in (7.23), we find it is

$$= x_i \int_0^{\sqrt{x_i}} \Big| \sum_{\substack{m \le z \\ P(m) \le x_i}} f(m) \Big|^2 \frac{dz}{z^2} \le x_i \int_0^{+\infty} \Big| \sum_{\substack{m \le z \\ P(m) \le x_i}} f(m) \Big|^2 \frac{dz}{z^2}.$$

The idea of inserting the constraint on the largest prime factor is taken from the proof of [25, Proposition 2]. Continuing arguing as in there, by appealing to Parseval's identity, Lemma 7.2.2, we rewrite the above as

$$\frac{x_i}{2\pi} \int_{-\infty}^{+\infty} \Big| \frac{\mathcal{S}_{x_i}(1/2+it)}{1/2+it} \Big|^2 dt,$$

where

$$\mathcal{S}_{x_i}(1/2+it) := \prod_{p \le x_i} \Big(1 + \frac{f(p)}{p^{1/2+it}} \Big),$$

in the Rademacher case, or

$$\mathcal{S}_{x_i}(1/2+it) := \prod_{p \le x_i} \Big(1 - \frac{f(p)}{p^{1/2+it}} \Big)^{-1},$$

in the Steinhaus case. We would like to stress that this maneuver, to pass from an L^2-integral of the partial sums of f to an L^2-integral of a product of independent random variables, is taken from Harper [25, Proof of Proposition 2]. It differentiates from what was done in Lau–Tenenbaum–Wu [43] in the fact that, once they arrived at a similar point, they kept working with the L^2-integral of partial sums of f, since such procedure would have not led them to a stronger result.

In conclusion, we may find

$$\mathbb{P}_1 \le \mathbb{P}\bigg(\bigg\{\sup_{X_{\ell-1}<x_i\le X_\ell} \frac{\sqrt{\log\log x_i}}{\log x_i}\bigg(\frac{\log x_i}{\log X_{\ell-1}}\bigg)^{1/(\ell-1)^K} \times \int_{-\infty}^{+\infty}\bigg|\frac{\mathcal{S}_{x_i}(1/2+it)}{1/2+it}\bigg|^2 dt > cT\bigg\}\bigg), \tag{7.24}$$

for a certain $c > 0$.

As it will be clear in a moment, the factors $(\frac{\log x_i}{\log X_{\ell-1}})^{1/(\ell-1)^K} \ge 1$ have been introduced to make the sequence of random variables

$$Y_{x_i} := \frac{1}{\log x_i}\bigg(\frac{\log x_i}{\log X_{\ell-1}}\bigg)^{1/(\ell-1)^K} \int_{-\infty}^{+\infty}\bigg|\frac{\mathcal{S}_{x_i}(1/2+it)}{1/2+it}\bigg|^2 dt$$

a submartingale sequence with respect to the filtration $\mathcal{F}_i := \sigma(\{f(p) : \ p \le x_i\})$. Without them only the integral alone would have given rise to a submartingale sequence and a direct application of Doob's maximal inequality to handle the supremum in (7.24) would have only led to an extremely large upper bound for $\mathbb{P}_1$. For instance, if we split the supremum over points x_i such that $\log x_i \in [2^{(\ell-1)^K+j-1}, 2^{(\ell-1)^K+j})$, for $j = 1, \dots, \ell^K - (\ell-1)^K$, and apply Doob's inequality to the event that the supremum of the submartingale sequence given by the integral is large on them, we would get

$$\begin{aligned} &\ll \sum_{j=1}^{\ell^K-(\ell-1)^K} \frac{\sqrt{\ell^K}}{T2^{(\ell-1)^K+j}}\mathbb{E}\bigg[\int_{-\infty}^{+\infty}\bigg|\frac{\mathcal{S}_{e^{2^{(\ell-1)^K+j}}}(1/2+it)}{1/2+it}\bigg|^2 dt\bigg] \\ &\ll \frac{\ell^{3K/2}}{T} \ll_\varepsilon \ell^{3K/2-8}, \end{aligned}$$

by Lemma 7.2.3 and Mertens' theorem, for if $T = \varepsilon^2\ell^8$, which is clearly not summable on ℓ.

We now prove that Y_{x_i} forms a submartingale sequence. In fact, each Y_{x_i} is certainly $\mathcal{F}_i$-measurable and L^1-bounded, since

$$\mathbb{E}[|\mathcal{S}_{x_i}(1/2+it)|^2] \ll \log x_i,$$

by Lemma 7.2.3 and Mertens' theorem. Moreover, we clearly have

$$\mathbb{E}[Y_{x_i}|\mathcal{F}_{i-1}] = \frac{1}{\log x_{i-1}}\bigg(\frac{\log x_{i-1}}{\log X_{\ell-1}}\bigg)^{1/(\ell-1)^K}\frac{\log x_{i-1}}{\log x_i}\bigg(\frac{\log x_i}{\log x_{i-1}}\bigg)^{1/(\ell-1)^K} \times \int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{x_{i-1}}(1/2+it)|^2}{|1/2+it|^2}\mathbb{E}\bigg[\frac{|\mathcal{S}_{x_i}(1/2+it)|^2}{|\mathcal{S}_{x_{i-1}}(1/2+it)|^2}\bigg]dt.$$

By Lemma 7.2.3, the expectation inside the integral equals

$$\exp\bigg(\sum_{x_{i-1}<p\le x_i}\frac{1}{p}+O\bigg(\frac{1}{x_{i-1}}\bigg)\bigg) = \exp\bigg(\frac{\varepsilon}{i}+O\bigg(\frac{1}{i^2}\bigg)\bigg),$$

by a strong form of Mertens' theorem (with error term following from the reminder in the Prime Number Theorem), if i is sufficiently large with respect to ε. On the other hand,

$$\frac{\log x_{i-1}}{\log x_i} = \bigg(1-\frac{1}{i}\bigg)^{\varepsilon} = \exp\bigg(-\frac{\varepsilon}{i}+O\bigg(\frac{1}{i^2}\bigg)\bigg)$$

and

$$\bigg(\frac{\log x_i}{\log x_{i-1}}\bigg)^{1/(\ell-1)^K} \ge \bigg(\frac{\log x_i}{\log x_{i-1}}\bigg)^{\log 2/\log\log x_i} = \exp\bigg(\frac{\log 2}{i\log i}+O_\varepsilon\bigg(\frac{1}{i^2\log i}\bigg)\bigg).$$

We deduce that

$$\mathbb{E}[Y_{x_i}|\mathcal{F}_{i-1}] \ge \frac{1}{\log x_{i-1}}\bigg(\frac{\log x_{i-1}}{\log X_{\ell-1}}\bigg)^{1/(\ell-1)^K}\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{x_{i-1}}(1/2+it)|^2}{|1/2+it|^2}dt = Y_{x_{i-1}},$$

if i is sufficiently large with respect to ε.

7.7.2 Conditioning and low moments estimates

We can now see (7.24) as the probability that the supremum of a normalized submartingale sequence is large. This is the field where Doob's maximal inequality operates. However, as preannounced before, an immediate application of Lemma 7.2.6 turns out to be inefficient. In fact, it would only lead to a bound of

$$\ll \frac{\sqrt{\ell^K}}{T}\mathbb{E}[Y_{X_\ell}] \ll \frac{\sqrt{\ell^K}}{T}2^{(\ell^K-(\ell-1)^K)/(\ell-1)^K} \ll_\varepsilon \frac{\sqrt{\ell^K}}{T} \ll_\varepsilon \ell^{K/2-8},$$

where by abuse of notation we indicated with X_ℓ the largest $x_i \le X_\ell$, we used Lemma 7.2.3 and Mertens' theorem, for if $T = \varepsilon^2\ell^8$, and

$$2^{(\ell^K-(\ell-1)^K)/(\ell-1)^K} \ll_\varepsilon 1,$$

getting a final bound clearly not summable on ℓ.

To improve the application of Doob's inequality, we first need to introduce a conditioning on the values of f at the small primes. More specifically, we will condition on the following event:

$$\Sigma_\ell := \left\{ \int_{-\infty}^{+\infty} \frac{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}{|1/2+it|^2}dt \le \frac{\sqrt{T}2^{(\ell-1)^K}}{\sqrt{(\ell-1)^K}} \right\}. \tag{7.25}$$

We note that the bound in (7.25) is simply $\sqrt{T}(\log X_{\ell-1})/\sqrt{\log\log X_{\ell-1}}$.

Therefore, it represents a $(\log\log X_{\ell-1})$-power saving compared to the expectation of the above integral. This is exactly what we need to succeed in this proof of Theorem 7.1.1, and in particular we could have taken any power $q \in (0,1)$ of $(\log\log X_{\ell-1})$ instead of $1/2$.

First of all, we need to check that Σ_ℓ holds with a probability sufficiently close to 1. In fact, this is the more delicate part of our argument because we need access to deep information about the distribution of the Euler product of a random multiplicative function. By Markov's inequality for the power 1/2, we get

$$\begin{aligned}\mathbb{P}(\overline{\Sigma_\ell}) &\le \left(\frac{\sqrt{(\ell-1)^K}}{\sqrt{T}2^{(\ell-1)^K}}\right)^{1/2}\mathbb{E}\left[\left(\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}{|1/2+it|^2}dt\right)^{1/2}\right]\\ &\ll \frac{1}{T^{1/4}},\end{aligned}$$

whenever ℓ is sufficiently large with respect to ε, which is good enough for

Lemma 7.5.1. Here, we have used Harper's low moment result, which gives

$$\mathbb{E}\Big[\Big(\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}{|1/2+it|^2}dt\Big)^{1/2}\Big] \ll \Big(\frac{\log X_{\ell-1}}{\sqrt{\log\log X_{\ell-1}}}\Big)^{1/2}$$
$$\ll \frac{2^{\frac{(\ell-1)^K}{2}}}{(\ell-1)^{K/4}}.$$

This follows from [25, Key Proposition 1] and [25, Key Proposition 2], as it is done in [25] in the paragraph entitled: "Proof of the upper bound in Theorem 1, assuming Key Propositions 1 and 2".

Now, by conditioning on the event Σ_ℓ, we get that (7.24) is at most

$$\mathbb{P}\Big(\Big\{\sup_{X_{\ell-1}<x_i\le X_\ell} Y_{x_i} > \frac{cT}{\sqrt{\ell^K}}\Big\}\Big|\Sigma_\ell\Big) + \frac{1}{T^{1/4}}. \tag{7.26}$$

By Doob's maximal inequality, Lemma 7.2.6, the probability in (7.26) is

$$\ll \frac{\sqrt{\ell^K}}{T}\mathbb{E}[Y_{X_\ell}|\Sigma_\ell]$$
$$\le \frac{\sqrt{\ell^K}}{T\log X_\ell}\Big(\frac{2^{\ell^K}}{2^{(\ell-1)^K}}\Big)^{1/(\ell-1)^K}\mathbb{E}\Big[\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{X_\ell}(1/2+it)|^2}{|1/2+it|^2}dt\Big|\Sigma_\ell\Big].$$

By standard properties of the conditional expectation, we can rewrite the above expectation as

$$= \mathbb{E}\Big[\mathbb{E}\Big[\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}{|1/2+it|^2}\frac{|\mathcal{S}_{X_\ell}(1/2+it)|^2}{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}dt\Big|\mathcal{F}_\ell\Big]\Big|\Sigma_\ell\Big]$$
$$= \mathbb{E}\Big[\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}{|1/2+it|^2}\mathbb{E}\Big[\frac{|\mathcal{S}_{X_\ell}(1/2+it)|^2}{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}\Big|\mathcal{F}_\ell\Big]dt\Big|\Sigma_\ell\Big],$$

where $\mathcal{F}_\ell := \sigma(\{f(p):\ p \le X_{\ell-1}\})$.

By Lemma 7.2.3 and Mertens' theorem, we get

$$\mathbb{E}\Big[\frac{|\mathcal{S}_{X_\ell}(1/2+it)|^2}{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}\Big|\mathcal{F}_\ell\Big] \ll \frac{\log X_\ell}{\log X_{\ell-1}},$$

which inserted back gives an overall bound for the probability in (7.26) of

$$\ll_\varepsilon \frac{\sqrt{\ell^K}}{T2^{(\ell-1)^K}}\mathbb{E}\bigg[\int_{-\infty}^{+\infty}\frac{|\mathcal{S}_{X_{\ell-1}}(1/2+it)|^2}{|1/2+it|^2}dt\bigg|\Sigma_\ell\bigg].$$

Finally, reminding of the definition (7.25) of the event Σ_ℓ, the above expression is $\ll_\varepsilon 1/\sqrt{T}$, since $\ell^K/(\ell-1)^K \ll_\varepsilon 1$. This is good enough for Lemma 7.5.1, since $T \geq 1$, and concludes its proof.

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