Instability of Boundary Layers with the Navier Boundary Condition

Lorenzo Quarisa⬤ and José L. Rodrigo

Communicated by D. Gerard-Varet

Abstract. We study the L^∞ stability of the Navier-Stokes equations in the half-plane with a viscosity-dependent Navier friction boundary condition around shear profiles which are linearly unstable for the Euler equation. The dependence from the viscosity is given in the Navier boundary condition as $\partial_y u = \nu^{-\gamma} u$ for some $\gamma \in \mathbb{R}$, where u is the tangential velocity. With the no-slip boundary condition, which corresponds to the limit $\gamma \to +\infty$, a celebrated result from E. Grenier (Comm. Pure Appl. Math. 53:1067–1091, 2000) provides an instability of order $\nu^{1/4}$. M. Paddick (Differ. Integral Equ. 27:893–930, 2014) proved the same result in the case $\gamma = 1/2$, furthermore improving the instability to order one. In this paper, we extend these two results to all $\gamma \in \mathbb{R}$, obtaining an instability of order ν^ϑ, where in particular $\vartheta = 0$ for $\gamma \le 1/2$ and $\vartheta = 1/4$ for $\gamma \ge 3/4$. When $\gamma \ge 1/2$, the result denies the validity of the Prandtl boundary layer expansion around the chosen shear profile.

Mathematics Subject Classification. 35Q30, 76D10.

Keywords. Boundary layers, Prandtl expansion, Inviscid limit, Navier friction boundary condition, Nonlinear instability.

Contents

1. Introduction

A central problem in mathematical fluid dynamics is the approximation of inviscid flows with low viscosity flows, especially when boundaries are present [6,13]. When a boundary is present, the theory is especially

suspectible to the type of boundary condition prescribed for the viscous flow. In this paper, we will focus on a *Navier boundary condition*, which allows slip at the boundary with a slip length depending on a power of the viscosity.

Assume that our spatial domain is two-dimensional with a flat boundary, e.g. $\mathbb{R} \times \mathbb{R}_+$ or $\mathbb{T} \times \mathbb{R}_+$, and let $\nu > 0$ be the viscosity. A viscous flow $\mathbf{u}^\nu = (u^\nu(t, x, y), v^\nu(t, x, y))$ is assumed to satisfy the *Navier-Stokes equations* with the *no-slip boundary condition*

$$\begin{cases} \partial_t \mathbf{u}^\nu + \mathbf{u}^\nu \cdot \nabla \mathbf{u}^\nu + \nabla p^\nu = \nu \Delta \mathbf{u}^\nu; \\ \nabla \cdot \mathbf{u}^\nu = 0; \\ \mathbf{u}^\nu = 0 \qquad\qquad\qquad\qquad \text{at } y = 0; \end{cases} \tag{1.1}$$

whereas an inviscid flow $\mathbf{u}^E = (u^E(t, x, y), v^E(t, x, y))$ satisfies the *Euler equations*

$$\begin{cases} \partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E + \nabla p^E = 0; \\ \nabla \cdot \mathbf{u}^E = 0; \\ v^E = 0 \qquad\qquad\qquad\qquad \text{at } y = 0. \end{cases} \tag{1.2}$$

The main obstacle to the convergence of $\mathbf{u}^\nu$ to $\mathbf{u}^E$ as $\nu \to 0$ is the difference between the boundary conditions 1.1_3 and 1.2_3.

To overcome this issue, in 1904 Prandtl [9] proposed the existence of a boundary layer with a *size* of order $\sqrt{\nu}$ where the fluid undergoes a transition from non-viscous flow to match the boundary condition $(1.1)_3$. In the boundary layer, some new equations for the flow may be derived, called the *Prandtl equations*. These are formally obtained by applying the change of variables $y \mapsto \tilde{y} := y/\sqrt{\nu}$, $(u^\nu, v^\nu) \mapsto (u^P, v^P) := (u^\nu, v^\nu/\sqrt{\nu})$ to (1.1):

$$\begin{cases} \partial_t u^P + u^P \partial_x u^P + v^P \partial_{\tilde{y}} u^P + \partial_x p^P = \partial_{\tilde{y}\tilde{y}} u^P; \\ \partial_x u^P + \partial_{\tilde{y}} v^P = 0; \\ \lim_{\tilde{y} \to +\infty} u^P = u^E \mid_{y=0}; \\ u^P = v^P = 0; \qquad\qquad\qquad\qquad \text{at } \tilde{y} = 0. \end{cases} \tag{1.3}$$

Moreover, Prandtl's model predicts that, given

$$\mathbf{u}^b = (u^b, \sqrt{\nu} v^b) := (u^P - u^E \mid_{y=0}, \sqrt{\nu} v^P)$$

then the following *boundary layer expansion* holds:

$$\mathbf{u}^\nu(t, x, y) \sim \mathbf{u}^E(t, x, y) + \mathbf{u}^b \left(t, x, \frac{y}{\sqrt{\nu}} \right) \qquad \text{as } \nu \to 0. \tag{1.4}$$

A long-standing problem is whether the above formula is mathematically valid. There are different questions which may be formulated in this context. For instance, the problem of the well-posedness of the Prandtl equations (1.3) is still open depending on the functional setting - see for instance the review [6], Section 3.4. In this paper, we will focus on the instability in time of (1.4).

Grenier [2] showed the nonlinear instability of the above expansion around a certain class of shear profiles. Namely, fix a profile $U_s \in C^\infty(\mathbb{R}_+)$, such that $\lim_{y \to +\infty} U_s(y) = U_\infty \in \mathbb{R}$. Suppose that U_s is linearly unstable for the Euler equations, meaning there exists an exponentially growing solution to the linearized Euler equations around U_s (see Definition 1). Now let $u_s(t, \tilde{y})$ be the unique evolution of U_s given by the heat equation

$$\begin{cases} \partial_t u_s(t, \tilde{y}) = \partial_{\tilde{y}\tilde{y}} u_s(t, \tilde{y}); \\ u_s(t, 0) = 0; \\ u_s(0, \tilde{y}) = U_s(\tilde{y}); \end{cases} \tag{1.5}$$

which upon substituting $\tilde{y} = y/\sqrt{\nu}$ is just the Navier-Stokes equations (1.1) in the special case of a shear flow $\mathbf{u}^\nu(t, x, y) = (u_s(t, y), 0)$.

The result proven in [2] is the following:

Theorem 1.1. *Given a smooth shear profile U_s, linearly unstable for the Euler equation, and an arbitrary integer $N > 0$, there exists a family of solutions $\{u^\nu(t,x,y)\}_{\nu>0}$ to the Navier-Stokes equations and constants $C, \delta > 0$ such that*

$$\|u_s(0, y/\sqrt{\nu}) - u^\nu(0,x,y)\|_{L^\infty} \le C\nu^N,$$

but

$$\|u_s(T^\nu, y/\sqrt{\nu}) - u^\nu(T^\nu, x, y)\|_{L^\infty} \ge \delta\nu^{1/4} \tag{1.6}$$

after a time $T^\nu \sim \sqrt{\nu} \log \nu \searrow 0$ as $\nu \to 0$.

This tells us that the boundary layer expansion (1.4) is unstable, as the Navier-Stokes solutions u_s and u^ν have the same boundary layer expansion at time $t = 0$, but they differ by an order of $\nu^{1/4}$ at time T^ν.

The above result assumes that the viscous flow satisfies the no-slip boundary condition $(1.1)_3$. It is therefore natural to ask if the instability remains when we replace the no-slip condition $(1.1)_3$ with boundary conditions which produce weaker boundary layers, such as the Navier boundary condition, which was originally proposed by Navier in 1823 [7]. This boundary condition, like the no-slip condition, forbids penetration of the fluid through the boundary, but allows a slip which is proportional to the normal derivative of the tangential velocity. In this paper, we will focus on the following viscosity-dependent condition:

$$\begin{cases} \partial_y u(t,x,0) = \nu^{-\gamma} u(t,x,0); \\ v(t,x,0) = 0. \end{cases} \tag{1.7}$$

When $\gamma = 0$, so that ν^γ - known as the *slip length* - is independent of the viscosity, Iftimie and Sueur [4] proved the validity of the following boundary layer expansion with an *amplitude* of order $\sqrt{\nu}$, which is the factor multiplying $\mathbf{u}^b$:

$$\mathbf{u}^\nu(t,x,y) = \mathbf{u}^E(t,x,y) + \sqrt{\nu}\mathbf{u}^b\left(t,x,\frac{y}{\sqrt{\nu}}\right) + O(\nu).$$

However if $\gamma > 0$, the boundary layers become significant again. Notice that in the limit $\gamma \to +\infty$ we recover the no-slip case. The corresponding appropriate boundary layer expansion, as investigated in [12], is the following for $\gamma \ge 0$:

$$\mathbf{u}^\nu(t,x,y) \sim \mathbf{u}^E(t,x,y) + \nu^{\max\{1/2-\gamma;0\}}\mathbf{u}^b\left(t,x,\frac{y}{\sqrt{\nu}}\right), \tag{1.8}$$

which corresponds to the expansion found in [4] when $\gamma = 0$. The factor $\nu^{\max\{1/2-\gamma;0\}}$ represents the amplitude of the boundary layer. Hence it makes sense to identify $\gamma = 1/2$ as the *critical exponent*, where we transition from order one amplitude ($\gamma \ge 1/2$) to a small amplitude which vanishes with the viscosity ($\gamma < 1/2$).

In [8], Paddick considers the case $\gamma = 1/2$ and proves an instability result akin to Theorem 1.1. Remarkably, the result in [8] is stronger, as a full instability of order one is obtained. This is in spite of the fact that similar techniques are used but the boundary layer is weaker, so one would expect better stability.

The main disadvantage of the exponents $\gamma \ne 1/2$ is that the evolution of the shear profile U_s will have to explicitly depend on the viscosity. Indeed, following (1.5) we must define a family of flows u_s^ν satisfying

$$\begin{cases} \partial_t u_s^\nu(t,\tilde{y}) = \partial_{\tilde{y}\tilde{y}} u_s^\nu(t,\tilde{y}); \\ \partial_{\tilde{y}} u_s^\nu(t,0) = \nu^{1/2-\gamma} u_s^\nu(t,0); \\ u_s^\nu(0,\tilde{y}) = U_s(\tilde{y}). \end{cases} \tag{1.9}$$

Because of the $\nu^{1/2-\gamma}$ factor appearing in the boundary condition $(1.9)_2$, it would be impossible for a single flow to satisfy (1.9) for all ν if $\gamma \ne 1/2$.

In the main result of this paper, Theorem 4.1, we extend Paddick's and Grenier's instability results to all exponents $\gamma \in \mathbb{R}$, while the invalidity of the boundary layer expansion (1.8) can be generalized to all $\gamma \geq 1/2$. Indeed, the initial condition $U_s(\tilde{y})$ for u_s^ν is a function of $y/\sqrt{\nu}$, so the shear flow u_s^ν can only satisfy the boundary layer expansion (1.8) at time $t = 0$ if the amplitude of the boundary layer is of order one, i.e. if $\gamma \geq 1/2$. In general, we obtain an L^∞ instability result for the Navier-Stokes equations which is of order ν^ϑ, where ϑ is a continuous and increasing function of γ given by

$$\vartheta := \begin{cases} \frac{1}{4} & \gamma \geq \frac{3}{4}; \\ \gamma - \frac{1}{2} & \frac{1}{2} < \gamma < \frac{3}{4}; \\ 0 & \gamma \leq \frac{1}{2}. \end{cases}$$

This interpolates Grenier's original result (limit for $\gamma \to +\infty$) and Paddick's for $\gamma = 1/2$. From $\gamma = 1/2$ the order of the instability decays until $\gamma = 3/4$, where it stabilizes at $\nu^{1/4}$, as in Grenier's case. The reason why this occurs is simply a consequence of the boundary layer expansion (1.8), and the use in the proof of an isotropic change of variables mapping γ to $2\gamma - 3/2$.

For $\gamma \neq 1/2$, as discussed above, we have to replace the single shear flow u_s in the statement of Theorem 1.1 with a family of viscosity-dependent shear flows $u_s^\nu(t, y/\sqrt{\nu})$, defined as the solution of (1.9). We refer to Sect. 4.1 for the details and the precise statement.

The main consequence of this choice is that Grenier's method must be complemented with some uniform-in-ν estimates on u_s^ν (Lemmas 4.2 and 4.3). Because we need these uniform bounds to hold in Sobolev spaces of arbitrarily high orders, the shear flows u_s^ν must satisfy the compatibility conditions of all orders at $(t, \tilde{y}) = (0, 0)$. This is only possible if all the derivatives of U_s vanish at $\tilde{y} = 0$, which will have to be added as an assumption. This is a heavy limitation, as it eliminates the possibility of using analytic shear flows. However, this assumption still includes shear flows that are smooth, or in an arbitrary non-analytic Gevrey class. In Sect. 3, we apply a result from [5] to show that there exist flows satisfying this assumption which are linearly unstable for the Euler equation.

We remark that there are also positive results for the validity of the formula (1.8). In [4], the validity is proven when $\gamma = 0$; in the more recent paper [11], the authors establish it for $\gamma \in (0, 1/2]$ and initial data in the Gevrey class $(2\gamma)^{-1}$, which for $\gamma = 1/2$ is just the analytic class. However, our result proves the invalidity of the boundary layer expansions only when $\gamma \geq 1/2$, and is therefore not in contradiction with these results.

We also point out that more recently E. Grenier and T. Nguyen proved [3] in a stronger result of order one instability in L^∞ for the no-slip condition. In light of this, it is likely that the order one instability can be extended for the full range $\gamma \geq 1/2$, using similar techniques. This problem is currently under scrutiny by the authors of this paper.

Organization of the paper. In Sect. 2, we gather some preliminary results on the heat equation with a mixed boundary condition, which are required in various moments of the proof of our main result. In Sect. 3, we prove that shear profiles satisfying all the assumptions of our main result do exist. Finally, in Sect. 4, we state and prove our main result.

2. Some Bounds for the Heat Equation with a Mixed Boundary Condition

In the proof of our main result Theorem 4.1, we will construct an instability using an asymptotic expansion in the viscosity ν where some of the terms satisfy some (possibly inhomogeneous) heat equations with boundary conditions involving the solution and its first derivative. Specifically, let $u_0 \in C^\infty(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ and $a \geq 0$ (which will later represent a real power of the viscosity). We consider the unique smooth and bounded solution of the problem

$$\begin{cases} \partial_t u(t, y) = \partial_{yy} u(t, y) & (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+; \\ \partial_y u(t, 0) = a u(t, 0) & y = 0; \\ u(0, y) = u_0(y) & t = 0. \end{cases} \tag{2.1}$$

In this section, we will state two types of bounds for the solutions of these problems.

1. Bounds concerning the behavior as the parameter in the boundary condition approaches the limiting cases $a \to +\infty$ and $a \to 0^+$ (Sect. 2.1);
2. Bounds establishing a controlled exponential growth in time with given inhomogeneous data satisfying the same bounds (Sect. 2.2).

We will leave most of the proofs and details to the Appendix A.

Recall the *heat kernel* on the real line

$$K(t,y) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}, \qquad t > 0, y \in \mathbb{R}. \tag{2.2}$$

Then the unique solution $u = u^a(t,y)$ to (2.1) is given by

$$u^a(t,y) = K(t,y) \star u_0^a(y), \tag{2.3}$$

where the convolution is on $\mathbb{R}$ and u_0^a is the continuous extension of u_0 to $\mathbb{R}$ such that $\partial_y u_0^a - a u_0^a$ is an odd function. This extension can be explicitly computed (see the Appendix A.1).

Notice that if $a = 0$, this is simply the even extension of u_0, which reduces (2.1) to a Neumann problem. Conversely if $a \to +\infty$, the contribution of the first derivative vanishes. Therefore, we define u_0^∞ to be the odd extension of u_0, and correspondingly $u^\infty := K \star u_0^\infty$, which is the solution to the Dirichlet problem.

2.1. Asymptotic Behavior in the Limits $a \to +\infty$ and $a \to 0$

In this section, we want to establish convergence results of the type $u^a \to u^\infty$ when $a \to +\infty$ (Proposition 2.1) and $u^a \to u^0$ as $a \to 0^+$ (Proposition 2.2). The convergence holds in L^2 as well as L^∞ based norms with all the derivatives, but only under certain assumptions on the initial condition u_0.

For the limit $a \to \infty$, we will work under the following assumption:

$$\begin{cases} \partial^k u_0(0) = 0, \\ \partial^{k+1} u_0 \in L^1(\mathbb{R}_+), \end{cases} \qquad \forall k \geq 0. \tag{2.4}$$

The first condition is equivalent to the requirement that the extension u_0^a is smooth at the origin (see the Appendix). This is crucial as it allows us to consider the derivatives of u_0^a in Sobolev spaces of all orders.

The second condition, together with $u_0 \in C_b^\infty(\mathbb{R}_+)$, implies that $u_0 \in W^{\infty,p}$ for any $p \in [1, \infty]$, in particular $u_0 \in H^s$ for all $s \geq 0$. Furthermore, since $u_0 \in W^{k+1,1}(\mathbb{R}_+)$ implies $\lim_{y \to +\infty} \partial^k u_0(y) = 0$, we deduce that

$$\lim_{y \to +\infty} u_0(y) = c \in \mathbb{R}, \qquad \lim_{y \to +\infty} \partial^k u_0(y) = 0 \,\forall k \geq 1.$$

Unfortunately, these assumptions also imply that we must exclude non-trivial profiles u_0 which are analytic on $\mathbb{R}_+$. However, we can include profiles in the Gevrey class G^ρ for all $\rho > 1$, such as $u_0(y) = e^{-y^{-(\rho-1)^{-1}}}$.

Using the explicit expression of the k-th order derivatives of u_0^a and u_0^∞ (see the Appendix A.1) we obtain the following.

Proposition 2.1. *Assuming* (2.4), *for all* $k \in \mathbb{Z}_{\geq 0}$ *and* $p \in [1, \infty]$ *we have*

$$\|u^a - u^\infty\|_{W^{k,p}} = O(a^{-1}), \qquad as\ a \to +\infty. \tag{2.5}$$

Consider now the limit $a \to 0^+$. We must replace (2.4) with the new assumption:

$$\partial^k u_0(0) = 0; \qquad \partial^k u_0 \in L^1(\mathbb{R}_+) \qquad \forall k \geq 0. \tag{2.6}$$

In other words, we additionally require that u_0 is integrable over $\mathbb{R}_+$. In particular, $\lim_{y \to +\infty} u_0(y) = 0$. This is necessary to obtain uniform convergence of u_0^a to u_0^0. In spite of this further condition, the order of convergence for L^2-based norms is only $O(a^{1/2})$, but this will suffice for our purposes.

Proposition 2.2. *Assume* (2.6). *Then for all $k \geq 0$,*

$$\|u^a - u^0\|_{H^k} = O(a^{1/2}),$$
$$\|u^a - u^0\|_{W^{k,\infty}} = O(a),$$

as $a \to 0^+$.

Proof. See the Appendix A.1.   □

Example 1. The order of convergence of $a^{1/2}$ from Proposition 2.2 is optimal, at least for $k = 0$. Indeed, suppose $u_0(y) = e^{-y}$, which is in $L^1(\mathbb{R}_+)$ with all its derivatives. Then

$$u_0^a(y) = \begin{cases} e^{-y} & y \geq 0; \\ e^y - \frac{2a}{a-1}(e^y - e^{ay}) & y < 0. \end{cases}$$

Then as $a \to 0^+$,

$$\left(\int_{-\infty}^{\infty} |u_0^a(y) - u_0^0(y)|^2 \, \mathrm{d}y \right)^{1/2} = \frac{2a}{|a-1|} \left(\frac{\frac{1}{2}(a-1)^2}{a(a+1)} \right)^{1/2} = \sqrt{2a}.$$

To be precise, we need to multiply $u_0(y) = e^{-y}$ by a smooth cut-off function χ with $\chi^{(k)}(0) = 0$ for all $k \geq 0$ and $\chi(y) = 1$ for all $y \geq 1$. Then, assumption (2.6) is fulfilled and the same estimates hold.

2.2. Bounds in Time for the Inhomogeneous Problem

Let $a, b \geq 0$, $(a, b) \neq (0, 0)$. Let $u_0, f \in H^k(\mathbb{R}_+)$ and $r \in H^k(\mathbb{R}_+ \times \mathbb{R}_+)$ for all $k \in \mathbb{Z}_{\geq 0}$. In this subsection, we will consider the following inhomogeneous problem:

$$\begin{cases} \partial_t u(t,y) = \partial_{yy} u(t,y) + r(t,y) & t \geq 0, y \geq 0 \\ au(t,0) - b\partial_y u(t,0) = f(t) & t \geq 0 \\ u(0,y) = u_0(y) & y \geq 0. \end{cases} \tag{2.7}$$

As long as the data r, f satisfy an exponential-type bound in time, then the unique smooth and bounded solution u will satisfy the same bound, without any additional growth in time. Both an L^2-based estimate (Proposition 2.3) and a pointwise estimate with exponential decay at infinity (Proposition 2.4) will be needed in the Proof of Theorem 4.1. Similar results were implicitly used in Grenier's and Paddick's works, but they were never stated explicitly.

Hereafter, we will denote with C_k an arbitrary positive constant depending on k, which may vary from line to line. These constants are always independent from t and y.

Proposition 2.3 (H^k *estimates*). *Suppose there exist $\alpha > 0$, $\beta \geq 0$ such that*

$$\|\partial_y^k r(t)\|_{L^2(\mathbb{R}_+)} + |f^{(k)}(t)| \leq C_k \frac{e^{\alpha t}}{(1+t)^\beta} \qquad \forall t \geq 0, k \in \mathbb{Z}_{\geq 0}.$$

Let $u(t,y)$ be the classical solution of (2.7). *Then, for all $k \in \mathbb{Z}_{\geq 0}$, we have*

$$\|u(t)\|_{H^k(\mathbb{R}_+)} \leq CC_k \frac{e^{\alpha t}}{(1+t)^\beta} \qquad \forall t \geq 0, \tag{2.8}$$

where C is independent from r, f or u_0, and also from a and b as long as either a or b is bounded away from 0.

Proof. See the Appendix A.2.   □

Proposition 2.4 (Pointwise estimates). *Suppose there exist $\alpha > 0$, $\beta \geq 0$, and a constant $\lambda > 0$ such that*

$$\begin{cases} |f^{(k)}(t)| \leq C_k \frac{e^{\alpha t}}{(1+t)^\beta}, \\ |\partial_y^k r(t,y)| \leq C_k \frac{e^{\alpha t}}{(1+t)^\beta} e^{-\lambda y}, \qquad \forall t \geq 0, k \in \mathbb{Z}_{\geq 0}. \\ |\partial_y^k u_0(y)| \leq C_k e^{-\lambda y}, \end{cases}$$

Let $u(t,y)$ be unique smooth and bounded solution of (2.7). *Then there exists a constant $\mu > 0$ independent from t such that*

$$|\partial_y^k u(t,y)| \leq C C_k \frac{e^{\alpha t}}{(1+t)^\beta} e^{-\mu y} \qquad \forall t \geq 0, y \geq 0, k \in \mathbb{Z}_{\geq 0},$$

where C is independent from f, r or u_0, and also from a and b as long as either a or b is bounded away from 0.

Proof. By Proposition 2.3, using Sobolev embeddings we know that

$$|u(t,0)| \leq C C_k \frac{e^{\alpha t}}{(1+t)^\beta} \qquad \forall t \geq 0.$$

We can now see $u(t,y)$ as the solution of (2.7) in the special case of the Dirichlet problem $(a,b) = (1,0)$, with a boundary condition $f(t) = u(t,0)$ satisfying the above estimate. Then u is given by $u = u_1 + u_2 + u_3$, where

$$u_1(t,y) = -2\partial_y K(t,y) \star f(t) = -2 \int_0^t \partial_y K(s,y) u(t-s,0)\,\mathrm{d}s,$$

$$u_2(t,y) = K(t,y) \star u_0^\infty(y),$$

$$u_3(t,y) = K(t,y) \star_{t,y} \tilde{r}(t,y) = \int_0^t \int_{\mathbb{R}} K(t-s, y-x) \tilde{r}(s,x)\,\mathrm{d}x\,\mathrm{d}s,$$

and u_0^∞ and $\tilde{r}$ are the odd extensions in the y variable of u_0 and r respectively. We then have

$$\left(\frac{e^{\alpha t}}{(1+t)^\beta} \right)^{-1} |\partial_y^k u_1(t,y)| \leq \int_0^t 2|\partial_y^{k+1} K(s,y)| \frac{e^{-\alpha s}}{(1-s/(1+t))^\beta}\,\mathrm{d}s$$

$$\leq C_{\alpha,\beta} \int_0^\infty 2|\partial_y^{k+1} K(s,y)| e^{-\frac{\alpha}{2}s}\,\mathrm{d}s$$

$$= C_{\alpha,\beta} \int_0^\infty \frac{p(s,y)}{s^{(3+2k)/2}} e^{-\frac{y^2}{8s}} e^{-\frac{1}{8s}\left(y^2 + 4\alpha s^2\right)}\,\mathrm{d}s$$

$$\leq C_{\alpha,\beta} q(y) e^{-\sqrt{\alpha} y/2} \leq C e^{-\sqrt{\alpha} y/3}.$$

where p and q are polynomials, and in the second-to-last step we simply used the inequality $-A^2 - B^2 \leq -2AB$. Consider now u_2 and u_3. We have

$$\partial_y^k u_2(t,y) = \partial_y^k K(t,y) \star u_0^a(y) \leq \frac{e^{\alpha t}}{(1+t)^\beta} \int_{\mathbb{R}} \partial_x^k K(t,x) e^{-\lambda|y-x|}\,\mathrm{d}x \leq C \frac{e^{\alpha t}}{(1+t)^\beta} e^{-\lambda y},$$

and finally

$$\partial_y^k u_3(t,y) := \partial_x^k K(t,y) \star_{t,y} \tilde{r}(t,y) \leq \int_{\mathbb{R}} \int_0^t \partial_x^k K(t-s,x) \frac{e^{\alpha s}}{(1+s)^\beta}\,\mathrm{d}s\, e^{-\lambda|y-x|}\,\mathrm{d}x$$

$$\leq C_{\alpha,\beta}' \frac{e^{\alpha t}}{(1+t)^\beta} \int_{\mathbb{R}} e^{-\mu x} e^{-\lambda|y-x|}\,\mathrm{d}x$$

$$\leq C_{\alpha,\beta}' e^{-\mu' y} \frac{e^{\alpha t}}{(1+t)^\beta}.$$

where $\mu' < \min\{\mu; \lambda\}$.

Since all the partial solutions u_1, u_2, u_3 satisfy the required estimate, then the full solutions $u = u_1 + u_2 + u_3$ also does. $\qquad\square$

3. A Class of Linearly Unstable Shear Flows

The goal of this section is to prove that there exists an ample class of shear flows U_s satisfying either assumption (2.4) or (2.6) which are linearly unstable for the Euler equation in the sense of Definition 1. We rely on a sufficient condition for linear instability is given by Z. Lin in [5]. We state his main result in the half-line case. First of all, for a shear flow $U_s \in C^2(\mathbb{R}_+)$ admitting an inflexion point y_0, define the **inflexion value** $U_0 := U_s(y_0)$ and the function

$$K(y) := \frac{-U_s''(y)}{U_s(y) - U_0}.$$

We say that U_s is in class $\mathcal{K}^+$ if K is a bounded and strictly positive function on $(0, +\infty)$. We stress that it is not necessary that $K(0) > 0$. This is important because a profile satisfying (2.4) or (2.6) will necessarily have $K(0) = 0$.

It is not difficult to construct profiles in the class $\mathcal{K}^+$. In essence, all that is required is that if the value of U_s increases after an inflexion point, then the second derivative U_s'' must be negative until the next inflexion point is reached if it exists, and vice versa. A simple example is the function $\sin y$ or $\cos y$.

Notice To overcome this that if $U_s \in \mathcal{K}^+$ then, while it can admit multiple inflexion points, the inflexion value U_0 must be unique. Indeed by definition the sign of U_s'' must change at any inflexion point $\bar{y}$, hence the sign of $U_s - U_0$ must also change at $\bar{y}$ if we want $K > 0$.

Secondly, if $\bar{y}$ is an inflexion point, then, assuming that $U_s \in C^3(\mathbb{R}_+)$ we have

$$K(\bar{y}) = -\lim_{y \to \bar{y}} \frac{U_s'''(y)}{U_s'(y)} > 0,$$

which implies that $U_s'(\bar{y}) \neq 0$. If the sign of this derivative is positive, then $U_s(y) - U_0 > 0$ in a right neigbhorhood of $\bar{y}$, and therefore $U_s'' < 0$. If, instead, $U_s'(\bar{y}) < 0$, then $U_s(y) - U_0 < 0$ and $U_s'' > 0$ in a right neighborhood of $\bar{y}$. These signs cannot change until the next inflection point (if it exists at all).

The following result is a restatement of Theorem 1.2 from [5] for shear profiles defined on the half-line. It is similar to Theorem 1.5(i) in [5], which covers the case of the full line. This version of the statement can be retrieved in [8], Theorem 4.2, however here we have explicitly removed the requirement that $K(0) > 0$. The proof presented below focuses on clarifying why this requirement can be removed.

Theorem 3.1. *Let $U_s \in C^2(\mathbb{R}_+)$, $U(y) \to U_\infty \in \mathbb{R}$ as $y \to +\infty$, and assume $U(y)$ takes the value U_∞ at most a finite amount of times. Suppose that $U_s \in \mathcal{K}^+$ and $\lim_{y \to +\infty} K(y) = 0$. If the operator $-\partial_{yy} - K$ on $H_0^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)$ has a strictly negative eigenvalue, then U_s is linearly unstable for the Euler equation.*

Proof. The proof is the same as Theorem 1.5(i) in [5], except that for each $n \in \mathbb{Z}_{\geq 0}$ we consider the interval $I_n = [n^{-1}, n]$, on which K is strictly positive, taking n large enough so that I_n contains an inflexion point of U_s. $\qquad\square$

The next result is a generalization of the argument used in [8] to prove that the profiles

$$u_\delta(y) = \arctan(y - \delta) + c$$

for $\delta, c \in \mathbb{R}$ satisfy the assumptions of Theorem 3.1. Of course, the above family of profiles in unsuitable to us, as they do not satisfy (2.4) or (2.6).

Proposition 3.2. *Suppose that $U_s \in \mathcal{K}^+$ and U_s satisfies assumption (2.4) or (2.6). Let $U_\infty := \lim_{y \to +\infty} U_s$ and suppose that U_s takes the value U_∞ at most a finite amount of times. Then U_s satisfies all the assumptions of Theorem 3.1, and is therefore linearly unstable for the Euler equation.*

Proof. By (2.4) or (2.6), we know that all the derivatives of U_s are integrable over $\mathbb{R}_+$ and vanish at infinity.

The operator $-\partial_{yy} - K^2$ having a strictly negative eigenvalue is equivalent to the quadratic form

$$Q(\phi) := \int_{\mathbb{R}_+} \left(|\phi'|^2 - K|\phi|^2 \right), \qquad \phi \in H_0^1(\mathbb{R}_+),$$

taking a negative value for some function ϕ. We will construct such a function from the profile U_s.

1. First of all, define for all $\eta > 0$

$$U_{s,\eta}(y) := U_s(y + y_0 - \eta) - U_0, \qquad y \in [\eta - y_0, +\infty).$$

This implies that $U_{s,\eta}(y)$ has an inflexion point at $y = \eta$, and the inflexion value is $U_{s,\eta}(\eta) = 0$.

2. Next, define the functions

$$w_\eta^n := \begin{cases} 0 & y \leq \eta \\ U_{s,\eta}\chi(y/n) & y \geq \eta, \end{cases}$$

where χ is a smooth cut-off function supported in $[0,2]$, with $\chi = 1$ in $[0,1]$. Then $w_\eta^n \in H_0^1(\mathbb{R}_+)$. Since $K \in L^\infty(\mathbb{R}_+)$, and by (2.4) or (2.6) all the derivatives of U_s are integrable over $\mathbb{R}_+$ and vanish at infinity, we have

$$\lim_{n \to +\infty} Q(w_\eta^n) = \int_\eta^{+\infty} \left(|U_{s,\eta}'|^2 - K|U_{s,\eta}|^2 \right) =: Q(\eta).$$

Let us show that $Q(y_0) = 0$.

$$\begin{aligned}
Q(y_0) &= \int_{y_0}^\infty \left(|U_s'(y)|^2 + \frac{U_s''(y)}{U_s - U_0}|U_s(y) - U_0|^2 \right) \, \mathrm{d}y \\
&= \int_{y_0}^\infty \left(|U_s'(y)|^2 + U_s''(y)(U_s(y) - U_0) \right) \, \mathrm{d}y \\
&= -U_0 \int_{y_0}^\infty U_s'' + [U_s U_s']_{y_0}^\infty \\
&= (-U_0 + U_\infty) \lim_{y \to +\infty} U_s'(y) = 0.
\end{aligned}$$

If we compute the derivative of Q as a real variable function, we obtain that for any $\eta > 0$,

$$Q'(\eta) = \int_\eta^\infty \left(-2U_{s,\eta}'' U_{s,\eta} + 2K U_{s,\eta}' U_{s,\eta} \right) - |U_{s,\eta}'(\eta)|^2.$$

In particular,

$$Q'(y_0) = -4 \int_{y_0}^\infty U_s' U_s'' - |U_s'(y_0)|^2 = -2[(U_s')^2]_{y_0}^\infty - (U_s'(y_0))^2 = (U_s'(y_0))^2 > 0.$$

3. We know that $Q(y_0) = 0$ and $Q'(y_0) > 0$. Therefore, for some $\eta_0 \in (0, y_0)$ we must have $Q(\eta_0) < 0$. Hence, for some $n \in \mathbb{Z}_{\geq 0}$ we have $Q(w_{\eta_0}^n) < 0$. This concludes the proof.

$\square$

Example 2. An explicit example of a flow satisfying the assumptions of Proposition 3.2 is given by

$$U_s(y) = e^{-y^{-\frac{1}{\rho-1}}}, \qquad \rho > 1.$$

Indeed, its first three derivatives are

$$U_s'(y) = \frac{1}{\rho - 1} y^{-\frac{\rho}{\rho-1}} e^{-y^{-\frac{1}{\rho-1}}},$$

$$U_s''(y) = \frac{1 - \rho y^{\frac{1}{\rho-1}}}{(\rho - 1)^2 y^{\frac{2\rho}{\rho-1}}} e^{-y^{-\frac{1}{\rho-1}}},$$
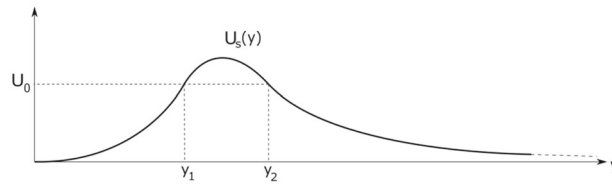
Fig. 1. A linearly unstable shear profile satisfying assumption (2.6)

$$U_s'''(y) = \frac{1 + 2\rho^2 y^{\frac{2}{\rho-1}} - \rho y^{\frac{1}{\rho-1}} \left(3 + y^{\frac{1}{\rho-1}} \right)}{(\rho - 1)^3 y^{\frac{3\rho}{\rho-1}}} e^{-y^{-\frac{1}{\rho-1}}},$$

so that $U_s' > 0$, there is a unique inflexion point at $y = y_0 := \rho^{1-\rho}$, with $U_s'' > 0$ for $y \in (0, y_0)$ and $U_s'' < 0$ for $y > y_0$, and $U_s'''(y_0) = -\dfrac{\rho^{3\rho-1}}{e^\rho (\rho - 1)^3} < 0$. This implies that $U_s \in \mathcal{K}^+$. Indeed $K(y) > 0$ for $y \neq 0, y_0$. Moreover, by De L'Hopital's rule we have

$$\lim_{y \to y_0} K(y) = -\lim_{y \to y_0} \frac{U_s^{(3)}(y)}{U_s'(y)} = -\frac{U_s^{(3)}(y_0)}{U_s'(y_0)} > 0.$$

By (4.3), we have $\lim_{y \to +\infty} U_s(y) = U_\infty \in \mathbb{R}$, and $\lim_{y \to +\infty} K(y) = 0$. Since $U_s' > 0$, it never actually takes the value U_∞. One can argue by induction that all the derivatives are bounded, integrable and vanish at $y = 0$. Thus, this is a linearly unstable shear flow satisfying assumption (2.4).

Remark 1. The above flow belongs to the Gevrey class $G^\rho(\mathbb{R}_+)$.

The flow from the previous example does not satisfy assumption (2.6), as $U_\infty \neq 0$. To construct a linearly unstable flow with $U_s(0) = U_\infty = 0$, we need at least two inflexion points y_1, y_2, with $U_s(y_1) = U_s(y_2) = U_0$. One can then easily construct a smooth profile U_s with $U_s > 0$ on $(0, \infty)$ satisfying all the requirements by requiring the following:

1. $U_s' > 0, U_s'' > 0$ on $(0, y_1)$;
2. $U_s'' < 0$ on (y_1, y_2), and U_s' changes its sign somewhere in (y_1, y_2),
3. $U_s' < 0, U_s'' > 0$ on $(y_2, +\infty)$. (Fig. 1)

4. Grenier's Instability with a Viscosity-dependent Navier Boundary Condition

Let $\gamma \in \mathbb{R}$. Let $U_s : \mathbb{R}_+ \to \mathbb{R}$ be a smooth shear flow, which we will later require to be linearly unstable for the Euler equation in the sense of Definition 1 (see Sect. 4.3). Consider the following Navier-Stokes equations with the Navier boundary condition:

$$\begin{cases} \partial_t \mathbf{u}^\nu + \mathbf{u}^\nu \cdot \nabla \mathbf{u}^\nu + \nabla p^\nu = \nu \Delta \mathbf{u}^\nu; \\ \nabla \cdot \mathbf{u}^\nu = 0; \\ \partial_y u^\nu = \nu^{-\gamma} u^\nu & \text{at } y = 0; \\ v^\nu = 0 & \text{at } y = 0. \end{cases} \tag{4.1}$$

Let $\tilde{y} = y/\sqrt{\nu}$, and let $u_s^\nu = u_s^\nu(t, \tilde{y})$ be the solution to the heat equation

$$\begin{cases} \partial_t u_s^\nu(t, \tilde{y}) = \partial_{\tilde{y}\tilde{y}} u_s^\nu(t, \tilde{y}) & (t, \tilde{y}) \in \mathbb{R}_+ \times \mathbb{R}_+; \\ \partial_Y u_s^\nu(t, 0) = \nu^{1/2 - \gamma} u_s^\nu(t, 0) & t \in \mathbb{R}_+; \\ u_s^\nu(0, \tilde{y}) = U_s(\tilde{y}) & \tilde{y} \in \mathbb{R}_+. \end{cases} \tag{4.2}$$

We will also use u_s^ν to denote the shear flow $(u_s^\nu, 0)$ which is therefore a solution to (4.1) in the original variables (t, x, y).

If we take the limit as $\nu \to 0$ in (4.2), we expect convergence of u_s^ν to the solution of the Dirichlet or Neumann problem for the heat equation, respectively if $\gamma > 1/2$ or $\gamma < 1/2$. In order to establish our main result, we will need the convergence results Proposition 2.1 and Proposition 2.2 respectively. For those to hold, we must require the following assumption on the profile U_s, depending on the sign of $\gamma - 1/2$:

$$\begin{cases} \lim_{y \to +\infty} U_s(y) = U_\infty \in \mathbb{R}; \\ U_s^{(k)}(0) = 0 \; \forall k \in \mathbb{Z}_{\geq 0}; \\ U_s^{(k)} \in L^1(\mathbb{R}_+) \; \forall k \geq 1 \qquad \text{if } \gamma > 1/2; \\ U_s^{(k)} \in L^1(\mathbb{R}_+) \; \forall k \geq 0 \qquad \text{if } \gamma < 1/2. \end{cases} \tag{4.3}$$

Notice that under this assumption, since all the derivatives of U_s vanish at the origin, U_s satisfies the compatibility conditions of (4.2) for all orders, and for all $\nu > 0$. Thus u_s^ν is smooth up to the boundary.

In Sect. 3 we confirmed the existence of profiles satisfying (4.3) which are linearly unstable for the Euler equation. These profiles cannot be analytic, but they can be found in the Gevrey classes G^ρ for any $\rho > 1$ (see Example 2). When $\rho \leq 2$, for these flows the Prandtl equation is well-posed (see [1]). Hence, by the next result, instability of the boundary layer expansion for the Navier boundary condition can occur even when the Prandtl equation is well-posed, in line with the no-slip case.

Finally, we remark that as the case $\gamma = 1/2$ has already been treated in [8], throughout this paper we will focus on the case $\gamma \neq 1/2$.

We are now ready to state the main result of this paper.

Theorem 4.1. *For $\nu > 0$, let $u_s^\nu = (u_s^\nu, 0) \in C^\infty(\mathbb{R}_+)$ be a family of shear flows defined by (4.2). Then for any $N \in \mathbb{Z}_{\geq 1}$ there exists a family of solutions $\mathbf{u}^\nu = (u^\nu, v^\nu)$ to (4.1), constants $C, \delta > 0$ and times $\tilde{T}^\nu \searrow 0$ such that for all $\nu > 0$,*

$$\|\mathbf{u}^\nu(0, x, y) - u_s^\nu(0, y/\sqrt{\nu})\|_{L^\infty} \leq C\nu^N \tag{4.4}$$

$$\|\mathbf{u}^\nu(\tilde{T}^\nu, x, y) - u_s^\nu(\tilde{T}^\nu, y/\sqrt{\nu})\|_{L^\infty} \geq \delta\nu^\vartheta, \tag{4.5}$$

where ϑ is a continuous and increasing function of γ given by

$$\vartheta := \begin{cases} \frac{1}{4} & \gamma \geq \frac{3}{4}; \\ \gamma - \frac{1}{2} & \frac{1}{2} < \gamma < \frac{3}{4}; \\ 0 & \gamma \leq \frac{1}{2}. \end{cases} \tag{4.6}$$

Moreover, for all $s > 2\vartheta + 1$, we have

$$\|\mathbf{u}^\nu(\tilde{T}^\nu, x, y) - u_s^\nu(\tilde{T}^\nu, y/\sqrt{\nu})\|_{\dot{H}^s} \to +\infty. \tag{4.7}$$

Notice that $(t, x, y) \mapsto u_s^\nu(t, y/\sqrt{\nu})$ satisfies (4.1). This shows that the Navier-Stokes equations (4.1) are unstable around the family of shear flows u_s^ν. When $\gamma \geq 1/2$, the above result also shows the instability of the boundary layer expansion

$$\mathbf{u}^\nu(t, x, y) \sim \mathbf{u}^E(t, x, y) + \mathbf{u}^b\left(t, x, \frac{y}{\sqrt{\nu}}\right) \qquad \text{as } \nu \to 0, \tag{4.8}$$

with $\lim_{\tilde{y} \to +\infty} \mathbf{u}^b(t, x, \tilde{y}) = 0$. Notice that such a boundary layer expansion is necessarily unique. Indeed, suppose there was another expansion for $\mathbf{u}^\nu$ given by $\tilde{\mathbf{u}}^E, \tilde{\mathbf{u}}^b$, so that

$$\mathbf{u}^E(t, x, y) + \mathbf{u}^b\left(t, x, \frac{y}{\sqrt{\nu}}\right) \sim \tilde{\mathbf{u}}^E(t, x, y) + \tilde{\mathbf{u}}^b\left(t, x, \frac{y}{\sqrt{\nu}}\right).$$

Taking the limit $\nu \to 0$ pointwise, we obtain $\mathbf{u}^E = \tilde{\mathbf{u}}^E$, and hence $\mathbf{u}^b = \tilde{\mathbf{u}}^b$. In this case, the inequality (4.4) tells us that at time $t = 0$ the boundary layer expansion (4.8) of $\mathbf{u}^\nu$ and u_s^ν must coincide. However, (4.5) tells us that at time $\tilde{T}^\nu$ the boundary layer expansions diverge by at least $O(\nu^\vartheta)$.

On the other hand, when $\gamma < 1/2$, Theorem 4.1 does not yield any information about boundary layer expansions. Indeed, the expected boundary layer expansion would be

$$\mathbf{u}^\nu(t,x,y) \sim \mathbf{u}^E(t,x,y) + \nu^{\min\{1/2-\gamma;1/2\}}\mathbf{u}^b\left(t,x,\frac{y}{\sqrt{\nu}}\right).$$

However, the shear flows u_s^ν cannot satisfy the above formula at $t = 0$, as $u_s^\nu \mid_{t=0} = U_s(y/\sqrt{\nu})$, which appears with a coefficient of order one with respect to the viscosity. This is also why our result is not in contradiction with the result of boundary layer expansion validity by Iftimie and Sueur [4] when $\gamma = 0$ (viscosity-independent slip length), or with the result of convergence of Navier-Stokes to Euler by Paddick [8] which establishes L^2 convergence of order $\nu^{\frac{1-\gamma}{2}}$ for all $\gamma < 1$.

4.1. General Strategy

As the case $\gamma = 1/2$ has already been treated in [8], we will focus our proof on the two cases $\gamma > 1/2$ and $\gamma < 1/2$. As in [2] and [8], we start by applying the isotropic scaling

$$(t,x,y) \mapsto \left(\frac{t}{\sqrt{\nu}}, \frac{x}{\sqrt{\nu}}, \frac{y}{\sqrt{\nu}}\right).$$

From now on, we will work exclusively with the new variables, which we will still denote with (t, x, y). After the scaling, 4.1 is transformed into the following:

$$\begin{cases} \partial_t \mathbf{u}^\nu + \mathbf{u}^\nu \cdot \nabla \mathbf{u}^\nu + \nabla p^\nu = \sqrt{\nu}\Delta \mathbf{u}^\nu; \\ \nabla \cdot \mathbf{u}^\nu = 0; \\ \partial_y u^\nu = \nu^{1/2-\gamma} u^\nu & \text{at } y = 0; \\ v^\nu = 0 & \text{at } y = 0. \end{cases} \quad (4.9)$$

Notice that the Navier-Stokes equations are preserved, except the new viscosity is $\sqrt{\nu}$, while in the boundary condition, γ becomes $\gamma - 1/2$ (or $2\gamma - 1$ with respect to $\sqrt{\nu}$). For instance, the exponent $\gamma = 1/2$, considered by Paddick, is trasformed into $\gamma = 0$. If $\gamma > 1/2$, then the boundary condition converges to the no-slip condition $u^\nu = 0$ as $\nu \to 0$, whereas it converges to the condition $\partial_y u^\nu = 0$ if $\gamma < 1/2$. Because of this, we will consider the two ranges of γ as separate cases in Sects. 4.2 and 4.7.

The new boundary layer expansion for $\mathbf{u}^\nu$, as per 1.8, becomes

$$\mathbf{u}^\nu(t,x,y) \sim \mathbf{u}^E(t,x,y) + \nu^\zeta \mathbf{u}^b(t,x,Y), \quad (4.10)$$

where $Y := y/\nu^{1/4}$, and ζ is a non-negative number representing the amplitude of the boundary layer, defined as

$$\zeta := \frac{1}{4} - \vartheta = \begin{cases} 0 & \gamma \geq \frac{3}{4}; \\ \frac{3}{4} - \gamma & \frac{1}{2} < \gamma < \frac{3}{4}; \\ \frac{1}{4} & \gamma \leq \frac{1}{2}; \end{cases} \quad (4.11)$$

Thus the critical exponent becomes $\gamma = 3/4$. The Euler equation is invariated, as are L^∞ norms, whereas spatial L^2 norms are increased by a factor of $\nu^{-1/2}$. The shear flows are written as $u_s = u_s(\sqrt{\nu}t, y)$ in the new coordinates, which means that their dependence from ν is now smooth.

An **approximate solution** to 4.9 is a solution up to some error function $\mathbf{R}^{\mathrm{app}}$, which can be made arbitrarily small. Taking inspiration from the boundary layer expansion 4.10, we will construct our approximate solution as

$$(u^{\mathrm{app}}, v^{\mathrm{app}})(t,x,y) = (u_s^\nu, 0)(\sqrt{\nu}t, y) + (u^I, v^I)(t,x,y) + \nu^\zeta(u^b, \nu^{1/4}v^b)(t,x,Y). \quad (4.12)$$

In first order, $u_s^\nu(\sqrt{\nu}t) + \mathbf{u}^I(t)$ will therefore satisfy the Euler equations, while $\mathbf{u}^b = (u^b(t,x,Y), \nu^{1/4}v^b(t,x,Y))$ will satisfy a Stokes equation. The boundary conditions will be chosen appropriately so that

$\mathbf{u}^{\text{app}}(t, x, y)$ will satisfy the Navier boundary condition up to a small error:

$$\begin{cases} \partial_y u^{\text{app}} = \nu^{1/2 - \gamma} u^{\text{app}} + r_1^{\text{app}}; \\ v^{\text{app}} = r_2^{\text{app}}; \end{cases} \qquad \text{at } y = 0.$$

A heuristic justification for the value of ζ can be given as follows. Let $u^E(t) := u^I(t) + u_s^\nu(\sqrt{\nu} t)$. Plugging the ansatz 4.12 into 4.9$_3$ and multipying by $\nu^{\gamma - 1/2}$, we obtain

$$\nu^{\gamma - 1/2} \partial_y u^E + \nu^{\gamma + \zeta - 3/4} \partial_Y u^b - \nu^\zeta u^b = u^E,$$

up to a small error which can be assumed to be smaller than all the other terms. As $\nu \to 0$, for the above relation to hold, the left-hand side must be asymptotic to a constant. But this is only possible if at least one between $\gamma + \zeta - 3/4$ and ζ is zero, and the other is greater or equal to zero, which leads to ζ being defined as in 4.11.

Fix $\delta > 0$. We first construct an approximate solution to 4.9 $\mathbf{u}^{\text{app}}$, a bounded subset $\Omega_A \subset \mathbb{R}_+^2$ and times T^ν with $\tilde{T}^\nu := \sqrt{\nu} T^\nu \to 0$ such that

$$\| (\mathbf{u}^{\text{app}} - u_s^\nu) |_{t=0} \|_{L^2} \leq C \nu^N; \tag{4.13}$$

$$\| \mathbf{u}^{\text{app}}(T^\nu) - u_s^\nu(\sqrt{\nu} T^\nu) \|_{L^2(\Omega_A)} \geq 2\delta \nu^\vartheta. \tag{4.14}$$

Starting from $\mathbf{u}^{\text{app}}$, we then construct an exact solution $\mathbf{u}^\nu$ to 4.9 such that

$$\mathbf{u}^\nu(0) = \mathbf{u}^{\text{app}}(0); \tag{4.15}$$

$$\| (\mathbf{u}^\nu - \mathbf{u}^{\text{app}}) |_{t=T^\nu} \|_{L^2} \leq \delta \nu^\vartheta; \tag{4.16}$$

where ϑ is defined as in 4.6. Once we have these, then

$$\begin{aligned} \| \mathbf{u}^\nu(T^\nu) - u_s^\nu(\sqrt{\nu} T^\nu) \|_{L^\infty} &\geq \| \mathbf{u}^\nu(T^\nu) - u_s^\nu(\sqrt{\nu} T^\nu) \|_{L^\infty(\Omega_A)} \\ &\gtrsim \| \mathbf{u}^{\text{app}}(T^\nu) - u_s^\nu(\sqrt{\nu} T^\nu) \|_{L^2(\Omega_A)} - \| (\mathbf{u}^\nu - \mathbf{u}^{\text{app}})(T^\nu) \|_{L^2} \\ &\geq 2\delta \nu^\vartheta - \delta \nu^\vartheta = \delta \nu^\vartheta. \end{aligned}$$

The estimates at time $t = 0$, 4.15 and 4.13, will hold by construction. The estimate 4.16 will be deduced from energy estimates, as in Sects. 4.6 and 4.14 will follow from the construction of $\mathbf{u}^{\text{app}}$. Note that 4.14 and 4.16 do not imply the instability of Theorem 4.1 in the L^2 norms, as after scaling back to the original variables, we lose a $\sqrt{\nu}$ factor.

4.2. Structure of the Approximate Solution

From now on, we will focus on the case $\gamma > 1/2$ and refer to Sect. 4.7 for the modifications needed to treat the case $\gamma < 1/2$. We want to construct $\mathbf{u}^\nu$ starting from an approximate solution $\mathbf{u}^{\text{app}}$. This is built according to 4.12, where $\mathbf{u}^I = (u^I, v^I)$ is constructed so that $u_s^\nu + \mathbf{u}^I$ satisfies the Navier-Stokes equations with an error $\mathbf{R}^{\text{app}}$ and the slip boundary condition, and $\mathbf{u}^b = (u^b, \nu^{1/4} v^b)$ corrects the boundary condition. Ultimately, $\mathbf{u}^{\text{app}}(t) - u_s^\nu(\sqrt{\nu} t)$ will satisfy the Navier-Stokes equations 4.9 with the Navier boundary condition up to a small error $\mathbf{r}^{\text{app}}$.

The standard procedure for the construction is to expand the terms $\mathbf{u}^I$ and $\mathbf{u}^b$ as power sums with respect to the viscosity (4.19). In this subsection we will derive the equations satisfied by the terms in these sums, and prove that all the terms appearing in these equations are bounded with respect to the viscosity (Lemmas 4.2 and 4.3).

Let $n \in \mathbb{Z}$, $n \geq 2$ be such that

$$\begin{cases} 2^{-n} \leq \gamma - \frac{3}{4} & \text{if } \gamma \geq \frac{3}{4}; \\ 2^{-n} \leq \gamma - \frac{1}{2} & \text{if } \frac{1}{2} < \gamma < \frac{3}{4}; \end{cases} \tag{4.17}$$

Note that in each case, $2^{-n} \leq \vartheta$.

Denote $\mathbf{w}(t) := \mathbf{u}^{\mathrm{app}}(t) - u_s^\nu(\sqrt{\nu}t)$. Constructing u^{app} is then equivalent to constructing $\mathbf{w}$, which must satisfy the equations

$$\begin{cases} \partial_t \mathbf{w} + (U_s \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)U_s + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p = \sqrt{\nu}\Delta\mathbf{w} + \nu^{2^{-n}}S\mathbf{w}; \\ \nabla \cdot \mathbf{w} = 0; \\ \partial_y \mathbf{w} \cdot \tau = \nu^{1/2-\gamma}\mathbf{w} \cdot \tau; \\ \mathbf{w} \cdot n = 0; \end{cases} \qquad (4.18)$$

where

$$S\mathbf{w}(t) := \frac{U_s - u_s^\nu(\sqrt{\nu}t)}{\nu^{2^{-n}}} \cdot \nabla\mathbf{w}(t) + \mathbf{w}(t) \cdot \nabla\left(\frac{U_s - u_s^\nu(\sqrt{\nu}t)}{\nu^{2^{-n}}}\right).$$

The reason behind this definition is that as we will prove in Lemma 4.2, $S\mathbf{w} = O(1)$ as $\nu \to 0$.

We are going to implement the ansatz 4.12 for the construction of $\mathbf{u}^{\mathrm{app}}$. The term $\mathbf{u}^I$ will be constructed so that it solves 4.18, but without the Navier boundary condition 4.18₃. This way, 4.18 reduces to a linearized Euler equation around U_s in first order approximation. The term $\mathbf{u}^b$ will be constructed so that $\mathbf{u}^I + \nu^\zeta \mathbf{u}^b$ fully solves 4.18, correcting the boundary condition. Since $\mathbf{u}^b$ is a function of $\left(t, x, y/\nu^{1/4}\right)$, it will satisfy a Stokes equation with a Dirichlet, Neumann or mixed boundary condition, depending on the value of γ.

Of course, the approximate solution $\mathbf{u}^{\mathrm{app}}$ we are constructing also needs to satisfy 4.14. To achieve this, in general, the function $\mathbf{w}$ will not satisfy 4.18 exactly but will leave a remainder $\mathbf{R}^{\mathrm{app}}$, which needs to be small enough so that 4.16 holds by energy estimates.

Choose $M \in \mathbb{N}$, which may be arbitrarily high. Since we want $(\mathbf{u}^{\mathrm{app}} - u_s^\nu)\,|_{t=0} = O(\nu^N)$ as $\nu \to 0$, we will use the following ansatz:

$$\mathbf{u}^I = \nu^N \sum_{j=0}^M \nu^{j2^{-n}}\mathbf{u}_j^I; \qquad \mathbf{u}^b = \nu^N \sum_{j=0}^M \nu^{j2^{-n}}\mathbf{u}_j^b. \qquad (4.19)$$

This, by the definition of n, ensures that the terms of order $\nu^{\gamma-1/2}$ and $\nu^{\gamma-3/4}$ can be moved to a higher order, so that they only appear in the remainder of the respective equations. Recall that $\mathbf{u}^b$ is then multiplied by a factor ν^ζ where $\zeta = \max\{3/4 - \gamma; 0\}$.

The construction of $\mathbf{w}$ will start from $\mathbf{u}_0^I$, which exhibits the required estimates, but does not solve the required equation. However, from $\mathbf{u}_0^I$ we can construct an approximate solution $\mathbf{u}^I$ of 4.18 by adding lower order terms in ν. On top of that we need a boundary term $\mathbf{u}^b$, which corrects the boundary condition so that the Navier boundary condition is satisfied. Such a term is a function of $\left(t, x, y/\nu^{1/4}\right)$, therefore its L^2 norm scales as $\sim \nu^{1/8}$. As a result, 4.14 will hold:

$$\|\mathbf{u}^{\mathrm{app}}(T^\nu) - u_s(\sqrt{\nu}T^\nu)\|_{L^2(\Omega_A)} \geq$$

$$\geq \nu^N \left(\|\mathbf{u}_0^I\|_{L^2(A)} - \sum_{j=1}^M \nu^{j2^{-n}}\|\mathbf{u}_j^I\|_{L^2} - \sum_{j=0}^M \nu^{j2^{-n}}\|\mathbf{u}_j^b\|_{L^2(y)}\right)$$

$$\geq \nu^N \left(\|\mathbf{u}_0^I\|_{L^2(A)} - \nu^{2^{-n}}\sum_{j=1}^M \nu^{(j-1)2^{-n}}\|\mathbf{u}_j^I\|_{L^2} - \nu^{1/8}\sum_{j=0}^M \nu^{j2^{-n}}\|\mathbf{u}_j^b\|_{L^2(Y)}\right)$$

$$\gtrsim \frac{\nu^N}{2}\|\mathbf{u}_0^I\|_{L^2(A)} \qquad \text{as } \nu \to 0.$$

We can now study the equations satisfied by $\mathbf{u}_j^I$ and $\mathbf{u}_j^b$. Each of the equations that follow should be paired with an initial condition $\mathbf{u}_{j,0}^I(x, y)$ or $\mathbf{u}_{j,0}^b(x, Y)$. Because we are constructing an approximate solution, we can choose the initial condition arbitrarily, as long as it satisfies the following conditions:

- it is compatible with the boundary condition and remainder;

- together with its derivatives, it is in L^2 and decays exponentially at infinity: in other words,

$$|\partial_x^\ell \partial_Y^k \mathbf{u}_{j,0}^I(x,y)| \leq |g_{j,k,\ell}^I(x)| e^{-\lambda_j y},$$
$$|\partial_x^\ell \partial_Y^k \mathbf{u}_{j,0}^b(x,Y)| \leq |g_{j,k,\ell}^b(x)| e^{-\lambda_j Y}, \tag{4.20}$$

for some $\lambda_j > 0, g_{j,k,\ell}^* \in L^2(\mathbb{R})$.

The equations satisfied by $\mathbf{u}_j^I$ involve all the terms of order in the interval $N + [j2^{-n}, (j+1)2^{-n})$. We obtain the following inhomogeneous linearized Euler equations:

$$\begin{cases} \partial_t \mathbf{u}_j^I + U_s \cdot \nabla \mathbf{u}_j^I + \mathbf{u}_j^I \cdot \nabla U_s + \nabla p_j^I = \mathbf{R}_j^I; \\ \nabla \cdot \mathbf{u}_j^I = 0; \\ v_j^I = -\nu^{a+1/4-2^{-n}} v_{j-1}^b; \end{cases} \tag{4.21}$$

where

$$\mathbf{R}_j^I = S\mathbf{u}_{j-1}^I + \Delta \mathbf{u}_{j-2^{n-1}}^I + \sum_{j_1+j_2=j-2^n N} \mathbf{u}_{j_1}^I \cdot \nabla \mathbf{u}_{j_2}^I. \tag{4.22}$$

Notice that $\zeta + 1/4 - 2^{-n} \geq \zeta \geq 0$, and $0 \leq \vartheta - 2^{-n} < 2^{-n}$. Thus $u_s + \mathbf{u}^I$ satisfies the Navier-Stokes equations with the slip boundary condition 4.21_3 and an error $\mathbf{R}^I$ consisting of all the terms of order greater or equal to $N + (M+1)2^{-n}$. Therefore,

$$\mathbf{R}^I = \nu^N \sum_{j \geq M+1} \nu^{j2^{-n}} \mathbf{R}_j^I. \tag{4.23}$$

Notice that the above sum is actually finite, as $\mathbf{R}_j^I = 0$ for $j > M + 2^n N$.

The equations satisfied by $\mathbf{u}_j^b = (u_j^b, \nu^{1/4} v_j^b)$ involve all the terms where u_j^b is of order between $N + \zeta + j2^{-n}$ and $N + \zeta + (j+1)2^{-n}$.

$$\begin{cases} \partial_t \mathbf{u}_j^b - \partial_{YY} \mathbf{u}_j^b + \nabla_{x,Y} p_j^b = \mathbf{R}_j^b; \\ \nabla_{x,Y} \cdot \mathbf{u}_j^b = 0; \\ \text{B.C.}(u_j^b, \gamma); \\ \lim_{Y \to +\infty} v_j^b = 0. \end{cases} \tag{4.24}$$

where B.C.(u_j^b, γ) is the appropriate boundary condition on u_j^b, depending on γ, which we will derive in the next subsection, and

$$\mathbf{R}_j^b = \left[\frac{u_s^\nu}{\nu^{2^{-n}}} \cdot \nabla \mathbf{u}_{j-1}^b + \mathbf{u}_{j-1}^b \cdot \nabla \left(\frac{u_s^\nu}{\nu^{2^{-n}}} \right) \right] - \partial_{xx} \mathbf{u}_b^{j-2^{n-1}} + \nu^\zeta \sum_{k+\ell=j-2^n N} \mathbf{u}_k^b \cdot \nabla \mathbf{u}_\ell^b$$
$$+ \sum_{j_1+j_2=j-2^n N} \left(u_{j_1}^I \partial_x \mathbf{u}_{j_2}^b + u_{j_1}^I \partial_y \mathbf{u}_{j_2+2^{n-2}}^b + \mathbf{u}_{j_2}^b \cdot \nabla \mathbf{u}_{j_1}^I \right) \tag{4.25}$$

First notice that all the terms in the above equation appear with index strictly smaller than j. A special remark must be given for the term containing $\partial_y \mathbf{u}_{j_2+2^{n-2}}^b$. The extra 2^{n-2} indices arise to compensate for the differentiation by y causing a loss of a $\nu^{1/4}$ factor. However, as $j_2 \leq j - 2^n N$, we know that

$$j_2 + 2^{n-2} \leq j + 2^{n-2}(1 - 4N) < j,$$

which holds since $N \geq 1$.

Secondly, v_j^b can be derived from u_j^b using the divergence free condition 4.24_2:

$$v_j^b(t,x,Y) = -\int_Y^{+\infty} \partial_x u_j^b(t,x,Z)\,\mathrm{d}Z; \tag{4.26}$$

using this expression, the condition at infinity 4.24_4 is automatically satisfied.

Thus $u_s + \mathbf{u}^I + \nu^\zeta \mathbf{u}^b$ satisfies the Navier-Stokes equations with error $\mathbf{R}^{\mathrm{app}} = \mathbf{R}^I + \nu^\zeta \mathbf{R}^b$, where $\mathbf{R}^b$ consists of all the terms of order greater or equal to $N + a + (M+1)2^{-n}$:

$$\mathbf{R}^b = \nu^N \sum_{j \geq M+1} \nu^{j2^{-n}} \mathbf{R}_j^b. \tag{4.27}$$

As with $\mathbf{R}^I$, the sum is actually finite.

As in Proposition 2.3, we can then estimate the value of $\mathbf{u}_j^b$ in terms of $\mathbf{R}_j^b$ and the terms appearing at the boundary. To do this, we need to prove bounds on the individual terms appearing in $\mathbf{R}_j^b$. The only difference compared to [8] and [2] is in the terms

$$\frac{u_s^\nu(\sqrt{\nu}t, y) - U_s(y)}{\nu^{2^{-n}}}, \qquad \frac{u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y)}{\nu^{2^{-n}}}, \tag{4.28}$$

which need to be bounded uniformly in ν. More precisely, he first term needs to be bounded in H^s for all $s \geq 0$. For the second term a pointwise bound of the derivatives for all $t, Y \geq 0$ with slow growth as $Y \to +\infty$ suffices. Indeed we are ultimately multiplying these terms by a rapidly decaying function in Y, therefore some growth in Y is allowed.

Lemma 4.2. *Assume that $U_s \in C_b^\infty(\mathbb{R}_+)$, and U_s satisfies 4.3. Then the function*

$$\nu \mapsto \frac{u_s^\nu(\sqrt{\nu}t, y) - U_s(y)}{\nu^{2^{-n}}} \tag{4.29}$$

is bounded in $H^s(\mathbb{R}_+)$ for all $s \geq 0$, uniformly as $\nu \to 0$.

Proof. Replacing $\nu = 0$ in u_s^ν, we know that

$$\frac{u_s^0(\sqrt{\nu}t, y) - U_s(y)}{\sqrt{\nu}} = \partial_{yy} u_s^0(\sqrt{\nu}\tau, y), \quad \tau = \tau(t) \in [0,1],$$

where u_s^0 satisfies the Dirichlet problem for the heat equation. But

$$\|\partial_{yy} u_s^0(\sqrt{\nu}\tau)\|_{H^s} = \|K(\sqrt{\nu}\tau) \star \tilde{U}_s''\|_{H^s} \leq \|\tilde{U}_s''\|_{H^s},$$

where $\tilde{U}_s$ is the odd extension of U_s from $\mathbb{R}_+$ to $\mathbb{R}$, which is independent from ν. Notice that $n \geq 2$, so $2^{-n} \leq 1/2$.

We want to show that $\dfrac{(u_s^\nu - u_s^0)(\sqrt{\nu}t, y)}{\nu^{\gamma - 1/2}}$ is uniformly bounded in H^s, which allows us to conclude since $2^{-n} \leq \gamma - 1/2$ by definition. But we know from Proposition 2.1, with $a = \nu^{1/2 - \gamma}$, that

$$\|u_s^\nu(t) - u_s^0(t)\|_{H^s} \leq C_s \nu^{\gamma - 1/2}, \qquad \forall t \geq 0,$$

which immediately provides the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2. For profiles that do not satisfy assumption 4.3, the above estimate does not hold, even for $s = 0$. For instance, take $\gamma > 1/2$, $U_s(y) = 1$ for all $y \geq 0$, so that all its derivatives are integrable. Then letting $a := \nu^{1/2 - \gamma}$, we have for all $a > 0$,

$$\tilde{U}_s^a(y) = \chi_{[0,\infty)} + (-1 + 2ae^{ay})\chi_{(-\infty,0]}.$$

The evolution of the profile is given by

$$u_s^a(t, y) = \mathrm{Erf}\left(\frac{y}{2\sqrt{t}}\right) + ae^{a(at+y)}\mathrm{Erfc}\left(a\sqrt{t} + \frac{y}{2\sqrt{t}}\right).$$

Then for $y \geq 0$ we have

$$\|u_s^a(\sqrt{\nu}t, y) - U_s(y)\|_{L^2} \sim C\nu^{1/2 - \gamma}\|\mathrm{Erfc}\left(\frac{y}{2\sqrt{t}\nu^{1/4}}\right)\|_{L^2} \sim Ct^{1/4}\nu^{5/8 - \gamma}.$$

Then if $\gamma \geq 5/8$, the above quantity does not vanish as $\nu \to 0^+$. Moreover, for each additional derivative, we gain an additional factor of $\nu^{1/2 - \gamma}$ which prevents convergence for a wider range of γ.

Lemma 4.3. *Assume that $U_s \in C_b^\infty(\mathbb{R}_+)$, and U_s satisfies* 4.3. *Then*

$$\left| \frac{u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y)}{\nu^{2^{-n}}} \right| \leq C_1 Y + C_2 \qquad \forall Y \geq 0, \tag{4.30}$$

$$\left| \frac{\partial_Y^k u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y)}{\nu^{1/4}} \right| \leq C_3, \qquad \forall Y \geq 0,\ k \geq 1, \tag{4.31}$$

where $C_1, C_2, C_3 > 0$ do not depend on $t \geq 0$ or on $\nu \to 0$.

Proof. We have

$$|u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y)| \leq |u_s^\nu(\sqrt{\nu}t, 0)| + \nu^{1/4}Y \|\partial_y u_s^\nu(\sqrt{\nu}t)\|_{L^\infty}$$
$$\leq |u_s^0(\sqrt{\nu}t, 0)| + \nu^{1/4}Y \|\partial_y u_s^0(\sqrt{\nu}t)\|_{L^\infty} + O(\nu^{\gamma - 1/2})$$
$$\leq \nu^{1/4}Y \|\partial_y U_s^0\|_{L^\infty} + O(\nu^{\gamma - 1/2}).$$

Hence, 4.30 follows. For 4.31, recall that $Y = \nu^{-1/4}y$ and therefore

$$\partial_Y^k u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y) = \nu^{k/4} \partial_y u_s^\nu(\sqrt{\nu}t, y)$$
$$= \nu^{k/4}\left(\partial_y u_s^0(\sqrt{\nu}t, y) + O(\nu^{\gamma - 1/2}) \right)$$
$$= \nu^{k/4} \|\partial_y U_s^0\|_{L^\infty}.$$

$\square$

4.3. The Inviscid Linear Instability

We first start by constructing a term $\mathbf{u}_0^I(t, x, y)$ displaying the instability, but solving a linearized Euler equation. The construction follows from our assumption that the shear profile U_s is linearly unstable for the linearized Euler equation, and proceeds exactly as in [2] or [8]. We sketch it here for the reader's convenience.

Let us first specify exactly what we mean by linearly unstable. Consider the linearized Euler equation around the shear profile U_s:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla U_s + U_s \cdot \nabla \mathbf{u} + \nabla p = 0; \\ v = 0 \end{cases} \qquad y = 0.$$

Then there exists a (nontrivial) solution $\mathbf{u}$ in the form

$$\mathbf{u}(t, x, y) = e^{ik(x - ct)}(\phi'(y), -ik\phi(y)), \qquad k \in \mathbb{R}, c \in \mathbb{C},$$

if and only if ϕ is a (nontrivial) solution of the Rayleigh equation

$$\begin{cases} (U_s - c)(\partial_{yy} - k^2)\phi - U_s''\phi = 0; \\ \phi(0) = \lim_{y \to +\infty} \phi(y) = 0. \end{cases} \tag{4.32}$$

Definition 1. For each fixed $k \in \mathbb{R}$ we say that c is an *eigenvalue* for the Rayleigh equation if there exists a nontrivial solution to the Rayleigh equation 4.32. The shear profile U_s is called *linearly unstable for the Euler equation* when the associated Rayleigh equation admits an eigenvalue c with $\Im c > 0$.

For each wavenumber $k \in \mathbb{R}$, let $\sigma(k)$ be the supremum of the real parts of the associated eigenvalues of the Rayleigh equation. By Theorem 4.1 from [2], this supremum is always attained at some eigenvalue $\lambda_k \in \mathbb{C}$; moreover, $k \mapsto \sigma(k)$ is real analytic, non-negative, and

$$\lim_{k \to 0} \sigma(k) = \lim_{|k| \to +\infty} \sigma(k) = 0.$$

In particular, σ admits a maximum $\sigma_0 = \sigma(k_0) \geq 0$ over $\mathbb{R}$. Since U_s is by assumption linearly unstable for Euler, $\sigma(k)$ is not identically zero, so $\sigma_0 > 0$. Because $k \mapsto \sigma(k)$ is continuous, we have $\sigma(k) > 0$ in a neighborhood I of k_0. Thus for all $k \in I$, we have a maximally unstable solution of the Euler equation

$$\mathbf{u}_k(t, x, y) = e^{ikx + \lambda_k t}(\psi_k'(y), -ik\psi_k(y)),$$

where ψ_k solves the Rayleigh equation with wavenumber k and eigenvalue λ_k. We thus define

$$\mathbf{u}_0^I(t, x, y) := \int_{\mathbb{R}} \varphi(k)\mathbf{u}_k(t, x, y)\, \mathrm{d}k,$$

where φ is supported in a small enough neighborhood $I' \subset I$ of k_0. Notice that thanks to this cut off, we have $\mathbf{u}_0^I \in H^s(\mathbb{R} \times \mathbb{R}_+)$ for all $s \geq 0$. We remark that if the domain of the x variables is bounded instead, e.g. $x \in \mathbb{T}$, we could simply define $\mathbf{u}_0^I := \mathbf{u}_{k_0}$.

We thus obtain

$$\|\mathbf{u}_0^I(t)\|_{H^s}^2 \sim \int_{I'} e^{2\sigma(k)t}\, \mathrm{d}k.$$

To estimate this integral, we can use a Taylor expansion of $\sigma(k)$ around $\sigma(k_0)$. Since k_0 is a maximum and σ is real analytic and non-constant, we have

$$\sigma(k) \sim \sigma_0 - \mu\sigma^{(2m)}(k_0)(k - k_0)^{2m},$$

for some $\mu > 0$ and $m \geq 1$. Therefore,

$$\int_{I'} e^{2\sigma(k)t}\, \mathrm{d}k \sim e^{2\sigma_0 t} \int_{I'} e^{-2\mu t(k-k_0)^{2m}}\, \mathrm{d}k \sim C\frac{e^{2\sigma_0 t}}{t^{1/2m}}, \qquad \text{as } t \to +\infty.$$

In the remainder of the argument, in line with [8] we just assume $m = 1$ (i.e. σ_0 is a nondegenerate maximum), in order to simplify the notation. All the results still hold for arbitrary m.

In this case, for all $s \geq 0$ there exists $C_s > 0$ such that

$$\|\mathbf{u}_0^I(t)\|_{H^s} \leq C_s \frac{e^{\sigma_0 t}}{(1+t)^{1/4}}, \qquad \forall t \geq 0. \tag{4.33}$$

Additionally (see [8], Section 3.2.2), there exists a bounded subset $\Omega_A \subset \mathbb{R}_+^2$, with measure of order $\sqrt{1+t}$ such that

$$\|\mathbf{u}_0^I\|_{L^2(\Omega_A)} \geq C'\frac{e^{\sigma_0 t}}{(1+t)^{1/4}}, \qquad \forall t \geq 0. \tag{4.34}$$

Assuming $N \geq 1$ and $\nu \leq 1$, we can define times $T_\vartheta^\nu > 0$ such that

$$\frac{e^{\sigma_0 T_\vartheta^\nu}}{(1+T_\vartheta^\nu)^{1/4}} = \nu^{\vartheta - N}. \tag{4.35}$$

Notice that $\lim_{\nu \to 0^+} T_\vartheta^\nu = \infty$, but in the original variables, $\lim_{\nu \to 0^+} \sqrt{\nu}T_\vartheta^\nu = 0$. Moreover, for $\tau > 0$ small enough depending on ν, we have $T^\nu := T_\vartheta^\nu - \tau > 0$, and by 4.34

$$\|\nu^N \mathbf{u}_0^I\,|_{t=T^\nu}\,\|_{L^2(\Omega_A)} \geq \delta(\tau)\nu^\vartheta.$$

This proves 4.14, for some value δ depending on the choice of τ. In other words, we can always subtract a value τ as large as we want from T_ϑ^ν, and the instability will hold at $t = T_\vartheta^\nu - \tau$, as long as ν is small enough. We will choose the specific value of $\tau > 0$ later on.

4.4. Correction of the Boundary Condition

As discussed above, we want $\mathbf{u}^{\mathrm{app}}$ to satisfy the Navier boundary condition 4.9_3. But since it is constructed using an asymptotic expansion, we cannot in principle expect it to be satisfied exactly. Instead, it will leave a remainder $\mathbf{r}^{\mathrm{app}} = (r_1^{\mathrm{app}}, r_2^{\mathrm{app}})$, which will be determined in this subsection.

Let us first consider the equation for the first component:

$$\partial_y(u_s^\nu + u^I + \nu^\zeta u^b) - \nu^{1/2-\gamma}(u_s^\nu + u^I + \nu^\zeta u^b) = 0 \qquad \text{at } y = 0.$$

By assumption, the shear flow u_s^ν satisfies the Navier condition $\partial_y u_s^\nu - \nu^{1/2-\gamma} u_s^\nu = 0$, so it can be eliminated from the above condition and we are left with

$$u^b = \nu^{\gamma-3/4} \partial_Y u^b + \nu^{\gamma-1/2-\zeta} \partial_y u^I - \nu^{-\zeta} u^I.$$

The leading order terms of the above equation as $\nu \to 0$ depend on the choice of γ. The resulting equations are summarized in the following table. (Table 1)

Notice that regardless of the value of γ, the terms with u^I do not appear at a higher order compared to the terms with u^b. This means that we can obtain the same estimates for u_j^b as we do for u_j^I. It would no longer be true if the value of ζ was higher.

Let us plug the ansatze 4.19 for u^I and u^b into the boundary conditions. Then we can derive recursive equations which determine the value of u_j^b. As discussed in Sect. 2, if u_j^b or $\partial_y u_j^b$ appear with a coefficient vanishing with the viscosity, the solution cannot be bounded uniformly with respect to the viscosity. Hence we need to move such term in the next order. This is possible in each case thanks to our choice of n.

Notice that these are mixed boundary conditions where either the coefficient of $\partial_Y u_j^b$ or u_j^b is equal to 1, and in particular is bounded away from 0 and thus satisfies the assumptions of Proposition 2.4. The other coefficients remain bounded as $\nu \to 0$, therefore do not cause any issues when applying our estimates.

As for the second component v_j^b, from 4.21$_3$ it follows that for all $j \leq M$ we have

$$v_j^I + \nu^{a+1/4-2^{-n}} v_{j-1}^b = 0 \qquad \text{at } y = 0.$$

Therefore,

$$v^{\text{app}} = \nu^{N+a+1/4-2^{-n}+M2^{-n}} v_M^j =: r_2^{\text{app}} \qquad \text{at } y = 0. \tag{4.36}$$

4.5. Construction of $\mathbf{u}_j^I$ and $\mathbf{u}_j^b$

We will now $\mathbf{u}_j^I$ and $\mathbf{u}_j^b$ by induction on $j \in \mathbb{Z}_{\geq 0}$. The induction is organized as follows. We start from $\mathbf{u}_0^I$, which was constructed in Sect. 4.3. From $\mathbf{u}_0^I$ we can construct $\mathbf{u}_0^b$, applying the boundary condition derived in Sect. 4.4. Next, suppose we have constructed $\mathbf{u}_{j'}^I$ and $\mathbf{u}_{j'}^b$ for all $0 \leq j' \leq j$. In the equation 4.21 satisfied by $\mathbf{u}_{j+1}^I$, $\mathbf{R}_{j+1}^I$ only depends on $\mathbf{u}_{j'}^{\prime I}$ for $j' \leq j$, while the boundary condition depends on $\mathbf{u}_j^b$, all of which have already been constructed. Thus we can derive $\mathbf{u}_{j+1}^I$. Similarly, by 4.25, all the terms in $\mathbf{R}_{j+1}^b$

TABLE 1. Boundary condition satisfied by u_b depending on the value of γ

Range for γ	Range for ζ	Equation satisfied by u_b
$1/2 < \gamma < 3/4$	$\zeta = 3/4 - \gamma$	$\partial_Y u^b = \nu^{3/4-\gamma} u^b + u^I - \nu^{\gamma-1/2} \partial_y u^I$
$\gamma = 3/4$	$\zeta = 0$	$\partial_Y u^b - u^b = u^I - \nu^{1/4} \partial_y u^I$
$\gamma > 3/4$	$\zeta = 0$	$u^b = \nu^{\gamma-3/4} \partial_Y u^b + \nu^{\gamma-1/2} \partial_y u^I - u^I$

TABLE 2. Boundary condition satisfied by the u_j^b and final error in the boundary condition for u^b. Each term appearing with a negative index should be replaced with 0

Range	Equation satisfied by u_j^b	Error in the boundary condition
$\frac{1}{2} < \gamma < \frac{3}{4}$	$\partial_Y u_j^b = \nu^{3/4-\gamma-2^{-n}} u_{j-1}^b + u_j^I - \nu^{\gamma-1/2} \partial_y u_j^I$	$r_1^{\text{app}} = \nu^{3/4-\gamma-2^{-n}+M2^{-n}} u_M^b$
$\gamma = 3/4$	$\partial_Y u_j^b - u_j^b = u_j^I - \nu^{1/4} \partial_y u_j^I$	None
$\gamma > 3/4$	$u_j^b = \nu^{\gamma-3/4-2^{-n}} \partial_Y u_{j-1}^b + u_j^I - \nu^{\gamma-1/2} \partial_y u_j^I$	$r_1^{\text{app}} = \nu^{\gamma-3/4-2^{-n}+M2^{-n}} \partial_Y u_M^b$

only depend on $\mathbf{u}_{j'}^b$ for $j' < j+1$ and $\mathbf{u}_{j'}^I$ for $j' \leq j+1$, and the same goes for the boundary condition (see Sect. 4.4). Thus we obtain $\mathbf{u}_{j+1}^b$. By induction, we can construct $\mathbf{u}_j^I$ and $\mathbf{u}_j^b$ for all $j = 0, \ldots, M$.

In what follows, to ease the notation we introduce for all $j = 1, \ldots, M$ the quantity

$$k_j := 1 + \frac{j}{2^n N}.$$

Proposition 4.4. *For all $s \geq 0$ there exists a constant $C = C(s,j) > 0$ such that for all $t \geq 0$, and $j = 0, \ldots, M$, we have*

$$\|\mathbf{u}_j^I(t)\|_{H^s} \leq C \frac{e^{\sigma_0 k_j t}}{(1+t)^{k_j/4}}, \tag{4.37}$$

$$\|\mathbf{u}_j^b(t)\|_{H^s(Y)} \leq C \frac{e^{\sigma_0 k_j t}}{(1+t)^{k_j/4}}, \qquad \forall t \geq 0. \tag{4.38}$$

Moreover, there exist functions $h_{k,\ell,j}(x) \in L^2(\mathbb{R}), \mu_j > 0$ such that for all $k, \ell \in \mathbb{Z}_{\geq 0}$, and for all $t \geq 0, (x,y) \in \mathbb{R} \times \mathbb{R}_+$,

$$|\partial_x^k \partial_Y^\ell \mathbf{u}_j^b(t,x,Y)| \leq |h_{k,\ell,j}(x)| \frac{e^{\sigma_0 k_j t}}{(1+t)^{k_j/4}} e^{-\mu_j Y}, \qquad j = 0, \ldots, M. \tag{4.39}$$

Proof. The proof is by induction. The estimate 4.37 for $j = 0$ is simply 4.33, whereas 4.39 and 4.38 follow from Proposition 2.4. Now suppose 4.37 and 4.38 hold for $j < J$. We first look to obtain 4.37 for $j = J$. We know that $\mathbf{u}_J^I$ satisfies 4.21 and 4.22. We want to find an H^s estimate on the remainder $\mathbf{R}_J^I$. Recall that, for all $f, g \in H^s$, for any $s \geq 0$ by Sobolev embeddings we have

$$\|fg\|_{H^s} \leq C_s \left(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\right) \leq C\|f\|_{H^{s+2}}\|g\|_{H^{s+2}}.$$

Therefore, using Lemma 4.2,

$$\|\mathbf{R}_J^I\|_{H^s} \lesssim \|\mathbf{u}_{J-1}^I\|_{H^{s+1}} + \|\mathbf{u}_{J-2}^I\|_{H^{s+2}} + \sum_{j_1+j_2=J-2^n N} \|\mathbf{u}_{j_1}^I\|_{H^{s+2}}\|\mathbf{u}_{j_2}^I\|_{H^{s+3}}$$

$$\lesssim \|\mathbf{u}_{j-1}^I\|_{H^s} + \frac{e^{\sigma_0(k_{j_1}+k_{j_2})t}}{(1+t)^{(k_{j_1}+k_{j_2})/4}}$$

$$\lesssim \frac{e^{\sigma_0 k_J t}}{(1+t)^{k_J/4}},$$

where we used the equality

$$k_{j_1} + k_{j_2} = k_{j_1+j_2+2^n N}.$$

Finally, using well-known spectral estimates on the linearized Euler equations (see Theorem 3.1 from [8]), since $\mathbf{u}_{J-1}^b$ satisfies 4.38, we deduce that

$$\|\mathbf{u}_J^I\|_{H^{s-2}} \lesssim \|\mathbf{R}_J^I\|_{H^s} \lesssim \frac{e^{\sigma_0 k_J t}}{(1+t)^{k_J/4}}.$$

Since the estimate works for all $s \geq 0$, we conclude that 4.37 holds for the index j.

Next, we look to prove the estimates for $\mathbf{u}_J^b$. It suffices to prove 4.39 as 4.38 immediately follows from it. Moreover, it is enough to prove 4.39 for the first component u_J^b, as we can then deduce them for the second component v_J^b by the divergence-free condition, as in 4.26. Recall that u_J^b satisfies 4.24 and 4.25 with the boundary conditions given in Table 2. For each $x \in \mathbb{R}$, this is a heat equation in Y with a remainder R_J^b as in 4.25, an initial condition satisfying 4.20, and a boundary condition given by a sum of u_j^I, u_j^b and their derivatives for $j \leq J$. For the u_j^I terms, by taking the trace at the boundary in 4.37 we deduce that

$$|\partial_x^\ell \partial_Y^k u_j^I(t,x,0)| \leq |f_{k,\ell,J}(x)| \frac{e^{\sigma_0 k_J t}}{(1+t)^{k_J/4}},$$

where $f_{k,\ell,J}(x) \in L^2(\mathbb{R})$. Together with the inductive assumption 4.39, we deduce that the inhomogeneous part of the boundary condition as a whole satisfies the same estimate.

Now consider the remainders R_J^b. The only potential danger is the terms in the remainders containing $u_s(\sqrt{\nu}t, y)\nu^{-1/8}$ which need to be replaced with $u_s^\nu(\sqrt{\nu}t, y)\nu^{-2^{-n}}$. More precisely these terms are of the form

$$\frac{u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y)}{\nu^{2^{-n}}} \cdot \nabla_{x,Y} u_{J-1}^b + u_{J-1}^b \cdot \nabla_{x,Y} \left(\frac{u_s^\nu(\sqrt{\nu}t, \nu^{1/4}Y)}{\nu^{2^{-n}}} \right). \tag{4.40}$$

By Proposition 4.3, since $n \geq 2$, these terms and their derivatives can be bound pointwise by

$$|C_1 Y + C_2| \cdot |\nabla_{x,Y} u_{J-1}^b(t, x, Y)| + C_3 \cdot |u_{J-1}^b(t, x, Y)|, \tag{4.41}$$

Thus assuming u_{J-1}^b satisfies 4.39, then the terms in 4.40 also do, after decreasing the value of μ_{J-1} by an arbitrarily small quantity to accomodate for the extra linear growth in Y. Therefore the remainder R_J^b satisfies the estimate 4.39. Since the estimate holds for Y-derivatives of all orders, we deduce that

$$|\partial_x^\ell \partial_y^k R_J^b(t, x, Y)| \leq |R_{k,\ell,J}(x)| \frac{e^{\sigma_0 k_J t}}{(1+t)^{k_J/4}} e^{-\mu_J Y},$$

where $R_{k,\ell,J} \in L^2(\mathbb{R})$. By Proposition 2.4, we conclude that

$$|\partial_x^\ell \partial_y^k u_J^b(t, x, Y)| \leq |h_{k,\ell,J}(x)| \frac{e^{\sigma_0 k_J t}}{(1+t)^{k_J/4}} e^{-\mu_J Y},$$

where, given the bound $g_j^b(x) \in L^2(\mathbb{R})$ for the initial conditions of $\mathbf{u}_j^b$ as in 4.20,

$$|h_{k,\ell,J}(x)| \leq C \left(|R_{k,\ell,J}(x)| + |f_{k,\ell,J}(x)| + |g_j^b(x)| \right),$$

and thus $h_{k,\ell,J}(x) \in L^2(\mathbb{R})$.

Hence, by induction, 4.39 is verified for all $0 \leq J \leq M$. Since it holds for derivatives of all orders, the H^s estimate immediately follows. $\qquad\square$

Corollary 4.5. *For all $s \geq 0$, we have*

$$\|\mathbf{R}^I\|_{H^s} \leq C_s \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1+\frac{M+1}{2^n N}} \qquad \forall t \leq T_\vartheta^\nu. \tag{4.42}$$

Proof. By 4.37 and 4.23, we know that there exists an $M' > M$ such that

$$\|\mathbf{R}^I(t)\|_{H^s} \leq C_s \nu^N \sum_{j=M+1}^{M'} \nu^{j2^{-n}} \frac{e^{\sigma_0 k_j t}}{(1+t)^{k_j/4}} = C_s \sum_{j=M+1}^{M'} \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{k_j}$$

$$\leq C_s \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1+\frac{M+1}{2^n N}} + C_s \sum_{j=M+2}^{M'} \nu^{\vartheta k_j}$$

$$\leq C_s \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1+\frac{M+1}{2^n N}}.$$

$\qquad\square$

This estimate will be necessary in order to prove corresponding estimates on the approximate solution $\mathbf{u}^{\mathrm{app}}$.

In the same way as for $\mathbf{R}^I$ in 4.42, we deduce the following estimate for $\mathbf{R}^b$.

Corollary 4.6. *For all $\ell, k \in \mathbb{Z}_{\geq 0}$ there exist constants $C_{k,\ell}, \mu > 0$ such that for all $t \leq T_\vartheta^\nu$ we have*

$$|\partial_x^\ell \partial_Y^k \mathbf{R}^b(t, x, Y)| \leq C_{k,\ell} \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1+\frac{M+1}{2^n N}} e^{-\mu y}. \tag{4.43}$$

Putting 4.42 and 4.43 together, we obtain the estimates for the remainder $\mathbf{R}^{\mathrm{app}} = \mathbf{R}^I + \nu^\zeta \mathbf{R}^b$: for all $s \geq 0$, we have

$$\|\mathbf{R}^{\mathrm{app}}(t)\|_{H^s} \leq C_s \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1 + \frac{M+1}{2^n N}}. \tag{4.44}$$

Now let us consider the remainder $\mathbf{r}^{\mathrm{app}}$ in the Navier boundary condition. For the first component, we have by 4.38,

$$\|r_1^{\mathrm{app}}(t)\|_{L^2(y=0)} \leq C_M \nu^{N + \frac{M+1}{2^n}} \frac{e^{\sigma_0 k_{M+1} t}}{(1+t)^{k_{M+1}/4}} = C_M \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1 + \frac{M+1}{2^n N}}. \tag{4.45}$$

For the second component r_2^{app}, by 4.36, discarding the $\nu^{a + 1/4 - 2^{-n}}$ factor we similarly obtain, using 4.39,

$$|\partial_x^\ell \partial_y^k r_2^{\mathrm{app}}(t, x, 0)| \leq C_{k,\ell} \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{1 + \frac{M+1}{2^n N}}. \tag{4.46}$$

4.6. Energy Estimates

Define $\mathbf{v} := \mathbf{u}^\nu - \mathbf{u}^{\mathrm{app}}$. Then $\mathbf{v} + \mathbf{u}^{\mathrm{app}}$ solves Navier-Stokes with an error $\mathbf{R}^{\mathrm{app}}$, so that $\mathbf{v}$ solves the equation

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{u}^{\mathrm{app}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}^{\mathrm{app}} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \sqrt{\nu}\Delta \mathbf{v} - \mathbf{R}^{\mathrm{app}}; \\ \nabla \cdot \mathbf{v} = 0; \\ \mathbf{v}\mid_{t=0} = 0; \end{cases}$$

with some boundary condition which we will specify later.

We want to prove 4.16, so we need to find an upper bound on the L^2 norm of v at time $t = T^\nu$. Deriving the standard energy estimate:

$$\frac{1}{2} \partial_t \|\mathbf{v}\|_{L^2}^2 - \sqrt{\nu} \int_{\partial\Omega} \partial_i v_j v_j n_i \leq \left(\|\nabla \mathbf{u}^{\mathrm{app}}\|_{L^\infty} + \beta \right) \|\mathbf{v}\|_{L^2}^2 + \frac{1}{4\beta} \|\mathbf{R}^{\mathrm{app}}\|_{L^2}^2.$$

Because the domain is flat, we see that

$$\int_{\partial\Omega} \partial_i v_j v_j n_i = -\int_{y=0} v_1 \partial_y v_1 - \int_{y=0} v_2 \partial_y v_2.$$

Suppose now that $\mathbf{u}^{\mathrm{app}}$ satisfies the Navier boundary condition with an error $\mathbf{r}^{\mathrm{app}}$, i.e.

$$\begin{cases} \partial_y u^{\mathrm{app}} = \sqrt{\nu}^{1-2\gamma} u^{\mathrm{app}} + r_1^{\mathrm{app}}; \\ v^{\mathrm{app}} = r_2^{\mathrm{app}}. \end{cases}$$

Then $\mathbf{v}$ satisfies the same boundary condition with error $-\mathbf{r}^{\mathrm{app}}$:

$$\begin{cases} \partial_y v_1 = \sqrt{\nu}^{1-2\gamma} v_1 - r_1^{\mathrm{app}}; \\ v_2 = -r_2^{\mathrm{app}}. \end{cases}$$

Hence, for any $\alpha > 0$,

$$\int_{y=0} v_1 \partial_y v_1 = \sqrt{\nu}^{1-2\gamma} \int_{y=0} |v_1|^2 - \int_{y=0} r_1^{\mathrm{app}} v_1 \geq -\frac{1}{4} \nu^{\gamma - 1/2} \int_{y=0} |r_1^{\mathrm{app}}|^2. \tag{4.47}$$

Since $\gamma \geq 1/2$, then $\nu^{\gamma - 1/2} \leq C$ for all ν small enough and the energy estimate becomes

$$\frac{1}{2} \partial_t \|\mathbf{v}\|_{L^2}^2 \leq \left(\|\nabla \mathbf{u}^{\mathrm{app}}\|_{L^\infty} + \beta \right) \|\mathbf{v}\|_{L^2}^2 + C \left(\|\mathbf{R}^{\mathrm{app}}\|_{L^2}^2 + \int_{y=0} |r_1^{\mathrm{app}}|^2 \right) - \int_{y=0} r_2^{\mathrm{app}} \partial_y r_2^{\mathrm{app}}. \tag{4.48}$$

Define
$$P := 1 + \frac{M+1}{2^n N}.$$

Combining 4.44, 4.45 and 4.46, we have

$$\|\mathbf{R}^{\mathrm{app}}(t)\|^2 + \|r^{\mathrm{app}}(t)\|^2_{L^2(y=0)} + \int_{y=0} |r_2^{\mathrm{app}} \partial_y r_2^{\mathrm{app}}| \leq C_M \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{2P}. \tag{4.49}$$

Next, choose M large enough so that

$$P\sigma_0 - 1 \geq \|\nabla \mathbf{u}^{\mathrm{app}}\|_{L^\infty} + \beta \qquad \forall \nu > 0.$$

For this to work, we need $\|\nabla \mathbf{u}^{\mathrm{app}}\|_{L^\infty}$ to be bounded uniformly in ν. The only potential issue is with $\mathbf{u}^b$, as it depends on $y/\sqrt{\nu}$. Using 4.38 we have, for $t \leq T^\nu$ as defined in 4.35,

$$\|\nabla \mathbf{u}^{\mathrm{app}}(t)\|_{L^\infty} - \|\partial_y u_s^\nu(\sqrt{\nu}t)\|_{L^\infty} \leq \nu^{N+\zeta-1/4} \sum_{j=0}^M \nu^{j2^{-n}} \frac{e^{\sigma_0 k_j t}}{(1+t)^{k_j/4}}$$

$$\leq C\nu^{\zeta-1/4} \sum_{j=0}^M \nu^{\vartheta\left(1+\frac{j}{2^n N}\right)} \leq C\nu^{\vartheta+\zeta-1/4}.$$

The power of ν appearing above is non-negative if and only if $\vartheta \geq 1/4 - \zeta$. Hence $\vartheta = 1/4 - \zeta$ is the best value we can get in the instability. Of course, $\|\partial_y u_s^\nu(\sqrt{\nu}t)\|_{L^\infty}$ is bounded uniformly in ν by Proposition 2.1 or Proposition 2.2.

Now 4.48 becomes

$$\partial_t \|\mathbf{v}(t)\|^2_{L^2} \leq (2P\sigma_0 - 1)\|\mathbf{v}(t)\|^2_{L^2} + C_M \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^{2P}.$$

To conclude the proof, we apply Lemma A.2 with $\varphi(t) = \|w(t)\|^2_{L^2}$, $\lambda = 2P\sigma_0 - 1$, $\mu = 2P\sigma_0$. We obtain

$$\|\mathbf{v}(t)\|_{L^2} = \|\mathbf{u}^\nu(t) - \mathbf{u}^{\mathrm{app}}(t)\|_{L^2} \leq C_M' \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}} \right)^P.$$

Choosing τ large enough in the definition of $T^\nu = T_\vartheta^\nu - \tau$, the above quantity is smaller than $\delta\nu^\vartheta$ for $t \leq T^\nu$, and we have verified 4.5; 4.7 follows by the embedding $L^\infty \hookrightarrow \dot{H}^s$ for $s > 1$. Therefore, Theorem 4.1 is proven.

4.7. Modifications for the Case $\gamma < 1/2$

In this Section, we explain the changes that must be applied to Sect. 4.2 in order to treat the case $\gamma < 1/2$.

The construction of the approximate solution, as detailed in the first part of Sect. 4.2, remains the same, except that the value of n must be changed so that it satisfies the following inequality instead:

$$2^{-n} \leq \frac{1}{4} - \frac{\gamma}{2}.$$

In Lemmas 4.2 and 4.3 we need to prove the boundedness of terms containing u_s^ν as $\nu \to 0$, which in the case of $\gamma > 1/2$ involves applying Proposition 2.1 since $a = \nu^{1/2-\gamma} \to \infty$ as $\nu \to 0$. When $\gamma < 1/2$ we have $a \to 0$ as $\nu \to 0$, so we simply replace $\gamma - 1/2$ with $1/4 - \gamma/2 \geq 2^{-n}$ in the proofs, and then use Proposition 2.2 instead.

In Sect. 4.4, the boundary conditions must be adjusted as follows. For $\mathbf{u}^b$ we obtain

$$\partial_Y u^b = \nu^{3/4-\gamma} u^b + \nu^{1/2-\gamma} u^I - \partial_y u^I,$$

which leads to

$$\partial_Y u_j^b = \nu^{3/4-\gamma-2^{-n}} u_{j-1}^b + \nu^{1/2-\gamma} u^I - \partial_y u^I,$$

with the error r_1^{app} in the boundary condition given by the same expression as the case $1/2 < \gamma < 3/4$:

$$r_1^{\mathrm{app}} = \nu^{3/4 - \gamma - 2^{-n} + M 2^{-n}} u_M^b. \tag{4.50}$$

Proposition 4.4 carries over after adjusting the boundary conditions, because Propositions 2.3 and 2.4 can be applied: the u_j^b are still satisfying a heat equation with mixed boundary conditions and the inhomogeneous terms vanish (or remain bounded) with the viscosity.

Finally, in the proof of the energy estimate in Sect. 4.6, the coefficient $\nu^{\gamma - 1/2}$ is now divergent as $\nu \to 0$. However, we have

$$0 > \gamma - \frac{1}{2} \geq 2\left(\gamma - \frac{1}{2}\right) \geq 2\left(\gamma - \frac{3}{4} + 2^{-n}\right),$$

hence for $\nu \leq 1$, the right-hand side of (4.47) is bounded from below by

$$-\frac{1}{4}\nu^{2(\gamma - 3/4 + 2^{-n})} \int_{y=0} |\, r_1^{\mathrm{app}}\,|^2\,.$$

Therefore by (4.50),

$$\nu^{\gamma - 3/4 + 2^{-n}} \|r_1^{\mathrm{app}}(t)\|_{L^2(y=0)} \leq \|u_M^b\|_{L^2(y=0)} \leq C_M \left(\nu^N \frac{e^{\sigma_0 t}}{(1+t)^{1/4}}\right)^{1 + \frac{M+1}{2^n N}}. \tag{4.51}$$

We can then use the above equation to obtain 4.49 and proceed as in the case $\gamma > 1/2$.

Declarations

Conflicts of interest There is no conflict of interest.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Proofs of the Results from Section 2

In this Section, we fill out the details we left out from Sect. 2.

A.1. Results from Section 2.1

Here we will prove Propositions 2.1 and 2.2, while also providing a justification for the assumptions (2.4) and (2.6).

Let us begin by giving the explicit expression of the extension u_0^a of u_0 which makes $\partial_y u_0^a - a u_0^a$ an odd function. We have

$$u_0^a(-y) = e^{-ay}\left(u_0(0) + \int_0^y e^{a\tilde{y}}\left(u_0'(\tilde{y}) - a u_0(\tilde{y})\right) \mathrm{d}\tilde{y}\right), \qquad y \geq 0. \tag{A.1}$$

Integrating by parts, (A.1) can be rewritten as

$$u_0^a(-y) = u_0(y) - 2a \int_0^y e^{-a(y-\tilde{y})} u_0(\tilde{y}) \, \mathrm{d}\tilde{y}, \tag{A.2}$$

or

$$u_0^a(-y) = -u_0(y) + 2e^{-ay} u_0(0) + 2 \int_0^y e^{-a(y-\tilde{y})} u_0'(\tilde{y}) \, \mathrm{d}\tilde{y}. \tag{A.3}$$

Differentiating both sides of (A.3) and integrating by parts repeatedly, we can find an explicit expression for the derivatives of u_0^a:

$$(-1)^k \partial^k u_0^a(-y) = -u_0^{(k)}(y) + 2e^{-ay} \sum_{j=0}^k (-a)^{k-j} u_0^{(j)}(0) + 2 \int_0^y e^{-a(y-\tilde{y})} u_0^{(k+1)}(\tilde{y}) \, \mathrm{d}\tilde{y} \tag{A.4}$$

$$= u_0^{(k)}(y) + 2e^{-ay} \sum_{j=0}^{k-1} (-a)^{k-j} u_0^{(j)}(0) - 2a \int_0^y e^{-a(y-\tilde{y})} u_0^{(k)}(\tilde{y}) \, \mathrm{d}\tilde{y}, \tag{A.5}$$

where $y > 0, k \in \mathbb{N}$. Taking $y \to 0$ in the expression (A.5) we get

$$(-1)^k \lim_{y \to 0^+} \partial^k u_0^a(-y) = u_0^{(k)}(0) + 2 \sum_{j=0}^{k-1} (-a)^{k-j} u_0^{(j)}(0).$$

This tells us that u_0^a is continuous, but for higher derivatives the gap between $\partial^k u_0^a$ and the $\partial^k u_0^\infty$ increases with a. In general, we have

$$\partial^k u_0^a(-y) - \partial^k u_0^\infty(-y) = 2e^{-ay} \sum_{j=0}^k (-a)^{k-j} u_0^{(j)}(0) + 2 \int_0^y e^{-a(y-\tilde{y})} u_0^{(k+1)}(\tilde{y}) \mathrm{d}\tilde{y}.$$

In particular, due to the discontinuity at $y = 0$, the derivative $\partial^k u_0^a$ does not even belong to $L^p(\mathbb{R})$ unless $u_0^{(j)}(0) = 0$ for all $j = 0, \ldots, k-1$. Requiring the weaker condition $u_0^{(2k+1)}(0) = a u_0^{(2k)}(0)$ for all $k \in \mathbb{Z}_{\geq 0}$, which corresponds to enforcing the compatibility conditions of all orders, would also get rid of the problematic terms. However, that can only be true for multiple values of a if $u_0^{(2k+1)}(0) = u_0^{(2k)}(0) = 0$. This is the justification for the assumption (2.4).

Proof of Proposition 2.1. The first condition in the assumption (2.4) implies that (A.4) can be simplified to

$$(-1)^k \partial^k u_0^a(-y) = -u_0^{(k)}(y) + 2 \int_0^y e^{-a(y-\tilde{y})} u_0^{(k+1)}(\tilde{y}) \, \mathrm{d}\tilde{y}. \tag{A.6}$$

By (A.6), for any $k \in \mathbb{Z}_{\geq 0}$ we have

$$\|\partial^k u_0^a - \partial^k u_0^\infty\|_{L^\infty} \leq 2\|\partial^{k+1} u_0\|_{L^\infty} \sup_{y \geq 0} \int_0^y e^{-a(y-\tilde{y})} \, \mathrm{d}\tilde{y}$$

$$\leq \frac{2}{a} \|\partial^{k+1} u_0\|_{L^\infty} = O(a^{-1}),$$

and

$$\|\partial^k u_0^a - \partial^k u_0^\infty\|_{L^1} \leq 2 \int_0^\infty \int_0^y e^{-a(y-\tilde{y})} |\partial^{k+1} u_0(\tilde{y})| \, \mathrm{d}\tilde{y} \, \mathrm{d}y$$

$$= 2 \int_0^\infty e^{a\tilde{y}} \int_{\tilde{y}}^\infty e^{-ay} \, \mathrm{d}y \, |\partial^{k+1} u_0(\tilde{y})| \, \mathrm{d}\tilde{y}$$

$$= \frac{2}{a} \|\partial^{k+1} u_0\|_{L^1} = O(a^{-1}).$$

By interpolation, we get that the initial conditions satisfy $\|u_0^a - u_0^\infty\|_{W^{k,p}} = O(a^{-1})$ as $a \to +\infty$. Let us call u^a and u^∞, respectively, the evolutions of the initial data u_0^a and u_0^∞ under (2.1), given by convolution with the kernel K. Then

$$\|u^a - u^\infty\|_{W^{k,p}} = \|K \star (u_0^a - u_0^\infty)\|_{W^{k,p}} \leq \|u_0^a - u_0^\infty\|_{W^{k,p}} = O(a^{-1}).$$

$\square$

If we look at (A.2), we immediately notice that, under the assumption (4.3), there is an issue with uniform convergence: we have, for all $y > 0$,

$$|u_0^a(-y) - u_0^0(-y)| \leq 2a \int_0^y e^{-a(y-\tilde{y})} |u_0(\tilde{y})| \, \mathrm{d}y.$$

However, the right hand side does not vanish as $a \to 0^+$ unless $u_0 \in L^1(\mathbb{R}_+)$, which evades assumption (2.4).

On the other hand, this limit is better behaved compared to $a \to +\infty$ in other aspects. Looking at (A.5), we notice that this time the terms at the boundary

$$2e^{-ay} \sum_{j=0}^{k-1} (-a)^{k-j} u_0^{(j)}(0)$$

all vanish uniformly in y as $a \to 0^+$. However, we still need to require $u_0^{(j)} = 0$ for all j, otherwise u_0^a (and hence $u_0^a - u_0^0$) does not belong to H^k for $k \geq 2$. This justifies assumption (2.6).

Proof of Proposition 2.2. For the L^2-based estimates we have, by Young's inequality for convolutions:

$$\|\partial^k u_0^a - \partial^k u_0^0\|_{L^2} = 2a \|e^{-a\tilde{y}} \star_{[0,y]} \partial^k u_0^0\|_{L^2} \leq 2a \|e^{-ay}\|_{L^2} \|\partial^k u_0^0\|_{L^1}$$
$$= \sqrt{2a} \|\partial^k u_0^0\|_{L^1} = O(a^{1/2}).$$

Hence For the L^∞-based estimates, from (A.5) we deduce that for all $k \geq 0$,

$$\|(-1)^k \partial^k u_0^a(y) - \partial_k u_0^0(y)\|_{L^\infty} \leq 2e^{-ay} \sum_{j=0}^{k-1} a^{k-j} |u_0^{(j)}(0)| + 2a \int_0^y e^{-a(y-\tilde{y})} |u_0^{(k)}(\tilde{y})| \, \mathrm{d}\tilde{y}$$

$$\leq O(a) + 2a \|u_0^{(k)}\|_{L^1} = O(a).$$

The estimates then extend with the same order to $u^a(t, y)$ by applying the heat kernel as in the Proof of Proposition 2.1. $\square$

A.2. Results from Section 2.2

In this subsection, we will prove Proposition 2.3. Let us begin with two technical lemmas.

Lemma A.1. *Let $\alpha > 0$ and $\beta \geq 0$. Then there is a constant $C_{\alpha,\beta} > 0$ such that*

$$\int_0^t \frac{e^{\alpha s}}{(1+s)^\beta} \, \mathrm{d}s \leq C_{\alpha,\beta} \frac{e^{\alpha t}}{(1+t)^\beta}, \qquad \forall t \geq 0.$$

Proof. Let $\varphi(t) := \frac{e^{\alpha t}}{(1+t)^\beta}$. Since $t \mapsto \frac{\int_0^t \varphi(s) \, \mathrm{d}s}{\varphi(t)}$ is continuous over $[0, \infty)$, we just need to prove that

$$\lim_{t \to +\infty} \frac{\int_0^t \varphi(s) \, \mathrm{d}s}{\varphi(t)} < \infty.$$

In fact, by L'Hopital's rule, we see that

$$\lim_{t \to +\infty} \frac{\int_0^t \varphi(s) \, \mathrm{d}s}{\varphi(t)} = \lim_{t \to +\infty} \frac{\varphi(t)}{\varphi(t)(\alpha - \beta(1+t)^{-1})} = \frac{1}{\alpha}.$$

$\square$

The following result is taken from [8], Lemma 3.4. We further include a proof.

Lemma A.2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ satisfy*

$$\varphi'(t) \leq \lambda \varphi(t) + C \frac{e^{\alpha t}}{(1+t)^\beta}, \qquad \forall t \geq 0, \tag{A.7}$$

where $0 \leq \lambda < \alpha$, $\beta > 0$ and $C > 0$ can depend on α, β but not on t. Then

$$\varphi(t) \leq C C_{\alpha - \lambda, \beta} \frac{e^{\alpha t}}{(1+t)^\beta}, \qquad \forall t \geq 0,$$

where $C_{\alpha - \lambda, \beta}$ is the constant from Lemma A.1.

Proof. Integrating (A.7) from 0 to t, we get

$$\varphi(t) \leq \varphi(0) + \int_0^t \lambda \varphi(s) \, \mathrm{d}s + C \int_0^t \frac{e^{\alpha s}}{(1+s)^\beta} \, \mathrm{d}s.$$

Hence, by an application of the Gronwall's lemma (see [10], page 19) and Lemma A.1 - since $\alpha - \lambda > 0$ - we have

$$\begin{aligned} \varphi(t) &\leq C e^{\lambda t} \left(\varphi(0) + \int_0^t e^{(\alpha - \lambda)s} (1+s)^{-\beta} \, \mathrm{d}s \right) \\ &\leq C e^{\lambda t} \cdot C_{\alpha - \lambda, \beta} \frac{e^{(\alpha - \lambda)t}}{(1+t)^\beta} \\ &\leq C C_{\alpha - \lambda, \beta} \frac{e^{\alpha t}}{(1+t)^\beta}. \end{aligned}$$

$\square$

We can now prove Proposition 2.3.

Proof of Proposition 2.3. We only consider the case $k = 0$, as the others follow by applying the result to $\partial_y^{2k} u$, using the compatibility conditions to derive the appropriate boundary conditions and interpolating for odd derivatives. Differentiating the energy, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{L^2}^2 = -\|\partial_y u(t)\|_{L^2}^2 - u(t,0) \partial_y u(t,0) + \langle r(t), u(t) \rangle_{L^2}.$$

Now there are two different cases, depending on whether a or b are bounded away from 0.

- $b \gg 0, a \geq 0$. We can replace $\partial_y u(t,0) = \frac{1}{b} (a u(t,0) - f(t))$, so that

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u(t)\|_{L^2}^2 &\leq -\|\partial_y u(t)\|_{L^2}^2 - \frac{a}{b} |u(t,0)|^2 + \frac{1}{b} u(t,0) f(t) + \langle r(t), u(t) \rangle_{L^2} \\ &\leq -\|\partial_y u(t)\|_{L^2}^2 + \frac{\varepsilon}{b} |u(t,0)|^2 + \frac{1}{4\varepsilon b} |f(t)|^2 + \langle r(t), u(t) \rangle_{L^2} \\ &\leq -\|\partial_y u(t)\|_{L^2}^2 + C \frac{\varepsilon}{b} \left(\|u(t)\|_{L^2}^2 + \|\partial_y u(t)\|_{L^2}^2 \right) + \dots \\ &\qquad \dots + \frac{1}{4\varepsilon b} |f(t)|^2 + \langle r(t), u(t) \rangle_{L^2} \\ &\leq \varepsilon \left(\frac{C}{b} + 1 \right) \|u(t)\|_{L^2}^2 + \frac{1}{4\varepsilon b} |f(t)|^2 + \frac{1}{4\varepsilon} \|r(t)\|_{L^2}^2 \\ &\leq \varepsilon \left(\frac{C}{b} + 1 \right) \|u(t)\|_{L^2}^2 + \frac{C_0}{4\varepsilon} \left(\frac{1}{b} + 1 \right) \frac{e^{\alpha t}}{(1+t)^\beta}, \end{aligned}$$

where we have chosen $\varepsilon > 0$ small enough so that $\varepsilon \dfrac{C}{b} < 1$ where C is the constant of the embedding $H^1(\mathbb{R}_+) \hookrightarrow L^\infty(\mathbb{R}_+)$. Further requiring $2\varepsilon \left(\dfrac{C}{b} + 1 \right) < \alpha$, using the fact that b is bounded away from 0, we can apply Lemma A.2 to obtain

$$\|u(t)\|_{L^2}^2 \leq C_{\alpha,\beta} \frac{e^{\alpha t}}{(1+t)^\beta}.$$

- $a \gg 0, b \geq 0$. In this case starting from the energy inequality we replace $u(t,0) = \frac{1}{a}(b\partial_y u(t,0) + f(t,0))$. Thus for any $\varepsilon < a$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 \leq -\|\partial_y u(t)\|_{L^2}^2 - \frac{b}{a}|\partial_y u(t,0)|^2 - \frac{1}{a}\partial_y u(t,0)f(t) + \langle r(t), u(t)\rangle_{L^2}$$
$$\leq \frac{1}{4\varepsilon a}|f(t,0)|^2 + \varepsilon\|u(t)\|_{L^2}^2 + \frac{1}{4\varepsilon}\|r(t)\|_{L^2}^2.$$

Additionally requiring $2\varepsilon < \alpha$, as in the previous case, we apply Lemma A.2 to obtain the required estimate, where the constant from (2.8) is independent from a and b since $a \gg 0$.

$\square$

References

[1] Dietert, H., Gerard-Varet, D.: Well-posedness of the Prandtl equation without any structural assumption. Ann. PDE **5**, 8 (2019)
[2] Grenier, E.: On the nonlinear instability of Euler and Prandtl equations. Comm. Pure Appl. Math. **53**(9), 1067–1091 (2000)
[3] Grenier, E., Nguyen, T.T.: L^∞ instability of Prandtl Layers. Ann. PDE **5**, 18 (2019)
[4] Iftimie, D., Sueur, F.: Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. Arch. Ration. Mech. Anal. **199**(1), 145–175 (2011)
[5] Lin, Z.: Instability of some ideal plane flows. SIAM J. Math. Anal. **35**(2), 318–356 (2003)
[6] Maekawa, Y., Mazzucato, A.: The Inviscid Limit and Boundary Layers for Navier-Stokes Flows. In: Giga Y., Novotný A. (eds) Handbook of Mathematical Analysis in Mechanics of Viscous Fluids (2018)
[7] Navier, C.L.M.H.: Mémoire sur les lois du mouvement des fluides. Mem. Acad. R. Sci. Paris. **6**, 389–416 (1823)
[8] Paddick, M.: Stability and instability of Navier boundary layers. Differ. Integral Equ. **27**(9/10), 893–930 (2014)
[9] Prandtl, L.: Über Flüssigkeitsbewegung bei sehr kleiner Reibung, pp. 484–491. Verhandlung des III Internationalen Mathematiker-Kongresses, Heidelberg (1904)
[10] Piccinini, L.C., Stampacchia, G., Vidossich, G.: Ordinary Differential Equations in $\mathbb{R}^n$ - Problems and Methods. Springer, Berlin (1984)
[11] Tao, T., Wang, W., Zhang, Z.: Zero-viscosity limit of the Navier-Stokes equations with the Navier friction condition. SIAM J. Math. Anal. **52**(2), 1040–1095 (2020)
[12] Wang, X., Wang, Y., Xin, Z.: Boundary layers in incompressible Navier-Stokes equations with Navier boundary conditions for the vanishing viscosity limit. Commun. Math. Sci. **8**(4), 965–998 (2010)
[13] Weinan, E.: Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. Acta Math. Sin. **16**(2), 207–218 (2000)

Lorenzo Quarisa and José L. Rodrigo
Mathematics Institute
University of Warwick
Coventry CV47AL
United Kingdom
e-mail: lorenzo.quarisa@warwick.ac.uk