

Virtual resolutions of points in $\mathbb{P}^1 \times \mathbb{P}^1$ Megumi Harada^a, Maryam Nowroozi^{b,*}, Adam Van Tuyl^a^a Department of Mathematics and Statistics, McMaster University, Hamilton, ON, L8S 4L8, Canada^b Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom

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ABSTRACT

We explore explicit virtual resolutions, as introduced by Berkesch, Erman, and Smith, for ideals of finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Specifically, we describe a virtual resolution for a sufficiently general set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$ that only depends on $|X|$. We also improve an existence result of Berkesch, Erman, and Smith in the special case of points in $\mathbb{P}^1 \times \mathbb{P}^1$; more precisely, we give an effective bound for their construction that gives a virtual resolution of length two for any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

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1. Introduction

Minimal free resolutions of ideals in the coordinate ring of a toric variety are well-studied in commutative algebra and algebraic geometry as a method of obtaining information about the corresponding subvariety. It has been observed, however, that minimal free resolutions can be longer than necessary in order to obtain useful information. The theory of virtual resolutions was introduced by Berkesch, Erman, and Smith [2] as a useful analogue to the theory of minimal free resolutions which often provide *shorter* resolutions. In this sense, virtual resolutions are a useful new tool to study varieties in toric varieties, including multi-projective spaces.

The theory of virtual resolutions is quite new, so there are abundant open questions, and a lack of (families of) concrete examples. On the other hand, since their introduction in [2], they have been actively studied. In their original work [2], Berkesch, Erman, and Smith gave multiple strategies for constructing virtual resolutions. Loper [15] identified two algebraic conditions that characterize when a chain complex

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is virtual. Kennedy [13] gave an equivalent algebraic definition for a complex to be a virtual resolution. The concept of being virtually Cohen-Macaulay, introduced in [2], is studied in the context of simplicial complexes by Kenshur, Lin, McNally, Xu, and Yu [14], and more generally in the recent paper of Berkesch, Klein, Loper, and Yang [3]. Yang [19] also studied virtual resolutions of monomial ideals on toric varieties. More recently, Booms-Peot and Cobb [4] have examined when a generalized Eagon-Northcott complex is a virtual resolution.

In this paper we focus on virtual resolutions of finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. By focusing on this family, we can take advantage of many known algebraic and geometric properties about them, e.g., as developed in [6,8,9,11,12]. At the same time, we complement Gao, Li, Loper, and Matto's paper [15] which studies points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are virtual complete intersections. The contribution of this manuscript is two-fold, as we now describe.

Our first main result gives explicit constructions of a family of virtual resolutions of length two for a set X of points in sufficiently general position (as defined in Section 2) in $\mathbb{P}^1 \times \mathbb{P}^1$. Note that we have suppressed the virtual resolution in the statement below, but the complete details can be found in Section 3.

Theorem 1.1 (Theorem 3.1). *Let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ in sufficiently general position. Assume $|X| > 1$ and suppose that there exists an $(i, j) \in \mathbb{N}^2$ with $i < j$ such that $|X| = (i + 1)(j + 1)$. Then there is a virtual resolution of S/I_X of length two which can be explicitly described in terms of i and j .*

We note that, in general, the minimal free resolutions of such X are known to have length three, so our result shows that we can shorten the resolution if we allow for virtual resolutions. The main tool in our construction is a “trimming” result of Berkesch, Erman, and Smith [2, Theorem 1.3] (stated below as Theorem 2.4), which roughly states that well-chosen (graded) subcomplexes of a (graded) minimal free resolution \mathcal{F}_\bullet of S/I_X is in fact a virtual resolution of S/I_X . We also employ results of Giuffrida, Maggioni, and Ragusa [8,9] which gives concrete information about the second-difference function $\Delta^2 H_X$ (defined in Section 2) of the Hilbert function and its relation to the minimal free resolution \mathcal{F}_\bullet of S/I_X . This information is used to prove that for certain choices of these graded subcomplexes, the resulting virtual resolution is of length two.

Our second main result improves on another result of Berkesch, Erman, and Smith (in the special case of points in $\mathbb{P}^1 \times \mathbb{P}^1$), in the following sense. In addition to the “trimming” result mentioned above, Berkesch, Erman, and Smith give another method for constructing virtual resolutions in [2, Theorem 4.1]. Let $S = \mathbb{k}[x_0, x_1, y_0, y_1]$ be the bihomogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, their theorem states that for some positive integer a , the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is a virtual resolution of length two of the original ring S/I_X . Their result is an existence result in the sense that they do not give a concrete value for the positive integer a . Our contribution is to give an explicit lower bound for the value of a for any set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely, let $\pi_1(X)$ denote the set of distinct first coordinates which appear in the points in X . Then we show:

Theorem 1.2 (Theorem 4.2). *Let X be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then for all $a \geq |\pi_1(X)| - 1$, the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is a virtual resolution of S/I_X of length two.*

Our results suggest several other directions for future work. For instance, it would be of interest to explore whether the methods of the current manuscript can be extended to cover more general multiprojective spaces $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_s}$, possibly starting with the smallest such generalizations such as $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It would also be of interest to explore virtual resolutions of subvarieties of multiprojective spaces of dimension > 0 .

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2. Background

In this section we recall some relevant background on virtual resolutions and finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Although it is not the most general setting for the theory of virtual resolutions, for the purposes of this manuscript we restrict attention to the case of the ring $S = \mathbb{k}[x_0, x_1, y_0, y_1]$, the \mathbb{Z}^2 -graded coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$, where \mathbb{k} is a field of characteristic 0. The \mathbb{Z}^2 -grading is defined by setting $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$ for $i = 0, 1$. An ideal I of S is a *bihomogeneous ideal* if the generators of I are bihomogeneous with respect to this bigrading on S . We use \preceq to denote the partial ordering on \mathbb{N}^2 defined by $(i_1, j_1) \preceq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$.

The original definition of virtual resolutions given in [2] uses the geometric language of sheaves. We do not give the sheaf-theoretic definition, but we will instead use an equivalent algebraic definition. To state it, we need some notation. Let B denote the irrelevant ideal $\langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle = \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1 \rangle$ of $\mathbb{P}^1 \times \mathbb{P}^1$ in S . For any S -module M , let

$$\Gamma_B(M) := \{m \in M \mid B^t m = 0 \text{ for some } t \in \mathbb{N}\},$$

i.e., $\Gamma_B(M)$ consists of elements annihilated by some power of the irrelevant ideal. With this notation, we can state the definition for a complex to be a virtual resolution. Note that the homology groups $H_i(\mathcal{F}_\bullet)$ of a complex \mathcal{F}_\bullet of S -modules are naturally S -modules.

Definition 2.1. Let M be a finitely generated S -module. Let

$$\mathcal{F}_\bullet := \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0$$

be a complex of finitely generated free S -modules satisfying:

- (1) for each $i > 0$ there is some power t such that $B^t H_i(\mathcal{F}_\bullet) = 0$, and
- (2) $H_0(\mathcal{F}_\bullet)/\Gamma_B(H_0(\mathcal{F}_\bullet)) \cong M/\Gamma_B(M)$.

Then \mathcal{F}_\bullet is a *virtual resolution* of M .

Remark 2.2. A proof of the equivalence of Definition 2.1 and the original geometric definition for a virtual resolution given in [2] can be found in Kennedy’s thesis [13, Theorem 4.9].

Given a bihomogeneous ideal J of S , the *B-saturation* of J is the ideal defined as

$$J : B^\infty := \bigcup_{t \in \mathbb{N}} (J : B^t).$$

An ideal J is *B-saturated* if $J : B^\infty = J$. The following theorem gives one way to construct a virtual resolution of S/I .

Theorem 2.3. Let I be a *B-saturated bihomogeneous ideal* of S . If J is a bihomogeneous ideal of S with $J : B^\infty = I$, then the minimal free resolution of S/J is a virtual resolution of S/I .

Proof. If $\mathcal{F}_\bullet := \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 = S$ is a minimal free resolution of S/J , then $H_i(\mathcal{F}_\bullet) = 0$ for all $i > 0$. Since $\Gamma_B(S/J) = (J : B^\infty)/J = I/J$,

$$H_0(\mathcal{F}_\bullet)/\Gamma_B(H_0(\mathcal{F}_\bullet)) = (S/J)/(I/J) \cong S/I.$$

Because I is B -saturated, $\Gamma_B(S/I) = (I : B^\infty)/I = (0)$, and thus $H_0(\mathcal{F}_\bullet)/\Gamma_B(H_0(\mathcal{F}_\bullet)) \cong (S/I)/(\Gamma_B(S/I))$. So \mathcal{F}_\bullet is a virtual resolution of S/I by Definition 2.1. \square

For Section 3, we use a result of Berkesch, Erman, and Smith [2] which constructs virtual resolutions out of a minimal free resolution \mathcal{F}_\bullet by deleting certain summands from the F_i . The rough idea of this construction is that we can “throw away” some summands in \mathcal{F}_\bullet , and what remains will be a virtual resolution. We will (informally) refer to this process as *trimming* a minimal free resolution (to obtain a virtual one).

The notion of \underline{d} -regularity, as introduced by Maclagan and Smith [16], is used in the statement of the next theorem. We will not give the definition here because, for the purposes of this paper, it will suffice to use a characterization of the multigraded regularity for points in $\mathbb{P}^1 \times \mathbb{P}^1$ as given in Theorem 2.11 below. We now state a special case of the result in [2, Theorem 1.3], which suffices for our results.

Theorem 2.4. *Let M be a finitely generated \mathbb{Z}^2 -graded B -saturated S -module and let $\underline{d} \in \mathbb{Z}^2$. Suppose M is \underline{d} -regular. If G is the free subcomplex of a minimal free resolution of M consisting of all summands generated in degree at most $\underline{d} + (1, 1)$ with respect to \preceq , then G is a virtual resolution of M .*

In the next section we will apply Theorem 2.4 to the bihomogeneous coordinate rings S/I_X where X is a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Note that going forward, we will assume that all sets of points are finite. As observed above, Theorem 2.4 requires information about the *multigraded regularity* of S/I_X , denoted $\text{reg}_B(S/I_X)$, defined to be the set of all \underline{d} for which S/I_X is \underline{d} -regular. Before giving the relevant statement, which is Theorem 2.11, we need some preliminaries and background concerning Hilbert functions and points in $\mathbb{P}^1 \times \mathbb{P}^1$. See [12] for details.

A point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ has bihomogeneous defining ideal I_P given by

$$I_P = \langle a_1x_0 - a_0x_1, b_1y_0 - b_0y_1 \rangle \subseteq S,$$

which is a prime ideal. For a set of s distinct points $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, the associated bihomogeneous ideal of X is $I_X = I_{P_1} \cap \cdots \cap I_{P_s}$. The coordinate ring of X is S/I_X (see [12, Theorem 3.1, Corollary 3.2] for more details). Let $\text{K-dim}(S/I_X)$ denote the Krull dimension of (S/I_X) . The following facts hold:

Lemma 2.5. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinate ring S/I_X .*

- (1) *There exists $L \in S$ with $\deg(L) = (0, 1)$ such that \bar{L} is a non-zero-divisor in S/I_X .*
- (2) $1 \leq \text{depth}(S/I_X) \leq \text{K-dim}(S/I_X) = 2$.
- (3) *The ideal I_X is B -saturated.*

Proof. For statement (1), [12, Lemma 3.5] proves the case $\deg(L) = (1, 0)$ and the proof can be easily adapted to our case. Statement (2) is [12, Lemma 4.2]. For statement (3), each ideal I_{P_i} is B -saturated, so I_X is B -saturated since the intersection of B -saturated ideals is also saturated. \square

Remark 2.6. By an appropriate change of coordinates, we can assume without loss of generality that the non-zero-divisor of Lemma 2.5 (1) is y_0 , i.e., we can assume that if $P = A \times B \in X$, then $B \neq [0 : 1]$.

We now recall the definition of Hilbert functions, their first and second differences, and some useful facts which we use below. First, for any bihomogeneous ideal $I \subseteq S$, the Hilbert function $H_{S/I}$ of S/I is the function $H_{S/I} : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by

$$H_{S/I}(i, j) := \dim_{\mathbb{k}}(S/I)_{i,j} = \dim_{\mathbb{k}} S_{i,j} - \dim_{\mathbb{k}}[I]_{i,j}.$$

If I_X is the defining ideal of a subvariety X of $\mathbb{P}^1 \times \mathbb{P}^1$, we denote H_{S/I_X} by H_X . If we let $m_{i,j} := H_X(i, j)$ for $(i, j) \in \mathbb{N}^2$, we can view H_X as an infinite matrix $(H_X(i, j)) = (m_{i,j})_{(i,j) \in \mathbb{N}^2}$. The first difference function of H_X is the function $\Delta H_X : \mathbb{N}^2 \rightarrow \mathbb{N}$ given by

$$\Delta H_X(i, j) = m_{i,j} - m_{i-1,j} - m_{i,j-1} + m_{i-1,j-1},$$

where we use convention that $H(i, j) = m_{i,j} = 0$ if $i < 0$ or $j < 0$. If $c_{i,j} := \Delta H_X(i, j)$, then ΔH_X can also be represented by the infinite matrix $(c_{i,j})_{(i,j) \in \mathbb{N}^2}$. The second difference function $\Delta^2 H_X$ is defined to be $\Delta(\Delta H_X)$, and so

$$\Delta^2 H_X(i, j) = c_{i,j} - c_{i-1,j} - c_{i,j-1} + c_{i-1,j-1}.$$

We use the notation $\Delta^2 H_X = (d_{i,j})_{(i,j) \in \mathbb{N}^2}$ in a manner similar to H_X and ΔH_X .

The functions H_X and $\Delta^2 H_X$ capture geometric and algebraic information about a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$. To explain this connection, we introduce some further notation. Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ denote the projection map to the first coordinate. Given a set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, the set $\pi_1(X) = \{A_1, \dots, A_\ell\}$ is the set of distinct first coordinates appearing in X . For each $A_i \in \pi_1(X)$, let $\alpha_i = |\pi_1^{-1}(A_i) \cap X|$, i.e., the number of points of X whose first coordinate is A_i . After relabeling, we can assume that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_\ell$.

We set $\alpha_X = (\alpha_1, \dots, \alpha_\ell)$. Note that the vector α_X is a partition of the integer $|X|$, i.e., $|X| = \alpha_1 + \dots + \alpha_\ell$. The conjugate of α_X , is the partition $\alpha_X^* = (\alpha_1^*, \dots, \alpha_{\alpha_1}^*)$ where $\alpha_i^* = |\{j \in \{1, \dots, \ell\} \mid \alpha_j \geq i\}|$.

We note that $|\pi_1(X)| = \alpha_1^* \geq \alpha_2^* \geq \dots \geq \alpha_{\alpha_1}^*$. The conjugate α_X^* of α_X , which is related to geometric information about X , is then encoded into the Hilbert function. The next theorem (see [12, Theorem 3.29]) describes one such relationship:

Theorem 2.7. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of points with associated tuple $\alpha_X^* = (\alpha_1^*, \dots, \alpha_{\alpha_1}^*)$. For all $i \geq |\pi_1(X)| - 1$,*

$$H_{S/I_X}(i, j) = \alpha_1^* + \dots + \alpha_{j+1}^*$$

where $\alpha_r^* = 0$ if $r > \alpha_1$.

We will require the following corollary.

Corollary 2.8. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of points with associated tuple $\alpha_X^* = (\alpha_1^*, \dots, \alpha_{\alpha_1}^*)$, and let $L \in S$ correspond to the non-zero-divisor of Lemma 2.5. For all $i \geq |\pi_1(X)| - 1$,*

$$H_{S/(I_X + \langle L \rangle)}(i, j) = \alpha_{j+1}^*$$

where $\alpha_r^* = 0$ if $r > \alpha_1$.

Proof. We have a short exact sequence

$$0 \rightarrow S/(I_X + \langle L \rangle)(0, -1) \xrightarrow{\times \bar{L}} S/I_X \rightarrow S/(I_X + \langle L \rangle) \rightarrow 0.$$

But $I_X : \langle L \rangle = I_X$ since L is a non-zero-divisor. The maps in the short exact sequence have degree $(0, 0)$. Thus, since Hilbert functions are additive on short exact sequences,

$$H_{S/(I_X + \langle L \rangle)}(i, j) = H_{S/I_X}(i, j) - H_{S/I_X}(i, j - 1).$$

The conclusion follows from the formula for H_{S/I_X} given in Theorem 2.7. \square

When $\alpha_X = (\alpha_1)$, i.e., the points of X all have the same first coordinate, [12, Lemma 3.26] gives more information about I_X and its Hilbert function:

Lemma 2.9. *Suppose $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of points of the form*

$$X = \{A \times B_1, A \times B_2, \dots, A \times B_\alpha\},$$

with $A = [a_0 : a_1]$ and $B_i = [b_{i,0} : b_{i,1}]$ for $i = 1, \dots, \alpha$. Then

$$(1) \ H_X(i, j) = \begin{cases} j + 1 & \text{if } j < \alpha \\ \alpha & \text{if } j \geq \alpha. \end{cases}$$

$$(2) \ I_X = \langle L_A, L_{B_1} L_{B_2} \cdots L_{B_\alpha} \rangle \text{ where } L_A = a_1 x_0 - a_0 x_1 \text{ and } L_{B_i} = b_{i,1} y_0 - b_{i,0} y_1 \text{ for } i = 1, \dots, \alpha.$$

Remark 2.10. By swapping the roles of the A 's and B 's and the roles of the i 's and j 's, a similar result holds for points of the form $X = \{A_1 \times B, A_2 \times B, \dots, A_\alpha \times B\}$.

We say that a set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ has the *generic Hilbert function* if

$$H_X(i, j) = \min\{|X|, (i + 1)(j + 1)\} \text{ for all } (i, j) \in \mathbb{N}^2.$$

As we saw above, to use Theorem 2.4, we need the multigraded regularity of S/I_X . If X has the generic Hilbert function, the multigraded regularity is easy to compute. The next result is simply a specialization of [16, Proposition 6.7] to the case of points in $\mathbb{P}^1 \times \mathbb{P}^1$:

Theorem 2.11. *If $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of points with the generic Hilbert function, then*

$$\text{reg}_B(S/I_X) = \{(i, j) \mid (i + 1)(j + 1) \geq |X|\}.$$

We now introduce some notation and results concerning the minimal free resolutions of S/I_X . By Lemma 2.5, since $1 \leq \text{depth}(S/I_X) \leq 2$, the Auslander-Buchsbaum Theorem (see [5, Theorem 19.9]) implies that the \mathbb{Z}^2 -graded minimal free resolution of S/I_X has either length 2 or 3. Following the notation of Giuffrida, Maggioni, and Ragusa [8] we can write this resolution as

$$0 \rightarrow \bigoplus_{\ell=1}^p S(-a_{3,\ell}, -a'_{3,\ell}) \hookrightarrow \bigoplus_{\ell=1}^n S(-a_{2,\ell}, -a'_{2,\ell}) \rightarrow \bigoplus_{\ell=1}^m S(-a_{1,\ell}, -a'_{1,\ell}) \rightarrow S \rightarrow S/I_X \rightarrow 0 \quad (2.1)$$

With the above notation, we define

$$\begin{aligned} \alpha_{r,s} &:= \#\{(a_{1,i}, a'_{1,i}) = (r, s)\} \text{ and } \beta_{r,s} := \#\{(a_{2,i}, a'_{2,i}) = (r, s)\} \\ &\text{and } \gamma_{r,s} := \#\{(a_{3,i}, a'_{3,i}) = (r, s)\}. \end{aligned}$$

Thus $\alpha_{r,s}$ is the number of generators of I_X of degree (r, s) , and $\beta_{r,s}$ and $\gamma_{r,s}$ represent the number of first and second syzygies of degree (r, s) , respectively. Giuffrida, Maggioni, and Ragusa [8, Proposition 3.3]

showed a number of relations between $H_X, \Delta H_X$ and $\Delta^2 H_X$ and the invariants in (2.1); we record one such relation in the next theorem.

Theorem 2.12. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with second difference Hilbert function $\Delta^2 H_X = (d_{i,j})$. If $(r, s) \succ (0, 0)$, then $d_{r,s} = -\alpha_{r,s} + \beta_{r,s} - \gamma_{r,s}$.*

We will use another result of Giuffrida, Maggioni, and Ragusa, which (under a certain hypothesis on the set X of points) relates the degrees of the minimal generators of I_X , in particular, the values $\alpha_{i,j}$ defined above, to the integers $d_{i,j}$ in the second difference matrix $\Delta^2 H = (d_{i,j})$. The version stated here is a special case of [9, Theorem 4.3]:

Theorem 2.13. *Let $s \geq 1$ be an integer. There exists a dense open subset U of $(\mathbb{P}^1 \times \mathbb{P}^1)^s$ such that for every $(P_1, \dots, P_s) \in U$, the set of points $X = \{P_1, \dots, P_s\}$ satisfies:*

- (1) X has generic Hilbert function, and
- (2) for any $(i, j) \in \mathbb{N}^2$ such that $d_{i,j} < 0$ and $d_{i,s} > 0$ for some $s > j$, or, $d_{i,j} < 0$ and $d_{r,j} > 0$ for some $r > i$, then we have $\alpha_{i,j} = -d_{i,j}$, i.e., the number of minimal generators of I_X of degree (i, j) is $-d_{i,j}$.

Moreover, every non-zero $\alpha_{i,j}$ arises in the manner described in (2).

Motivated by the above result, a set of s points $X = \{P_1, \dots, P_s\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is said to be in *sufficiently general position* if (P_1, \dots, P_s) belongs to the open set of Theorem 2.13.

Example 2.14. We illustrate some of the ideas of this section with the following running example. Let X be a set of six points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are in sufficiently general position. The Hilbert function of S/I_X is thus the generic Hilbert function. Thus we know H_X , and consequently $\Delta^2 H_X$; these functions are written as infinite matrices below (where rows and columns are index by $\{0, 1, \dots\}$):

$$H_X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 6 & 6 & 6 & 6 \\ 3 & 6 & 6 & 6 & 6 & 6 & 6 \\ 4 & 6 & 6 & 6 & 6 & 6 & 6 \\ \dots & & & & & & \\ 5 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ & & & & \vdots & & \end{bmatrix} \quad \text{and} \quad \Delta^2 H_X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 2 \\ 0 & 0 & -3 & 4 & 0 & 0 & -1 \\ 0 & -2 & 4 & -2 & 0 & 0 & 0 \\ \dots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ & & & & \vdots & & \end{bmatrix}.$$

Theorem 2.13 says we can determine the number of the minimal generators of I_X of any particular bidegree by looking for negative entries in the matrix $\Delta^2 H_X = (d_{i,j})$ which also have a positive entry either to the right of, or below, the negative entry. For our example of six points, these entries are:

- $d_{0,6} = -1 < 0$ and $d_{1,6} = 2 > 0$
- $d_{3,1} = -2 < 0$ and $d_{6,1} = 2 > 0$
- $d_{1,3} = -2 < 0$ and $d_{1,6} = 2 > 0$
- $d_{6,0} = -1 < 0$ and $d_{6,1} = 2 > 0$.
- $d_{2,2} = -3 < 0$ and $d_{2,4} = 4 > 0$

So, the ideal I_X has one minimal generator of degree $(0, 6)$, two minimal generators of degree $(1, 3)$, three minimal generators of degree $(2, 2)$, two minimal generators of degree $(3, 1)$, and one minimal generator of degree $(6, 0)$. This agrees with the bigraded minimal free resolution of S/I_X obtained by using Macaulay2:

$$\begin{array}{ccccccc}
 & & & S(-3, -2)^4 & & & \\
 & & & \oplus & & S(-3, -1)^2 \oplus S(-2, -2)^3 & \\
 S(-3, -3)^2 & & & & & & \\
 \oplus & & & S(-2, -3)^4 & & & \\
 0 \rightarrow S(-6, -2) \rightarrow & & & \oplus & \rightarrow & S(-1, -3)^2 & \rightarrow S \rightarrow S/I_X \rightarrow 0. \quad (2.2) \\
 \oplus & & & S(-1, -6)^2 & & \oplus & \\
 S(-2, -6) & & & \oplus & & S(0, -6) \oplus S(-6, 0) & \\
 & & & S(-6, -1)^2 & & &
 \end{array}$$

Very roughly speaking, a set of s points X in $\mathbb{P}^1 \times \mathbb{P}^1$ is in sufficiently general position if the points are “random” enough. In other words, there should be no subset Y of X that is contained on some curve of smaller than expected degree. One way to interpret Theorem 2.13 is that for a collection of points X that is random enough, the number of generators of I_X and their degrees can be read off the Hilbert function. As the next example shows, simply having the correct Hilbert function does not allow us to count the generators and their degrees from H_X .

Example 2.15. Consider the five points

$$X = \{[1 : 1] \times [1 : 1], [1 : 2] \times [1 : 2], [1 : 3] \times [1 : 3], [1 : 4] \times [1 : 4], [1 : 6] \times [1 : 8]\}.$$

These five points are not in sufficiently general position. Indeed, note that for this set of points, the bihomogeneous element $x_0y_1 - x_1y_0$ of degree $(1, 1)$ passes through the first four points; for a sufficiently general set of points, we would expect no four points of a set of five points to lie on a curve of degree $(1, 1)$.

This set of points also gives us an example of a set of points that satisfies the first condition of Theorem 2.13, but not the second. The Hilbert function of S/I_X is the generic Hilbert function, where the corresponding second difference function is given by

$$H_X = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 5 & & \\ 2 & 4 & 5 & 5 & 5 & 5 & & \\ 3 & 5 & 5 & 5 & 5 & 5 & & \\ 4 & 5 & 5 & 5 & 5 & 5 & \cdots & \\ 5 & 5 & 5 & 5 & 5 & 5 & & \\ 5 & 5 & 5 & 5 & 5 & 5 & & \\ & & & \vdots & & & & \end{bmatrix} \quad \text{and} \quad \Delta^2 H_X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & & \\ 0 & 0 & -1 & -1 & 0 & 2 & & \\ 0 & -1 & 0 & 2 & 0 & -1 & & \\ 0 & -1 & 2 & -1 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \\ -1 & 2 & -1 & 0 & 0 & 0 & & \\ & & & \vdots & & & & \end{bmatrix}.$$

The ideal of I_X contains a generator of degree $(2, 2)$, namely,

$$288x_0^2y_0^2 - 600x_0^2y_0y_1 + 41x_1^2y_0y_1 + 420x_0^2y_1^2 - 161x_0x_1y_1^2 + 12x_1^2y_1^2.$$

However, if X was in sufficiently general position, then the ideal I_X would have no generator of degree $(2, 2)$ by Theorem 2.13 because the $(2, 2)$ entry of $\Delta^2 H_X$ is not negative.

Remark 2.16. Gao, Li, Loper, and Mattoo [7] defined a set of points to be a *virtual complete intersection* if S/I_X has a virtual resolution of the form $K(f, g)$, where $K(f, g)$ denotes the Koszul complex of two bihomogeneous forms f and g . Any finite set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$ that is in sufficiently general position is also a virtual complete intersection. To see this fact, first note that for such a set of points, the first coordinates, respectively second coordinates, of X are distinct, so each horizontal and vertical ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ that intersects X meets at only one point. Because X has the same number of points on each vertical and horizontal ruling, X is now a virtual complete intersection by [7, Theorem 3.1]. In the next section, we will give alternative virtual resolutions for these sets of points.

3. Virtual resolutions via “trimming”

In this section, we give a family of explicitly described virtual resolutions for points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, using the trimming result of Berkesch, Erman, and Smith. Our main result is the following.

Theorem 3.1. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ in sufficiently general position. Assume $|X| > 1$ and suppose that there exists an $(i, j) \in \mathbb{N}^2$ with $i < j$ such that $|X| = (i + 1)(j + 1)$. Then there is a virtual resolution of S/I_X of length two. In fact, the virtual resolution can be described explicitly as follows. Let $|X| = (i + 2)q + r$ with $q, r \in \mathbb{N}$ and $0 \leq r < (i + 2)$.*

Then the virtual resolution has the form

$$\begin{array}{ccccccc}
 & & & & S(-i, -j - 1)^{i+1} & & \\
 & & & & \oplus & & \\
 0 \rightarrow & S(-i - 1, -j - 1)^{2i+2} & \rightarrow & S(-i - 1, -q)^{i+2-r} & \rightarrow & S & \rightarrow S/I_X \rightarrow 0, \\
 & & & \oplus & & & \\
 & & & & S(-i - 1, -q - 1)^r & &
 \end{array}$$

where if $r = 0$, the module $S(-i - 1, -q - 1)^0$ does not appear.

Remark 3.2. If $|X| = 1$, then I_X is a complete intersection and the minimal free resolution of S/I_X

$$0 \rightarrow S(-1, -1) \rightarrow S(-1, 0) \oplus S(0, -1) \rightarrow S \rightarrow S/I_X \rightarrow 0$$

is already of length two. For this reason, we omit this case in the theorem above.

Remark 3.3. For a set X of points with $|X| > 1$ in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$, there always exists at least one tuple (i, j) that satisfies the hypotheses, namely $(i, j) = (0, |X| - 1)$.

As a first step toward the proof of Theorem 3.1, we show that its hypotheses completely determine a submatrix of the second difference Hilbert function $\Delta^2 H_X$. In fact, we can compute this submatrix quite explicitly, as recorded in the lemma below.

Lemma 3.4. *Assume the hypotheses and notation of Theorem 3.1.*

If $i > 0$ and $q < j$, then the $(i + 2) \times (j + 2)$ submatrix of $\Delta^2 H_X$ consisting of the first $(i + 2)$ rows and the first $(j + 2)$ columns of $\Delta^2 H_X$ is of the form:

$$\begin{array}{cccccccc}
 & 0 & 1 & \cdots & q & q + 1 & \cdots & j & j + 1 \\
 0 & \left(\begin{array}{cccccccc}
 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -i - 1 \\
 0 & 0 & \cdots & -i - 2 + r & -r & \cdots & 0 & 2i + 2
 \end{array} \right)
 \end{array}$$

Note that we take the convention that if $j = q + 1$, then we ignore the column labeled j and take the column labeled $q + 1$, and if $i = 1$, then we ignore the row labeled $i - 1$ and take the row labeled 0.

If $i > 0$ and $q = j = i + 1$, then the $(i + 2) \times (i + 3)$ submatrix of $\Delta^2 H_X$ consisting of the first $(i + 2)$ rows and the first $(i + 3) = (j + 2)$ columns of $\Delta^2 H_X$ is of the form:

$$\begin{matrix} & 0 & 1 & \cdots & j-2 & j-1 & j & j+1 \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ i-1 \\ i \\ i+1 \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -i-1 \\ 0 & 0 & \cdots & 0 & 0 & -i-2 & 2i+2 \end{pmatrix} \end{matrix}.$$

We make the convention that if $i = 1$, then we ignore the row labeled $i - 1$ and take the row labeled 0 . Finally, if $i = 0$, then the first two rows and $j + 2$ columns of $\Delta^2 H_X$ are of the form:

$$\begin{matrix} & 0 & 1 & \cdots & q-1 & q & \cdots & j & j+1 \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -2+r & -r & \cdots & 0 & 2 \end{pmatrix} \end{matrix}.$$

Proof. We first consider the case $i > 0$ and $q < j$. Note that $i(j + 2) < |X| = (i + 1)(j + 1)$. Indeed, $i(j + 2) \geq (i + 1)(j + 1)$ implies that $i \geq j + 1$, contradicting our assumption that $i < j$. So, because X has the generic Hilbert function, $H_X(i - 1, j + 1) = i(j + 2)$.

More generally, under the assumptions $i > 0$ and $q < j$ we can compute the first $(i + 2)$ rows and first $(j + 2)$ columns of H_X to be the following:

$$\begin{matrix} & 0 & 1 & \cdots & q-1 & q & \cdots & j-1 & j & j+1 \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ i-1 \\ i \\ i+1 \end{matrix} & \begin{pmatrix} 1 & 2 & \cdots & q & q+1 & \cdots & j & j+1 & j+2 \\ 2 & 4 & \cdots & 2q & 2(q+1) & \cdots & 2j & 2(j+1) & 2(j+2) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ i & (i)2 & \cdots & iq & i(q+1) & \cdots & ij & i(j+1) & i(j+2) \\ i+1 & (i+1)2 & \cdots & (i+1)q & (i+1)(q+1) & \cdots & (i+1)j & |X| & |X| \\ i+2 & (i+2)2 & \cdots & (i+2)q & |X| & \cdots & |X| & |X| & |X| \end{pmatrix} \end{matrix}.$$

Note we are using the fact that our hypotheses imply that

$$H_X(i + 1, q - 1) = \min\{(i + 2)q, |X|\} = \min\{(i + 2)q, (i + 2)q + r\} = (i + 2)q$$

and that $H_X(i + 1, q) = |X|$. The conclusion of the lemma for this special case now follows from the definition of $\Delta^2 H_X$ and the above description of H_X .

The proofs for the case that $i > 0$ and $q = j$ and the case that $i = 0$ are similar and left to the reader. For the case $i > 0$ and $q = j$, note that since $i < j$, $|X| = (i + 2)q + r = (i + 1)(j + 1)$ only occurs if $r = 0$ and $q = j = i + 1$. \square

Using the notation introduced in Section 2, we require the following calculations that compute the number of generators, syzygies, and second syzygies of I_X for specific degrees.

Lemma 3.5. Assume the hypotheses and notation of Theorem 3.1, and let $\Delta^2 H_X = (d_{r,s})$. If $(a, b) \preceq (i + 1, j + 1)$, then

- (1) $\alpha_{a,b} = -d_{a,b}$ if and only if $d_{a,b} < 0$, and $\alpha_{a,b} = 0$ otherwise
- (2) $\beta_{a,b} = \begin{cases} 2i + 2 & \text{if } (a, b) = (i + 1, j + 1) \\ 0 & \text{otherwise} \end{cases}$
- (3) $\gamma_{a,b} = 0$.

Proof. We first prove the formula for $\alpha_{a,b}$, the number of minimal generators of I_X of degree (a, b) . Because X is in sufficiently general position, by Theorem 2.13, $\alpha_{a,b}$ can be read off of $\Delta^2 H_X$. Since we are only interested in all $(a, b) \preceq (i + 1, j + 1)$, we can use Lemma 3.4 to compute all $\alpha_{a,b}$ in this range. In particular, we note that for each $d_{a,b}$ with $d_{a,b} < 0$ in this range, there exists an entry in $\Delta^2 H_X$ that is either to the right or below it which is positive. Applying Theorem 2.13 then gives the desired result.

To prove the formulas for $\beta_{a,b}$ and $\gamma_{a,b}$, we remind the reader that

$$d_{r,s} = -\alpha_{r,s} + \beta_{r,s} - \gamma_{r,s} \text{ for all } (0, 0) \neq (r, s) \in \mathbb{N}^2$$

by Theorem 2.12. We also recall the general fact that if $\beta_{r,s} > 0$ (respectively $\gamma_{r,s} > 0$), then there is some $\alpha_{a,b} > 0$ (respectively, $\beta_{a,b} > 0$) with $(a, b) \prec (r, s)$. This is because $\beta_{r,s}$ records a syzygy among generators of degree strictly less than (r, s) , and similarly, $\gamma_{r,s}$ records a second syzygy among the syzygies of degrees strictly less than (r, s) .

For the discussion below, we always restrict to $(a, b) \preceq (i + 1, j + 1)$. As we have already observed, $\alpha_{a,b} > 0$ exactly at the locations (a, b) such that $d_{a,b} < 0$ and $\alpha_{a,b} = 0$ elsewhere. Let (a, b) be such that $\alpha_{a,b} > 0$ and assume that b is the smallest index with $\alpha_{a,b} > 0$. An examination of the different cases of $\Delta^2 H_X$ in Lemma 3.4 yields the conclusion that $\alpha_{u,v} = 0$ for any $(u, v) \prec (a, b)$, i.e., I_X has no generators of these degrees. This then implies that $\beta_{u,v} = 0$ and hence also $\gamma_{u,v} = 0$ for such (u, v) .

We now examine the locations at which $\alpha_{a,b} > 0$ more closely.

For $(a, b) = (i, j + 1)$, we have

$$-\alpha_{i,j+1} = d_{i,j+1} = -\alpha_{i,j+1} + \beta_{i,j+1} - \gamma_{i,j+1}.$$

So, $\beta_{i,j+1} = \gamma_{i,j+1}$. But now $\beta_{i,j+1} = 0$ since, as we saw above, there is no $\alpha_{a,b} > 0$ with $(a, b) \prec (i, j + 1)$. Consequently, $\gamma_{i,j+1} = 0$. This argument also works to show that $\beta_{i+1,v} = \gamma_{i+1,v} = 0$ as well, for v the minimal index with $\alpha_{i+1,v} > 0$.

Suppose we are in a case when $\alpha_{i+1,v+1} > 0$ where v is the minimal index as above. For $(a, b) = (i + 1, v + 1)$, we again have

$$-\alpha_{i+1,v+1} = d_{i+1,v+1} = -\alpha_{i+1,v+1} + \beta_{i+1,v+1} - \gamma_{i+1,v+1},$$

i.e., $\beta_{i+1,v+1} = \gamma_{i+1,v+1}$. But since we have now seen that there is no $\beta_{a,b} > 0$ for any $(a, b) \prec (i + 1, v + 1)$, we must have $\gamma_{i+1,v+1} = 0$. But this implies $\beta_{i+1,v+1} = 0$ also.

Looking again at the possibilities for $\Delta^2 H_X$ as given in Lemma 3.4, we see that it remains to show that $\beta_{i+1,k} = \gamma_{i+1,k} = 0$ for some values of $k \leq j + 1$. We may proceed inductively. Let k be the minimal index for which we wish to prove the claim and assume $k < j + 1$ so that $d_{i+1,k} = 0$. We already know $\alpha_{i+1,k} = 0$ and hence, by a similar argument as that above, $\beta_{i+1,k} = \gamma_{i+1,k}$. But we have already shown that all $\beta_{a,b} = 0$ for $(a, b) \prec (i + 1, k)$ and hence $\gamma_{i+1,k} = 0$, which implies $\beta_{i+1,k} = 0$. Now by induction the same holds for $\beta_{i+1,k+1}$ and $\gamma_{i+1,k+1}$ and so on, provided that $k < j$.

Finally, we must analyze the case $(a, b) = (i + 1, j + 1)$, where we have $d_{i+1,j+1} = 2i + 2$. Since $\alpha_{i+1,j+1} = 0$, we have

$$2i + 2 = d_{i+1,j+1} = \beta_{i+1,j+1} - \gamma_{i+1,j+1}.$$

But as before, we have already seen that there is no $\beta_{a,b} > 0$ for any $(a, b) \prec (i + 1, j + 1)$. Hence $\gamma_{i+1,j+1} = 0$ and $\beta_{i+1,j+1} = 2i + 2$, as desired. This completes the proof. \square

We now come to the proof of Theorem 3.1.

Proof of Theorem 3.1. Our hypotheses imply that $H_X(i, j) = \min\{(i + 1)(j + 1), |X|\} = (i + 1)(j + 1)$. Thus, by Theorem 2.11, we have that $(i, j) \in \text{reg}_B(S/I_X)$. Hence, when we trim the bigraded minimal free resolution of S/I_X in degrees at most $(i + 1, j + 1)$ using the partial order \preceq , we obtain a virtual resolution of S/I_X by Theorem 2.4.

The bigraded minimal free resolution of S/I_X can be written as

$$0 \rightarrow \bigoplus_{(r,s) \in \mathbb{N}^2} S(-r, -s)^{\gamma_{r,s}} \rightarrow \bigoplus_{(r,s) \in \mathbb{N}^2} S(-r, -s)^{\beta_{r,s}} \rightarrow \bigoplus_{(r,s) \in \mathbb{N}^2} S(-r, -s)^{\alpha_{r,s}} \rightarrow S \rightarrow S/I_X \rightarrow 0.$$

If we now trim this resolution with respect to $(i + 1, j + 1)$, i.e., we keep those summands generated in degrees $\preceq (i + 1, j + 1)$, then the conclusion follows from Lemma 3.5 which explicitly calculates $\alpha_{a,b}, \beta_{a,b}$ and $\gamma_{a,b}$ for all $(a, b) \preceq (i + 1, j + 1)$. \square

Example 3.6. Suppose X is a set of 6 points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$. Then the associated Hilbert function and second difference function can be found in Example 2.14. We first note that $|X| = (1 + 1)(2 + 1)$, so we will take $(i, j) = (1, 2)$. We also have $|X| = 2 \cdot 3 + 0$, so $r = 0$ and $q = 2$ in this case. So, by Theorem 3.1,

$$0 \rightarrow S(-2, -3)^4 \rightarrow S(-2, -2)^3 \oplus S(-1, -3)^2 \rightarrow S \rightarrow S/I_X \rightarrow 0$$

is a virtual resolution of S/I_X . To see why this is true in this special case, note that $H_X(1, 2) = 6$ so $(1, 2) \in \text{reg}_B(S/I_X)$. We are thus trimming the resolution of S/I_X in Example 2.14 using $(1, 2) + (1, 1) = (2, 3)$.

Remark 3.7. Because of the symmetry of H_X , we can make a statement similar to Theorem 3.1 that assumes $j < i$. In this case, the bidegrees need to be swapped and the roles of i and j are also swapped.

Remark 3.8. Computer experiments with Macaulay2 suggest that one might be able to get additional virtual resolutions from $\Delta^2 H_X$. In particular, we have observed that if X is in sufficiently general position with $\Delta^2 H_X = (d_{i,j})$, and if

$$(i, j) \in \min\{(a, b) \mid H_X(a, b) = |X|\},$$

then there exists a virtual resolution of S/I_X of length two of the following form:

$$0 \rightarrow \bigoplus_{\substack{(0,0) \prec (a,b) \preceq (i+1, j+1) \\ d_{a,b} > 0}} S(-a, -b)^{d_{a,b}} \rightarrow \bigoplus_{\substack{(a,b) \preceq (i+1, j+1) \\ d_{a,b} < 0}} S(-a, -b)^{|d_{a,b}|} \rightarrow S \rightarrow S/I_X \rightarrow 0.$$

Note that if $(i + 1)(j + 1) = |X|$, then $(i, j) \in \min\{(a, b) \mid H_X(a, b) = |X|\}$. The above formula agrees with Theorem 3.1.

4. Bounding a result of Berkesch, Erman, and Smith

In Section 3 we used Berkesch, Erman, and Smith’s trimming construction to find a short virtual resolution of S/I_X for a set of points X in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$. For zero-dimensional schemes in

$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, Berkesch, Erman, and Smith also proved the existence of short virtual resolutions using the minimal free resolution of a specific module. We state below a special case of their result (see [2, Theorem 4.1]) for the case of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 4.1. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then for all $a \gg 0$, the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is a virtual resolution of S/I_X of length two.*

The purpose of this section is to improve the above result of Berkesch, Erman, and Smith by giving an explicit bound on the a that satisfies the claim in Theorem 4.1. Specifically, we prove the following result. (Recall that $\pi_1(X)$ denotes the set of distinct first coordinates appearing in X .)

Theorem 4.2. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. For all $a \geq |\pi_1(X)| - 1$, the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is a virtual resolution of S/I_X of length two.*

The key lemma, whose proof we postpone until the end of this section, is the following ideal decomposition.

Lemma 4.3. *Let X be any set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\pi_1(X) = \{A_1, \dots, A_\ell\}$ and $\alpha_k := |\pi_1^{-1}(A_k) \cap X|$. Suppose that y_0 is a non-zero-divisor on S/I_X . Then for any integer $a \geq |\pi_1(X)| - 1$, the equality*

$$\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle = \bigcap_{A_k \in \pi_1(X)} \langle y_0, y_1^{\alpha_k}, L_{A_k} \rangle \cap \langle \langle x_0, x_1 \rangle^a, y_0 \rangle \tag{4.1}$$

is a primary decomposition of $\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle$, where $L_{A_k} = a_{k,1}x_0 - a_{k,0}x_1$ if $A_k = [a_{k,0} : a_{k,1}]$.

Assuming the above lemma, we prove our main result of this section, Theorem 4.2.

Proof of Theorem 4.2. We first claim that for all integers a , the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is a virtual resolution of S/I_X . To see this, note that by Lemma 2.5, the ideal I_X is B -saturated. Moreover, we have the identity

$$(I_X \cap \langle x_0, x_1 \rangle^a) : B^\infty = I_X \quad \text{for all integers } a \geq 0. \tag{4.2}$$

Indeed, because $(I_X \cap \langle x_0, x_1 \rangle^a) \subseteq I_X$, we get $(I_X \cap \langle x_0, x_1 \rangle^a) : B^\infty \subseteq I_X : B^\infty = I_X$, where we are using the fact that I_X is B -saturated by Lemma 2.5 (3). For the reverse inclusion, it is enough to note that $I_X B^a \subseteq I_X \cap B^a \subseteq (I_X \cap \langle x_0, x_1 \rangle^a)$. Our claim now follows from Theorem 2.3.

It now suffices to show that for any $a \geq |\pi_1(X)| - 1$, the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ has length 2. We first note that the Krull dimension of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is 2. This can be seen geometrically, as follows. If we ignore the bigrading, the zero-locus $Z = Z(I_X \cap \langle x_0, x_1 \rangle^a)$ defines an algebraic set in \mathbb{A}^4 . By the Nulstellenstaz,

$$Z = Z(\sqrt{I_X \cap \langle x_0, x_1 \rangle^a}) = Z(\sqrt{I_X} \cap \sqrt{\langle x_0, x_1 \rangle^a}) = Z(I_X \cap \langle x_0, x_1 \rangle)$$

since $I_X = \sqrt{I_X} = \bigcap_{i=1}^s I_{P_i}$ is the intersection of prime ideals. The ideals I_{P_i} for $i = 1, \dots, s$ and the ideal $\langle x_0, x_1 \rangle$ define planes in \mathbb{A}^4 . So, Z is union of planes in \mathbb{A}^4 . Hence $\text{K-dim}(S/(I_X \cap \langle x_0, x_1 \rangle^a)) = \dim Z = 2$. Because $\text{depth}(M) \leq \text{K-dim}(M)$ for any S -module M (e.g., see [18, Lemma 2.3.6]), we conclude that $\text{depth}(S/(I_X \cap \langle x_0, x_1 \rangle^a)) \leq 2$.

To complete the proof, it suffices to show that for all $a \geq |\pi_1(X)| - 1$, the depth is exactly two. Indeed, if we can show this fact, then the Auslander-Buchsbaum Theorem implies that the projective dimension of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is two, implying (by definition) that its minimal free resolution is of length 2, as desired.

We now show that for all $a \geq |\pi_1(X)| - 1$, the depth is exactly two. By Remark 2.6, we can assume that y_0 is a non-zero-divisor of S/I_X . In fact, y_0 is also a non-zero-divisor of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ because if $y_0H \in (I_X \cap \langle x_0, x_1 \rangle^a)$, we have $y_0H \in I_X$ and $y_0H \in \langle x_0, x_1 \rangle^a$, and this can only happen if $H \in I_X \cap \langle x_0, x_1 \rangle^a$.

Next, we claim that the depth of $S/(I_X \cap \langle x_0, x_1 \rangle^a, y_0)$ is not zero. This would show that there is a regular sequence of at least 2 in $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ (the first element of which is y_0), which would prove that the depth is ≥ 2 as claimed. Because $a \geq |\pi_1(X)| - 1$, Lemma 4.3 gives a primary decomposition of $\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle$. The zerodivisors of $S/\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle$ are the elements in the union of the associated primes (e.g., see [18, Lemma 2.1.19]), that is, the elements of the set

$$\left(\bigcup_{A_k \in \pi_1(X)} \langle y_0, y_1, L_{A_k} \rangle \right) \cup \langle x_0, x_1, y_0 \rangle. \tag{4.3}$$

Let L be any element with $\deg L = (1, 0)$ such that L does not vanish at any point in $\pi_1(X)$. Then $L + y_1$ is not in the union (4.3), so $L + y_1$ is a non-zero-divisor of $S/\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle$. As we saw above, this implies that $\text{depth}(S/\langle I_X \cap \langle x_0, x_1 \rangle^a \rangle) \geq 2$. Putting this together with the fact that $\text{depth}(S/\langle I_X \cap \langle x_0, x_1 \rangle^a \rangle) \leq \text{K-dim}(S/\langle I_X \cap \langle x_0, x_1 \rangle^a \rangle) = 2$ we obtain that the depth is 2, as desired. This now completes the proof of the theorem. \square

It remains to prove Lemma 4.3, which we now do.

Proof of Lemma 4.3. We start with some observations about our set of points X and associated defining ideal I_X . By definition of the α_k 's, for each $A_k \in \pi_1(X)$, there exist α_k many points $\{B_{k,1}, \dots, B_{k,\alpha_k}\}$ in \mathbb{P}^1 such that

$$X_k := \{A_k \times B_{k,1}, A_k \times B_{k,2}, \dots, A_k \times B_{k,\alpha_k}\} \subseteq X$$

and all other points in X have first coordinate differing from A_k . Thus X_k is exactly the subset of X consisting of points with first coordinate equal to A_k . Consequently, $I_X = \bigcap_{k=1}^\ell I_{X_k}$. By Lemma 2.9, for each k , $I_{X_k} = \langle F_k, G_k \rangle$ where $\deg F_k = (1, 0)$ and $\deg G_k = (0, \alpha_k)$. Furthermore, by Remark 2.6, we can assume that no second coordinate has the form $[0 : 1]$, and consequently, by Lemma 2.9 we have $y_0 \nmid G_k$ for all k .

In general, for any three ideals I, J, K , we always have $(I \cap J) + K \subseteq (I + K) \cap (J + K)$. It follows that for all integers $a \geq 0$

$$\begin{aligned} \langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle &= \langle I_{X_1} \cap \dots \cap I_{X_\ell} \cap \langle x_0, x_1 \rangle^a, y_0 \rangle \\ &\subseteq \langle I_{X_1}, y_0 \rangle \cap \dots \cap \langle I_{X_\ell}, y_0 \rangle \cap \langle \langle x_0, x_1 \rangle^a, y_0 \rangle. \end{aligned} \tag{4.4}$$

From Lemma 2.9 we know that for any k we have

$$\langle I_{X_k}, y_0 \rangle = \langle L_{A_k}, L_{B_{k,1}} \cdots L_{B_{k,\alpha_k}}, y_0 \rangle = \langle y_0, y_1^{\alpha_k}, L_{A_k} \rangle \tag{4.5}$$

where the second equality holds because $L_{B_{k,1}} \cdots L_{B_{k,\alpha_k}}$ is a homogeneous polynomial of degree α_k only in the variables y_0 and y_1 , and the coefficient of $y_1^{\alpha_k}$ is not zero. Consequently, combining (4.4) and (4.5), it follows that the LHS of (4.1) is contained in its RHS, for any $a \geq 0$.

We now wish to show the reverse containment, i.e. the RHS of (4.1) is contained in the LHS, for all $a \geq |\pi_1(X)| - 1$. Since both ideals are bihomogeneous, it suffices to show the inclusion on each (i, j) -th graded piece, that is, to show that

$$\left[\bigcap_{A_k \in \pi_1(X)} \langle y_0, y_1^{\alpha_i}, L_{A_k} \rangle \cap \langle \langle x_0, x_1 \rangle^a, y_0 \rangle \right]_{i,j} \subseteq [\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle]_{i,j} \tag{4.6}$$

for all $(i, j) \in \mathbb{N}^2$. We now consider two cases: (1) $0 \leq i < a$ and (2) $a \leq i$.

We first argue for case (1). Suppose that $0 \leq i < a$ and suppose that F lies in the LHS of (4.6). In particular, $F \in \langle \langle x_0, x_1 \rangle^a, y_0 \rangle$. Since $0 \leq i < a$ by assumption, we conclude that $F \in \langle y_0 \rangle$. But then $F \in [\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle]_{i,j}$ also, so (4.6) holds.

For case (2), we have $a \leq i$, so $[\langle x_0, x_1 \rangle^a]_{i,j} = S_{i,j}$. Consequently

$$\langle \langle x_0, x_1 \rangle^a, y_0 \rangle_{i,j} = S_{i,j} \quad \text{and} \quad [\langle I_X \cap \langle x_0, x_1 \rangle^a, y_0 \rangle]_{i,j} = [\langle I_X, y_0 \rangle]_{i,j}.$$

Thus, to prove (4.6) in this case it suffices to check that

$$\left[\bigcap_{A_k \in \pi_1(X)} \langle y_0, y_1^{\alpha_k}, L_{A_k} \rangle \right]_{i,j} \subseteq [\langle I_X, y_0 \rangle]_{i,j}. \tag{4.7}$$

We now analyze the ideals appearing in the intersection in the LHS of (4.7) individually. Fix a k with $1 \leq k \leq \ell$. We consider cases. Suppose first that $\alpha_k \leq j$. In this case $[\langle y_0, y_1^{\alpha_k}, L_{A_k} \rangle]_{i,j} = S_{i,j}$ since every monomial of degree (i, j) with $j \geq \alpha_k$ is either divisible by y_0 or $y_1^{\alpha_k}$. On the other hand, suppose that $j < \alpha_k$. Then we have

$$[\langle y_0, y_1^{\alpha_i}, L_{A_i} \rangle]_{i,j} = [\langle y_0, L_{A_i} \rangle]_{i,j}$$

since no monomial containing y^{α_k} contributes to the (i, j) -th graded piece of the ideal if $\alpha_k > j$. In summary, we have shown that to prove (4.7) it suffices to show

$$\left[\bigcap_{\substack{A_k \in \pi_1(X) \\ \alpha_k > j}} \langle y_0, L_{A_k} \rangle \right]_{i,j} \subseteq [\langle I_X, y_0 \rangle]_{i,j}. \tag{4.8}$$

Because the vector space on the RHS of (4.8) is a subspace of the vector space on the LHS, it is enough to show that the two vector spaces in (4.8) have the same dimension.

The ideal on the LHS of (4.8) is the ideal of the points

$$Y = \{A_k \times [0 : 1] \mid \alpha_k > j\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1.$$

The number of points in this set is the number of $\alpha_k \in \alpha_X$ with $\alpha_k \geq j + 1$ which is by definition α_{j+1}^* . Moreover, by Lemma 2.9 (and Remark 2.10), we have $H_Y(i, j) = \alpha_{j+1}^*$ since $i \geq a \geq |\pi(X)| - 1 = \alpha_1^* - 1$ by assumption and $\alpha_1^* \geq \alpha_{j+1}^*$ by properties of partitions. On the other hand, because y_0 is a non-zero-divisor and since $i \geq |\pi_1(X)| - 1$, by Corollary 2.8 we also have

$$H_{S/\langle I_X, y_0 \rangle}(i, j) = \alpha_{j+1}^*.$$

Since $H_Y(i, j) = \dim_{\mathbb{k}} S_{i,j} - \dim_{\mathbb{k}} [I_Y]_{i,j}$ and $H_{S/\langle I_X, y_0 \rangle}(i, j) = \dim_{\mathbb{k}} S_{i,j} - \dim_{\mathbb{k}} [\langle I_X, y_0 \rangle]_{i,j}$, we can conclude that the two vector spaces in (4.8) have the same dimension, as desired.

To complete the proof, we show that all the ideals on the RHS are primary. Recall that a monomial ideal $Q = \langle m_1, \dots, m_t \rangle$ in a polynomial ring $\mathbb{k}[z_1, \dots, z_n]$ is a primary ideal if and only if, after a permutation

of the variables, $Q = \langle z_1^{a_1}, \dots, z_r^{a_r}, m'_1, \dots, m'_p \rangle$ and the only variables that divide m'_1, \dots, m'_p are z_1, \dots, z_r (e.g., see [18, Proposition 6.1.7]).

It follows that the ideal $\langle \langle x_0, x_1 \rangle^a, y_0 \rangle$ is a primary monomial ideal, since the generators are x_0^a, x_1^a, y_0 and monomials of the form $x_0^b x_1^c$ with $b + c = a$ and $b, c > 0$. For the ideals $\langle y_0, y_1^{\alpha_k}, L_{A_k} = a_{k,1}x_0 - a_{k,0}x_1 \rangle$, if $a_{k,0} = 0$, then this is also a primary monomial ideal. If $a_{k,0} \neq 0$, then

$$S/\langle y_0, y_1^{\alpha_k}, a_{k,1}x_0 - a_{k,0}x_1 \rangle = S/\langle y_0, y_1^{\alpha_k}, \frac{a_{k,1}}{a_{k,0}}x_0 - x_1 \rangle \cong \mathbb{k}[x_0, y_0, y_1]/\langle y_0, y_1^{\alpha_k}, x_0 \rangle.$$

Since $\langle y_0, y_1^{\alpha_k}, x_0 \rangle$ is primary, so is $\langle y_0, y_1^{\alpha_k}, L_{A_i} \rangle$. \square

The *virtual dimension* of a graded S -module M , denoted $\text{vdim}(M)$, is the minimum length of a virtual resolution of M . By [2, Proposition 2.5], we always have $\text{vdim}(M) \geq \text{codim}(M)$. We say M is *virtually Cohen-Macaulay* if $\text{vdim}(M) = \text{codim}(M)$. Our main result implies the following corollary.

Corollary 4.4. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then S/I_X is virtually Cohen-Macaulay.*

Proof. For any finite set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$, we have $\text{codim}(S/I_X) = 2$. The result now follows from Theorem 4.2 which shows $\text{vdim}(S/I_X) \leq 2$. \square

Remark 4.5. The above fact was already observed in [2] (in particular, see the comments after [2, Theorem 4.1]), although the terminology of virtually Cohen-Macaulay was not used. It was also observed in [2, Corollary 4.2] that the ring $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is Cohen-Macaulay for $a \gg 0$. This result can also be recovered from Theorem 4.2; in fact, our proof shows that one can take $a \geq |\pi_1(X)| - 1$. The virtually Cohen-Macaulay property is further explored in [3].

We conclude with some examples which illustrate our results. Our first example shows that the uniform bound of Theorem 4.2 is optimal.

Example 4.6. Let X be a set of six points with generic Hilbert function, as in Example 2.14. For this example, $|\pi_1(X)| - 1 = 5$, and thus, by Theorem 4.2, the minimal graded free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ has length two for all $a \geq 5$. A direct computation using a computer algebra system shows that $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ has a minimal graded free resolution of length three for $0 \leq a < 5$. For example, when $a = 4$, Macaulay2 yields the minimal free resolution:

$$0 \rightarrow S \rightarrow S^6 \rightarrow S^6 \rightarrow S \rightarrow S/(I_X \cap \langle x_0, x_1 \rangle^4) \rightarrow 0,$$

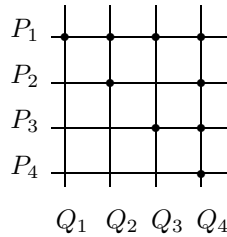
where we have dropped the bigraded shifts for readability. Thus, the uniform bound for all points in $\mathbb{P}^1 \times \mathbb{P}^1$ of Theorem 4.2 cannot be improved.

For a specific set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$, however, it is possible that the smallest integer a such that $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ has a minimal resolution of length two may be strictly smaller than $|\pi_1(X)| - 1$. For example, if X has the property that S/I_X is a Cohen-Macaulay ring (such X are classified in [6,8,11,12]), then $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ has a minimal graded free resolution of length two for all $a \geq 0$. Below, we give another example of this phenomenon.

Example 4.7. Let P_1, P_2, P_3, P_4 be four distinct points in \mathbb{P}^1 and let Q_1, Q_2, Q_3, Q_4 be another collection of four distinct points in \mathbb{P}^1 (we allow for the case that $P_i = Q_j$ for some i and j). Consider the following set of nine points in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$X = \{P_1 \times Q_1, P_1 \times Q_2, P_1 \times Q_3, P_1 \times Q_4, P_2 \times Q_2, P_2 \times Q_4, P_3 \times Q_3, P_3 \times Q_4, P_4 \times Q_4\}.$$

We can visualize these points as follows:



Consequently, $|\pi(X)| - 1 = 4 - 1 = 3$.

On the other hand, using Macaulay2, we find that the smallest a for which $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ has minimal resolution of length two is $a = 2 < |\pi_1(X)| - 1$.

We round out this paper with some observations and questions.

Remark 4.8. It is well-known that for a set of points in Z in \mathbb{P}^1 , the Castelnuovo-Mumford regularity of R/I_Z (where I_Z is the defining ideal of Z in $R = \mathbb{k}[x_0, x_1]$) is given by $\text{reg}(R/I_Z) = |Z| - 1$. If we combine this fact with Theorem 4.2, we can rewrite the bound in terms of the regularity of the set of points in the projection, i.e., $a \geq \text{reg}(R/I_Z)$ where $Z = \pi_1(X) \subseteq \mathbb{P}^1$. This suggests that one way to generalize Theorem 4.2 to $\mathbb{P}^n \times \mathbb{P}^m$ (or even $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$) is to consider the regularity of R/I_Z where $Z = \pi_1(X) \subseteq \mathbb{P}^n$ (or the multigraded regularity in the case of more than two projective spaces).

Remark 4.9. Example 4.7 shows that for some specific configurations of points in $\mathbb{P}^1 \times \mathbb{P}^1$, the bound in Theorem 4.2 can be improved. For example, if we know that the set of points has the property that S/I_X is a Cohen-Macaulay ring, then we can take $a = 0$. Moreover, these points can be classified geometrically. The points in Example 4.7 do not have the Cohen-Macaulay property, but they are not as “sparse” as a set of points with generic Hilbert function. It would be interesting to determine what conditions on the geometry of the points allow us to improve our bounds on the existence of short virtual resolutions.

References

- [1] A. Almousa, J. Bruce, M.C. Loper, M. Sayrafi, The virtual resolutions package for Macaulay2, *J. Softw. Algebra Geom.* 10 (2020) 51–60.
- [2] C. Berkesch, D. Erman, G.G. Smith, Virtual resolutions for a product of projective spaces, *Algebr. Geom.* 7 (2020) 460–481.
- [3] C. Berkesch, P. Klein, M.C. Loper, J. Yang, Homological and combinatorial aspects of virtually Cohen–Macaulay sheaves, *Trans. Lond. Math. Soc.* 8 (2021) 413–434.
- [4] C. Booms-Peot, J. Cobb, Virtual criterion for generalized Eagon–Northcott complexes, *J. Pure Appl. Algebra* 226 (12) (2022) 107138.
- [5] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.
- [6] G. Favacchio, E. Guardo, J. Migliore, On the arithmetically Cohen-Macaulay property for sets of points in multiprojective spaces, *Proc. Am. Math. Soc.* 146 (2018) 2811–2825.
- [7] J. Gao, Yutong Li, M.C. Loper, A. Mattoo, Virtual complete intersections in $\mathbb{P}^1 \times \mathbb{P}^1$, *J. Pure Appl. Algebra* 225 (1) (2021) 106473.
- [8] S. Giuffrida, R. Maffioni, A. Ragusa, On the postulation of 0-dimensional subschemes on a smooth quadric, *Pac. J. Math.* 155 (2) (1992) 251–282.
- [9] S. Giuffrida, R. Maffioni, A. Ragusa, Resolutions of generic points lying on a smooth quadric, *Manuscr. Math.* 91 (4) (1996) 421–444.
- [10] D.R. Grayson, M.E. Stillman, Macaulay2, a software system for research in algebraic geometry, <http://www.math.uiuc.edu/Macaulay2/>.
- [11] E. Guardo, Fat points schemes on a smooth quadric, *J. Pure Appl. Algebra* 162 (2001) 183–208.

- [12] E. Guardo, A. Van Tuyl, *Arithmetically Cohen-Macaulay Sets of Points in $\mathbb{P}^1 \times \mathbb{P}^1$* , Springer Briefs in Mathematics, Springer, Cham, 2015.
- [13] J. Kennedy, *An algebraic condition for a complex to be virtual*, MSc Thesis, McMaster University, 2020.
- [14] N. Kenshur, F. Lin, S. McNally, Z. Xu, T. Yu, *On virtually Cohen-Macaulay simplicial complexes*, preprint, arXiv: 2007.09443, 2020.
- [15] M.C. Loper, *What makes a complex a virtual resolution?*, *Trans. Am. Math. Soc. Ser. B* 8 (2021) 885–898.
- [16] D. Maclagan, G.G. Smith, *Multigraded Castelnuovo-Mumford regularity*, *J. Reine Angew. Math.* 571 (2004) 179–212.
- [17] M. Nowroozi, *Virtual Resolutions of Points in Sufficiently General Position in $\mathbb{P}^1 \times \mathbb{P}^1$* , MSc Thesis, McMaster University, 2021.
- [18] R. Villarreal, *Monomial Algebras*, second edition, *Monographs and Research Notes in Mathematics*, CRC Press, Boca Raton, FL, 2015.
- [19] J. Yang, *Virtual resolutions of monomial ideals on toric varieties*, *Proc. Am. Math. Soc. Ser. B* 8 (2021) 100–111.