Brownian Motions in Random Environments

by

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Declarations

This work, which is being submitted to the University of Warwick for the degree of doctor of philosophy, has been carried out by the author, under the supervision of Dr Jon Warren, and has not been submitted for any previous degree.

Chapter 1 is an introduction to random walks in random environments and their continuum counterparts. I do not claim originality for the material, only in the presentation.

The results in Chapter 2 are original and have been submitted for publication [BW21], in collaboration with my supervisor Dr Jon Warren.

Chapters 3 and 4 are original and have not been submitted for publication.

Abstract

In this thesis, we focus on continuum versions of random walks in random environments in one spatial dimension; these can be thought of as modelling the trajectory of a particle in a turbulent fluid. We study the density of a cloud of particles all moving through the same environment. In Section 1.1, we review random walks in random environments; in the following sections, we discuss their continuum counterparts and our results for them.

CHAPTER 1

Introduction

1.1 Random Walks in Random Environments

A random walk in a random environment (RWRE) is simply a random walk whose transition probabilities are given by an environment consisting of random variables attached to each point in space and time. We will focus on one-dimensional random walks in dynamic random environments, that is, random walks on $\mathbb{Z}$ with random transition probabilities that depend on both time and space. We define the *random environment* as a family of $[0,1]$ valued random variables $\boldsymbol{\omega} = (\omega_{x,t})_{x,t\in\mathbb{Z}}$, with law and expectation $\mathbb{P}$, and $\mathbb{E}$ respectively. Throughout we will assume that the environment is independent and identically distributed in time and has a translation invariant distribution in space, i.e. for $\boldsymbol{\omega}_t := (\omega_{x,t})_{x\in\mathbb{Z}}$, the random variables $(\boldsymbol{\omega}_t)_{t\in\mathbb{Z}}$ are i.i.d. and $\boldsymbol{\omega}_t$ is equal in distribution to $(\omega_{x+z,t})_{x\in\mathbb{Z}}$ for any $z \in \mathbb{Z}$. The simplest such setting is where the environment consists of i.i.d. random variables, but we will also consider the case with spatial correlations that decay with distance.

We then define the *random walk in a random environment* via the transition probabilities:

$$P^{\boldsymbol{\omega}}(X(t+1) = x+1 |\ X(t) = x) = \omega_{x,t};$$
$$P^{\boldsymbol{\omega}}(X(t+1) = x-1 |\ X(t) = x) = 1 - \omega_{x,t}.$$

Where $P^{\boldsymbol{\omega}}$ and $E^{\boldsymbol{\omega}}$ denote, respectively, the law of the RWRE conditional on the environment and its expectation. By considering the random transition probabilities $P^\omega(X_t = y|\ X_0 = x)$, we can also consider this model as describing how a cloud of particles is spread through a turbulent fluid after time t. If $\mathbb{P}(\omega_{x,t} \in \{0,1\}) < 1$ then a cluster of particles can break apart, representing the individual particles' independent molecular diffusivity.

Much of the previous work on random walks in random environments has focused on the case where the environment is *static*, that is, ω_t is constant in t, as opposed to the case we consider, where the environment is *dynamic*. Many of the results do not apply in our case, as they require an ellipticity condition on the environment, more precisely they require an $\varepsilon > 0$ such that $P^{\boldsymbol{\omega}}(X_{t+1} = x + v|X_t = x) > \varepsilon$ for all x and v with $|v| = 1$. Whilst the process (t, X_t) is a random walk in a static random environment on $\mathbb{Z}^2$, the ellipticity assumption is clearly not satisfied. However, several large deviation results exist, for example, [RASY11] gives a quenched large deviation principle for a more general class of RWRE covering the case considered here and [RASY16], by the same authors, provide formulae for both the averaged and quenched large deviation principles under the same assumptions on the environment we have made here: i.i.d. in time and translation invariant in space.

An important idea for studying the random walks in random environments we are considering are the *n-point motions*, we run n random walks independently through a sampling of the environment, and then average out the environment; this breaks the particles' independence. That is, if $X(t) = (X^1(t), ..., X^n(t))$ is the n-point motion then

$$\mathbb{P}(X(t+1) = y|X(t) = x) = \mathbb{E}\left[\prod_{i=1}^{n} P^{\boldsymbol{\omega}}(X^i(t+1) = y_i|X^i(t) = x_i)\right].$$

Alternatively we can view the n-point motions as describing the behaviour of n particles thrown into the fluid. The n-point motions have a natural consistency property: any k coordinates of the n point motion have the same distribution as the k-point motion; they are also Markov processes because of our assumption that the environment is independent in time. The individual coordinates of the n-point motions behave as independent simple random walks on $\mathbb{Z}$ when far enough apart. However, when they are close enough to see correlated parts of the environment they have an interaction. In the case where the $\omega_{x,t}$ are i.i.d., the probability that a cluster of n particles in the same location all move upwards is greater than the probability they all move up when in distinct locations. This can be seen directly by applying Jensen's inequality. The probability of n particles all moving up when in the same location is simply $\mathbb{E}[\omega^n]$, where ω is a copy of an environment variable.

Thus, as a consequence of Jensen's inequality we have the desired inequality,

$$\mathbb{E}[\omega^n] \geq \mathbb{E}[\omega]^n.$$

The right hand side is the probability n particles all move up, given they are all in distinct locations. Similar behaviour occurs when the environment has spatial correlations that decay rapidly with distance. However, the interactions no longer simply occur when particles meet, but when they become close enough to see correlated parts of the environment. A group of particles situated at the same site, x, at time t can break into at most two groups. For a general environment, the probability of a group of n particles breaking into two groups of size k and l, with the k moving to $x+1$ and the l to $x-1$, is

$$\mathbb{E}[\omega_{x,t}^k(1-\omega_{x,t})^l].$$

Hence, the distribution of ω can be viewed as controlling the rate at which groups of particles break up and the size of the groups they tend to break into. When the environment has spatial correlations of some finite length similar behaviour is observed, however the particles no longer need to meet to interact.

As an example, we can consider the case where the environment random variables are chosen to be i.i.d. in space and time with $\mathbb{P}(\omega_{x,t} \in \{0,1\}) = 1$. In this case, the n-point motions become coalescing simple symmetric random walks, and their behaviour in a given realisation of the environment is deterministic; from the particle point of view, this means there is no molecular diffusivity. The environment can be viewed as a discretisation of the Brownian web, as described in [SSS16], a collection of paths of coalescing Brownian motions starting from every point in space and time. Another example, at the opposite end of the scale in terms of environment strength, is given by taking the environment to be deterministic and given by $\omega_{t,x} = \frac{1}{2}$ for every t and x. The n-point motions are then simply independent simple symmetric random walks.

An important example of RWREs is the Beta random walk in a random environment (Beta RWRE), where the $\omega_{t,x}$ are i.i.d. and have a Beta distribution. This case is studied by Barraquand and Corwin in [BC17], where they find Fredholm determinant expressions for the cumulative distribution of the n-point motions. This was done by showing an equality in law between the partition functions of an exactly solvable polymer model, and the cumulative probabilities of the Beta random walk in a random environment. A difference equation for the moments of said partition functions (at fixed times but mixed in space) was then solved via the Bethe ansatz, the proof using a non-commutative binomial formula from [Pov13]. Using their formulae for the cumulative distributions, Barraquand and Corwin showed that the Beta RWRE is in the KPZ universality class. Additionally, Balázs, Rassoul-Agha and Seppäläinen showed in [BRAS19] that conditioning the Beta RWRE to escape at an atypical velocity led to a wandering exponent of $\frac{2}{3}$ in agreement with the characteristic scaling of the KPZ universality class. After a brief review of the KPZ equation and its universality class, we discuss this and other relevant results in the next section.

1.2 The KPZ equation and Universality

The stochastic heat equation is the stochastic partial differential equation

$$\partial_t z = \frac{\nu}{2}\Delta z + \kappa z \dot{W}, \tag{1.2.1}$$

driven by a space-time white noise $\dot{W}$. The logarithm of the stochastic heat equation, $h = \frac{\nu}{2\delta}\log z$, is the Cole-Hopf solution to the KPZ equation:

$$\partial_t h = \frac{\nu}{2}\Delta h + \delta(\partial_x h)^2 + \kappa \dot{W}. \tag{1.2.2}$$

The KPZ equation is the canonical model for random surface growth and lies at the centre of the KPZ universality class [KPZ86], a class of models with fluctuations given by the Tracy-Widom distributions of random matrix theory and common scaling exponents, for more detailed information on the KPZ universality class see the survey articles [Cor12], [Qua11]. The KPZ universality class is expected to contain evolving interface models whose dynamics, like those of the KPZ equation itself, have the following three features: smoothing, slope dependent growth and a space-time uncorrelated driving noise. This is known as strong KPZ universality to distinguish it from the universality of the KPZ equation itself, which is known as weak KPZ universality. It has been shown that a large class of continuous surface growth models lie in the weak KPZ universality class [HQ18]. In addition, it has been shown that certain observables of some discrete models converge to those of the KPZ equation, via convergence to the stochastic heat equation. The earliest such result was for the height function of the weakly asymmetric exclusion process [BG97]. More recently, convergence to the KPZ equation has been shown for the free energy of directed random polymers in the intermediate disorder limit [AKQ14]. Following this result, weak KPZ universality has been shown for a generalisation of ASEP [CST18], a class of weakly asymmetric non-simple exclusion processes [DT16], the Higher-Spin Exclusion process [CT17] and the related Stochastic 6-vertex model

[CGST20]. Most relevant for us, in [CG16] Corwin and Gu showed that the transition probabilities for the random walk in a random environment evaluated in the large deviation regime, after rescaling, converge to the solution to the stochastic heat equation.

The models mentioned in the above paragraph rely on discrete versions of the Cole-Hopf transform (called Gärtner transforms) for their proofs so that they may instead show convergence towards the much more manageable stochastic heat equation. Another approach, which avoids the need for a Gärtner transform, is to show convergence to a so called energy solution of the KPZ equation, as was used in [GJ10] for weakly asymmetric, conservative particle systems with respect to the stationary states. The existence and uniqueness of energy solutions of the KPZ equation were shown in [GP15]; in addition, the authors showed that the energy solution to the KPZ equation differs from the Cole-Hopf solution by a linear drift term.

Returning to random walks in random environments, Barraquand and Corwin [BC17] showed that when the transition probabilities are chosen to be Beta distributed, the model becomes exactly solvable. They then used their formulae to show that the tail probabilities of the Beta Random walk in a random environment have Tracy-Widom GUE fluctuations of size $N^{1/3}$, placing the model in the strong KPZ universality class. Furthermore, the same result is expected to hold for the density of the transition probabilities evaluated at a point in the tail, not just for the cumulative tail probabilities [TLD16].

In the next sections, we will introduce the continuum analogues of random walks in random environments. We are interested in two cases: the first is where the environment variables are independent and identically distributed, the second is where the environment variables have correlations that decay rapidly with distance. To get interesting behaviours in the diffusive scaling limit, we need to adjust the distributions of the environment as we scale; in both cases, diffusive scaling dampens the effects of the environment, meaning we need to reinforce it. There are two ways to strengthen the effect of the environment: the first is to make the environment random variables closer to being Bernoulli $0, 1$ random variables so that the n-point motions behave like coalescing random walks with a small probability of breaking apart; the second is to increase the distance at which the environment variables remain correlated so that the correlation length remains fixed as we diffusively scale. In the following sections, we will discuss the two cases and outline our results for them.

1.3 Sticky Brownian Motions

We will begin by describing Brownian motion with a sticky point at 0 before discussing Brownian motions with sticky interactions as scaling limits of random walks in random environments. A Brownian motion with a sticky point at 0 with parameter $\theta > 0$ is a diffusion on $\mathbb{R}$ on natural scale with speed measure $2dx + \frac{2}{\theta}\delta_0(dx)$.The process can be constructed from a standard Brownian motion via a time change. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$, and $(\mathcal{L}^0_t(B))_{t\geq 0}$ be its local time at 0. Let $\theta > 0$ be a parameter and define the continuous, strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, as follows

$$\alpha(t) := t + \frac{1}{\theta}\mathcal{L}^0_t(B).$$

The function, α, has a continuous and strictly increasing inverse, α^{-1}. The process $(X_t)_{t\geq 0} := (B_{\alpha^{-1}(t)})_{t\geq 0}$ is called a Brownian motion with a sticky point at 0 with parameter θ. Because the local time of Brownian motion only increases when the Brownian motion is at 0, the resulting sticky Brownian motion spends a positive amount of time at 0. θ determines how sticky the point at 0 is, the larger θ is the less time the sticky Brownian motion spends at 0, when $\theta \to 0$ the behaviour approaches that of a Brownian motion absorbed at 0. In [Bas14] and [EP14] it was shown that sticky Brownian motion is the unique weak solution to the following system of stochastic differential equations.

$$\begin{aligned} dX_t &= \mathbb{1}_{X_t \neq 0} dB_t, \\ \mathbb{1}_{X_t = 0} dt &= \frac{1}{\theta} d\mathcal{L}^0_t(X). \end{aligned}$$

We are interested in a diffusion in $\mathbb{R}^n$ where each coordinate is a Brownian motion, and the difference between each pair of coordinates is Brownian motion with a sticky point at 0. Such a process was first defined on the circle using Dirichlet forms [LJR04b], and they were shown to arise as diffusive scaling limits of the n-point motions of RWRE on $\mathbb{Z}/n\mathbb{Z}$ when the random environment consisted of i.i.d. Beta($\frac{\theta}{n}, \frac{\theta}{n}$) random variables [LJL04]. Later Howitt and Warren [HW09] proved the following result, which we state in a reformulation proved in [SSS10], giving a condition for the convergence of the n-point motions of a general RWRE to have sticky Brownian motion as the diffusive scaling limit.

Theorem 1.3.1. *Suppose $(X(t))_{t>0}$ is the n-point motion of a RWRE, where the random environment consists of i.i.d. random variables with law μ^ε satisfying the* *following:*

$$\int_0^1 (2q-1)\mu^\varepsilon(dq) = 0,$$
$$\frac{1}{\varepsilon} q(1-q)\mu^\varepsilon(dq) \implies \nu(dq), \quad \textit{as } \varepsilon \to 0,$$

where ν is a finite measure on $[0,1]$. Then the laws of the processes $(\varepsilon X(\varepsilon^2 t))_{t\geq 0}$ converge weakly to a diffusion we call sticky Brownian motions with characteristic measure ν.

Howitt and Warren also showed that the sticky Brownian motions with characteristic measure ν exist and are the unique solution to a martingale problem, which we will state in full in Chapter 2. For the sticky Brownian motions, there is a positive probability that more than two particles can all meet at the same time, and the interaction between pairs does not determine the interaction between multiple particles. Instead, the characteristic measure determines this interaction through the values $\theta(k,l) = \int_0^1 q^k(1-q)^l \nu(dq)$ which can be thought of as the rate, in a certain excursion theoretic sense, at which a cluster of $k+l$ particles breaks up into two clusters of k and l particles.

Recall that Barraquand and Corwin [BC17] showed that when the environment is given by i.i.d. random variables with the Beta distribution, the RWRE becomes exactly solvable using the Bethe ansatz. It is easy to check that if for a $\theta > 0$, μ^ε is given by the Beta$(\theta\varepsilon, \theta\varepsilon)$ distribution, Theorem 1.3.1 is satisfied with $\nu = \frac{\theta}{2}dx$, where dx is the Lebesgue measure on $[0,1]$. This suggests that the sticky Brownian motions with characteristic measure $\frac{\theta}{2}dx$ should inherit the exact solvability of the discrete model. In Chapter 2, we first prove an explicit formula for an invariant measure for an ordered version of the sticky Brownian motions. We then prove an exact formula for the transition density for the ordered process with respect to the invariant measure.

Working independently of us, Barraquand and Rychnovsky [BR20] derived exact formulae for the tail probabilities of the sticky Brownian motions with characteristic measure $\frac{\theta}{2}dx$ by taking appropriate limits of the exact formulae for the Beta RWRE. Using their formulae, they showed that the tail of the Howitt-Warren flows has Tracy-Widom GUE fluctuations of size $t^{1/3}$. Further, they conjectured the tails of the Howitt-Warren flows converge, as the stickiness is removed and under suitable rescaling, to the stochastic heat equation, based on the convergence of the moments. Barraquand and Le Doussal then showed that the same convergence of moments holds in a moderate deviation regime, distance $t^{\frac{3}{4}}$ away from the origin as $t \to \infty$, for a fixed stickiness [BD20].

In order to state our results, we first must introduce some notation. Let $\mathbb{W}^n := \{x \in \mathbb{R}^n | \ x_1 > x_2 > ... > x_n\}$ and $\overline{\mathbb{W}^n} := \{x \in \mathbb{R}^n | \ x_1 \geq ... \geq x_n\}$ denote the principal Weyl chamber and its closure respectively. The images of the principal Weyl chamber under a permutation are called simply Weyl chambers; however, we may sometimes refer to the principal Weyl chamber as just the Weyl chamber. Let Π_n denote the collection of ordered partitions, $(\pi_1, ..., \pi_k)$, of $\{1, .., n\}$ such that if $a \in \pi_j$, $b \in \pi_k$ and $j < k$, then $a < b$. That is the elements of the partition each consist of intervals intersected with $\mathbb{Z}$ and are indexed according to the size of their elements.

To each partition $\pi \in \Pi_n$, we associate a subset of $\overline{\mathbb{W}^n}$ defined by

$$\overline{\mathbb{W}^n_\pi} := \{x \in \overline{\mathbb{W}^n} | \ x_\alpha = x_\beta \text{ if and only if there is a } \pi_i \in \pi \text{ such that } \alpha, \beta \in \pi_i\}.$$

In other words, all the points in $\overline{\mathbb{W}^n}$ whose coordinates are equal if and only if their indices are in the same element of π. Notice for $\pi = \{\{1\}, ..., \{n\}\}$, $\overline{\mathbb{W}^n_\pi} = \mathbb{W}^n$. In addition, $\overline{\mathbb{W}^n} = \cup_{\pi \in \Pi_n} \overline{\mathbb{W}^n_\pi}$, and the sets $\overline{\mathbb{W}^n_\pi}$ are disjoint. There is a natural continuous bijection $I^\pi : \overline{\mathbb{W}^n_\pi} \to \mathbb{W}^{|\pi|}$, given by $I^\pi(x) = (x_{p_1}, ..., x_{p_{|\pi|}})$ for any choice of $p_i \in \pi_i$. We can now define a Borel measure on $\overline{\mathbb{W}^n_\pi}$ as the pushforward of the Lebesgue measure λ on $\mathbb{W}^{|\pi|}$, $\lambda^\pi := (I^\pi)^{-1}_* \lambda$. The measure can be extended to a Borel measure on $\overline{\mathbb{W}^n}$ via the formula $\lambda^\pi(A) := \lambda^\pi(A \cap \overline{\mathbb{W}^n_\pi})$.

Definition 1.3.2. *For $\theta > 0$ the Borel measure $m^{(n)}_\theta$ on $\overline{\mathbb{W}^n}$ is defined as*

$$m^{(n)}_\theta := \sum_{\pi \in \Pi_n} \theta^{|\pi| - n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \lambda^\pi.$$

Suppose $\theta > 0$ and $X = (X(t))_{t \geq 0}$ is the process of sticky Brownian motions in $\mathbb{R}^n$ with characteristic measure $\frac{\theta}{2} \mathbb{1}_{[0,1]} dx$ and initial condition x under $\mathbb{P}_x$. Then we define $Y = (Y(t))_{t \geq 0}$ as the process obtained by ordering the coordinates of X, i.e. for each $t \geq 0$ $Y(t) = (Y^1(t), ..., Y^n(t)) = (X^{\sigma(1)}(t), ..., X^{\sigma(n)}(t))$ for some $\sigma \in S_n$ such that $Y^1(t) \geq ... \geq Y^n(t)$.

Theorem 1.3.3. *For every bounded Lipschitz continuous function $f : \overline{\mathbb{W}^n} \to \mathbb{R}$, $x \in \overline{\mathbb{W}^n}$ and $t > 0$*

$$\mathbb{E}_x[f(Y_t)] = \int u_t(x, y) f(y) m^{(n)}_\theta(dy).$$

Where $\mathbb{E}_x$ is the expectation under the measure $\mathbb{P}_x$ and $u_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined *for each $t > 0$ by*

$$u_t(x, y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \frac{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)} k_{\sigma(\alpha)}}{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)} k_{\sigma(\alpha)}} dk,$$

where S_n denotes the group of permutations on $\{1, ..., n\}$ and $k_\sigma = (k_{\sigma(1)}, ..., k_{\sigma(n)})$.

For a fixed characteristic measure ν, the sticky Brownian motions in $\mathbb{R}^n$, for $n \in \mathbb{N}$, form a family of Feller processes. Further, the family is consistent in the following sense; any k distinct coordinates of the sticky Brownian motions in $\mathbb{R}^n$ are equal in law to the sticky Brownian motions in $\mathbb{R}^k$. In [SSS10] Le Jan and Raimond showed that any consistent family of Feller processes are the n-point motions of a *stochastic flow of kernels*, which is a family of random probability kernels indexed by start and end times. They are analogous to the random transition probabilities for the RWRE, and their n-point motions are defined in a similar way to those for RWRE. The flows corresponding to the sticky Brownian motions are called *Howitt-Warren flows*, they are studied in depth in [SSS10] where, among many other results, it is shown that the sticky Brownian motions can be constructed as processes that are independent conditional on a random environment. The random environment is constructed from the Brownian web by marking special points where paths are allowed to branch into two separate paths; the sticky Brownian motions follow a path in the web until they meet such a point where they independently choose one of the two possible paths to follow, in a manner determined by the characteristic measure. Another result of [SSS10] shows that the Howitt-Warren flows, for deterministic times and a fixed starting point, are almost surely purely atomic measures. We can use Theorem 1.3.3 to show that in the large time limit, the size of a random atom in the Howitt-Warren flow behaves as $\frac{1}{\sqrt{t}}$ multiplied by an exponential random variable with rate θ, in Section 2.5.3 we provide further details and a precise statement of this result.

In the next section, we describe the other continuum model we are interested in, which arises when we consider spatially correlated random environments instead of i.i.d. ones.

1.4 Brownian Motions with White Noise Drifts

Another interpretation of the random walk, $(X_t)_{t \geq 0}$, in a random environment, $\boldsymbol{\omega} = (\omega_{x,t})_{x,t \in \mathbb{Z}}$, is as a random walk with a random velocity. In this setting, we assume that the environment is independent in time and correlated in space with a finite correlation length. Clearly $E^{\boldsymbol{\omega}}[X_{t+1} - X_t | \, X_t = x] = 2\omega_{x,t} - 1$, we can therefore decompose X_t into two components, where one component is mean 0 for each realisation of the environment, and the other is some function of the environment and the path of X up to time $t-1$,

$$X_t = \sum_{k=0}^{t-1} w(k, X_k) + \beta_t.$$

Here $w(t,x) := (2\omega_{x,t} - 1)$ and $\beta_t := X_t - \sum_{k=0}^{t-1} w(k, X_k)$ so that $E^{\boldsymbol{\omega}}[\beta_t] = 0$ almost surely. It is easy to see that we also have the following equalities holding almost surely $E^{\boldsymbol{\omega}}[(\beta_t - \beta_{t-1})^2 |\ X_t = x] = 1 - w(t-1, x)^2$ and $E^{\boldsymbol{\omega}}[(\beta_t - \beta_{t-1})w(t-1, X_{t-1})] = 0$. Furthermore, the coordinate processes of the n-point motions can each be decomposed in the same way, and the resulting collection of β processes are uncorrelated for any realisation of the environment. Below we introduce a continuum version of this model via an SDE mirroring the above decomposition of the RWRE.

Suppose W_ρ is a Gaussian field on $\mathbb{R}_{>0} \times \mathbb{R}$ with correlations $\mathbb{E}[W_\rho(s,x)W_\rho(t,y)] = (s \wedge t)\tilde{\rho}(x-y)$, where $\tilde{\rho} = \int \rho(\cdot - y)\rho(y)dy$ for some symmetric function $\rho \in C_c^\infty(\mathbb{R})$ ($C_c^\infty(\mathbb{R})$ denotes the set of smooth compactly supported functions on $\mathbb{R}$). We consider the SDE

$$dX_t = \mu W_\rho(dt, X_t) + \sigma dB_t, \tag{1.4.1}$$

where B is an independent standard Brownian motion on $\mathbb{R}$ and $\mu, \sigma > 0$ are parameters. Both integrals are to be interpreted in the Itô sense, see [Kun94b] for definitions, applying Theorem 3.4.1 in [Kun94b] shows the SDE has a unique solution. Just as in the discrete setting X consists of two components, one representing the effect of the environment and the other the randomness of the walk itself. Following this analogy, in the continuum version σ^2 plays the role of the quantity $1 - \mathbb{E}[w(t-1,x)^2]$, the variance of a single step of β in the discrete version. The function ρ simply gives the spatial correlation structure of the environment, and we take W_ρ to be Brownian in time to mimic the independence in time of the discrete environment. The parameter μ controls the strength of the environment; it plays the role of the quantity $\mathbb{E}[w(t,x)^2]^{\frac{1}{2}}$ in the discrete setting. Note that X_t simply behaves as a Brownian motion with diffusivity $\sigma^2 + \mu^2\tilde{\rho}(0)$. However, we do get interesting behaviour when we instead consider the behaviour conditional on the environment. This behaviour can be studied through the associated stochastic flow of kernels, $(U_{s,t})_{s \le t}$, which can be thought of as the density of an infinite number of particles in the same environment. $(U_{s,t})_{s\le t}$ is given by the simple relation

$$U_{s,t}(x, A) = \mathbb{P}^B(X_t \in A |\ X_s = x), \tag{1.4.2}$$

where $\mathbb{P}^B$ is the law of the Brownian motion B.

We are interested in the behaviour of large numbers of particles in the same fluid; n particles are described by taking n solutions to the stochastic differential equation (1.4.1) with respect to a common Gaussian field but independent Brownian motions. The resulting process in $\mathbb{R}^n$ is the n-point motion of U and can be thought of as an interacting particle system. Because the correlations of the velocity field are rapidly decaying in space, the particles behave independently when separated and then become correlated when close. However, the additional noise provided by B allows the particles to break apart. The result is a system of particles with an attractive local interaction; the strength of this interaction is determined by the molecular diffusivity σ^2 and the correlation length of W_ρ in space. The longer the correlation length and the smaller the additional diffusivity, the stronger the effect of the interaction. It is possible to take σ^2 to 0 along with the correlation length whilst preserving the interaction. The limiting process is given by sticky Brownian motions with an explicit splitting measure [War15].

The model is an example of the compressible Kraichnan model for turbulence, where the velocity field is simplified to be white in time; see the review [FGmcV01]. In our case, we take the spatial correlations to be of short length and smooth in space, similar to the case considered in [GH04], where the authors showed that removing the molecular diffusivity, at the same time as reducing the correlation length of the velocity field, led to sticky interactions between pairs of particles in the limiting process. For the model we consider, this result was extended to a full description of the interactions between any number of particles in the limiting process [War15]. This result suggests the convergence of the random transition density associated with the Brownian motion running through the random drift field W towards a Howitt-Warren flow. In [DG21], the authors study the same model and show that the fluctuations of the density of the flow of kernels solve an stochastic partial differential equation (SPDE). In addition, they show the density is well approximated as $t \to \infty$ by the product of the heat kernel and the stationary solution to that SPDE.

We study the fluctuations of the density in the tail of the random transition density, that is, at a distance t away from the origin. In particular, we will show that the fluctuations are governed by the KPZ equation when the environment noise is small. This work was motivated by the non-rigorous arguments in [DT17], as well as the results for RWRE mentioned in Section 1.2. Further, we conjecture that the KPZ equation also appears when the independent diffusivity of the particles is small instead of the environment; in this regime, the behaviour of particles is much different and closer to that of sticky Brownian motions.

It is known that the kernels $(U_{s,t})_{s\leq t}$ have continuous densities, u, with respect to the Lebesgue measure, so that $u(s,t;x,y)dy = U_{s,t}(x,dy)$, [DG21]. Further, the density solves the following SPDE [LJR04a] [DG21] when the noise is interpreted in the Itô sense; for an introduction to SPDEs see, [Wal86].

$$\partial_t u = \tfrac{\nu}{2}\Delta u - \mu\partial_y\left(u\dot{W}_\rho\right). \tag{1.4.3}$$

Here, $\nu = \sigma^2 + \|\rho\|_2^2$ and $\dot{W}_\rho$ is the formal time derivative of W_ρ. It should be noted that, apart from the $\|\rho\|_2^2$ term in front of the Laplacian, this SPDE is just the Fokker-Planck equation for a Brownian motion with diffusivity σ^2 moving through a velocity field $\dot{W}_\rho$. The additional $\|\rho\|_2^2$ term acts like an Itô correction to the noise term and does not have a physical meaning. Indeed, the solution only exists as a continuous function if $\sigma > 0$. If instead $\sigma = 0$, the solution to the SDE 1.4.1 is entirely determined by the environment, and the resulting flow of kernels (1.4.2) is almost surely for any fixed s, t and x a point mass so that there is a flow of maps solution to the SDE [LJR04a].

We evaluate u at distances of order λt away from the origin and rescale to define the tilted kernel v with tilt λ via the formula

$$v(t,y) = e^{\frac{\lambda^2}{2}\nu t + \lambda(y-x)} u(0,t;x,y+\lambda\nu t). \tag{1.4.4}$$

Then function v satisfies the SPDE

$$\partial_t v = \tfrac{\nu}{2}\Delta v + \lambda\mu v\dot{W}_\rho - \mu\partial_y\left(u\dot{W}_\rho\right). \tag{1.4.5}$$

This SPDE is a perturbation of the stochastic heat equation (1.2.1), and it suggests that by choosing ρ as an approximation to a delta function, and $\mu = \frac{1}{\lambda}$ to be small, we can recover the stochastic heat equation itself as a limit. Indeed, the SPDE suggests that the stochastic heat equation should arise under these choices, even if we also take σ to 0. This turns out to be misleading; in the following subsection, we will discuss the various scaling regimes and our results for them.

1.4.1 Fluctuations in the Tail

First, we introduce a new parameter controlling the correlation length of the Gaussian field. Suppose $\rho \in C_c^\infty(\mathbb{R})$ is non-negative and symmetric, then for $n \in \mathbb{N}$, let $\rho_n := n\rho(n\cdot)$ and $W_n := W_{\rho_n}$. The variable n governs the correlation length of the underlying Gaussian field, and $\mu > 0$ is an additional parameter governing the strength of the environment on the particle. As before, we take the molecular diffusivity to simply be some $\sigma > 0$ and the tilt to be $\lambda > 0$.

Remark 1.4.1. *Instead of changing the underlying Gaussian field by varying n, we can diffusively rescale the density. For a set of parameters (σ, μ, ρ), the diffusively rescaled density $u^{(\varepsilon)}(s, t; x, y) = \frac{1}{\varepsilon} u(\frac{s}{\varepsilon^2}, \frac{t}{\varepsilon^2}; \frac{x}{\varepsilon}, \frac{y}{\varepsilon})$ is equal in distribution to the density with parameters $(\sigma, \sqrt{\varepsilon}\mu, \varepsilon^{-1}\rho(\varepsilon^{-1}\cdot))$. We can also diffusively rescale the tilted density (1.4.4) in the same way, which changes the parameters $(\sigma, \mu, \rho, \lambda)$ to $(\sigma, \sqrt{\varepsilon}\mu, \varepsilon^{-1}\rho(\varepsilon^{-1}\cdot), \varepsilon^{-1}\lambda)$.*

Apart from looking at the SPDE (1.4.5), we can also use moment calculations using the n-point motions to guess the right choice of scalings. We will discuss the moment calculations in further detail in Chapter 3. For now, we describe the scaling regimes under which we get convergence of the moments of v towards the moments of a stochastic heat equation. To begin, we choose all parameters to depend on n, for the tilt we set $\lambda = n^\beta$ where $\beta > 0$, for the remaining parameters we set $\frac{n\mu^2}{\sigma^2} = n^{2\alpha}$ for some $\alpha \in \mathbb{R}$. The quantity $\frac{n\mu^2}{\sigma^2}$ can be thought of as a measure of the interaction strength between the n-point motions; it is the ratio between the diffusivity contributed by the environment and the molecular diffusivity. Recall that each particle has its own independent molecular diffusivity but moves through the same environment. We add the condition that if $\alpha \geq 0$, then $n\mu^2$ is held constant, and if $\alpha \leq 0$, then σ is held constant. The result is figure (1.1), for which the line represents the choices of parameters for which we conjecture the stochastic heat equation appears as the limit of the tilted kernels, defined by 1.4.4, as $n \to \infty$ with the preceding choice of parameters.

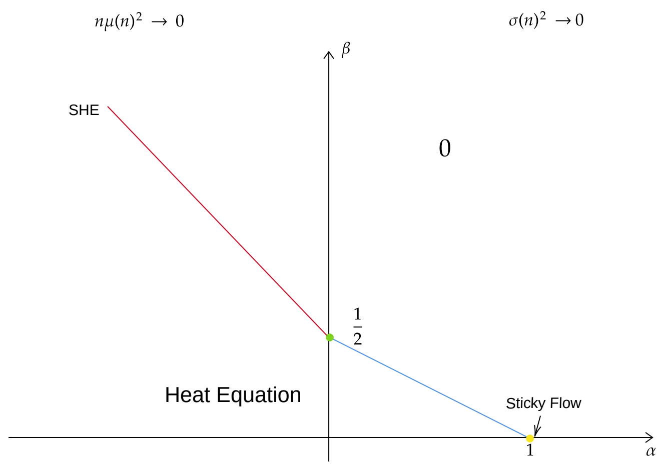


Figure 1.1: Above the line, we expect the limit to be 0 in probability. Below the line, the limit is the heat equation. On the $\beta = 0$ axis the limit is a sticky flow for $\alpha = 1$, for $\alpha > 1$ it is the Arratia flow.

Below, we summarise the distinct regimes in which we conjecture the appearance of the stochastic heat equation based on our above moment calculations. We also state the conjectured coefficients of the limiting stochastic heat equations, using the parameter labels as in (1.2.1).

1. **Weak environment**, where we have $n\mu(n)^2$ vanishing in the limit and $\sigma(n)$ is constant, so $\alpha < 0$ in figure 3.1. We also take $\lambda(n) = \mu(n)^{-1}$, so that $\beta = \frac{1}{2} - \alpha$. This regime agrees with the scalings suggested by the SPDE; the coefficients for the SHE are $\nu = \sigma^2$ and $\kappa = 1$.

2. **Weak diffusivity**, where we have $n\mu(n)^2$ constant and $\sigma(n)$ vanishing in the limit, so we have $\mu(n) = n^{-\frac{1}{2}}$ and $\alpha > 0$ in figure 3.1. Here we require $\frac{\lambda(n)^2}{n\sigma(n)}$ to converge, which disagrees with the scalings suggested by the SPDE, so we set $\lambda(n) = n\sigma(n) = n^{\frac{1-\alpha}{2}}$. Hence, we have $\beta = \frac{1-\alpha}{2}$ in Figure 3.1; the coefficients for the SHE are $\nu = \tilde{\rho}(0)$ and $\kappa = \left(\frac{\pi\|\rho\|_2^2}{\|\rho'\|_2}\right)^{\frac{1}{2}}$. Since we need $\lambda(n) \to \infty$ as $n \to \infty$, we require $\alpha \in (0,1)$. Note that for $\alpha = 1$ the limit is a sticky flow.

3. **Fixed diffusivity and environment**, where $n\mu(n)^2$ and $\sigma(n)$ are held constant, i.e. $\mu(n) = n^{-\frac{1}{2}}$ and $\sigma(n) = \sigma > 0$, we also take $\lambda(n) = \mu(n)^{-1} = n^{\frac{1}{2}}$. In the diagram, this is the green dot where the line hits the β axis, when $\alpha = 0$. This regime agrees with the scalings suggested by the SPDE, however the limiting SHE disagrees, instead of $\kappa = 1$, the limiting SHE has $\kappa > 1$.

In Chapter 3, we discuss the weak environment regime. In this setting, we can use an explicit formula for the chaos expansion of the density u from [LJR04a] to prove convergence when $\alpha < -1$. We get the following result for the tilted densities (1.4.4) with choice of parameters $(\sigma, \mu, \rho, \lambda) = (\sigma, n^{-\frac{1}{2}-\alpha}, \rho_n, n^{\frac{1}{2}-\alpha})$ for $\alpha < -1$.

Theorem 1.4.2. *Let $z_x \in C((0,T); C(\mathbb{R}))$* [1] *be the solution to the stochastic heat equation (3.1.19) with diffusivity ν, driving noise W defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and initial condition $\delta(x-y)$, where x is taken as a constant and y is the space variable. Then for every $f \in C_c^\infty(\mathbb{R})$ and $t > 0$*

$$\int v(t,x,y)f(y)dy \to \int z_x(t,y)f(y)dy, \qquad in\ L^2(\Omega).$$

In the weak diffusivity regime, we can no longer use the chaos expansion, as the limiting stochastic heat equation is not driven by the same noise. Because of this, we need a way of determining the solution to the stochastic heat equation without reference to the underlying noise. To achieve this, we would like to use the martingale

[1] Where, for topological spaces X and Y, the space $C(X;Y)$ denotes the space of continuous functions from X to Y endowed with the topology of uniform convergence on compact sets. As before $C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ and we also endow it with the topology of uniform convergence on compact sets.

problem, which characterises the solution to the stochastic heat equation, leading to a second problem. The martingale problem for the stochastic heat equation requires the squaring of the solution. However, when $\sigma \to 0$, the behaviour of the densities is so bad that their square diverges. This is not that surprising as we only expect the densities to converge weakly, but it does stop us from showing limit points satisfy the martingale problem directly. For this reason, instead of working with the density itself, we work with a smoothed version of the density, removing the smoothing slowly enough that the square of the smoothed density remains well behaved. Pursuing this, we get the following result. Once again v_n is the tilted density, this time with parameters $(\sigma, \mu, \rho, \lambda) = (n^{-\alpha}, n^{-\frac{1}{2}}, \rho_n, n^{\frac{1-\alpha}{2}})$ for $\alpha \in (0, 1)$, and $\psi : \mathbb{R} \to \mathbb{R}$ is assumed to be a mollifier.

Theorem 1.4.3. *Suppose $m = m(n)$ is a real valued sequence such that $m(n) \to \infty$ as $n \to \infty$ and $m(n)n^{-\frac{1}{2}} \to 0$ as $n \to \infty$, and there is a weakly convergent subsequence of the sequence of random variables $(v_n(\cdot) * \psi_m)_{n=1}^{\infty} \subset C((0,T), C(\mathbb{R}))$ with limit v such that there is a constant $C > 0$ with $\mathbb{E}[v(t,y)^2] \leq Cp_t(x-y)$ for every $t > 0$ and $y \in \mathbb{R}$, where p_t^{ν} denotes the heat kernel with diffusivity ν. Then v is equal in distribution to the solution to the stochastic heat equation, with initial condition δ_x:*

$$\partial_t z_x = \frac{\|\rho\|_2^2}{2} \Delta z_x + \frac{\sqrt{\pi}\|\rho\|_2}{\|\rho'\|_2^{1/2}} z_x \dot{W}. \tag{1.4.6}$$

This will be proved in Chapter 4. Note that this does not show that the sequence $v_n * \psi_m$ is convergent in $C((0,T), C(\mathbb{R}))$, only that all limit points satisfy the stochastic heat equation. Proving convergence requires we prove tightness of the sequence of smoothed densities in the appropriate space, which we have not shown; however, this would be a natural next step for further work.

CHAPTER 2

The Bethe Ansatz for Sticky Brownian Motions

2.1 Introduction

In this chapter, we study the sticky Brownian motions introduced in Section 1.3. The process consisting of sticky Brownian motions is a diffusion in $\mathbb{R}^n$, the coordinates of which evolve as independent one-dimensional Brownian motions when they are distinct and have an attractive, so called sticky interaction when they are equal. The diffusion can be interpreted as the evolving positions of n particles on the real line, which interact when they meet. In particular, the difference between two coordinates is described by a one-dimensional sticky Brownian motion, recently studied as the weak solution to an SDE in [Bas14], and [EP14]. Recall from Section 1.3 that sticky Brownian motion with parameter $\theta > 0$ is a diffusion in $\mathbb{R}$ on natural scale with speed measure $m(dx) = 2dx + \frac{2}{\theta}\delta_0(dx)$. The diffusion in $\mathbb{R}^n$ can visit the diagonal $\{x \in \mathbb{R}^n | \; x_1 = ... = x_n\}$ for a set of times with positive Lebesgue measure, quite unlike a standard Brownian motion in $\mathbb{R}^n$. The interaction between coordinates at such times is not determined solely by specifying the parameter θ describing the stickiness. It was shown in [HW09] that the possible interactions can be specified by a finite measure on $[0, 1]$ called the characteristic, or splitting, measure. The diffusions are consistent, in that for any $k < n$, any k coordinates of the sticky Brownian motions in $\mathbb{R}^n$ with characteristic measure ν, are sticky Brownian motions in $\mathbb{R}^k$ with the same characteristic measure, ν. An example of such a diffusion was originally investigated by Le Jan and Raimond [LJR04b] using Dirichlet forms (on the torus rather than Euclidean space), and then the more general case was studied by Howitt and Warren [HW09] via a martingale problem which we describe later.

The consistency property means that we can also consider such systems of sticky Brownian motions to be the n-point motions of a stochastic flow of kernels. A flow of kernels $(K_{s,t}(x, dy))_{s\leq t}$ is essentially a random family of transition probability measures for a Markov process. Le Jan and Raimond introduced flows of kernels in [LJR04a] as a generalisation of flows of maps to study stationary evolutions of turbulent fluids. The n-point motions can then be thought of as describing the behaviour of n particles thrown into the fluid. Stochastic flows of kernels whose n-point motions are described by sticky Brownian motions are called Howitt-Warren flows in [SSS10], where their properties are studied in detail. As discussed in the previous chapter, sticky behaviour arises in certain limits of the Kraichnan model for turbulent advection, as shown by Gawedzki and Horvai, [GH04]. Warren then proved the convergence of n particles towards sticky Brownian motions with an explicit characteristic measure [War15]. Sun, Swart and Schertzer studied Howitt-Warren flows, constructing them directly as flows of mass in the Brownian web [SSS10] by marking special separation points and attaching extra random variables to them that tells the mass following a path in the web how to split. The law of these additional random variables is described by the characteristic measure. Amongst other results, they showed that the Howitt-Warren flows are almost surely purely atomic at deterministic times.

In this chapter, we will derive the Kolmogorov backwards equation for the sticky Brownian motions with ordered coordinates from the martingale problem characterisation. In the case that the characteristic measure is uniform, we apply the Bethe ansatz to find an exact formula for the transition density of this process. The choice of uniform characteristic measure seems to be essential, only in this case is the diffusion exactly solvable by the Bethe ansatz. Further, this seems to be the only case the diffusion is reversible, at least with respect to a measure we can write down explicitly. Note that we are finding the transition density for the process with ordered coordinates. Whilst it is possible to retrieve the transition density of the original process for two particles, it is unclear if it is possible for an arbitrary number of particles. Our method is similar to that used by Tracy and Widom for the delta Bose gas [TW08]; however, the importance of interactions between more than two particles adds significant complexity. A discussed in section 1.3, sticky Brownian motions with a uniform characteristic measure arise as the scaling limit of the exactly solvable random walk in a Beta random environment model. The Beta RWRE has close connections to the KPZ universality class, see [BC17] and [BRAS19]. Barraquand and Rychnovsky [BR20], working independently of us, derived exact solutions for the point to half-line probabilities of sticky Brownian motions with uniform characteristic measures by taking limits of the exact formulae for the Beta RWRE. An asymptotic analysis then led to the discovery of GUE Tracy-Widom fluctuations in the large deviations of sticky Brownian motions.

The main part of this chapter will be the proof of Theorem 1.3.3, that sticky Brownian motions are exactly solvable via Bethe ansatz. Let us briefly sketch the method, which is based on [TW08]. Let $(Y_t)_{t\geq 0}$ be the ordered sticky Brownian motions discussed above Theorem 1.3.3; we begin by showing that if u satisfies the below PDE, then $\int_{\mathbb{R}} u_t(x,y)f(y)dy = \mathbb{E}_x[f(Y_t)]$.

$$\begin{cases} \frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t, & \text{for all } x \in \overline{\mathbb{W}^n}; \\ \theta\left(\frac{\partial u}{\partial x_b} - \frac{\partial u}{\partial x_a}\right) = (b-a)\frac{\partial^2 u}{\partial x_a \partial x_b}, & \text{when } x_a = x_b, \text{ for some } a < b. \end{cases} \tag{2.1.1}$$

We then show that the PDE can be solved with the Bethe ansatz, which we construct by first considering the $n = 2$ problem. Using a similar idea to how the transition density for a reflected Brownian motion in one dimension can be found, we try to combine solutions with permuted coordinates so that the boundary conditions are satisfied.

$$u_t(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^2} e^{-\frac{1}{2}t|k|^2}\left(A(k)e^{ik\cdot(x-y)} + B(k)e^{ik\cdot((x_2,x_1)-y)}\right)dk. \tag{2.1.2}$$

Notice that when $x_1 = x_2$, the exponential terms become equal. Thus, the boundary conditions will be satisfied if we have

$$(i\theta(k_2-k_1)+k_1k_2)\,A(k) + (i\theta(k_1-k_2)+k_1k_2)\,B(k) = 0.$$

It turns out that setting $A(k) = 1$ and $B(k) = \frac{i\theta(k_2-k_1)+k_1k_2}{i\theta(k_2-k_1)-k_1k_2}$ ensures the correct initial condition is satisfied. The Bethe ansatz then suggests we guess the following solution for general n:

$$u_t(x,y) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2}\sum_{\sigma\in S_n} e^{ik_\sigma\cdot(x-y_\sigma)}\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\frac{i\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})+k_{\sigma(\beta)}k_{\sigma(\alpha)}}{i\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})-k_{\sigma(\beta)}k_{\sigma(\alpha)}}dk,$$

where S_n denotes the group of permutations on $\{1,...,n\}$ and $k_\sigma = (k_{\sigma(1)},...,k_{\sigma(n)})$. The construction ensures that the boundary conditions for $x_a = x_{a+1}$ are always satisfied; in Section 2.4.2, we will prove the remaining conditions. Next, we make some remarks on the Bethe ansatz.

Remark 2.1.1. *Note that the function u_t is well defined (the integral always con-*

verges), because for every $t > 0$, $x, y, k \in \mathbb{R}^n$, and every permutation $\sigma \in S_n$

$$\left| e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \frac{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)} k_{\sigma(\alpha)}}{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)} k_{\sigma(\alpha)}} \right| = 1.$$

The function above is not continuous at points where there are distinct α, β such that $k_\alpha = k_\beta = 0$ (where the denominator vanishes), but since the modulus is constant the value can simply chosen to be 1 here, and it does not affect the integral because such points have measure zero. It is easily seen that we can pass derivatives under the integral, and thus we have $u_t(\cdot, y) \in C_0^2(\mathbb{R}^n)$ for all $t > 0$ and $y \in \mathbb{R}^n$. In particular, $u_t(\cdot, y) \in C_0^2(\overline{\mathbb{W}^n})$ for all $t > 0$ and $y \in \overline{\mathbb{W}^n}$ when restricted to $\overline{\mathbb{W}^n}$. As we will show later, it is also the case that $u_t(x, y) = u_t(y, x)$ for all $t > 0$ and $x, y \in \mathbb{R}^n$.

Remark 2.1.2. *Another representation $u_t(x, y)$ is in terms of a product of eigenfunctions of the generator of the ordered sticky Brownian motions. For each $k \in \mathbb{R}^n$ we have an eigenfunction given by*

$$E_k(x) := \sum_{\sigma \in S_n} e^{ik_\sigma \cdot x} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \frac{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)} k_{\sigma(\alpha)}}{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)} k_{\sigma(\alpha)}}.$$

The transition density is given by

$$u_t(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{W}^n} e^{-\frac{1}{2}t|k|^2} E_k(x) \overline{E_k(y)} dk.$$

The proof the above expression for $u_t(x, y)$ agrees with the one previously given is straightforward, and so omitted.

Furthermore, we prove that $m_\theta^{(n)}$ (Definition 1.3.2) is in fact the stationary measure of the ordered sticky Brownian motions, and that they are reversible with respect to $m_\theta^{(n)}$.

The Howitt-Warren flows are almost surely purely atomic; it is possible to interpret the values of the transition densities of the ordered n-point motions, the process Y above, as the moments of the size of the atom at a given location. Using this interpretation, we consider the fluctuations of the sizes of the atoms as $t \to \infty$ and find them to be exponentially distributed when taken to be $\sim \sqrt{t}$ away from the origin, with parameter determined by θ. This result is similar to the Gamma fluctuations found in the same regime for the point to point probabilities of the Beta random walk in a random environment by Thierry and Le Doussal [TLD16]. In the same paper, the authors found that in the large deviation regime, the fluctuations have Tracy-Widom GUE fluctuations, just as for the point to half-line probabilities. It thus seems reasonable to conjecture the same fluctuations appear in the size of atoms of the Howitt-Warren flows, but we do not pursue the necessary asymptotic analysis here.

The outline of the chapter is as follows: In Section 2.2 we define the diffusion via a martingale problem, in Section 2.3 we derive the Kolmogorov backwards equation for the ordered n-point motions, and show that the generator of the process is symmetric with respect to the measure $m_\theta^{(n)}$ when restricted to a certain class of C^2 functions. In Section 2.4 we show that the backwards equation is solvable by the Bethe ansatz, and as a consequence, we show that the ordered n point motions are reversible with respect to $m_\theta^{(n)}$. Finally, in Section 2.5 we introduce stochastic flows of kernels and apply our results to Howitt-Warren flows.

2.2 A Consistent Family of Sticky Brownian Motions

We introduce the *Howitt-Warren martingale problem* in $\mathbb{R}^n$ with drift $\beta \in \mathbb{R}$ and characteristic measure ν (a finite measure on $[0,1]$), as formulated in [HW09]. Solutions are processes in $\mathbb{R}^n$ representing the positions of n particles each moving as one dimensional Brownian motions with drift β. When two or more particles meet, they undergo sticky interactions determined by ν. The solutions are consistent, in the sense that if X is the solution to martingale problem in $\mathbb{R}^n$ with characteristic measure ν and drift β, then for any choice of distinct $i_1, ..., i_k \in \{1, ..., n\}$ with $k < n$, $(X^{i_j})_{j=1}^k$ is a solution to the martingale problem in $\mathbb{R}^k$ with characteristic measure ν and drift β.

To each point $x \in \mathbb{R}^n$ we associate a partition of the set $\{1, ..., n\}$, $\pi(x)$, where $i, j \in \{1, ..., n\}$ are in the same component of $\pi(x)$ if and only if $x_i = x_j$. Next, for each pair of disjoint subsets $I, J \subset \{1, ..., n\}$, we define the vectors $v_{I,J} \in \mathbb{R}^n$ as

$$(v_{I,J})_i = \begin{cases} 1, & \text{if } i \in I; \\ -1, & \text{if } i \in J; \\ 0, & \text{otherwise.} \end{cases}$$

Note that I and J are allowed to be empty. Then we define the set of vectors $\mathcal{V}(x)$ as

$$\mathcal{V}(x) := \{v_{I,J} : \quad I \cup J \in \pi(x),\ I \cap J = \emptyset.\}.$$

$\mathcal{V}(x)$ keeps track of the directions in which the process can infinitesimally move. We'll use this to describe the interactions. Define the parameters $\theta(k,l)$ for $k, l \geq 1$

by

$$\theta(k,l) := \int_0^1 x^{k-1}(1-x)^{l-1}\nu(dx). \tag{2.2.1}$$

For $k,l \geq 0$, first set $\theta(1,0) - \theta(0,1) = \beta$ and $\theta(0,0) = 0$, imposing the consistency property $\theta(k,l) = \theta(k+1,l) + \theta(k,l+1)$ for all $k,l \geq 0$ gives definition to all $k,l \geq 0$.

Definition 2.2.1. *Let D_n be the collection of functions $f : \mathbb{R}^n \to \mathbb{R}$ which are continuous and are such that for all Weyl chambers $A \subset \mathbb{R}^n$ the restriction of f to A is linear, so that if $A \subset \mathbb{R}$ is a Weyl chamber and $x, y \in A$ then $f(x+y) = f(x) + f(y)$.*

For functions $f \in D_n$ we define the operator $\mathcal{A}_n^\theta$ by

$$\mathcal{A}_n^\theta f(x) := \sum_{v_{I,J} \in \mathcal{V}(x)} \theta(|I|,|J|) \nabla_{v_{I,J}} f(x).$$

Where $\nabla_{v_{I,J}}$ denotes the one sided derivative in direction $v_{I,J}$.

Definition 2.2.2. *Let $(X(t))_{t\geq 0} = \big((X^1(t), ..., X^n(t))\big)_{t\geq 0} \subset \mathbb{R}^n$ be a continuous square integrable semi-martingale with initial condition $X(0) = x \in \mathbb{R}^n$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then $(X(t))_{t\geq 0}$ is a solution to the **Howitt-Warren martingale problem** with drift β and characteristic measure ν if for any $i, j \in \{1, ..., n\}$:*

$$\langle X^i, X^j \rangle(t) = \int_0^t \mathbb{1}_{\{X^i(s) = X^j(s)\}} ds,$$

and the following process is a martingale with respect to the filtration generated by X, for every function $F \in D_n$,

$$F(X(t)) - \int_0^t \mathcal{A}_n^\theta F(X(s)) ds.$$

Note that the first condition implies that $\langle X^i, X^i \rangle(t) = t$, and it follows from the second condition and the definition of $\mathcal{A}_n^\theta$ that $X^i(t) - \beta t$ is a martingale for each i. Hence each coordinate must be a Brownian motion with drift β. The well posedness of this martingale problem and that the solutions do indeed form a consistent family of Feller processes is shown in [HW09].

2.3 The Backwards Equation

2.3.1 The Generator of Ordered Sticky Brownian Motions

Define the functions $F^{(i)} : \mathbb{R}^n \to \mathbb{R}$ by $F^{(i)}(x) = x_j$ where x_j is the i^{th} largest coordinate of x, and $F : \mathbb{R}^n \to \overline{\mathbb{W}^n}$ by $F(x) := (F^{(1)}(x), ..., F^{(n)}(x))$. Note that these functions are in D_n. Further, suppose $X = (X(t))_{t\geq 0}$ is a solution to the Howitt-Warren martingale problem in $\mathbb{R}^n$ with characteristic measure ν, drift $\beta = 0$ and initial condition $x \in \overline{\mathbb{W}^n}$. Then define the process $Y = (Y(t))_{t\geq 0}$ by $Y(t) := F(X(t))$. Note that we defined Y from x started inside the Weyl chamber. This process lies entirely in the Weyl chamber $\overline{\mathbb{W}^n}$, making it admissible to the Bethe ansatz. This section aims to identify the Kolmogorov Backwards equation for Y and from it the invariant measure for Y.

Remark 2.3.1. *Before talking about its Kolmogorov backward equation, we need to know Y is a Markov process. For this, we refer to Dynkin's criterion [RP81]. In particular, we only need to show that $\mathbb{E}_x[f \circ F(X_t)] = \mathbb{E}_{F(x)}[f(Y_t)]$ for every $x \in \mathbb{R}^n$. The equality holds by definition for $x \in \overline{\mathbb{W}^n}$; for $x \in \mathbb{R}^n \setminus \overline{\mathbb{W}^n}$, we need to show that for any permutation $\sigma \in S_n$ $\sigma(X(t)) := (X^{\sigma(1)}(t), ..., X^{\sigma(n)}(t))$ remains a solution to the same Howitt-Warren martingale problem, but with initial condition $\sigma(x)$. It is clear $\sigma(X)$ remains a continuous square-integrable semi-martingale and has initial condition $\sigma(x)$. Further, it is immediate that $\sigma(X)$ has the correct quadratic variations. Finally, because the function σ is a continuous, linear, and maps Weyl chambers to Weyl chambers, $\{F \circ \sigma : F \in D_n\} = D_n$, the martingale problem is still satisfied by $\sigma(X)$. For each $x \in \mathbb{R}^n$, there exists a permutation $\sigma \in S_n$ such that $\sigma(x) \in \overline{\mathbb{W}^n}$, and by definition, $\sigma(x) = F(x)$. By uniqueness of solutions to the martingale problem, we have $\mathbb{E}_x[f \circ F(X_t)] = \mathbb{E}_x[f \circ F \circ \sigma^{-1} \circ \sigma(X_t)] = \mathbb{E}_{\sigma(x)}[f \circ F \circ \sigma^{-1}(X_t)]$ but clearly $F \circ \sigma^{-1} = F$. Hence $\mathbb{E}_x[f \circ F(X_t)] = \mathbb{E}_{\sigma(x)}[f \circ F(X_t)] = \mathbb{E}_{F(x)}[f(Y_t)]$ as required; thus, $Y = F(X)$ is a Markov process.*

We proceed by deriving the action of the generator of Y on certain C^2 functions.

Definition 2.3.2. *Let $\mathcal{D}_\theta$ denote the set of functions $f \in C_0^2(\overline{\mathbb{W}^n})$ such that for any $a, b \in \{1, ..., n\}$ with $a < b$, $x_a = x_b$ implies*

$$\frac{1}{2} \sum_{\substack{a\leq i,j\leq b: \\ i\neq j}} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \sum_{i=a}^{b} \frac{\partial f}{\partial x_i}(x) \sum_{k=0}^{b-a+1} \binom{b-a+1}{k} \theta(k, b-a+1-k)\, \mathrm{sign}(k-i+a-1). \quad (2.3.1)$$

Where sign(0) *is taken to be* 1 *here.*

Proposition 2.3.3. *Suppose $f \in \mathcal{D}_\theta$ then, denoting the generator of the process Y by $\mathcal{G}_\theta$ (in the sense of [RY13]), we have*

$$\mathcal{G}_\theta f = \frac{1}{2}\Delta f.$$

The same calculations will also give us a backwards equation for the process.

Proposition 2.3.4. *Suppose $g \in C^2(\mathbb{R}_{>0} \times \overline{\mathbb{W}^n})$, and $g(t,\cdot) \in \mathcal{D}_\theta$ for all $t > 0$. Further, suppose that g satisfies the PDE*

$$\frac{\partial g}{\partial t} = \frac{1}{2}\Delta g, \text{ for all } t > 0,\ x \in \overline{\mathbb{W}^n}. \tag{2.3.2}$$

With the initial condition $g(0,x) = f(x)$ for a function $f \in C_b(\overline{\mathbb{W}^n})$. To be precise, we require that $g(t,\cdot) \to f$ uniformly as $t \to 0$. Then for each $t > 0$ $(g(t-s, Y(s)))_{s\in[0,t]}$ is a continuous local martingale.

Proof of Proposition 2.3.3. Since X solves the martingale problem, and $F^{(i)} \in D_n$, Y is a semi-martingale. For $f \in C_0^2(\overline{\mathbb{W}^n})$, Itô's formula gives

$$\begin{aligned}&\mathbb{E}_x[f(Y(t))] = f(x)\\ &\quad + \sum_{i=1}^n \mathbb{E}_x\left[\int_0^t \frac{\partial f}{\partial x_i}(Y(s))dY^i(s)\right] + \frac{1}{2}\sum_{i,j=1}^n \mathbb{E}_x\left[\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))d\langle Y^i, Y^j\rangle(s)\right].\end{aligned}$$

We need to calculate the quadratic covariations for Y. Before we proceed to the rather abstract proof we'll provide a heuristic for what the answer should be. We know from the martingale problem for the sticky Brownian motions, Definition 2.2.2, that for the unordered process the bracket is given by

$$\langle X^i, X^j\rangle(t) = \int_0^t \mathbb{1}_{X^i(s)=X^j(s)}ds.$$

However, we also know that the ordered process behaves the same as the unordered process, apart from a drift when particles meet that maintains the order of the particles. Thus, making the reasonable assumption that this drift does not contribute to the bracket, we should expect

$$\langle Y^i, Y^j\rangle(t) = \int_0^t \mathbb{1}_{Y^i(s)=Y^j(s)}ds.$$

This heuristic turns out to be correct, which we show below.

Let $P_i = \{A \subset \{1,...,n\} \mid |A| = n-i+1\}$ be the set of subsets of $\{1,...,n\}$ with exactly

$n-i+1$ elements; define $f_A : \mathbb{R}^n \to \mathbb{R}$ as $f_A(x) = \max_{a\in A} x_a$ and $g_i : \mathbb{R}^{P_i} \to \mathbb{R}$ as $g_i((y_A)_{A\in P_i}) = \min_{A\in P_i} y_A$. Then $F^{(i)}(x) = g_i\left((f_A(x))_{A\in P_i}\right)$, f_A is a convex function and g_i is a concave function. Referring to [Gri13, Proposition 8] we can write the local martingale part of $F^{(i)}(X)$ in terms of a linear combination of stochastic integrals with respect to the X^i. In particular, we can write

$$f_A(X_t) = f_A(x) + \sum_{a\in A} \int_0^t \mathbb{1}_{B_a^A}(X_s) dX_s^a + C_t.$$

Where C_t has finite variation, and $B_a^A = \{x : \min_{k\in A}\{k : \max_{j\in A} x_j = x_k\} = a\}$. Notice that for a fixed x and A there is only one a such that $\mathbb{1}_{B_a^A}(x)$ is non zero.

Now we put an ordering on the set P_i. The specific ordering does not matter; we just need to be able to minimise over the indices of elements in $\mathbb{R}^{P_i}$. Suppose $A, B \in P_i$ are distinct, define $(a_j)_{j=1}^{n-i+1}$ and $(b_j)_{j=1}^{n-i+1}$ as the elements of A and B respectively in increasing order. We say $A < B$ if for $l := \min\{k \in \mathbb{N} : b_k \neq a_k,\ 1 \le k \le n-i+1\}$ we have $a_l < b_l$; if instead $b_l < a_l$, then $B < A$. This ordering is a total ordering for P_i. Supposing Z is a semi-martingale taking values in $\mathbb{R}^{P_i}$ with decomposition $Z_t = Z_0 + M_t + K_t$ where M is a local martingale and K a process with finite variation. Then, using that for $y \in \mathbb{R}^{P_i}$ $-g_i(-y) = -\max_{A\in P_i}(-y_A)$, we have

$$-g_i(-Z_t) = -g_i(-Z_0) + \sum_{A\in P_i} \int_0^t \mathbb{1}_{B_A}(Z_s) dZ_s^A + D_t,$$

here D has finite variation and $B_A := \{z \in \mathbb{R}^{P_i} : \min\{B \in P_i : \inf_{C\in P_i} z_C = z_B\} = A\}$ with the minimum understood in terms of the ordering we just defined on P_i. That is, B_A is the subset of $z \in \mathbb{R}^{P_i}$ such that $z_A \le z_B$ for any $B \in P_i$, and for any $B < A$ (according to the ordering defined in the previous paragraph) $z_B > z_A$. Notice that for a fixed z there is only one A such that $\mathbb{1}_{B_A}(z)$ is non zero. The local martingale part of $Y^i = g_i((f_A(X))_{A\in P_i})$ is given by

$$\sum_{A\in P_i} \sum_{a\in A} \int_0^t \mathbb{1}_{B_a^A}(X_s) \mathbb{1}_{B_A}((f_C(X_s))_{C\in P_i}) dX_s^a.$$

Giving that the quadratic covariation processes are

$$\begin{aligned}&\langle Y^i, Y^j \rangle_t \\ &= \sum_{\substack{A\in P_i,\\ B\in P_j}} \sum_{\substack{a\in A,\\ b\in B}} \int_0^t \mathbb{1}_{B_a^A}(X_s) \mathbb{1}_{B_A}((f_C(X_s))_{C\in P_i}) \mathbb{1}_{B_b^B}(X_s) \mathbb{1}_{B_B}((f_C(X_s))_{C\in P_i}) \mathbb{1}_{\{X_s^a = X_s^b\}} ds.\end{aligned}$$

Recall $f_C(x) = \max_{c\in C} x_c$ so that $\mathbb{1}_{B_A}((f_C(x))_{C\in P_i})$ is non zero precisely when A is the subset of $\{1, ..., n\}$ with indices corresponding to the first $i-1$ largest coordinates of X_s removed, call this set $A_i(X_s)$. Then $\mathbb{1}_{B_a^{A_i(X_s)}}(X_s)$ is non zero if and only if a is the smallest element of $\{1, ..., n\}$ such that X_s^a is equal to the ith largest coordinate of X_s, i.e. Y_s^i. Hence we have

$$\langle Y^i, Y^j \rangle_t = \int_0^t \mathbb{1}_{\{Y_s^i = Y_s^j\}} ds.$$

The martingale problem also tell us that for each i

$$Y^i(t) - \int_0^t \mathcal{A}_n^\theta F^{(i)}(X(s)) ds$$

is a martingale. Recall $f \in C_0^2(\overline{\mathbb{W}^n})$, thus $\frac{\partial f}{\partial x_i}$ is bounded on $\overline{\mathbb{W}^n}$ so that the stochastic integral with respect to the martingale part of Y is a true martingale. Thus, we can rewrite the expectation as

$$\begin{aligned} \mathbb{E}_x[f(Y(t))] = f(x) + \sum_{i=1}^n \mathbb{E}_x \left[\int_0^t \frac{\partial f}{\partial x_i}(Y(s)) \mathcal{A}_n^\theta F^{(i)}(X(s)) ds \right] \\ + \frac{1}{2} \sum_{i,j=1}^n \mathbb{E}_x \left[\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s)) \mathbb{1}_{\{Y^i(s) = Y^j(s)\}} ds \right]. \end{aligned} \tag{2.3.3}$$

By evaluating $\mathcal{A}_n^\theta F^{(i)}$, and then differentiating equation (2.3.3) in time, we can determine the generator of Y.

Let $x \in \mathbb{R}^n$ and denote $y = F(x) \in \overline{\mathbb{W}^n}$. We have

$$\mathcal{A}_n^\theta F^{(i)}(x) = \sum_{v \in \mathcal{V}(x)} \theta(v) \nabla_v F^{(i)}(x), \tag{2.3.4}$$

where ∇_v is the directional derivative in direction v. Recall $v \in \mathcal{V}(x)$ is defined by the disjoint subsets $I, J \subset \{1, ..., n\}$ such that $I \cup J \in \pi(x)$. With $v_i = 1$ if $i \in I$, -1 if $i \in J$, and 0 otherwise. For each element, B, of the partition $\pi(x)$ there is a corresponding element, C, of the partition $\pi(y)$ such that for each $i \in B$ there is a $j_i \in C$ with $x_i = y_{j_i}$, and the j_i can be chosen so that the mapping $i \mapsto i_j$ is injective. Letting C denote the element of $\pi(y)$ corresponding to $I \cup J \in \pi(x)$, it is clear that if $i \notin C$ then $\nabla_v F^{(i)}(x) = 0$, and for $i \in C$ the derivative is either 1 or -1 depending only on the sizes of I and J. Since $y \in \overline{\mathbb{W}^n}$ there is an $a \in \{1, ..., n\}$ and

$m > 0$ such that $C = \{a, ..., a + m - 1\}$. Hence line (2.3.4) is equal to

$$\sum_{k=0}^{m} \binom{m}{k} \theta(k, m-k) \operatorname{sign}(k - i + a - 1),$$

where sign(0) is taken to be 1 here. In particular, this means that when y_i is distinct from all other coordinates, the above equals $\theta(1, 0) - \theta(0, 1) = \beta = 0$.

$$\sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(y) \mathcal{A}_n^{\theta} F^{(i)}(x)$$
$$= \sum_{C \in \pi(y)} \sum_{i \in C} \frac{\partial f}{\partial y_i}(y) \sum_{k=0}^{|C|} \binom{|C|}{k} \theta(k, |C| - k) \operatorname{sign}(k - i + \inf C - 1), \qquad (2.3.5)$$

where each of the partial derivatives are evaluated at y. Putting (2.3.5) into (2.3.3) we can compute the limit

$$\lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}_x \left[f(Y(t)) \right] - f(x) \right)$$
$$= \lim_{t \to 0} \frac{1}{2t} \int_0^t \mathbb{E}_x[\Delta f(Y(s))] ds$$
$$+ \frac{1}{t} \Bigg(\int_0^t \mathbb{E}_x \Bigg[\sum_{C \in \pi(y)} \sum_{i \in C} \frac{\partial f}{\partial y_i}(y) \sum_{k=0}^{|C|} \binom{|C|}{k} \theta(k, |C| - k) \operatorname{sign}(k - i + \inf C - 1) \Bigg]$$
$$+ \frac{1}{2} \sum_{i \neq j} \mathbb{E}_x \left[\frac{\partial^2 f}{\partial y_i \partial y_j}(Y(s)) \mathbb{1}_{\{Y^i(s) = Y^j(s)\}} \right] ds \Bigg).$$

In particular, if we have $f \in \mathcal{D}_\theta$ then the term in the bracket cancels to 0, leaving only the term on the first line. Recalling that $F : \mathbb{R}^n \to \overline{\mathbb{W}^n}$ is continuous and $Y(t) = F(X(t))$, we can use the Feller property of X. Since $\Delta f \in C_0(\overline{\mathbb{W}^n})$, $\Delta f \circ F \in C_0(\mathbb{R}^n)$ (since $F(x) \to \infty$ as $|x| \to \infty$). Hence, $\frac{1}{2t} \int_0^t \mathbb{E}_x[\Delta f(Y(s))] ds$ converges uniformly to $\frac{1}{2} \Delta f(y)$ as $t \to 0$ and thus for $f \in \mathcal{D}_\theta$

$$\lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}_x \left[f(Y(t)) \right] - f(x) \right) = \frac{1}{2} \Delta f(y), \text{ with respect to the uniform norm.}$$

Hence, if $f \in \mathcal{D}_\theta$, it is in the domain of the generator of Y, and $\mathcal{G}_\theta f = \frac{1}{2} \Delta f$. □

We now use the above calculations to prove Proposition 2.3.4.

Proof. Proof of Proposition 2.3.4 By applying Itô's formula as we did in the preceding proof, we see that for any function g satisfying the assumptions of the proposition there is an adapted process $(M(u))_{u \in [0,t]}$ that is a continuous local martingale on

$[0, s]$ for each $s < t$ such that

$$g(t-s, Y(s)) = -\int_0^s \frac{\partial g}{\partial t}(t-u, Y(u))du + \int_0^s \Delta g(t-u, Y(u))du + M(s),$$
$$= M(s).$$

Now we just need to show that $M(s)$ is a local martingale on $[0, t]$. Since $g(t, \cdot) \to f$ uniformly as $t \to 0$ we have

$$|M(s)| = |g(t-s, Y(s))| \leq \underbrace{\|g(t-s, \cdot) - f\|_\infty}_{\to 0 \text{ as } s \to t} + \|f\|_\infty.$$

Thus, there is an $\varepsilon > 0$ such that $M(s)$ is bounded on $[t-\varepsilon, t]$. Therefore $M(s) - M(t-\varepsilon)$ is a martingale on $[t-\varepsilon, t]$. It follows that $M(s)$ is a local martingale on $[0, t]$. Clearly $M(0) = g(t, x)$, and $M(t) = f(Y(t))$ since

$$\begin{aligned} |M(s) - f(Y(t))| &= |g(t-s, Y(s)) - f(Y(t))| \\ &\leq \|g(t-s, \cdot) - f\|_\infty + |f(Y(s)) - f(Y(t))|. \end{aligned}$$

The first term vanishes as $s \to t$ due to the uniform convergence of g to f, and the second almost surely due to the continuity of f and Y. □

Hence, we can find the transition probabilities of Y by looking for the Green's function for (2.3.2), providing solutions are sufficiently regular to make $g(t-s, Y(s))$ a true martingale. In general, it is not clear that there should be solutions to (2.3.2); it is not even clear whether $\mathcal{D}_\theta$ is non-trivial. In the rest of the paper we focus on the case of a uniform characteristic measure: $\nu = \frac{1}{2}\theta \mathbb{1}_{[0,1]}dx$. Since we know ν, we can calculate the constants $\theta(k, l)$. By definition we have

$$\begin{aligned} \theta(k, l) &= \frac{\theta}{2}\int_0^1 x^{k-1}(1-x)^{l-1}dx, \\ &= \frac{\theta}{2}\frac{(l-1)!(k-1)!}{(k+l-1)!}. \end{aligned} \tag{2.3.6}$$

In this case, we also have $\theta(k, 0) = \theta(0, k)$ for all $k \in \mathbb{N}$. Hence, for the characteristic measure $\nu = \frac{1}{2}\theta \mathbb{1}_{[0,1]}dx$, (2.3.1) can be rewritten as

$$\frac{1}{2}\sum_{\substack{a \leq i,j \leq b: \\ i \neq j}} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = -\frac{\theta}{2}\sum_{i=a}^{b} \frac{\partial f}{\partial x_i}(x)a(b-a+1, i), \quad \text{whenever } x_a = x_b. \tag{2.3.7}$$

Where the coefficients are defined

$$a(b-a+1, i-a+1) := \sum_{k=1}^{b-a} \frac{b-a+1}{k(b-a+1-k)} \operatorname{sign}(k-i+a-1). \qquad (2.3.8)$$

In the following section, this particular form of the constants $\theta(k,l)$ will allow us to replace the conditions in line (2.3.7) with far simpler conditions. In particular, we will find each of the second derivatives in terms of the first derivatives.

Remark 2.3.5. *If we try to derive the Kolmogorov Backwards equation for the original process X, we run into problems: the action of the generator of X within the set of C_0^2 functions does not determine the process. We can see this by considering a pair of sticky Brownian motions with parameter $\theta > 0$ X^1, X^2. We have by Itô's formula for all $f \in C_0^2(\mathbb{R}^2)$*

$$\begin{aligned}\mathbb{E}_x[f(X^1(t), X^2(t))] =& f(x_1, x_2) + \frac{1}{2}\int_0^t \mathbb{E}_x[\Delta f(X^1(s), X^2(s))]ds \\ &+ \int_0^t \mathbb{E}_x[\mathbb{1}_{\{X^1(s)=X^2(s)\}} \frac{\partial^2 f}{\partial x_1 \partial x_2}(X^1(s), X^2(s))]ds.\end{aligned}$$

So that f is in the domain of the generator if $\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = 0$ whenever $x_1 = x_2$. But this does not depend on the parameter θ, and thus the generator restricted to this set cannot determine the law of the sticky Brownian motions.

2.3.2 Rearranging the Boundary Conditions

Henceforth we consider the case where the characteristic measure is uniform, i.e. $\nu(dx) = \frac{\theta}{2}\mathbb{1}_{[0,1]}dx$. Let us first note that if we set $|C| = 2$ in (2.3.7), we see $f \in \mathcal{D}_\theta$ satisfies

$$\frac{\partial^2 f}{\partial x_a \partial x_{a+1}} = \theta\left(\frac{\partial f}{\partial x_{a+1}} - \frac{\partial f}{\partial x_a}\right), \quad \text{whenever } x_a = x_{a+1}.$$

We will show that we can replace the full boundary conditions with equivalent ones of the above form, that is

Lemma 2.3.6.

$$\mathcal{D}_\theta = \left\{ f \in C_0^2(\overline{\mathbb{W}^n}) | \text{ for } a < b \text{ , if } x_a = x_b \text{ then } \frac{\theta}{b-a}\left(\frac{\partial f}{\partial x_b} - \frac{\partial f}{\partial x_a}\right) = \frac{\partial^2 f}{\partial x_a \partial x_b} \right\}.$$

Remark 2.3.7. *Essentially we are solving for the second derivatives of functions in $\mathcal{D}_\theta$, given their first derivatives. Whilst this should be possible for any characteristic measure, our method relies on the special form of the parameters $\theta(k,l)$ in the case of the uniform characteristic measure.*

Proof. Note that because we are in the Weyl chamber, $x_a = x_b$ implies $x_a = x_{a+1} = \cdots = x_b$. Thus the condition for $x_a = ... = x_{b-1}$ must also hold when $x_a = ... = x_b$ etc. Using an inductive argument, we prove that the original conditions (2.3.7) are equivalent to the new conditions. That is, we prove that the new condition for $x_a = x_b$ is equivalent to the old conditions, assuming the new conditions for $x_c = x_d$ are satisfied for all $a \leq c < d \leq b$ such that $d - c < b - a$.

Hence we assume that the boundary conditions (2.3.7) for $x_c = x_d$ are satisfied for all $a \leq c < d \leq b$ and that for all $a \leq c < d \leq b$ with $d - c < b - a$

$$\frac{\partial^2 f}{\partial x_c \partial x_d}(x) = \frac{\theta}{d-c}\left(\frac{\partial f}{\partial x_d}(x) - \frac{\partial f}{\partial x_c}(x)\right), \quad \text{if } x_c = ... = x_d. \tag{2.3.9}$$

Without loss of generality, we can relabel $(x_a, ..., x_b)$ as $(x_1, ..., x_m)$ for $m = b-a+1$. Then for $u \in \mathcal{D}_\theta$, we can rewrite the sum over mixed derivatives.

$$\frac{1}{2}\sum_{i\neq j} \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{2}\sum_{\substack{i\neq j\\ i,j\neq m}} \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=2}^{m-1} \frac{\partial^2 f}{\partial x_k \partial x_m} + \frac{\partial^2 f}{\partial x_1 \partial x_m}.$$

Using equations (2.3.7) and (2.3.9), when $x_1 = ... = x_m$ we have the equality

$$\begin{aligned}\frac{\partial^2 f}{\partial x_1 \partial x_m} =& -\frac{\theta}{2}\sum_{j=1}^{m} \frac{\partial f}{\partial y_j}(y) \sum_{k=1}^{m-1} \frac{m}{k(m-k)} \operatorname{sign}(k-j) - \sum_{i<j} \frac{\theta}{j-i}\left(\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}\right)\\ &+ \frac{\theta}{m-1}\left(\frac{\partial f}{\partial x_m} - \frac{\partial f}{\partial x_1}\right).\end{aligned} \tag{2.3.10}$$

We have the following equalities

$$\begin{aligned}\sum_{i<j} \frac{\theta}{j-i}\left(\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}\right) =& \sum_{j=2}^{m}\sum_{i=1}^{j-1} \frac{\theta}{j-i}\frac{\partial f}{\partial x_j} - \sum_{j=2}^{m-1}\sum_{i=1}^{j-1} \frac{\theta}{j-i}\frac{\partial f}{\partial x_i}\\ =& \sum_{j=2}^{m}\sum_{i=1}^{j-1} \frac{\theta}{j-i}\frac{\partial f}{\partial x_j} + \sum_{j=1}^{m-1}\sum_{i=j+1}^{m} \frac{\theta}{j-i}\frac{\partial f}{\partial x_j}\\ =& \theta\sum_{j=1}^{m} \frac{\partial f}{\partial x_j}\sum_{i\neq j} \frac{1}{j-i}.\end{aligned}$$

So that we are finished if for each $j \in \{1, ..., n\}$

$$\frac{1}{2}\sum_{k=1}^{m-1} \frac{m}{k(m-k)} \operatorname{sign}(k-j) + \sum_{i\neq j} \frac{1}{j-i} = 0.$$

Noting that we have $\frac{m}{k(m-k)} = \frac{1}{k} + \frac{1}{m-k}$, we get

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{m-1}\frac{m}{k(m-k)}\operatorname{sign}(k-j) =& \frac{1}{2}\sum_{k=j}^{m-j}\frac{m}{k(m-k)} \\ =& \frac{1}{2}\sum_{k=j}^{m-j}\left(\frac{1}{k}+\frac{1}{m-k}\right) \\ =& \sum_{k=j}^{m-j}\frac{1}{k}. \end{aligned} \tag{2.3.11}$$

In addition

$$\begin{aligned}\sum_{i\neq j}\frac{1}{j-i} =& \sum_{i=1}^{j-1}\frac{1}{j-i} - \sum_{i=j+1}^{m}\frac{1}{i-j} \\ =& \sum_{k=1}^{j-1}\frac{1}{k} - \sum_{k=1}^{m-j}\frac{1}{k} = -\sum_{k=j}^{m-j}\frac{1}{k}.\end{aligned}$$

With the convention that, when $a < b$, $\sum_{k=b}^{a} c_k = -\sum_{k=a}^{b} c_k$. Putting this into line (2.3.10), we see

$$\frac{\partial^2 f}{\partial x_1 \partial x_m} = \frac{\theta}{m-1}\left(\frac{\partial f}{\partial x_m} - \frac{\partial f}{\partial x_1}\right).$$

As noted previously, for $m = 2$ both conditions are equivalent; thus, by induction the old conditions imply the new conditions. Finally it is easy to see that assuming the new conditions hold on $x_c = x_d$ for all $a \le c < d \le b$ and the old conditions on $x_c = x_d$ for all $a \le c < d \le b$ such that $d - c < b - a$, we can follow the above argument in reverse to prove the new conditions imply the old ones. Hence the equivalence of the two sets of conditions is proven. □

As a consequence we can reframe proposition 2.3.4. For $g \in C_0^2(\mathbb{R}_{>0}\times\overline{\mathbb{W}^n})$ satisfying the PDE

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{1}{2}\Delta g, \text{ for } x \in \overline{\mathbb{W}^n}; \\ \frac{\partial^2 u}{\partial x_a \partial x_b} = \frac{\theta}{b-a}\left(\frac{\partial g}{\partial x_b} - \frac{\partial g}{\partial x_a}\right), \text{ if } b > a \text{ and } x_a = x_b. \end{cases} \tag{2.3.12}$$

with initial condition $g(t,\cdot) \to f$ uniformly as $t \to 0$, where $f \in C_b(\overline{\mathbb{W}^n})$, we have $g(t,x) = \mathbb{E}_x\left[f(Y(t))\right]$. This rearrangement will simplify the combinatorics required to show that we can solve the PDE with the Bethe ansatz.

2.3.3 Invariant Measure

In this section, we prove an integration by parts formula for the generator of the ordered n-point motion of the Howitt-Warren flow with uniform characteristic measure. First, we introduce some useful notation.

Recall that for $\pi \in \Pi_n$, $\overline{\mathbb{W}^n_\pi}$ consists of all $x \in \overline{\mathbb{W}^n}$ such that if i and j are in the same element of π, then $x_i = x_j$. Thus, by replacing the multiple indices in each block of π with a single index, as the corresponding x_i are all equal, we can map $\overline{\mathbb{W}^n_\pi}$ into $\overline{\mathbb{W}^{|\pi|}}$, providing a natural bijection between $\mathbb{W}^{|\pi|}$ and $\overline{\mathbb{W}^n_\pi}$ which we'll denote $I^\pi : \overline{\mathbb{W}^n_\pi} \to \mathbb{W}^{|\pi|}$. To be precise, let $\underline{\pi_i} = \min\{a \in \pi_i\}$ and set $I^\pi(x)_i = x_{\underline{\pi_i}}$. For a function $u : \overline{\mathbb{W}^n} \to \mathbb{R}$, denote by $u_\pi : \mathbb{W}^{|\pi|} \to \mathbb{R}$ the function defined by $u_\pi(x) := u \circ (I^\pi)^{-1}(x)$ for all $x \in \mathbb{W}^{|\pi|}$. For $u, v \in C^1(\overline{\mathbb{W}^n})$ such that the below integrals converge, we define

$$(u,v)_\theta := \sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \int_{\overline{\mathbb{W}^{|\pi|}}} \nabla u_\pi \cdot \nabla v_\pi dx. \tag{2.3.13}$$

Now we can state the integration by parts formula for the measure $m_\theta^{(n)}$ from definition 1.3.2.

Proposition 2.3.8. *Suppose $u \in \mathcal{D}_\theta$ and $v \in C_b^1(\overline{\mathbb{W}^n})$, such that there exists $a, c > 0$ such that $|\nabla u(x)| \leq a e^{-c|x|}$. We have*

$$\int_{\overline{\mathbb{W}^n}} \Delta u(x) v(x) m_\theta^{(n)}(dx) = -(u,v)_\theta, \tag{2.3.14}$$

whenever the above integrals are finite.

Proof. Since $u \in \mathcal{D}_\theta$ we can relate Δu_π and $(\Delta u)_\pi$. Clearly we have

$$\Delta u_\pi = \sum_{\pi_\iota \in \pi} \sum_{j,k \in \pi_\iota} \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)_\pi.$$

Hence

$$\begin{aligned}\Delta u_\pi - (\Delta u)_\pi =& \sum_{\pi_\iota \in \pi} \sum_{\substack{j,k \in \pi_\iota \\ j \neq k}} \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)_\pi \\ =& 2 \sum_{\pi_\iota \in \pi} \sum_{\substack{j,k \in \pi_\iota \\ j < k}} \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)_\pi.\end{aligned}$$

Clearly, the second sum is empty whenever $|\pi_\iota| = 1$, so we can exclude those terms from the first sum. Using equations (2.3.7), (2.3.8) and the notations $\underline{\pi_\iota} := \inf \pi_\iota$, $\overline{\pi_\iota} := \sup \pi_\iota = |\pi_\iota| + \underline{\pi_\iota} - 1$, the previous expression is equal to

$$-\theta \sum_{\substack{\pi_\iota \in \pi: \\ |\pi_\iota|>1}} \sum_{j \in \pi_\iota} \left(\frac{\partial u}{\partial x_j} \right)_\pi a(|\pi_\iota|, j - \underline{\pi_\iota} + 1).$$

Now we consider the left hand side of equation (2.3.14). Using Definition 1.3.2, this is equal to

$$\sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \int_{\overline{\mathbb{W}^n_\pi}} \Delta u(x) v(x) \lambda^\pi(dx). \tag{2.3.15}$$

We can rewrite the integral in the summand above in terms of a Lebesgue integral over a lower dimensional space, the result is the integral is equal to

$$\begin{aligned} &\int_{\mathbb{W}^{|\pi|}} (\Delta u)_\pi(x) v_\pi(x) dx \\ &= \int_{\mathbb{W}^{|\pi|}} \left(\Delta u_\pi(x) + \theta \sum_{\substack{\pi_\iota \in \pi: \\ |\pi_\iota|>1}} \sum_{j \in \pi_\iota} \left(\frac{\partial u}{\partial x_j} \right)_\pi a(|\pi_\iota|, j - \underline{\pi_\iota} + 1) \right) v_\pi(x) dx. \end{aligned} \tag{2.3.16}$$

Since the Weyl chamber has a piecewise smooth boundary, we can apply Green's identity to the first term in each integral. Applying the identity on $\overline{\mathbb{W}^{|\pi|}} \cap \{x \in \overline{\mathbb{W}^n} : |x| < R\}$ and then taking $R \to \infty$, the exponential bound on $|\nabla u|$ together with the boundedness of v ensures the only boundary term to survive in the limit will be the integral over $\partial \overline{\mathbb{W}^{|\pi|}}$.

The smooth part of the boundary of the Weyl chamber $\mathbb{W}^{|\pi|}$ can be written in terms of the disjoint union of $\overline{\mathbb{W}^{|\pi|}_{\tilde{\pi}}}$ over the set $M_\pi := \{\tilde{\pi} \in \Pi_{|\pi|} : \ |\tilde{\pi}| = |\pi| - 1\}$. Note that if $|\pi| = 1$ this union is empty, and the boundary integral vanishes. Each $\tilde{\pi}$ in M_π consists of $|\pi| - 2$ singletons and one set $\{l, l+1\}$ for some $l \in \{1, ..., |\pi|\}$. Further, the outward unit normal on $\overline{\mathbb{W}^{|\pi|}_{\tilde{\pi}}}$ is given by

$$\underline{n}(x)_r = \begin{cases} -\frac{1}{\sqrt{2}}, & \text{if } r = l; \\ \frac{1}{\sqrt{2}}, & \text{if } r = l+1; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the boundary measure is given by $\sum_{\tilde\pi\in M_\pi}\sqrt{2}\lambda^{\tilde\pi}$, so that (2.3.16) equals

$$\Bigg(\sum_{\tilde\pi\in M_\pi}\int_{\overline{\mathbb{W}^{|\pi|}_{\tilde\pi}}}\left(\frac{\partial u_\pi}{\partial y_{l+1}}-\frac{\partial u_\pi}{\partial y_l}\right)\,v_\pi d\lambda^{\tilde\pi}-\int_{\mathbb{W}^{|\pi|}}\nabla u_\pi(x)\cdot\nabla v_\pi(x)dx$$
$$+\,\theta\sum_{\substack{\pi_\iota\in\pi:\\|\pi_\iota|>1}}\sum_{j\in\pi_\iota}\int_{\mathbb{W}^{|\pi|}}\left(\frac{\partial u}{\partial x_j}\right)_\pi a(|\pi|,j-\underline{\pi_\iota}+1)v_\pi(x)dx\Bigg),$$

where l depends on $\tilde\pi$ and is defined as above. We have written the partial derivatives of u_π with respect to y to emphasise the fact that u_π is a function on $\overline{\mathbb{W}^{|\pi|}}$ rather than $\overline{\mathbb{W}^n}$. Hence, to complete the proof it is enough to show that the first and third terms cancel. Rewriting the integrals with respect to $\lambda^{\tilde\pi}$, the first term is equal to

$$\sum_{\tilde\pi\in M_\pi}\int_{\mathbb{W}^{|\tilde\pi|}}\left(\frac{\partial u_\pi}{\partial y_{l+1}}-\frac{\partial u_\pi}{\partial y_l}\right)_{\tilde\pi}\,(v_\pi)_{\tilde\pi}d\lambda^{\tilde\pi}.$$

Clearly, this is equal to

$$\sum_{\tilde\pi\in M_\pi}\int_{\mathbb{W}^{|\tilde\pi|}}\left(\left(\sum_{j\in\pi_{l+1}}\frac{\partial u}{\partial x_j}\right)_\pi-\left(\sum_{j\in\pi_l}\frac{\partial u}{\partial x_j}\right)_\pi\right)_{\tilde\pi}(x)\ (v_\pi)_{\tilde\pi}(x)dx.$$

Summing this over $\pi\in\Pi_n$ with the appropriate coefficients, we see that (2.3.15) is equal to

$$\sum_{\pi\in\Pi_n}\sum_{\tilde\pi\in M_\pi}\theta^{|\pi|-n}\left(\prod_{\pi_\iota\in\pi}\tfrac{1}{|\pi_\iota|}\right)$$
$$\int_{\mathbb{W}^{|\pi|-1}}\sum_{j\in\pi_{l+1}\cup\pi_l}\left(\left(\frac{\partial u}{\partial x_j}\right)_\pi\right)_{\tilde\pi}(x)\,\mathrm{sign}(j-\underline{\pi_{l+1}})\ (v_\pi)_{\tilde\pi}(x)dx.$$

Notice that for each $\pi\in\Pi_n$ and $\tilde\pi\in M_\pi$ we can rewrite the summand in terms of a new partition, $\hat\pi$, formed from π by merging two adjacent blocks to form the $\pi_{l+1}\cup\pi_l$ block. Further, because the partitions are in Π_n, there are exactly $|\pi_{l+1}\cup\pi_l|-1$ partitions that yield $\hat\pi$ by merging two blocks to form $\pi_{l+1}\cup\pi_l$. Rewriting the sum in terms of $\hat\pi$ we get

$$\sum_{\hat\pi\in\Pi_n}\theta^{|\hat\pi|+1-n}\left(\prod_{\hat\pi_\iota\in\hat\pi}\tfrac{1}{|\hat\pi_\iota|}\right)\sum_{\substack{\hat\pi_\iota\in\hat\pi:\\|\hat\pi_\iota|>1}}$$
$$\int_{\mathbb{W}^{|\hat\pi|}}\sum_{k=1}^{|\hat\pi_\iota|-1}\tfrac{|\hat\pi_\iota|}{k(|\hat\pi_\iota|-k)}\sum_{j\in\hat\pi_\iota}\left(\frac{\partial u}{\partial x_j}\right)_{\hat\pi}(x)\,\mathrm{sign}(j-\underline{\hat\pi_\iota}-k)\ v_{\hat\pi}(x)dx.$$

Here, the sum over j is over the partitions whose blocks have been merged to get $\hat{\pi}$, with k corresponding to the size of the lower block. The extra factor $\frac{|\hat{\pi}_\iota|}{k(|\hat{\pi}_\iota|-k)}$ is simply a correction to the product to write it in terms of $\hat{\pi}$ rather than the π partition whose blocks we merged.

Recalling that sign$(0) = 1$ here, equation (2.3.8) yields that the above is precisely equal to

$$-\sum_{\pi\in\Pi_n} \theta^{|\pi|+1-n} \left(\prod_{\pi_\iota\in\pi} \frac{1}{|\pi_\iota|}\right) \sum_{\substack{\pi_\iota\in\pi:\\ |\pi_\iota|>1}} \sum_{j\in\pi_\iota} \int_{\mathbb{W}^{|\pi|}} \left(\frac{\partial u}{\partial x_j}\right)_\pi (x) a(|\pi|, j - \underline{\pi_\iota} + 1) v_\pi(x) dx.$$

Hence (2.3.15) is equal to

$$\begin{aligned} &- \sum_{\pi\in\Pi_n} \theta^{|\pi|-n} \prod_{\pi_\iota\in\pi} \frac{1}{|\pi_\iota|} \int_{\mathbb{W}^{|\pi|}} \nabla u_\pi(x) \cdot \nabla v_\pi(x) dx \\ =& - (u, v)_\theta. \end{aligned}$$

□

Thus, if we denote by $L^2(m_\theta^{(n)})$ the L^2 space on $\overline{\mathbb{W}^n}$ with respect to the measure $m_\theta^{(n)}$ and the standard L^2 inner product, then the generator is symmetric on $\mathcal{D}_\theta \cap L^2(m_\theta^{(n)})$. This symmetry suggests the process is reversible with respect to this measure, but because our calculations are only done for $u \in \mathcal{D}_\theta$, and we do not know how rich the set $\mathcal{D}_\theta$ is, this is not enough for a proof. However, taking $v = 1$, the right-hand side of (2.3.14) vanishes, giving us the following helpful corollary.

Corollary 2.3.9. *For $u \in \mathcal{D}_\theta$ such that there are $a, c > 0$ with $|\nabla u(x)| \leq a e^{-c|x|}$ we have*

$$\frac{1}{2} \int \Delta u(x) m_\theta^{(n)}(dx) = 0.$$

In the next section, we find the Green's function for the backwards equation, and thus the transition density for the process (with respect to the measure $m_\theta^{(n)}$). Using this we can prove that $m_\theta^{(n)}$ is the stationary measure, and that Y is reversible with respect to $m_\theta^{(n)}$.

2.4 Bethe Ansatz for Sticky Brownian Motions

We are trying to find a solution to the PDE (2.3.12), which we restate below, for each fixed $y \in \mathbb{W}^n$ and θ some positive constant, with the initial condition $u_0(x, y) =$

$\delta(x-y)$, where δ is the Dirac delta distribution.

$$\begin{cases} \frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t, & \text{for all } x \in \overline{\mathbb{W}^n}; \\ \theta\left(\frac{\partial u}{\partial x_b} - \frac{\partial u}{\partial x_a}\right) = (b-a)\frac{\partial^2 u}{\partial x_a \partial x_b}, & \text{when } x_a = x_b, \text{ for some } a < b. \end{cases} \tag{2.4.1}$$

The Bethe ansatz suggests that if we define

$$S_{\alpha,\beta}(k) := \frac{i\theta\,(k_\beta - k_\alpha) + k_\alpha k_\beta}{i\theta\,(k_\beta - k_\alpha) - k_\alpha k_\beta}, \tag{2.4.2}$$

then the solution is given by the following equation,

$$u_t(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k)dk, \tag{2.4.3}$$

where S_n denotes the group of permutations on $\{1,...,n\}$ and $k_\sigma = (k_{\sigma(1)},...,k_{\sigma(n)})$.

The idea here is similar to that used to find the transition density of a reflected Brownian motion. Since we are considering a process with ordered coordinates, we combine solutions to the interior equation with permuted coordinates, the permutations representing possible orderings of the original process. The more complicated boundary conditions require us to combine our solutions in a more complicated way. In particular, we take linear combinations in Fourier space in such a way that the boundary conditions where $b-a=1$ are satisfied; this is how we find the form of (2.4.2). In fact, it forces this ansatz onto us, leaving no freedom to deal with the additional conditions which correspond to $b-a>1$ in (2.4.1).

Barraquand and Rychnovsky conjectured in [BR20] that the Backwards equation for the system of sticky Brownian motions was the heat equation with the boundary conditions corresponding to $b-a=1$ in (2.4.1), based on the Bethe ansatz answer for the system. It is important to note that for any other choice of characteristic measure ν with $\nu([0,1]) = \frac{\theta}{2}$, the boundary conditions corresponding to $b-a=1$ would be the same, so we do not expect these boundary conditions alone to give uniqueness of the PDE. However, to simplify the boundary conditions in Definition 2.3.2 to those in (2.4.1), we assume the solution to be C^2 in space. It is possible that the $b-a=1$ boundary conditions do determine the solution under this additional regularity assumption and the transition densities for all of the other systems of sticky Brownian motions are not C^2 in space.

It is clear that (2.4.3) satisfies the first condition in (2.4.1) and our choice of (2.4.2) guarantees the second condition holds when $b-a=1$. However, when $b-a>1$, it is not clear that they are still satisfied. Fortunately, and surprisingly, the second condition turns out to be satisfied in its entirety. Moreover, we can show the initial condition holds; hence, we obtain our main result, which we restate here:

Theorem 2.4.1. *Suppose $\theta > 0$, and $X = (X(t))_{t\geq 0}$ is a solution to the Howitt-Warren martingale problem in $\mathbb{R}^n$ with characteristic measure $\frac{\theta}{2}\mathbb{1}_{[0,1]}dx$ and zero drift. Let $Y = (Y(t))_{t\geq 0}$ be the process obtained by ordering the coordinates of $(X(t))_{t\geq 0}$. Then for every bounded and Lipschitz continuous function $f : \overline{\mathbb{W}^n} \to \mathbb{R}$, $x \in \overline{\mathbb{W}^n}$ and $t > 0$*

$$\mathbb{E}_x[f(Y_t)] = \int u_t(x,y)f(y)m_\theta^{(n)}(dy).$$

Where u is as in (2.4.3), $m_\theta^{(n)}$ is defined in Definition 1.3.2.

In the following section, we shall prove Theorem (2.4.1), first we show the boundary conditions are satisfied and then the initial condition. To ensure we can perform the necessary exchanges of integral and derivative we start with some bounds for the Bethe ansatz.

2.4.1 Bounds for Dominated Convergence

Lemma 2.4.2. *For every $x \in \overline{\mathbb{W}^n}$ and $t > 0$ we have $u_t(x,\cdot) \in L^1(m_\theta^{(n)})$, where $u_t(x,\cdot)$ is defined as in (2.4.3). Further, for each $x \in \overline{\mathbb{W}^n}$ and $t > 0$, there exist $a, c > 0$ such that $|\nabla_y u_t(x,y)| \leq ae^{-c|y|}$ for all $y \in \overline{\mathbb{W}^n}$. The same statement holds if we instead consider the x derivative and vary x with y being fixed. Similarly for each $x \in \overline{\mathbb{W}^n}$ and $s > 0$ we can find $a, c > 0$ such that $|u_t(x,y)|,\ |\partial_t u_t(x,y)| \leq ae^{-c|y|}$ for all $t > s$ and $y \in \overline{\mathbb{W}^n}$.*

We leave the proof of this lemma to the end of Section 2.4.3, as it is a simplification of the methods used in that section.

The second part of the above lemma provides the necessary bounds to justify passing derivatives through the first integral in $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$. Further, it is easy to see we can apply Dominated convergence to find

$$\frac{\partial u_t}{\partial x_a} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2}\sum_{\sigma\in S_n} ik_{\sigma(a)}e^{ik_\sigma\cdot(x-y_\sigma)}\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k)dk,$$

$$\frac{\partial^2 u_t}{\partial x_a \partial x_b} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2}\sum_{\sigma\in S_n} k_{\sigma(a)}k_{\sigma(b)}e^{ik_\sigma\cdot(x-y_\sigma)}\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k)dk.$$

This allows us to not only confirm that $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ solves the heat equation but also to reduce the boundary conditions to a combinatorial problem.

2.4.2 Boundary Conditions

Proposition 2.4.3.

$$\int u_t(x,y)f(y)m_\theta^{(n)}(dy) \in \mathcal{D}_\theta.$$

Using the same ideas as in the previous subsection, we can derive sufficient bounds to show $\int u_t(x,y)f(y)m_\theta^{(n)}(dy) \in C_0^2(\overline{\mathbb{W}^n})$. Hence, we just need to show it satisfies the correct boundary conditions from the PDE (2.4.1). The proof will follow from several lemmas. To begin, we derive the combinatorial identity that implies the above proposition.

Fix $a, b \in \{1, ..., n\}$ with $a < b$, then for $t > 0$ we can differentiate under the integral as noted in the previous subsection to see that the corresponding boundary condition is satisfied if for all $a < b$, $x_a = x_b$ implies

$$\sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x - y_\sigma)} \left(i\theta(k_{\sigma(b)} - k_{\sigma(a)}) + (b-a)k_{\sigma(b)}k_{\sigma(a)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k) = 0. \tag{2.4.4}$$

This can be simplified by splitting the summand into parts dependent on $\sigma(a), ..., \sigma(b)$ and on the remaining values σ takes. Noting that we have $x_a = ... = x_b$

$$\prod_{c=a}^{b} e^{ik_{\sigma(c)}(x_c - y_{\sigma(c)})} = \prod_{c=a}^{b} e^{ik_{\sigma(c)}(x_a - y_{\sigma(c)})} = \prod_{\tilde{c} \in \{\sigma(a), ..., \sigma(b)\}} e^{ik_{\tilde{c}}(x_a - y_{\tilde{c}})}.$$

Notice that the last expression above depends only on the set $\{\sigma(a), ..., \sigma(b)\} = \sigma(\{a, ..., b\})$, and not the order of the values σ takes on $\{a, .., , b\}$. Thus, the exponential factor of the summand in (2.4.4) only depends on $\sigma(\{a, ..., b\})$ and not $\sigma(a), ..., \sigma(b)$ themselves. Now we split the product

$$\prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k) = \prod_{\substack{\alpha < a \le \beta \le b: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k) \prod_{\substack{a \le \alpha \le b < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k)$$
$$\prod_{\substack{\alpha, \beta \in \{a, ..., b\}^c: \\ \alpha < \beta, \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k) \prod_{\substack{a \le \alpha < \beta \le b: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k).$$

Note that $S_{\sigma(\beta),\sigma(\alpha)}$ does not depend on α and β, but on $\sigma(\alpha)$ and $\sigma(\beta)$. Suppose, for a given permutation σ, $S_{\sigma(\beta),\sigma(\alpha)}$ appears in the first product, then for any permutation τ with $\sigma(c) = \tau(c)$ for every $c \in \{a, ..., b\}^c$, we have $\sigma(\beta) \in \{\sigma(a), ..., \sigma(b)\} = \{\tau(a), ..., \tau(b)\}$. Thus, there exists $\gamma \in \{a, ..., b\}$ such that $\tau(\gamma) = \sigma(\beta)$, and so we have $\tau(\alpha) = \sigma(\alpha) > \sigma(\beta) = \tau(\gamma)$ and $\alpha < a \leq \gamma$. Hence, $S_{\tau(\gamma),\tau(\alpha)} = S_{\sigma(\beta),\sigma(\alpha)}$ appears in the product for τ. This shows the first product doesn't depend on $\{\sigma(a), ..., \sigma(b)\}$, and similarly the second doesn't either. The third product clearly doesn't depend on $\{\sigma(a), ..., \sigma(b)\}$, leaving only the fourth product. Finally, we note that the fourth product doesn't depend on the values σ takes outside $\{a, ..., b\}$. Hence we can split the sum into a sum over possibilities for the permutation outside $\{a, ..., b\}$ and then a sum over possibilities inside $\{a, .., b\}$. Pulling the parts depending only on the values of σ outside $\{a, ..., b\}$ out of the second sum we see that it is sufficient for the second sum to vanish; thus, our condition will hold if

$$\sum_{\sigma \in S_m} \left(i\theta(k_{\sigma(m)} - k_{\sigma(1)}) + (m-1)k_{\sigma(m)}k_{\sigma(1)}\right) \prod_{\substack{1 \leq \alpha < \beta \leq m: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k) = 0,$$

where we have relabelled $k_a, ..., k_b$ to $k_1, ..., k_m$. Hence, it is enough to prove the following

Proposition 2.4.4. *For every $n \in \mathbb{N}$ we have the identity*

$$\sum_{\sigma \in S_n} \left(i\theta\left(k_{\sigma(n)} - k_{\sigma(1)}\right) + (n-1)k_{\sigma(n)}k_{\sigma(1)}\right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)} = 0. \tag{2.4.5}$$

First, we simplify the left hand side by pulling out the common denominator. Recalling (2.4.2)

$$\begin{aligned}
&\prod_{\sigma(\beta) < \sigma(\alpha)} \left(i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}\right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)} \\
= &\prod_{\sigma(\beta) < \sigma(\alpha)} \left(i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}\right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \frac{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)}k_{\sigma(\beta)}}{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)}k_{\sigma(\beta)}} \\
= &\prod_{\substack{\beta < \alpha: \\ \sigma(\beta) < \sigma(\alpha)}} \left(i\theta(k_{\sigma(\beta)} - k_{\sigma(\alpha)}) - k_{\sigma(\alpha)}k_{\sigma(\beta)}\right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \left(i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)}k_{\sigma(\beta)}\right).
\end{aligned}$$

Thus, multiplying both sides of (2.4.5) by $\prod_{\sigma(\beta)<\sigma(\alpha)} \left(i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}\right)$ (since permutations are bijections, this does not depend on σ) gives the equivalent equation

$$0 = \sum_{\sigma \in S_n} \left(i\theta \left(k_{\sigma(n)} - k_{\sigma(1)} \right) + (n-1) k_{\sigma(n)} k_{\sigma(1)} \right)$$
$$\prod_{\substack{\beta<\alpha: \\ \sigma(\beta)<\sigma(\alpha)}} \left(i\theta (k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \left(i\theta (k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)} \right).$$

We can get rid of the $i\theta$ factors by replacing each k_j with $i\theta k_j$, since $\theta > 0$ this change of variables is invertible. We are left with the following equivalent equation, which we will prove for $k \in \mathbb{C}^n$.

$$0 = \sum_{\sigma \in S_n} \left(\left(k_{\sigma(n)} - k_{\sigma(1)} \right) + (n-1) k_{\sigma(n)} k_{\sigma(1)} \right)$$
$$\prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)<\sigma(\beta)}} \left((k_{\sigma(\beta)} - k_{\sigma(\alpha)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \left((k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)} \right).$$

Where we have cancelled off a factor of $(i\theta)^{2\left(\binom{n}{2}+1\right)}$.

We'll now split the equation into two parts and simplify before showing they cancel. Making the following rearrangements, and defining the polynomial B

$$\prod_{\substack{\beta<\alpha: \\ \sigma(\beta)<\sigma(\alpha)}} \left((k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \left((k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)} \right). \tag{2.4.6}$$
$$= \prod_{\alpha<\beta} \operatorname{sign}(\sigma(\beta) - \sigma(\alpha)) \left(k_{\sigma(\beta)} - k_{\sigma(\alpha)} - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right)$$
$$= \operatorname{sign}(\sigma) \prod_{\alpha<\beta} \left(k_{\sigma(\beta)} - k_{\sigma(\alpha)} - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) =: \operatorname{sign}(\sigma) B(k_\sigma).$$

We proceed by considering the expressions

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)(n-1) k_{\sigma(n)} k_{\sigma(1)} B(k_\sigma); \tag{2.4.7}$$
$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(k_{\sigma(n)} - k_{\sigma(1)} \right) B(k_\sigma). \tag{2.4.8}$$

It is clear that both (2.4.7) and (2.4.8) are polynomials in the k_j; we will now make some more general statements about polynomials of this form.
It is clear that if $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, then

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) f(k_\sigma) B(k_\sigma) \tag{2.4.9}$$

is an alternating polynomial. To see this suppose $a < b$ and we exchange k_a and k_b in the above expression. Then k_σ becomes $k_{(a,b)\circ\sigma}$ giving

$$\begin{aligned}\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_{(a,b)\circ\sigma}) B(k_{(a,b)\circ\sigma}) &= -\sum_{\sigma\in S_n} \operatorname{sign}((a,b)\circ\sigma) f(k_{(a,b)\circ\sigma}) B(k_{(a,b)\circ\sigma}) \\ &= -\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_\sigma) B(k_\sigma).\end{aligned}$$

In particular, whenever we have $k_\alpha = k_\beta$, for $\alpha \neq \beta$, any such polynomial must vanish. Hence we must be able to take the Vandermonde determinant, $\prod_{\alpha<\beta}(k_\beta - k_\alpha)$, out as a factor; since this is itself alternating, whatever remains must be symmetric. Thus for any polynomial $f : \mathbb{R}^n \to \mathbb{R}^n$, there exists a symmetric polynomial $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_\sigma) B(k_\sigma) = g(k) \prod_{\alpha<\beta} (k_\beta - k_\alpha). \tag{2.4.10}$$

In the case of (2.4.7) and (2.4.8), the polynomial f is also multilinear (no variable appears with exponent higher than one), and depends only on two variables. The following lemma will allow us to make further statements about g based on these assumptions.

Lemma 2.4.5. *If $i, j \in \{2, ..., n-1\}$ with $i \neq j$, and $\kappa \in \mathbb{R}^n$ such that we fix $\kappa_i = -1$ and $\kappa_j = 1$, then $B(\kappa)$ has degree at most $n-2$ when considered as a polynomial of κ_1 or κ_n.*

Proof. Recalling the formula for $B(k)$, (2.4.6), we have

$$\begin{aligned}B(\kappa) = &\prod_{\substack{\alpha<\beta: \\ \alpha,\beta\neq i,j}} (\kappa_\beta - \kappa_\alpha - \kappa_\alpha\kappa_\beta) \prod_{\alpha\neq i,j} (\operatorname{sign}(j-\alpha)(1-\kappa_\alpha) - \kappa_\alpha) \\ &\times \prod_{\alpha\neq i,j} (\operatorname{sign}(i-\alpha)(-1-\kappa_\alpha) + \kappa_\alpha)\ (2\operatorname{sign}(j-i)+1)\,.\end{aligned}$$

The first product contains $(n-3)$ factors with κ_1 and κ_n each. The second and third contribute a factor of the form:

$$(1-2\kappa_1)(-1)$$

for κ_1, and a factor of the form

$$(-1)(2\kappa_n + 1) \tag{2.4.11}$$

for κ_n. Leaving a total of $n-2$ factors involving κ_1 and κ_n each, which proves the statement. □

Now we can apply the above lemma to the expressions we are interested in.

Lemma 2.4.6. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a multilinear polynomial, then there exists constants C_0, C_1 and C_2 such that*

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) f(k_{\sigma(1)}, k_{\sigma(n)}) B(k_\sigma) = \prod_{\alpha<\beta} (k_\beta - k_\alpha)$$
$$\times \left(C_0 + \sum_{m=1}^{\lfloor n/2 \rfloor} \left(C_1 \sum_{\alpha_1 < \ldots < \alpha_{2m}} k_{\alpha_1} \ldots k_{\alpha_{2m}} + C_2 \sum_{\alpha_1 < \ldots < \alpha_{2m+1}} k_{\alpha_1} \ldots k_{\alpha_{2m+1}} \right) \right).$$

Proof. The discussion preceding Lemma 2.4.5 shows that we at least have equation (2.4.10), and that g must be symmetric. To get the form given in the statement, we will show that g is also multilinear. This tells us we can write it as a linear combination of elementary symmetric polynomials, and then that the coefficients in this combination are of the form given above. Both of these arguments proceed by considering the exponents of the variables k_j.

To show multilinearity, we note that for each k_j, $\prod_{\alpha<\beta}(k_\beta - k_\alpha)$ contains $n-1$ linear factors of k_j. Furthermore, each $B(k_\sigma)$ also contains exactly $n-1$ linear factors of k_j. But f is multilinear, so in the summand $\operatorname{sign}(\sigma) f(k_\sigma) B(k_\sigma)$ the largest possible power of k_j is n. Hence the largest possible power of k_j in $g(k)$ is 1. This holds for each j; thus, $g(k)$ is multilinear.

Since $g(k)$ is multilinear and symmetric it must be of the form

$$g(k) = C_0 + \sum_{m=1}^{n} C_m \sum_{\alpha_1 < \ldots < \alpha_m} k_{\alpha_1} \ldots k_{\alpha_m}.$$

Now we show that the constants C_m satisfy $C_1 = C_{2m+1}$ and $C_2 = C_{2m}$ for all $m \leq n/2$. Setting $\kappa = (k_1, \ldots, k_{n-2}, -1, 1)$, we have the equality

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) f(\kappa_{\sigma(1)}, \kappa_{\sigma(n)}) B(\kappa_\sigma) = 2g(\kappa) \prod_{\alpha<\beta<n-1} (k_\beta - k_\alpha) \prod_{\gamma=1}^{n-2} (1 - k_\gamma)(-1 - k_\gamma). \tag{2.4.12}$$

Since g is symmetric polynomial, if one of its terms contains k_{n-1} but not k_n, there is a term otherwise equal where k_{n-1} is replaced with k_n, and vice versa. Using κ as defined in the previous proof, in $g(\kappa)$, these terms cancel, leaving only the terms that contain both or neither. For κ, we have set $k_{n-1}k_n = -1$, so we have the following.

$$g(\kappa) = C_0 + \sum_{m=1}^{n-2} (C_m - C_{m+2}) \sum_{\alpha_1 < \ldots < \alpha_m < n-1} k_{\alpha_1} \ldots k_{\alpha_m}.$$

The next step is to consider the exponents on the left hand side of (2.4.12) for each term in the sum, and show $g(\kappa)$ must be constant. First $B(k_\sigma)$ contains $(n-1)$ linear factors of each k_j, so the only way a k_j with exponent n can appear is if it also occurs in $f(\kappa_{\sigma(1)}, \kappa_{\sigma(n)})$; hence, only if $j = \sigma(n)$ or $\sigma(1)$. But the previous lemma tells us that $B(\kappa_\sigma)$ has degree $n-2$ as a polynomial of $\kappa_{\sigma(1)}$ or $\kappa_{\sigma(n)}$. Thus the highest possible power of any of the k_j on the left hand side of (2.4.12) is $n-1$. However, the right hand side still contains $n-1$ linear factors of each k_j outside of $g(\kappa)$, so $g(\kappa)$ must be constant. Hence, $C_m = C_{m+2}$ for every $m > 0$, proving the result. □

Remark 2.4.7. *Using the general formula for the sum of elementary symmetric polynomials on n variables, $\prod_{j=1}^n (1 + x_j)$, together with the above lemma, gives us that for a multilinear polynomial $f : \mathbb{R}^2 \to \mathbb{R}$, there are constants C_m and D_m such that*

$$\begin{aligned}
&\sum_{\sigma \in S_n} \operatorname{sign}(\sigma) f(k_{\sigma(1)}, k_{\sigma(n)}) B(k_\sigma) \\
&= \prod_{\alpha<\beta} (k_\beta - k_\alpha) \Bigg(C_0 + \frac{1}{2} C_1 \left(\prod_{j=1}^n (1 + k_j) + \prod_{j=1}^n (1 - k_j) - 2 \right) \\
&\qquad + \frac{1}{2} C_2 \left(\prod_{j=1}^n (1 + k_j) - \prod_{j=1}^n (1 - k_j) \right) \Bigg) \\
&= \prod_{\alpha<\beta} (k_\beta - k_\alpha) \left(D_0 + D_1 \prod_{j=1}^n (1 + k_j) + D_2 \prod_{j=1}^n (1 - k_j) \right) \\
&= \det\left(k_i^{j-1}\right) \left(D_0 + D_1 \det\left((1 + k_j)\delta_{ij}\right) + D_2 \det\left((1 - k_j)\delta_{ij}\right)\right).
\end{aligned}$$

Now we can return to our original expressions (2.4.7) and (2.4.8). These two lemmas imply that we have constants $C_0^{(n)}, \tilde{C}_0^{(n)}, C_1, \tilde{C}_1^{(n)}, C_2^{(n)}$ and $\tilde{C}_2^{(n)}$ such that

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)(k_{\sigma(n)} - k_{\sigma(1)}) B(k_\sigma) = \prod_{\alpha<\beta} (k_\beta - k_\alpha) \tag{2.4.13}$$
$$\left(C_0^{(n)} + \sum_{m=1}^{\lfloor n/2 \rfloor} \left(C_1^{(n)} \sum_{\alpha_1 < \ldots < \alpha_{2m}} k_{\alpha_1} \ldots k_{\alpha_{2m}} + C_2^{(n)} \sum_{\alpha_1 < \ldots < \alpha_{2m+1}} k_{\alpha_1} \ldots k_{\alpha_{2m+1}} \right) \right).$$

$$\sum_{\sigma\in S_n}\text{sign}(\sigma)(n-1)k_{\sigma(n)}k_{\sigma(1)}B(k_\sigma)=\prod_{\alpha<\beta}(k_\beta-k_\alpha) \tag{2.4.14}$$
$$\left(\tilde{C}_0^{(n)}+\sum_{m=1}^{\lfloor n/2\rfloor}\left(\tilde{C}_1^{(n)}\sum_{\alpha_1<...<\alpha_{2m}}k_{\alpha_1}...k_{\alpha_{2m}}+\tilde{C}_2^{(n)}\sum_{\alpha_1<...<\alpha_{2m+1}}k_{\alpha_1}...k_{\alpha_{2m+1}}\right)\right).$$

The next lemma provides a link between these constants for different values of n that will allow us to find their value inductively.

Lemma 2.4.8. *For $m=0,1,2$ and $n\geq 3$, we have that $C_m^{(n)}=(n-1)C_m^{(n-1)}$ and $\tilde{C}_m^{(n)}=(n-1)\tilde{C}_m^{(n-1)}$.*

Proof. Take $k_n=0$ in (2.4.13) we get the equality

$$\sum_{\sigma\in S_n}\text{sign}(\sigma)(k_{\sigma(n)}-k_{\sigma(1)})B(k_\sigma)|_{k_n=0}=\prod_{\alpha=1}^{n-1}(-k_\alpha)\prod_{\alpha<\beta<n}(k_\beta-k_\alpha)$$
$$\left(C_0^{(n)}+\sum_{m=1}^{\lfloor n/2\rfloor}\left(C_1^{(n)}\sum_{\alpha_1<...<\alpha_{2m}<n}k_{\alpha_1}...k_{\alpha_{2m}}+C_2^{(n)}\sum_{\alpha_1<...<\alpha_{2m+1}<n}k_{\alpha_1}...k_{\alpha_{2m+1}}\right)\right).$$

Recalling how we defined the polynomial B in line (2.4.6), we see that the left-hand side of the above equality is equal to

$$\sum_{\substack{\sigma\in S_n:\\ \sigma(1),\sigma(n)\neq n}}(k_{\sigma(n)}-k_{\sigma(1)})\left(\prod_{\alpha=1}^{n-1}(-k_\alpha)\right)D_\sigma(k) \tag{2.4.15}$$
$$+\sum_{\substack{\sigma\in S_n:\\ \sigma(1)=n}}k_{\sigma(n)}\left(\prod_{\alpha=1}^{n-1}(-k_\alpha)\right)D_\sigma(k)-\sum_{\substack{\sigma\in S_n:\\ \sigma(n)=n}}k_{\sigma(1)}\left(\prod_{\alpha=1}^{n-1}(-k_\alpha)\right)D_\sigma(k).$$

Where we have used the shorthand

$$\begin{aligned}D_\sigma(k)&=\prod_{\alpha<\beta<n}\left(k_\beta-k_\alpha-\text{sign}\left(\sigma^{-1}(\beta)-\sigma^{-1}(\alpha)\right)k_\beta k_\alpha\right)\\ &=\prod_{\substack{\alpha<\beta:\\ \alpha,\beta\neq\sigma^{-1}(n)}}\text{sign}(\sigma(\beta)-\sigma(\alpha))(k_{\sigma(\beta)}-k_{\sigma(\alpha)}-k_{\sigma(\alpha)}k_{\sigma(\beta)}).\end{aligned}$$

Note that $\sigma^{-1}(n)$ plays no role in the terms of the first sum on line (2.4.15). Thus, we can relabel each permutation in that sum to one in S_{n-1}, with each one occurring $n-2$ times. For example, when $n=4$, we would replace the permutations $\binom{1\ 2\ 3\ 4}{1\ 4\ 3\ 2}$ and $\binom{1\ 2\ 3\ 4}{1\ 2\ 4\ 3}$ with $\binom{1\ 2\ 3}{1\ 3\ 2}$ and $\binom{1\ 2\ 3}{1\ 2\ 3}$ respectively. Note that this replacement does not change $\text{sign}\left(\sigma^{-1}(\beta)-\sigma^{-1}(\alpha)\right)$, and thus does not change the summand. We can do the same with the two sums on the next line, these have no repeats as $\sigma^{-1}(n)$ must be 1 or n depending on the sum. Under this relabelling, $D_\sigma(k)$ becomes $\text{sign}(\sigma)B(k_\sigma)$. Thus, we get

$$(n-2)\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\text{sign}(\sigma)(k_{\sigma(n-1)}-k_{\sigma(1)})B(k_\sigma)$$
$$+\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\text{sign}(\sigma)k_{\sigma(n-1)}B(k_\sigma)-\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\text{sign}(\sigma)k_{\sigma(1)}B(k_\sigma).$$

Which is equal to

$$(n-1)\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\text{sign}(\sigma)(k_{\sigma(n-1)}-k_{\sigma(1)})B(k_\sigma).$$

Applying equation (2.4.13) in the $n-1$ case, we get that the above is equal to

$$(n-1)\prod_{\alpha=1}^{n-1}(-k_\alpha)\prod_{\alpha<\beta<n}(k_\beta-k_\alpha)\Bigg(C_0^{(n-1)}$$
$$+\sum_{m=1}^{\lfloor (n-1)/2\rfloor}\Bigg(C_1^{(n-1)}\sum_{\alpha_1<\ldots<\alpha_{2m}}k_{\alpha_1}\ldots k_{\alpha_{2m}}+C_2^{(n-1)}\sum_{\alpha_1<\ldots<\alpha_{2m+1}}k_{\alpha_1}\ldots k_{\alpha_{2m+1}}\Bigg)\Bigg).$$

Comparing coefficients with what we started with, it is clear that $C_m^{(n)}=(n-1)C_m^{(n-1)}$ for $m=0,1,2$ as required.

The proof for the $\tilde{C}_m^{(n)}$ follows the same lines as above. $\square$

Finally, we just need to establish the values $C_0^{(2)},C_1^{(2)},C_2^{(2)},\tilde{C}_0^{(2)},\tilde{C}_1^{(2)}$ and $\tilde{C}_2^{(2)}$ to find all the remaining values by induction. (2.4.7) in the $n=2$ case is

$$k_1k_2(k_2-k_1-k_1k_2)+k_1k_2(k_2-k_1+k_1k_2)=2(k_2-k_1)k_1k_2.$$

Thus $C_0^{(2)}=0$, $C_1^{(2)}=0$ and $C_2^{(2)}=2$. Combining the two lemmas above this implies for $m=0,1$ $C_0^{(n)}=0$ for every n, and $C_2^{(n)}=2(n-1)!$ for every n. (2.4.8) in the $n=2$ case is

$$(k_2-k_1)(k_2-k_1-k_1k_2)+(k_1-k_2)(k_2-k_1+k_1k_2)=-2(k_2-k_1)k_1k_2.$$

Thus $\tilde{C}_0^{(2)}=0$, $\tilde{C}_1^{(2)}=0$ and $\tilde{C}_2^{(2)}=-2$. Combining the two lemmas above this implies for $m=0,1$ $\tilde{C}_m^{(n)}=0$ for every n, and $\tilde{C}_2^{(n)}=-2(n-1)!$ for every n. In particular, this shows that the sum of (2.4.7) and (2.4.8) is 0, proving Proposition 2.4.4.

As a consequence, we have proven Proposition 2.4.3, concluding this subsection.

2.4.3 Initial Condition

Proposition 2.4.9. *For any bounded Lipschitz continuous function $f : \overline{\mathbb{W}^n} \to \mathbb{R}$, we have*

$$\int u_t(\cdot, y) f(y) m_\theta^{(n)}(dy) \to f \text{ uniformly, as } t \to 0.$$

Where the definitions of $m_\theta^{(n)}$ and u_t are given in definition 1.3.2 and Definition (2.4.3) respectively.

First we'll show

Lemma 2.4.10.

$$\int u_t(x, y) m_\theta^{(n)}(dy) = 1 \quad \text{for all } x \in \overline{\mathbb{W}^n},\ t > 0.$$

Proof. Lemma 2.4.2 allows us to calculate the time derivative by passing it through the integral

$$\begin{aligned}\frac{\partial}{\partial t} \int u_t(x, y) m_\theta^{(n)}(dy) &= \int \frac{1}{2} \Delta u_t(x, y) m_\theta^{(n)}(dy) \\ &= 0.\end{aligned}$$

The first equality is clear from the definition of u. The second equality follows from Corollary 2.3.9 and Lemma 2.4.2. This shows the integral is constant, to finish we shall show convergence to 1 as $t \to \infty$. Scaling k by $t^{-\frac{1}{2}}$ and y by $t^{\frac{1}{2}}$ we see the following

$$\begin{aligned}&\int u_t(x, y) m_\theta^{(n)}(dy) \\ &= \int \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k) dk \ m_\theta^{(n)}(dy) \\ &= \sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \frac{1}{(2\pi)^n t^{\frac{1}{2}(n-|\pi|)}} \int \int_{\mathbb{R}^n} e^{-\frac{1}{2}|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x/\sqrt{t} - y_\sigma)} \\ &\quad \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} S_{\sigma(\beta),\sigma(\alpha)}(k/\sqrt{t}) dk \ \lambda^\pi(dy).\end{aligned}$$

We can justify applying Dominated convergence to this by referring to lemma 2.4.2 to take $t \to \infty$. It is clear that $S_{\sigma(\beta),\sigma(\alpha)}(\frac{k}{\sqrt{t}}) \to 1$ as $t \to \infty$ for almost every k. Notice that all terms with $|\pi| < n$ in the sum over partitions vanish in the limit, leaving only the partition consisting exclusively of singletons. For this partition, λ^π is just the Lebesgue measure on the Weyl chamber. Thus, we have

$$\begin{aligned}\int u_t(x,y)m_\theta^{(n)}(dy) &= \frac{n!}{(2\pi)^n}\int_{\mathbb{W}^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}|k|^2-ik\cdot y}dkdy,\\ &= 1.\end{aligned}$$

The $n!$ comes from the sum over permutations, the resulting integral in k is just the Fourier transform of a Gaussian; hence, the integral over the Weyl chamber is easily calculated. □

Now we can write

$$\int u_t(x,y)f(y)m_\theta^{(n)}(dy) - f(x) = \int u_t(x,y)\left(f(y)-f(x)\right)m_\theta^{(n)}(dy).$$

It follows directly from the definition of $m_\theta^{(n)}$ that

$$\begin{aligned}&\left|\int u_t(x,y)\left(f(y)-f(x)\right)m_\theta^{(n)}(dy)\right|\\ &\leq \sum_{\pi\in\Pi_n}\theta^{|\pi|-n}\prod_{\pi_i\in\pi}\tfrac{1}{|\pi_i|}\left|\int u_t(x,y)\left(f(y)-f(x)\right)\lambda^\pi(dy)\right|. \end{aligned}\tag{2.4.16}$$

Thus we can restrict our attentions to the integral with respect to λ^π for a fixed $\pi\in\Pi_n$.

Let us briefly outline the proof. We begin by rearranging $u_t(x,y)$ into a more convenient form, and then we split the sum over permutations, so that we first sum over permutations σ for which the images $(\sigma(\pi_\iota))_{\iota=1}^{|\pi|}$ are fixed. We then bound $u_t(x,y)$ by making contour shifts, following the same idea used to calculate the Fourier transform of the Gaussian density. This step is complicated by the presence of poles in the integral defining $u_t(x,y)$; however, our previous step gives us some control over where the poles appear, and we can further use that x and y are both in the Weyl chamber to derive Gaussian bounds on $u_t(x,y)$. In the final step, we combine these bounds with the Lipschitz property for f to derive the desired uniform convergence. This requires bounding of the contribution from $\overline{\mathbb{W}^\pi}$ to $\int|u_t(x,y)|m_\theta^{(n)}(dy)$ and some care in considering what happens when x is near, but not in, $\overline{\mathbb{W}^\pi}$ to ensure we get uniform convergence.

To begin our rearrangements, we prove that $u_t(x,y)$ is symmetric under swaps of x and y.

Lemma 2.4.11. *For every $x, y \in \overline{\mathbb{W}^n}$ and $t > 0$*

$$u_t(x,y) = u_t(y,x).$$

Proof. Recall that u is defined in (2.4.3) as

$$u_t(x,y) = \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} \tfrac{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)}k_{\sigma(\beta)}}{i\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)}k_{\sigma(\beta)}} dk.$$

If we first take the sum outside the integral, then perform the change of variables in the k integral, $k \to -k_{\sigma^{-1}}$, this becomes

$$\tfrac{1}{(2\pi)^n} \sum_{\sigma \in S_n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2 + ik_{\sigma^{-1}} \cdot (x_{\sigma^{-1}} - y)} \prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} \tfrac{i\theta(k_\beta - k_\alpha) + k_\alpha k_\beta}{i\theta(k_\beta - k_\alpha) - k_\alpha k_\beta} dk.$$

Notice that we can relabel the product as follows

$$\prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} \tfrac{i\theta(k_\beta - k_\alpha) + k_\alpha k_\beta}{i\theta(k_\beta - k_\alpha) - k_\alpha k_\beta} = \prod_{\substack{\alpha<\beta: \\ \sigma^{-1}(\alpha)>\sigma^{-1}(\beta)}} \tfrac{i\theta(k_{\sigma^{-1}(\alpha)} - k_{\sigma^{-1}(\beta)}) + k_{\sigma^{-1}(\alpha)}k_{\sigma^{-1}(\beta)}}{i\theta(k_{\sigma^{-1}(\alpha)} - k_{\sigma^{-1}(\beta)}) - k_{\sigma^{-1}(\alpha)}k_{\sigma^{-1}(\beta)}}.$$

Hence by relabelling the sum to be over $\sigma^{-1} \in S_n$, we see that we get $u_t(y,x)$ as desired. □

Now we proceed with the proof of the proposition, we can rewrite the summand of (2.4.16) (ignoring the constants) as

$$\begin{aligned} &\left| \int u_t(y,x) \left(f(y) - f(x)\right) \lambda^\pi(dy) \right| \\ &= \left| \int \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (y - x_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} S_{\sigma(\beta),\sigma(\alpha)}(k) \; dk \left(f(y) - f(x)\right) \tfrac{\lambda^\pi(dy)}{(2\pi)^n} \right|. \end{aligned} \tag{2.4.17}$$

For a partition $\pi \in \Pi_n$ and permutation $\sigma \in S_n$ define the set of ordered pairs $\sigma(\pi) := \{(\pi_1, \sigma(\pi_1)), ..., (\pi_{|\pi|}, \sigma(\pi_{|\pi|}))\}$ (where $\sigma(A)$ denotes the image of A under σ). We can rewrite the sum in the above integral as follows

$$\sum_{\substack{\tau \in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} \sum_{\substack{\sigma \in S_n: \\ \sigma(\pi) = \tau(\pi)}} e^{ik_\sigma \cdot (y - x_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} S_{\sigma(\beta),\sigma(\alpha)}(k).$$

Let's consider $e^{ik_\sigma\cdot(y-x_\sigma)} = e^{-ik\cdot x}\prod_{j=1}^n e^{ik_{\tau(j)}y_j}$. We'll use the notation $\overline{\pi_\iota} := \sup \pi_\iota$, and $\underline{\pi_\iota} := \sup \pi_\iota$. We know that for each $\pi_\iota \in \pi$, $\alpha, \beta \in \pi_\iota$ implies $y_\alpha = y_\beta$ λ^π-a.e. Hence $\prod_{j=1}^n e^{ik_{\sigma(j)}y_j} = \prod_{\pi_\iota\in\pi}\prod_{\alpha\in\pi_\iota} e^{ik_{\sigma(\alpha)}y_{\overline{\pi_\iota}}}$ λ^π-a.e. But since $\sigma(\pi) = \tau(\pi)$ this is just equal to $\prod_{\pi_\iota\in\pi}\prod_{\alpha\in\pi_\iota} e^{ik_{\tau(\alpha)}y_{\overline{\pi_\iota}}}$ which equals $e^{ik_\tau\cdot y}$. Hence we can pull the exponential out of the second sum to make the previous expression equal λ^π-a.e. to

$$\sum_{\substack{\tau\in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} e^{ik_\tau\cdot(y-x_\tau)} \sum_{\substack{\sigma\in S_n:\\ \sigma(\pi)=\tau(\pi)}} \prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}} S_{\sigma(\beta),\sigma(\alpha)}(k).$$

Now we can consider the product; in particular, we can show that when α and β are in different elements of π then the appearance of $S_{\sigma(\beta),\sigma(\alpha)}(k)$ in the product depends only on τ, and not the specific σ. Suppose $\alpha < \beta$ are in different elements of π and that $\sigma(\beta) < \sigma(\alpha)$. Since π is an ordered partition, there exists $\iota < j$ such that $\alpha \in \pi_\iota$ and $\beta \in \pi_j$. Now since $\sigma(\pi) = \tau(\pi)$, there must exist $\gamma \in \pi_\iota$ and $\delta \in \pi_j$ (thus $\gamma < \delta$) such that $\tau(\gamma) = \sigma(\alpha) > \sigma(\beta) = \tau(\delta)$. Hence, for each such $\alpha < \beta$ such that $\sigma(\beta) < \sigma(\alpha)$, where α and β are in different elements of π, there are $\gamma < \delta$ in different elements of π such that $\tau(\delta) < \tau(\gamma)$. Similarly, we can go in the other direction, so that if α and β are in different elements of π then $(\sigma(\beta), \sigma(\alpha))$ is an inversion for σ if and only if it is an inversion for τ (That is if α and β are in different elements of π then $\alpha < \beta$ with $\sigma(\beta) < \sigma(\alpha)$ occurs if and only if $\tau^{-1}(\sigma(\alpha)) < \tau^{-1}(\sigma(\beta))$ with $\sigma(\beta) < \sigma(\alpha)$). Hence, we can split off the part of the product where α and β are in different elements of π and rewrite entirely in terms of τ. Thus, the previous expression is equal to

$$\sum_{\substack{\tau\in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} e^{ik_\tau\cdot(y-x_\tau)} \left(\prod_{\iota<j}\prod_{\substack{\alpha\in\pi_\iota,\ \beta\in\pi_j:\\ \tau(\beta)<\tau(\alpha)}} S_{\tau(\beta),\tau(\alpha)}(k)\right) \sum_{\substack{\sigma\in S_n:\\ \sigma(\pi)=\tau(\pi)}} A^o_{\sigma,\pi}(k). \quad (2.4.18)$$

Where $A^o_{\sigma,\pi}$ is shorthand for the summand of the second sum and defined as follows.

$$A^o_{\sigma,\pi}(k) := \prod_{\pi_\iota\in\pi}\prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta),\\ \alpha,\beta\in\pi_\iota}} S_{\sigma(\beta),\sigma(\alpha)}(k)$$

We can calculate the second sum using the formula in the following lemma.

Lemma 2.4.12. *Suppose $m \in \mathbb{N}$ and $\theta > 0$ then for all $k \in \mathbb{R}^m$ such that $k_\alpha \neq 0$*

for all $\alpha \in \{1, \dots, m\}$

$$\sum_{\sigma \in S_m} \prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} S_{\sigma(\beta),\sigma(\alpha)}(k) = m! \prod_{\alpha<\beta} \frac{i\theta(k_\beta - k_\alpha)}{i\theta(k_\beta - k_\alpha) - k_\alpha k_\beta}.$$

Proof. First, we prove the following equality holds for all $\xi \in \mathbb{C}^m$

$$\sum_{\sigma \in S_m} \left(\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} (-1) \right) \left(\prod_{\alpha<\beta} \xi_{\sigma(\alpha)} - \xi_{\sigma(\beta)} - 1 \right) = m! \prod_{\alpha<\beta} (\xi_\alpha - \xi_\beta). \tag{2.4.19}$$

It is clear that the left hand side is a degree $\binom{m}{2}$ polynomial, which we shall denote $P(\xi)$. Thus if we can prove that $P(\xi)$ is also alternating, it must be a constant multiple of the right hand side. We then just need to check the constant to finish the proof.

To prove the left hand side is alternating it is enough to consider swaps of consecutive variables, e.g. ξ_j and ξ_{j+1} for some $j \in \{1, ..., m-1\}$. Let $s_j = (j, j+1) \in S_n$, i.e. the permutation that swaps j and $j+1$ leaving everything else fixed. Clearly, for all $\sigma \in S_n$

$$\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} (-1) = - \prod_{\substack{\alpha<\beta: \\ \sigma\circ s_j(\beta)<\sigma\circ s_j(\alpha)}} (-1). \tag{2.4.20}$$

It follows, by relabelling the sum in its definition, that $P(\xi_{s_j}) = -P(\xi)$. Hence, P is an alternating polynomial, and there is a $c \in \mathbb{R}$ such that

$$\sum_{\sigma \in S_m} \left(\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} (-1) \right) \left(\prod_{\alpha<\beta} \xi_{\sigma(\alpha)} - \xi_{\sigma(\beta)} - 1 \right) = c \prod_{\alpha<\beta} (\xi_\alpha - \xi_\beta).$$

To finish, we just have to note that if we expand the bracket on the left hand side we get $m! \prod_{\alpha<\beta}(\xi_\alpha - \xi_\beta)$ plus additional terms of lower degree. But we know that the left hand side, P, is a constant multiple of $\prod_{\alpha<\beta}(\xi_\beta - \xi_\alpha)$; thus, the lower degree terms must cancel. This proves (2.4.19).

To prove the lemma, we just need to divide both sides of (2.4.19) by $\prod_{\alpha<\beta}(\xi_\alpha - \xi_\beta - 1)$ then set $\xi_j = i\theta/k_j$ for each j. An application of the following equality on the left hand side, and some simple rearrangements, give the desired identity.

$$\left(\prod_{\alpha<\beta} \xi_\alpha - \xi_\beta - 1\right) = \left(\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} \xi_{\sigma(\beta)} - \xi_{\sigma(\alpha)} - 1\right)\left(\prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)<\sigma(\beta)}} \xi_{\sigma(\alpha)} - \xi_{\sigma(\beta)} - 1\right).$$

□

Hence, we get that (2.4.18) is equal to

$$\left(\prod_{\iota=1}^{|\pi|} |\pi_\iota|!\right) \sum_{\substack{\tau\in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} e^{ik_\tau\cdot(y-x_\tau)} T^{\tau,\pi}(k).$$

Where $T^{\tau,\pi}:\mathbb{C}^n \to \mathbb{C}^n$ is defined (for a.e. $k\in\mathbb{C}^n$) as follows

$$T^{\tau,\pi}(k) := \left(\prod_{\iota<j} \prod_{\substack{\alpha\in\pi_\iota,\ \beta\in\pi_j:\\ \tau(\beta)<\tau(\alpha)}} S_{\tau(\beta),\tau(\alpha)}(k)\right)\left(\prod_{\pi_\iota\in\pi} \prod_{\substack{\alpha<\beta:\\ \alpha,\beta\in\pi_\iota}} \frac{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right).$$

This rearrangement, together with the triangle inequality, gives us that (2.4.17) is bounded above by

$$\frac{\prod_{\iota=1}^{|\pi|}|\pi_\iota|!}{(2\pi)^n} \sum_{\substack{\tau\in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} \int \left| \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+ik_\tau\cdot(y-x_\tau)} T^{\tau,\pi}(k)dk\right| |f(y)-f(x)|\lambda^\pi(dy). \tag{2.4.21}$$

Now we can move on to the next step, which we briefly motivate. We want to get control on the k integral in the above expression, and we need the bound to be integrable in y with respect to λ^π and to be vanishing as $t\to 0$ whenever $y\neq x$. Note that we can rewrite the exponent appearing in the integrand as follows

$$-\frac{1}{2}t|k|^2 + ik_\tau\cdot(y-x_\tau) = -\frac{1}{2}t\sum_{\alpha=1}^n (k_{\tau(\alpha)} - \frac{i}{t}(y_\alpha - x_{\tau(\alpha)}))^2 - \frac{(y_\alpha - x_{\tau(\alpha)})^2}{2t}.$$

Suggesting that we should use Cauchy's residue theorem to shift the $k_{\tau(\alpha)}$ contour from $\mathbb{R}$ to $C_\alpha := \{z\in\mathbb{C}:\ z-\frac{i}{t}(y_\alpha - x_{\tau(\alpha)})\in\mathbb{R}\}$ for each $\alpha\in\{1,...,n\}$, and then parametrise the resulting contour integral as an integral over $\mathbb{R}$. Supposing we can do this without encountering any poles, the exponent becomes

$$-\frac{1}{2}t\sum_{\alpha=1}^{n}\tilde{k}_{\tau(\alpha)}^{2}-\frac{(y_\alpha - x_{\tau(\alpha)})^2}{2t}.$$

Where $\tilde{k}_{\tau(\alpha)} \in \mathbb{R}$ is our new integration variable. The second term of the summand gives us the necessary control in the y variable, and the first term should allow us to control the resulting k integral. However, this approach is complicated by $T^{\tau,\pi}$ which contribute poles that hinder our contour shifting. We end up not being able to shift the integration contours for all of the k variables without encountering poles; however, we are still able to make some of the desired contour shifts. To see which shifts can be made, we need check where these poles occur. Note that by definition

$$T^{\tau,\pi}(k) := \left(\prod_{\iota<j}\prod_{\substack{\alpha<\beta:\\ \tau(\beta)<\tau(\alpha),\\ \alpha\in\pi_\iota,\ \beta\in\pi_j}}\frac{i\theta(k_{\tau(\alpha)}-k_{\tau(\beta)})+k_{\tau(\alpha)}k_{\tau(\beta)}}{i\theta(k_{\tau(\alpha)}-k_{\tau(\beta)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right)\left(\prod_{\pi_\iota\in\pi}\prod_{\substack{\alpha<\beta:\\ \alpha,\beta\in\pi_\iota}}\frac{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right) \tag{2.4.22}$$

The following lemma provides us with the desired information.

Lemma 2.4.13. *We'll use the notation* $\mathbb{H} = \{x + iy \in \mathbb{C}|\ x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}$ *for the upper half complex plane. The function* $(z, w) \mapsto i\theta(z - w) - zw$ *has no zeroes in the set* $\mathbb{H} \times -\mathbb{H}$.

Proof. For $w \in -\mathbb{H}$ there are $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0}$ such that $w = a - bi$. It is easily checked that $i\theta(z - w) - zw = 0$ if and only if we have

$$z = \frac{\theta^2 a - i\theta((\theta + b)b + a^2)}{(\theta + b)^2 + a^2} \in -\mathbb{H}.$$

Thus, there are no zeroes inside $\mathbb{H} \times -\mathbb{H}$ as claimed. □

Observing the structure of the products in (2.4.22), we define the set $E^{\tau,\pi} \subset \mathbb{C}^n$ defined as $\times_{k=1}^{n} E_k^{\tau,\pi}$ where $E_k^{\tau,\pi}$ is the upper half complex plane if there is a $\pi_\iota \in \pi$ such that $k = \sup \pi_\iota$ and $\tau(\alpha) < \tau(k)$ for all $\alpha < k$, the lower half complex plane if there is a π_ι such that $k = \inf \pi_\iota$ and $\tau(\beta) > \tau(k)$ for all $\beta > k$, the whole complex plane if both of these conditions are satisfied, and the real line if neither are satisfied. Lemma 2.4.13 shows the denominator of $T^{\tau,\pi}$ as in (2.4.22) has no zeroes in the set $E^{\tau,\pi}$ (2.4.22), and thus we can perform our contour shifts as long as the contours remain within this set. To simplify our notation slightly we'll henceforth write $\overline{\pi_\iota} := \sup \pi_\iota$ and $\underline{\pi_\iota} := \inf \pi_\iota$ for each $\pi_\iota \in \pi$.

We'll now use these ideas to get a following bound on the k integral in line (2.4.21). First, we need to find a family of indices for which contour shifts can be made, that is a collection of α such that $E_\alpha^{\tau,\pi}$ contains at least one complex half plane.

Lemma 2.4.14. *Suppose $\pi \in \Pi_n$ and $\tau \in S_n$ such that $\tau|_{\pi_\iota}$ is increasing for every $\pi_\iota \in \pi$. For each $\pi_\iota \in \pi$ there are $a_\iota \leq \iota \leq b_\iota$ such that $\tau(\underline{\pi_{b_\iota}}) \leq \tau(\overline{\pi_{a_\iota}})$, and the following properties hold*

- $\tau(\underline{\pi_{b_\iota}}) < \tau(\beta)$ *for every* $\beta > \underline{\pi_{b_\iota}}$;
- *and* $\tau(\overline{\pi_{a_\iota}}) > \tau(\alpha)$ *for all* $\alpha < \overline{\pi_\iota}$.

Further, given such a (a_ι, b_ι), we can define $m_\iota := \sup\{\tau(\alpha)|\ \overline{\pi_{a_\iota}} \leq \alpha \leq \underline{\pi_{b_\iota}}\}$ and $l_\iota := \inf\{\tau(\beta)|\ \overline{\pi_{a_\iota}} \leq \beta \leq \underline{\pi_{b_\iota}}\}$ if $\overline{\pi_{a_\iota}} < \underline{\pi_{b_\iota}}$; and $m_\iota := \tau(\overline{\pi_{a_\iota}})$ and $l_\iota := \tau(\underline{\pi_{b_\iota}})$ if $\overline{\pi_{a_\iota}} \geq \underline{\pi_{b_\iota}}$. The following properties hold for m_ι and l_ι:

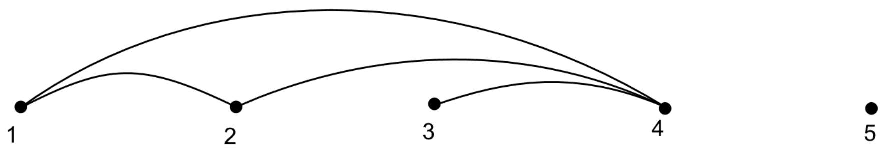
- *there are $\pi_c, \pi_d \in \pi$ such that $\tau^{-1}(m_\iota) = \overline{\pi_d}$ and $\tau^{-1}(l_\iota) = \underline{\pi_c}$;*
- *for all $\alpha < \tau^{-1}(m_\iota)$ we have $\tau(\alpha) < m_\iota$;*
- *and for all $\beta > \tau^{-1}(l_\iota)$ we have $\tau(\beta) > l_\iota$.*

Proof. First we define $\mu_\iota := \overline{\pi_{a_\iota}}$ where $a_\iota := \inf\{a \leq \iota :\ \tau(\overline{\pi_a}) \geq \tau(\underline{\pi_\iota})\}$, and then from it we define $\nu_\iota := \underline{\pi_{b_\iota}}$ where $b_\iota := \sup\{b \geq \iota :\ \tau(\mu_\iota) \geq \tau(\underline{\pi_b})\}$. μ_ι and ν_ι are introduced for convenience and will be used throughout this section. In Figure 2.1 we provide an example of a permutation and partition and the resulting values of μ_ι and ν_ι.

It is easy to see that the a_ι and b_ι satisfy the first two properties we claimed for them, namely that $a_\iota \leq \iota \leq b_\iota$ and $\tau(\nu_\iota) = \tau(\underline{\pi_{b_\iota}}) \leq \tau(\overline{\pi_{a_\iota}}) = \tau(\mu_\iota)$.

Now we show $\tau(\nu_\iota) < \tau(\beta)$ for all $\beta > \nu_\iota$, and $\tau(\mu_\iota) > \tau(\alpha)$ for all $\alpha < \mu_\iota$. Starting with μ_ι, if there is an $\alpha < \mu_\iota$ such that $\tau(\mu_\iota) < \tau(\alpha)$ then by definition of μ_ι α must be in a different element of π to μ_ι, say π_c, with $c < a_\iota$. Since τ is increasing on every element of π this means we must have $\tau(\overline{\pi_c}) > \tau(\alpha) > \tau(\mu_\iota) = \tau(\overline{\pi_{a_\iota}})$ which contradicts the definition of a_ι, so no such α exists. By a similar argument, there is no $\beta > \nu_\iota$ such that $\tau(\nu_\iota) > \tau(\beta)$.

It just remains to prove the second set of statements, those about m_ι and l_ι. Suppose we are given (a_ι, b_ι), as in the first part of the lemma, and once more define $\mu_\iota := \overline{\pi_{a_\iota}}$ and $\nu_\iota := \underline{\pi_{b_\iota}}$. The first property for m_ι and l_ι follows immediately from the fact that $\tau|_{\pi_j}$ is increasing for all $\pi_j \in \pi$, the definitions of m_ι and l_ι, and from $\pi \in \Pi_n$.



π_1	π_2	π_3	π_4
(1,4)	(1,4)	(1,4)	(5,5)

Figure 2.1: The bottom row displays the values of (μ_ι, ν_ι) for the permutation $\tau = (1, 3, 4)$ in S_5 and partition $\pi = (\{1\}, \{2, 3\}, \{4\}, \{5\})$. Lines connect pairs of indices which τ inverts, so $i < j$ are connected if $\tau(j) < \tau(i)$. μ_ι is the largest element of the leftmost block of π connected to π_ι by a line. Similarly, ν_ι is the least element of the rightmost block of π connected to π_ι by a line.

For the second and third statements we consider two cases separately: $\mu_\iota < \nu_\iota$ and $\nu_\iota \leq \mu_\iota$. For the latter case we have $m_\iota = \tau(\nu_\iota)$ and $l_\iota = \tau(\mu_\iota)$, so the statements are the same as those we just proved. If instead we have $\mu_\iota < \nu_\iota$ we can argue the second statement as follows. Clearly for all α such that $\mu_\iota \leq \alpha \leq \nu_\iota$ we have $\tau(\alpha) < m_\iota$, thus we only need to check that $\alpha < \mu_\iota$ implies $\tau(\alpha) < m_\iota$. Suppose this is false, i.e. there is an $\alpha < \mu_\iota$ such that $\tau(\alpha) > m_\iota$. Since $m_\iota > \tau(\mu_\iota)$ this implies $\tau(\alpha) > \tau(\mu_\iota)$, since we also have $\alpha < \mu_\iota$ this is a contradiction as we know from previously that $\tau(\mu_\iota) > \tau(\alpha)$ whenever $\mu_\iota > \alpha$. A similar argument proves the third statement, thereby proving the lemma. $\square$

In the following proposition, we will assume we have a $\pi \in \Pi_n$ with a family $(a_\iota, b_\iota)_{\pi_\iota \in \pi}$ given by the above lemma, and adopt the notation of the above proof, namely $\mu_\iota := \overline{\pi_{a_\iota}}$ and $\nu_\iota := \underline{\pi_{b_\iota}}$. The above lemma ensures that whenever $\alpha = \mu_\iota, \tau^{-1}(m_\iota)$ the set $E_\alpha^{\tau,\pi}$ contains the upper half complex plane, and if $\beta = \nu_\iota, \tau^{-1}(l_\iota)$ then $E_\beta^{\tau,\pi}$ contains the lower half complex plane.

Proposition 2.4.15. *Suppose $\pi \in \Pi_n$ and $\tau \in S_n$ such that $\tau|_{\pi_\iota}$ is increasing for every $\pi_\iota \in \pi$, and for each $\pi_\iota \in \pi$ we have $a_\iota \leq \iota \leq b_\iota$ as in the above lemma. There is a constant $C > 0$, depending only on π and n, such that the following bound holds* *for all* $x, y \in \overline{\mathbb{W}^n}$

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+ik_\tau\cdot(y-x_\tau)}T^{\tau,\pi}(k)dk\right|\\ \leq& Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|}e^{-\frac{|y-\chi|^2}{12nt}}\prod_{\pi_\iota\in\pi}e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}.\end{aligned} \tag{2.4.23}$$

Where $\chi = \chi(x) \in \mathbb{R}^n$ *is defined by* $\chi_\alpha := \chi^\iota := \frac{1}{2}(x_{\tau(\mu_\iota)} + x_{\tau(\nu_\iota)})$ *for all* $\alpha \in \pi_\iota$.

We begin the proof with the following intermediate bound.

Lemma 2.4.16. *Let* $\Gamma_{\alpha,x,y} = C_\alpha$ *if* $x, y \in \overline{\mathbb{W}^n}$ *are such that the* C_α *contour lies in* $E^{\tau,\pi}_\alpha$, *and* $\mathbb{R}$ *otherwise.*

$$\begin{aligned}\left|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+ik_\tau\cdot(y-x_\tau)}T^{\tau,\pi}(k)dk\right| \leq& e^{-\frac{|y-\chi|^2}{12nt}}\left(\prod_{\pi_\iota\in\pi}e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right)\\ &\int_{\times_{\alpha=1}^n\Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n Re(k_{\tau(\alpha)})^2}\left|T^{\tau,\pi}(k)\right|dk.\end{aligned} \tag{2.4.24}$$

Proof. We can apply Cauchy's residue theorem to shift the contours of the integral into the complex plane, onto the contours $\Gamma_{\alpha,x,y}$ to be precise. This is possible because we have defined the contours $\Gamma_{\alpha,x,y}$ in such a way that they are either $\mathbb{R}$, and thus no deformation is required, or the integrand is analytic in whichever half plane they occupy. The result is the following equality

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+ik_\tau\cdot(y-x_\tau)}T^{\tau,\pi}(k)dk =& e^{-\frac{1}{2t}\sum_{\alpha:\Gamma_{\alpha,x,y}\neq\mathbb{R}}(y_\alpha-x_{\tau(\alpha)})^2}\\ \times\int_{\times_{\alpha=1}^n\Gamma_{\alpha,x,y}} & e^{-\frac{1}{2}t\sum_{\alpha=1}^n \mathrm{Re}(k_{\tau(\alpha)})^2+i\sum_{\alpha:\Gamma_{\alpha,x,y}=\mathbb{R}}k_{\tau(\alpha)}(y_\alpha-x_{\tau(\alpha)})}T^{\tau,\pi}(k)dk.\end{aligned} \tag{2.4.25}$$

Note that on the right hand side, when the contour for k_j is not the real line, we have rewritten the exponential by completing the square: $k_j^2 - \frac{2i}{t}k_j(y_{\tau^{-1}(j)} - x_j) = (k_i - \frac{i}{t}(y_{\tau^{-1}(i)} - x_i))^2 + \frac{1}{t^2}(y_{\tau^{-1}(i)} - x_i)^2$.

From (2.4.25), we see that to prove lemma 2.4.16 we must bound the exponential appearing in front of the integral, which means we need to consider which contour shifts have been made. That is, we want to check when the condition for $\Gamma_{\alpha,x,y} = C_\alpha$ is true, for $\alpha = \mu_\iota, \nu_\iota$. Thus, we want to check when C_α lies inside $E^{\tau,\pi}_\alpha$. We know from lemma 2.4.14 that $E^{\tau,\pi}_{\mu_\iota}$ contains the upper half complex plane, and $E^{\tau,\pi}_{\nu_\iota}$ contains the lower half complex plane. Thus, $C_{\mu_\iota} \subset E^\tau_{\mu_\iota}$ when $y_{\mu_\iota} \geq x_{\tau(\mu_\iota)}$, and

$C_{\nu_\iota} \subset E^{\tau}_{\nu_\iota}$ when $y_{\nu_\iota} \leq x_{\tau(\nu_\iota)}$. Hence we have the following inequalities:

$$-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 \leq -\frac{1}{2t}\mathbb{1}_{\{(y_{\mu_\iota}\geq x_{\tau(\mu_\iota)})\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2;$$
$$-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \leq -\frac{1}{2t}\mathbb{1}_{\{(y_{\nu_\iota}\leq x_{\tau(\nu_\iota)})\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2.$$

There are two cases of interest, the first is when $\mu_\iota < \nu_\iota$. In this case, the two indices are in different elements of π. The second case is when $\mu_\iota \geq \nu_\iota$, for which the two indices are in the same element of π. Let us deal now with the first case.

By definition we have $\tau(\mu_\iota) \geq \tau(\nu_\iota)$; thus, since $x, y \in \overline{\mathbb{W}^n}$, it follows that we always have $y_{\nu_\iota} \leq y_{\mu_\iota}$. Hence, if we have both $y_{\nu_\iota} > x_{\tau(\nu_\iota)}$ and $y_{\mu_\iota} < x_{\tau(\mu_\iota)}$, it follows that $x_{\tau(\nu_\iota)} < x_{\tau(\mu_\iota)}$, but since $x \in \overline{\mathbb{W}^n}$ this is a contradiction. Hence, for all $x, y \in \overline{\mathbb{W}^n}$ at least one of $y_{\mu_\iota} \geq x_{\tau(\mu_\iota)}$ and $y_{\nu_\iota} \leq x_{\tau(\nu_\iota)}$ must be true. This means we have the following equality

$$-\frac{1}{2t}\mathbb{1}_{\{(y_{\mu_\iota}\geq x_{\tau(\mu_\iota)})\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{(y_{\nu_\iota}\leq x_{\tau(\nu_\iota)})\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2$$
$$= \begin{cases} -\frac{1}{2t}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2, \text{ if } y_{\mu_\iota} \geq x_{\tau(\mu_\iota)} \text{ and } y_{\nu_\iota} > x_{\tau(\nu_\iota)}; \\ -\frac{1}{2t}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2, \text{ if } y_{\mu_\iota} < x_{\tau(\mu_\iota)} \text{ and } y_{\nu_\iota} \leq x_{\tau(\nu_\iota)}; \\ -\frac{1}{2t}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2, \text{ if } y_{\nu_\iota} \leq x_{\tau(\nu_\iota)} \text{ and } y_{\mu_\iota} \geq x_{\tau(\mu_\iota)}. \end{cases} \tag{2.4.26}$$

Let $\chi^\iota := \frac{1}{2}(x_{\tau(\mu_\iota)} + x_{\tau(\nu_\iota)})$, we can rewrite the first line as

$$-\frac{1}{2t}\left((y_{\mu_\iota} - \chi^\iota)^2 + \frac{1}{4}(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right) + \frac{1}{2t}(y_{\mu_\iota} - \chi^\iota)(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)}).$$

We have $\tau(\mu_\iota) > \tau(\nu_\iota)$ and $x \in \overline{\mathbb{W}^n}$, so that $(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)}) < 0$. From $y \in \overline{\mathbb{W}^n}$ and $\mu_\iota < \nu_\iota$, it follows that $y_{\mu_\iota} \geq \frac{1}{2}(y_{\mu_\iota} + y_{\nu_\iota})$, which, under the conditions of the first line, is bounded below by $\chi^\iota = \frac{1}{2}(x_{\tau(\mu_\iota)} + x_{\tau(\nu_\iota)})$. Thus $y_{\mu_\iota} - \chi^\iota > 0$, and the last term above is negative. We also have $y_{\nu_\iota} - \chi_\iota \geq y_{\nu_\iota} - x_{\tau(\nu_\iota)} > 0$ in this case; thus, using $y_{\nu_\iota} \leq y_{\mu_\iota}$, we get $-(y_{\mu_\iota} - \chi^\iota)^2 \leq -(y_{\nu_\iota} - \chi^\iota)^2$. It follows that the above expression is bounded above by

$$-\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + \frac{1}{2}(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right).$$

The same ideas yield the same bound on the cases of the second and third lines, so that the above expression is a bound for (2.4.26).

Now we need to look at the contour shifts for m_ι and l_ι. Recall $m_\iota = \sup\{\tau(\alpha) |\ \mu_\iota \leq$

$\alpha \le \nu_\iota\}$ and $l_\iota = \inf\{\tau(\beta)|\ \mu_\iota \le \beta \le \nu_\iota\}$. Note that it is quite possible for $m_\iota = \tau(\mu_\iota)$ or for $l_\iota = \tau(\nu_\iota)$. We need to check when $C_{\tau^{-1}(m_\iota)} \subset E^{\tau,\pi}_{\tau^{-1}(m_\iota)}$. From Lemma 2.4.14 we know $E^{\tau,\pi}_{\tau^{-1}(m_\iota)}$ contains the upper half complex plane. Therefore $\Gamma_{\tau^{-1}(m_\iota),x,y} = C_{\tau^{-1}(m_\iota)}$ if $y_{\tau^{-1}(m_\iota)} \ge x_{m_\iota}$. Therefore we have

$$-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(m_\iota),x,y}\neq\mathbb{R}\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \le -\frac{1}{2t}\mathbb{1}_{\{(y_{\tau^{-1}(m_\iota)}\ge x_{m_\iota})\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2.$$

We can combine this with our previous bound to get the following

$$\begin{aligned}
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \\
&\qquad -\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(m_\iota),x,y}\neq\mathbb{R},\ m_\iota\neq\tau(\mu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&\le -\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right) \\
&\qquad -\frac{1}{2t}\mathbb{1}_{\{y_{\tau^{-1}(m_\iota)}\ge x_{m_\iota},\ m_\iota\neq\tau(\mu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2. \qquad (2.4.27)
\end{aligned}$$

We aim to show this is bounded above, for some positive constants C_1, C_2, by

$$-\frac{C_1}{t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right) - \frac{C_2}{t}(x_{m_\iota} - \chi^\iota)^2.$$

Thus we consider the various cases for the indicator in (2.4.27).

If $m_\iota = \tau(\mu_\iota)$, then it follows from $x_{m_\iota} \le \chi_\iota \le x_{\tau(\nu_\iota)}$ that $(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2 \le (x_{m_\iota} - \chi^\iota)^2$, so that our desired bound is easily seen.

In the case that $m_\iota \neq \tau(\mu_\iota)$ and $y_{\tau^{-1}(m_\iota)} \ge x_{m_\iota}$, if we further assume $y_{\tau^{-1}(m_\iota)} \ge \chi^\iota$ then it follows

$$\begin{aligned}
&-(y_{\mu_\iota} - \chi^\iota)^2 - (y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&= -(y_{\mu_\iota} - y_{\tau^{-1}(m_\iota)})^2 - (\chi^\iota - x_{m_\iota})^2 + 2(y_{\mu_\iota} - x_{m_\iota})(\chi^\iota - y_{\tau^{-1}(m_\iota)}) \\
&\le -(y_{\mu_\iota} - y_{\tau^{-1}(m_\iota)})^2 - (\chi^\iota - x_{m_\iota})^2 \le -(\chi^\iota - x_{m_\iota})^2.
\end{aligned}$$

Where the last line is true because $y \in \overline{\mathbb{W}^n}$, so that $y_{\mu_\iota} \ge y_{\tau^{-1}(m_\iota)}$; thus, our assumptions imply the last term on the third line above is negative. If instead, $x_{m_\iota} \le y_{\tau^{-1}(m_\iota)} < \chi^\iota$, then $y \in \overline{\mathbb{W}^n}$ implies that $y_{\nu_\iota} \le y_{\tau^{-1}(m_\iota)}$; thus, $0 > y_{\tau^{-1}(m_\iota)} - \chi^\iota \ge y_{\nu_\iota} - \chi^\iota$. Hence

$$\begin{aligned}
&-(y_{\nu_\iota} - \chi^\iota)^2 - (y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&\le -(y_{\tau^{-1}(m_\iota)} - \chi^\iota)^2 - (y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&= -2(y_{\tau^{-1}(m_\iota)} - \frac{1}{2}(x_{m_\iota} + \chi^\iota))^2 - \frac{1}{2}(x_{m_\iota} - \chi^\iota)^2 \le -\frac{1}{2}(x_{m_\iota} - \chi^\iota)^2.
\end{aligned}$$

Hence, when $y_{\tau^{-1}(m_\iota)} \geq x_{m_\iota}$ we have the bound on (2.4.27)

$$-\frac{1}{8t}(x_{m_\iota} - \chi^\iota)^2. \tag{2.4.28}$$

If instead we have $m_\iota \neq \tau(\mu_\iota)$ and $y_{\tau^{-1}(m_\iota)} < x_{m_\iota}$, then we have $x, y \in \overline{\mathbb{W}^n}$; therefore, $y_{\nu_\iota} \leq y_{\tau^{-1}(m_\iota)} < x_{m_\iota} \leq \chi^\iota$. Thus, $-(y_{\nu_\iota} - \chi^\iota)^2 \leq -(x_{m_\iota} - \chi^\iota)^2$, so that expression (2.4.28) is a bound on (2.4.27) for any $x, y \in \overline{\mathbb{W}^n}$, as desired. Following the same steps for l_ι, we get the analogous bound

$$\begin{aligned}
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \\
&\qquad -\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(l_\iota),x,y}\neq\mathbb{R},\ l_\iota\neq\tau(\nu_\iota)\}}(y_{\tau^{-1}(l_\iota)} - x_{l_\iota})^2 \\
&\leq -\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right) \\
&\qquad -\frac{1}{2t}\mathbb{1}_{\{y_{\tau^{-1}(l_\iota)}\geq x_{l_\iota},\ l_\iota\neq\tau(\nu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&\leq -\frac{1}{8t}(x_{l_\iota} - \chi^\iota)^2.
\end{aligned} \tag{2.4.29}$$

Combining the bounds in (2.4.27), (2.4.28), and (2.4.29) we get the following bound when $\nu_\iota > \mu_\iota$,

$$\begin{aligned}
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \\
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(m_\iota),x,y}\neq\mathbb{R},\ m_\iota\neq\tau(\mu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(l_\iota),x,y}\neq\mathbb{R},\ l_\iota\neq\tau(\nu_\iota)\}}(y_{\tau^{-1}(l_\iota)} - x_{l_\iota})^2 \\
&\leq -\frac{1}{12t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2\right) - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right) \\
&\leq -\frac{1}{12(\overline{\pi_{b_\iota}} - \underline{\pi_{a_\iota}})t}\sum_{\alpha=\underline{\pi_{a_\iota}}}^{\overline{\pi_{b_\iota}}}(y_\alpha - \chi^\iota)^2 - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right).
\end{aligned} \tag{2.4.30}$$

Where for the last line, we have used that from by definition $\mu_\iota = \overline{\pi_{a_\iota}}$ and $\nu_\iota = \underline{\pi_{b_\iota}}$, and that under λ^π, we have that for any $\pi_j \in \pi$, if $\alpha, \beta \in \pi_j$ then $y_\alpha = y_\beta$ a.e. as well as having that $y \in \overline{\mathbb{W}^n}$, so that $(y_{\mu_\iota} - \chi^\iota) \geq (y_\alpha - \chi_\iota) \geq (y_{\nu_\iota} - \chi_\iota)$ for all $\underline{\pi_{a_\iota}} \leq \alpha \leq \overline{\pi_{b_\iota}}$; thus, either $-(y_\alpha - \chi^\iota)^2 \leq -(y_{\mu_\iota} - \chi^\iota)^2$ or $-(y_\alpha - \chi^\iota)^2 \leq -(y_{\nu_\iota} - \chi^\iota)^2$.

Before we use this to get the bound on (2.4.21), we need to deal with the second case: $\nu_\iota \leq \mu_\iota$.

In the second case, μ_ι and ν_ι are both in π_ι, and therefore, under λ^π we have $y_{\mu_\iota} = y_{\nu_\iota}$ almost everywhere. Further, since τ is increasing on every element of π, it follows that $m_\iota := \tau(\mu_\iota) = \sup\{\tau(\alpha) |\ \nu_\iota \leq \alpha \leq \mu_\iota\}$ and $l_\iota := \tau(\nu_\iota) = \inf\{\tau(\beta) |\ \nu_\iota \leq \beta \leq \mu_\iota\}$.

Following the same steps as before, if we assume both $y_{\nu_\iota} > x_{\tau(\nu_\iota)}$ and $y_{\mu_\iota} < x_{\tau(\mu_\iota)}$ then since $y_{\mu_\iota} = y_{\nu_\iota}$, it follows that $x_{\tau(\nu_\iota)} < x_{\tau(\mu_\iota)}$, which is a contradiction since $\tau(\nu_\iota) < \tau(\mu_\iota)$ and $x \in \overline{\mathbb{W}^n}$. Thus, at least one of $y_{\nu_\iota} \leq x_{\tau(\nu_\iota)}$ and $y_{\mu_\iota} \geq x_{\tau(\mu_\iota)}$ must hold for all $x, y \in \overline{\mathbb{W}^n}$. With similar ideas to those used above, we find

$$\begin{aligned}
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \\
\leq& -\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{m_\iota} - x_{l_\iota})^2\right) \\
\leq& -\frac{1}{12t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2\right) - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right) \qquad (2.4.31) \\
\leq& -\frac{1}{12(\mu_\iota - \nu_\iota)t}\sum_{\alpha=\nu_\iota}^{\mu_\iota}(y_\alpha - \chi^\iota)^2 - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right). \qquad (2.4.32)
\end{aligned}$$

Where the idea behind the bounds is similar, but this time we use $y_{\mu_\iota} = y_{\nu_\iota}$, and we used that $x_{l_\iota} \geq \chi^\iota \geq x_{m_\iota}$ for the second inequality. The constants appearing in the denominator have been chosen to be consistent with (2.4.30), and so are not optimal.

Applying the bounds (2.4.30) and (2.4.32) to (2.4.25) leads to the following inequality

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2 + ik_\tau\cdot(y - x_\tau)} T^{\tau,\pi}(k)dk\right| \leq& e^{-\frac{1}{12nt}|y-\chi|^2}\left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2 + (x_{l_\iota}-\chi^\iota)^2\right)}\right) \\
&\int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n \mathrm{Re}(k_{\tau(\alpha)})^2} \left|T^{\tau,\pi}(k)\right| dk,
\end{aligned} \qquad (2.4.33)$$

where we have used $\mu_\iota - \nu_\iota, \overline{\pi_{b_\iota}} - \underline{\pi_{a_\iota}} < n$ for all ι to get the form of the Gaussian bound given above. □

We complete the proof of Proposition 2.4.15 with the following lemma.

Lemma 2.4.17. *There is a constant $C > 0$, depending only on π and n, such that*

$$\int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n Re(k_{\tau(\alpha)})^2} \left|T^{\tau,\pi}(k)\right| dk \leq Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|}. \qquad (2.4.34)$$

Proof. Now we bound the k integral in the above expression, for which we need to collect some bounds on the factors appearing in the products (2.4.22). We need to make sure the bound covers the new contours; therefore, it is sufficient to bound for $k \in E^{\tau,\pi}$. This can be done for the factors in the first product by bounding for all

$h_a, h_b \geq 0$ and $k_a, k_b \in \mathbb{R}$

$$\begin{aligned}
&\left|\frac{i\theta((k_a+ih_a)-(k_b-ih_b))+((k_a+ih_a))((k_b-ih_b))}{i\theta((k_a+ih_a)-(k_b-ih_b))-((k_a+ih_a))((k_b-ih_b))}\right| \\
=&\left|\frac{i\theta(k_a-k_b)-\theta(h_b+h_a)+i(k_bh_a-k_ah_b)+k_ak_b+h_ah_b}{i\theta(k_a-k_b)-\theta(h_b+h_a)-i(k_bh_a-k_ah_b)-k_ak_b-h_ah_b}\right| \\
=&\left(\frac{\theta^2(k_a-k_b)^2+(k_bh_a-k_ah_b)^2-2\theta(k_b^2h_a+k_a^2h_b)+\theta^2(h_b+h_a)^2-2\theta h_ah_b(h_b+h_a)+(k_ak_b+h_ah_b)^2}{\theta^2(k_a-k_b)^2+(k_bh_a-k_ah_b)^2+2\theta(k_b^2h_a+k_a^2h_b)+\theta^2(h_b+h_a)^2+2\theta h_ah_b(h_b+h_a)+(k_ak_b+h_ah_b)^2}\right)^{\frac{1}{2}} \\
\leq&1, \quad \text{because } h_a, h_b \geq 0. \qquad (2.4.35)
\end{aligned}$$

Here, the k variables are the real part of the integration variables, and the h variables are the imaginary part. Hence, we have that for all $k \in E^{\tau,\pi}$

$$\begin{aligned}
&\int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n \mathrm{Re}(k_{\tau(\alpha)})^2} \left|T^{\tau,\pi}(k)\right| dk \\
\leq& \prod_{\pi_\iota \in \pi} \int_{\times_{\alpha\in\pi_\iota}\Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha\in\pi_\iota}\mathrm{Re}(k_{\tau(\alpha)})^2} \prod_{\substack{\alpha<\beta: \\ \alpha,\beta\in\pi_\iota}} \left|\frac{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right| dk. \qquad (2.4.36)
\end{aligned}$$

Similar to the previous argument it suffices to bound for $k_a, k_b \in \mathbb{R}$ and $h_a, h_b \geq 0$

$$\begin{aligned}
&\left|\frac{i\theta((k_a+ih_a)-(k_b-ih_b))}{i\theta((k_a+ih_a)-(k_b-ih_b))-((k_a+ih_a))((k_b-ih_b))}\right| \\
=&\left(\frac{\theta^2(k_a-k_b)^2+\theta^2(h_b+h_a)^2}{\theta^2(k_a-k_b)^2+(k_bh_a-k_ah_b)^2+2\theta(k_b^2h_a+k_a^2h_b)+\theta^2(h_b+h_a)^2+2\theta h_ah_b(h_b+h_a)+(k_ak_b+h_ah_b)^2}\right)^{\frac{1}{2}} \\
\leq&\begin{cases} 1, \\ \theta\left(\frac{|k_a-k_b|}{((k_bh_a-k_ah_b)^2+(k_ak_b+h_ah_b)^2)^{\frac{1}{2}}}\right)+\theta\left(\frac{|h_b+h_a|}{((k_bh_a-k_ah_b)^2+(k_ak_b+h_ah_b)^2)^{1/2}}\right) \end{cases} \\
\leq&\begin{cases} 1, \\ 2\theta\left(\frac{1}{|k_a|}+\frac{1}{|k_b|}\right). \end{cases} \qquad (2.4.37)
\end{aligned}$$

The last line follows by expanding the brackets in the denominator, removing some non-negative terms, and then applying the triangle inequality.

Returning to (2.4.36), we can divide each contour integral into two parts: one where $|\mathrm{Re}(k_\alpha)| < \varepsilon/\sqrt{t}$ and another where $|\mathrm{Re}(k_\alpha)| \geq \varepsilon/\sqrt{t}$; this gives the following

$$\begin{aligned}
&\prod_{\pi_\iota\in\pi}\int_{\times_{\alpha\in\pi_\iota}\Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha\in\pi_\iota}\mathrm{Re}(k_{\tau(\alpha)})^2} \\
&\quad\prod_{\alpha\in\pi_\iota}\left(\mathbb{1}_{\{|\mathrm{Re}(k_\alpha)|<\varepsilon/\sqrt{t}\}}+\mathbb{1}_{\{|\mathrm{Re}(k_\alpha)|\geq\varepsilon/\sqrt{t}\}}\right)\prod_{\substack{\alpha<\beta: \\ \alpha,\beta\in\pi_\iota}}\left|\frac{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{i\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right| dk.
\end{aligned}$$

We can simplify as follows: expanding the first product, in each term where an indicator for $|k_\alpha| < \varepsilon/\sqrt{t}$ appears, we bound all factors in the second product which depend on k_α by 1 using the first line bound in (2.4.37); it is then easy to see that the contribution from the k_α integral to that term is at most $2\varepsilon/\sqrt{t}$. We can then bound any remaining terms in the second product by the second line bound in (2.4.37) (remembering that the k in that estimate represents the real part of the complex integration variable), the resulting integral depends only on the number of k_α for which $|\mathrm{Re}(k_\alpha)| \geq \varepsilon/\sqrt{t}$. Relabelling the remaining variables, we see that the previous expression is equal to the one below.

$$\prod_{\pi_\iota \in \pi} \sum_{j=1}^{|\pi_\iota|} \binom{|\pi_\iota|}{j} 2^{|\pi_\iota|-j+\binom{j}{2}} \theta^{\binom{j}{2}} (\varepsilon/\sqrt{t})^{|\pi_\iota|-j} \int_{\substack{\mathbb{R}^j: \\ |k_\alpha| \geq \varepsilon/\sqrt{t},\ \forall \alpha}} e^{-\frac{1}{2}t|k|^2} \prod_{\alpha<\beta} \left(\frac{1}{|k_\alpha|} + \frac{1}{|k_\beta|} \right) dk.$$

Rescaling the k variables by $\frac{1}{\sqrt{t}}$, we see that this equals

$$\prod_{\pi_\iota \in \pi} \sum_{j=1}^{|\pi_\iota|} \binom{|\pi_\iota|}{j} 2^{|\pi_\iota|-j+\binom{j}{2}} \theta^{\binom{j}{2}} \varepsilon^{|\pi_\iota|-j} t^{\frac{1}{2}(\binom{j}{2}-j)} \int_{\substack{\mathbb{R}^j: \\ |k_\alpha| \geq \varepsilon,\ \forall \alpha}} e^{-\frac{1}{2}|k|^2} \prod_{\alpha<\beta} \left(\frac{1}{|k_\alpha|} + \frac{1}{|k_\beta|} \right) dk. \tag{2.4.38}$$

Since the product runs through all pairs of $\alpha, \beta \in \{1, ..., j\}$, upon expanding the brackets, every term will involve at least $j-1$ of the k_γ. Further, at most one has exponent -1, with the rest having exponent at most -2. It is clear from repeated integration by parts, that for each $y \neq 1$ there is some constant $C > 0$ such that

$$\int_{|x|\geq\varepsilon} \frac{1}{|x|^y} e^{-\frac{1}{2}|x|^2} dx \leq C\varepsilon^{1-y}, \qquad \text{when } \varepsilon \in (0,1).$$

For $y = 1$, we instead have that there is a constant $C > 0$ such that

$$\int_{|x|\geq\varepsilon} \frac{1}{|x|} e^{-\frac{1}{2}|x|^2} dx \leq C|\log(\varepsilon)|, \qquad \text{when } \varepsilon \in (0,1).$$

Hence, since the sum of all the powers of all the k_γ in each term of the expanded brackets is $\binom{j}{2}$, and because the product runs through all pairs of indices so that in each term in the expansion there can be at most one k_γ appearing with power 1, there is some constant $C > 0$ depending only on n and π such that for all $\varepsilon \in (0,1)$, (2.4.38) is bounded by

$$\leq C \prod_{\pi_\iota \in \pi} \sum_{j=1}^{|\pi_\iota|} \varepsilon^{|\pi_\iota|-j+j-1-\binom{j}{2}} |\log(\varepsilon)| t^{\frac{1}{2}(\binom{j}{2}-j)}.$$

Which, if we set $\varepsilon = \sqrt{t}$ is clearly bounded above by

$$Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|}.$$

Which is the desired upper bound. □

Proof of Proposition 2.4.15. This proposition follows by combining the above lemma and Lemma 2.4.16. □

We now apply this bound to complete the proof of the main proposition of the subsection.

Proof of Proposition 2.4.9. Proposition 2.4.15 implies that (2.4.21) is bounded by

$$Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} \int e^{\frac{1}{12nt}|y-\chi|^2} \prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}|f(y)-f(x)|\lambda^\pi(dy) \tag{2.4.39}$$

We can replace the function $f:\overline{\mathbb{W}^n}\to\mathbb{R}$ with its symmetric extension $\overline{f}:\mathbb{R}^n\to\mathbb{R}$, that is the function $\overline{f}:\mathbb{R}^n\to\mathbb{R}$ such that for any $\sigma\in S_n$, $x\in\mathbb{R}^n$ we have $\overline{f}(x_\sigma)=\overline{f}(x)$ and $\overline{f}|_{\overline{\mathbb{W}^n}}=f$. Then, after rescaling y by $\sqrt{t}$, (2.4.39) is bounded by

$$C|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} \int_{\overline{\mathbb{W}^{|\pi|}}} e^{-\frac{1}{12n}|y|^2}\left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right)|\overline{f}(\sqrt{t}\underline{y}+\chi)-\overline{f}(x_\tau)|dy.$$

In the above, $\chi\in\mathbb{R}^n$ is defined by $\chi_\alpha := \chi^\iota$ when $\alpha\in\pi_\iota$, $\underline{y}$ is defined by $\underline{y}_\alpha = y_\iota$ for all $\alpha\in\pi_\iota$, and we have used $\overline{f}(x)=\overline{f}(x_\tau)$. We have also rewritten the integral with respect to λ^π as an integral with respect to the Lebesgue measure. Since f is a Lipschitz function, it is straightforward to show that $\overline{f}$ is also Lipschitz; therefore, the above expression is bounded by

$$C|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} \int_{\overline{\mathbb{W}^{|\pi|}}} e^{-\frac{1}{12n}|y|^2}\left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right)\left(\sqrt{t}|\underline{y}|+|\chi-x_\tau|\right)dy.$$

The integrand is non negative and $|\underline{y}| \leq |\pi||y|$; therefore, this is bounded above (for a new constant C) by

$$
\begin{aligned}
&C|\log(t)|^{|\pi|} \sum_{\substack{\tau \in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} \\
&\int_{\mathbb{R}^{|\pi|}} e^{-\frac{1}{12n}|y|^2} \left(\prod_{\pi_\iota \in \pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)} \right) \left(\sqrt{t}|\underline{y}| + |\chi - x_\tau| \right) dy \\
\leq &C|\log(t)|^{|\pi|} \sum_{\substack{\tau \in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} \qquad (2.4.40) \\
&\int_{\mathbb{R}^{|\pi|}} (|y|+1) e^{-\frac{1}{12n}|y|^2} dy \left(\sqrt{t} + |\chi - x_\tau| e^{-\frac{1}{24nt}\sum_{\pi_\iota \in \pi}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)} \right). \qquad (2.4.41)
\end{aligned}
$$

Now we note that $|\chi - x_\tau| \leq \sum_{\alpha=1}^n |\chi_\alpha - x_{\tau(\alpha)}|$, but for all $\alpha \in [\mu_\iota, \nu_\iota]$ (or $[\nu_\iota, \mu_\iota]$) we have $x_{m_\iota} \leq x_{\tau(\alpha)}, \chi_\alpha \leq x_{l_\iota}$ (or $x_{l_\iota} \leq x_{\tau(\alpha)}, \chi_\alpha \leq x_{m_\iota}$). Hence, either $|\chi_\alpha - x_{\tau(\alpha)}| \leq |\chi_\alpha - x_{m_\iota}|$ or $|\chi_\alpha - x_{\tau(\alpha)}| \leq |\chi_\alpha - x_{l_\iota}|$. Note that that for any $c > 0$ and $x \in \mathbb{R}$ we have the inequality $|x|e^{-c|x|^2} \leq (2ec)^{-\frac{1}{2}}$. Hence, (2.4.41) is bounded by

$$
\begin{aligned}
&C\sqrt{t}|\log(t)|^{|\pi|} \sum_{\substack{\tau \in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} \int_{\mathbb{R}^{|\pi|}} (|y|+1) e^{-\frac{1}{12n}|y|^2} dy \\
\leq &C\sqrt{t}|\log(t)|^{|\pi|}.
\end{aligned}
$$

Where we have bounded the integral independently of $|\pi|$, and the constant C has changed between lines. Summing over $\pi \in \Pi_n$, and using that since Π_n is a finite set the constants C in the above expression have a finite maximum, we get, for a new constant $C > 0$ depending only on n,

$$
\begin{aligned}
\sup_{x \in \overline{\mathbb{W}^n}} \left| \int u_t(x,y) f(y) m_\theta^{(n)}(dy) - f(x) \right| \leq &C\sqrt{t} \sum_{\pi \in \Pi_n} |\log(t)|^{|\pi|} \\
\leq &C\sqrt{t}\log(t)^n \to 0, \quad \text{as } t \to 0.
\end{aligned}
$$

The last inequality is valid for $t < 1/e$. Hence, we have the desired uniform convergence, and Proposition 2.4.9 is proven. □

As a consequence of Proposition 2.4.3 and Proposition 2.4.9, we can apply Proposition 2.3.4 to our function $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ to prove $\int u_{t-s}(Y_s,y)f(y)m_\theta^{(n)}(dy)$ is a local martingale. Suppose that $f \in C_c^\infty(\overline{\mathbb{W}^n})$, i.e. f has an extension to an open set U containing $\overline{\mathbb{W}^n}$ that is smooth and compactly supported. Then since $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ converges uniformly to f as $t \to 0$, and f is bounded, there must be some $\varepsilon > 0$ such that $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ is bounded for $t \in [0,\varepsilon]$ and $x \in \overline{\mathbb{W}^n}$. We also have $|\int u_t(x,y)f(y)m_\theta^{(n)}(dy)| \leq \frac{1}{(2\pi t)^{n/2}} \int |f(y)|m_\theta^{(n)}(dy)$, which is bounded for $t \in [\varepsilon, \infty)$. Hence, $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ is bounded as a function of $(t,x) \in \mathbb{R}_{>0} \times \overline{\mathbb{W}^n}$. It follows that $\int u_{t-s}(Y_s,y)f(y)m_\theta^{(n)}(dy)$ is a true martingale; thus, $\mathbb{E}_x[f(Y_t)] = \int u_t(x,y)f(y)m_\theta^{(n)}(dy)$. In particular, if $f(x) \geq 0$ for all $x \in \overline{\mathbb{W}^n}$, then $\int u_t(x,y)f(y)m_\theta^{(n)}(dy) \geq 0$. Since this holds for every $f \in C_c^\infty(\overline{\mathbb{W}^n})$, we have that for each $t > 0, x \in \overline{\mathbb{W}^n}$ $u_t(x,y) \geq 0$ $m_\theta^{(n)}$ almost everywhere.

Returning to the case where f is merely bounded and Lipschitz, we can use the non-negativity of $u_t(x,y)$ and Lemma 2.4.10 to get the bound $|\int u_t(x,y)f(y)m_\theta^{(n)}(dy)| \leq \|f\|_\infty$. Hence, the local martingale $\int u_{t-s}(Y_t,y)f(y)m_\theta^{(n)}(dy)$ is in fact a true martingale for $s \in [0,t]$, and so $\mathbb{E}_x[f(Y_t)] = \int u_t(x,y)f(y)m_\theta^{(n)}(dy)$. Thus, the proof of Theorem 2.4.1 is completed. □

As a consequence we can also prove the following.

Theorem 2.4.18. *$m_\theta^{(n)}$ is a stationary measure for Y, and Y is reversible with respect to $m_\theta^{(n)}$.*

Proof. For f a bounded, integrable, Lipschitz continuous function, we have for all $t > 0$

$$\frac{d}{dt}\int \mathbb{E}_x[f(Y_t)]m_\theta^{(n)}(dx) = \frac{d}{dt}\int\int u_t(x,y)f(y)m_\theta^{(n)}(dy)m_\theta^{(n)}(dx) \tag{2.4.42}$$

$$= 0. \tag{2.4.43}$$

With the first equality a consequence of Theorem 2.4.1 and the second equality a consequence of Corollary 2.3.9 and Fubini's theorem. The necessary bounds to pass the derivatives through the integrals, and then apply Fubini follow in the same way as Lemma 2.4.2. The same bounds, together with the uniform convergence we just proved, gives

$$\lim_{t\to 0}\int \mathbb{E}_x[f(Y_t)]m_\theta^{(n)}(dx) = \int f(x)m_\theta^{(n)}(dx). \tag{2.4.44}$$

We can extend this to any $L^1(m_\theta^{(n)})$ function by a density argument, proving that $m_\theta^{(n)}$ is the stationary measure for Y.

If f and g are bounded, Lipschitz continuous, and integrable; Fubini's theorem gives

$$\begin{aligned}\int \mathbb{E}_x[f(Y_t)]g(x)m_\theta^{(n)}(dx) &= \int\int u_t(x,y)f(y)m_\theta^{(n)}(dy)g(x)m_\theta^{(n)}(dx)\\ &= \int\int u_t(x,y)g(x)m_\theta^{(n)}(dx)f(y)m_\theta^{(n)}(dy)\\ &= \int \mathbb{E}_y[g(Y_t)]f(y)m_\theta^{(n)}(dy).\end{aligned}$$

Where we have used the symmetry $u_t(x,y) = u_t(y,x)$ in the last line. Hence, Y is reversible with respect to $m_\theta^{(n)}$. □

Finally, we return to prove Lemma 2.4.2.

Proof of Lemma 2.4.2. The proof of this lemma is a simplified version of the methods we apply in Section 2.4.3; as such, we omit the main details to avoid repetition and instead sketch the proof. Following the arguments used to prove Proposition 2.4.15, with $\pi = \{\{1\},\{2\},...,\{n\}\}$, we can derive a Gaussian bound on the summand in (2.4.3). We can then adapt the arguments in Lemma 2.4.17 to bound the resulting contour integrals, which will have additional factors of k due to the derivatives. In fact, the proof can be significantly simplified in this case as we do not need to consider the $t \to 0$ limit, and therefore we do not need to ensure we get the optimal exponent for t. The above arguments give us a bound in the form of a finite sum of Gaussian kernels, multiplied by a negative power of t, from which the above bounds follow easily (note that for the bound on the x derivatives, we can apply the bound on the y derivatives, as $u_t(x,y) = u_t(y,x)$ which we prove later in Lemma 2.4.11). □

2.5 Stochastic Flows of Kernels

2.5.1 Random Walks in Random Environments

We will begin by recalling the definitions for the discrete counterparts of Howitt-Warren flows and sticky Brownian motions: Random walks in space-time i.i.d. random environments on $\mathbb{Z}$ and their n-point motions, discussed in Section 1.1. A random walk in a random environment on $\mathbb{Z}$ is simply a random walk on $\mathbb{Z}$ whose transition probabilities are themselves random variables. We define the *Random Environment* as a family of i.i.d $[0,1]$ valued random variables $\boldsymbol{\omega} = (\omega_{t,x})_{t,x\in\mathbb{Z}}$ with law and expectation $\mathbb{P}$ and $\mathbb{E}$, respectively. We then define a random walk running through realisation of the environment with transition probabilities:

$$P^{\boldsymbol{\omega}}(X(t+1)=x+1|\ X(t)=x)=\omega_{x,t};$$
$$P^{\boldsymbol{\omega}}(X(t+1)=x-1|\ X(t)=x)=1-\omega_{x,t}.$$

Where $P^{\boldsymbol{\omega}}$ denotes the law of the RWRE, and $E^{\boldsymbol{\omega}}$ its expectation, both of which depend on the environment. By considering the random transition probabilities $P^{\omega}(X_t = y|\ X_0 = x)$ we can also consider this model as a random flow of a fluid, where the quantities describe how a point mass at x is spread through the fluid at time t.

An important idea for studying such models are the *n-point motions*, we run n random walks independently through a sampling of the environment, and then average out the environment. The averaging over the law of the environment will break the particles' independence. That is, if $X(t) = (X^1(t), ..., X^n(t))$ is the n-point motion, then

$$\mathbb{P}(X(t+1)=y|X(t)=x)=\mathbb{E}\left[\prod_{i=1}^{n} P^{\boldsymbol{\omega}}(X^i(t+1)=y_i|X(t)=x_i)\right].$$

Alternatively, we can view the n-point motions as describing the behaviour of n particles thrown into the fluid. Notice now that since the environment is i.i.d, the coordinate processes of the n-point motion behave independently when they are apart. However, when they meet, they interact. In particular, it is a simple consequence of Jensen's inequality that they are more likely to move in the same direction when together than when apart:

$$\mathbb{E}[\omega^n]+\mathbb{E}[(1-\omega)^n]\geq\mathbb{E}[\omega]^n+\mathbb{E}[1-\omega]^n,$$

ω being a copy of an environment variable. A group of particles situated at the same site, x, at time t can break into at most two groups. The probability of a group of n particles breaking into two groups of size k and l, with the k moving to $x+1$ and the l to $x-1$, is

$$\mathbb{E}[\omega_{x,t}^k(1-\omega_{x,t})^l].$$

Hence, the distribution of ω can be viewed as controlling the rate at which groups of particles break up, and the size of the groups they tend to break into. At the extreme ends, if the environment variables are chosen to be $\{0,1\}$ valued Bernoulli random variables, then the n-point motions become coalescing simple random walks. On the other hand, it the environment variables are chosen to be deterministic with value $\frac{1}{2}$, then the n-point motions will simply be independent simple random walks. Thus, the strength of the effect of the environment on the interaction between the n-point motions is related to how probable it is that the environment variables take values near 0 or 1.

If we take the diffusive scaling limit of these n-point motions in an environment having a fixed distribution, then the contribution of the environment is overcome in the limit, and we simply end up with independent Brownian motions (assuming the environment variables are mean 1/2 so there is no drift).

It was shown by Howitt and Warren [HW09] that by changing the distribution of the ω as we take the diffusive scaling limit, we can obtain Brownian motions which still interact; specifically, they are sticky when they meet, see also Schertzer, Sun and Swart [SSS10]. To preserve the interaction into the diffusive scaling limit the strength of the interaction has to be increased; this means taking the laws of the environment random variables to be closer to that of a Bernoulli random variable. This requirement is made explicit in the second condition of Howitt and Warren's theorem, stated below.

Theorem 2.5.1. *Suppose $X(t)$ is the n-point motion of a RWRE, where the environment variables have law $\mu^{(\varepsilon)}$ satisfying the following:*

$$\frac{1}{\varepsilon}\int_0^1 (1-2q)\mu^{(\varepsilon)}(dq) \to \beta, \quad \textit{as } \varepsilon \to 0;$$
$$\frac{1}{\varepsilon} q(1-q)\mu^{\varepsilon}(dq) \implies \nu(dq), \quad \textit{as } \varepsilon \to 0.$$

Then the laws of the processes $(\varepsilon X(\varepsilon^2 t))_{t\geq 0}$ converge weakly to the law of a solution to the Howitt-Warren martingale problem with drift β and characteristic measure ν.

In the special case of $\nu(dx) = \theta/2dx$, where dx is the Lebesgue measure the above result shows the solution to the Howitt-Warren martingale problem is the scaling limit of the Beta random walk in a random environment. That is, choose $\mu^{(\varepsilon)}(dq) = \frac{\Gamma(2\theta\varepsilon)}{\Gamma(\theta\varepsilon)\Gamma(\theta\varepsilon)} q^{\theta\varepsilon-1}(1-q)^{\theta\varepsilon-1}dq$, then for any function $C_b([0,1])$ the Dominated Convergence Theorem implies

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^1 f(q)q(1-q)\mu^{(\varepsilon)}(dq) &= \frac{\Gamma(2\theta\varepsilon)}{\varepsilon\Gamma(\theta\varepsilon)\Gamma(\theta\varepsilon)}\int_0^1 f(q)q^{\theta\varepsilon}(1-q)^{\theta\varepsilon}dq \\ &\to \frac{\theta}{2}\int_0^1 f(q)dq,\end{aligned}$$

using $\Gamma(x) = \frac{\Gamma(x+1)}{x} \sim \frac{1}{x}$ as $x \to 0$. Hence $\frac{1}{\varepsilon}q(1-q)\mu^{(\varepsilon)} \Rightarrow \frac{\theta}{2}dx$; since we also have $\int_0^1 (1-2q)\mu^{(\varepsilon)}(dq) = 0$ for all $\varepsilon > 0$ the theorem implies the convergence of the n-point motions of the Beta random walk in a random environment to solutions of the Howitt-Warren martingale problem with characteristic measure $\frac{\theta}{2}\mathbb{1}_{[0,1]}dx$ and zero drift. This is the key motivator for looking for exact solutions in the sticky Brownian motion case and was used by Barraquand and Rychnovsky in [BR20] to find Fredholm determinant expressions in the sticky Brownian motions case by taking limits of those found for the Beta random walk in a random environment in [BC17].

2.5.2 The Howitt-Warren process

We now briefly introduce stochastic flows of kernels, these are essentially random transition probabilities $(K_{s,t}(x,dy))_{s\leq t}$, with the following additional assumptions: independent increments in the sense that for any $t_0 < ..., t_n$ the random kernels $K_{t_0,t_1}, ..., K_{t_{n-1},t_n}$ are independent; stationarity, that is the law of $K_{s,t}$ depends only on $t-s$. They can be thought of as the continuum version of the random environment that is i.i.d. in space and time we considered in the previous section.

The n-point motions of a stochastic flow of kernels are the family of Markov processes $(X_n)_{n=1}^{\infty}$ with X_n taking values in $\mathbb{R}^n$ with transition probabilities

$$\mathbb{P}(X_n(t) \in E|\ X_n(s) = x) = \mathbb{E}\Big[\int_E \prod_{i=1}^{n} K_{s,t}(x_i, dy_i)\Big], \quad \text{for } x \in \mathbb{R}^n,\ E \in \mathcal{B}(\mathbb{R}^n).$$

Notice that this is very similar to the definition of the n-point motions in the RWRE case, with K taking the place of the random transition probabilities.

Le Jan and Raimond [LJR04a] have shown that any consistent family of Feller processes are the n-point motions of some stochastic flow of kernels. A family of Feller processes $(X_n)_{n=1}^{\infty}$, $X_n : \mathbb{R}_{>0} \to \mathbb{R}^n$ is consistent, if for any $k \leq n$ and any choice of k coordinates from X_n: $(X_n^{i_1}, ..., X_n^{i_k})$ is equal in law to X_k. For a more complete introduction to stochastic flows of kernels we refer to [LJR04a]. When the family of n-point motions, $(X_n)_{n=1}^{\infty}$, are sticky Brownian motions characterised by a Howitt-Warren martingale problem the resulting flow of kernels is called a Howitt-Warren flow. These flows have been studied extensively by Schertzer, Sun, and Swart [SSS10].

Definition 2.5.2. *The stochastic flow of kernels whose n-point motions solve the Howitt-Warren martingale problem, as stated in Definition 2.2.2, with characteristic measure ν and drift β is called the Howitt-Warren flow with characteristic measure ν and drift β.*

Rather than look at the flow directly, we want to consider the *Howitt-Warren process* a measure valued process that describes how an initial mass is carried by the flow. In our case, we are interested in the case where all mass starts at the origin; thus, we consider the Howitt-Warren process with initial condition δ_0. That is, for the Howitt-Warren flow $(K_{s,t})_{s\leq t}$ with characteristic measure ν and drift β we define the Howitt-Warren process started from δ_0 with characteristic measure ν and drift β to be the measure valued process given by

$$\rho_t(A) := K_{0,t}(0, A), \quad \text{for every Borel set } A \subset \mathbb{R}. \tag{2.5.1}$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is a symmetric function, then $\mathbb{E}_x[f(X(t))] = \mathbb{E}_x[f(Y(t))]$ for all $x \in \overline{\mathbb{W}^n}$. Hence, we have the following corollary of our main result, Theorem 2.4.1, that allows us to study the Howitt-Warren process.

Corollary 2.5.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a symmetric function, and its restriction to $\overline{\mathbb{W}^n}$ is a bounded, Lipschitz continuous function, then for a Howitt-Warren flow $(K_{s,t})_{s\leq t}$ with characteristic measure $\frac{\theta}{2}dx$ and drift zero we have*

$$\mathbb{E}\left[\int f(y)\prod_{i=1}^{n} K_{s,t}(x_i, dy_i)\right] = \int u_{t-s}(x,y)f(y)m_\theta^{(n)}(dy) \quad \textit{for all } x \in \overline{\mathbb{W}^n}.$$

From which it clearly follows that for the Howitt-Warren process started from δ_0 with characteristic measure $\frac{\theta}{2}\mathbb{1}_{[0,1]}$ and drift 0, we have that

$$\mathbb{E}\left[\int f(y)\rho_t^{\otimes n}(dy)\right] = \int u_t(0,y)f(y)m_\theta^{(n)}(dy). \tag{2.5.2}$$

This allows us to study the process directly, via u, which we will pursue further in the next subsection.

2.5.3 Atoms of the Howitt-Warren process

Schertzer, Swart, and Sun proved [SSS10, Theorem 2.8] that any Howitt-Warren process is almost surely purely atomic for fixed times t. Thus, almost surely we can write the Howitt-Warren process at time t as a linear combination of delta measures $\rho_t(dy) = \sum_i w_i \delta_{y_i}(dy)$, where the w_i and y_i are both random. One can think of the Howitt-Warren process as the density of an infinite number of sticky Brownian motions evolving in time. Thus, the fact that the process is atomic shows that when the number of particles is very large, the sticky behaviour leads to the formation of large clusters of particles. This is very different from the behaviour of large numbers of independent Brownian motions.

We can think of the collection of pairs (y_i, w_i) as a point process on $\mathbb{R} \times \mathbb{R}_{>0}$. Note that the Howitt-Warren process conserves mass, so that for any $t > 0$ we will have $\sum_i w_i = 1$. However, due to another result of [SSS10], the total number of points will be infinite a.s. This point process has an associated intensity measure γ_t on $\mathbb{R} \times \mathbb{R}_{>0}$ defined by

$$\gamma_t(A_1 \times A_2) = \mathbb{E}\left[\sum_i \mathbb{1}_{y_i \in A_1,\ w_i \in A_2}\right].$$

We will use this intensity to study the behaviour of the weight of a single atom at a given point in space. See [Bor09] for an introduction to point processes. For any $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ that is bounded and Lipschitz continuous, we have the equalities

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}_{>0}} f(y)w^n \ \gamma_t(dy, dw) =&\mathbb{E}\left[\sum_i f(y_i)w_i^n\right] \\ =&\mathbb{E}\left[\int_{\mathbb{D}^n} f^{\otimes n}(y)\rho_t^{\otimes n}(dy)\right] \\ =&\int_{\mathbb{D}^n} f^{\otimes n}(y)u_t^{(n)}(0, y)m_\theta^{(n)}(dy) \\ =&n^{-1}\theta^{1-n}\int_{\mathbb{R}} f(y)u_t^{(n)}(0, (y, ..., y))dy. \qquad (2.5.3)\end{aligned}$$

Above, $\mathbb{D}^n := \{(y, ..., y) \in \mathbb{R}^n :\ y \in \mathbb{R}\}$, and we have written $u_t^{(n)}$ for the transition density u_t on $\mathbb{R}^n$, which we do for the rest of the section to indicate the dependency on dimension. The first equality can be seen by approximating by simple functions, the second is direct from the definitions, the third is a consequence of Corollary 2.5.3 and the fourth from Definition 1.3.2.

Equality (2.5.3) also shows that the measure $\gamma_t(dy, dw)$ can be written in the form $\gamma_t(y, dw)dy$, and that we have for each $n \in \mathbb{N}$ and almost every $y \in \mathbb{R}$ the equality

$$\int_{\mathbb{R}_{>0}} w^n\gamma_t(y, dw) = n^{-1}\theta^{1-n}u_t^{(n)}(0, (y, ..., y)). \qquad (2.5.4)$$

We will study the asymptotic behaviour of the measure $\gamma_t(y, dw)$ for certain choices of y. We can interpret $\gamma_t(y, dw)$ as describing the distribution of the size of an atom at y. However, $\gamma_t(y, dw)$ is not a probability distribution; the measure of any neighbourhood of $w = 0$ is infinite. Introducing size biasing, and instead considering the measure $w\gamma_t(y, dw)$, we do get a finite measure which describes the size of an atom picked at random from ρ_t, using the sizes of the atoms as probabilities and conditioning the chosen atom to be at y.

Proposition 2.5.4. *For each $x \in \mathbb{R}$, we have as $t \to \infty$*

$$t^{-\frac{1}{2}}\sqrt{2\pi}e^{\frac{x^2}{2}}w\gamma_t\left(\sqrt{t}x, \frac{dw}{\sqrt{t}}\right) \Rightarrow \theta\sqrt{2\pi}e^{\frac{x^2}{2}}e^{-\theta\sqrt{2\pi}e^{\frac{x^2}{2}}w}dw.$$

Where the right hand side is the exponential distribution with rate $\theta\sqrt{2\pi}e^{\frac{x^2}{2}}$.

Proof. Note that the measure on the left hand side in the proposition has been normalised and is a probability measure. Thus, it is enough to show pointwise convergence of the moment generating functions on a neighbourhood of 0. With Theorem 2.4.1, we can rewrite the expression for the moments derived in line (2.5.4) as follows.

$$\int_{\mathbb{R}_{>0}} w^n\sqrt{2\pi}t^{-\frac{1}{2}}e^{\frac{x^2}{2}}w\gamma_t\left(\sqrt{t}x, \tfrac{dw}{\sqrt{t}}\right) = \sqrt{2\pi}e^{\frac{x^2}{2}}t^{\frac{n+1}{2}}\int_{\mathbb{R}_{>0}} w^{n+1}\gamma_t\left(\sqrt{t}x, dw\right)$$
$$=\sqrt{2\pi}e^{\frac{x^2}{2}}t^{\frac{n+1}{2}}\frac{u_t^{(n+1)}((0,\ldots,0),\sqrt{t}(x,\ldots,x))}{(n+1)\theta^n}$$
$$=\sqrt{2\pi}\frac{e^{\frac{x^2}{2}}t^{\frac{n+1}{2}}}{(n+1)\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}t|k|^2-i\sqrt{t}k\cdot\underline{x}}\sum_{\sigma\in S_{n+1}}\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\frac{i\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})+k_{\sigma(\beta)}k_{\sigma(\alpha)}}{i\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})-k_{\sigma(\beta)}k_{\sigma(\alpha)}}dk$$
$$=\sqrt{2\pi}\frac{n!e^{\frac{x^2}{2}}t^{\frac{n+1}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}t|k|^2-i\sqrt{t}k\cdot\underline{x}}\prod_{\alpha<\beta}\frac{i\theta(k_\beta-k_\alpha)}{i\theta(k_\beta-k_\alpha)-k_\alpha k_\beta}dk$$
$$=\sqrt{2\pi}\frac{n!e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-k\cdot\underline{x}}\prod_{\alpha<\beta}\frac{i\theta(k_\beta-k_\alpha)}{i\theta(k_\beta-k_\alpha)-t^{-\frac{1}{2}}k_\alpha k_\beta}dk.$$

To go from the first to the second line we have used line (2.5.4) and to go from the third to the fourth line we have used the summation formula from Lemma 2.4.12. We can now write the moment generating function in terms of the moments.

$$\sqrt{2\pi}t^{-\frac{1}{2}}e^{\frac{x^2}{2}}\int_{\mathbb{R}_{>0}} e^{\lambda w}w\gamma_t\left(\sqrt{t}x, \frac{dw}{\sqrt{t}}\right)$$
$$=\sum_{n=0}^{\infty}\sqrt{2\pi}\frac{\lambda^n e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-k\cdot\underline{x}}\prod_{\alpha<\beta}\frac{i\theta(k_\beta-k_\alpha)}{i\theta(k_\beta-k_\alpha)-t^{-\frac{1}{2}}k_\alpha k_\beta}dk.$$

To take $t \to \infty$, we want to apply the Dominated Convergence Theorem to pass the limit through both the sum and the integral. Similarly to what we have seen previously, line (2.4.37) to be precise, the modulus of the product within the integral is bounded above by 1. With this bound we find that the modulus of the n^{th} term of the series is bounded above for all $t > 0$ by $\frac{\lambda^n e^{x^2/2}}{\theta^n}$, which is summable for $|\lambda| < \theta$, and so we can take the limit $t \to \infty$ through the sum. Further the bound on the integral allows us to take the limit through the integral. Hence, for $|\lambda| < \theta$, we have

$$\lim_{t\to\infty}\sum_{n=0}^{\infty}\sqrt{2\pi}\frac{\lambda^n e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-ik\cdot\underline{x}}\prod_{\alpha<\beta}\frac{i\theta(k_\beta-k_\alpha)}{i\theta(k_\beta-k_\alpha)-t^{-\frac{1}{2}}k_\alpha k_\beta}dk$$
$$=\sum_{n=0}^{\infty}\sqrt{2\pi}\frac{\lambda^n e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-ik\cdot\underline{x}}dk=\sum_{n=0}^{\infty}\left(\frac{\lambda e^{-\frac{x^2}{2}}}{\theta\sqrt{2\pi}}\right)^n.$$

This is exactly the moment generating function of an exponential random variable with parameter $\theta\sqrt{2\pi}e^{x^2/2}$, and thus the statement is proved. □

We note that this result is analogous to Thiery and Le Doussal's result in [TLD16], where they found that the fluctuations of the transition probabilities of the Beta RWRE were Gamma distributed in the large t limit. In a remark, Sun, Swart and Schertzer showed that the stationary distribution of the Howitt-Warren process with a uniform interaction measure is given by a Poisson point process with intensity measure $dx\frac{1}{w}e^{-w}dw$ [SSS10]. This remark was based on a similar result by Le Jan and Raimond for sticky flows on the circle [LJR04b]. In the same work, the authors show that when the Howitt-Warren process is started from a distribution with infinite mass, it converges towards the stationary solution. The following corollary concerns the case when the starting mass is instead finite.

Corollary 2.5.5.

$$t^{-\frac{1}{2}}w\gamma_t\left(\sqrt{t}x,\frac{dw}{\sqrt{t}}\right)dx \Rightarrow \theta e^{-\theta\sqrt{2\pi}e^{\frac{x^2}{2}}w}dxdw.$$

Proof. This statement follows from the previous proposition by a simple application of the Dominated convergence theorem. □

We also have the following Fredholm determinant formula, which is analogous to formula (52) in [TLD16].

Proposition 2.5.6.

$$1+\sum_{n=1}^{\infty}\int_{\mathbb{R}_{>0}}\frac{(\lambda w)^n}{n!(n-1)!}\gamma_t(y,dw)=\theta\det\left(I+\frac{\lambda}{\theta 2\pi}K\right). \tag{2.5.5}$$

Above, the determinant is a Fredholm determinant, and K is an integral operator on $L^2(\mathbb{R})$ with kernel

$$K(x,y)=\frac{xye^{-\frac{1}{4}t(x^2+y^2)}}{i\theta(y-x)+xy}. \tag{2.5.6}$$

Proof. Equation (2.5.4) and the summation formula in Lemma 2.4.12 give the equality

$$\int_{\mathbb{R}>0} w^n \gamma_t(y, dw) = \frac{(n-1)!}{\theta^{n-1}(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2 - ik\cdot \underline{y}} \prod_{\alpha<\beta} \frac{i\theta(k_\beta - k_\alpha)}{i\theta(k_\beta - k_\alpha) - k_\alpha k_\beta} dk.$$

The proof is completed by the following identity, which is a consequence of the equalities (A.1) and (D.1) in [TLD16]

$$\sum_{\sigma\in S_n} \prod_{\alpha<\beta} \frac{i\theta(k_{\sigma(\beta)} - k_{\sigma(\alpha)})}{i\theta(k_{\sigma(\beta)} - k_{\sigma(\alpha)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)}} = n! \det_{1\le\alpha,\beta\le n} \left[\frac{k_\beta k_\alpha}{i\theta(k_\beta - k_\alpha) + k_\alpha k_\beta} \right]. \quad (2.5.7)$$

□

It would be interesting to use the above formula to analyse the behaviour of γ_t in the large deviation regime: $\frac{y}{t}$ converges to a non zero number as $t \to \infty$, where we expect the appearance of GUE Tracy-Widom fluctuations. Unfortunately, the above Fredholm determinant is not in an ideal form for asymptotic analysis. We would instead want an analogue of the conjectured formula (92) in [TLD16]. In [BR20], Barraquand and Rychnovsky considered the tails of the Howitt-Warren process, $\rho_t([tx, \infty])$, and derived a Fredholm determinant formula for the Laplace transform via a scaling limit from the Beta random walk in a random environment, with which they were able to prove the existence of GUE fluctuations. In a non-rigorous work Thiery and Le Doussal [TLD16] show the existence of GUE fluctuations for the transition probabilities of the Beta RWRE evaluated at a point. This suggests the following conjecture for the fluctuations of the individual atoms.

Conjecture 2.5.7. *If $X_{x,t}$ is a random variable on $\mathbb{R}$ with law $\sqrt{2\pi t}e^{-t\frac{x^2}{2}} w\gamma_t(tx, dw)$, then there are functions $J : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{\log(X_{x,t}) + J(x)t}{t^{1/3}\sigma(x)} < z \right) = F_{GUE}(z), \quad (2.5.8)$$

where F_{GUE} is the cumulative function for the Tracy-Widom GUE distribution.

CHAPTER 3

Brownian motions with White Noise Drifts

3.1 Introduction

We consider a model of turbulent advection describing particle trajectories in one-dimensional turbulent fluids. The particles each have their own independent molecular diffusivity σ^2 and are transported through a Gaussian random drift field W, which is Brownian in time and smooth in space but rapidly decorrelating. The field W represents the effect of the fluid on the particles. Associated with this model is a stochastic flow of kernels [LJR04a], which can be thought of as the density of a cloud of particles in the fluid, or as the random transition density of a single particle running through a realisation of the drift field W. Among other results in [LJR04a], the authors show a more general class of flows of kernels solve stochastic partial differential equations (SPDEs) similar to a Fokker-Planck equation for a Brownian motion with drift. In [DG21] the authors study the specific case we are interested in and show that the fluctuations of the density of the flow of kernels are well approximated, at large times, by the product of the heat kernel and the stationary solution to the SPDE. The two-dimensional version of this SPDE has been studied recently in [HK20], wherein the authors discuss the existence, uniqueness and regularity of solutions before studying the rate at which the solution dissipates.

As discussed in Section 1.4, the model is a continuum analogue of the random walks in random environments from Section 1.1. It is also an example of the compressible Kraichnan model for turbulence, where the velocity field is simplified to be white in time, see the review [FGmcV01]. In our case, we make some further simplifications; in the usual Kraichnan model, the spatial correlations are chosen to mimic physically observed scaling phenomena, whereas we assume the spatial correlations are taken to be of short length and smooth in space. This is similar to the case considered in [GH04], where the authors showed that removing the molecular diffusivity, at the same time as reducing the correlation length of the velocity field, led to sticky interactions between pairs of particles in the limiting process. For the model we consider, this result was extended by Warren [War15], who proved that the limiting process for n particles was the n point motion of a Howitt-Warren flow with an explicit interaction measure. This suggests the convergence of the associated flow of kernels towards the Howitt-Warren flow.

We are interested in the fluctuations of the density in the tail, that is, $O(t)$ away from the origin. In particular, we will show that the fluctuations are governed by the KPZ equation when the environment noise is taken to be small. This work was motivated by analogous results in [CG16] for a discrete version of our model, which we will discuss in more detail shortly, and by the non-rigorous arguments in [DT17]. Further, we conjecture that the KPZ equation also appears when the independent diffusivity of the particles is taken to be small instead of the environment.

Similar results exist for sticky Brownian motions and the associated Howitt-Warren flows in the integrable case studied in Chapter 2. This is not the same case that arises in the limit of our model, but universality leads us to expect the results to hold more generally. Barraquand and Rychnovsky [BR20] showed that the tail of the Howitt-Warren flows has Tracy-Widom GUE fluctuations of size $t^{1/3}$. Further, they conjectured the tails of the Howitt-Warren flows converge, as the stickiness is removed and under suitable rescaling, to the stochastic heat equation, based on the convergence of the moments. Barraquand and Le Doussal [BD20] then showed that the same convergence of moments happens in a moderate deviation regime, $t^{\frac{3}{4}}$ away from the origin, for a fixed stickiness.

In this chapter, we show that the flow of kernels associated with our model converges to the solution to the stochastic heat equation when observed far from the origin, whilst the strength of the random environment and the correlation length are both taken to 0 at suitable speeds. This result is analogous to the result of Corwin and Gu [CG16] for the RWRE, and we follow similar ideas for the proof. We make use of two distinct descriptions for the flow of kernels: the first is the family of SDEs describing the motion of n particles in the fluid, the second is the SPDE which has an explicit Wiener chaos expansion, given in [LJR04a]. The first step in the proof is to show the termwise L^2 convergence of the chaos expansion towards the chaos expansion of the solution to the stochastic heat equation; the second step is to show the convergence of the second moments, for which we use the associated two-point motion.

Before moving on to the proofs, we introduce the model in detail. We will also derive the correct rescalings to get convergence to the stochastic heat equation by performing a non-rigorous moment calculation. From this calculation, we conjecture other scaling regimes under which we expect to also get convergence to the stochastic heat equation.

3.1.1 The Model

Let $\rho \in C_c^\infty(\mathbb{R})$ be a non-negative, symmetric function that is non-increasing on $\mathbb{R}_{>0}$ and satisfies $\int \rho(x)dx = 1$ ($C_c^\infty(\mathbb{R})$ denotes the space of smooth compactly supported functions on $\mathbb{R}$). Then, suppose that W_ρ is a cylindrical Brownian motion on $L^2(\mathbb{R};\mathbb{R})$ with covariance $\mathbb{E}[W_\rho(s,x)W_\rho(t,y)] = (s \wedge t)\tilde{\rho}(x-y)$, where $\tilde{\rho}$ is defined as the self convolution of ρ, i.e. $\tilde{\rho} := \rho \star \rho$. For a sequence of independent standard Brownian motions, $(B^k)_{k\in\mathbb{N}}$, and an orthonormal basis of $L^2(\mathbb{R})$, $(e_k)_{k\in\mathbb{N}}$, we can write $W_\rho(t,x) = \sum_{k=1}^\infty \rho * e_k(x)B^k(t)$. For parameters $\mu, \sigma > 0$ we are interested in the solutions to the SDE

$$dX(t) = \mu W_\rho(dt, X(t)) + \sigma dB(t). \tag{3.1.1}$$

In the above SDE, B is a Brownian motion on $\mathbb{R}$ independent of W_ρ and both stochastic integrals are understood in the Itô sense, see [Kun94b] for definitions. The solution is distributed as a Brownian motion with diffusivity $\mu^2 \int \rho(x)^2dx + \sigma^2$, which can be checked directly by calculating the quadratic variation and recalling Lévy's characterisation of Brownian motion. Alternatively, the solution can be thought of as a Brownian motion with diffusivity σ^2, running through the time dependent Gaussian random field W_ρ, which we can think of as a random velocity field. We then consider the transition function $(U_{s,t})_{s\leq t}$ of the process $(X(t))_{s\leq t}$ conditional on W:

$$U_{s,t}(x, A) := \mathbb{P}^B(X(t) \in A|\ X(s) = x), \tag{3.1.2}$$

where $\mathbb{P}_B$ is the law of the Brownian motion B. We will later use $\mathbb{P}$ to denote the joint law of W and B and $\mathbb{E}$ its expectation. Note that the family of kernels $(U_{s,t})_{s<t}$ depend only on the field W, and form a stochastic flow of kernels as introduced by Le Jan and Raimond in [LJR04a], which we briefly discussed in Section 2.5 in relation to sticky Brownian motions. Further, because X is itself a Brownian motion, if we

average $U_{s,t}$ over the law of W we get the heat kernel.

$$\mathbb{E}[U_{s,t}(x,A)] = P^{\nu}_{s,t}(x,A), \tag{3.1.3}$$

here P^{ν} denotes the heat kernel with diffusivity ν.

It is known [DG21] that the family of probability kernels $(U(s,t,\cdot,dx))_{s<t}$ are a stochastic flow of kernels, as described in Section 2.5 of the previous chapter. Furthermore, the kernels have continuous densities with respect to the Lebesgue measure, $u(s,t,x,\cdot)$, which solve the stochastic partial differential equations

$$\partial_t u(s,t,x,y) = \frac{\nu}{2}\Delta_y u(s,t,x,y) - \mu\partial_y \left(u(s,t,x,y)\dot{W}_\rho(t,y)\right), \tag{3.1.4}$$

together with the initial condition $u(s,s,x,y) = \delta(x-y)$, where δ is the Dirac delta. Here, $\nu = \mu^2 \int \rho(x)^2 dx + \sigma^2$ and $\dot{W}_\rho$ is the (formal) time derivative of W_ρ, so that it is white in time and smoothly correlated in space. By solution, we mean it is a generalised solution in the sense of [Kun94a], which we describe by recalling [DG21, Proposition 2.1] briefly below.

Proposition 3.1.1. *The process $u(0,t,x,\cdot)$, considered as a time-indexed family of tempered distributions on $\mathbb{R}$, is the unique solution to (3.1.4). That is, for every $s,x \in \mathbb{R}$ and every Schwartz function $f : \mathbb{R} \to \mathbb{R}$ the following equality holds almost surely for every $t > s$*

$$\begin{aligned}&\int_{\mathbb{R}} u(s,t,x,y)f(y)dy\\ =&f(x) + \frac{\nu}{2}\int_s^t \int_{\mathbb{R}} u(s,r,x,y)f''(y)dydr + \mu \int_s^t \int_{\mathbb{R}} u(s,r,x,y)f'(y)W_\rho(dr,y)dy.\end{aligned} \tag{3.1.5}$$

Here, the stochastic integral is interpreted in the Itô sense; for a general introduction to SPDEs see [Wal86] or [DPZ92].

Proof. The proof is essentially an application of [Kun94a, Theorem 3.1] and can be found in [DG21]. □

The same SPDE, in a slightly different formulation, was derived for the flow of kernels (3.1.2) in [LJR04a, Section 5]. The solution to the SPDE can be constructed directly in terms of a Wiener chaos expansion, [LJR02, Theorem 3.2]. Note that this is the same as the Fokker-Planck equation for a Brownian motion with diffusivity σ^2 moving through a velocity field $\mu\dot{W}_\rho$, where the coefficient of the Laplacian part is ν instead of σ^2 because of the Itô correction from the stochastic integral. A version of the corresponding Backwards equation was studied in [ANV00], with an added non-linearity, where existence of solutions was shown. Notice that ν is exactly the diffusivity of the Brownian motion X solving the SDE (3.1.1).

3.1.2 The Stochastic Heat Equation as a Limit

We are interested in the fluctuations of u far from the origin; hence, we look at the rescaled quantities

$$v(t,x,y) := e^{\frac{\nu}{2}\lambda^2 t+\lambda(y-x)}u(0,t,x,y+\lambda\nu t) \tag{3.1.6}$$

Where $\lambda \in \mathbb{R}$ is a parameter, and the prefactor is motivated by fixing the average of v. From equation (3.1.3), we see

$$\begin{aligned}\mathbb{E}\left[v(t,x,y)\right] =& e^{\frac{\nu}{2}\lambda^2 t+\lambda(y-x)}\mathbb{E}\left[u(0,t,x,y+\lambda\nu t)\right] && (3.1.7)\\ =& e^{\frac{\nu}{2}\lambda^2 t+\lambda(y-x)}\frac{1}{\sqrt{2\pi\nu t}}e^{-\frac{(x-y-\lambda\nu t)^2}{2\nu t}} && (3.1.8)\\ =& \frac{1}{\sqrt{2\pi\nu t}}e^{-\frac{(x-y)^2}{2\nu t}}. && (3.1.9)\end{aligned}$$

Note, this is independent of λ. It is easily shown that v satisfies its own SPDE. Let $f \in C_c^\infty(\mathbb{R})$ and define $g(t,y) := e^{-\frac{\nu}{2}\lambda^2 t+\lambda(y-x)}f(y-\lambda\nu t)$. By definition, we have $\int_{\mathbb{R}} v(t,x,y)f(y)dy = \int_{\mathbb{R}} u(0,t,x,y)g(t,y)dy$. Thus, a straightforward calculation using equation (3.1.5) leads to the equality

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\partial_s g(s,y)u(0,s,x,y)dyds + f(x)\\ =&\int_{\mathbb{R}} v(t,y)f(y)dy - \frac{\nu}{2}\int_0^t\int_{\mathbb{R}}\partial_y^2 g(s,y)u(0,s,x,y)dyds\\ &-\mu\int_0^t\int_{\mathbb{R}} e^{-\frac{\nu}{2}\lambda^2 t+\lambda(y-x)}(\lambda f(y-\lambda\nu t)+f(y-\lambda\nu t))u(0,s,x,y)W_\rho(ds,y)dy.\end{aligned}$$

Noticing that $\partial_t g(t,y) = -\frac{\nu}{2}\partial_y^2 g(t,y) + \frac{\nu}{2}e^{-\frac{\nu}{2}\lambda^2 t+\lambda(y-x)}f''(y-\lambda\nu t)$, we get

$$\begin{aligned}&\int_{\mathbb{R}} v(t,y)f(y)dy - f(x) && (3.1.10)\\ =&\frac{\nu}{2}\int_0^t\int_{\mathbb{R}} v(s,y)f''(y)dyds + \mu\int_0^t\int_{\mathbb{R}} v(s,y)\left(\lambda f(y)+f'(y)\right)W_\rho(ds,y+\lambda\nu t)dy. && (3.1.11)\end{aligned}$$

Finally, we note that the field W_ρ is translation invariant in space, it follows that v is equal in distribution to the solution to the SPDE below.

$$\partial_t v = \frac{\nu}{2}\Delta_y v + \lambda\mu v\dot{W}_\rho - \mu\partial_y\left(v\dot{W}_\rho\right),\ v(0,\cdot) = \delta_x. \tag{3.1.12}$$

We aim to vary μ, λ, σ and ρ in such a way that v converges to the stochastic heat equation in the limit. The SPDE (3.1.12) suggests we should have $\lambda = \frac{1}{\mu}$ and take μ to 0 in the limit so that the middle term in the SPDE remains fixed, whilst the final term hopefully vanishes. We also need $\dot{W}_\rho$ to converge to a space-time white noise. Thus, take ρ to be a mollifier and replace it with $\rho_n(x) := n\rho(nx)$, so that the desired limit is achieved as $n \to \infty$. The quantity $\frac{1}{n}$ determines the correlation length of the random environment W_{ρ_n}; the larger this correlation length is the farther the distance within which particles begin to interact; thus, a larger correlation length increases the effect of the environment. For simplicity, we let the remaining parameters depend on n, with $\lambda(n) = \frac{1}{\mu(n)}$ and $\mu(n) \to 0$ as $n \to \infty$. The final constraint is on the diffusivity, given below; we require it to converge to a positive constant in the limit.

$$\nu(n) = \mu(n)^2 \int \rho_n(x)^2 dx + \sigma(n)^2 \tag{3.1.13}$$

$$= n\mu(n)^2 \int \rho(x)^2 dx + \sigma(n)^2. \tag{3.1.14}$$

The requirement that the diffusivity, (3.1.13), converges suggests two other cases of interest: In the first we choose $\sigma(n)$ such that $\sigma(n) \to 0$ as $n \to \infty$, and take $\mu(n) = n^{-\frac{1}{2}}$; For the second we take $\mu(n) = n^{-\frac{1}{2}}$ and keep $\sigma(n) > 0$ constant to keep the diffusivity $\nu(n)$ fixed. Interestingly, for these two cases, looking at the moments suggests different scaling regimes to looking at the SPDE. In the next section we study the moments of v under various parameter choices and conjecture three regimes where the stochastic heat equation should appear.

3.1.3 Convergence of Moments

We can use the N point motions of u to find the moments of v when tested against a function. The N point motions of u are the Markov processes in $\mathbb{R}^N$ with transition function defined for $x \in \mathbb{R}^N$ and $A \subset \mathbb{R}^N$ by $P^N_{t-s}(x, A) = \mathbb{E}[\int_A \prod_{i=1}^N u(s, t, x_i, y_i)dy]$. They can also be described as the coordinates of the solution to a system of SDEs. Let $B = (B^1, ..., B^N)$ be a standard Brownian motion in $\mathbb{R}^N$, and $W_n := W_{\rho_n}$ be an independent Brownian motion on $L^2(\mathbb{R}; \mathbb{R})$, as described in Section 3.1.1, and suppose both are defined on a filtered probability space $(\Omega, \mathcal{A}, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the filtration generated by (B, W_n). We consider the following system of SDEs.

$$dX^i(t) = \mu W_n(dt, X^i(t)) + \sigma dB^i(t). \tag{3.1.15}$$

As a consequence of [Kun94b, Theorem 3.4.1], the system of SDEs has a unique solution adapted to the filtration generated by (W_n, B). Indeed, each X^i depends only on (W, B^i), so that the X^i are conditionally independent given W. We also have that the solution is a continuous martingale under $\mathbb{P}$, and calculating the quadratic variation of X^i, we find $\langle X^i\rangle(t) = \nu t$. It follows from Levy's characterisation of Brownian motion that each X^i is a Brownian motion. The fact that the process X is the N point motion for the flow of kernels u follows directly from the definitions and that the X^i are conditionally independent with respect to W_n.

In the following, the probability measure $\mathbb{P}_x$ with expectation $\mathbb{E}_x$ denotes the law of $X = (X^1, X^2)$, which is the solution to (3.1.15) with initial condition (x, x).

We can also check under which scalings the variance converges to that of the stochastic heat equation. For arbitrary $\lambda, \mu, \sigma, \rho$, it follows directly from the definitions of v, (3.1.6), and u, (3.1.2), that we have the following equality.

$$\begin{aligned}
&\mathbb{E}\left[\left(\int v(t,x,y) f(y) dy\right)^2\right] \\
=&\mathbb{E}\left[\left(\int_{\mathbb{R}} e^{\frac{\nu}{2}\lambda^2 t + \lambda(y-x)} u(0,t,x,y+\lambda\nu t) f(y) dy\right)^2\right] \\
=&\mathbb{E}_x\left[e^{-\nu\lambda^2 t + \lambda(X^1(t) - x + X^2(t) - x)} f(X^1(t) - \lambda\nu t) f(X^2(t) - \lambda\nu t)\right].
\end{aligned} \tag{3.1.16}$$

By Girsanov's theorem, see for example [RY13], under $\tilde{\mathbb{P}}_x := \mathcal{E}(X^1 + X^2)(t) \cdot \mathbb{P}_x$, where $\mathcal{E}(X^1 + X^2) := e^{\lambda(X^1(\cdot) - x + X^2(\cdot) - x) - \frac{1}{2}\lambda^2\langle X^1 + X^2\rangle(\cdot)}$ is the exponential martingale, the processes $(X^i(t) - \langle X^i, \lambda(X^1 + X^2)\rangle(t))_{t\geq 0}$ are local martingales for $i = 1, 2$. We denote the expectation under $\tilde{\mathbb{P}}_x$ by $\tilde{\mathbb{E}}_x$. We can easily find the quadratic variations from (3.1.15): $\langle X^1\rangle(t) = \langle X^2\rangle(t) = \nu t$ so that $(X^i(t) - \langle X^i, \lambda(X^1 + X^2)\rangle(t))_{t\geq 0}$ are Brownian motions with diffusivity ν under $\tilde{\mathbb{P}}_x$. It follows that (3.1.16) can be rewritten as

$$\tilde{\mathbb{E}}_x\left[e^{\lambda^2\langle Y^1, Y^2\rangle(t)} f(Y^1(t)) f(Y^2(t))\right], \tag{3.1.17}$$

where $Y^i(t) := X^i(t) - \lambda\nu t$. We have the following description of the process Y.

Proposition 3.1.2. *Under $\tilde{\mathbb{P}}_x$, the process $(Y)_{t>0}$ is a diffusion with generator $\mathcal{G}$*

which acts on $f \in C_c^\infty(\mathbb{R}^2)$ as follows

$$\mathcal{G}f(y) = \frac{1}{2}\sum_{i,j=1}^{2}(\sigma^2\delta_{i,j} + \mu^2\tilde{\rho}(y_i - y_j))\frac{\partial^2 f}{\partial y_i \partial y_j}(y) + \sum_{i=1}^{2}\lambda\mu^2\tilde{\rho}(y_1 - y_2)\frac{\partial f}{\partial y_i}(y). \tag{3.1.18}$$

$\delta_{i,j}$ denotes the Kronecker delta.

Proof. From the discussion above, we know that the processes $(M^i(t))_{t\geq 0} := (Y^i(t) - \lambda\mu^2\int_0^t \tilde{\rho}(Y_s^1 - Y_s^2)ds)_{t\geq 0}$ are continuous local martingales under $\tilde{\mathbb{P}}_x$. Further, we know $\langle Y^i\rangle(t) = \nu t$, and thus, the M^i are Brownian motions by Lévy's characterisation. For any $f \in C_c^\infty(\mathbb{R}^2)$, we have from the above discussion and Itô's formula that, under $\tilde{\mathbb{P}}_x$,

$$\begin{aligned} f(Y(t)) =& f((x,x)) + \int_0^t \nabla f(Y(s)) \cdot dY(s) \\ &+ \frac{\nu}{2}\int_0^t \Delta f(Y(s))ds + \mu^2\int_0^t \frac{\partial^2 f}{\partial y_1 \partial y_2}(Y(s))\tilde{\rho}(Y^1(s) - Y^2(s))ds \\ =& f((x,x)) + \int_0^t \nabla f(Y(s)) \cdot dM(s) + \lambda\mu^2\sum_{i=1}^{2}\int_0^t \frac{\partial f}{\partial y_i}(Y(s))\tilde{\rho}(Y^1(s) - Y^2(s))ds \\ &+ \frac{\nu}{2}\int_0^t \Delta f(Y(s))ds + \mu^2\int_0^t \frac{\partial^2 f}{\partial y_1 \partial y_2}(Y(s))\tilde{\rho}(Y^1(s) - Y^2(s))ds. \end{aligned}$$

Notice that the right hand side is a constant plus a stochastic integral plus $\int_0^t \mathcal{G}f(Y(s))ds$. Since we know that the M^i are Brownian motions and $f \in C_c^\infty(\mathbb{R}^2)$, the expectation of the stochastic integral is just 0. Therefore, we have the equality

$$\tilde{\mathbb{E}}_x\left[f(Y(t))\right] = f((x,x)) + \tilde{\mathbb{E}}_x\left[\int_0^t \mathcal{G}f(Y(s))ds\right],$$

which proves the statement. □

We can compare (3.1.17) to the corresponding second moment for the stochastic heat equation with diffusivity η and initial condition δ_x

$$\partial_t z = \frac{\eta}{2}\Delta_y z + \kappa z\dot{W}. \tag{3.1.19}$$

From [BC95], we have that the variance can be written in terms of a pair of independent Brownian motions with diffusivity η.

$$\mathbb{E}\left[\left(\int z(t,y)f(y)dy\right)^2\right] = \mathbb{E}_x\left[e^{\frac{\kappa^2}{2\eta}\mathcal{L}_t^0(B^1-B^2)}f(B_t^1)f(B_t^2)\right]. \tag{3.1.20}$$

In the above equation, B^1, B^2 are a pair of independent Brownian motions in $\mathbb{R}$, each with diffusivity η, both starting from x under $\mathbb{E}_x$. $\mathcal{L}^0_\cdot(B^1 - B^2)$ is the local time of $B^1 - B^2$ at 0, see [RY13] for details. From Proposition 3.1.2, we have

$$\begin{aligned}\langle Y^1 - Y^2\rangle(t) &= 2\nu t - 2\mu^2 \int_0^t \tilde{\rho}(Y^1(s) - Y^2(s))ds \\ &= 2\int_0^t \sigma^2 + \mu^2(\tilde{\rho}(0) - \tilde{\rho}(Y^1(s) - Y^2(s)))ds.\end{aligned}$$

Hence, the occupation times formula gives

$$\begin{aligned}\langle Y^1, Y^2\rangle(t) =& \frac{\mu^2}{2}\int_0^t \frac{\tilde{\rho}(Y^1(s) - Y^2(s))}{\sigma^2 + \mu^2(\tilde{\rho}(0) - \tilde{\rho}(Y^1(s) - Y^2(s)))} d\langle Y^1 - Y^2\rangle(s) \\ =& \frac{\mu^2}{2}\int \frac{\tilde{\rho}(z)}{\sigma^2 + \mu^2(\tilde{\rho}(0) - \tilde{\rho}(z))}\mathcal{L}^z_t(Y^1 - Y^2)dz. \qquad (3.1.21)\end{aligned}$$

The above equality provides a link between equations (3.1.17) and (3.1.20), which we can exploit to show the convergence of (3.1.17) towards (3.1.20).

Once more replace ρ with ρ_n, and let μ, σ, and λ depend on n. We shall proceed formally, and leave the details for later. If we can show $\lambda(n)^2\langle Y^1, Y^2\rangle(t)$ is converging to a multiple of the local time at 0, whilst $\lambda(n) \to \infty$ as $n \to \infty$, then the bracket $\langle Y^1, Y^2\rangle(t)$ must be vanishing in the limit, suggesting (Y^1, Y^2) is converging to a pair of independent Brownian motions with diffusivity $\lim_{n\to\infty} \nu(n)$, which we need to be positive. Consequently, the limit of the expectation (3.1.17) is of the same form as the variance of the stochastic heat equation (3.1.20).

From (3.1.21) we see

$$\lambda(n)^2\langle Y^1, Y^2\rangle(t) = \frac{\lambda(n)^2\mu(n)^2}{2}\int \frac{\tilde{\rho_n}(z)}{\sigma(n)^2 + \mu^2(\tilde{\rho_n}(0) - \tilde{\rho_n}(z))}\mathcal{L}^z_t(Y^1 - Y^2)dz \qquad (3.1.22)$$

$$= \frac{\lambda(n)^2\mu(n)^2}{2}\int \frac{\tilde{\rho}(z)}{\sigma(n)^2 + n\mu(n)^2(\tilde{\rho}(0) - \tilde{\rho}(z))}\mathcal{L}^{z/n}_t(Y^1 - Y^2)dz. \qquad (3.1.23)$$

This equality suggests several choices of scalings depending on the behaviour of the denominator in the above integral. The first is to keep both terms in the denominator fixed, i.e. choose a constant $\sigma(n) = \sigma$ constant and set $\mu(n) = n^{-\frac{1}{2}}$. We then need to fix the prefactor, so we substitute $\lambda(n) = \mu(n)^{-1} = \sqrt{n}$. In this case, we have $\nu(n) = \mu(n)^2 \int \rho_n(x)^2 dx + \sigma(n)^2$, $\mu(n) = n^{-\frac{1}{2}}$ and $\sigma(n) = \sigma$, so that $\nu(n) = \int \rho(x)^2 dx + \sigma^2 = \nu$ is constant and positive. If we substitute (3.1.23) into (3.1.17) with these choices and take $n \to \infty$, then we expect that (3.1.17) converges to

$$\mathbb{E}_x\left[e^{\frac{1}{2}C_\rho \mathcal{L}_t^0(Y^1-Y^2)}f(Y^1(t))f(Y^2(t))\right], \tag{3.1.24}$$

where Y^1, Y^2 are independent Brownian motions with diffusivity ν under P_x and $C_\rho = \int \frac{\tilde{\rho}(z)}{\sigma^2+\tilde{\rho}(0)-\tilde{\rho}(z)}dz$. We will call this the fixed diffusivity and environment regime. Since we assumed that ρ is non negative, this is also true of $\tilde{\rho}$ and thus

$$C_\rho = \int \frac{\tilde{\rho}(z)}{\sigma^2+\tilde{\rho}(0)-\tilde{\rho}(z)}dz > \frac{1}{\sigma^2+\tilde{\rho}(0)}\int \tilde{\rho}(z)dz = \frac{1}{\sigma^2+\tilde{\rho}(0)} = \frac{1}{\nu}, \tag{3.1.25}$$

using that $\int \tilde{\rho}(z)dz = \int\int \rho(z-y)\rho(y)dydz = (\int \rho(y)dy)^2 = 1$, and that by symmetry of ρ we have $\tilde{\rho}(0) = \int \rho(-y)\rho(y)dy = \int \rho(y)^2dy$. This suggests that if the SHE is appearing in the limit then it appears with parameter $\kappa = (C_\rho\nu)^{\frac{1}{2}} > 1$, which differs from the parameter suggested by our consideration of the SPDE (3.1.12) at the end of Section 3.1.2, where the discussion lead us to anticipate $\kappa = 1$.

The second regime we can consider involves choosing a constant $\sigma(n) = \sigma$, but allowing $n\mu(n)^2$ to vanish as $n \to \infty$. Letting $n\mu(n)^2 = n^{2\alpha}$ for $\alpha < 0$, we again need to fix the prefactor, so we must choose $\lambda(n) = \mu(n)^{-1} = n^{\frac{1}{2}-\alpha}$. Since $n\mu(n)^2 \to 0$ as $n \to \infty$, we have $\nu(n) \to \sigma^2 > 0$ as $n \to \infty$. In this case, we get that the expectation should converge to

$$\mathbb{E}_x\left[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(B^1-B^2)}f(B^1(t))f(B^2(t))\right], \tag{3.1.26}$$

here, B^1 and B^2 are independent Brownian motions with diffusivity σ^2 under $\mathbb{P}_x$. This suggests v is converging towards the SHE, (3.1.19), with $\eta = \sigma^2$ and $\kappa = 1$. Notice that these parameters agree with our prediction from the SPDE itself, at the end of Section 3.1.2. We'll call this the weak environment regime.

The third is to take $\sigma(n) \to 0$ as we take $n \to \infty$. We will set $\sigma(n) = n^{-\alpha}$ for some $\alpha > 0$. In order to ensure $\nu(n)$, (3.1.13), converges to something positive, we must have $\mu(n) = n^{-\frac{1}{2}}$. Thus, (3.1.23) becomes

$$\frac{\lambda(n)^2}{2n\sigma(n)^2}\int \frac{\tilde{\rho}(z)}{1+\sigma(n)^{-2}(\tilde{\rho}(0)-\tilde{\rho}(z))}\mathcal{L}_t^{z/n}(Y^1-Y^2)dz \tag{3.1.27}$$

$$=\frac{\lambda(n)^2}{2n^{1-\alpha}}\int \frac{\tilde{\rho}(n^{-\alpha}z)}{1+n^{2\alpha}(\tilde{\rho}(0)-\tilde{\rho}(n^{-\alpha}z))}\mathcal{L}_t^{n^{-\alpha-1}z}(Y^1-Y^2)dz. \tag{3.1.28}$$

With $\sigma(n) \to 0$, the above integral is converging as $n \to \infty$, so that the above expression converges as $n \to \infty$ as long as $\frac{\lambda(n)^2}{n\sigma(n)}$ converges. This suggests we make the choice $\lambda(n) = n^{\frac{1-\alpha}{2}}$. Note that this choice disagrees with the $\lambda(n) = \mu(n)^{-1}$

requirement suggested by the SPDE! (3.1.28) then becomes

$$\frac{1}{2}\int \frac{\tilde{\rho}(n^{-\alpha}z)}{1+n^{2\alpha}(\tilde{\rho}(0)-\tilde{\rho}(n^{-\alpha}z))}\mathcal{L}_t^{n^{-(\alpha+1)}z}(Y^1-Y^2)dz. \tag{3.1.29}$$

Which we can expect to converge to $\frac{\pi\tilde{\rho}(0)}{2\sqrt{-\tilde{\rho}''(0)}}\mathcal{L}_t^0(B^1-B^2)$, where B^1 and B^2 are independent Brownian motions with diffusivity $\tilde{\rho}(0)=\int\rho(y)^2dy$, both starting at x. Therefore, in this regime we expect (3.1.17) to converge to

$$\mathbb{E}_x\left[e^{\frac{\pi\tilde{\rho}(0)}{2\sqrt{-\tilde{\rho}''(0)}}\mathcal{L}_t^0(B^1-B^2)}f(B^1(t))f(B^2(t)))\right]. \tag{3.1.30}$$

We'll call this the weak diffusivity regime. This limit suggests the limiting SHE has $\kappa=\frac{\sqrt{\pi}\tilde{\rho}(0)}{(-\tilde{\rho}''(0))^{\frac{1}{4}}}=\left(\frac{\pi\|\rho\|_2^2}{\|\rho'\|_2}\right)^{\frac{1}{2}}$. However, with this choice of scalings, both noise terms in the SPDE (3.1.12) have vanishing coefficients. Despite this, we conjecture the limit is not the deterministic heat equation as looking at the SPDE suggests.

It is possible to generalise the above argument to show convergence of all moments towards the corresponding moments of the stochastic heat equation. However, since the moments do not determine the distribution of the stochastic heat equation, this would be insufficient for a proof of the convergence of v towards the stochastic heat equation.

If we set $\frac{n\mu(n)^2}{\sigma(n)^2}=n^{2\alpha}$ and $\lambda(n)=n^{\beta}$, we get the phase diagram in Figure 3.1 describing the conjectured $n\to\infty$ limit of v under the different choices of parameters. For (α,β) below the SHE line, the limit is the heat equation. This can be shown via moment convergence using a straightforward extension of the arguments used above, which we cover more rigorously for the weak environment regime in Section 3.2.2. Above the line we conjecture the limit is 0 in probability; we expect that most of the mass is collecting into large spikes, which occur at a given point with low probability.

Below we summarise the distinct regimes in which we conjecture the appearance of the stochastic heat equation, based on our above moment calculations.

1. **Weak environment**, where we have $n\mu(n)^2$ vanishing in the limit and $\sigma(n)$ is constant, so $\alpha<0$ in figure 3.1. We also take $\lambda(n)=\mu(n)^{-1}$, so that $\beta=\frac{1}{2}-\alpha$. The coefficients for the SHE are $\eta=\sigma^2$ and $\kappa=1$.
2. **Weak diffusivity**, where we have $n\mu(n)^2$ constant and $\sigma(n)$ vanishing in the limit, so we have $\mu(n)=n^{-\frac{1}{2}}$ and $\alpha>0$ in figure 3.1. Here we require $\frac{\lambda(n)^2}{n\sigma(n)}$ to converge, so we set $\lambda(n)=n^{\frac{1-\alpha}{2}}$. Hence, we have $\beta=\frac{1-\alpha}{2}$ in figure 3.1;

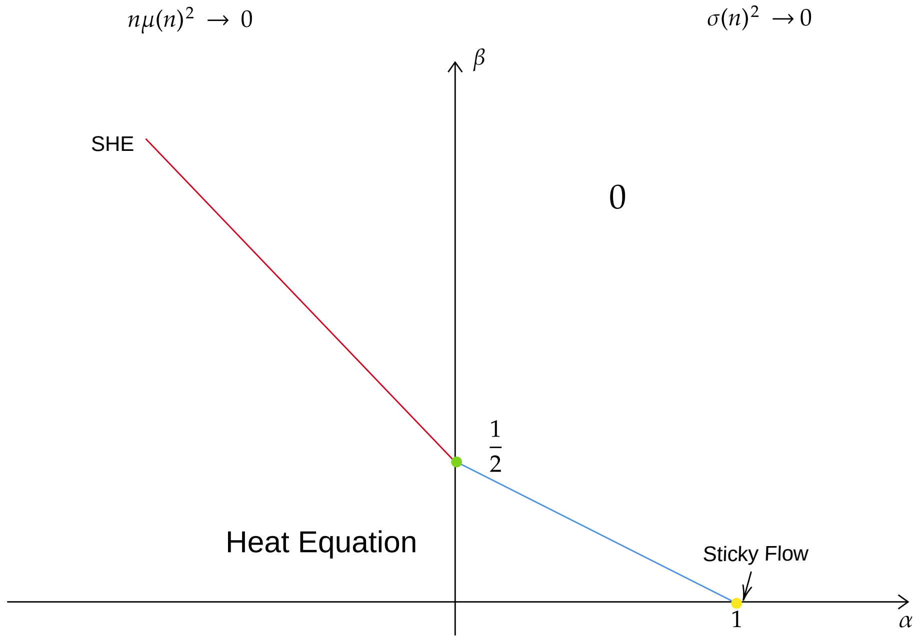


Figure 3.1: Above the line we expect the limit to be 0 in probability, below the line the limit is the heat equation. On the $\beta = 0$ axis the limit is a sticky flow for $\alpha \geq 1$, for $\alpha > 1$ it is the Arratia flow.

the coefficients for the SHE are $\eta = \tilde{\rho}(0)$ and $\kappa = \left(\frac{\pi\|\rho\|_2^2}{\|\rho'\|_2}\right)^{\frac{1}{2}}$. Since we need $\lambda(n) \to \infty$ as $n \to \infty$, we require $\alpha \in (0, 1)$. Note that for $\alpha = 1$ the limit is a sticky flow.

3. **Fixed diffusivity and environment**, where $n\mu(n)^2$ and $\sigma(n)$ are held constant, i.e. $\mu(n) = n^{-\frac{1}{2}}$ and $\sigma(n) = \sigma > 0$, we also take $\lambda(n) = \mu(n)^{-1} = n^{\frac{1}{2}}$. In the diagram, this is the green dot where the line hits the β axis and $\alpha = 0$. In this regime, the limiting SHE should have $\kappa = C_\rho > 1$ and $\eta = 1$.

Interestingly, it is only the weak environment regime which agrees with the choice of scalings suggested by looking at the SPDE, which we discussed at the end of Section 3.1.2. A possible reason for this comes from observing that the termwise limits of the chaos expansion of v in the other two regimes have second moments that are strictly less than the limit of the second moments of v. One interpretation of this is that the limit, if it exists, is no longer a functional of the noise W. In this chapter, we will consider the Weak environment regime, where we can make use of the chaos expansion to study the limit.

Remark 3.1.3. *For fixed parameters $(\rho, \sigma, \mu, \lambda)$, it is not difficult to show that the diffusively scaled tilted density $\varepsilon^{-1}v(\varepsilon^{-2}t, \varepsilon^{-1}y)$ is equal in distribution to the tilted* *density with parameters* $(\rho_{\varepsilon^{-1}}, \sigma, \varepsilon^{\frac{1}{2}}\mu, \varepsilon^{-1}\lambda)$. *Thus, if we set* $\mu = \varepsilon^{1/2}$ *and fix the parameters* ρ, σ *and* λ, *whilst diffusively scaling* v *as described, then we are in the weak environment regime corresponding to* $(\alpha, \beta) = (-\frac{1}{2}, 1)$ *in figure 3.1. This is the regime studied by Corwin and Gu, in the discrete setting of random walks in random environments, [CG16].*

3.2 Weak Environment Scaling

In this section, we'll consider the weak environment scaling, and therefore fix $\sigma > 0$. A special case of this regime was studied by Corwin and Gu for the discrete analogue of our model, a random walk in a dynamic random environment. For an i.i.d space-time random environment, the probability density function of the RWRE has been shown to have fluctuations that are governed by the stochastic heat equation in a [CG16] in the diffusive scaling limit, on trajectories that are at t distance from the origin. This was shown by proving the termwise convergence of discrete Wiener chaos expansions to their continuum counterpart for the stochastic heat equation.

Following the proof for the discrete case, we will show termwise convergence of the Wiener chaos expansion towards that of the stochastic heat equation. The method requires some restrictive assumptions on the choice of scaling parameters, which we do not believe to be necessary for the convergence to hold, namely the environment coefficient, $\mu(n)$, must be chosen to decay faster than $\frac{1}{n}$, whilst the moment calculations of the previous section suggest that the result should hold as long as $\mu(n)$ decays more quickly than $\frac{1}{\sqrt{n}}$. In the discrete setting, the result was proven for the case corresponding to $\mu(n) = \frac{1}{n}$, see remark 3.1.3.

Let W be a cylindrical Brownian motion on $L^2(\mathbb{R};\mathbb{R})$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so that we have the formal covariance $\mathbb{E}[W(t,x)W(s,y)] = (s \wedge t)\delta_0(x-y)$. Let $\rho \in C_c^\infty(\mathbb{R})$ be as described at the beginning of Section 3.1.1 and set $\rho_n(x) := n\rho(nx)$. It is easily shown that ρ_n satisfies the same assumptions as ρ. Define $W_n := W_{\rho_n} = W * \rho_n$, where the convolution is in the space variable. Note that for $f, g \in L^2([0,T] \times \mathbb{R})$ we have $\mathbb{E}[(f, \dot{W}_n)(g, \dot{W}_n)] = \int_0^T \int_{\mathbb{R}} f * \rho_n(s,y) g * \rho_n(s,y) dy ds$, where, once again, the convolutions are both in the space variable and $\dot{W}_n$ denotes the formal time derivative of W_n, so that the integral is defined via a stochastic integral. This is simply equal to $\int_0^T \int_{\mathbb{R}} f(s,y_1)g(s,y_2)\tilde{\rho}_n(y_1-y_2)dyds$ where $\tilde{\rho}_n(x) := \int \rho_n(x-y)\rho_n(y)dy$. For a constant $\beta > 1$, set $\mu = n^{-\beta}$. Let u_n be the density of the flow of kernels, as in Section 3.1.1. Denoting the formal time derivative of W_n

by $\dot{W}_n$, u_n is the solution to the SPDE

$$\partial_t u_n = \frac{\nu_n}{2}\Delta u_n - n^{-\beta}\partial_y\left(u_n\dot{W}_n\right). \tag{3.2.1}$$

In the above equation, $\nu_n = n^{-2\beta}\int \rho_n(y)^2 dy + \sigma^2 = n^{1-2\beta}\int \rho(y)^2 dy + \sigma^2$. Thus, $n^{1-2\beta}\int \rho(y)^2 dy$ governs the strength of the random environment in the above equation, so that in the limit as $n \to \infty$ the effect of the environment should disappear.

We define the tilted density as in equation (3.1.6), with our choice of parameters.

$$v_n(s,t,x,y) := e^{\frac{\nu_n}{2}n^{2\beta}(t-s)+n^{\beta}(y-x)}u_n(s,t,x,y+n^{\beta}\nu_n(t-s)). \tag{3.2.2}$$

We will most often suppress the s variable and use the notation $v_n(t,x,y) = v_n(0,t,x,y)$. In the following, we will assume we defined u_n with respect to the tilted field defined by $\tilde{W}_n(t,x) := W_n(t,x-n^{\beta}\nu_n t)$, note that this does not affect the distribution of the stochastic integrals. The SPDE for v_n is then

$$\partial_t v_n(t,x,y) = \frac{\nu_n}{2}\Delta_y v_n(t,x,y) + v_n\dot{W}_n(t,y) - n^{-\beta}\partial_y\left(v_n(t,x,y)\dot{W}_n(t,y)\right). \tag{3.2.3}$$

Note that $v_n(0,x,y) = e^{n^{\beta}(y-x)}u(0,0,x,y) = \delta(x-y)$, where the multiplication makes sense because the exponential is a smooth function.

For X the solution to the SDE (3.1.1) with initial condition x, we have

$$\int u_n(0,t,x,y)f(y)dy = \mathbb{E}^B[f(X_t)]. \tag{3.2.4}$$

Where $\mathbb{E}^B$ is the expectation over the law of the Brownian motion B. So that the rescaled quantity v_n satisfies

$$\int v_n(t,x,y)f(y)dy = e^{-\frac{\nu_n}{2}n^{2\beta}t}\mathbb{E}^B[e^{n^{\beta}(X_t-x)}f(X_t-n^{\beta}\nu_n t)]. \tag{3.2.5}$$

We aim to show that as $n \to \infty$, this converges in an L^2 sense to the solution to the stochastic heat equation with diffusivity σ^2 integrated against f. The proof proceeds in two parts: in Section 3.2.1, we'll show termwise convergence of the chaos expansion, and in Section 3.2.2, we'll show the second moment converges to the correct limit. Together this suffices to prove the desired L^2 convergence.

3.2.1 Chaos expansion

It was shown by Le Jan and Raimond [LJR04a], that for any $f \in C_c^{\infty}(\mathbb{R})$ the random variables $\int_{\mathbb{R}} f(y)u_n(s,t;x,y)dy$ are given by an explicit Wiener chaos expansion.

This can be seen as a Le Jan and Raimond originally considered this chaos expansion in [LJR02], where they used it to show that strong solutions to non-Lipschitz SDEs are given by random Markovian kernels. The aim of this section is to show the term by term convergence of the chaos expansion of v_n towards that of the stochastic heat equation. We'll begin with a brief introduction to chaos expansions, see [Jan97] for full details.

Let $\Lambda_k := \{t \in \mathbb{R}^n | 0 \leq t_1 < ... < t_n\}$, for $f \in L^2(\Lambda^k \times \mathbb{R}^k)$ we can consider the stochastic integral

$$\int_{\Lambda^k}\int_{\mathbb{R}^k} f(t_1,...,t_k,y_1,...,y_k)W(dt_1,dy_1)...W(dt_k,dy_k) = \int_{\Lambda^k}\int_{\mathbb{R}^k} f(t,y)W^{\otimes k}(dt,dy),$$

which can be defined by iteration. For $f \in L^2(\Lambda_k \times \mathbb{R}^k)$ and $g \in L^2(\Lambda_j \times \mathbb{R}^j)$ we have

$$\mathbb{E}\left[\int_{\Lambda^k}\int_{\mathbb{R}^k} f(t,y)W^{\otimes k}(dt,dy)\int_{\Lambda^j}\int_{\mathbb{R}^j} g(t,y)W^{\otimes j}(dt,dy)\right] = \langle f,g\rangle_{L^2}\delta_{j,k}.$$

Where $\delta_{j,k}$ is the Kronecker delta. In fact, the stochastic integrals above provide an isometry between L^2 random variables measurable with respect to $\sigma(W)$ and $\otimes_{k=1}^{\infty} L^2(\Lambda_k \times \mathbb{R}^k)$. For linear SPDEs it is often possible to find an explicit chaos expansion for the solution, which we now demonstrate for the stochastic heat equation. Let z be the solution to the stochastic heat equation with driving noise W and initial condition δ_x, where x is a constant and y is the space variable, i.e.

$$\partial_t z = \frac{\sigma^2}{2}\Delta z + z\dot{W}. \tag{3.2.6}$$

We will denote by P_t^ν the Heat operator with diffusivity ν, and by p_t^ν the corresponding kernel; that is,

$$P_t^\nu f(x) = \int_{\mathbb{R}} p_t^\nu(x-y)f(y)dy = \frac{1}{\sqrt{2\pi t\nu_n}}\int_{\mathbb{R}} e^{\frac{(x-y)^2}{2t\nu_n}} f(y)dy.$$

Following the survey [Qua11], we can write the equation in Duhamel form.

$$z(t,y) = p_t^{\sigma^2}(y-x) + \int_0^t\int_{\mathbb{R}} p_{t-s}^{\sigma^2}(y-y')z(s,y')W(ds,dy').$$

Iterating this equation we see that z can be written explicitly in terms of a chaos expansion, for further details on the chaos representation of the stochastic heat equation see the review [Qua11]. Let $\Lambda_k(t) := \{t \in \Lambda_k | t_k < t\}$, then

$$z(t,y) = p_t^{\sigma^2}(y-x)$$
$$+\sum_{k=1}^{\infty}\int_{\Lambda_k(t)}\int_{\mathbb{R}^k} p_{s_1}^{\sigma^2}(y_1-x)p_{s_2-s_1}^{\sigma^2}(y_2-y_1)...p_{t-s_k}^{\sigma^2}(y-y_k)W(ds_1,dy_1)...W(ds_k,dy_k).$$

When z is integrated against a test function $f \in C_c^2(\mathbb{R})$ we can write

$$\int z(t,y)f(y)dy$$
$$=P_t^{\sigma^2}f(x)+\sum_{k=1}^{\infty}\int_{\Lambda_k(t)} P_{s_1}^{\sigma^2}\left(W(ds_1)P_{s_2-s_1}^{\sigma^2}\left(...W(ds_k)P_{t-s_k}^{\sigma^2}f\right)\right)(x). \tag{3.2.7}$$

Now we move on to the chaos expansion for the flow of kernels from Le Jan and Raimond [LJR04a]

$$\int u_n(t,x,y)f(y)dy = P_t^{\nu_n}f(x)+\sum_{k=1}^{\infty}J_{0,t}^k f(x). \tag{3.2.8}$$

Where $J_{s,t}^k f(x)$ is defined via the inductive formula below.

$$J_{s,t}^k f(x) = n^{-\beta}\int_s^t J_{s,u}^{k-1}\left(\tilde{W}_n(du,\cdot)\partial_y P_{t-u}^{\nu_n}f\right)(x). \tag{3.2.9}$$

With $J_{s,t}^1 f(x) = n^{-\beta}\int_s^t P_{u-s}^{\nu_n}\left(\tilde{W}_n(du,\cdot)\partial_y P_{t-u}^{\nu_n}f\right)(x)$. The stochastic integral is constructed as follows, for an orthonormal basis of $L^2(\mathbb{R})$, $(e_k)_{k\in\mathbb{N}}$ there is a sequence of independent Brownian motions $(W^k)_{k\in\mathbb{N}}$ such that $W_n(t,y)=\sum_{k=1}^{\infty}\rho_n * e_k(y)W_t^k$, the above expression is given by the following sum

$$n^{-\beta}\sum_{k=1}^{\infty}\int_s^t J_{s,u}^{k-1}\left(\rho_n * e_k(\cdot+n^{\beta}\nu_n u)\partial_y P_{t-u}^{\nu_n}f\right)(x)dW_u^k. \tag{3.2.10}$$

The above stochastic integrals are defined using the usual Itô integration.

By iterating equation (3.2.9) we can write $J_{s,t}^k f(x)$ in closed form.

$$n^{-k\beta}\int_{s\le s_1\le\cdots\le s_k\le t} P_{s_1-s}^{\nu_n}W_n(ds_1)\partial_y P_{s_2-s_1}^{\nu_n}W_n(ds_2)\dots W_n(ds_k)\partial_y P_{t-s_k}^{\nu_n}f(x).$$

It is clear that if $f \in C_c^2(\mathbb{R})$ then $e^{n^{\beta}(\cdot-x)}f$ is as well; thus, we can find the chaos expansion for $\int_{\mathbb{R}} u_n(0,t;x,y+n^{\beta}\nu_n t)e^{n^{\beta}(y-x)}f(y)dy$, from which we can find the chaos expansion for v_n. Ultimately, we get the following equality.

$$\int v_n(t,x,y)f(y)dy = P_t^{\nu_n} f(x) + \sum_{k=1}^{\infty} \mathcal{I}_{0,t}^{k,n} f(x). \tag{3.2.11}$$

Where $\mathcal{I}$ is defined through the following recursion, and the stochastic integrals are defined as in (3.2.10).

$$\mathcal{I}_{s,t}^{k,n} f(x) = \int_s^t \mathcal{I}_{s,u}^{k-1,n} \left(W_n(du,\cdot) \left(1 + n^{-\beta}\partial_y\right) P_{t-u}^{\nu_n} f\right)(x). \tag{3.2.12}$$

This is seen by noting that since

$$\int v_n(t,x,y)f(y)dy = \int_{\mathbb{R}} u_n(0,t;x,y+n^{\beta}\nu_n t)e^{n^{\beta}(y-x)+\frac{\nu_n}{2}n^{2\beta}t} f(y)dy,$$

we must have

$$\begin{aligned}\int v_n(t,x,y)f(y)dy =& e^{n^{\beta}(y-x)+\frac{\nu_n}{2}n^{2\beta}t} P_t^{\nu_n}(f(\cdot - n^{\beta}\nu_n t))(x) \\ &+ \sum_{k=1}^{\infty} e^{n^{\beta}(y-x)+\frac{\nu_n}{2}n^{2\beta}t} J_{0,t}^k(f(\cdot - n^{\beta}\nu_n t)(x).\end{aligned}$$

We can define $\mathcal{I}_{s,t}^{k,n} f(x) := e^{n^{\beta}(y-x)+\frac{\nu_n}{2}n^{2\beta}t} J_{0,t}^k(f(\cdot - n^{\beta}\nu_n t)(x)$. It is easily checked that the first term on the right hand side is exactly $P_t^{\nu_n} f(x)$. It is then easy to derive the recursion relation (3.2.12) from the recursion for J (3.2.9) by using an induction argument.

Our aim is to show that the Wiener chaos expansion for v_n, (3.2.11), is converging term by term in L^2 to the Wiener chaos expansion of the solution to the stochastic heat equation in line (3.2.7). To show this, we will expand each term in the chaos expansion (3.2.11) using the following basic equality.

$$\partial_x P_t^{\nu_n} f(x) = P_t^{\nu_n} f'(x). \tag{3.2.13}$$

This follows immediately by integrating by parts, after a straightforward application of the dominated convergence theorem to pass the derivative through the integral. This allows us to rewrite the recursion formula, (3.2.12).

$$\begin{aligned}\mathcal{I}_{s,t}^{k,n} f(x) &= \int_s^t \mathcal{I}_{s,u}^{k-1,n} \left(W_n(du,\cdot) P_{t-u}^{\nu_n}(f + n^{-\beta} f')\right)(x) &(3.2.14)\\ &= \sum_{j=1}^{\infty} \int_0^t \mathcal{I}_{s,u}^{k-1,n} \left((\rho_n * e_j) P_{t-u}^{\nu_n}(f + n^{-\beta} f')\right)(x) dW_u^j. &(3.2.15)\end{aligned}$$

Iterating this equation, we get an explicit formula for $\mathcal{I}^{k,n}_{s,t}$.

$$\begin{aligned}
&\mathcal{I}^{k,n}_{s,t} f(x) \\
&= \sum_{j_1,\dots,j_k=1}^{\infty} \int_{s\leq u_k \leq \dots \leq u_1 \leq t} (\rho_n * e_{j_k}) P^{\nu_n}_{u_k - u_{k-1}} \qquad (3.2.16)\\
&\quad \Big((1+n^{-\beta}\partial_y)(\rho_n * e_{j_{k-1}}) P^{\nu_n}_{u_{k-1}-u_{k-2}} \dots P^{\nu_n}_{t-u_1}(f + n^{-\beta} f')\Big)(x) dW^{j_1}_{u_1} \dots dW^{j_k}_{u_k}.
\end{aligned}$$

We show this converges to the corresponding term in the chaos expansion for the stochastic heat equation in the following proposition.

Proposition 3.2.1. *For each $x \in \mathbb{R}$, $t > 0$ and $k \in \mathbb{N}$ the following convergence holds, as $n \to \infty$, in the L^2 sense*

$$\mathcal{I}^{k,n}_{0,t} f(x) \to I^k_{0,t} f(x).$$

Above $I^k_{0,t} f(x)$ denotes the k^{th} term in the chaos expansion for the stochastic heat equation, integrated against f, in terms of W as written in line (3.2.7).

Proof. We begin the proof by expanding the expression on line (3.2.16), using the fact that derivatives commute with the heat operator. Writing out the heat operators in full, and denoting the k^{th} spatial derivative of W_n by $W_n^{(k)}$, so that $W_n^{(k)} = \sum_{j=1}^{\infty} (\rho_n^{(k)} * e_j) W^k$, where $f_n^{(k)}$ denotes the $k^t h$ derivative of the function f, we can rewrite (3.2.16) as

$$\begin{aligned}
&\sum_{j=0}^{k} n^{-j\beta} \sum_{d \in D_{j,k}} N_d \int_{0\leq s_1 \leq \dots \leq s_k \leq t} \int p^{\nu_n}_{s_1}(x, y_1) W_n(ds_1, y_1) \int p^{\nu_n}_{s_2 - s_1}(y_1, y_2) \\
&\quad W_n^{(d_1)}(ds_2, y_2) \int p^{\nu_n}_{s_3 - s_2}(y_2, y_3) \dots W_n^{(d_{k-1})}(ds_k, y_k) \qquad (3.2.17)\\
&\quad \int p^{\nu_n}_{t-s_k}(y_k, y_{k+1}) f^{(d_k)}(y_{k+1}) dy_{k+1} \dots dy_1.
\end{aligned}$$

Here, $D_{j,k} := \{d = (d_1, \dots, d_k) \in \mathbb{N}_0^k |\ \sum_{i=1}^k d_i = j,\ \sum_{i=1}^l d_i \leq l\}$ and $N_d \in \mathbb{N}$ depends only on $d \in D_{j,k}$ and accounts for repetitions; it is easy to see that when $d = 0 \in \mathbb{N}_0^k$, we have $N_d = 1$.

Now we'll show that all terms of the above sum for which we do not have $d = 0$ vanish in the limit in the L^2 sense. We'll start by looking at the variance of the stochastic integrals in each term of the sum.

It is easily shown that we have $\mathbb{E}[W^{(i)}(s,x) W^{(i)}(t,y)] = s \wedge t \int_{\mathbb{R}} \rho_n^{(i)}(x-z) \rho_n^{(i)}(z - y) dz = (s \wedge t)(-1)^i\ \tilde{\rho}_n^{(2i)}(x-y)$, where we have used integration by parts to get the second equality. Therefore, we have the following equality for the stochastic integrals.

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg(\int_{0\le s_1\le\cdots\le s_k\le t}\int p^{\nu_n}_{s_1}(x,y_1)W_n(ds_1,y_1)\int p^{\nu_n}_{s_2-s_1}(y_1,y_2)W_n^{(d_1)}(ds_2,y_2)\\
&\quad\ldots W_n^{(d_{k-1})}(ds_k,y_k)\int p^{\nu_n}_{t-s_k}(y_k,y_{k+1})f^{(d_k)}(y_{k+1})dy_{k+1}\ldots dy_1\Bigg)^2\Bigg]\\
=&(-1)^{\sum_{i=1}^{k-1}d_i}\int_{0\le s_1\le\cdots\le s_k\le t}\int p^{\nu_n}_{s_1}(x,y_1)p^{\nu_n}_{s_1}(x,y_1')\tilde{\rho}_n(y_1-y_1')\\
&\quad\cdots\int p^{\nu_n}_{s_k-s_{k-1}}(y_{k-1},y_k)p^{\nu_n}_{s_k-s_{k-1}}(y_{k-1}',y_k')\tilde{\rho}_n^{(2d_{k-1})}(y_k-y_k')\\
&\quad\int p^{\nu_n}_{t-s_k}(y_k,y_{k+1})p^{\nu_n}_{t-s_k}(y_k',y_{k+1}')f^{(d_k)}(y_{k+1})f^{(d_k)}(y_{k+1}')dydy'ds.
\end{aligned}$$

The above expression can be rewritten as

$$\begin{aligned}
&\Bigg|\int_{0\le s_1\le\ldots s_k\le t}\int_{\mathbb{R}^{k+1}}\int_{\mathbb{R}^{k+1}}\left(\prod_{i=1}^{k}p^{\nu_n}_{s_i-s_{i-1}}(y_{i-1},y_i)p^{\nu_n}_{s_i-s_{i-1}}(y_{i-1}',y_i')\tilde{\rho}_n^{(2d_{i-1})}(y_i-y_i')\right)\\
&\quad p^{\nu_n}_{t-s_k}(y_k,y_{k+1})p^{\nu_n}_{t-s_k}(y_k',y_{k+1}')f^{(d_k)}(y_{k+1})f^{(d_k)}(y_{k+1}')dydy'ds\Bigg|.
\end{aligned}$$

Where we have set $y_0=y_0'=x$ and $s_0=d_0=0$. Since $f,\rho\in C_c^\infty(\mathbb{R})$, we can perform a change of variables by replacing y_i with $y_i/n+y_i'$ for each $1\le i\le k$, we can also use the bound $p_t^{\nu_n}(y)\le(2\pi\nu_n t)^{-\frac{1}{2}}$ to get that the above expression is bounded by

$$\frac{n^{2(j-d_k)}}{(2\pi\nu_n)^{\frac{k+1}{2}}}\int_{0\le s_1\le\ldots s_k\le t}\int_{\mathbb{R}^{k+1}}\int_{\mathbb{R}^{k+1}}\left(\prod_{i=1}^{k}(s_i-s_{i-1})^{-\frac{1}{2}}p^{\nu_n}_{s_i-s_{i-1}}(y_{i-1}',y_i')|\tilde{\rho}^{(2d_{i-1})}(y_i)|\right) \tag{3.2.18}$$

$$(t-s_k)^{-\frac{1}{2}}p^{\nu_n}_{t-s_k}(y_k',y_{k+1}')|f^{(d_k)}(y_{k+1})f^{(d_k)}(y_{k+1}')|dydy'ds, \tag{3.2.19}$$

where the extra powers of n come from rewriting the derivatives of the ρ_n in terms of ρ, and using the equality $\sum_{i=1}^{k-1}d_i=j-d_k$. Since $f,\rho\in C_c^\infty(\mathbb{R})$, there are constants $C,\tilde{C}>0$ depending only on f,ρ and k such that the above expression is bounded by

$$Cn^{2(j-d_k)}\int_{0\le s_1\le\cdots\le s_k\le t}\frac{1}{\sqrt{s_1}\sqrt{s_2-s_1}\sqrt{s_3-s_2}\ldots\sqrt{s_k-s_{k-1}}\sqrt{t-s_k}}ds \tag{3.2.20}$$

$$=\tilde{C}t^{\frac{1}{2}(k-1)}n^{2(j-d_k)}. \tag{3.2.21}$$

Hence, we get the following bound on the variance of each term in line (3.2.17)

$$\tilde{C}N_d^2 t^{\frac{1}{2}(k-1)} n^{2(j-d_k)-2j\beta}. \tag{3.2.22}$$

Where the constant $\tilde{C}$ has changed between lines, but remains positive and dependent only on ρ, f, j and k. It's clear from $0 \leq d_k \leq j$ that $2(j-d_k)-2j\beta \leq 2j(1-\beta)$, which is strictly negative if $j > 0$ and $\beta > 1$. This proves, for $\beta > 1$, that all terms in the sum (3.2.17) apart from that with $j = 0$ vanish as $n \to \infty$. Since the sum is finite, and the constants are dependent only on ρ, f, d and k, this shows that their sum vanishes as well.

Remark 3.2.2. *The above step is the only part of the proof of the main result where the assumption $\beta > 1$ is required. It should be stressed that the result is expected to hold for all $\beta > \frac{1}{2}$, and that we believe the need for $\beta > 1$ is a technical requirement of the proof only.*

Hence, only when we have $j = 0$ in line (3.2.17) does the term survive to the L^2 limit, in this case $D_{j,k}$ only contains a single element, $d = 0$, for which $N_d = 1$. Thus, the only term in (3.2.17) that survives into the $n \to \infty$ limit is the one below.

$$\int_{0\leq s_1 \leq \cdots \leq s_k \leq t} \int p_{s_1}^{\nu_n}(x, y_1) W_n(ds_1, y_1) \int p_{s_2-s_1}^{\nu_n}(y_1, y_2) W_n(ds_2, y_2)$$
$$\int p_{s_3-s_2}^{\nu_n}(y_2, y_3) \dots W_n(ds_k, y_k) \int p_{t-s_k}^{\nu_n}(y_k, y_{k+1}) f(y_{k+1}) dy_{k+1} \dots dy_1. \tag{3.2.23}$$

We just need to show that this converges to the corresponding term in the chaos expansion for the stochastic heat equation. Recalling that we defined $W_n = \rho_n * W$, this amounts to showing the stochastic integral with respect to the mollified noise converges to the stochastic integral with respect to the original noise, which we now show.

Once again we need to show L^2 convergence, Itô's isometry gives us that the variance of the difference between (3.2.23) and the limit (where $W_n(ds, y)dy$ is replaced with

$W(ds, dy))$ is given by the following expression.

$$\int_{0\le s_1\le\cdots\le s_k\le t}\int\int p^{\nu_n}_{s_1}(x,y_1)p^{\nu_n}_{s_1}(x,y_1')\dots p^{\nu_n}_{t-s_k}(y_k,y_{k+1})p^{\nu_n}_{t-s_k}(y_k',y_{k+1}')$$
$$\left(\prod_{i=1}^{k}\tilde{\rho}_n(y_i-y_i')\right)f(y_{k+1})f(y_{k+1}')dydy'ds$$
$$+\int_{0\le s_1\le\cdots\le s_k\le t}\int\int p^{\sigma^2}_{s_1}(x,y_1)p^{\sigma^2}_{s_1}(x,y_1)\dots p^{\sigma^2}_{t-s_k}(y_k,y_{k+1})p^{\sigma^2}_{t-s_k}(y_k,y_{k+1})$$
$$f(y_{k+1})f(y_{k+1}')dydy'$$
$$-2\int_{0\le s_1\le\cdots\le s_k\le t}\int\int p^{\nu_n}_{s_1}(x,y_1)p^{\sigma^2}_{s_1}(x,y_1')\dots p^{\nu_n}_{t-s_k}(y_k,y_{k+1})p^{\sigma^2}_{t-s_k}(y_k',y_{k+1}')$$
$$\left(\prod_{i=1}^{k}\rho_n(y_i-y_i')\right)f(y_{k+1})f(y_{k+1}')dydy'ds.$$

Following the same lines as before, we make the change of variables $y_i \mapsto y_i n + y_i'$. A straightforward application of the dominated convergence theorem shows that the resulting expression converges to 0, as required. Hence, we have term by term convergence of the chaos expansion for v to the chaos expansion for the solution to the stochastic heat equation (3.2.7). That is, the $L^2(\Omega)$ limit of the k^{th} term in the chaos expansion for v (3.2.11) is given by

$$\int_{0\le s_1\le\cdots\le s_k\le t}\int p^{\sigma^2}_{s_1}(x,y_1)W(ds_1,dy_1)\int p^{\sigma^2}_{s_2-s_1}(y_1,y_2)W(ds_2,dy_2) \tag{3.2.24}$$
$$\int p^{\sigma^2}_{s_3-s_2}(y_2,y_3)\dots W(ds_k,dy_k)\int p^{\sigma^2}_{t-s_k}(y_k,y_{k+1})f(y_{k+1})dy_{k+1}. \tag{3.2.25}$$

Here, p is the heat kernel with diffusivity σ^2. This is the same as the k^{th} term of the stochastic heat equation. □

Termwise convergence of the chaos expansions is not enough on its own to prove convergence, we must also show that nothing is escaping description by the chaos expansion in the limit, leading us to consider the second moments.

3.2.2 The 2-point motions

We want to show the convergence of the second moment to the corresponding quantity for the SHE. Once again, let z be the solution to the stochastic heat equation (3.1.19) with initial condition δ_x. Bertini and Cancrini, [BC95], proved a formula for the moments of the stochastic heat equation, below we state the formula for the second moment of the solution integrated against a test function.

$$\mathbb{E}\left[\left(\int z(t,y)f(y)dy\right)^2\right] = \mathbb{E}_x\left[e^{\frac{\kappa^2}{2\eta}\mathcal{L}_t^0(B^1-B^2)}f(B^1(t))f(B^2(t))\right]. \quad (3.2.26)$$

$\mathcal{L}_t^0$ denotes local time at 0 up to time t, and B^1 and B^2 are independent Brownian motions in $\mathbb{R}$ with diffusivity η, both starting from x.

We're interested in looking at the variance of the rescaled quantity (3.2.2) integrated against a test function $f \in C_c^2(\mathbb{R})$, i.e. the quantity given in line (3.2.11). This reduces to computing an expectation of the 2-point motions. That is, if X_n^1 and X_n^2 are the 2-point motion of the flow of kernels solving the SPDE (3.2.1) then

$$\begin{aligned}&\mathbb{E}\left[\int v_n(t,x_1,y_1)f(y_1)dy_1 \int v_n(t,x_2,y_2)f(y_2)dy_2\right]\\ =&\mathbb{E}_x\left[e^{-n^{2\beta}\nu_n t+n^\beta\left(X_n^1(t)-x_1+X_n^2(t)-x_2\right)}f(X_n^1(t)-n^\beta\nu_n t)f(X_n^2(t)-n^\beta\nu_n t)\right]. \quad (3.2.27)\end{aligned}$$

Denote the law of (X_n^1, X_n^2) by $\mathbb{P}_x^n$. We have that the quadratic variations are given by

$$\langle X_n^1, X_n^2\rangle(t) = n^{1-2\beta}\int_0^t \tilde{\rho}(n(X_n^1(s)-X_n^2(s)))ds,$$

Thus, Girsanov's theorem tells us that under the measure

$$\mathbb{Q}_x^n := e^{-n^{2\beta}\nu_n t+n^\beta\left(X_n^1(t)-x_1+X_n^2(t)-x_2\right)-n\int_0^t\tilde{\rho}(n(X_n^1(s)-X_n^2(s)))ds}\cdot\mathbb{P}_x^n,$$

we have that $X_n^i(t) - n^\beta\nu_n t - n^{1-\beta}\int_0^t\tilde{\rho}(n(X_n^1(s)-X_n^2(s)))ds$ is a continuous local martingale under $\mathbb{Q}_x^n$. Denoting the expectation with respect to $\mathbb{Q}_x^n$ by $\mathbb{E}_x^{\mathbb{Q}^n}$ we get that line (3.2.27) is equal to

$$\begin{aligned}\mathbb{E}_x^{\mathbb{Q}^n}\Bigg[&e^{\int_0^t\tilde{\rho}_n(Y_n^1(s)-Y_n^2(s))ds}f(Y_n^1(t)+n^{-\beta}\int_0^t\tilde{\rho}_n(Y_n^1(s)-Y_n^2(s))ds)\\ &f(Y_n^2(t)+n^{-\beta}\int_0^t\tilde{\rho}_n(Y_n^1(s)-Y_n^2(s))ds)\Bigg].\end{aligned}$$

Where $Y_n^i(t) := X_n^i(t) - n^\beta\nu_n t - n^{-\beta}\int_0^t\tilde{\rho}_n(Y_n^1(s)-Y_n^2(s))ds$, so that $Y_n^i(t)$ is a continuous local martingale under $\mathbb{Q}_x^n$. In fact, $\langle Y_n^i\rangle(t) = \langle X_n^i\rangle(t) = \nu_n t$; thus, the Y_n^i are both Brownian motions by Levy's characterisation.

Since Y_n^1 and Y_n^2 are both Brownian motions (with diffusivity $\nu_n \to \sigma^2$ as $n\to\infty$), it follows easily that the sequences Y_n^1 and Y_n^2 are both tight in $C([0,T];\mathbb{R})$ and therefore the sequence of $C([0,T];\mathbb{R}^2)$ valued random variables, (Y_n^1, Y_n^2), is tight.

Taking a weakly converging subsequence, the Skorokhod representation theorem gives us a sequence of processes $(Z_n)_{n\in\mathbb{N}}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $C([0,T];\mathbb{R}^2)$, which is converging almost surely to some limit Z, with Z_n equal in distribution to Y_n under $\mathbb{Q}_x^n$ for all $n \in \mathbb{N}$. We start by showing that Z is an $\mathbb{R}^2$ Brownian motion, the first step is the following lemma.

Lemma 3.2.3. *The following convergence holds in $L^1(\Omega)$ for each $t > 0$, and in $L^1(\Omega \times [0,T])$*

$$n^{\beta}\langle Z_n^1, Z_n^2\rangle(t) = n^{-\beta}\int_0^t \tilde{\rho}_n(Z_n^1(s) - Z_n^2(s))ds \to 0, \quad \textit{as } n \to \infty.$$

Proof. Since $\tilde{\rho}$ is symmetric and decreasing on $\mathbb{R}_{>0}$, we have that for any $\alpha \in (0,1)$,

$$\begin{aligned}
&n^{-\beta}\int_0^t \tilde{\rho}_n(Z_n^1(s) - Z_n^2(s))ds \\
=&n^{1-\beta}\int_0^t \tilde{\rho}\left(n(Z_n^1(s) - Z_n^2(s))\right)\left(\mathbb{1}_{\{|Z_n^1(s)-Z_n^2(s)|\le n^{-\alpha}\}} + \mathbb{1}_{\{|Z_n^1(s)-Z_n^2(s)|> n^{-\alpha}\}}\right)ds \\
\le&n^{1-\beta}\tilde{\rho}(n^{1-\alpha})t + n^{1-\beta}\int_0^t \tilde{\rho}(n(Z_n^1(s) - Z_n^2(s)))\mathbb{1}_{\{|Z_n^1(s)-Z_n^2(s)|\le n^{-\alpha}\}}ds. \qquad (3.2.28)
\end{aligned}$$

For any $\alpha \in (0,1)$, $n^{1-\beta}\tilde{\rho}(n^{1-\alpha}) \to 0$ as $n \to \infty$. It remains to bound the second term.

$$\begin{aligned}
&n^{1-\beta}\int_0^t \tilde{\rho}(n(Z_n^1(s) - Z_n^2(s)))\mathbb{1}_{\{|Z_n^1(s)-Z_n^2(s)|\le n^{-\alpha}\}}ds \\
\le&\frac{n^{1-\beta}}{2}\int_0^t \frac{\mathbb{1}_{\{|Z_n^1(s)-Z_n^2(s)|\le n^{-\alpha}\}}}{\sigma^2 + n^{1-2\beta}(\tilde{\rho}(0) - \tilde{\rho}(n(Z_n^1(s) - Z_n^2(s))))}d\langle Z_n^1 - Z_n^2\rangle(s) \\
\le&\frac{n^{1-\beta}}{2\sigma^2}\int_{-n^{-\alpha}}^{n^{-\alpha}} \mathcal{L}_t^y(Z_n^1 - Z_n^2)dy. \qquad (3.2.29)
\end{aligned}$$

Where the the last line follows from and application of the occupation times formula, see [RY13, Corollary 1.6, Chapter VI]. From (3.2.28) and (3.2.29) we get

$$\begin{aligned}
&\mathbb{E}_x[|n^{1-\beta}\int_0^t \tilde{\rho}\left(n(Z_n^1(s) - Z_n^2(s))\right)ds|] \\
\le&n^{1-\beta}\tilde{\rho}(n^{1-\alpha})t + \frac{n^{1-\beta}}{2\sigma^2}\mathbb{E}_x[\int_{-n^{-\alpha}}^{n^{-\alpha}} \mathcal{L}_t^y(Z_n^1 - Z_n^2)dy] \\
=&n^{1-\beta}\tilde{\rho}(n^{1-\alpha})t + \frac{n^{1-\beta}}{2\sigma^2}\int_{-n^{-\alpha}}^{n^{-\alpha}} \mathbb{E}_x[|Z_n^1(t) - Z_n^2(t) - y| - |x_1 - x_2 - y|]dy \\
\le&n^{1-\beta}\tilde{\rho}(n^{1-\alpha})t + \frac{n^{1-\beta-\alpha}}{\sigma^2}\sqrt{\frac{2\nu_n t}{\pi}}.
\end{aligned}$$

Choosing α with $1-\beta < \alpha < 1$ and taking $n \to \infty$ we get the desired convergence.

□

As a consequence we prove the limit of Z_n is as desired.

Lemma 3.2.4.

$$Z_n \Rightarrow Z.$$

Where Z is a Brownian motion in $\mathbb{R}^2$ with diffusivity σ^2.

Proof. Notice that for any $k \geq 0$, choice of functions $f, h_1, ..., h_k \in C_0^2(\mathbb{R}^2)$, and any $0 \leq t_1 < ... < t_k \leq t < t+s$ we have the following

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(f(Z_n(t+s)) - f(Z_n(t)) - \int_t^{t+s} \frac{\sigma^2}{2}\Delta f(Z_n(u))du\right)\prod_{i=1}^k h_i(Z_n(t_i))\right]\right| \\
\leq &\left|\mathbb{E}\left[\left(\int_t^{t+s} \nabla f(Z_n)\cdot dZ_n + \int_t^{t+s} \frac{\partial^2 f}{\partial x_1 \partial x_2}(Z_n) d\langle Z^1, Z^2\rangle(u)\right)\prod_{i=1}^k h_i(Z_n(t_i))\right]\right| \\
&+ \left|\mathbb{E}\left[\left(n^{1-2\beta}\tilde{\rho}(0)\int_t^{t+s} \Delta f(Z_n)du\right)\prod_{i=1}^k h_i(Z_n(t_i))\right]\right|
\end{aligned}$$

Since f and the h_i are all in $C_0^2(\mathbb{R}^2)$, it follows that they are bounded, further that $\frac{\partial^2 f}{\partial x_1 \partial x_2}$ is bounded. Giving, for some constant $C > 0$ depending only on f and the h_i, the bound

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_t^{t+s} \nabla f(Z_n)\cdot dZ_n \prod_{i=1}^k h_i(Z_n(t_i))\right]\right| + C\mathbb{E}\left[\left|n^{1-2\beta}\int_t^{t+s} \tilde{\rho}(n(Z_n^1 - Z_n^2))du\right|\right] \\
&+ \tilde{C}n^{1-2\beta}\tilde{\rho}(0)s.
\end{aligned}$$

We can use the martingale property for Z_n to show the first term is just 0, and the previous lemma gives that the middle term vanishes in the limit, the last term vanishes because $\beta > \frac{1}{2}$. Since the action of the generator of a Brownian motion with diffusivity σ^2 on $\mathbb{R}^2$ is given by $\frac{\sigma^2}{2}\Delta$, an application of [EK09, Theorem 8.2] yields the convergence of the finite dimensional distributions of Z_n to those of a Brownian motion. Since we already have that the sequence is tight, the result is proven. □

Now we want to show the convergence of $\int_0^t \tilde{\rho}_n((Z_n^1 - Z_n^2))ds$ to a multiple of the local time $\mathcal{L}_t^0(Z^1 - Z^2)$.

Lemma 3.2.5.

$$\int_0^t \tilde{\rho}_n(Z_n^1 - Z_n^2)ds \to \frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1 - Z^2), \quad \textit{in } L^2(\Omega) \textit{ for each } t > 0.$$

Proof. We start by once again applying the occupation times formula to get the equalities

$$\begin{aligned}\int_0^t \tilde{\rho}_n(Z_n^1 - Z_n^2)ds =& \frac{n}{2}\int_0^t \frac{\tilde{\rho}(n(Z_n^1(s) - Z_n^2(s)))}{\sigma^2 + n^{1-2\beta}(\tilde{\rho}(0) - \tilde{\rho}(n(Z_n^1(s) - Z_n^2(s)))}d\langle Z_n^1 - Z_n^2\rangle(s)\\ =& \frac{1}{2}\int \frac{\tilde{\rho}(y)}{\sigma^2 + n^{1-2\beta}(\tilde{\rho}(0) - \tilde{\rho}(y))}\mathcal{L}_t^{yn^{-1}}(Z_n^1 - Z_n^2)dy.\end{aligned}$$

Hence, we need to show that the sequence of local times converges at least locally uniformly. Let $M_n := Z_n^1 - Z_n^2$, we have from Tanaka's formula

$$\begin{aligned}&|\mathcal{L}_t^{yn^{-1}}(M_n) - \mathcal{L}_t^0(M_n)|\\ =&\left||M_n(s) - yn^{-1}| - |M_n(s)| + \int_0^t \left(\operatorname{sign}(M_n(s) - yn^{-1}) - \operatorname{sign}(M_n(s))\right) dM_n(s)\right|\\ \leq&|y|n^{-1} + \left|\int_0^t \left(\operatorname{sign}(M_n(s) - yn^{-1}) - \operatorname{sign}(M_n(s))\right) dM_n(s)\right|\\ =&|y|n^{-1} + 2\left|\int_0^t \operatorname{sign}(y)\mathbb{1}_{\{|M_n(s)| \leq |y|n^{-1}\}}dM_n(s)\right|.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\mathbb{E}_x[|\mathcal{L}_t^{y/n}(M_n) - \mathcal{L}_t^0(M_n)|^2]^{1/2} \leq \frac{|y|}{n} + \left(\mathbb{E}_x[2\int_0^t \mathbb{1}_{\{|M_n(s)| \leq |y|/n\}}d\langle M_n\rangle(s)]\right)^{1/2}\\ =&\frac{|y|}{n} + \left(\mathbb{E}_x[\int_{-|y|/n}^{|y|/n} \mathcal{L}_t^z(M_n)dz]\right)^{1/2}\\ \leq&\frac{|y|}{n} + \left(\int_{-|y|/n}^{|y|/n} \mathbb{E}_x[|M_n(t) - z| - |x_1 - x_2 - z|]dz\right)^{1/2}\\ \leq&\frac{|y|}{n} + \left(\frac{2|y|}{n}\sqrt{\frac{2\nu_n t}{\pi}}\right)^{1/2}. \qquad (3.2.30)\end{aligned}$$

Where the last line follows from the fact that $M_n = Z_n^1 - Z_n^2$ where Z_n^1 and Z_n^2 are Brownian motions with diffusivity ν_n. This vanishes as $n \to \infty$. Next we need to show the local time is converging. Defining $M = Z^1 - Z^2$, analogously to M_n, we have the bound

$$
\begin{aligned}
&\left| |M(t)| - |M_n(t)| + \int_0^t \operatorname{sign}(M(s)) dM(s) - \int_0^t \operatorname{sign}(M_n(s)) dM_n(s) \right| \\
\leq &|M(t) - M_n(t)| + \left| \int_0^t \left(\operatorname{sign}(M(s)) - \operatorname{sign}(M_n(s))\right) dM(s) \right| \\
&+ \left| \int_0^t \operatorname{sign}(M_n(s)) d(M - M_n)(s) \right|.
\end{aligned}
$$

From which it follows that

$$
\begin{aligned}
&\mathbb{E}_x[|\mathcal{L}_t^0(M) - \mathcal{L}_t^0(M_n)|^2]^{1/2} \\
\leq &\mathbb{E}_x[|M(t) - M_n(t)|^2]^{1/2} + \mathbb{E}_x \left[\left| \int_0^t \left(\operatorname{sign}(M(s)) - \operatorname{sign}(M_n(s))\right) dM(s) \right|^2 \right]^{1/2} \\
&+ \mathbb{E}_x \left[\left| \int_0^t \operatorname{sign}(M_n(s)) d(M - M_n)(s) \right|^2 \right]^{1/2} \\
\leq &\mathbb{E}_x[|Z(t) - Z_n(t)|^2]^{1/2} + \mathbb{E}_x \left[2 \int_0^t \left(\operatorname{sign}(M(s)) - \operatorname{sign}(M_n(s))\right)^2 ds \right]^{1/2} \\
&+ \mathbb{E}_x \left[\langle M - M_n \rangle (t) \right]^{1/2} \\
\leq &2\mathbb{E}_x[|Z(t) - Z_n(t)|^2]^{1/2} + \mathbb{E}_x \left[2 \int_0^t \left(\operatorname{sign}(M(s)) - \operatorname{sign}(M_n(s))\right)^2 ds \right]^{1/2} \\
= &2\mathbb{E}_x[|Z(t) - Z_n(t)|^2]^{1/2} + \left(8 \int_0^t \mathbb{P}_x \left(M(s)\ M_n(s) < 0 \right) ds \right)^{1/2}. \qquad (3.2.31)
\end{aligned}
$$

For any $\varepsilon > 0$ the following equality is true.

$$
\begin{aligned}
&\mathbb{P}_x \left(M(s)\ M_n(s) < 0 \right) \\
\leq &\mathbb{P}_x \left(M(s) \in [-\varepsilon,\ \varepsilon], |M(s) - M_n(s)| < \varepsilon \right) + \mathbb{P}_x \left(|M(s) - M_n(s)| \geq \varepsilon \right).
\end{aligned}
$$

Since $\frac{1}{\sqrt{2}} M = \frac{1}{\sqrt{2}} (Z^1 - Z^2)$ is a Brownian motion, the first probability is bounded above by $\frac{2\varepsilon}{\sqrt{2\pi s}}$. The second probability vanishes as $n \to \infty$, because $M_n \to M$ almost surely; thus, also in probability. Hence, dominated convergence gives that the second probability in (3.2.31) vanishes. Further, since we can get uniform bounds on the fourth moments of $|Z(t) - Z_n(t)|$, the almost sure convergence of $Z_n(t)$ to $Z(t)$ gives $L^2(\Omega)$ convergence of $Z_n(t)$ to $Z(t)$. Therefore,

$$
\mathbb{E}_x[|\mathcal{L}_t^0(M) - \mathcal{L}_t^0(M_n)|^2]^{1/2} \to 0, \quad \text{as } n \to \infty. \qquad (3.2.32)
$$

Since we also have $Z_n \to Z$ a.s. in $C([0,T])$ by construction, the same idea gives

$$\left(\int_0^T \mathbb{E}_x[|\mathcal{L}_t^0(M) - \mathcal{L}_t^0(M_n)|^2]dt\right)^{1/2} \to 0, \quad \text{as } n \to \infty.$$

Hence,using the occupation times formula once again, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\left(\int_0^t \tilde{\rho}_n(Z_n^1 - Z_n^2)ds - \frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1 - Z^2)\right)^2\right]^{1/2} \\
\leq &\mathbb{E}_x\left[\left(\int_0^t \tilde{\rho}_n(M_n(s))ds - \frac{1}{2\sigma^2}\int \frac{\tilde{\rho}_n(y)}{\nu_n - n^{-2\beta}\tilde{\rho}_n(y)}dy\mathcal{L}_t^0(M)\right)^2\right]^{1/2} \\
&+ \left|\frac{1}{2}\int \frac{\tilde{\rho}(y)}{\sigma^2 + n^{1-2\beta}(\tilde{\rho}(0) - \tilde{\rho}(y))}dy - \frac{1}{2\sigma^2}\right|\mathbb{E}_x[\mathcal{L}_t^0(M)^2]^{1/2} \\
= &\mathbb{E}_x\left[\left(\frac{1}{2}\int \frac{\tilde{\rho}(y)}{\nu_n - \tilde{\rho}(y)}\left(\mathcal{L}_t^{y/n}(M_n) - \mathcal{L}_t^0(M)\right)dy\right)^2\right]^{1/2} \\
&+ \left|\frac{1}{2}\int \frac{\tilde{\rho}(y)}{\sigma^2 + n^{1-2\beta}(\tilde{\rho}(0) - \tilde{\rho}(y))}dy - \frac{1}{2\sigma^2}\right|\mathbb{E}_x[\mathcal{L}_t^0(M)^2]^{1/2} \\
\leq &\left(\frac{1}{4}\int \frac{\tilde{\rho}(y)}{\nu_n - \tilde{\rho}(y)}dy\int \frac{\tilde{\rho}(y)}{\nu_n - \tilde{\rho}(y)}\mathbb{E}_x\left[\left(\mathcal{L}_t^{y/n}(M_n) - \mathcal{L}_t^0(M)\right)^2\right]dy\right)^{1/2} \\
&+ \frac{n^{1-2\beta}}{2\sigma^2}\int \frac{\tilde{\rho}(x)(\tilde{\rho}(0) - \tilde{\rho}(x))}{\sigma^2 + n^{1-2\beta}(\tilde{\rho}(0) - \tilde{\rho}(x))}dx\mathbb{E}_x[\mathcal{L}_t^0(M)^2]^{1/2}.
\end{aligned}$$

Where the last inequality is a consequence of Jensen's inequality. Since $|y|\tilde{\rho}(y)$ is integrable and because the expectation vanishes due to (3.2.32) and (3.2.30), we can apply dominated convergence to see that the first term above vanishes. The second also vanishes, because the integral is bounded above by $\tilde{\rho}(0)/\sigma^2$ and the expectation is finite. Thus, the result is proven. □

Finally, we show the desired convergence of the expectation.

Proposition 3.2.6. *For $f \in C_c^\infty(\mathbb{R})$, we have*

$$\begin{aligned}
&\mathbb{E}_x\left[e^{\int_0^t \tilde{\rho}_n(Z_n^1(s) - Z_n^2(s))ds}f(Z_n^1(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t))f(Z_n^2(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t)))\right] \\
\to &\mathbb{E}_x\left[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1 - Z^2)}f(Z^1(t))f(Z^2(t))\right], \quad \textit{as } n \to \infty.
\end{aligned}$$

As a direct consequence, we have

$$\mathbb{E}\left[\int v_n(t,y)f(y)dy^2\right] \to \mathbb{E}\left[\int z(t,y)f(y)dy^2\right], \quad \textit{as } n \to \infty.$$

Proof. First note

$$
\begin{aligned}
&\Big|\mathbb{E}_x\Big[e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds} f(Z_n^1(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t)) f(Z_n^2(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t))) \\
&- e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)} f(Z^1(t)) f(Z^2(t))\Big]\Big| \\
\leq &\Big|\mathbb{E}_x\Big[\Big(e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds} - e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\Big) \\
&\quad f(Z_n^1(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t)) f(Z_n^2(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t)))\Big]\Big| \\
&+ \Big|\mathbb{E}_x\Big[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)} \\
&\quad \Big(f(Z_n^1(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t)) f(Z_n^2(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t))) - f(Z^1(t)) f(Z^2(t))\Big)\Big]\Big|.
\end{aligned}
$$

From Lemma 3.2.3 we know $n^\beta\langle Z_n^1, Z_n^2\rangle(t) \to 0$ as $n \to \infty$ in $L^1(\Omega)$; thus, there is a subsequence converging almost surely. Hence, as long as $e^{\mathcal{L}_t^0(Z^1-Z^2)}$ has finite expectation, the dominated convergence theorem gives that the second expectation vanishes on this subsequence. But, since every subsequence must contain a subsequence for which this convergence holds, it follows that

$$
\begin{aligned}
&\Big|\mathbb{E}_x\Big[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)} \\
&\quad \Big(f(Z_n^1(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t)) f(Z_n^2(t) + n^\beta\langle Z_n^1, Z_n^2\rangle(t))) - f(Z^1(t)) f(Z^2(t))\Big)\Big]\Big| \\
&\to 0, \quad \text{as } n \to \infty. \qquad (3.2.33)
\end{aligned}
$$

To complete the above argument, we need to show the exponential of the local time has finite expectation, later we will also need to bound the exponential moments. Denoting $M_n = Z_n^1 - Z_n^2$ as previously, it is a consequence of the monotone convergence theorem that

$$
\mathbb{E}_x\left[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(M_n)}\right] = \sum_{k=0}^{\infty} \frac{\sigma^{2k} p^k}{2^k k!} \mathbb{E}_x[\mathcal{L}_t^0(M_n)^k]. \qquad (3.2.34)
$$

Hence, we need a bound on the moments of the local time. For this we apply

Tanaka's formula, setting $M_n(0) = x_1 - x_2 = m_0$ we have

$$\begin{aligned}
&\mathbb{E}_x\left[|\mathcal{L}_t^y(M_n)|^k\right]^{\frac{1}{k}} \\
=&\mathbb{E}_x\left[\left||M_n(t)-y| - |m_0 - y| - \int_0^t \operatorname{sign}(M_n(s)-y)dM_n(s)\right|^k\right]^{\frac{1}{k}} \\
\leq&\mathbb{E}_x\left[|M_n(t)-m_0|^k\right]^{\frac{1}{k}} + \mathbb{E}_x\left[\left|\int_0^t \operatorname{sign}(M_n(s)-y)dM_n(s)\right|^k\right]^{\frac{1}{k}} \\
\leq&2\mathbb{E}_x\left[\left|Z_n^1(t)-x_1\right|^k\right]^{\frac{1}{k}} + \sum_{i=1}^2 \mathbb{E}_x\left[\left|\int_0^t \operatorname{sign}(M_n(s)-y)dZ_n^i(s)\right|^k\right]^{\frac{1}{k}}. \qquad (3.2.35)
\end{aligned}$$

Where the last line comes from the fact that $M_n = Z_n^1 - Z_n^2$, and that Z_n^1 and Z_n^2 are both Brownian motions with diffusivity ν_n for all $n \in \mathbb{N}$. Since the first term is just the moments of a Brownian motion they are easily calculated. For the moments of the stochastic integrals, we first note that they are clearly continuous local martingales with quadratic variation

$$\begin{aligned}
\langle\int_0^{\cdot} \operatorname{sign}(M_n(s)-y)dZ_n^i(s)\rangle(t) &= \nu_n \int_0^t \operatorname{sign}(M_n(s)-y)^2 ds \\
&= \nu_n t.
\end{aligned}$$

With the last line following from the fact that $\mathbb{P}_x(M_n(s) - y = 0) = \mathbb{P}_x(Z_n^1(s) = Z_n^2(s) - y) = 0$ for all $s > 0$. In particular, the stochastic integral is also a Brownian motion with diffusivity ν_n; thus, (3.2.35) is given by

$$4\mathbb{E}_x\left[\left|Z_n^1(t)-x_1\right|^k\right]^{\frac{1}{k}} = 4\pi^{-\frac{1}{2k}}\sqrt{2\nu_n t}\,\Gamma\left(\frac{k+1}{2}\right)^{\frac{1}{k}}. \qquad (3.2.36)$$

Giving us the bound

$$\mathbb{E}_x\left[|\mathcal{L}_t^y(M_n)|^k\right] \leq (4\sqrt{2})^k \pi^{-\frac{1}{2}}(\nu_n t)^k\,\Gamma\left(\frac{k+1}{2}\right). \qquad (3.2.37)$$

Putting this bound into (3.2.34), we get

$$\mathbb{E}_x\left[e^{pC_\rho \mathcal{L}_t^0(M_n)}\right] \leq \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(2\sqrt{2}\sigma^2 pC_\rho \nu_n t)^k}{k!}\,\Gamma\left(\frac{k+1}{2}\right).$$

Which is finite for all $p \in \mathbb{R}$, proving (3.2.33) is satisfied.

It remains to show

$$\Big|\mathbb{E}_x\Big[\Big(e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds} - e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\Big)$$
$$f(Z_n^1(t)+n^\beta\langle Z_n^1,Z_n^2\rangle(t))f(Z_n^2(t)+n^\beta\langle Z_n^1,Z_n^2\rangle(t)))\Big]\Big|$$
$$\to 0, \quad \text{as } n\to\infty.$$

Using the boundedness of $f\in C_c^\infty(\mathbb{R})$, we see

$$\Big|\mathbb{E}_x\Big[\Big(e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds} - e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\Big)$$
$$f(Z_n^1(t)+n^\beta\langle Z_n^1,Z_n^2\rangle(t))f(Z_n^2(t)+n^\beta\langle Z_n^1,Z_n^2\rangle(t)))\Big]\Big|$$
$$\le C\mathbb{E}_x\left[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\left|e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds-\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}-1\right|\right].$$

Once again, we want to use almost sure convergence along a subsequence. However, in this case we cannot apply dominated convergence directly, so we split the expectation.

$$=C\mathbb{E}_x\left[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\left|e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds-\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}-1\right|\mathbb{1}_{\{e_n(t)\le\varepsilon\}}\right]$$
$$+C\mathbb{E}_x\left[e^{\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\left|e^{\int_0^t \tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds-\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}-1\right|\mathbb{1}_{\{e_n(t)>\varepsilon\}}\right].$$

Where $e_n(t):=|n\int_0^t\tilde{\rho}(n(Z_n^1(s)-Z_n^2(s)))ds-\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)|$. Lemma 3.2.5 gives that there is a subsequence along which $n\int_0^t\tilde{\rho}(n(Z_n^1(s)-Z_n^2(s)))ds$ converges almost surely to $\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)$; hence, dominated convergence shows that the first term in the above expression vanishes as $n\to\infty$ along such a subsequence. Again, this argument gives that every subsequence contains a subsequence for which the desired convergence holds, so it holds for the whole sequence. Applying the triangle inequality, and then the generalised Hölder inequality, we see the second term is bounded by

$$C\left(\mathbb{E}_x\left[e^{\frac{1}{\sigma^2}\mathcal{L}_t^0(Z^1-Z^2)}\right]^{\frac{1}{2}}+\mathbb{E}_x\left[e^{2\int_0^t\tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds}\right]^{\frac{1}{2}}\right)\mathbb{P}\left(e_n(t)>\varepsilon\right)^{\frac{1}{2}}. \quad (3.2.38)$$

Hence, we need a bound on the exponential moments of $\int_0^t\tilde{\rho}_n(Z_n^1(s)-Z_n^2(s))ds$. Using the occupation times formula for the first line and then Jensen's inequality for the second, we have

$$\begin{aligned}
&\left(\int_0^t \tilde{\rho}_n(Z_n^1(s) - Z_n^2(s))ds\right)^k \\
=&\left(\frac{1}{2}\int \frac{\tilde{\rho}(y)}{\sigma^2 + n^{\alpha-1}(\tilde{\rho}(0) - \tilde{\rho}(y))}\mathcal{L}_t^{yn^{-\alpha}}(Z_n^1(t) - Z_n^2(t))dy\right)^k \\
\le&\frac{\sigma^{2(1-k)}}{2^k}\int \frac{\tilde{\rho}(y)}{\sigma^2 + n^{\alpha-1}(\tilde{\rho}(0) - \tilde{\rho}(y))}\mathcal{L}_t^{yn^{\alpha-1}}(Z_n^1(t) - Z_n^2(t))^k dy.
\end{aligned}$$

Where, for the application of Jensen's inequality, we used that

$$\int \frac{\tilde{\rho}(y)}{\sigma^2 + n^{\alpha-1}(\tilde{\rho}(0) - \tilde{\rho}(y))}dy \le \sigma^{-2}.$$

Hence, we can apply the bound (3.2.37) to find that the expectation is bounded independently of n. Lemma 3.2.5 implies $\int_0^t \tilde{\rho}_n(Z_n^1(s) - Z_n^2(s))ds$ converges in probability to $\frac{1}{2\sigma^2}\mathcal{L}_t^0(Z^1 - Z^2)$. So that the above bound, together with the previous bound on the exponential moments of $\mathcal{L}_t^0(Z^1 - Z^2)$, implies (3.2.38) vanishes as $n \to \infty$, proving the statement. □

Theorem 3.2.7. *Let $z \in C((0,T); C(\mathbb{R}))$*[1] *be the solution to the stochastic heat equation (3.2.6) with driving noise W and initial condition δ_x, where x is taken as a constant and y is the space variable. Then for every $f \in C_c^\infty(\mathbb{R})$ and $t > 0$*

$$\int v_n(t,x,y)f(y)dy \to \int z(t,y)f(y)dy, \qquad \text{in } L^2(\Omega).$$

Proof. This follows from the termwise $L^2(\Omega)$ convergence of the chaos expansion, and the convergence of the $L^2(\Omega)$ norm to the correct value, (3.2.26), that we have just shown. To see this, note that for $w_n(t) := \int v_n(t,x,y)f(y)dy$, and $z_f(t) := \int z(t,y)f(y)dy$ we have

$$\|w_n(t) - z_f(t)\|_{L^2(\Omega)}^2 = \|w_n(t)\|_{L^2(\Omega)}^2 - \|z_f(t)\|_{L^2(\Omega)}^2 + 2(z_f(t) - w_n(t), z_f(t))_{L^2(\Omega)}. \tag{3.2.39}$$

The first two terms cancel in the limit as $n \to \infty$ due to Proposition 3.2.6. We can write the last term as a sum using the chaos expansions. For any $N \in \mathbb{N}$, the first N terms of this sum will vanish due to the termwise convergence of the chaos terms. Applying Proposition 3.2.6 again, we see that the remaining terms

[1] Where, for topological spaces X and Y, the space $C(X;Y)$ denotes the space of continuous functions from X to Y endowed with the topology of uniform convergence on compact sets. As before $C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ and we also endow it with the topology of uniform convergence on compact sets.

in the chaos expansion are bounded, for n sufficiently large, by $(2\|z_f(t)\|_{L^2(\Omega)} + 1)\left(\sum_{k=N+1}\|I^k_{0,t}f\|^2_{L^2(\Omega)}\right)^{1/2}$, where $I^k_{0,t}f$ is the k^{th} term of the chaos expansion for $z_f(t)$ from (3.2.7). Since we have $\|z_f(t)\|^2_{L^2(\Omega)} = \sum_{k=0}^{\infty}\|I^k_{0,t}f\|^2_{L^2(\Omega)} < \infty$, taking $N \to \infty$ gives the desired result. □

In the next section, we outline possible further work related to the weak environment regime.

3.3 Further Work

Theorem 3.2.7 has several obvious extensions. The first of these is to remove the restriction that $\beta > 1$, which we made at the start of Section 3.2, and complete the analysis of the weak environment regime; we discuss this more at the end of Chapter 4. The next extension is to strengthen the type of convergence; in principle this should follow from v_n being tight in an appropriate topology, as Theorem 3.2.7 does serve to determine the limit. However, tightness results pose significant problems, due to the necessity of dealing with moments of v_n itself, rather than the integral of v_n against a nice function. Further, tightness requires control of higher moments than the second, and beyond the second moment the estimates become difficult to deal with, as the tilted diffusions become very complicated. Indeed, since Kolmogorov's criterion requires bounds on the moments of a Hölder semi-norm, we would need to control at least the fourth moments, because the stochastic heat equation is $\frac{1}{2}^-$ Hölder continuous in space and $\frac{1}{4}^-$ Hölder continuous in time [Qua11]. This means we need control on higher moments than the fourth moment of the time increments and the second of the space increments.

Another question we can ask is whether the RWRE model we discussed in Section 1.1 has an analogue of the weak environment regime. The weak environment regime for the RWRE corresponds to letting the distribution of the random environment be very close to a deterministic one. That is, for a sequence of space-time i.i.d. random environments $\boldsymbol{\omega}^{(n)} = (\omega^{(n)}_{t,x})_{t,x\in\mathbb{Z}}$, we let $\omega^{(n)}_{t,x} = \frac{1}{2} + n^a w_{t,x}$, where $(w_{t,x})_{t,x\in\mathbb{Z}}$ is an i.i.d. collection of mean zero $[-\frac{1}{2}, \frac{1}{2}]$ valued random variables. As in Section 1.1, $P^{\boldsymbol{\omega},n}$ refers to the law of the RWRE in a given realisation of the environment, $\boldsymbol{\omega}^{(n)}$.

For $a \in [-\frac{1}{2}, 0)$ and some b determined by a, we are interested in the fluctuations of the quantity

$$nP^{\boldsymbol{\omega},n}(X(\lceil n^2 t\rceil_2) = \lceil n^{2+b}\lambda t + ny\rceil_2 | X(0) = 0). \tag{3.3.1}$$

Here $\lceil\cdot\rceil_2 := 2\lceil\frac{\cdot}{2}\rceil$ and $\lambda > 0$ is a fixed parameter, for $b = 0$ we add the restriction that $\lambda < 1$ to ensure that we are evaluating $P^{\omega,n}$ away from the edge of its support. After a suitable rescaling, we conjecture that the above quantity has fluctuations described by the stochastic heat equation in the $n \to \infty$ limit. We conjecture the statement holds when b is given by $b = -\frac{1}{2} - a$. This is an adaptation to the condition for α and β in the weak environment regime displayed in Figure 3.1. The reason the conditions are different is because we are diffusively scaling at the same time as we are looking far from the origin. So that a and b are not directly analogous to α and β. Instead, (a, b) is analogous to $(\alpha, \beta - 1)$. The effect of diffusive scaling on the continuum model is discussed in Remark 3.1.3.

As we discussed in Remark 3.1.3, Corwin and Gu [CG16] showed that when $(a, b) = (-\frac{1}{2}, 0)$ quantity (3.3.1) converges to the solution to the stochastic heat equation, when rescaled so that the mean converges to the solution to the heat equation. Note that the mean is given by the transition probabilities for the simple symmetric random walk, making the convergence of the mean to the solution to the heat equation relatively straightforward. Hence, the appearance of the SHE in the weak environment is already known. We know of no results for other values of a and b. It should be said that for $b > 0$, no such result can hold, as the RWRE can only move one step in space for every one step in time. Therefore, if $b > 0$, then quantity (3.3.1) would eventually be 0 almost surely. We can however ask if there are analogous results for the case $b \in (-\frac{1}{2}, 0)$. This corresponds to setting $\beta \in (0, 1)$ in the continuous model, so that proving the result for $b \in (-\frac{1}{2}, 0)$ would provide further evidence for the conjecture on the left side of Figure 3.1.

CHAPTER 4

Weak Diffusivity Scaling

In this chapter, we investigate the weak diffusivity regime for the stochastic partial differential equations introduced in the previous chapter, as we described in Section 3.1.2. In the following section, we will briefly recall the setup of our model from Section 3.1.1 and the details of the weak diffusivity regime. After this, we will discuss the major qualitative differences between the weak diffusivity and the weak environment regimes and then the results we have for the weak diffusivity regime.

4.1 Setup

Just as in the previous chapter, we start with a cylindrical Brownian motion on $L^2(\mathbb{R};\mathbb{R})$, W, and take a mollifier $\rho : \mathbb{R} \to \mathbb{R}$ satisfying the same assumptions as at the start of section 3.1.1, and set $W_n := W * \rho_n := W * (n\rho(n\cdot))$. Suppose that $(\sigma_n)_{n=1}^{\infty}$ is a null sequence of positive numbers and let $\nu_n = \frac{1}{2}(\rho * \rho(0) + \sigma_n^2)$. We're interested in the weak diffusivity regime described in section (3.1.3). In this setting, the tilted kernel v_n defined in equation (3.1.6), with $\lambda = (n\sigma(n))^{\frac{1}{2}}$, solves the SPDE,

$$\partial_t v_n = \frac{\nu_n}{2}\Delta \nu_n + \sigma_n^{1/2} v_n \dot{W}_n - n^{-\frac{1}{2}}\partial_y \left(v_n \dot{W}_n\right), \tag{4.1.1}$$

together with the initial condition $v_n(0, y) = \delta_x(y)$. An explicit form for v_n can be found in terms of the solution to an SDE. Given a standard Brownian motion on the real line, B, let X be the solution to the SDE:

$$dX(t) = n^{-\frac{1}{2}} W_n(dt, X(t)) + \sigma_n dB(t). \tag{4.1.2}$$

Where the stochastic integrals are understood in the Itô sense, see [Kun94b] for definitions. Let $\mathbb{E}^B$ denote the expectation over the law of the Brownian motion B. Then v_n is given explicitly by the following formula

$$v_n(t, y)dy = e^{n\sigma_n \frac{\nu_n}{2} t + (n\sigma_n)^{\frac{1}{2}}(y-x)} U_{0,t}(x, dy + (n\sigma_n)^{\frac{1}{2}} \nu_n t). \tag{4.1.3}$$

In the above equation, U is the flow of kernels associated to the SDE (4.1.2) by the formula $U_{s,t}(x, dy) = \mathbb{P}(X(t) \in dy | W, X(s) = x)$, and $x \in \mathbb{R}$ is assumed to be fixed.

The relation (4.1.3) allows us to rewrite the moments of V_n in terms of the solutions to a system of SDEs (3.1.15) called the n-point motions, which we introduced in Section 3.1.3. In particular, for the second moment, we have the following equality

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}} v_n(t, y) f(y)^2 dy\right] \\
=&\mathbb{E}_x\left[e^{\nu_n t + (n\sigma_n)^{\frac{1}{2}}(X^1(t) + X^2(t) - 2x)} f(X^1(t) - (n\sigma_n)^{\frac{1}{2}} \nu_n t) f(X^2(t) - (n\sigma_n)^{\frac{1}{2}} \nu_n t)\right] \\
&= \tilde{\mathbb{E}}_x\left[e^{\sigma_n \int_0^t \tilde{\rho}_n(Y^1(s)) - Y^2(s) ds} f(Y^1(t)) f(Y^2(t))\right].
\end{aligned} \tag{4.1.4}$$

Where, following the discussion at the start of Section 3.1.3, we have performed a change of measure in the second equality and the processes $Y^i(t) = X^i(t) - (n\sigma_n)^{\frac{1}{2}} \nu_n t$ were described in Proposition 3.1.2. From this, we can show the convergence of the moments towards the solution of the stochastic heat equation. This was briefly discussed in Section 3.1.3, and will be discussed further in Section 4.2.

4.1.1 The Stochastic Partial Differential Equation

In [LJR04a, Proposition 5.4], Le Jan and Raimond show that the flow of kernels $(U_{s,t})_{s<t}$, defined in (3.1.2), is equivalent to a flow of Markovian operators. The same authors showed the flow of Markovian operators were the solution to a SPDE in a certain weak sense, [LJR02, Theorem 3.2]. Recently, Dunlap and Gu, [DG21], showed the flow of kernels satisfy the same SPDE as a generalised solution in the sense of [Kun94a], . Further, they showed that the flow of kernels have continuous densities with respect to the Lebesgue measure.

As discussed in the introduction for the previous chapter (in particular, see equation (3.1.10)), the tilted density v_n, from equation (4.1.3), is the solution to equation

(4.1.1) in the following sense. Let $(v_n(t), f) = \int_{\mathbb{R}} v_n(t,y)f(y)dy$. For all $f \in C_c^\infty(\mathbb{R})$

$$(v_n(t), f) = f(x) + \frac{\nu_n}{2}\int_0^t (v_n(s), f'')ds + \underbrace{\sigma_n^{\frac{1}{2}}\int_0^t (v_n(s), fW_n(ds))}_{=\star} + \underbrace{n^{-\frac{1}{2}}\int_0^t (v_n(s), f'W_n(ds))}_{=\dagger}. \tag{4.1.5}$$

Where both stochastic integrals are understood as in the previous chapter.

Using the same ideas as in Section 3.2.2, we can show that $\mathbb{E}[\dagger^2] \to 0$ as $n \to \infty$. We use Itô's isometry for the space-time white noise to turn the second moment of the stochastic integral into a deterministic integral, which we calculate using equation (4.1.4), from Proposition 3.1.2.

$$\begin{aligned}
\mathbb{E}[\dagger^2] &= n^{-1}\mathbb{E}\left[\left(\int_0^t v_n(s), f'W_n(ds)\right)^2\right] \\
&= n^{-1}\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} (v_n(s)f') * \rho_n(y)^2 dyds\right)\right] \\
&= n^{-1}\tilde{\mathbb{E}}_x\left[\int_0^t e^{\sigma_n\int_0^s \tilde{\rho}(Y^1(r)-Y^2(r))dr} f'(Y^1(s))f'(Y^2(s))\tilde{\rho}_n(Y^1(s)-Y^2(s))ds\right] \\
&\leq \|f'\|_\infty^2 n^{-1}\tilde{\mathbb{E}}_x\left[\int_0^t e^{\sigma_n\int_0^s \tilde{\rho}(Y^1(r)-Y^2(r))dr}\tilde{\rho}_n(Y^1(s)-Y^2(s))ds\right] \\
&= \|f'\|_\infty^2 (n\sigma_n)^{-1}\tilde{\mathbb{E}}_x\left[e^{\sigma_n\int_0^t \tilde{\rho}(Y^1(r)-Y^2(r))dr} - 1\right].
\end{aligned}$$

In a similar way to the arguments for the weak environment setting from Section 3.2.2, we can show the expectation on the final line above is converging; thus, we have that $\mathbb{E}[\dagger^2]$ vanishes as $n \to \infty$.

Further, using similar ideas to the above argument which we cover in Section 4.2, it can be shown that $\mathbb{E}[\star^2]$ converges to $\mathbb{E}[(\kappa\int_0^t (z_x(s), fW(ds)))^2]$, where z_x is the solution to the SHE, (3.1.19), with parameters $\kappa = \sqrt{\pi}\frac{\|\rho\|_2}{\|\rho'\|_2^{1/2}}$ and $\eta = \|\rho\|_2^2$. Thus, we hope that $\star$ must converge to the noise term in the SHE. This is counter-intuitive because the coefficient in front of $\star$ is converging to 0, and if v_n is converging, then it seems reasonable to expect that stochastic integral should also converge. Thus, the whole term should converge to 0 and not to the noise term in the SHE. However, this assumes that v_n is converging in a "nice" topology, for example uniformly on compact sets, which we do not believe to be the case. As $\sigma(n)$ gets very small, the mass of the density v_n collects into large spikes, which leads to the increasingly irregular behaviour of the product $(v_n(s)f, W_n(s))$.

The discussion in the previous paragraph presents a second problem, it is not clear that (v_n, W_n) is converging to (z_x, W) where z_x is the solution to the SHE driven by W. This means we cannot show convergence to the to the solution of the stochastic heat equation directly, because we do not have a way of coupling the sequence v_n with the limiting noise W. Instead, we must show convergence in distribution. To show convergence in distribution, we need to find a way to determine the limit point as a solution to the SHE without reference to the noise W. This can be done with a martingale problem.

4.1.2 The Martingale Problem for the Stochastic Heat Equation

We begin by stating an equivalent formulation for the stochastic heat equation via a martingale problem. Uniqueness for the martingale problem was proven in [KS88]. We use the formulation from [BG97], adapted for a delta initial condition, as in [ACQ11]. The survey paper [Qua11] contains a short review of the martingale problem approach to the SHE.

Definition 4.1.1. *$z_x \in C((0,T); C(\mathbb{R}))$ is the solution to the martingale problem for the stochastic heat equation, $\partial_t z_x = \frac{\eta}{2}\Delta z_x + \kappa v \dot{W}$, if there exists a constant $C > 0$ such that $\mathbb{E}[z_x(t,y)^2] \leq C p_t(x-y)^2$ for all $t \in (0,T)$ and $y \in \mathbb{R}$ and, for every $\phi \in C_c^\infty(\mathbb{R})$, $N(z_x, \phi)$ and $\Lambda(z_x, \phi)$, defined below, are martingales with respect to a common filtration.*

$$N^{\nu_n,\kappa}(z_x,\phi)(t) := \int_{\mathbb{R}} \phi(y) z_x(t,y)dy - \phi(x) - \frac{\eta}{2}\int_0^t \int_{\mathbb{R}} \phi''(y) z_x(s,y)dyds, \quad \textit{and} \tag{4.1.6}$$

$$\Lambda^{\nu_n,\kappa}(z_x,\phi)(t) := N_t(\phi)^2 - \kappa^2 \int_0^t \int_{\mathbb{R}} \phi(y)^2 z_x(s,y)^2 dyds. \tag{4.1.7}$$

It follows from (4.1.5) that the following processes are martingales with respect to the filtration generated by W.

$$N^n(v_n,\phi)(t) := \int_{\mathbb{R}} \phi(y) v_n(t,y)dy - \phi(x) - \frac{\nu_n}{2}\int_0^t \int_{\mathbb{R}} \phi''(y) v_n(s,y)dyds, \quad \text{and} \tag{4.1.8}$$

$$\Lambda^n(v_n,\phi)(t) := N_t^n(\phi)^2 - \int_0^t \int_{\mathbb{R}} \left(\left(v_n(s,\cdot)\sigma_n^{1/2}\phi + n^{-1/2}\phi'\right) * \rho_n(y)\right)^2 dyds. \tag{4.1.9}$$

Where we have used that for any square integrable $g : \mathbb{R} \to \mathbb{R}$ the following equality holds

$$\int_{\mathbb{R}} g * \rho_n(y)^2 dy = \int_{\mathbb{R}^2} g(y_1) g(y_2) \tilde{\rho}_n(y_1 - y_2) dy,$$

where $\tilde{\rho}_n := \rho_n * \rho_n$. We want to show that the sequence v_n is tight in a suitable topology, namely one where the mappings $v \mapsto N(v, \phi)$ and $v \mapsto \Lambda(v, \phi)$ are continuous, and then derive the martingale properties for any subsequential limits of the v_n by using the above martingales. The main problems are: in what topology might the sequence v_n be tight that is strong enough to derive the martingale property for $\Lambda(v, \phi)$ from the martingale property from $\Lambda^n(v_n, \phi)$.

One potential answer to this problem is to add some smoothing to v_n and hope that the resultant process will be tight in a strong enough topology to determine its limit points as the solution to our martingale problem. In the next section, we will show that, for a limit point, (4.1.7) is a martingale. We will show this by using that (4.1.9) has the martingale property with respect to the filtration generated by W.

4.2 Convergence to the Stochastic Heat Equation

Let $\psi \in C_c^\infty(\mathbb{R})$ be a symmetric mollifier and v_n be as in Section 4.1. The aim of this section will be to prove the following theorem.

Theorem 4.2.1. *Suppose $m = m(n)$ is a real valued sequence such that $m(n) \to \infty$ as $n \to \infty$ and $m(n)n^{-\frac{1}{2}} \to 0$ as $n \to \infty$. Suppose further that there is a weakly convergent subsequence of the sequence of random variables $(v_n(\cdot) * \psi_m)_{n=1}^\infty \subset C((0,T), C(\mathbb{R}))$, with limit v, such that there is a constant $C > 0$ with $\mathbb{E}[v(t,y)^2] \leq Cp_t^\nu(x-y)^2$ for every $t > 0$ and $y \in \mathbb{R}$, where p_t^ν denotes the heat kernel with diffusivity ν. Then v is equal in distribution to the solution to the stochastic heat equation, with initial condition δ_x:*

$$\partial_t z_x = \frac{\|\rho\|_2^2}{2} \Delta z_x + \frac{\sqrt{\pi}\|\rho\|_2}{\|\rho'\|_2^{1/2}} z_x \dot{W}. \tag{4.2.1}$$

Proof. Throughout the proof, we will write $\kappa = \frac{\sqrt{\pi}\|\rho\|_2}{\|\rho'\|_2^{1/2}}$. To prove the statement we will show that any subsequential limit of the process must satisfy the martingale problem (4.1.6). For a limit point, v, of $v_n * \psi_m$ we can easily show $N^{\nu,\kappa}(v, \phi)$ satisfies the martingale property, with respect to the filtration generated by v, from the martingale property for $N^n(v_n, \psi_m * \phi)$, (4.1.8). This leaves checking the martingale property for $\Lambda^{\nu,\kappa}(v, \phi)$. To prove the martingale property, we use the properties of the flow of kernels. By independence of increments, and from the SPDE for v_n, we have (for the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the noise W)

$$
\begin{aligned}
&\mathbb{E}[\Lambda^{\nu_n,\kappa}(v_n * \psi_m, \phi)(t)|\mathcal{F}_s] \\
=&\Lambda^{\nu_n,\kappa}(v_n * \psi_m, \phi)(s) \\
&+\mathbb{E}\left[\int_s^t \int_{\mathbb{R}} \left((v_n(r;x,\cdot)(\sigma_n^{\frac{1}{2}}\psi_m * \phi + n^{-\frac{1}{2}}\psi_m' * \phi)\right) * \rho_n(z)^2 dzdr|\mathcal{F}_s\right] \\
&-\mathbb{E}\left[\kappa^2 \int_s^t \int_{\mathbb{R}} (v_n(r;x,\cdot) * \psi_m)(z)^2\phi(z)^2 dzdr|\mathcal{F}_s\right].
\end{aligned}
$$

Note that this equality comes from applying the martingale property for $\Lambda^n(v_n, \psi_m * \phi)$.

It is sufficient to show the difference between the second and third terms vanishes in the limit. Using the flow and independent increment properties for $v_n(s,t;x,y)$ (which carry over from the original flow of kernels U, as defined in Section 3.1.1, see [LJR04a]), we can rewrite this difference as

$$
\begin{aligned}
&\int_{\mathbb{R}^2} v_n(0,s;x,w_1)v_n(0,s;x,w_2) \\
&\mathbb{E}\Bigg[\sigma_n \int_s^t \int_{\mathbb{R}} (v_n(s,r)(\psi_m * \phi)) * \rho_n(z)^2 dz - \kappa^2 \int_{\mathbb{R}}\int_{\mathbb{R}} (v_n(s,r) * \psi_m)(z)^2 \phi(z)^2 dzdr \qquad (4.2.2)\\
&+2\sqrt{\frac{\sigma_n}{n}}\,(v_n(s,r)(\psi_m * \phi)) * \rho_n(z)\left(v_n(s,r)(\psi_m' * \phi)\right) * \rho_n(z) \qquad (4.2.3)\\
&+\frac{1}{n}\left(v_n(s,r)(\psi_m' * \phi)\right) * \rho_n(z)^2\Bigg]dw. \qquad (4.2.4)
\end{aligned}
$$

The stationarity property for stochastic flows of kernels also carries over to the tilted density v_n. Therefore, the expectation in the above expression can be rewritten in terms of $v_n(0, r-s; x, \cdot)$. For line (4.2.2), we have from (4.1.4)

$$
\int_0^{t-s} \mathbb{E}\Bigg[\int_{\mathbb{R}} \sigma_n \left(v_n(r;x,\cdot)(\psi_m * \phi)\right) * \rho_n(z)^2 - \kappa^2 (v_n(r;x,\cdot) * \psi_m)(z)^2 \phi(z)^2 dz\Bigg]dr. \qquad (4.2.5)
$$

$$
\begin{aligned}
=&\sigma_n \int_0^{t-s} \tilde{\mathbb{E}}_w\left[e^{\sigma_n \int_0^r \tilde{\rho}_n(Y^1-Y^2)(\tau)d\tau}\tilde{\rho}_n(Y^1-Y^2)(r)(\phi * \psi_m)^{\otimes 2}(Y(r))\right]dr \qquad (4.2.6)\\
&-\kappa^2 \int_0^{t-s}\int_{\mathbb{R}} \tilde{\mathbb{E}}_w\left[e^{\sigma_n \int_0^r \tilde{\rho}_n(Y^1-Y^2)(\tau)d\tau}\psi_m(z-Y^1(r))\psi_m(z-Y^2(r))\right]\phi(z)^2 dzdr. \qquad (4.2.7)
\end{aligned}
$$

In the above, we use the shorthand $g^{\otimes 2}$ for the tensor product of the function g with itself. The process $Y = (Y^1, Y^2)$ and the tilted expectation $\tilde{\mathbb{E}}_w$ are defined in the same way as in Section 3.2.2 for the weak environment setting, its generator for general parameters was discussed in Proposition 3.1.2. The proof of the theorem is completed by showing that the above expression, together with the terms on line (4.2.3) and (4.2.4), vanish as $n \to \infty$. The proof that the above expression vanishes will follow in the remains of this section. Here, we will show that lines (4.2.3) and (4.2.4) both vanish as $n \to \infty$. First note that since $\psi, \phi \in C_c^\infty(\mathbb{R})$, and $\int_{\mathbb{R}} \psi(y)dy = 1$ we have

$$\psi_m * \phi(z) = \int_{\mathbb{R}} \psi_m(y)\phi(y-z)dy \leq \|\phi\|_\infty.$$

Note that $\phi' \in C_c^\infty(\mathbb{R})$ as well, so we can apply the same bound to $\psi_m' * \phi$. It follows easily that the contribution to the expectation from line (4.2.3) is at most

$$\begin{aligned}
&2\sqrt{\frac{\sigma_n}{n}}\mathbb{E}\left[\int_0^{t-s}\int_{\mathbb{R}^2} v_n(r;x,y_1)v_n(r;x,y_2)\tilde{\rho}_n(y_1-y_2)dydr\right] \\
=&2\sqrt{\frac{\sigma_n}{n}}\tilde{E}_x\left[\int_0^{t-s} e^{\sigma_n\int_0^r \tilde{\rho}(Y^1(\tau)-Y^2(\tau))d\tau}\tilde{\rho}_n(Y^1(r)-Y^2(r))dr\right] \\
=&\frac{1}{\sqrt{n\sigma_n}}\tilde{E}_x\left[e^{\sigma_n\int_0^{t-s}\tilde{\rho}(Y^1(r)-Y^2(r))dr}-1\right].
\end{aligned}$$

Where we used (4.1.4) to get the second line. We will show in Lemma 4.2.2, below, that the expectation is bounded. Thus line (4.2.3) vanishes as $n \to \infty$, repeating the same steps for line (4.2.4) shows that it will also vanish, finishing this part of the proof.

Before we complete the above proof by calculating the limit of the final expression, we first collect some useful facts about the diffusion $Y = (Y^1, Y^2)$.

Lemma 4.2.2. *For any $t > 0$, $w, y \in \mathbb{R}$ and $k \in \mathbb{N}$ we have the following inequality for $Z = Y^1 - Y^2$*

$$\tilde{\mathbb{E}}_w\left[\mathcal{L}_t^y(Z)^k\right] \leq 4(4\sqrt{2\nu_n t})^k \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}}.$$

Proof. Because the finite variation components of Y^1 and Y^2 are equal we may write $Z = Y^{1,m} - Y^{2,m}$ where $Y^{1,m}$ and $Y^{2,m}$ are the martingale parts, under $\tilde{\mathbb{E}}$, of Y^1 and Y^2 respectively. Because the quadratic variation of both Y^1 and Y^2 is given by

$\nu_n t$, the martingale parts are both Brownian motions. Tanaka's formula yields

$$\begin{aligned}\tilde{\mathbb{E}}_w\left[\mathcal{L}_t^y(Z)^k\right] \leq & 2^k\left(\tilde{\mathbb{E}}_w\left[|Z(t)|^k\right]+\tilde{\mathbb{E}}_w\left[\left|\int_0^t \operatorname{sign}(Z(s)-y)dZ_s\right|^k\right]\right)\\ \leq & 4^k\Bigg(\tilde{\mathbb{E}}_w\left[|Y^{1,m}(t)|^k\right]+\tilde{\mathbb{E}}_w\left[|Y^{2,m}(t)|^k\right]\\ & +\ \tilde{\mathbb{E}}_w\left[\left|\int_0^t \operatorname{sign}(Z(s)-y)dY^{1,m}(s)\right|^k\right]\\ & +\tilde{\mathbb{E}}_w\left[\left|\int_0^t \operatorname{sign}(Z(s)-y)dY^{2,m}(s)\right|^k\right]\Bigg).\end{aligned}$$

To finish the proof, we just have to note that the two stochastic integrals on the last line are continuous local martingales with quadratic variation process $\nu_n t$, and so are Brownian motions by Levy's characterisation. The inequality follows by calculating the moments of the Brownian motions. □

Lemma 4.2.3. *For $Y = (Y^1, Y^2)$ we have*

$$\left(Y, \sigma_n \int_0^{\cdot} \tilde{\rho}_n(Y^1 - Y^2)(s)ds\right) \Rightarrow \left(B, \frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathcal{L}^0_{\cdot}(B^1 - B^2)\right).$$

Here, B is a Brownian motion on $\mathbb{R}^2$, with diffusivity $\tilde{\rho}(0)$ and $\mathcal{L}^0_{\cdot}(B^1 - B^2)$ the local time of $B^1 - B^2$ at 0.

Proof. To show that $Y \Rightarrow B$ we refer to [EK09, Theorem 8.2], and use that we know both the quadratic variations for the process Y and the semi-martingale decomposition from the discussion preceding Proposition 3.1.2, we recall both below.

$$\langle Y^i, Y^j\rangle(t) = \sigma_n^2\delta_{i,j} + \int_0^t \tilde{\rho}\Big(n\big(Y^i - Y^j\big)(s)\Big)ds; \tag{4.2.8}$$

$$Y^i - \sqrt{n\sigma_n}\int_0^{\cdot} \tilde{\rho}\Big(n\big(Y^1 - Y^2\big)(s)\Big)ds \text{ is a Brownian motion on } \mathbb{R}\text{, for } i = 1, 2. \tag{4.2.9}$$

The necessary bounds on $\int_0^{\cdot} \tilde{\rho}\left(n(Y^1 - Y^2)(s)\right)ds$ come from the occupation times formula, see [RY13], and the previous lemma.

To prove the joint convergence of the local time we apply the Skorokhod representation theorem to realise Y on a probability space where the convergence to a Brownian motion holds almost surely. The convergence is then a straightforward consequence of the Tanaka formula for the local time. □

We can now return to the unproven statement at the end of the proof of Theorem 4.2.1. We begin by computing the limit of line (4.2.6). We will then show it is cancelled by the limit of line (4.2.7), thereby proving the statement.

Lemma 4.2.4. *The following convergence holds, for any $t > 0$, as $n \to \infty$.*

$$\sigma_n \int_0^t \tilde{\mathbb{E}}_w \left[e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr} \tilde{\rho}_n(Y^1 - Y^2)(s)(\phi * \psi_m)^{\otimes 2}(Y(s))\right] ds \tag{4.2.10}$$
$$\to \frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathbb{E}_w \left[\int_0^t e^{\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathcal{L}_s^0(B^1-B^2))} \phi(B^1(s))\phi(B^2(s))d\mathcal{L}_s^0(B^1 - B^2)\right].$$

Proof. Rescaling the integral over $\mathbb{R}^2$ and applying Fubini's theorem, (4.2.10) is equal to

$$\int_{\mathbb{R}^2} \tilde{\mathbb{E}}_w \left[\int_0^t e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr} \sigma_n\tilde{\rho}_n(Y^1 - Y^2)(s)\phi^{\otimes 2}(\tfrac{z}{m} + Y(s))ds\right] \psi^{\otimes 2}(z)dz. \tag{4.2.11}$$

Applying Itô's formula to $e^{\sigma_n \int_0^t \tilde{\rho}_n(Y^1-Y^2)(s)ds}\phi(z_1/m + Y^1(s))\psi(z_2/m + Y^2(s))$, it becomes clear that the above expression is equal to

$$\int_{\mathbb{R}^2} \tilde{\mathbb{E}}_w \Bigg[e^{\sigma_n \int_0^t \tilde{\rho}_n(Y^1-Y^2)(s)ds}\phi^{\otimes 2}(\tfrac{z}{m} + Y(t)) - \phi^{\otimes 2}(\tfrac{z}{m} + (x, x)) \tag{4.2.12}$$
$$- \int_0^t e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr} \phi'(z_1/m + Y^1(s))\phi(z_2/m + Y^2(s))dY^1(s) \tag{4.2.13}$$
$$- \int_0^t e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr} \phi(z_1/m + Y^1(s))\phi'(z_2/m + Y^2(s))dY^2(s) \tag{4.2.14}$$
$$-\frac{1}{2}\int_0^t e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr} \sum_{i,j=1}^{2} \frac{\partial^2\phi^{\otimes 2}}{\partial y_i \partial y_j}(\tfrac{z}{m} + Y(s))d\langle Y^i, Y^j\rangle(s)\Bigg]\psi^{\otimes 2}(z)dz. \tag{4.2.15}$$

Of course, $\langle Y^i, Y^j\rangle$ is known and given at the start of the proof of Lemma 4.2.3. We'll find the limit of the expectation, line by line, as $n \to \infty$, and then apply DCT to get the full limit.

Because ϕ is smooth and compactly supported, and for each $n \in \mathbb{N}$ the exponential term is bounded, the expectation on line (4.2.13) is equal to (ignoring the minus sign)

$$\tilde{\mathbb{E}}_w\left[\sqrt{n\sigma_n}\int_0^t e^{\sigma_n\int_0^s\tilde{\rho}_n(Y^1-Y^2)(r)dr}(\phi'\otimes\phi)(\tfrac{z}{m}+Y(s))\tilde{\rho}\Big(n\big(Y^1-Y^2\big)(s)\Big)ds\right] \tag{4.2.16}$$

$$\leq C(n\sigma_n)^{-\frac{1}{2}}\tilde{\mathbb{E}}\left[e^{\sigma_n\int_0^t\tilde{\rho}_n(Y^1-Y^2)(s)ds}-1\right]. \tag{4.2.17}$$

Where we have just used that ϕ and its derivative is bounded. Expanding the exponential via a Taylor expansion, and using the occupation times formula to rewrite the time integral in terms of $\mathcal{L}_t^y(Y^1-Y^2)$ we can show the expectation is bounded as a consequence of Lemma 4.2.2. Since $(n\sigma)^{-1}\to 0$ as $n\to\infty$, the above expectation vanishes in the limit. Thus, line (4.2.13), and by the same argument (4.2.14), disappear when we take $n\to\infty$. Similarly the $i\neq j$ parts of the sum on line (4.2.15) vanish. Putting everything together, the combined limit of lines (4.2.12), (4.2.13), (4.2.14), and (4.2.15) is, for B a Brownian motion on $\mathbb{R}^2$ with diffusivity $\tilde{\rho}(0)$,

$$\mathbb{E}_w\left[e^{\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathcal{L}_t^0(B^1-B^2))}\phi^{\otimes 2}(B(t))-\phi(x)^2-\frac{\tilde{\rho}(0)}{2}\int_0^t e^{\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathcal{L}_s^0(B^1-B^2))}\Delta\phi^{\otimes 2}(B(s))ds\right].$$

Applying Itô's formula to the semi-martingale $(B^1,B^2,\mathcal{L}_\cdot^0(B^1-B^2))$ we get that the above expression is equal to the following.

$$\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathbb{E}_w\left[\int_0^t e^{\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathcal{L}_s^0(B^1-B^2))}\phi^{\otimes 2}(B(s))d\mathcal{L}_s^0(B^1-B^2)\right]. \tag{4.2.18}$$

This is just equal to $\mathbb{E}[\kappa^2\int_0^t\int_{\mathbb{R}}z_x(t,y)^2\phi(y)^2dy]$, where z_x is the solution to the SHE with initial condition $\delta_x(y)$. □

Now we need to show the same limit is achieved by the second line, (4.2.7), which we do in the following lemma.

Lemma 4.2.5. *For each $t>0$ the following convergence holds as $n\to\infty$*

$$\tilde{\mathbb{E}}_w\left[\kappa^2\int_0^t\int_{\mathbb{R}}e^{\sigma_n\int_0^s\tilde{\rho}_n(Y^1-Y^2)(r)dr}\psi_m(z-Y^1(s))\psi_m(z-Y^2(s))\phi(z)^2dzds\right]$$
$$\to\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathbb{E}_w\left[\int_0^t e^{\frac{\pi\|\rho\|_2^2}{2\|\rho'\|_2}\mathcal{L}_s^0(B^1-B^2))}\phi(B^1(s))\phi(B^2(s))d\mathcal{L}_s^0(B^1-B^2)\right].$$

Proof. Once again we want to use Itô's formula to perform an integration by parts, in this case to get around the singular term $\psi_m(z-Y^1(s))\psi_m(z-Y^2(s))$. First, we translate the z integral up by $Y^1(s)$ and rescale, to get the following

$$\tilde{\mathbb{E}}_w\left[\kappa^2\int_0^t\int_{\mathbb{R}} e^{\sigma_n\int_0^s\tilde{\rho}_n(Y^1-Y^2)(r)dr}\psi_m(\tfrac{z}{m}+Y^1(s)-Y^2(s))\phi(\tfrac{z}{m}+Y^1(s))^2\psi(z)dzds\right]. \tag{4.2.19}$$

From (4.2.8), we know the quadratic variation of Y^1-Y^2 to be given by

$$\langle Y^1-Y^2\rangle(t)=2\int_0^t \sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}\big(n(Y^1-Y^2)(s)\big)ds. \tag{4.2.20}$$

Thus, it follows from the occupation times formula for continuous semi-martingales that

$$\int_0^t\psi_m(\tfrac{z}{m}+Y^1-Y^2)(s)ds=\frac{1}{2}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)}\mathcal{L}_t^{\frac{y_1}{m}}(Y^1-Y^2)dy_1. \tag{4.2.21}$$

Using this we can rewrite the time integral as a Riemann-Stieltjes integral in terms of the local time. It is easy to see that we can still rearrange the integrals however we wish. We end up with the following expression.

$$\tilde{\mathbb{E}}_w\Bigg[\frac{\kappa^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)} \int_0^t e^{\sigma_n\int_0^s\tilde{\rho}_n(Y^1-Y^2)(r)dr}\phi(\tfrac{z}{m}+Y^1(s))^2d\mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)dy_1\psi(z)dz\Bigg]. \tag{4.2.22}$$

In the following, we will show that the limit as $n\to\infty$ of the above expression is the same as that of the one below

$$\tilde{\mathbb{E}}_w\Bigg[\frac{\kappa^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)} \int_0^t e^{\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)}\phi(\tfrac{z}{m}+Y^1(s))^2d\mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)dy_1\psi(z)dz\Bigg]. \tag{4.2.23}$$

To show this we will estimate the difference. We begin with the following equality, which is the consequence of the occupation times formula.

$$\sigma_n\int_0^t\tilde{\rho}_n(Y^1-Y^2)(s)ds=\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}\left(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2)\right)}\mathcal{L}_t^{\sigma_n/ny_2}(Y^1-Y^2)dy_2. \tag{4.2.24}$$

We can use equality (4.2.24), to estimate

$$\left|\int_0^t \left(e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr} - e^{\frac{1}{2}\int_{\mathbb{R}} \frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))} dy_2 \mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)}\right)\right. \tag{4.2.25}$$

$$\left.\phi(\tfrac{z}{m}+Y^1(s))^2 d\mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)\right|. \tag{4.2.26}$$

Using the simple inequality $|e^x - e^y| \leq |x-y|e^{x+y}$ for $x, y \geq 0$, together with (4.2.24), the above expression is bounded above by

$$\frac{\|\phi\|_\infty^2}{2}\int_0^t \int_{\mathbb{R}} \frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))} |\mathcal{L}_s^{\sigma_n/n y_2}(Y^1-Y^2) - \mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)| dy_2 \tag{4.2.27}$$

$$e^{\sigma_n \int_0^s \tilde{\rho}_n(Y^1-Y^2)(r)dr + \frac{1}{2}\int_{\mathbb{R}} \frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))} dy_2 \mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)} d\mathcal{L}_s^{y_1/n}(Y^1-Y^2). \tag{4.2.28}$$

The exponential is an increasing function of time, and so can be bounded above by its value at t, we want to do the same thing for the difference between the two local times. In the following we will write $Z := Y^1 - Y^2$ and estimate this difference. The following bound for $|\mathcal{L}_s^z(Z) - \mathcal{L}_s^y(Z)|$ follows easily from the triangle inequality

$$\begin{aligned}&\left||Z_s - z| - |z| - |Z_s - y| + |y| - \int_0^s \operatorname{sign}(Z_r - z)dZ_r + \int_0^t \operatorname{sign}(Z_r - y)dZ_r\right| \\ \leq &2|z-y| + \left|\int_0^s \operatorname{sign}(Z_r - z) - \operatorname{sign}(Z_r - y)dZ_r\right|.\end{aligned}$$

Below, we estimate the modulus of the difference between (4.2.22) and (4.2.23) by first rewriting it in terms of (4.2.28), and then applying the bound we just derived.

To shorten notation we will let $Z := Y^1 - Y^2$.

$$\Bigg|\tilde{\mathbb{E}}_w\Bigg[\frac{\kappa^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)}$$
$$\int_0^t\left(e^{\sigma_n\int_0^s\tilde{\rho}_n(Y^1-Y^2)(r)dr}-e^{\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)}\right)$$
$$\phi(\tfrac{z}{m}+Y^1(s))^2 d\mathcal{L}_s^{\frac{y_1}{m}}(Y^1-Y^2)dy_1\psi(z)dz\Bigg]\Bigg| \tag{4.2.29}$$
$$\leq\tilde{\mathbb{E}}_w\Bigg[\frac{\kappa^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)}\int_0^t\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}$$
$$\left(2\left|\frac{\sigma_n y_2}{n}-\frac{y_1}{m}\right|+\left|\int_0^s\operatorname{sign}(Z_r-\frac{\sigma_n y_2}{n})-\operatorname{sign}(Z_r-\frac{y_1}{m})dZ_r\right|\right)dy_2$$
$$e^{\sigma_n\int_0^s\tilde{\rho}_n(Z)(r)dr+\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_s^{\frac{y_1}{m}}(Z)}d\mathcal{L}_s^{\frac{y_1}{m}}(Z)dy_1\psi(z)dz\Bigg]. \tag{4.2.30}$$

This is then bounded above by

$$\leq\tilde{\mathbb{E}}_w\Bigg[\frac{\kappa^2\|\phi\|_\infty^2}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)}$$
$$\left(2\left|\frac{\sigma_n y_2}{n}-\frac{y_1}{m}\right|+\sup_{0\leq s\leq t}\left|\int_0^s\operatorname{sign}(Z_r-\frac{\sigma_n y_2}{n})-\operatorname{sign}(Z_r-\frac{y_1}{m})dZ_r\right|\right)dy_2$$
$$\mathcal{L}_t^{\frac{y_1}{m}}(Z)e^{\sigma_n\int_0^t\tilde{\rho}_n(Z)(r)dr+\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_t^{\frac{y_1}{m}}(Z)}dy_1\psi(z)dz\Bigg]. \tag{4.2.31}$$

Where we used that used that the exponential term increases in time. For the next steps we want to move the expectation inside all the integrals, and then apply Hölder's inequality. Now we collect the necessary bounds to control the above expression, beginning with

$$\tilde{\mathbb{E}}_w\left[\mathcal{L}_t^{\frac{y_1}{m}}(Z)e^{\sigma_n\int_0^t\tilde{\rho}_n(Z(r))dr+\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_t^{\frac{y_1}{m}}(Z)}\right] \tag{4.2.32}$$
$$\leq\tilde{\mathbb{E}}_w\left[\left|\mathcal{L}_t^{\frac{y_1}{m}}(Z)\right|^3\right]^{\frac{1}{3}}\tilde{\mathbb{E}}_w\left[e^{3\sigma_n\int_0^t\tilde{\rho}_n(Z)(r)dr}\right]^{\frac{1}{3}}\tilde{\mathbb{E}}_w\left[e^{\frac{3}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_t^{\frac{y_1}{m}}(Z)}\right]^{\frac{1}{3}}. \tag{4.2.33}$$

Lemma 4.2.2 gives us a uniform in n bound on this expression. Next we want a bound on

$$\tilde{\mathbb{E}}_w\Bigg[\mathcal{L}_t^{\frac{y_1}{m}}(Z)e^{\sigma_n\int_0^t\tilde{\rho}_n(Z(r))dr+\frac{1}{2}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_t^{\frac{y_1}{m}}(Z)} \tag{4.2.34}$$

$$\sup_{0\le s\le t}\left|\int_0^s \operatorname{sign}(Z_r-\frac{\sigma_n y_2}{n})-\operatorname{sign}(Z_r-\frac{y_1}{m})dZ_r\right|\Bigg] \tag{4.2.35}$$

$$\le\tilde{\mathbb{E}}_w\left[\mathcal{L}_t^{\frac{y_1}{m}}(Z)^4\right]^{\frac{1}{4}}\tilde{\mathbb{E}}_w\left[e^{4\sigma_n\int_0^t\tilde{\rho}_n(Z(r))dr}\right]^{\frac{1}{4}}\tilde{\mathbb{E}}_w\left[e^{2\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))}dy_2\mathcal{L}_t^{\frac{y_1}{m}}(Z)}\right]^{\frac{1}{4}} \tag{4.2.36}$$

$$\tilde{\mathbb{E}}_w\left[\sup_{0\le s\le t}\left|\int_0^s \operatorname{sign}(Z_r-\frac{\sigma_n y_2}{n})-\operatorname{sign}(Z_r-\frac{y_1}{m})dZ_r\right|^4\right]^{\frac{1}{4}}. \tag{4.2.37}$$

The expectations on line (4.2.36) are uniformly bounded in n, again as a consequence of Lemma 4.2.2. Using the Burkholder-Davis-Gundy inequality the expectation on line (4.2.37) is bounded above, for some constant $C>0$, by

$$C\tilde{\mathbb{E}}_w\left[\int_0^s\left(\operatorname{sign}(Z_r-\frac{\sigma_n y_2}{n})-\operatorname{sign}(Z_r-\frac{y_1}{m})\right)^2 d\langle Z\rangle(r)^2\right]^{\frac{1}{4}} \tag{4.2.38}$$

$$=\sqrt{2}C\tilde{\mathbb{E}}_w\left[\int_0^s \mathbb{1}_{\{Z(r)\in(\frac{\sigma_n y_2}{n},\frac{y_1}{m})\}}\left(\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(nZ(r))\right)dr^2\right]^{\frac{1}{4}} \tag{4.2.39}$$

$$=\sqrt{2}C\tilde{\mathbb{E}}_w\left[\left|\int_{\frac{\sigma_n y_2}{n}}^{\frac{y_1}{m}}(\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(ny_3))\mathcal{L}_t^{y_3}(Z)dy_3\right|^2\right]^{\frac{1}{4}} \tag{4.2.40}$$

Where the final equality follows from the occupation times formula for continuous martingales. Since $\tilde{\rho}$ is non-negative and has its maximum at 0, we can replace the bracket in the integrand with $\nu_n=\sigma_n^2+\tilde{\rho}(0)$. Following this with an application of Jensen's inequality to the spatial integral, we get the upper bound

$$\sqrt{2\nu_n}C\left(\left|\frac{y_1}{m}-\frac{\sigma_n y_2}{n}\right|\int_{\frac{\sigma_n y_2}{n}}^{\frac{y_1}{m}}\tilde{\mathbb{E}}_w\left[\mathcal{L}_t^{y_3}(Z)^2\right]dy_3\right)^{\frac{1}{4}}. \tag{4.2.41}$$

The variance of the local time is bounded uniformly in n by Lemma 4.2.2. Hence, we have that there is a constant $C>0$, independent of n, such that line (4.2.31) is bounded above by

$$\frac{\kappa^2 C}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z+y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)}\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}\left(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2)\right)}$$
$$\left(\left|\tfrac{\sigma_n y_2}{n}-\tfrac{y_1}{m}\right|+\left|\tfrac{\sigma_n y_2}{n}-\tfrac{y_1}{m}\right|^{1/2}\right)dy_2 dy_1\psi(z)dz. \tag{4.2.42}$$

Since $\tilde{\rho}$ has compact support, there is some constant L such that $|\sigma_n y_2\tilde{\rho}(\sigma_n y_2)| \leq L\tilde{\rho}(\sigma_n y_2)$. Using the triangle inequality alongside the simple inequality $|a+b|^{1/2} \leq |a|^{1/2}+|b|^{1/2}$, we get

$$\int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}\left(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2)\right)}\left(\left|\tfrac{\sigma_n y_2}{n}-\tfrac{y_1}{m}\right|+\left|\tfrac{\sigma_n y_2}{n}-\tfrac{y_1}{m}\right|^{1/2}\right)dy_2$$
$$\leq \int_{\mathbb{R}}\frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}\left(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2)\right)}dy_2\left(\frac{L}{n}+\sqrt{\frac{L}{n}}+\frac{|y_1|}{m}+\sqrt{\frac{|y_1|}{m}}\right).$$

It is not hard to show that the y_2 integral is converging to a finite limit as $n\to\infty$, from which it follows that the integral is bounded. Therefore, line (4.2.42) is bounded above by the following expression

$$C\int_{\mathbb{R}}\frac{\tilde{\psi}(y_1)}{\sigma_n^2+\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)}\left(\frac{1}{n}+\sqrt{\frac{1}{n}}+\frac{|y_1|}{m}+\sqrt{\frac{|y_1|}{m}}\right)dy_1.$$

Where C has absorbed all the other constants and we have written $\tilde{\psi}(y)=\int_{\mathbb{R}}\psi(z+y)\psi(z)dz$. $\tilde{\rho}$ is a smooth symmetric function that has a global maximum at 0 with $\tilde{\rho}''(0)<0$; it follows that there is an $\varepsilon>0$ such that for some $\delta>0$ we have $\tilde{\rho}''(y)<-\delta<0$ for all $y\in(-\varepsilon,\varepsilon)$. We also have that for all $\varepsilon>0$ there is some $\gamma>0$ such that $\tilde{\rho}(0)-\tilde{\rho}(y)>\gamma$ for all $|y|>\varepsilon$. By using Taylor's theorem on $\tilde{\rho}(0)-\tilde{\rho}(\frac{n}{m}y_1)$, in combination with the two facts we just stated, we get the following bound on the above expression.

$$C\Bigg(\int_{-\frac{m}{n}\varepsilon}^{\frac{m}{n}\varepsilon}\frac{\tilde{\psi}(y_1)}{\sigma_n^2+\frac{\delta}{2}(\frac{n}{m}y_1)^2}\left(\frac{1}{n}+\sqrt{\frac{1}{n}}+\frac{|y_1|}{m}+\sqrt{\frac{|y_1|}{m}}\right)dy_1$$
$$+\int_{|y|>\frac{m}{n}\varepsilon}\frac{\tilde{\psi}(y_1)}{\eta}\left(\frac{1}{n}+\sqrt{\frac{1}{n}}+\frac{|y_1|}{m}+\sqrt{\frac{|y_1|}{m}}\right)dy_1\Bigg).$$

Once again absorbing all constants into $C>0$, and using that $\psi\in C_c^\infty(\mathbb{R})$ to bound the right hand integral, we get that the above expression is bounded above by

$$C\left(\frac{m}{n^{\frac{3}{2}}\sigma_n}+\frac{1}{\sqrt{m}}\right).$$

Where we have used that $n \in \mathbb{N}$ to get rid of higher order terms. Since we have chosen m to be such that $mn^{-\frac{1}{2}}$ is vanishing as $n \to \infty$, the above expression vanishes as $n \to \infty$.

It follows that the limit as $n \to \infty$ of line (4.2.22) is the same as the limit of line (4.2.23), as claimed. Now we compute the limit of line (4.2.23), which we recall below.

$$\frac{\kappa^2}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\psi(z+y_1)\psi(z)}{\sigma_n^2 + \tilde{\rho}(0) - \tilde{\rho}(\frac{n}{m} y_1)}$$
$$\tilde{\mathbb{E}}_w \left[\int_0^t e^{\frac{1}{2} \int_{\mathbb{R}} \frac{\tilde{\rho}(\sigma_n y_2)}{1+\sigma_n^{-2}(\tilde{\rho}(0)-\tilde{\rho}(\sigma_n y_2))} dy_2 \mathcal{L}_s^{\frac{y_1}{m}}(Y^1 - Y^2)} \phi(\tfrac{z}{m} + Y^1(s))^2 d\mathcal{L}_s^{\frac{y_1}{m}}(Y^1 - Y^2) \right] dy_1 dz. \tag{4.2.43}$$

We start by computing the limit of the expectation, we can compute the limit in the same way as we did in the proof of Lemma 4.2.4. We perform the integration by parts (with Itô's formula) on the time integral in the above expression, then compute the limit of each term. Applying Itô's formula to that limit we see it is given by the following expectation, where B^1, B^2 are independent Brownian motions with diffusivity $\tilde{\rho}(0)$ in $\mathbb{R}$, both starting from x under $\mathbb{E}_x$.

$$\mathbb{E}_x \left[\int_0^t e^{\frac{\pi \|\rho\|_2^2}{2\|\rho'\|_2} \mathcal{L}_s^0(B^1 - B^2)} \phi(B^1(s))^2 d\mathcal{L}_s^0(B^1 - B^2) \right].$$

It is fairly straightforward, using the estimates on the moments of the local time we have already derived and the fact that $\psi \in C_c^\infty(\mathbb{R})$, to see that we can apply the Dominated convergence theorem to line (4.2.43), to get that the the limit of line (4.2.43) as $n \to \infty$ is simply the above expression multiplied by $\frac{\kappa^2}{2\tilde{\rho}(0)} = \frac{\pi \|\rho\|^2}{2\|\rho'\|_2}$. That the above expression is equal to the one on line (4.2.18) follows easily from the support properties of the local time, see [RY13] for further details. □

With this Lemma, the proof of Theorem 4.2.1 is completed. □

In the next section we will discuss some potential further work related to the weak diffusivity regime.

4.3 Further Work

The most obvious next step is to complete the proof of convergence in Theorem 4.2.1 by proving the boundedness condition, $\mathbb{E}[z_x(t,y)^2] \leq Cp_t(x-y)^2$, for the limit point and tightness of $v_n * \psi_m$ as a sequence of random variables taking values in

$C((0,T),C(\mathbb{R}))$. One potential method for tightness would be to attempt to apply Kolmogorov's criterion, as discussed for the weak environment setting in Section 3.3. Whilst the bounds for both tightness and the boundedness condition should follow from a calculation using the n-point motions, they become complicated for the same reasons as discussed in Section 3.3 for the weak environment regime. Another approach is to apply the Aldous' criterion for tightness, which should allow us to use second moments of the time increments, rather than higher moments. However, we still need higher moments for the space increments.

If tightness can be proven, the next question of interest would be whether the same methods can also be used to prove convergence to the stochastic heat equation for the fixed environment and diffusivity regime and the whole weak environment regime, extending the arguments from Chapter 1.4, which only apply when the effect of the environment is taken to 0 sufficiently quickly, $\beta > 1$ in Figure 3.1. This would complete proof of the conjectured SHE line from Figure 3.1.

Another interesting question is whether or not there is an analogue of the weak diffusivity regime for the RWRE model. As has been previously mentioned, a particular instance of the weak environment regime was proven by Corwin and Gu, [CG16], but no result analogous to the weak diffusivity regime exists that we know of.

Following the notation used at the start of Section 1.4, let $\boldsymbol{w}^{(n)} = (w_{t,x}^{(n)})_{t,x\in\mathbb{Z}}$ be a sequence of i.i.d. space-time random environments, where the $w_{t,x}^{n}$ are mean zero and take values in $[-1,1]$. The random walk in random environment is the simple random walk defined by the random transition probabilities

$$P^{\boldsymbol{w},n}\left(X(t+1) = x \pm 1 | X(t) = x\right) = \frac{1}{2}(1 \pm w_{t,x}^{(n)}).$$

We are interested in the case where the sequence of random variables $w_{t,x}^{(n)}$ is converging to a $\{-1,1\}$ valued Bernoulli random variable. In this case, the path of the Random walk in the random environment is almost entirely determined by the environment, just as in the weak diffusivity setting for the continuous model.

The quantity $\mathbb{E}[(w_{t,x}^{(n)})^2] - 1$ is the discrete analogue to the molecular diffusivity, σ^2. Thus, we suppose that $\mathbb{E}[(w_{t,x}^{(n)})^2] - 1 = n^{-2\alpha}$ for some $\alpha > 0$ and study the fluctuations of the quantity

$$nP^{\boldsymbol{w},n}\left(X(\lceil n^2 t\rceil_2) = \lceil n^{\beta+1}t + ny\rceil_2 | X(0) = 0\right). \tag{4.3.1}$$

where $\lceil\cdot\rceil_2 := 2\lceil\frac{\cdot}{2}\rceil$. If $\beta = \frac{1-\alpha}{2}$, after suitable rescaling by some exponential factor, we expect the fluctuations as $n \to \infty$ to be described by the stochastic heat equation.

Another related result which would be of interest is whether or not similar statements hold for Howitt-Warren flows, when viewed as the continuum analogues of quantity 4.3.1. Such a conjecture was made by Barraquand and Rychnovsky, [BR20], where they showed convergence of the moments of uniform Howitt-Warren flows (the case studied in Chapter 2) towards the moments of the solution to the SHE. Whilst convergence of the moments does not imply full convergence, because the moments of the SHE do not determine its distribution, it is highly suggestive. The universality of the KPZ equation, discussed in Section 1.2, suggests the convergence should hold for more general Howitt-Warren flows.

The final part of Figure 3.1 to discuss is the behaviour above the line, where we conjecture the limit to be 0. This is based on the behaviour of the second moments, which can be rewritten in terms of the two point motions. Recall that the tilted density (3.1.6) can be rewritten in terms of the tilted two point motions (3.1.17). For (α, β) above the SHE line in Figure 3.1, the exponential term in (3.1.17) is no longer converging, in fact it appears to diverge to infinity. This means that the second moments are diverging to infinity whilst the first moments remain fixed. As a consequence, we believe the mass of the tilted density, 3.1.6, is collecting into large spikes, which occur at a given location with low probability. This suggests that the tilted density should converge to 0 in probability. However, this is simply a heuristic, and nowhere near a full proof. One starting point might be to consider the RWRE instead, and compare the model with polymers, where similar results already exist; see the review [Com17] for further details.

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