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Linear Non-Autonomous Heat Flow in $L^1_0(\mathbb{R}^d)$ and Applications to Elliptic Equations in $\mathbb{R}^d$

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Abstract
We study solutions of the equation $u_t - \Delta u + \lambda u = f$, for initial data that is 'large at infinity' as treated in our previous papers on the unforced heat equation. When $f = 0$ we characterise those (u_0, λ) for which solutions converge to 0 as $t \to \infty$, as not every $\lambda > 0$ is able to achieve that for all initial data. When $f \neq 0$ we give conditions to guarantee that the solution is given by the usual 'variation of constants formula' $u(t) = \mathrm{e}^{-\lambda t} S(t)u_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)\,\mathrm{d}s$, where $S(\cdot)$ is the heat semigroup. We use these results to treat the elliptic problem $-\Delta u + \lambda u = f$ when f is allowed to be 'large at infinity', giving conditions under which a solution exists that is given by convolution with the usual Green's function for the problem. Many of our results are sharp when $u_0, f \geq 0$.

Keywords Heat equation · Large solutions · Blow-up · Global solutions · Regularity of elliptic problem

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1 Introduction

In previous papers, [4–6], we considered solutions of the heat equation

$$u_t - \Delta u = 0, \qquad x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x) \tag{1.1}$$

for some optimal classes of initial data that allow large values at infinity; despite the large growth of these functions at infinity, solutions can still be written using the heat kernel,

$$u(x,t) = S(t)u_0(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d,\ t > 0. \tag{1.2}$$

The paper [4] arose from the aim to identify the optimal class of initial data that gives rise to a global classical solution and to analyse the long-time behaviour of such solutions. However, our analysis also exposed some surprising features of this canonical linear model that are more in line with nonlinear systems. In particular we showed that the equations can exhibit finite-time blowup, and that solutions that exist for all times can have wild oscillatory behaviour as $t \to \infty$. All of these phenomena are due to a mechanism of *mass coming from infinity*, as initial data have stored a lot of mass as $|x| \to \infty$. As time evolves this mass diffuses to bounded sets in $\mathbb{R}^d$ producing the nonlinear-like behaviour of solutions.

Those classes of initial data are given by the family of spaces

$$L^1_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)|\,\mathrm{d}x < \infty \right\},$$

which are Banach spaces when equipped with the norm

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)|\,\mathrm{d}x.$$

Similar spaces of measures can be also considered, see (2.3) in Sect. 2.

A relevant feature of the solutions thus considered is that we cannot work in general with a single space $L^1_\varepsilon(\mathbb{R}^d)$ as the solution can only be estimated, for $t > 0$, in another space $L^1_{\varepsilon(t)}(\mathbb{R}^d)$ with $\varepsilon(t) := \frac{1}{1-4\varepsilon t}$, see Sect. 2. These estimates are optimal.

We proved then that global solutions are obtained provided that the initial data belongs to the Frechet space

$$u_0 \in L^1_0(\mathbb{R}^d) := \bigcap_{\varepsilon>0} L^1_\varepsilon(\mathbb{R}^d). \tag{1.3}$$

This sufficient condition is also necessary for non-negative $0 \le u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Several properties of the heat semigroup in this space have been analysed in [5], including suitable $L^p - L^q$ type estimates using the family of spaces

$$L^p_\varepsilon(\mathbb{R}^d) := \{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)|^p\,\mathrm{d}x < \infty \}$$

with norm

$$\|f\|_{L^p_\varepsilon} := \left(\frac{\varepsilon}{\pi}\right)^{d/2p} \left(\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2} |f(x)|^p\,\mathrm{d}x \right)^{1/p}$$

(with a suitable definition for $p = \infty$) and the Frechet spaces

$$L^p_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^p_\varepsilon(\mathbb{R}^d), \qquad \text{and} \qquad L^\infty_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^\infty_\varepsilon(\mathbb{R}^d).$$

Those estimates in [5] have been improved later in [6]. For example, for $u_0 \in L^p_0(\mathbb{R}^d)$ with $1 \le p < \infty$ then for any $1 \le p \le q < \infty$, then the solution (1.2) satisfies $u(t) \in L^q_0(\mathbb{R}^d)$ for all $t > 0$ and for any $\delta > 0$, $t > 0$ we have

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le c_{p,q}\Big(\frac{1+4p\delta t}{4p\delta t}\Big)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)} \tag{1.4}$$

for $\delta_p(t) = \frac{p\delta}{1+4p\delta t}$. Notice again that to estimate the solution in one space, at a given time t, we need to estimate the initial data in another space that depends on t. See Sect. 2 for full details. This improved estimates from [6] will be crucial in what follows.

We use these estimates here, starting in Sect. 3, where we give conditions on the initial data for the solution of the problem

$$u_t - \Delta u + \lambda u = 0,\ x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x). \tag{1.5}$$

to decay to zero as $t \to \infty$. In contrast with the heat flow in more standard spaces, we will see below that not every $\lambda > 0$ is capable of producing decay in all solutions of (1.5). Actually, we will prove that for functions that grow slower than any exponential (slower than $\mathrm{e}^{c|x|}$, for every $c > 0$) as $|x| \to \infty$, the additional dissipative term λu in the equation is able to bring the solution to zero as $t \to \infty$. For functions that grow like $\mathrm{e}^{c|x|}$ then we need to take $\lambda > c^2$ and this result is sharp. Finally for functions that grow faster than any exponential (faster than $\mathrm{e}^{c|x|}$, for every $c > 0$), there is no $\lambda > 0$ that can bring the solution to zero as $t \to \infty$.

In Sect. 4 we analyse solutions of the non-homogeneous problem

$$u_t - \Delta u + \lambda u = f,\ x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x) \in L^1_\varepsilon(\mathbb{R}^d) \tag{1.6}$$

with $\lambda \in \mathbb{R}$ and a given f, in $[0, T]$, such that

$$[0, T] \ni t \mapsto f(t) \in L^1_{\varepsilon(t)}(\mathbb{R}^d)$$

for some $\varepsilon : [0, T] \to (0, \infty)$. We also consider the case in which u_0 and f are only measures.

In more standard settings we would expect that the solution of (1.6) is given by

$$u(t) = \mathrm{e}^{-\lambda t} S(t)u_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)\, \mathrm{d}s. \tag{1.7}$$

Here we show that (1.7) actually provides a suitable solution of (1.6) in the non-standard setting of this paper. Since the term $\mathrm{e}^{-\lambda t} S(t)u_0$, has already been dealt with in [4, 5] and Sect. 3 we will concentrate in Sect. 4 on the expression

$$U(t) := U(t, f) = \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)\, \mathrm{d}s. \tag{1.8}$$

In Theorem 4.1 we prove that if $f \in L^1((0, T), L^1_{\varepsilon(\cdot)}(\mathbb{R}^d))$ then there exist $0 < T_0 \le T$ and $\delta(t) > 0$ for $0 < t < T_0$ such that $U(t) \in L^1_{\delta(t)}(\mathbb{R}^d)$ and for any $\tau < T_0$

$$\sup_{0\le t\le\tau} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le C(\tau)\|f\|_{L^1((0,\tau),L^1_{\varepsilon(\cdot)})}. \tag{1.9}$$

In particular, $U \in C([0, T_0), L^1_{\delta(\cdot)}(\mathbb{R}^d))$, in the sense that for any fixed $0 < t < T_0$ and $\tilde{\delta} > \delta(t)$ we have, as $s \to t$, $U(s) \to U(t)$ in $L^1_{\tilde{\delta}}(\mathbb{R}^d)$ and

$$\lim_{t\to 0} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} = 0, \tag{1.10}$$

and is a very weak solution of (1.6) in $(0, T_0)$.

Notice again that this result allows $f(t)$ to belong to different spaces, varying with time, and that $U(t)$ is also estimated in a space that changes with time.

Then we prove in Theorem 4.4 that stronger assumptions on f allows us to obtain a strong solution of (1.6). Indeed we show that if $\varepsilon(s) = \varepsilon > 0$ constant and $f \in C^1([0, T], L^1_\varepsilon(\mathbb{R}^d))$, then for $0 < t < T_0$ and any $\delta > \delta(t) = \frac{\varepsilon}{1-4\varepsilon t}$ we have that U is differentiable, at t, in $L^1_\delta(\mathbb{R}^d)$, $-\Delta U(t) \in L^1_\delta(\mathbb{R}^d)$ and satisfies (1.6). Further regularising estimates on U and ∇U are obtained in Sect. 4.2.

In Sect. 4.3 we give conditions for U to be a global solution, that is, defined for as long f is defined, i.e. $T_0 = T$. We also obtain global estimates of U and its gradient in terms of f. Then, in Sect. 4.4, we analyse the possibility that U blows up in finite time. This is possible due to the blow–up results for the solutions of (1.1) in [4] described above.

Our Theorem 4.12 roughly states that if $f \geq 0$ and $T_0 < T$, then U cannot be defined at any point $x \in \mathbb{R}^d$ beyond T_0. Then Proposition 4.14 considers the case when $0 \leq f \in L^1_\varepsilon(\mathbb{R}^d)$ is independent of t and then characterized the points $x \in \mathbb{R}^d$ such that the pointwise limit

$$\lim_{t \to T} U(x, t) \tag{1.11}$$

exists, where $T = \frac{1}{4\varepsilon}$ is the time at which $S(t)f$ ceases to exist.

Finally in Sect. 5 we use these results to find conditions on f that guarantee the existence of solutions of the elliptic problem

$$-\Delta u_* + \lambda u_* = f \tag{1.12}$$

with $\lambda \geq 0$ and $f \in L^1_0(\mathbb{R}^d)$ (we also consider the case when f is only a measure). First observe that since we are dealing with functions that can be very large at infinity, there is no uniqueness for (1.12) for any value of λ. Indeed as observed in [4] for any $\lambda \in \mathbb{R}$, we have nontrivial solutions $\varphi \in L^1_0(\mathbb{R}^d)$ of $-\Delta\varphi + \lambda\varphi = 0$. Indeed, existence for (1.12) will not be achieved for every $f \in L^1_0(\mathbb{R}^d)$, as some restriction on the behavior of the $L^1_\varepsilon(\mathbb{R}^d)$ norms of f, for ε small will be required.

If we had a standard semigroup in a Banach space, we would expect that a solution of (1.12) would be given by

$$u_* = u_*(f) = A^{-1}f = \int_0^\infty e^{-\lambda t} S(t) f \, dt.$$

In Theorem 5.1 we show that, for $\lambda > 0$ (or $\lambda = 0$ and $d \geq 3$), if $f \in L^1_0(\mathbb{R}^d)$ satisfies

$$\int_{0^+} \frac{e^{-\frac{\lambda}{4s}}}{s^2} \|f\|_{L^1_s(\mathbb{R}^d)} \, ds < \infty \tag{1.13}$$

then $u_* \in L^1_0(\mathbb{R}^d)$, is a very weak solution of (1.12) and moreover $u_* \in L^q_0(\mathbb{R}^d)$ for any $1 \leq q < \frac{d}{(d-2)_+}$. A slighter stronger condition than (1.13) allows us to obtain that $\nabla u_* \in L^1_0(\mathbb{R}^d)$, u_* is a weak solution of (1.12), and moreover, $\nabla u_* \in L^r_0(\mathbb{R}^d)$ for any $1 \leq r < \frac{d}{(d-1)_+}$. Our results also imply that the mapping $f \mapsto u_*$ is continuous in suitable norms.

In Theorem 5.5 we prove then that u_* can be represented as the convolution of f with the Green's function, as in the classical results for the Poisson's equation. But, even more, conversely it states that, for $f \geq 0$, if a solution of (1.12) can be represented using the Green's function, then f must satisfy (1.13) and the solution must be u_*. More precisely, we prove the following. Assume that $0 \le f \in L^1_{\rm loc}(\mathbb{R}^d)$ and we define for $\lambda > 0$ (or $\lambda = 0$ and $d \ge 3$)

$$U(x) = \int_{\mathbb{R}^d} G_\lambda(x-y) f(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^d$$

where G_λ is the Green's function for $-\Delta + \lambda I$ in $\mathbb{R}^d$. Then, if there exist $x_0 \in \mathbb{R}^d$ such that $U(x_0) < \infty$ then $f \in L^1_0(\mathbb{R}^d)$, satisfies (1.13), $U(x)$ is finite for a.e. $x \in \mathbb{R}^d$, $U \in L^1_0(\mathbb{R}^d)$ and $U = u_*$ as above.

The rest of Sect. 5 is devoted to completing the elliptic theory for (1.12) by proving results of the type: if $f \in L^p_0(\mathbb{R}^d)$ $1 \le p < \infty$ (plus some condition like (1.13)) then $\nabla u_* \in L^r_0(\mathbb{R}^d)$ for any $r < \frac{pd}{(d-p)_+}$. If $p > d$ then we can take $r = \infty$ as well. In particular if $p > \frac{d}{2}$ then $u_* \in C(\mathbb{R}^d)$. See Corollary 5.12.

2 Some Previous Results

In two previous papers [4, 5] we have developed a theory to treat solutions of the heat equation

$$u_t - \Delta u = 0,\ x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x), \tag{2.1}$$

on the whole of $\mathbb{R}^d$ for initial data that is large at infinity (and may be a Radon measure). In [4] we identified the largest class of initial data for which solutions can be given in terms of the heat kernel

$$u(x,t,u_0) = S(t)u_0(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}u_0(y), \tag{2.2}$$

where $u_0 \in \mathcal{M}_{\rm loc}(\mathbb{R}^d)$ is a Radon measure.

Our analysis makes use of various spaces of functions that are large at infinity. For every $\varepsilon > 0$ we define the space $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ of measures as

$$\mathcal{M}_\varepsilon(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}_{\rm loc}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\,\mathrm{d}|\mu(x)| < \infty \right\}; \tag{2.3}$$

i.e. $\mathrm{e}^{-\varepsilon|x|^2} \in L^1(\mathrm{d}|\mu|)$, with the norm

$$\|\mu\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} := \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\,\mathrm{d}|\mu(x)|. \tag{2.4}$$

For every $\varepsilon > 0$ set

$$\rho_\varepsilon(x) = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{-\varepsilon|x|^2}, \qquad \int_{\mathbb{R}^d} \rho_\varepsilon(x)\,\mathrm{d}x = 1, \tag{2.5}$$

and then, for $0 < \varepsilon_1 < \varepsilon_2$,

$$\rho_{\varepsilon_2}(x) \le \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{d/2} \rho_{\varepsilon_1}(x); \qquad x \in \mathbb{R}^d. \tag{2.6}$$

Hence, the norms (2.4) satisfy for $0 < \varepsilon_1 < \varepsilon_2$

$$\|\mu\|_{\mathcal{M}_{\varepsilon_2}(\mathbb{R}^d)} \le \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{d/2} \|\mu\|_{\mathcal{M}_{\varepsilon_1}(\mathbb{R}^d)}. \tag{2.7}$$

We also define similarly a family of weighted L^p spaces. Indeed for $1 \leq p < \infty$ we define

$$L^p_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \rho_\varepsilon(x)|f(x)|^p \, \mathrm{d}x < \infty \right\} \tag{2.8}$$

with the norm $\|f\|_{L^p_\varepsilon(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \rho_\varepsilon(x)|f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$; and for $p = \infty$,

$$L^\infty_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \rho_\varepsilon(x)|f(x)| < \infty \right\} \tag{2.9}$$

with the norm $\|f\|_{L^\infty_\varepsilon(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} \rho_\varepsilon(x)|f(x)|$. The spaces $L^p_\varepsilon(\mathbb{R}^d)$ are Banach spaces for every $1 \leq p \leq \infty$ and every $\varepsilon > 0$, see [4]. Obviously $L^1_\varepsilon(\mathbb{R}^d) \subset \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and if $f \in L^1_\varepsilon(\mathbb{R}^d)$ then $\|f\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = \|f\|_{L^1_\varepsilon(\mathbb{R}^d)}$.

The fundamental existence result from [4] guarantees that initial data in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ gives rise to a smooth solution on $(0, 1/4\varepsilon)$.

Theorem 2.1 *Suppose that $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$, set $T(\varepsilon) = 1/4\varepsilon$, and let $u(x,t) = S(t)u_0(x)$ be given by (2.2). Then*

(i) *$u(t) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ for $t \in (0, T(\varepsilon))$. Also $u \in C^\infty(\mathbb{R}^d \times (0, T(\varepsilon)))$ and satisfies*

$$u_t - \Delta u = 0 \quad \text{for all} \quad x \in \mathbb{R}^d, \ 0 < t < T(\varepsilon).$$

(ii) *For every $\varphi \in C_c(\mathbb{R}^d)$ and $0 \leq t < T(\varepsilon)$*

$$\int_{\mathbb{R}^d} \varphi \, u(t) = \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0. \tag{2.10}$$

In particular, $u(t) \to u_0$ as $t \to 0^+$ in the sense of measures, i.e.

$$\int_{\mathbb{R}^d} \varphi \, u(t) \to \int_{\mathbb{R}^d} \varphi \, \mathrm{d}u_0$$

for every $\varphi \in C_c(\mathbb{R}^d)$. Furthermore, if $u_0 \in L^1_\varepsilon(\mathbb{R}^d)$ then for any $\delta > \varepsilon$ we have

$$u(t) \to u_0 \quad \text{in} \quad L^1_\delta(\mathbb{R}^d) \quad \text{as} \quad t \to 0^+.$$

(iii) *If $0 \leq u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ is nonzero then $u(x,t) > 0$ for all $x \in \mathbb{R}^d$, $t \in (0, T(\varepsilon))$, i.e. the Strong Maximum Principle holds.*

Theorem 2.2 *Suppose that u, defined in $\mathbb{R}^d \times (0, T]$, is such that for some $\delta > 0$ and for each $0 < t < T$, $u(t) \in L^1_\delta(\mathbb{R}^d)$.*

(i) *Suppose furthermore that*

$$u, \nabla u, \Delta u \in L^1_{\mathrm{loc}}((0,T), L^1_\delta(\mathbb{R}^d)) \tag{2.11}$$

and satisfies $u_t - \Delta u = 0$ almost everywhere in $\mathbb{R}^d \times (0, T)$. Then we have

$$u(t) = S(t-s)u(s) \tag{2.12}$$

for any $0 < s < t < T$.

Assume hereafter that u satisfies (2.12) for any $0 < s < t < T$.

(ii) *Then for each* $0 < t < T$ *and every* $\varphi \in C_c(\mathbb{R}^d)$ *the following limit exist*

$$\lim_{s\to 0}\int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} u(t)\varphi.$$

(iii) *There exists* $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ *for some* $\varepsilon > 0$ *and such that* $u(t) = S(t)u_0$ *for* $0 < t < T$ *if and only if for every* $\varphi \in C_c(\mathbb{R}^d)$ *and* t *small enough*

$$\lim_{s\to 0}\int_{\mathbb{R}^d} u(s)S(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0. \tag{2.13}$$

(iv) *Condition* (2.13) *is satisfied provided that either one of the following holds:*

(iv-a) *For any function* $\phi \in C_0(\mathbb{R}^d)$ *such that* $|\phi(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$, *with* $\gamma > \varepsilon$ *we have, as* $t \to 0$

$$\lim_{t\to 0}\int_{\mathbb{R}^d} \phi u(t) \to \int_{\mathbb{R}^d} \phi \, \mathrm{d}u_0. \tag{2.14}$$

(iv-b) *For some* $\tau \le T$ *small and* $0 < t \le \tau$ *we have* $u(t) \in L^1_\varepsilon(\mathbb{R}^d)$ *with*

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}|u(x,t)| \, \mathrm{d}x \le M \quad t \in (0,\tau]; \tag{2.15}$$

i.e. $u \in L^\infty((0,\tau], L^1_\varepsilon(\mathbb{R}^d))$ *and for every* $\varphi \in C_c(\mathbb{R}^d)$, *as* $t \to 0$

$$\int_{\mathbb{R}^d} \varphi\, u(t) \to \int_{\mathbb{R}^d} \varphi \, \mathrm{d}u_0. \tag{2.16}$$

Proposition 2.3 *Suppose that* $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$, *set* $T(\varepsilon) = 1/4\varepsilon$, *and let* $u(x,t)$ *be given by* (2.2). *Then*

(i) *For* $0 < t < T(\varepsilon)$ *and for any* $\delta \ge \frac{1}{4(T(\varepsilon)-t)} > \varepsilon$ *we have* $u(t) \in L^1_\delta(\mathbb{R}^d)$. *Moreover if we set* $\varepsilon(t) := \frac{1}{4(T(\varepsilon)-t)} = \frac{\varepsilon}{(1-4\varepsilon t)}$ *then*

$$\|u(t)\|_{L^1_{\varepsilon(t)}(\mathbb{R}^d)} \le \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}. \tag{2.17}$$

(ii) *For* $0 \le s < t < T(\varepsilon)$, $u(t) = S(t-s)u(s)$.

(iii) *For any multi-index* $\alpha \in \mathbb{N}^d$, *for* $0 < t < T(\varepsilon)$ *and for any* $\delta > \frac{1}{4(T(\varepsilon)-t)} > \varepsilon$ *we have* $D^\alpha_x u(t) \in L^1_\delta(\mathbb{R}^d)$. *Moreover for any* $\gamma > 1$ *and* $0 < t < \frac{T(\varepsilon)}{\gamma}$ *and if we set* $\delta(t) := \frac{1}{4(T(\varepsilon)-\gamma t)} = \frac{\varepsilon}{(1-4\varepsilon\gamma t)}$ *then*

$$\|D^\alpha_x u(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le \frac{c_{\alpha,\gamma}}{t^{\frac{|\alpha|}{2}}}\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}. \tag{2.18}$$

(iv) *For any multi-index* $\alpha \in \mathbb{N}^d$, $m \in \mathbb{N}$ *and for each* $t_0 \in (0, T(\varepsilon))$ *there exists* $\delta(t_0) > \varepsilon$ *such that the mapping* $(0, T(\varepsilon)) \ni t \mapsto D^{\alpha,m}_{x,t} u(t)$ *is continuous in* $L^1_{\delta(t_0)}(\mathbb{R}^d)$ *at* $t = t_0$.

We now define spaces of initial data for which $S(t)$ in (2.2) is defined for all $t \ge 0$. We set

$$\mathcal{M}_0(\mathbb{R}^d) := \bigcap_{\varepsilon>0}\mathcal{M}_\varepsilon(\mathbb{R}^d) \qquad \text{and} \qquad L^p_0(\mathbb{R}^d) := \bigcap_{\varepsilon>0} L^p_\varepsilon(\mathbb{R}^d),\ 1 \le p \le \infty.$$

These are Fréchet spaces with the corresponding family of norms (see Lemma 3.2 in [5]) and satisfy

$$L^p_0(\mathbb{R}^d) \subset L^q_0(\mathbb{R}^d) \subset L^1_0(\mathbb{R}^d) \subset \mathcal{M}_0(\mathbb{R}^d), \qquad 1 \le q \le p \le \infty.$$

The following spaces naturally play a role in the next result.

Definition 2.4 The space $\mathcal{M}_{0,B}(\mathbb{R}^d)$ is the subspace of $\mathcal{M}_0(\mathbb{R}^d)$ consisting of measures such that

$$|||u_0|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)} := \sup_{\varepsilon>0} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} < \infty. \tag{2.19}$$

Analogously, $L^1_{0,B}(\mathbb{R}^d)$ is the subspace of $L^1_0(\mathbb{R}^d)$ consisting of functions such that

$$|||u_0|||_{L^1_{0,B}(\mathbb{R}^d)} := \sup_{\varepsilon>0} \|u_0\|_{L^1_\varepsilon(\mathbb{R}^d)} < \infty \tag{2.20}$$

and clearly $|||u_0|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)} = |||u_0|||_{L^1_{0,B}(\mathbb{R}^d)}$ for $u_0 \in L^1_{0,B}(\mathbb{R}^d)$.

The space $\mathcal{M}_{0,B}(\mathbb{R}^d)$ can be characterized as the space of all $u_0 \in \mathcal{M}_{\text{loc}}(\mathbb{R}^d)$ such that the solution of the heat Eq. (1.1) given by (1.2) is defined for all $t > 0$ and $u(\cdot, \cdot; u_0)$ is uniformly bounded in sets $\frac{|x|}{\sqrt{t}} \le R$, with $R > 0$; see Lemma 3.7 in [5].

From Propositions 3.3, 3.5, 4.8 and 4.9 in [5] we get the following.

Proposition 2.5 *If we define $S(t)u_0$ by (2.2) then*

$$S(t)\colon \mathcal{M}_0(\mathbb{R}^d) \to L^1_0(\mathbb{R}^d) \quad \textit{and} \quad S(t)\colon L^p_0(\mathbb{R}^d) \to L^p_0(\mathbb{R}^d).$$

These mappings are all linear, continuous, and order-preserving, and $\{S(t)\}_{t\ge 0}$ is a continuous semigroup on $\mathcal{M}_0(\mathbb{R}^d)$ and $L^p_0(\mathbb{R}^d)$ for every $1 \le p < \infty$.

In $L^p_0(\mathbb{R}^N)$ the semigroup is continuous at $t = 0$, while in $\mathcal{M}_0(\mathbb{R}^d)$ the semigroup is continuous in the sense of measures. Furthermore, for any multi-index $\alpha \in \mathbb{N}^d$, and any $1 \le p \le q \le \infty$ and $t > 0$, $D^\alpha_x S(t)\colon L^p_0(\mathbb{R}^d) \longrightarrow L^q_0(\mathbb{R}^d)$ is linear and continuous.

For each $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$, any $1 \le p < \infty$, and any multi-index $\alpha \in \mathbb{N}^d$, the solution curve

$$(0, \infty) \ni t \longmapsto D^\alpha_x S(t)u_0 \in L^p_0(\mathbb{R}^d)$$

is C^∞. If additionally, $u_0 \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ as in (2.19), that is

$$\sup_{\varepsilon>0} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} < \infty,$$

then the solution curve is $C^\omega((0,\infty), L^p_0(\mathbb{R}^d))$, i.e. analytic.

Also from Proposition 3.8 in [5] we have the following result concerning the solutions of the heat equation with initial data in $L^1_{0,B}(\mathbb{R}^d)$ or $\mathcal{M}_{0,B}(\mathbb{R}^d)$ as in Definition 2.4.

Proposition 2.6 *The spaces $L^1_{0,B}(\mathbb{R}^d)$ and $\mathcal{M}_{0,B}(\mathbb{R}^d)$ are invariant subspaces for the semigroup $S(t)$ in $\mathcal{M}_0(\mathbb{R}^d)$, which satisfies*

$$|||S(t)u_0|||_{L^1_{0,B}(\mathbb{R}^d)} \le |||u_0|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)}, \quad t > 0$$

with equality if $u_0 \ge 0$. For any multi-index $\alpha \in \mathbb{N}^d$ there exists a constant $c_\alpha > 0$ such that

$$\left|\left|\left| D^\alpha_x S(t)u_0 \right|\right|\right|_{L^1_{0,B}(\mathbb{R}^d)} \le \frac{c_\alpha}{t^{\frac{|\alpha|}{2}}} |||u_0|||_{\mathcal{M}_{0,B}(\mathbb{R}^d)}, \quad t > 0.$$

In particular $S(t)\colon L^1_{0,B}(\mathbb{R}^d) \to L^1_{0,B}(\mathbb{R}^d)$ is an analytic order-preserving contraction semigroup.

The following "L^p–L^q" type estimates from [6] will be crucial in what follows.

Proposition 2.7 *If $u_0 \in L^p_0(\mathbb{R}^d)$ with $1 \le p < \infty$ and for any $1 \le p \le q < \infty$, then $u(t) \in L^q_0(\mathbb{R}^d)$ for all $t > 0$ and for any $\delta > 0$, $t > 0$ and $\delta_p(t) = \frac{p\delta}{1+4p\delta t}$ we have*

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le c_{p,q} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)} \tag{2.21}$$

with $c_{p,q} = \left(\frac{q}{p}\right)^{d/2q}$. For $q = \infty$ we have

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le c_{p,\infty} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)} \tag{2.22}$$

with $c_{p,\infty} = \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}}$. Finally, for $p = \infty$ and $\delta(t) = \frac{\delta}{1+4\delta t}$,

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le (1+4\delta t)^d \|u_0\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)}. \tag{2.23}$$

In particular, $S(t) : L^p_0(\mathbb{R}^d) \longrightarrow L^q_0(\mathbb{R}^d)$, $1 \le p \le q \le \infty$, is continuous.

Moreover, for any multi-index $\alpha \in \mathbb{N}^d$ and for any $1 \le p \le q \le \infty$, then for any $\delta > 0$, $t > 0$, $\gamma > 1$ $\tilde{\delta}_p(t) = \frac{p\delta}{1+4p\gamma\delta t}$

$$\|D^\alpha_x u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le \frac{c_{\alpha,p,q,\delta,\gamma}}{t^{\frac{|\alpha|}{2}}} \left(\frac{1+4p\gamma\delta t}{t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)} \tag{2.24}$$

and

$$\|D^\alpha_x u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le \frac{c_{\alpha,p,\delta,\gamma}}{t^{\frac{|\alpha|}{2}}} \left(\frac{1+4p\gamma\delta t}{t}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}.$$

If $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$ the same estimates are valid, setting $p = 1$ and replacing $\|u_0\|_{L^1_{\delta(t)}(\mathbb{R}^d)}$ on the right-hand side by $\|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}$.

Notice, in particular, that taking $1 \le p = q < \infty$ we have the "contraction" type estimate

$$\|u(t)\|_{L^p_{p\delta}(\mathbb{R}^d)} \le \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}.$$

For non-negative initial data, the norm as a measure of the initial data is preserved. We make this precise in the following proposition.

Proposition 2.8 *Assume $0 \le u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ and let $u(x,t)$ be given by (2.2).*

(i) *For every $\delta > \varepsilon$ and $0 \le t \le \frac{1}{4\varepsilon} - \frac{1}{4\delta} = T(\varepsilon) - T(\delta)$*

$$\|u(t)\|_{L^1_\delta(\mathbb{R}^d)} = \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}$$

with $\delta(t) = \frac{\delta}{1+4\delta t}$. In particular, this estimate holds for any $\delta > 0$ and $t > 0$ if $0 \le u_0 \in \mathcal{M}_0(\mathbb{R}^d)$.

(ii) *For $0 \le t < T(\varepsilon) = \frac{1}{4\varepsilon}$ and for $\varepsilon(t) := \frac{1}{4(T(\varepsilon)-t)} = \frac{\varepsilon}{(1-4\varepsilon t)}$ we have*

$$\|u(t)\|_{L^1_{\varepsilon(t)}(\mathbb{R}^d)} = \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}.$$

The estimates in Proposition 2.7 hold for $u_0 \in L^p_0(\mathbb{R}^d)$. Similar ones were also obtained in [5] for the case of $u_0 \in L^p_\varepsilon(\mathbb{R}^d)$ which are therefore valid only in finite time intervals. More precisely, we have the following.

Proposition 2.9 *Assume that* $\varepsilon > 0$ *and* $T(\varepsilon) = \frac{1}{4\varepsilon}$. *For every* $\delta > \varepsilon$ *take* $0 \le t \le \frac{1}{4\varepsilon} - \frac{1}{4\delta} = T(\varepsilon) - T(\delta)$ *and define* $\delta(t) = \frac{\delta}{1+4\delta t}$ *and* $\delta_p(t) = \frac{p\delta}{1+4p\delta t}$.

(i) *For any* $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ *we have*

$$\|u(t)\|_{L^1_\delta(\mathbb{R}^d)} \le \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)},$$

with equality if $u_0 \ge 0$, *and also*

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le \left(\frac{1+4\delta t}{4\pi t}\right)^{d/2} \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}$$

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le q^{d/2q}\left(\frac{1+4\delta t}{4\delta t}\right)^{\frac{d}{2}(1-\frac{1}{q})} \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}$$

for $1 \le q < \infty$.

(ii) *For any* $u_0 \in L^\infty_\varepsilon(\mathbb{R}^d)$ *we have*

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le (1+4\delta t)^d \|u_0\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)}.$$

(iii) *For any* $u_0 \in L^p_{p\varepsilon}(\mathbb{R}^d)$ *with* $1 \le p < \infty$ *and for any* $1 \le p \le q < \infty$ *we have*

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le c_{p,q}\big(\frac{1+4p\delta t}{4p\delta t}\big)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}$$

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le c_{p,\infty}\big(\frac{1+4p\delta t}{4p\delta t}\big)^{\frac{d}{2p}} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}$$

which hold for $0 < t \le \frac{1}{p}\big(T(\varepsilon) - T(\delta)\big)$.

(iv) *Analogous derivative estimates to* (2.24) *also hold true.*

At the opposite extreme we will need estimates on solutions of the heat equation with rapidly decaying initial data.

Proposition 2.10 *If* $|\varphi(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$ *then* $u(t) = S(t)\varphi$ *satisfies*

$$|u(x,t)| \le \frac{A}{(1+4\gamma t)^{d/2}} \mathrm{e}^{-\frac{\gamma}{1+4\gamma t}|x|^2}, \qquad x \in \mathbb{R}^d, \quad t > 0.$$

Remark 2.11 Observe that $u(0,t,|u_0|) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y|^2}{4t}}\, \mathrm{d}|u_0(y)|$ and so

$$\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = u(0, \frac{1}{\varepsilon}, |u_0|) = S(\frac{1}{\varepsilon})|u_0|(0).$$

3 The Equation with an Additional Linear Term $+\lambda u$

In this section we consider a linear perturbation of Eq. (1.1). Our goal is to understand the effect of this term on the asymptotic behaviour of solutions. Thus, we consider the equation

$$u_t - \Delta u + \lambda u = 0, \; x \in \mathbb{R}^d, \; t > 0, \qquad u(x,0) = u_0(x). \tag{3.1}$$

In contrast with the heat flow in more standard spaces, we will see below that not every $\lambda > 0$ is capable of producing decay for all solutions of (3.1).

We first get the following result for (3.1) using Theorem 4.1 in [4] with $u(x,t) = e^{-\lambda t} v(x,t)$, and Proposition 2.5.

Proposition 3.1 *Suppose that v is defined in $\mathbb{R}^d \times (0,T)$ and is such that for some $\delta > 0$*

$$v, \nabla v, \Delta v \in L^1_{\mathrm{loc}}((0,T), L^1_\delta(\mathbb{R}^d))$$

and

$$v_t - \Delta v + \lambda v = 0, \quad a.e.\ x \in \mathbb{R}^d,\ 0 < t < T, \tag{3.2}$$

with $\lambda \in \mathbb{R}$. Suppose also that for every $\varphi \in C_c(\mathbb{R}^d)$ and t small enough v satisfies

$$\lim_{s\to 0} \int_{\mathbb{R}^d} v(s) S(t)\varphi = \int_{\mathbb{R}^d} S(t)\varphi \, \mathrm{d}u_0 \tag{3.3}$$

for some $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^N)$, for some $\varepsilon > 0$. Then

$$v(x,t) = e^{-\lambda t} u(x,t), \quad x \in \mathbb{R}^d,\ 0 < t < T,$$

where $u(x,t)$ is the solution of (1.1) with the same initial data.

In particular, (3.2) defines a semigroup in $L^1_0(\mathbb{R}^d)$ and in $\mathcal{M}_0(\mathbb{R}^N)$ given by

$$S_\lambda(t) = e^{-\lambda t} S(t), \quad t \geq 0,$$

which satisfies all the smoothing properties of $S(t)$ stated in Proposition 2.5 and for any $\delta > 0$

$$\|S_\lambda(t) u_0\|_{L^1_\delta(\mathbb{R}^d)} \leq e^{-\lambda t} \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}, \quad t > 0 \tag{3.4}$$

with $\delta(t) = \frac{\delta}{1+4\delta t}$; there is equality in (3.4) if $u_0 \geq 0$.

Since $S_\lambda(t) = e^{-\lambda t} S(t)$ for $t \geq 0$, we can deduce information on the boundedness and decay of solutions of (3.2) for large times using Proposition 5.1 in [5].

Corollary 3.2 (i) *If $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$ then $u(t) = S_\lambda(t) u_0$ is bounded/decays to zero in $L^q_0(\mathbb{R}^d)$ as $t \to \infty$, for any $1 \leq q \leq \infty$, provided that*

$$e^{-\frac{\lambda}{4\varepsilon}} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} \qquad \text{is bounded/tends to zero as } \varepsilon \to 0. \tag{3.5}$$

The converse is also true if $u_0 \geq 0$.

If $\lambda > 0$ and $u_0 \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ (as in Definition 2.4) then (3.5) holds.

(ii) *If $u_0 \in L^\infty_0(\mathbb{R}^d)$ then $u(t) = S_\lambda(t) u_0$ is bounded/decays to zero in $L^\infty_0(\mathbb{R}^d)$ as $t \to \infty$ if*

$$\frac{e^{-\frac{\lambda}{4\varepsilon}}}{\varepsilon^d} \|u_0\|_{L^\infty_\varepsilon(\mathbb{R}^d)} \qquad \text{is bounded/tends to zero as } \varepsilon \to 0.$$

(iii) *If $u_0 \in L^p_0(\mathbb{R}^d)$, $1 \leq p < \infty$, then $u(t) = S_\lambda(t) u_0$ is bounded/decays to zero in $L^q_0(\mathbb{R}^d)$, $p \leq q \leq \infty$ as $t \to \infty$, if*

$$e^{-\frac{\lambda}{4\varepsilon}} \|u_0\|_{L^p_\varepsilon(\mathbb{R}^d)} \qquad \text{is bounded/tends to zero as } \varepsilon \to 0.$$

Proof We use $S_\lambda(t) = e^{-\lambda t} S(t)$ and the estimates in Proposition 2.7 as follows. For (i) and (ii), for fixed δ, $t > 0$ denote $\varepsilon = \delta(t) = \frac{\delta}{1+4\delta t}$, that is, $t = \frac{1}{4\varepsilon} - \frac{1}{4\delta}$ and then $\mathrm{e}^{-\lambda t} \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)} = c(\delta)\mathrm{e}^{-\frac{\lambda}{4\varepsilon}} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}$ and $\mathrm{e}^{-\lambda t}(1+4\delta t)^d \|u_0\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)} = c(\delta)\frac{\mathrm{e}^{-\frac{\lambda}{4\varepsilon}}}{\varepsilon^d} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}$ respectively. If $u_0 \geq 0$, the converse in case of (i) follows from Proposition 2.8.

For (iii) we argue analogously with $\varepsilon = \delta_p(t) = \frac{p\delta}{1+4p\delta t}$, that is $t = \frac{1}{4\varepsilon} - \frac{1}{4p\delta}$ and then $c_{p,q}\mathrm{e}^{-\lambda t}\left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)} \leq c(\delta, p, q)\mathrm{e}^{-\frac{\lambda}{4\varepsilon}} \|u_0\|_{L^p_\varepsilon(\mathbb{R}^d)}$. □

Now we present some typical examples of elements in $\mathcal{M}_0(\mathbb{R}^d)$ and estimate their norms in order to check whether they satisfy the condition for decay in (3.5).

As we will see below for functions that grow strictly slower than any exponential as $|x| \to \infty$, i.e. slower than $\mathrm{e}^{c|x|}$ for every $c > 0$, the additional dissipative term λu in the equation is able to drive the solution to zero as $t \to \infty$. For functions that grow like $\mathrm{e}^{c|x|}$ then we need to take $\lambda > c^2$ and this result is sharp. Finally for functions that grow faster than $\mathrm{e}^{c|x|}$ for every $c > 0$, no $\lambda > 0$ can bring the solution to zero as $t \to \infty$.

We start with the cases of a delta function at a point $x_0 \in \mathbb{R}^d$ and trigonometric functions. Both proofs are immediate.

Example 3.3 For $f = \delta_{x_0}$, $x_0 \in \mathbb{R}^d$, we have

$$\|f\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{-\varepsilon|x_0|^2}.$$

Hence $u_0 = \delta_{x_0}$ satisfies (3.5) for $\lambda \geq 0$.

Example 3.4 For $f(x) = \mathrm{e}^{\mathrm{i}\omega x}$, $\omega \in \mathbb{R}^d$, we have

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \leq 1, \qquad \varepsilon > 0.$$

Therefore, $u_0 = f$ satisfies (3.5) for any $\lambda > 0$.

For functions that grow less quickly than an exponential, we have the following.

Example 3.5 If $f(x) = |x|^\beta$, $\beta > 0$, then $f \in L^1_0(\mathbb{R}^d)$ and

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} = \frac{A}{\varepsilon^{\frac{\beta}{2}}}$$

for some A depending on β. Therefore, $u_0 = f$ satisfies (3.5) for any $\lambda > 0$.

Proof. Just notice that taking polar coordinates

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_0^\infty \mathrm{e}^{-\varepsilon r^2} r^{\beta+d-1}\,\mathrm{d}r = \frac{1}{\varepsilon^{\frac{\beta}{2}}}\left(\frac{1}{\pi}\right)^{d/2} \int_0^\infty \mathrm{e}^{-y^2} y^{\beta+d-1}\,\mathrm{d}r. \quad \square$$

Example 3.6 For $f(x) = \mathrm{e}^{\omega x}$, $\omega \in \mathbb{R}^d$, we have

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} = \mathrm{e}^{\frac{|\omega|^2}{4\varepsilon}}, \qquad \varepsilon > 0.$$

Therefore, $u_0 = f$ satisfies (3.5) only for $\lambda > |\omega|^2$.

Proof. Completing the square

$$\varepsilon|x|^2 - \omega x = \left|\sqrt{\varepsilon}x - \frac{\omega}{2\sqrt{\varepsilon}}\right|^2 - \frac{|\omega|^2}{4\varepsilon}$$

leads to

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} = \mathrm{e}^{\frac{|\omega|^2}{4\varepsilon}} \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-|\sqrt{\varepsilon}x - \frac{\omega}{2\sqrt{\varepsilon}}|^2}\, \mathrm{d}x = \mathrm{e}^{\frac{|\omega|^2}{4\varepsilon}} \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2}\, \mathrm{d}x = \mathrm{e}^{\frac{|\omega|^2}{4\varepsilon}}. \quad \square$$

Example 3.7 If $f(x) = \mathrm{e}^{c|x|}$, $c > 0$, then $f \in L^1_0(\mathbb{R}^d)$ and

$$\exp\left(\frac{c^2}{4\varepsilon}\right) \le \|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \le A_1 \exp\left(\frac{A_2}{4\varepsilon}\right), \qquad \varepsilon > 0 \tag{3.6}$$

for any $A_2 > c^2$ and some $A_1 > 1$. Therefore, $u_0 = f$ satisfies (3.5) only for $\lambda > c^2$.

Proof Notice that for any $0 < \theta < 1$ we have

$$-\varepsilon|x|^2 + c|x| \le -(1-\theta)\varepsilon|x|^2 + A_{\theta,\varepsilon}, \qquad x \in \mathbb{R}^d,$$

where $A_{\theta,\varepsilon} = \frac{c^2}{4\theta\varepsilon} > 0$. Hence

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \le \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \mathrm{e}^{-\varepsilon|x|^2 + c|x|}\, \mathrm{d}x \le \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{A_{\theta,\varepsilon}} \int_{\mathbb{R}^d} \mathrm{e}^{-(1-\theta)\varepsilon|x|^2}\, \mathrm{d}x = \frac{1}{(1-\theta)^{d/2}} \mathrm{e}^{\frac{c^2}{4\theta\varepsilon}}$$

and the upper bound in (3.6) follows.

Conversely, taking polar coordinates and completing the square yields

$$\begin{aligned}
\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} &= \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_0^\infty \mathrm{e}^{-\varepsilon r^2 + cr} r^{d-1}\, \mathrm{d}r = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{c^2}{4\varepsilon}} \int_0^\infty \mathrm{e}^{-\varepsilon(r - \frac{c}{2\varepsilon})^2} r^{d-1}\, \mathrm{d}r \\
&\ge \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{c^2}{4\varepsilon}} \int_{\frac{c}{2\varepsilon}}^\infty \mathrm{e}^{-\varepsilon(r - \frac{c}{2\varepsilon})^2} r^{d-1}\, \mathrm{d}r \\
&\ge \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{c^2}{4\varepsilon}} \int_{\frac{c}{2\varepsilon}}^\infty \mathrm{e}^{-\varepsilon(r - \frac{c}{2\varepsilon})^2} (r - \frac{c}{2\varepsilon})^{d-1}\, \mathrm{d}r \\
&= \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{c^2}{4\varepsilon}} \int_0^\infty \mathrm{e}^{-\varepsilon z^2} z^{d-1}\, \mathrm{d}z = \mathrm{e}^{\frac{c^2}{4\varepsilon}},
\end{aligned}$$

which completes the proof of (3.6). The rest is immediate. $\square$

The next two examples show that the linear exponential above gives a threshold for whether or not the dissipative term λu can bring solutions down to zero as $t \to \infty$.

Firs we treat functions that grow more slowly than an exponential.

Example 3.8 Assume $f(x)$ is such that for every $\mu > 0$ there exists $c(\mu)$ such that

$$|f(x)| \le \mathrm{e}^{\mu|x| + c(\mu)}, \quad x \in \mathbb{R}^d$$

e.g. $f(x) = \mathrm{e}^{c|x|^\alpha}$, $c > 0$, $0 < \alpha < 1$. Then

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \le A_1 \exp\left(\frac{A_2}{4\varepsilon}\right), \qquad \varepsilon > 0$$

for any $A_2 > \mu^2$ and some $A_1 > 1$.

Therefore, $u_0 = f$ satisfies (3.5) for any $\lambda > 0$.

Proof The proof is immediate from the previous example. □

Now we consider functions that grow more quickly than a linear exponential.

Example 3.9 Assume $f(x)$ is such that for every $\mu > 0$ there exists $r(\mu) > 0$ such that

$$|f(x)| \geq \mathrm{e}^{\mu|x|}, \quad \text{for } |x| \geq r(\mu),$$

e.g. $f(x) = \mathrm{e}^{c|x|^\alpha}$, $1 < \alpha < 2$, $c > 0$.
Then for $0 < \varepsilon \leq \varepsilon_0 = \frac{\mu}{2r(\mu)}$ we have

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \geq \mathrm{e}^{\frac{\mu^2}{4\varepsilon}}.$$

Since $\mu > 0$ is arbitrary, (3.5) is not satisfied for any $\lambda > 0$.

Proof. Taking polar coordinates

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \geq \left(\frac{\varepsilon}{\pi}\right)^{d/2} \int_{r(\mu)}^{\infty} \mathrm{e}^{-\varepsilon r^2+\mu r} r^{d-1}\,\mathrm{d}r$$

and completing the square as above, we get

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \geq \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{\mu^2}{4\varepsilon}} \int_{r(\mu)}^{\infty} \mathrm{e}^{-\varepsilon(r-\frac{\mu}{2\varepsilon})^2} r^{d-1}\,\mathrm{d}r \geq \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{\mu^2}{4\varepsilon}} \int_{\frac{\mu}{2\varepsilon}}^{\infty} \mathrm{e}^{-\varepsilon(r-\frac{\mu}{2\varepsilon})^2} r^{d-1}\,\mathrm{d}r$$

provided that $r(\mu) \leq \frac{\mu}{2\varepsilon}$, which can be achieved for $0 < \varepsilon \leq \varepsilon_0$ as in the statement. Hence, for this range of ε we get

$$\|f\|_{L^1_\varepsilon(\mathbb{R}^d)} \geq \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{\frac{\mu^2}{4\varepsilon}} \int_{\frac{\mu}{2\varepsilon}}^{\infty} \mathrm{e}^{-\varepsilon(r-\frac{\mu}{2\varepsilon})^2} (r - \frac{\mu}{2\varepsilon})^{d-1}\,\mathrm{d}r = \mathrm{e}^{\frac{\mu^2}{4\varepsilon}}. \quad \square$$

4 The Variation of Constants Formula in $\mathcal{M}_0(\mathbb{R}^d)$

In this section our aim is to solve, in a suitable sense, the non-homogeneous equation

$$u_t - \Delta u + \lambda u = f,\ x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x) \in \mathcal{M}_\varepsilon(\mathbb{R}^d) \tag{4.1}$$

with $\lambda \in \mathbb{R}$ and a given f, in $[0, T]$, such that

$$[0,T] \ni t \mapsto f(t) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d).$$

In more standard settings we would expect that the solution of (4.1) is given by

$$u(t) = \mathrm{e}^{-\lambda t} S(t) u_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)\,\mathrm{d}s. \tag{4.2}$$

Here we show that (4.2) actually provides a suitable solution of (4.1) in the non-standard setting of this paper.

4.1 Existence and Uniqueness

Since part of the solution of the homogeneous equation, namely $e^{-\lambda t}S(t)u_0$, has already been dealt with in Proposition 3.1, we will concentrate below on the term

$$U(t) := U(t, f) = \int_0^t e^{-\lambda(t-s)} S(t-s) f(s)\, ds. \tag{4.3}$$

The following result gives sufficient conditions for U in (4.3) to be a "very weak" solution of (4.1). For this, we will use the following notation. For a given function $\varepsilon : [0, T] \to (0, \infty)$ we say that

$$f \in L^p((0, T), \mathcal{M}_{\varepsilon(\cdot)}(\mathbb{R}^d))$$

for some $1 \le p \le \infty$ if $f(t) \in \mathcal{M}_{\varepsilon(t)}(\mathbb{R}^d)$ for each $0 \le t \le T$ and

$$\|f\|_{L^p((0,T),\mathcal{M}_{\varepsilon(\cdot)}(\mathbb{R}^d))} := \left\{ \begin{matrix} \left(\int_0^T \|f(s)\|^p_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)}\, ds \right)^{1/p} & 1 \le p < \infty \\ \sup_{t\in[0,T]} \|f(t)\|_{\mathcal{M}_{\varepsilon(t)}} & p = \infty \end{matrix} \right\} < \infty.$$

Theorem 4.1 *Assume that $\varepsilon : [0, T] \to (0, \infty)$ is such that $\varepsilon(t) \ge \varepsilon_0$ for some $\varepsilon_0 > 0$ and $\limsup_{t\to 0^+} \varepsilon(t) < \infty$. For $t \in [0, T]$ define the non-increasing function*

$$e(t) = \inf_{0\le s\le t} \left(\frac{1}{4\varepsilon(s)} + s \right)$$

and determine $T_0 > 0$ by setting

$$T_0 := \sup\{t \in [0, T],\ e(t) > t\} \le T. \tag{4.4}$$

Finally for $0 \le t < T_0$ define

$$\delta(t) := \frac{1}{4(e(t) - t)},$$

which is increasing and satisfies $\delta(t) \ge \varepsilon(t)$.

Suppose that $f \in L^1((0, T), \mathcal{M}_{\varepsilon(\cdot)}(\mathbb{R}^d))$ and define

$$U(t) = \int_0^t e^{-\lambda(t-s)} S(t-s) f(s)\, ds.$$

Then for $0 \le t < T_0$ we have $U(t) \in L^1_{\delta(t)}(\mathbb{R}^d)$ with

$$\|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le C(t) \int_0^t \|f(s)\|_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)}\, ds,$$

where $C(t) = \left(\frac{\delta(t)}{\varepsilon_0}\right)^{d/2} \sup_{0\le s\le t} e^{-\lambda s}$. In particular, for any $\tau < T_0$

$$\sup_{0\le t\le \tau} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le C(\tau) \|f\|_{L^1((0,\tau),\mathcal{M}_{\varepsilon(\cdot)})}. \tag{4.5}$$

In particular, $U \in C([0, T_0), L^1_{\delta(\cdot)}(\mathbb{R}^d))$, in the sense that for any fixed $0 < t < T_0$ and $\tilde{\delta} > \delta(t)$ we have, as $s \to t$, $U(s) \to U(t)$ in $L^1_{\tilde{\delta}}(\mathbb{R}^d)$ and

$$\lim_{t\to 0} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} = 0, \tag{4.6}$$

and is a very weak solution of (4.1) *in* $(0, T_0)$*, that is, for every* $\varphi \in C_c^\infty(\mathbb{R}^d)$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} U(t)\varphi + \int_{\mathbb{R}^d} U(t)(-\Delta\varphi + \lambda\varphi) = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}f(t), \quad 0 < t < T_0. \tag{4.7}$$

Proof. Fix $t \in [0, T]$ and take $\delta > 0$ to be chosen below. Then from Proposition 2.9

$$\|U(t)\|_{L^1_\delta(\mathbb{R}^d)} \leq \int_0^t \mathrm{e}^{-\lambda(t-s)} \|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)} \,\mathrm{d}s \tag{4.8}$$

with $\tilde{\delta}(s) = \frac{\delta}{1+4\delta s}$ and $U(t)$ is well defined provided that

$$\tilde{\delta}(t-s) \geq \varepsilon(s) \quad \text{for all} \quad 0 \leq s \leq t \qquad \text{and} \qquad \int_0^t \|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)} \,\mathrm{d}s < \infty.$$

The second condition is satisfied using the integrability assumption on f and (2.7),

$$\|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)} \leq \left(\frac{\delta}{\varepsilon_0}\right)^{d/2} \|f(s)\|_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)}, \quad 0 < s < t. \tag{4.9}$$

Now observe that the condition $\tilde{\delta}(t-s) \geq \varepsilon(s)$ for all $0 \leq s \leq t$ is equivalent to requiring $e(t) \geq \frac{1}{4\delta} + t$. Since $\limsup_{t\to 0} \varepsilon(t) < \infty$, $e(t)$ is well defined and non increasing with $e(0) > 0$. Therefore, T_0 in (4.4) is well defined and is the only time at which $e(t)$ and t may cross in the interval $[0, T]$ since $e(t)$ is non increasing. In particular, as $e(t) \geq \frac{1}{4\delta} + t$, we also get $0 < t \leq T_0$.

Notice that the smallest choice for δ is $\delta = \delta(t)$ with $e(t) = \frac{1}{4\delta(t)} + t$ as in the stament. Then by this definition of $\delta(t)$ we have $\tilde{\delta}(t-s) = \frac{1}{4(e(t)-s)}$, $\delta(t) \geq \varepsilon(t)$ and $\delta(t)$ is increasing.

Hence with such choices we have for $0 < t < T_0$

$$\|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \leq C(t) \int_0^t \|f(s)\|_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)} \,\mathrm{d}s$$

with $C(t) = \left(\frac{\delta(t)}{\varepsilon_0}\right)^{d/2} \sup_{0\leq s\leq t} \mathrm{e}^{-\lambda s}$, which also implies that $\lim_{t\to 0} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} = 0$.

Now observe that for any $0 < \tau < T_0$ we have

$$U(t) = \mathrm{e}^{-\lambda(t-\tau)} S(t-\tau)U(\tau) + \int_\tau^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s) \,\mathrm{d}s, \quad \tau < t < T_0. \tag{4.10}$$

To see this, note that for $\varphi \in C_c^\infty(\mathbb{R}^d)$ we have, using (2.10),

$$I = \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda(t-\tau)} S(t-\tau)U(\tau)\varphi = \mathrm{e}^{-\lambda(t-\tau)} \int_{\mathbb{R}^d} U(\tau)S(t-\tau)\varphi.$$

Now

$$U(\tau) = \int_0^\tau \mathrm{e}^{-\lambda(\tau-s)} S(\tau-s) f(s) \,\mathrm{d}s$$

and Fubini's Theorem and the semigroup property lead to

$$I = \mathrm{e}^{-\lambda(t-\tau)} \int_0^\tau \mathrm{e}^{-\lambda(\tau-s)} \int_{\mathbb{R}^d} S(\tau-s) f(s) S(t-\tau)\varphi \,\mathrm{d}s = \int_0^\tau \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} S(t-s) f(s)\varphi \,\mathrm{d}s$$

that is

$$I = \int_{\mathbb{R}^d} \int_0^\tau \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)\varphi \,\mathrm{d}s.$$

Since this is for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ we obtain

$$\mathrm{e}^{-\lambda(t-\tau)} S(t-\tau)U(\tau) = \int_0^\tau \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)\,\mathrm{d}s$$

and (4.10) is proved.

Now, the same argument as in (4.6) gives that the norm in $L^1_{\delta(t)}(\mathbb{R}^d)$ of the integral term in (4.10) goes to zero as $t \to \tau$. On the other hand, by Theorem 2.1 (ii), since $U(\tau) \in L^1_{\delta(\tau)}(\mathbb{R}^d)$ we have that $\mathrm{e}^{-\lambda(t-\tau)} S(t-\tau)U(\tau)$ is continuous as $t \to \tau$ in $L^1_{\tilde{\delta}}(\mathbb{R}^d)$ for $\tilde{\delta} > \delta(\tau)$. This proves the time continuity , $U \in C([0, T_0), L^1_{\delta(\cdot)}(\mathbb{R}^d))$ as in the statement.

Finally we prove U is a very weak solution of (4.1). For this, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} U(t)\varphi = \int_0^t \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} S(t-s) f(s)\varphi\,\mathrm{d}s = \int_0^t \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} S(t-s)\varphi\,\mathrm{d}f(s).$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} U(t)\varphi &= \int_{\mathbb{R}^d} \varphi\,\mathrm{d}f(t) - \lambda \int_0^t \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} S(t-s)\varphi\,\mathrm{d}f(s) \\
&\quad + \int_0^t \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} \partial_t S(t-s)\varphi\,\mathrm{d}f(s) \\
&= \int_{\mathbb{R}^d} \varphi\,\mathrm{d}f(t) - \lambda \int_0^t \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} S(t-s)\varphi\,\mathrm{d}f(s) \\
&\quad + \int_0^t \mathrm{e}^{-\lambda(t-s)} \int_{\mathbb{R}^d} S(t-s)\Delta\varphi\,\mathrm{d}f(s).
\end{aligned}$$

Another use of Fubini's Theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} U(t)\varphi = \int_{\mathbb{R}^d} \varphi\,\mathrm{d}f(t) + \int_{\mathbb{R}^d} U(t)\big(\Delta\varphi - \lambda\varphi\big). \quad \square$$

Revisiting the proof of Theorem 4.1 we see that we can improve (4.8) to an equality if $f(t)$ is non-negative.

Corollary 4.2 *With the assumptions and notations of Theorem 4.1 assume moreover that $f(t) \geq 0$ for all t. Then for each $0 \leq t \leq T_0$ and $\delta \geq \delta(t) = \frac{1}{4(e(t)-t)}$*

$$\|U(t)\|_{L^1_\delta(\mathbb{R}^d)} = \int_0^t \mathrm{e}^{-\lambda(t-s)} \|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)}\,\mathrm{d}s$$

with $\tilde{\delta}(s) = \frac{\delta}{1+4\delta s}$. In particular for $\delta = \delta(t)$ then $\tilde{\delta}(t-s) = \frac{1}{4(e(t)-s)} \geq \varepsilon(s)$.

Proof To see this notice that

$$U(x,t) = \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)(x)\,\mathrm{d}s, \quad x \in \mathbb{R}^d$$

and then for any $\delta \geq \delta(t)$

$$\mathrm{e}^{-\delta|x|^2} U(x,t) = \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s) f(s)(x) \mathrm{e}^{-\delta|x|^2}\,\mathrm{d}s.$$

Integrating in $x \in \mathbb{R}^d$ and using Fubini's Theorem we get

$$\|U(t)\|_{L^1_\delta(\mathbb{R}^d)} = \int_0^t \mathrm{e}^{-\lambda(t-s)} \|S(t-s) f(s)\|_{\mathcal{M}_\delta(\mathbb{R}^d)}\,\mathrm{d}s$$

and using Proposition 2.8 we get

$$\|U(t)\|_{L^1_\delta(\mathbb{R}^d)} = \int_0^t e^{-\lambda(t-s)} \|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)} \, ds.$$

The rest is as in Theorem 4.1. □

Now we prove a uniqueness result for solutions of (4.1).

Proposition 4.3 *There exists at most one function v is defined in $\mathbb{R}^d \times (0, T)$ such that for some $\delta > 0$*

$$v, \nabla v, \Delta v \in L^1_{loc}((0, T), L^1_\delta(\mathbb{R}^d))$$

and satisfies, for every $\varphi \in C_c(\mathbb{R}^d)$ and t small enough

$$\lim_{s\to 0} \int_{\mathbb{R}^d} v(s) S(t)\varphi = 0$$

that is, (2.13) for $u_0 = 0$ (e.g. (2.14) or (2.15) and (2.16), with $u_0 = 0$) and

$$v_t - \Delta v + \lambda v = f, \quad a.e.\ x \in \mathbb{R}^d,\ 0 < t < T.$$

Proof If there exist two such functions, apply Theorem 2.2 to $e^{\lambda t}(v_1(t) - v_2(t))$. □

Now we prove that if f is suitably smooth in space and time then $U = U(f)$ in (4.3), actually satisfies the nonhomogeneous heat Eq. (4.1), with $u_0 = 0$. In particular U is the unique solution in Proposition 4.3. For simplicity we will consider $f \in L^1((0, T), \mathcal{M}_{\varepsilon(\cdot)})$ with $\varepsilon(s) = \varepsilon > 0$ constant. This is the case if for example f does not depend on time that will be analysed further below. In such a case in Theorem 4.1 we have $\varepsilon_0 = \varepsilon$, $e(t) = \frac{1}{4\varepsilon}$, $T_0 := \sup\{t \in [0, T],\ e(t) > t\} = \min\{\frac{1}{4\varepsilon}, T\}$ and $\delta(t) = \frac{1}{4(e(t)-t)} = \frac{\varepsilon}{1-4\varepsilon t}$.

Theorem 4.4 *We consider the same expressions as in Theorem 4.1, but now assume that $f \in L^1((0, T), \mathcal{M}_{\varepsilon(\cdot)})$, where $\varepsilon(s) = \varepsilon > 0$ constant, and $f \in C^1([0, T], L^1_\varepsilon(\mathbb{R}^d))$.*

Then for $0 < t < T_0$ and any $\delta > \delta(t) = \frac{\varepsilon}{1-4\varepsilon t}$ we have that U is differentiable, at t, in $L^1_\delta(\mathbb{R}^d)$, $-\Delta U(t) \in L^1_\delta(\mathbb{R}^d)$ and

$$U_t - \Delta U + \lambda U = f, \quad x \in \mathbb{R}^d,\ 0 < t < T_0,$$

holds.

Proof As in Proposition 3.1, we set $S_\lambda(t) = e^{-\lambda t} S(t)$. We proceed in several steps.

Step 1. From (4.10), $U(t+h) = S_\lambda(h)U(t) + \int_t^{t+h} S_\lambda(t+h-s) f(s)\, ds$, for $h > 0$, hence

$$\frac{U(t+h) - U(t)}{h} = \frac{S_\lambda(h) - I}{h} U(t) + \frac{1}{h}\int_t^{t+h} S_\lambda(t+h-s) f(s)\, ds. \tag{4.11}$$

We claim that the second term converges to $f(t)$, as $h \to 0$ in $L^1_\delta(\mathbb{R}^d)$ for any $\delta > \varepsilon$. For this, denote

$$J(h) = \frac{1}{h}\int_t^{t+h} S_\lambda(t+h-s) f(s)\, ds - f(t) = \frac{1}{h}\int_t^{t+h} (S_\lambda(t+h-s) f(s) - f(t))\, ds$$

and write

$$J(h) = \frac{1}{h}\int_t^{t+h} S_\lambda(t+h-s)[f(s) - f(t)] + (S_\lambda(t+h-s) - I) f(t)\, ds$$

Now for every $\mu > 0$ there exists h_0 such that for evey $h < h_0$, if $s \in (t, t+h)$ we have $\|f(s) - f(t)\|_{L^1_\varepsilon(\mathbb{R}^d)} \leq \mu$. Hence from (2.17) we have for any $\delta > \varepsilon$, if h_0 is small enough, $\|S_\lambda(t+h-s)[f(s) - f(t)]\|_{L^1_\delta(\mathbb{R}^d)} \leq \mu/2$, while from Theorem 2.1 (ii), $\|(S_\lambda(h) - I)f(t)\|_{L^1_\delta(\mathbb{R}^d)} \leq \mu/2$, and therefore $\|J(h)\|_{L^1_\delta(\mathbb{R}^d)} \leq \mu$.

Step 2. From Theorem 4.1 we have $U(t) \in L^1_{\delta(t)}(\mathbb{R}^d)$. Then we show that if for some $\delta \geq \delta(t)$ the term $\frac{S_\lambda(h)-I}{h}U(t)$ has a limit in $L^1_\delta(\mathbb{R}^d)$ then the limit must be $-\Delta U(t) + \lambda U(t)$. For this, for every $\varphi \in C^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{S_\lambda(h) - I}{h} U(t)\varphi = \int_{\mathbb{R}^d} U(t) \frac{S_\lambda(h) - I}{h}\varphi \to \int_{\mathbb{R}^d} U(t)(-\Delta + \lambda I)\varphi$$

see e.g. Section 6 in [5] for the heat flow in spaces of rapidly decaying functions.

With this, from (4.11) we obtain that $U(t)$ is differentiable in $L^1_\delta(\mathbb{R}^d)$, for some $\delta \geq \delta(t)$, iff $-\Delta U(t) \in L^1_\delta(\mathbb{R}^d)$ and in such a case (4.1) holds in $\mathbb{R}^d$.

Step 3. We prove now that U is differentiable in $L^1_\delta(\mathbb{R}^d)$ for $\delta > \delta(t)$. For this, we write, using (4.3),

$$\frac{U(t+h) - U(t)}{h} = \frac{1}{h}\Bigg[\int_0^h S_\lambda(t+h-s)f(s)\,\mathrm{d}s + \int_h^{t+h} S_\lambda(t+h-s)f(s)\,\mathrm{d}s - \int_0^t S_\lambda(t-s)f(s)\,\mathrm{d}s\Bigg]$$

in the middle term we change variables $s = r + h$ to get

$$\begin{aligned}\frac{U(t+h) - U(t)}{h} &= \frac{1}{h}\Bigg[\int_0^h S_\lambda(t+h-s)f(s)\,\mathrm{d}s \\ &\quad + \int_0^t S_\lambda(t-r)f(r+h)\,\mathrm{d}r - \int_0^t S_\lambda(t-s)f(s)\,\mathrm{d}s\Bigg] \\ &= S_\lambda(t)\frac{1}{h}\int_0^h S_\lambda(h-s)f(s)\,\mathrm{d}s + \int_0^t S_\lambda(t-s)\frac{f(s+h) - f(s)}{h}\,\mathrm{d}s.\end{aligned}$$

As in Step 1 above the first term converges, as $h \to 0$ in $L^1_\delta(\mathbb{R}^d)$, for any $\delta > \varepsilon$, to $S_\lambda(t)f(0)$ while since $f \in C^1([0,T], L^1_\varepsilon(\mathbb{R}^d))$, using (4.5), the second converges to $\int_0^t S_\lambda(t-s)f'(s)\,\mathrm{d}s$ in $L^1_{\delta(t)}(\mathbb{R}^d)$ and we get the result. □

4.2 Improved Regularity of *U*

We now show that when f has better integrability in time we get better regularity of U.

Corollary 4.5 *We consider the same expressions as in Theorem 4.1, but now assume that for some $1 < \sigma \leq \infty$, we have $f \in L^\sigma((0,T), \mathcal{M}_{\varepsilon(\cdot)})$.*

Then for any $\tau < T_0$

$$\sup_{0\leq t\leq\tau} \|U(t)\|_{L^q_{q\delta(t)}(\mathbb{R}^d)} \leq C(\tau)\|f\|_{L^\sigma((0,T),\mathcal{M}_{\varepsilon(\cdot)})} \tag{4.12}$$

for $\delta(t) = \frac{1}{4(e(t)-t)}$, $1 \leq q < \frac{d}{(d-2)_+}$ and $\frac{1}{\sigma} + \frac{d}{2} < 1 + \frac{d}{2q}$. Also, $C(\tau) = \left(\frac{\delta(\tau)}{\varepsilon_0}\right)^{d/2} c(\tau)$, and $c(\tau)$ is uniformly bounded in τ if $\lambda > 0$.

In particular $U \in C([0,T_0), L^q_{q\delta(\cdot)}(\mathbb{R}^d))$*, in the sense that for any fixed* $0 < t < T_0$ *and* $\tilde{\delta} > q\delta(t)$ *we have, as* $s \to t$*,* $U(s) \to U(t)$ *in* $L^q_{\tilde{\delta}}(\mathbb{R}^d)$ *and*

$$\lim_{t\to 0} \|U(t)\|_{L^q_{q\delta(t)}(\mathbb{R}^d)} = 0, \tag{4.13}$$

Proof Using Proposition 2.9, we get for fixed $0 < t < T_0$, for any $\delta \geq \delta(t)$ and $\tilde{\delta}(t) = \frac{\delta}{1+4\delta t}$

$$\|U(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_q \int_0^t \mathrm{e}^{-\lambda(t-s)} \phi_{1,q}(t-s) \|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)}\, \mathrm{d}s,$$

with $\phi_{1,q}(t) = \left(\frac{1+4\delta t}{4\delta t}\right)^{\frac{d}{2q'}}$. Using (4.9) and choosing $\delta = \delta(t)$ as in Theorem 4.1, if $1 < \sigma < \infty$,

$$\|U(t)\|_{L^q_{q\delta(t)}(\mathbb{R}^d)} \leq c \left(\frac{\delta(t)}{\varepsilon_0}\right)^{d/2} \left(\int_0^t \mathrm{e}^{-\sigma'\lambda s} \phi^{\sigma'}_{1,q}(s)\, \mathrm{d}s\right)^{\frac{1}{\sigma'}} \left(\int_0^T \|f(s)\|^{\sigma}_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)}\, \mathrm{d}s\right)^{\frac{1}{\sigma}}$$

provided that $\frac{\sigma' d}{2q'} < 1$, that is, $1 < \sigma' < \frac{2q'}{d}$, due to the singularity of the first integrand at $s = 0$. This choice is possible only if $q < \frac{d}{(d-2)_+}$ and, in such a case, if $\sigma > \frac{2q'}{2q'-d}$, that is, $\frac{1}{\sigma} < 1 - \frac{d}{2q'} = 1 - \frac{d}{2} + \frac{d}{2q}$. Also, if $\lambda > 0$ the first integral is bounded as $t \to \infty$. Hence we get (4.12).

If $\sigma = \infty$ we get (4.12) provided that $\frac{d}{2q'} < 1$, that is $q < \frac{d}{(d-2)_+}$.

For the time continuity, from (4.10), the estimates above imply that the integral term converges to zero in $L^q_{q\delta(t)}(\mathbb{R}^d)$ as $t \to \tau$. On the other hand, since $U(\tau) \in L^q_{q\delta(\tau)}(\mathbb{R}^d)$ we have that $\mathrm{e}^{-\lambda(t-\tau)} S(t-\tau)U(\tau)$ is continuous as $t \to \tau$ in $L^q_{\tilde{\delta}}(\mathbb{R}^d)$ for $\tilde{\delta} > q\delta(\tau)$, see Proposition 4.8 in [5]. □

With a similar argument we can also estimate the gradient of $U(f)$ as follows. In such a case, we also get that $U(f)$ is a weak solution of (4.1).

Proposition 4.6 *Under the notations of Theorem 4.1, assume moreover that for some* $\sigma > 2$*, we have* $f \in L^{\sigma}((0,T), \mathcal{M}_{\varepsilon(\cdot)})$*.*

For any $\gamma > 1$ *define the non increasing function* $e_\gamma(t) := \inf_{0<s<t} \left(\frac{1}{4\varepsilon(s)} + \gamma s\right)$ *and* $T_{0,\gamma}$ *as* $0 < T_{0,\gamma} = \sup\{t \in [0,T],\ e_\gamma(t) > \gamma t\} \leq T$*. Finally define* $\delta_\gamma(t) := \frac{1}{4(e_\gamma(t)-\gamma t)}$*.*

Then $e(t) \leq e_\gamma(t) \leq \gamma e(t)$*,* $T_{0,\gamma} \leq T_0$ *and* $\gamma\delta_\gamma(t) \geq \delta(t)$ *and* $T_{0,\gamma} \to T_0$ *as* $\gamma \to 1$*. Moreover for any* $\tau < T_0$ *and* $\gamma > 1$ *sufficiently close to* 1 *we have*

$$\sup_{0\leq t\leq \tau} \|\nabla U(t)\|_{L^1_{\delta_\gamma(t)}(\mathbb{R}^d)} \leq C(\tau) \|f\|_{L^{\sigma}((0,T),\mathcal{M}_{\varepsilon(\cdot)})} \tag{4.14}$$

and $C(\tau) = \left(\frac{\delta_\gamma(\tau)}{\varepsilon_0}\right)^{d/2} c(\tau)$*, and* $c(\tau)$ *is uniformly bounded in* τ *if* $\lambda > 0$*.*

Moreover, U *is a weak solution of* (4.1) *that is for every* $\varphi \in C^\infty_c(\mathbb{R}^d)$*,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} U(t)\varphi + \int_{\mathbb{R}^d} \nabla U(t) \nabla\varphi + \lambda \int_{\mathbb{R}^d} U(t)\varphi = \int_{\mathbb{R}^d} \varphi\, \mathrm{d}f(t), \quad 0 < t < T_0. \tag{4.15}$$

In particular, $\nabla U \in C([0,T_0), L^1_{\delta_\gamma(\cdot)}(\mathbb{R}^d))$*, in the sense that for any fixed* $0 < t < T_0$ *and* $\tilde{\delta} > \delta_\gamma(t)$ *we have, as* $s \to t$*,* $\nabla U(s) \to \nabla U(t)$ *in* $L^1_{\tilde{\delta}}(\mathbb{R}^d)$ *and*

$$\lim_{t\to 0} \|\nabla U(t)\|_{L^1_{\delta_\gamma(t)}(\mathbb{R}^d)} = 0, \tag{4.16}$$

Proof Now we use estimates for derivatives as in (2.24) and Proposition 2.9, to get for fixed $0 < t < T_0$, for some $\delta > 0$ to be chosen below, $\tilde{\delta}(t) = \frac{\delta}{1+4\gamma\delta t}$ with an arbitrary $\gamma > 1$, using (4.9),

$$\|\nabla U(t)\|_{L^1_{\tilde{\delta}}(\mathbb{R}^d)} \leq c \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{1/2}} \|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)} \, ds$$
$$\leq c \left(\frac{\delta}{\varepsilon_0}\right)^{d/2} \Big(\int_0^t \frac{e^{-\sigma'\lambda s}}{s^{\sigma'/2}} \, ds \Big)^{\frac{1}{\sigma'}} \Big(\int_0^T \|f(s)\|^{\sigma}_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)} \, ds \Big)^{\frac{1}{\sigma}},$$

(with obvious changes if $\sigma = \infty$) provided that $\tilde{\delta}(t-s) \geq \varepsilon(s)$ for $0 < s < t$ and this is finite if $\sigma > 2$. Also, if $\lambda > 0$ then the first integral is bounded uniformly as $t \to \infty$. The condition $\tilde{\delta}(t-s) \geq \varepsilon(s)$ for $0 < s < t$ holds iff

$$\gamma t + \frac{1}{4\delta} \leq e_\gamma(t) := \inf_{0<s<t} \big(\frac{1}{4\varepsilon(s)} + \gamma s\big).$$

This is satisfied provided that $0 < t < T_{0,\gamma}$ which is defined as

$$0 < T_{0,\gamma} = \sup\{t \in [0, T], \ e_\gamma(t) > \gamma t\} \leq T.$$

Hence we chose the smallest possible value of δ that is, $\delta = \delta_\gamma(t) := \frac{1}{4(e_\gamma(t)-\gamma t)}$. Hence we get (4.14).

Now observe that, with the notations in Theorem 4.1, $e(t) \leq e_\gamma(t) \leq \gamma e(t)$ and this implies $T_{0,\gamma} \leq T_0$ and $\delta_\gamma(t) \geq \gamma\delta(t)$. Also, notice that $T_{0,\gamma} \to T_0$ as $\gamma \to 1$.

Now, integrating by parts in (4.7) as in Lemma A.3 in [4], we get (4.15).

For the time continuity, from (4.10), the estimates above imply that the gradient of the integral term converges to zero in $L^1_{\delta_\gamma(t)}(\mathbb{R}^d)$ as $t \to \tau$. On the other hand, since $\nabla U(\tau) \in L^1_{\delta_\gamma(\tau)}(\mathbb{R}^d)$ we have that using properties of the convolution $e^{-\lambda(t-\tau)}\nabla S(t-\tau)U(\tau) = e^{-\lambda(t-\tau)}S(t-\tau)\nabla U(\tau)$ and so is continuous as $t \to \tau$ in $L^1_{\tilde{\delta}}(\mathbb{R}^d)$ for $\tilde{\delta} > \delta_\gamma(\tau)$. □

Similarly, we now obtain an L^r type estimate of the gradient of $U(f)$.

Corollary 4.7 *Under the notations of Theorem 4.1 and Proposition 4.6, assume moreover that we have $f \in L^\sigma((0,T), \mathcal{M}_{\varepsilon(\cdot)})$ for $\sigma > 2$.*

Then for any $\tau < T_0$ and $\gamma > 1$ sufficiently close to 1 we have

$$\sup_{0\leq t\leq\tau} \|\nabla U(t)\|_{L^r_{r\delta_\gamma(t)}(\mathbb{R}^d)} \leq C(\tau)\|f\|_{L^\sigma((0,T),\mathcal{M}_{\varepsilon(\cdot)})} \tag{4.17}$$

where $\delta_\gamma(t) := \frac{1}{4(e_\gamma(t)-\gamma t)}$ and $e_\gamma(t) := \inf_{0<s<t}\big(\frac{1}{4\varepsilon(s)} + \gamma s\big)$, $1 \leq r < \frac{d}{(d-1)_+}$ and $\frac{1}{\sigma} + \frac{d}{2} < \frac{1}{2} + \frac{d}{2r}$. Also, $C(\tau) = \left(\frac{\delta_\gamma(\tau)}{\varepsilon_0}\right)^{d/2} c(\tau)$, and $c(\tau)$ is uniformly bounded in τ if $\lambda > 0$.

In particular, $\nabla U \in C([0, T_0), L^r_{r\delta_\gamma(\cdot)}(\mathbb{R}^d))$, in the sense that for any fixed $0 < t < T_0$ and $\tilde{\delta} > r\delta_\gamma(t)$ we have, as $s \to t$, $\nabla U(s) \to \nabla U(t)$ in $L^r_{\tilde{\delta}}(\mathbb{R}^d)$ and

$$\lim_{t\to 0} \|\nabla U(t)\|_{L^r_{r\delta_\gamma(t)}(\mathbb{R}^d)} = 0, \tag{4.18}$$

Proof Again from derivative estimates in Proposition 2.9 we get for fixed $0 < t < T_0$, for some $\delta > 0$ to be chosen below, $\tilde{\delta}(t) = \frac{\delta}{1+4\gamma\delta t}$ with an arbitrary $\gamma > 1$, using (4.9),

$$\|\nabla U(t)\|_{L^r_{r\delta}(\mathbb{R}^d)} \leq c \left(\frac{\delta}{\varepsilon_0}\right)^{d/2} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\frac{1}{2}}} \Phi_{1,r}(t-s)\|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)} \, ds,$$

with $\Phi_{1,r}(t) = \left(\frac{1+4\gamma\delta t}{t}\right)^{\frac{d}{2r'}}$ provided that $\tilde{\delta}(t-s) \geq \varepsilon(s)$ for $0 < s < t$ which holds iff

$$\gamma t + \frac{1}{4\delta} \leq e_{\gamma}(t) = \inf_{0<s<t} \left(\frac{1}{4\varepsilon(s)} + \gamma s \right).$$

Then we chose $\delta = \delta_{\gamma}(t)$ as in Proposition 4.6 and hence

$$\|\nabla U(t)\|_{L^r_{r\delta_{\gamma}(t)}(\mathbb{R}^d)} \leq c \left(\frac{\delta_{\gamma}(t)}{\varepsilon_0} \right)^{d/2} \left(\int_0^t \frac{e^{-\sigma'\lambda s}}{s^{\frac{\sigma'}{2}}} \Phi^{\sigma'}_{1,r}(s)\, ds \right)^{\frac{1}{\sigma'}} \left(\int_0^T \|f(s)\|^{\sigma}_{\mathcal{M}_{\varepsilon(s)}(\mathbb{R}^d)}\, ds \right)^{\frac{1}{\sigma}}$$

provided that $\sigma'(\frac{1}{2} + \frac{d}{2r'}) < 1$, that is, $1 < \sigma' < \frac{2r'}{d+r'}$, due to the singularity of the first integrand at $s = 0$. Also if $\lambda > 0$ the first integral is uniformly bounded as $t \to \infty$. This choice is possible if $r < \frac{d}{(d-1)_+}$ and in such a case $\sigma > \frac{2r'}{r'-d}$, that is, $\frac{1}{\sigma} < \frac{1}{2} - \frac{d}{2r'} = \frac{1}{2} - \frac{d}{2} + \frac{d}{2r}$. Hence we get (4.17). If $\sigma = \infty$ we get (4.17) provided that $\frac{d}{r'} < 1$, that is $r < \frac{d}{(d-1)_+}$.

The time continuity is as in Proposition 4.6. □

In the next result, we assume f has better integrability in space and obtain further regularity for $U(f)$ and its gradient.

Proposition 4.8 *Under the notations of Theorem 4.1 and Proposition 4.6, assume* $f \in L^{\sigma}((0,T), L^p_{\varepsilon(\cdot)}(\mathbb{R}^d))$

(i) *Then for any* $\tau < T_0$ *and* $1 \leq p < \infty$ *and* $\delta(t) := \frac{1}{4(e(t)-t)}$,

$$\sup_{0\leq t\leq \tau} \|U(t)\|_{L^q_{\frac{q}{p}\delta(t)}(\mathbb{R}^d)} \leq C(\tau)\|f\|_{L^{\sigma}((0,T),L^p_{\varepsilon(\cdot)})} \tag{4.19}$$

for $p \leq q < \frac{pd}{(d-2p)_+}$ *and* $\frac{1}{\sigma} + \frac{d}{2p} < 1 + \frac{d}{2q}$. *Also,* $C(\tau) = \left(\frac{p\delta(\tau)}{\varepsilon_0}\right)^{d/2p} c(\tau)$, *and* $c(\tau)$ *is uniformly bounded in* τ *if* $\lambda > 0$.
If $p = \infty$ *and* $1 \leq \sigma \leq \infty$

$$\sup_{0\leq t\leq \tau} \|U(t)\|_{L^{\infty}_{\delta(t)}(\mathbb{R}^d)} \leq C(\tau)\|f\|_{L^{\sigma}((0,T),L^{\infty}_{\varepsilon(\cdot)})} \tag{4.20}$$

with $C(\tau) = \left(\frac{\delta(\tau)}{\varepsilon_0}\right)^{d/2} c(\tau)$, *and* $c(\tau)$ *is uniformly bounded in* τ *if* $\lambda > 0$.
Also $U \in C([0,T_0), L^q_{q\delta(\cdot)}(\mathbb{R}^d))$, *in the sense that for any fixed* $0 < t < T_0$ *and* $\tilde{\delta} > \frac{q}{p}\delta(t)$ *we have, as* $s \to t$, $U(s) \to U(t)$ *in* $L^q_{\tilde{\delta}}(\mathbb{R}^d)$ *and*

$$\lim_{t\to 0} \|U(t)\|_{L^q_{\frac{q}{p}\delta(t)}(\mathbb{R}^d)} = 0, \tag{4.21}$$

(ii) *For any* $\tau < T_0$ *and* $1 \leq p \leq r < \infty$ *and* $\delta_{\gamma}(t) := \frac{1}{4(e_{\gamma}(t)-\gamma t)}$ *with* $\gamma > 1$ *sufficiently close to* 1

$$\sup_{0\leq t\leq \tau} \|\nabla U(t)\|_{L^r_{r\delta_{\gamma}(t)}(\mathbb{R}^d)} \leq C(\tau)\|f\|_{L^{\sigma}((0,T),L^p_{\varepsilon(\cdot)})} \tag{4.22}$$

for $r < \frac{pd}{(d-p)_+}$ *and* $\frac{1}{\sigma} + \frac{d}{2p} < \frac{1}{2} + \frac{d}{2r}$ *and* $C(\tau) = \left(\frac{p\delta_{\gamma}(t)}{\varepsilon_0}\right)^{d/2p} c(\tau)$, *and* $c(\tau)$ *is uniformly bounded in* τ *if* $\lambda > 0$.
In particular, if $\frac{1}{\sigma} + \frac{d}{2p} < 1$ *we can take* $r > d$ *and then* $U(t)$ *is continuous in* $\mathbb{R}^d$ *for each* $t < T_0$.

Also, $\nabla U \in C([0,T_0), L^r_{r\delta_\gamma(\cdot)}(\mathbb{R}^d))$*, in the sense that for any fixed* $0 < t < T_0$ *and* $\tilde{\delta} > r\delta_\gamma(t)$ *we have, as* $s \to t$*,* $\nabla U(s) \to \nabla U(t)$ *in* $L^r_{\tilde{\delta}}(\mathbb{R}^d)$ *and*

$$\lim_{t\to 0} \|\nabla U(t)\|_{L^r_{r\delta_\gamma(t)}(\mathbb{R}^d)} = 0, \tag{4.23}$$

Proof (i) If $1 \le p \le q < \infty$, from Proposition 2.9 (see also (2.21)) we get for fixed $0 < t < T_0$, for some $\delta > 0$ to be chosen below and $\tilde{\delta}_p(s) = \frac{p\delta}{1+4p\delta s}$

$$\|U(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le \int_0^t \mathrm{e}^{-\lambda(t-s)} \phi_{p,q}(t-s) \|f(s)\|_{L^p_{\tilde{\delta}_p(t-s)}(\mathbb{R}^d)}\, \mathrm{d}s,$$

with $\phi_{p,q}(t) = c\big(\frac{1+4p\delta t}{4p\delta t}\big)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$. Now, analogously to (4.9), from (2.6) we have

$$\|f(s)\|_{L^p_{\tilde{\delta}_p(t-s)}(\mathbb{R}^d)} \le \left(\frac{p\delta}{\varepsilon_0}\right)^{d/2p} \|f(s)\|_{L^p_{\varepsilon(s)}(\mathbb{R}^d)}, \quad 0 < s < t. \tag{4.24}$$

provided that $\tilde{\delta}_p(t-s) \ge \varepsilon(s)$. In such a case

$$\|U(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le c\left(\frac{p\delta}{\varepsilon_0}\right)^{d/2p} \left(\int_0^t \mathrm{e}^{-\sigma'\lambda s}\phi^{\sigma'}_{p,q}(s)\,\mathrm{d}s\right)^{\frac{1}{\sigma'}} \left(\int_0^T \|f(s)\|^{\sigma}_{L^p_{\varepsilon(s)}(\mathbb{R}^d)}\,\mathrm{d}s\right)^{\frac{1}{\sigma}}$$

(with obvious changes if $\sigma = \infty$) which is finite provided that $1 < \sigma' < \frac{1}{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$ due to the singularity of the first integrand at $s = 0$. This choice is possible only if $\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) < 1$, that is $q < \frac{pd}{(d-2p)_+}$ and $\sigma > \frac{1}{1-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$, that is $\frac{1}{\sigma} < 1 - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})$. In such a case, if $\lambda > 0$, the first integral is bounded uniformly as $t \to \infty$. Now, the condition $\tilde{\delta}_p(t-s) \ge \varepsilon(s)$ translates again into

$$t + \frac{1}{4p\delta} \le e(t) = \inf_{0<s<t} \big(\frac{1}{4\varepsilon(s)} + s\big).$$

Hence we choose $\delta = \delta_p(t)$ such that $t + \frac{1}{4p\delta_p(t)} = e(t)$, that is,

$$\delta_p(t) = \frac{1}{4p(e(t)-t)} = \frac{1}{p}\delta(t)$$

and we get (4.19).

If $p = \infty$, we use Proposition 2.9 (see also (2.23)) and then for fixed $0 < t < T_0$, for some $\delta > 0$ to be chosen below and $\tilde{\delta}(s) = \frac{\delta}{1+4\delta s}$

$$\|U(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le \int_0^t \mathrm{e}^{-\lambda(t-s)}\phi_{\infty,\infty}(t-s)\|f(s)\|_{L^\infty_{\tilde{\delta}(t-s)}(\mathbb{R}^d)}\,\mathrm{d}s,$$

with $\phi_{\infty,\infty}(t) = c\big(1 + 4p\delta t\big)^d$. Hence

$$\|U(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le c(t)\int_0^t \|f(s)\|_{L^\infty_{\tilde{\delta}(t-s)}(\mathbb{R}^d)}\,\mathrm{d}s,$$

with $c(t) = \sup_{0\le s\le t} \mathrm{e}^{-\lambda s}\phi_{\infty,\infty}(s)$ and as in the proof of Theorem 4.1, the condition $\tilde{\delta}(t - s) \ge \varepsilon(s)$ is equivalent to $e(t) \ge \frac{1}{4\delta} + t$, so we chose $\delta = \delta(t) = \frac{1}{4(e(t)-t)}$. Therefore

$$\|U(t)\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)} \le c(t)\left(\frac{\delta(t)}{\varepsilon_0}\right)^{d/2} \int_0^T \|f(s)\|_{L^\infty_{\varepsilon(s)}(\mathbb{R}^d)}\,\mathrm{d}s$$

and we get (4.20).

The proof of the time continuity is as in Corollary 4.5.

(ii) For $1 \le p \le r < \infty$, from the gradient estimates in Proposition 2.9 (see also (2.24)), we have for for any $\tau < T_0$, $\gamma > 1$, $\delta > 0$ to be chosen below and $\tilde{\delta}_p(s) = \frac{p\delta}{1+4p\delta\gamma s}$

$$\|\nabla U(t)\|_{L^r_{r\delta}(\mathbb{R}^d)} \le \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\frac{1}{2}}} \Phi_{p,r}(t-s)\|f(s)\|_{L^p_{\tilde{\delta}_p(t-s)}(\mathbb{R}^d)}\, ds,$$

with $\gamma > 1$ and $\Phi_{p,r}(t)$ with $\Phi_{p,r}(t) = c\big(\frac{1+4p\gamma\delta t}{t}\big)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}$ provided that $\tilde{\delta}_p(t-s) \ge \varepsilon(s)$. In such a case, using (4.24),

$$\|\nabla U(t)\|_{L^r_{r\delta}(\mathbb{R}^d)} \le c\left(\frac{p\delta}{\varepsilon_0}\right)^{d/2p} \left(\int_0^t \frac{e^{-\sigma'\lambda s}}{s^{\frac{\sigma'}{2}}} \Phi^{\sigma'}_{p,r}(s)\, ds\right)^{\frac{1}{\sigma'}} \left(\int_0^T \|f(s)\|^{\sigma}_{L^p_{\varepsilon(s)}(\mathbb{R}^d)}\, ds\right)^{\frac{1}{\sigma}}$$

(with obvious changes if $\sigma = \infty$) which is finite provided that $1 < \sigma' < \frac{1}{\frac{1}{2}+\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}$ due to the singularity of the first integrand at $s = 0$. Also, if $\lambda > 0$ the first integral above is uniformly bounded as $t \to \infty$. This choice is possible only if $\frac{d}{2}(\frac{1}{p} - \frac{1}{r}) < \frac{1}{2}$, that is $r < \frac{pd}{(d-p)_+}$ and $\sigma > \frac{1}{\frac{1}{2}-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}$, that is, $\frac{1}{\sigma} < \frac{1}{2} - \frac{d}{2}(\frac{1}{p} - \frac{1}{r})$.

As in Proposition 4.6 the condition $\tilde{\delta}_p(t-s) \ge \varepsilon(s)$ translates into $\delta \ge \delta_\gamma(t) := \frac{1}{4(e_\gamma(t)-\gamma t)}$. Taking $\delta = \delta_\gamma(t)$ we get (4.22).

In particular, if $\frac{1}{\sigma} + \frac{d}{2p} < 1$ we can take $r > d$ such that $\frac{1}{\sigma} + \frac{d}{2p} < \frac{1}{2} + \frac{d}{2r} < 1$ and then for each $t < T_0$ we have $\nabla U(t) \in L^r_{loc}(\mathbb{R}^d)$ and therefore $U(t)$ is continuous in $\mathbb{R}^d$.

The proof of the time continuity is as in Proposition 4.6 and Corollary 4.7. $\square$

4.3 Global Solutions and Estimates

In this section we analyze when the function $U(f)$ above is a global solution, that is, defined for as long f is defined. We are also interested in obtaining global (in the same sense) estimates on $U(f)$ in terms of f.

To begin with we observe that in (4.5), we can even take $\tau = T_0$ provided that

$$T < e(T) = \inf_{0\le s\le T}\left(\frac{1}{4\varepsilon(s)} + s\right),$$

and then $T_0 = T$ and $\delta(T) < \infty$. This holds if and only if

$$\varepsilon(s) < \frac{1}{4(T-s)}, \quad 0 \le s < T. \tag{4.25}$$

Otherwise $\delta(t) \to \infty$ as $t \to T_0$.

If (4.25) holds, then (4.5), reads

$$\sup_{0\le t\le T} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^1((0,T),\mathcal{M}_{\varepsilon(\cdot)})} \tag{4.26}$$

with $C(T) = \left(\frac{1}{4\varepsilon_0(e(T)-T)}\right)^{d/2} \sup_{0\le s\le T} e^{-\lambda s}$. In the same way we obtain global estimates for U in (4.12), (4.19), (4.20), with constants $C(T) = \left(\frac{p}{4\varepsilon_0(e(T)-T)}\right)^{d/2p} c(T)$, where $c(T)$ is uniformly bounded in T if $\lambda > 0$.

Analogously, for the estimates on the gradients, we can take $\tau = T$ provided that for some $\gamma > 1$

$$T < e_\gamma(T) := \inf_{0<s<T} \Big(\frac{1}{4\varepsilon(s)} + \gamma s\Big)$$

that is,

$$\varepsilon(s) < \frac{1}{4(T-\gamma s)}, \quad 0 \le s < \frac{T}{\gamma},$$

and then $\delta_\gamma(T) := \frac{1}{4(e_\gamma(T)-\gamma T)} < \infty$. Then (4.14) reads

$$\sup_{0\le t\le T} \|\nabla U(t)\|_{L^1_{\delta_\gamma(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_{\varepsilon(\cdot)})} \tag{4.27}$$

with $C(T) = \Big(\frac{1}{4\varepsilon_0(e_\gamma(T)-\gamma T)}\Big)^{d/2} c(T)$, and $c(T)$ is uniformly bounded in T if $\lambda > 0$. In the same way we obtain global estimates for ∇U in (4.17), (4.22) with constants $C(T) = \Big(\frac{p}{4\varepsilon_0(e(T)-T)}\Big)^{d/2p} c(T)$, where $c(T)$ is uniformly bounded in T if $\lambda > 0$.

We concentrate below on the significant special case in which $f \in L^\sigma((0,T),\mathcal{M}_{\varepsilon(\cdot)})$ with $\varepsilon(s) = \varepsilon > 0$ constant. This is the case, if for example, f does not depend on time, which will be analysed further below.

In such a case, then $\varepsilon_0 = \varepsilon$, $e(t) = \frac{1}{4\varepsilon}$, $T_0 := \sup\{t \in [0,T],\ e(t) > t\} = \min\{\frac{1}{4\varepsilon}, T\}$ and $\delta(t) = \frac{1}{4(e(t)-t)} = \frac{\varepsilon}{1-4\varepsilon t}$.

Also, $e_\gamma(t) = \frac{1}{4\varepsilon}$, $0 < T_{0,\gamma} = \sup\{t \in [0,T],\ e_\gamma(t) > \gamma t\} = \min\{\frac{1}{4\varepsilon\gamma}, T\}$ and $\delta_\gamma(t) = \frac{1}{4(e_\gamma(t)-\gamma t)} = \frac{\varepsilon}{1-4\varepsilon\gamma t}$.

In particular, we can take $\tau = T$ in (4.26) and (4.27) if $\gamma\varepsilon T < \frac{1}{4}$, $\gamma > 1$. Therefore we have proved the following result.

Corollary 4.9 *Assume $f \in L^\sigma((0,T),\mathcal{M}_\varepsilon)$ for $\varepsilon > 0$, $1 \le \sigma \le \infty$ and $\varepsilon T < \frac{1}{4}$. Then we have, with the notations and ranges in (4.5), (4.12), (4.19), (4.20), (4.14), (4.17), (4.22),*

$$\sup_{0\le t\le T} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^1((0,T),\mathcal{M}_\varepsilon)}$$
$$\sup_{0\le t\le T} \|U(t)\|_{L^q_{q\delta(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_\varepsilon)}$$
$$\sup_{0\le t\le T} \|U(t)\|_{L^q_{\frac{q}{p}\delta(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),L^p_\varepsilon)}$$
$$\sup_{0\le t\le T} \|U(t)\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),L^\infty_\varepsilon)}$$

with constants $C(T) = \Big(\frac{p}{1-4\varepsilon T}\Big)^{d/2p} c(T)$, where $c(T)$ is uniformly bounded in T if $\lambda > 0$ and

$$\sup_{0\le t\le T} \|\nabla U(t)\|_{L^1_{\delta_\gamma(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_\varepsilon)}$$
$$\sup_{0\le t\le T} \|\nabla U(t)\|_{L^r_{r\delta_\gamma(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_\varepsilon)}$$
$$\sup_{0\le t\le T} \|\nabla U(t)\|_{L^r_{r\delta_\gamma(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),L^p_\varepsilon)}$$

with constants $C(T) = \left(\frac{p}{1-4\gamma\varepsilon T}\right)^{d/2p} c(T)$, *where* $\gamma > 1$ *and* $\gamma\varepsilon T < \frac{1}{4}$, $c(T)$ *is uniformly bounded in* T *if* $\lambda > 0$.

Assume furthermore that

$$f \in L^1((0,T), \mathcal{M}_0(\mathbb{R}^d))$$

in the sense that $f \in L^1((0,T), \mathcal{M}_\varepsilon(\mathbb{R}^d))$ for every $\varepsilon > 0$ and $\|f\|_{L^1((0,T),\mathcal{M}_\varepsilon(\mathbb{R}^d))} \le C < \infty$ independent of $\varepsilon > 0$. Then we have the following result.

Corollary 4.10 *We have* $U \in L^\infty((0,T), L^1_0(\mathbb{R}^d))$ *and for any* $\varepsilon > 0$ *small, with the notations and ranges in Corollary 4.9,*

$$U \in L^\infty((0,T), L^1_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|U(t)\|_{L^1_{2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^1((0,T),\mathcal{M}_\varepsilon)}$$
$$U \in L^\infty((0,T), L^q_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|U(t)\|_{L^q_{q2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_\varepsilon)}$$
$$U \in L^\infty((0,T), L^q_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|U(t)\|_{L^q_{\frac{q}{p}2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),L^p_\varepsilon)}$$
$$U \in L^\infty((0,T), L^\infty_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|U(t)\|_{L^\infty_{2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),L^\infty_\varepsilon)}$$

and

$$\nabla U \in L^\infty((0,T), L^1_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|\nabla U(t)\|_{L^1_{2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_\varepsilon)}$$
$$\nabla U \in L^\infty((0,T), L^r_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|\nabla U(t)\|_{L^r_{r2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),\mathcal{M}_\varepsilon)}$$
$$\nabla U \in L^\infty((0,T), L^r_0(\mathbb{R}^d)) \quad \text{and} \quad \sup_{0\le t\le T} \|\nabla U(t)\|_{L^r_{r2\varepsilon}(\mathbb{R}^d)} \le C(T)\|f\|_{L^\sigma((0,T),L^p_\varepsilon)}$$

where $\gamma > 1$, *and* $C(T)$ *is uniformly bounded in* T *if* $\lambda > 0$. *In such a case we can take* $T \to \infty$ *in the estimates above.*

Proof We take $0 < \varepsilon < \frac{1}{8T}$ and $\gamma > 1$ such that $0 < \varepsilon\gamma \le \frac{1}{8T}$ so the assumptions in Corollary 4.9 are satisfied and also $\varepsilon \le \delta(t) = \frac{\varepsilon}{1-4\varepsilon t} \le 2\varepsilon$ and $\varepsilon \le \delta_\gamma(t) = \frac{\varepsilon}{1-4\varepsilon\gamma t} \le 2\varepsilon$ for $t \in [0,T]$.

For example estimate (4.26) now gives, using (2.7),

$$\frac{1}{2^{d/2}} \sup_{0\le t\le T} \|U(t)\|_{L^1_{2\varepsilon}(\mathbb{R}^d)} \le \sup_{0\le t\le T} \|U(t)\|_{L^1_{\delta(t)}(\mathbb{R}^d)} \le C(T)\|f\|_{L^1((0,T),\mathcal{M}_\varepsilon)}$$

with $C(T) = \left(\frac{1}{1-4\varepsilon T}\right)^{d/2} \sup_{0\le s\le T} e^{-\lambda s} \le 2^{d/2} \sup_{0\le s\le T} e^{-\lambda s}$. All other estimates are obtained in the same way. □

4.4 Finite-Time Blowup

In this section we explore the possibility that $U(f)$ blows up in finite time. Before continuing, we prove the following result for solutions of (1.1) and (4.1) regarding translations of the initial data.

Lemma 4.11 (i) *Assume $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$. Then for any $y \in \mathbb{R}^d$ we have*

$$S(t)\tau_{-y}u_0 = \tau_{-y}S(t)u_0, \qquad 0 \le t \le \frac{1}{4\varepsilon}.$$

(ii) *Whenever $U = U(f)$ in (4.3) is defined,*

$$U(t, \tau_{-y}f) = \tau_{-y}U(t, f).$$

Proof. From Lemma 5.4 in [4] we have

$$\begin{aligned} S(t)\tau_{-y}u_0(x) &= \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-z|^2}{4t}} \, \mathrm{d}\tau_{-y}u_0(z) \\ &= \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x+y-z|^2}{4t}} \, \mathrm{d}u_0(z) = \tau_{-y}S(t)u_0(x). \end{aligned}$$

Using this we obtain

$$\begin{aligned} U(t, \tau_{-y}f) &= \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s)\tau_{-y}f(s)\,\mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-\lambda(t-s)} \tau_{-y}S(t-s)f(s)\,\mathrm{d}s = \tau_{-y}U(t, f). \end{aligned}$$ □

Recall from [4] that given $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^d)$ we can consider its 'optimal index'

$$0 \le \varepsilon_0(\mu) := \inf\{\varepsilon : \ \mu \in \mathcal{M}_\varepsilon(\mathbb{R}^d)\} = \sup\{\varepsilon : \ \mu \notin \mathcal{M}_\varepsilon(\mathbb{R}^d)\} \le \infty. \tag{4.28}$$

If $f \ge 0$ with $0 < \varepsilon_0(f) < \infty$ it follows Theorem 5.5 in [4] that $S(t)f$ cannot be defined at any point in $\mathbb{R}^d$ beyond the time $T(f) = \frac{1}{4\varepsilon_0(f)}$. Now we can prove a converse result for Theorem 4.1 for non-negative data.

Theorem 4.12 *Assume that for each $0 \le t \le T$, $f(t) \ge 0$ and has optimal index $\varepsilon_0(t) < \infty$. Assume furthermore that for some $x_0 \in \mathbb{R}^d$ and $0 < t \le T$ we have*

$$U(x_0, t) = \int_0^t \mathrm{e}^{-\lambda(t-s)} S(t-s)f(s)(x_0)\,\mathrm{d}s < \infty.$$

Then

(i) $t \le e_0(t) := \inf_{0 \le s \le t} \left(\frac{1}{4\varepsilon_0(s)} + s\right)$, *$\varepsilon_0(U(s)) < \infty$ for $0 < s < t$ and U is a very weak solution of (4.1) in $(0, t)$, with $U(0) = 0$.*
Therefore, U cannot be defined at any point $x \in \mathbb{R}^d$ beyond $0 < T_0 \le T$ where T_0 is characterized by

$$T_0 = \sup\{s \in [0, T], \ e_0(s) > s\} \le T.$$

(ii) *If*

$$t < e_0(t) := \inf_{0 \le s \le t} \left(\frac{1}{4\varepsilon_0(s)} + s\right)$$

then the optimal index of $U(t)$ satisfies

$$\frac{1}{4(e_0(t) - t)} \le \varepsilon_0(U(t)) < \infty \qquad \textit{and} \qquad \varepsilon_0(U(t)) \ge \varepsilon_0(t).$$

Proof (i) Since

$$U(x_0,t)=\int_0^t \mathrm{e}^{-\lambda(t-s)}S(t-s)f(s)(x_0)\,\mathrm{d}s<\infty$$

then for a.e. $0<s<t$ we have $S(t-s)f(s)(x_0)<\infty$ and then from Lemma 3.2 in [4], $f(s)\in\mathcal{M}_{\varepsilon(s)}$ for every $\varepsilon(s)>\frac{1}{4(t-s)}$. This implies that

$$\varepsilon_0(s)\le\frac{1}{4(t-s)},\quad 0<s<t,$$

that is

$$t\le e_0(t)=\inf_{0\le s\le t}\Big(\frac{1}{4\varepsilon_0(s)}+s\Big).$$

Since $e_0(t)$ is non decreasing, this gives the characterization of the maximal existence time T_0 for U.

On the other hand, we can write, using Remark 2.11,

$$\begin{aligned}U(x_0,t)&=\int_0^t \mathrm{e}^{-\lambda(t-s)}S(t-s)\tau_{-x_0}f(s)(0)\,\mathrm{d}s\\&=\int_0^t \mathrm{e}^{-\lambda(t-s)}\|\tau_{-x_0}f(s)\|_{\mathcal{M}_{\frac{1}{4(t-s)}}(\mathbb{R}^d)}\,\mathrm{d}s<\infty\end{aligned}$$

which implies that

$$\int_0^t\|\tau_{-x_0}f(s)\|_{\mathcal{M}_{\frac{1}{4(t-s)}}(\mathbb{R}^d)}\,\mathrm{d}s<\infty.$$

Hence, by Lemma 4.11, it follows that $U(t,\tau_{-x_0}f)=\tau_{-x_0}U(t,f)$ satisfies Theorem 4.1 with $\varepsilon(s)=\frac{1}{4(t-s)}$ in $[0,\tau]$ for any $0<s<\tau<t$. In particular $\varepsilon_0(U(s))<\infty$ for $0<s<t$ and U is a very weak solution of (4.1) in $(0,t)$ with $U(0)=0$.
(ii) For any $\delta>\varepsilon_0(U(t))$ we have, as in Corollary 4.2,

$$\|U(t)\|_{L^1_\delta(\mathbb{R}^d)}=\int_0^t \mathrm{e}^{-\lambda(t-s)}\|f(s)\|_{\mathcal{M}_{\tilde{\delta}(t-s)}(\mathbb{R}^d)}\,\mathrm{d}s<\infty$$

with $\tilde{\delta}(s)=\frac{\delta}{1+4\delta s}$ which implies that $\tilde{\delta}(t-s)\ge\varepsilon_0(s)$ for a.e. $0\le s\le t$. This condition then reads

$$\frac{1}{4\delta}+t\le e_0(t)=\inf_{0\le s\le t}\Big(\frac{1}{4\varepsilon_0(s)}+s\Big)$$

which gives $\delta\ge\frac{1}{4(e(t)-t)}$ and $\delta\ge\varepsilon_0(t)$ and the result follows. □

When f does not depend on time the above results become much simpler.

Corollary 4.13 *Assume that* $f\in\mathcal{M}_\varepsilon(\mathbb{R}^d)$*. Then*

(i) *Theorem 4.1 applies with* $e(t)=\frac{1}{4\varepsilon}=T(\varepsilon)$*,* $T<T(\varepsilon)$*,* $T_0=T$ *and* $\delta(t)=\delta>\varepsilon$ *and* U *is given by*

$$U(t)=\int_0^t \mathrm{e}^{-\lambda s}S(s)f\,\mathrm{d}s,\qquad 0\le t<T(\varepsilon).$$

In particular, if $f\ge 0$ *then* $U(t)$ *is increasing in* t*.*

(ii) *Moreover, if f has optimal index $0 < \varepsilon_0 < \infty$ then Theorem 4.12 applies with $e_0(t) = \frac{1}{4\varepsilon_0} = T(f)$, $T = T_0 = T(f)$. Hence U cannot be defined at any point $x \in \mathbb{R}^d$ beyond time $T(f)$.*

Notice that from the results in [4], if $\varepsilon_0 = \varepsilon_0(f) > 0$ then there is a convex set $K \subset \mathbb{R}^d$, with $K = \emptyset$ or $K = \mathbb{R}^d$ being possible, such that, as $t \to T(f) = T$, $S(t)f(x)$ has a finite limit if and only if $x \in K$. This set is characterized by the property $x \in K$ if and only if $\tau_{-x} f \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)$.

The following result identifies those points in $\mathbb{R}^d$ such that $U(x,t)$ has a pointwise limit as $t \to T$. These include the set K plus some others for which f satisfies some additional conditions.

Proposition 4.14 *Assume $0 \leq f$ is such that*

$$0 < \varepsilon_0(f) < \infty;$$

define $T(f) = \frac{1}{4\varepsilon_0(f)}$ and consider $U(t) = U(t, f)$ as in Corollary 4.13.

Then the pointwise limit

$$\lim_{t \to T(f)} U(x,t)$$

exists if and only if $x \in K$ or, if $x \notin K$,

$$\int_{\varepsilon_0}^{\infty} \frac{1}{r^2} \|\tau_{-x} f\|_{\mathcal{M}_r(\mathbb{R}^d)} \, \mathrm{d}r < \infty.$$

This in turn is equivalent to the asymptotic condition

$$\int_{|y|>1} \frac{\mathrm{e}^{-\varepsilon_0 |y|^2}}{|y|^2} \, \mathrm{d}\tau_{-x} f(y) < \infty \tag{4.29}$$

and a local condition, which for $d \geq 3$ is

$$\int_{|y|<1} \frac{1}{|y|^{d-2}} \, \mathrm{d}\tau_{-x} f(y) < \infty \tag{4.30}$$

and for $d = 2$ is

$$\int_{|y|<1} \ln|y| \, \mathrm{d}\tau_{-x} f(y) < \infty. \tag{4.31}$$

Proof. Let K be the convex set $\{x \in \mathbb{R}^d : \tau_{-x} f \in \mathcal{M}_{\varepsilon_0}(\mathbb{R}^d)\}$ mentioned above. Hence, for $x \in K$, as $t \to T(f) = T$,

$$U(x,t) = \int_0^t \mathrm{e}^{-\lambda s} S(s) f(x) \, \mathrm{d}s \to \int_0^T \mathrm{e}^{-\lambda s} S(s) f(x) \, \mathrm{d}s = U(T(f), x).$$

If $x \notin K$ we write

$$\begin{aligned} U(x,t) &= \int_0^t \mathrm{e}^{-\lambda s} S(s) f(x) \, \mathrm{d}s \\ &= \int_0^t \mathrm{e}^{-\lambda s} \frac{1}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4s}} \, \mathrm{d}f(y) \, \mathrm{d}s \end{aligned}$$

$$= \int_0^t e^{-\lambda s} \frac{1}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4s}} \, d\tau_{-x} f(y) \, ds$$
$$= \int_{\mathbb{R}^d} \int_0^t e^{-\lambda s} \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|y|^2}{4s}} \, ds \, d\tau_{-x} f(y).$$

This converges monotonically, as $t \to T$, to

$$I(x) = \int_{\mathbb{R}^d} F(y) \, d\tau_{-x} f(y), \qquad \text{where} \qquad F(y) = \int_0^T e^{-\lambda s} \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|y|^2}{4s}} \, ds. \tag{4.32}$$

Now notice that

$$I(x) = \int_0^T e^{-\lambda s} \frac{1}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{4s}} \, d\tau_{-x} f(y) \, ds = \int_0^T e^{-\lambda s} \|\tau_{-x} f\|_{\mathcal{M}_{1/4s}(\mathbb{R}^d)} \, ds$$

and changing variables with $r = 1/4s$ leads to

$$I(x) = \int_{\varepsilon_0}^{\infty} \frac{e^{\frac{-\lambda}{4r}}}{r^2} \|\tau_{-x} f\|_{\mathcal{M}_r(\mathbb{R}^d)} \, dr$$

which, since $\varepsilon_0 > 0$, is finite if and only if

$$\int_{\varepsilon_0}^{\infty} \frac{1}{r^2} \|\tau_{-x} f\|_{\mathcal{M}_r(\mathbb{R}^d)} \, dr < \infty.$$

To analyze $F(y)$ notice that changing variables with $r = 1/4s$ leads to

$$F(y) = \frac{1}{4\pi^{d/2}} \int_{\varepsilon_0}^{\infty} e^{\frac{-\lambda}{4r}} r^{d/2-2} e^{-r|y|^2} \, dr, \quad y \in \mathbb{R}^d$$

and since $\varepsilon_0 > 0$ the term $e^{\frac{-\lambda}{4r}}$ is bounded above and below independent of r so

$$F(y) \approx H(y) = \int_{\varepsilon_0}^{\infty} r^{d/2-2} e^{-r|y|^2} \, dr.$$

First, changing variables with $\omega = r|y|^2$ gives

$$H(y) = \frac{1}{|y|^{d-2}} \int_{\varepsilon_0 |y|^2}^{\infty} \omega^{d/2-2} e^{-\omega} \, d\omega$$

and so for $|y| \leq 1$ we get

$$H(y) \approx \begin{cases} \frac{1}{|y|^{d-2}}, & d \geq 3 \\ 1 + \ln|y| & d = 2. \end{cases}$$

On the other hand, for $|y| \geq 1$ we can write

$$H(y) = e^{-\varepsilon_0 |y|^2} \int_{\varepsilon_0}^{\infty} r^{d/2-2} e^{-(r-\varepsilon_0)|y|^2} \, dr$$

and changing variables with $\omega = (r - \varepsilon_0)|y|^2$ gives,

$$H(y) = \frac{e^{-\varepsilon_0 |y|^2}}{|y|^2} \int_0^{\infty} (\frac{\omega}{|y|^2} + \varepsilon_0)^{d/2-2} e^{-\omega} \, d\omega \approx \frac{e^{-\varepsilon_0 |y|^2}}{|y|^2}.$$

□

If $\varepsilon_0(f) = 0$ then all the above applies for any $T < \infty$. The asymptotic behaviour of $U(t)$ as $t \to \infty$, will be considered in the next section.

5 Poisson's Equation in $\mathcal{M}_0(\mathbb{R}^d)$

In this section our goal is to give solvability results for the elliptic equation

$$-\Delta u_* + \lambda u_* = f \tag{5.1}$$

with $\lambda \geq 0$ and $f \in \mathcal{M}_0(\mathbb{R}^d)$.

First observe that since we are dealing with functions that can be very large at infinity, there is no uniqueness for (5.1) for any value of λ. Indeed as observed in [4] for any $\lambda \in \mathbb{R}$, we have nontrivial solutions $\varphi \in L^1_0(\mathbb{R}^d)$ of $-\Delta\varphi + \lambda\varphi = 0$.

As we will show below, existence for (5.1) will not either be achieved for any $f \in \mathcal{M}_0(\mathbb{R}^d)$, or even any $f \in L^1_0(\mathbb{R}^d)$, as some restriction on the behavior of the $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ norms of f, for ε small will be required.

Now we describe our approach to solve (5.1). If we had a standard semigroup in a Banach space, denoting $A = -\Delta + \lambda I$, standard results, e.g. Lemma 2.1.6 in page 40 in [3], would give that using the semigroup generated by A, see Proposition 3.1, we should have in (5.1)

$$u_* = u_*(f) = A^{-1} f = \int_0^\infty e^{-\lambda t} S(t) f \, dt.$$

From classical results, e.g. [7], this is actually the solution of (5.1) if for example $f \in L^p(\mathbb{R}^d)$ with $1 \leq p \leq \infty$. This is also de case if $f \in L^p_U(\mathbb{R}^d)$ with $1 \leq p < \infty$; see [1]. Also we easily recognise here u_* as the (formal) limit as $t \to \infty$ of the function $U(t)$ in Corollary 4.13. However in the situation in this paper the convergence of the integral above requires $f \in \mathcal{M}_0(\mathbb{R}^d)$ and some restrictions on the possible growth of the $\mathcal{M}_\varepsilon(\mathbb{R}^d)$ norms of f for small enough ε as we now describe.

Theorem 5.1 *For $\lambda > 0$ (or $\lambda = 0$ and $d \geq 3$), assume $f \in \mathcal{M}_0(\mathbb{R}^d)$ satisfies*

$$\int_{0^+} \frac{e^{-\frac{\lambda}{4s}}}{s^2} \|f\|_{\mathcal{M}_s(\mathbb{R}^d)} \, ds < \infty. \tag{5.2}$$

Then the equation

$$-\Delta u_* + \lambda u_* = f$$

has a very weak solution $u_* \in L^1_0(\mathbb{R}^d)$, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u_*(-\Delta\varphi + \lambda\varphi) = \int_{\mathbb{R}^d} \varphi \, df.$$

In particular $-\Delta u_* + \lambda u_* = f$ in distributional sense and $\Delta u_* = \lambda u_* - f \in \mathcal{M}_0(\mathbb{R}^d)$. Moreover $u_* \in L^q_0(\mathbb{R}^d)$ for any $1 \leq q < \frac{d}{(d-2)_+}$.

If $f \geq 0$ and nontrivial, then $u_* > 0$ in $\mathbb{R}^d$.
If additionally

$$\int_{0^+} \frac{e^{-\frac{\lambda}{4\gamma s}}}{s^{3/2}} \|f\|_{\mathcal{M}_s(\mathbb{R}^d)} \, ds < \infty. \tag{5.3}$$

for some $\gamma > 1$ *then* $\nabla u_* \in L^1_0(\mathbb{R}^d)$ *and is a weak solution of* (5.1)*, that is, for every* $\varphi \in C^\infty_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \nabla u_* \nabla \varphi + \lambda \int_{\mathbb{R}^d} u_* \varphi = \int_{\mathbb{R}^d} \varphi \, \mathrm{d} f.$$

Moreover, $\nabla u_* \in L^r_0(\mathbb{R}^d)$ *for any* $1 \leq r < \frac{d}{(d-1)_+}$.

Proof Define

$$u_* = u_*(f) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) f \, dt. \tag{5.4}$$

From the estimates in Proposition 2.7 we get for any $\delta > 0$

$$\|u_*\|_{L^1_\delta(\mathbb{R}^d)} \leq \int_0^\infty \mathrm{e}^{-\lambda t} \|f\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)} \, \mathrm{d}t, \qquad \delta(t) = \frac{\delta}{1+4\delta t}. \tag{5.5}$$

Then observe that changing variables as $s = \delta(t)$, $t = \frac{1}{4\delta}\left(\frac{\delta - s}{s}\right)$, $\mathrm{d}t = \frac{-1}{4s^2}\mathrm{d}s$

$$\|u_*\|_{L^1_\delta(\mathbb{R}^d)} \leq C_\delta \int_0^\delta \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{s^2} \|f\|_{\mathcal{M}_s(\mathbb{R}^d)} \, \mathrm{d}s \tag{5.6}$$

which is finite by assumption (5.2).

In a similar way, for any $\delta > 0$ we get

$$\|u_*\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_q \int_0^\infty \mathrm{e}^{-\lambda t} \left(\frac{1+4\delta t}{4\delta t}\right)^{d/2q'} \|f\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)} \, \mathrm{d}t, \qquad \delta(t) = \frac{\delta}{1+4\delta t} \tag{5.7}$$

which is integrable at $t = 0$ if $\frac{d}{2q'} < 1$, that is $q < \frac{d}{(d-2)_+}$. Changing variables as $s = \delta(t)$, $t = \frac{1}{4\delta}\left(\frac{\delta - s}{s}\right)$, $\mathrm{d}t = \frac{-1}{4s^2}\mathrm{d}s$

$$\|u_*\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq C_\delta \int_0^\delta \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{s^2(\delta - s)^{\frac{d}{2q'}}} \|f\|_{\mathcal{M}_s(\mathbb{R}^d)} \, \mathrm{d}s \tag{5.8}$$

which is finite again by assumption (5.2).

Also, if $f \geq 0$ and nontrivial, then from part (iii) in Theorem 2.1 we have $S(t) f(x) > 0$ for all $x \in \mathbb{R}^d$ and $t > 0$. Then (5.4) implies $u_* > 0$ in $\mathbb{R}^d$.

Now we prove u_* is a very weak solution of (5.1). Indeed for every $\varphi \in C^\infty_c(\mathbb{R}^d)$, using Fubini, we get

$$\int_{\mathbb{R}^d} u_* \left(-\Delta\varphi + \lambda\varphi\right) = \int_0^\infty \mathrm{e}^{-\lambda t} \int_{\mathbb{R}^d} S(t) f(x) \left(-\Delta\varphi(x) + \lambda\varphi(x)\right) \mathrm{d}x \, \mathrm{d}t.$$

Now using part (ii) in Theorem 2.1 this equals

$$\int_{\mathbb{R}^d} u_* \left(-\Delta\varphi + \lambda\varphi\right) = \int_0^\infty \mathrm{e}^{-\lambda t} \int_{\mathbb{R}^d} S(t)(-\Delta\varphi(x) + \lambda\varphi(x)) \, \mathrm{d}f(x) \, \mathrm{d}t.$$

Now observe that for $\psi \in C^\infty_c(\mathbb{R}^d)$ we have

$$\int_0^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda t} |S(t)\psi|(x) \, \mathrm{d}|f(x)| \, \mathrm{d}t < \infty. \tag{5.9}$$

To see this observe that if ψ has support in the ball $B(0, R)$, we have, for any $t > 0$,

$$S(t)\psi(x) = \frac{1}{(4\pi t)^{d/2}} \int_{B(0,R)} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \psi(y)\,\mathrm{d}y;$$

using that for any $0 < \beta < 1$, $|x-y|^2 \geq (1-\beta)|x|^2 - (\frac{1}{\beta}-1)|y|^2 \geq (1-\beta)|x|^2 - (\frac{1}{\beta}-1)R^2$, we get

$$|S(t)\psi(x)| \leq \frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-(1-\beta(t))\frac{|x|^2}{4t} + (\frac{1}{\beta(t)}-1)\frac{R^2}{4t}} \int_{B(0,R)} |\psi(y)|\,\mathrm{d}y \tag{5.10}$$

for any $0 < \beta(t) < 1$ for $t > 0$ which we will choose below.

Setting, $\delta(t) = \frac{\delta}{1+4\delta t}$, (5.10) yields to

$$\begin{aligned} I &= \int_1^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda t} |S(t)\psi|(x)\,\mathrm{d}|f(x)|\,\mathrm{d}t \\ &= \int_1^\infty \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda t} \frac{1}{\rho_{\delta(t)}(x)} |S(t)\psi|(x) \rho_{\delta(t)}(x)\,\mathrm{d}|f(x)|\,\mathrm{d}t \\ &\leq c \int_1^\infty \frac{\mathrm{e}^{-\lambda t}\mathrm{e}^{(\frac{1}{\beta(t)}-1)\frac{R^2}{4t}}}{(4\pi t)^{d/2}} (\frac{\pi}{\delta(t)})^{\frac{d}{2}} \int_{\mathbb{R}^d} \mathrm{e}^{(\delta(t) - \frac{(1-\beta(t))}{4t})|x|^2} \rho_{\delta(t)}(x)\,\mathrm{d}|f(x)|\,\mathrm{d}t \end{aligned}$$

where ρ_δ are the exponential weights in (2.5). Now observe that we have $\delta(t) - \frac{1-\beta(t)}{4t} < 0$ provided that we choose

$$\beta(t) < 1 - 4t\delta(t) = \frac{1}{1+4\delta t}.$$

This choice leads to

$$I \leq c \int_1^\infty \mathrm{e}^{-\lambda t} \mathrm{e}^{(\frac{1}{\beta(t)}-1)\frac{R^2}{4t}} (\frac{1+4\delta t}{4\delta t})^{\frac{d}{2}} \|f\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}\,\mathrm{d}t.$$

Now $(\frac{1}{\beta(t)} - 1)\frac{R^2}{4t}$ is bounded above if and only if

$$\beta(t) \geq \frac{1}{1+4Kt}$$

for some sufficently large $K > \delta$. With such a choice, we get

$$I \leq c \int_1^\infty \mathrm{e}^{-\lambda t} \|f\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}\,\mathrm{d}t$$

which is finite, by (5.5).

On the other hand, for $0 < t < 1$ and $0 < \beta < 1$ write (5.10) as

$$|S(t)\psi(x)| \leq \frac{C}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{(1-\beta)}{4t}(|x|^2 - \frac{R^2}{\beta})}.$$

Then note that, for any $\gamma > 0$, $|x|^2 - \frac{R^2}{\beta} \geq (1-\beta)|x|^2 + \gamma\frac{R^2}{\beta}$ provided that $|x|^2 \geq (1+\gamma)\frac{R^2}{\beta^2}$.

Hence, for such x we get for any $0 \leq t \leq 1$

$$|S(t)\psi(x)| \leq \frac{C\mathrm{e}^{-\frac{(1-\beta)}{4t}\gamma\frac{R^2}{\beta}}}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{(1-\beta)^2}{4t}|x|^2} \leq C\mathrm{e}^{-\frac{(1-\beta)^2}{4}|x|^2}.$$

In particular, with $\gamma = 3$, setting $\alpha = \frac{(1-\beta)^2}{4}$ and using $\|S(t)\psi\|_{L^\infty(\mathbb{R}^d)} \le \|\psi\|_{L^\infty(\mathbb{R}^d)}$, for any $0 \le t \le 1$ we have

$$|S(t)\psi(x)| \le \begin{cases} \|\psi\|_{L^\infty(\mathbb{R}^d)}, & |x| \le \frac{2R}{\beta} \\ C\mathrm{e}^{-\alpha|x|^2}, & |x| \ge \frac{2R}{\beta}. \end{cases} \tag{5.11}$$

This gives

$$\int_0^1 \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda t} |S(t)\psi|(x)\, \mathrm{d}|f(x)|\, \mathrm{d}t \le c \int_{\mathbb{R}^d} \mathrm{e}^{-\alpha|x|^2}\, \mathrm{d}|f(x)| < \infty$$

and this ends the proof of (5.9).

Hence Fubini once more leads to

$$\int_{\mathbb{R}^d} u_* \,(-\Delta\varphi + \lambda\varphi) = \int_{\mathbb{R}^d} \int_0^\infty \mathrm{e}^{-\lambda t} S(t)(-\Delta\varphi(x) + \lambda\varphi(x))\, \mathrm{d}t\, \mathrm{d}f(x) \tag{5.12}$$

and, if $\lambda > 0$, from Lemma 5.2 below we get $G(x) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)(-\Delta\varphi(x) + \lambda\varphi(x))\, \mathrm{d}t = \varphi(x)$. For $\lambda = 0$ and $d \ge 3$ Lemma 5.2 implies $-\Delta G = -\Delta\varphi$. Since from the results in [4] the only Harmonic functions in $L^1_0(\mathbb{R}^d)$ are the constant ones then we conclude $G = \varphi + c$. But then $c = 0$ since $G \in L^p(\mathbb{R}^d)$ as in Lemma 5.2.

Hence from (5.12) for every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u_*(-\Delta\varphi + \lambda\varphi)\, \mathrm{d}x = \int_{\mathbb{R}^d} \varphi\, \mathrm{d}f$$

hence u_* is a very weak solution of (5.1). That is $-\Delta u_* + \lambda u_* = f$ in distributional sense. From this, $\Delta u_* = \lambda u_* - f \in \mathcal{M}_0(\mathbb{R}^d)$.

With the additional assumption on f in the statement, from the gradient estimates in Proposition 2.7 we get

$$\|\nabla u_*\|_{L^1_\delta(\mathbb{R}^d)} \le c \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{t^{1/2}} \|f\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}\, \mathrm{d}t, \qquad \tilde{\delta}(t) = \frac{\delta}{1 + 4\gamma\delta t},\ \gamma > 1. \tag{5.13}$$

Changing variables as $s = \tilde{\delta}(t)$, $t = \frac{1}{4\gamma\delta}\big(\frac{\delta - s}{s}\big)$, $\mathrm{d}t = \frac{-1}{4\gamma s^2}\mathrm{d}s$, we get, using (5.3),

$$\|\nabla u_*\|_{L^1_\delta(\mathbb{R}^d)} \le C_{\delta,\gamma} \int_0^\delta \frac{\mathrm{e}^{-\frac{\lambda}{4\gamma s}}}{s^{3/2}(\delta - s)^{1/2}} \|f\|_{\mathcal{M}_s(\mathbb{R}^d)}\, \mathrm{d}s < \infty. \tag{5.14}$$

Now for $\varphi \in C_c^\infty(\mathbb{R}^d)$ integrating by parts as in as in Lemma A.3 in [4],

$$\int_{\mathbb{R}^d} u_*(-\Delta\varphi) = \int_{\mathbb{R}^d} \nabla u_* \nabla\varphi$$

and u_* is a weak solution of (5.1).
In a similar way for any $\delta > 0$,

$$\|\nabla u_*\|_{L^r_{r\delta}(\mathbb{R}^d)} \le c \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{t^{\frac{1}{2}}} \Big(\frac{1 + 4\gamma\delta t}{t}\Big)^{d/2r'} \|f\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}\, \mathrm{d}t, \qquad \tilde{\delta}(t) = \frac{\delta}{1 + 4\delta\gamma t} \tag{5.15}$$

with $\gamma > 1$ which is integrable at $t = 0$ provided that $\frac{d}{r'} < 1$, that is $r < \frac{d}{(d-1)_+}$. Changing variables as $s = \tilde{\delta}(t)$, $t = \frac{1}{4\gamma\delta}\big(\frac{\delta-s}{s}\big)$, $\mathrm{d}t = \frac{-1}{4\gamma s^2}\mathrm{d}s$, we get

$$\|\nabla u_*\|_{L^r_{r\delta}(\mathbb{R}^d)} \le C_{\delta,\gamma}\int_0^\delta \frac{\mathrm{e}^{-\frac{\lambda}{4\gamma s}}}{s^{\frac{3}{2}}(\delta-s)^{\frac{d}{2r'}+\frac{1}{2}}}\|f\|_{\mathcal{M}_s(\mathbb{R}^d)}\,\mathrm{d}s < \infty. \tag{5.16}$$

□

Now we prove the result used above.

Lemma 5.2 *For* $\lambda > 0$ *and* $|g(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$, *the function*

$$G = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)g\,\mathrm{d}t \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$$

satisfies $-\Delta G + \lambda G = g$. *The same holds true for* $\lambda = 0$ *if* $d \ge 3$, *with* $G \in L^p(\mathbb{R}^d)$ *for* $\frac{d}{d-2} < p < \infty$.

Proof Indeed from Proposition 2.10 since $|g(x)| \le A\mathrm{e}^{-\gamma|x|^2}$, $x \in \mathbb{R}^d$, then $S(t)g$ satisfies

$$|S(t)g|(x) \le \frac{A}{(1+4\gamma t)^{d/2}}\mathrm{e}^{-\frac{\gamma}{1+4\gamma t}|x|^2}, \qquad x \in \mathbb{R}^d, \quad t > 0$$

and then

$$\|G\|_{L^p(\mathbb{R}^d)} \le \int_0^\infty \mathrm{e}^{-\lambda t}\|S(t)g\|_{L^p(\mathbb{R}^d)}\,\mathrm{d}t \le \int_0^\infty \frac{c\mathrm{e}^{-\lambda t}}{(1+4\gamma t)^{\frac{d}{2}(1-\frac{1}{p})}}\,\mathrm{d}t$$

which is finite for $1 \le p \le \infty$ if $\lambda > 0$. If $\lambda = 0$ then $G \in L^p(\mathbb{R}^d)$ for $p > \frac{d}{d-2}$ and using the bound above on $|S(t)g|(x)$ and integrating in time

$$|G(x)| \le \frac{c}{|x|^{d-2}}\int_0^{|x|^2\gamma} z^{\frac{d}{2}-2}\mathrm{e}^{-z}\,\mathrm{d}z, \qquad x \in \mathbb{R}^d.$$

Also denote, for $\lambda > 0$ (or $\lambda = 0$ and $d \ge 3$)

$$G_\varepsilon := \int_\varepsilon^{\frac{1}{\varepsilon}} \mathrm{e}^{-\lambda t} S(t)g\,\mathrm{d}t \longrightarrow G$$

in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$ and

$$\begin{aligned}-\Delta G_\varepsilon &= \int_\varepsilon^{\frac{1}{\varepsilon}} \mathrm{e}^{-\lambda t}\big(-\Delta S(t)g\big)\,\mathrm{d}t = -\int_\varepsilon^{\frac{1}{\varepsilon}} \mathrm{e}^{-\lambda t}\partial_t S(t)g\,\mathrm{d}t\\ &= \mathrm{e}^{-\lambda\varepsilon}S(\varepsilon)g - \mathrm{e}^{-\lambda\frac{1}{\varepsilon}}S(\frac{1}{\varepsilon})g - \lambda G_\varepsilon\end{aligned}$$

and then

$$-\Delta G_\varepsilon + \lambda G_\varepsilon = \mathrm{e}^{-\lambda\varepsilon}S(\varepsilon)g - \mathrm{e}^{-\lambda\frac{1}{\varepsilon}}S(\frac{1}{\varepsilon})g \longrightarrow g$$

in $L^p(\mathbb{R}^d)$ as $\varepsilon \to 0$. Hence, using the fact that $-\Delta$ is a closed operator in $L^p(\mathbb{R}^d)$ we get

$$-\Delta G + \lambda G = g.$$

□

Remark 5.3 Observe that by the definition of the norm, see (2.4), $\|f\|_{\mathcal{M}_s(\mathbb{R}^d)}$ is bounded for s in compact sets in $(0,\infty)$ and $\frac{e^{-\frac{\lambda}{s}}}{s^2} \to 0$ as $s \to 0$ if $\lambda > 0$. Therefore the condition on f in Theorem 5.1, (5.2), is a limitation in the growth of $\|f\|_{\mathcal{M}_s(\mathbb{R}^d)}$ as $s \to 0$. Note that this condition allows for

$$\|f\|_{\mathcal{M}_s(\mathbb{R}^d)} \approx s^m e^{\frac{\lambda}{4s}}, \quad s \to 0$$

for some $m > 1$ for $\lambda \geq 0$.

Observe that the integrability condition on f in Theorem 5.1, (5.2), can be written as follows. For $\delta > 0$ and $\delta(t) = \frac{\delta}{1+4\delta t}$ define as in (5.5)

$$I = \int_0^\infty e^{-\lambda t} \|f\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)} \, dt.$$

Then changing variables as $s = \delta(t)$, $t = \frac{1}{4\delta}\left(\frac{\delta - s}{s}\right)$, $dt = \frac{-1}{4s^2} ds$ we get as in (5.6),

$$I = \frac{e^{\frac{\lambda}{4\delta}}}{4} \int_0^\delta \frac{e^{-\frac{\lambda}{4s}}}{s^2} \|f\|_{\mathcal{M}_s(\mathbb{R}^d)} \, ds < \infty.$$

On the other hand, changing variables as $r = \frac{1+4\delta t}{4\delta} = \frac{1}{4\delta(t)}$ we get

$$I = e^{\frac{\lambda}{4\delta}} \int_{\frac{1}{4\delta}}^\infty e^{-\lambda r} \|f\|_{\mathcal{M}_{\frac{1}{4r}}(\mathbb{R}^d)} \, dr = e^{\frac{\lambda}{4\delta}} \int_{\frac{1}{4\delta}}^\infty e^{-\lambda r} S(t)|f|(0) \, dr.$$

Remark 5.4 Observe that (5.6) and (5.14) and (5.8) and (5.16) reflect continuous dependence of the solutions of the Poisson's equation (5.1), that is of the mapping

$$f \mapsto u_*$$

The next result states that the solution of (5.1) constructed in Theorem 5.1 can be represented as the convolution of f with the Green's function, as in the classical results for the Poisson's equation. But, even more, conversely it states that, for non-negative data, if a solution of (5.1) can be represented using the Green's function, then f must be as in Theorem 5.1.

Theorem 5.5 *Assume* $\lambda \geq 0$.

(i) *Under the assumptions* $\lambda > 0$ *(or* $\lambda = 0$ *and* $d \geq 3$*) and* (5.2) *of Theorem* 5.1*, the solution* $u_* = u_*(f)$ *in* (5.4) *is given by*

$$u_*(x) = \int_{\mathbb{R}^d} G_\lambda(x-y) \, df(y), \qquad x \in \mathbb{R}^d$$

where G_λ *is the Green's function for* $-\Delta + \lambda I$ *in* $\mathbb{R}^d$.

(ii) *Conversely, assume* $0 \leq f \in \mathcal{M}_{loc}(\mathbb{R}^d)$ *and define for* $\lambda > 0$ *(or* $\lambda = 0$ *and* $d \geq 3$*)*

$$U(x) = \int_{\mathbb{R}^d} G_\lambda(x-y) \, df(y), \qquad x \in \mathbb{R}^d.$$

Then, if there exist $x_0 \in \mathbb{R}^d$ *such that* $U(x_0) < \infty$ *then* $f \in \mathcal{M}_0(\mathbb{R}^d)$*, satisfies* (5.2)*,* $U(x)$ *is finite for a.e.* $x \in \mathbb{R}^d$*,* $U \in L^1_0(\mathbb{R}^d)$ *and* $U = u_*(f)$ *as in* (5.4) *in Theorem* 5.1.

Proof (i) Observe that $|f|$ satisfies the assumptions in Theorem 5.1 and then, clearly $|u_*(f)| \leq u_*(|f|) \in L^1_0(\mathbb{R}^d)$. Then plugging (2.2), that is

$$S(t)|f|(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \, \mathrm{d}|f|(y) \qquad x \in \mathbb{R}^d$$

into expression (5.4), that is $u_*(|f|) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)|f| \, dt$, then Fubini's theorem gives

$$u_*(|f|)(x) = \int_{\mathbb{R}^d} G_\lambda(x-y) \, \mathrm{d}|f(y)|$$

where

$$G_\lambda(z) = \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{|z|^2}{4t}} \, \mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} \big(S(t)\delta_0\big)(z) \, \mathrm{d}t \qquad z \in \mathbb{R}^d$$

is the Green's function of $-\Delta + \lambda I$, see [7], page 132. Since all integrals are absolutely convergent, we get part (i) for $u_*(f)$. The case $\lambda = 0$ and $d \geq 3$ follows as well and in this case

$$G_0(z) = \frac{C_d}{|z|^{d-2}}, \qquad z \in \mathbb{R}^d.$$

(ii) Since $G_\lambda(z) = \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{|z|^2}{4t}} \, \mathrm{d}t$, see [7], page 132, Fubini's theorem gives

$$U(x_0) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) f \,(x_0) \, \mathrm{d}t < \infty$$

and then Tonelli's theorem implies that for a.e. $0 < t < \infty$ we have

$$S(t)(f)\,(x_0) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x_0-y|^2}{4t}} \, \mathrm{d}f(y) < \infty.$$

Hence, Lemma 3.2 in [4] implies $f \in \mathcal{M}_0(\mathbb{R}^d)$.

Moreover, we can write

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x_0-y|^2}{4t}} \, \mathrm{d}f(y) = \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|z|^2}{4t}} \, \mathrm{d}\tau_{-x_0} f(z)$$

(see e.g. Lemma 5.4 in [4]) and then

$$U(x_0) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) \tau_{-x_0} f \,(0) \, \mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t} \|\tau_{-x_0} f\|_{\mathcal{M}_{\frac{1}{4t}}(\mathbb{R}^d)} \, \mathrm{d}t < \infty.$$

Changing variables as $s = \frac{1}{4t}$ we get

$$U(x_0) = \frac{1}{4} \int_0^\infty \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{s^2} \|\tau_{-x_0} f\|_{\mathcal{M}_s(\mathbb{R}^d)} \, \mathrm{d}s < \infty.$$

In particular, $g = \tau_{-x_0} f$ satisfies the assumption (5.2) in Theorem 5.1. Hence $u_*(\tau_{-x_0} f)$ in (5.4) belongs to $L^1_0(\mathbb{R}^d)$ and by part (i) we have, a.e. $x \in \mathbb{R}^d$,

$$u_*(\tau_{-x_0} f)(x) = \int_{\mathbb{R}^d} G_\lambda(x-y) \, \mathrm{d}\tau_{-x_0} f(y) = \int_{\mathbb{R}^d} G_\lambda(x + x_0 - y) \, \mathrm{d}f(y) = U(x + x_0),$$

see e.g. Lemma 5.4 in [4]. That is, $u_*(\tau_{-x_0} f) = \tau_{-x_0} U$ and then U is finite a.e. in $\mathbb{R}^d$ and $U \in L^1_0(\mathbb{R}^d)$.

Now, as in Corollary 4.2, since

$$U(x) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) f(x)\, \mathrm{d}t, \quad x \in \mathbb{R}^d$$

for any $\delta > 0$

$$\mathrm{e}^{-\delta |x|^2} U(x) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t) f(x) \mathrm{e}^{-\delta |x|^2}\, \mathrm{d}t.$$

Integrating in $x \in \mathbb{R}^d$ and using Fubini we get

$$\|U\|_{L^1_\delta(\mathbb{R}^d)} = \int_0^\infty \mathrm{e}^{-\lambda t} \|S(t) f\|_{\mathcal{M}_\delta(\mathbb{R}^d)}\, \mathrm{d}t$$

and using Proposition 2.8 we get

$$\|U\|_{L^1_\delta(\mathbb{R}^d)} = \int_0^\infty \mathrm{e}^{-\lambda t} \|f\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}\, \mathrm{d}t < \infty.$$

So, by (5.5) and (5.6), f satisfies (5.2); see Remark 5.3. □

Remark 5.6 Observe that the Green's function

$$G_\lambda(y) = \int_0^\infty \mathrm{e}^{-\lambda s} \frac{1}{(4\pi s)^{d/2}} \mathrm{e}^{-\frac{|y|^2}{4s}}\, \mathrm{d}s$$

decays, as $|y| \to \infty$ as $G_\lambda(y) \approx \mathrm{e}^{-\frac{\sqrt{\lambda}}{2}|y|}$.

For this observe that $\lambda s + \frac{|y|^2}{4s} \geq \sqrt{\lambda}|y|$ and if $|y| \geq 1$ we also have $\lambda s + \frac{|y|^2}{4s} \geq \lambda s + \frac{1}{4s}$. Hence

$$G_\lambda(y) \leq \mathrm{e}^{-\frac{\sqrt{\lambda}}{2}|y|} \int_0^\infty \frac{1}{(4\pi s)^{d/2}} \mathrm{e}^{-\frac{1}{2}(\lambda s + \frac{1}{4s})}\, \mathrm{d}s$$

Now we revise the whether or not the examples at the end of Sect. 3 satisfy condition (5.2).

Example 5.7 The Dirac delta at a point, $f = \delta_{x_0}$, $x_0 \in \mathbb{R}^d$, satisfies (5.2) for any $\lambda > 0$ and for $\lambda = 0$ if $d \geq 3$.

Example 5.8 The function $f(x) = \mathrm{e}^{c|x|^\alpha}$, $0 < \alpha < 2$, $c > 0$ satisfies (5.2) for $0 < \alpha < 1$ and any $\lambda > 0$ or $\alpha = 1$ and $\lambda > c^2$.

So by Theorem 5.5 for $1 < \alpha < 2$ there is no solution of the Poisson equation given by the Green's function.

Example 5.9 The function $f(x) = (1 + |x|^2)^\beta$, $\beta > 0$, satisfies (5.2) for any $\lambda > 0$.

The following corollary establishes the basin of attraction of $u_*(f)$ in $\mathcal{M}_0(\mathbb{R}^d)$.

Corollary 5.10 *Under the assumptions of Theorem 5.1, for every $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$ such that*

$$\mathrm{e}^{-\frac{\lambda}{4\varepsilon}} \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0$$

the solution of (4.1) given by (4.2) satisfies

$$u(t) \to u_*(f), \qquad t \to \infty$$

in $L^1_0(\mathbb{R}^d)$. The converse is also true if $u_0 \geq 0$.

Proof The result follows from (4.2) and Corollary 3.2, since with the notations in Corollary 4.13, we have $U(t) \to u_*$ in $L^1_0(\mathbb{R}^d)$, as $t \to \infty$. □

From the local regularity of u_* we obtain the following result.

Corollary 5.11 *Assume $0 \leq f \in \mathcal{M}_0(\mathbb{R}^d)$ is as in Theorem 5.1 and assume moreover that $f \in L^p_0(\mathbb{R}^d)$ with $p > \frac{d}{2}$.*

Then, with the notations in Corollary 4.13 we have that

$$U(t) \to u_*(f), \qquad t \to \infty,$$

uniformly in compact sets of $\mathbb{R}^d$.

Proof Since $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ with $p > \frac{d}{2}$ the local regularity of weak solutions of $-\Delta v + \lambda v = f$ imply that $u_*(f) \in C(\mathbb{R}^d)$. On the other hand, from Proposition 4.8 with $\sigma = \infty$ we get $U(t) \in C(\mathbb{R}^d)$.

Since $U(t) \to u_*(f)$ monotonically as $t \to \infty$, then Dini's criterium (c.f. [2, p. 194]) implies the convergence is uniform in compact sets. □

Now using the estimates in Sect. 4.2 we obtain the following further regularity of the solution of (5.1) given in Theorem 5.1.

Corollary 5.12 *With the notations above, assume $f \in L^p_0(\mathbb{R}^d)$ with $1 \leq p < \infty$ and*

$$\int_{0^+} \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{s^2} \|f\|_{L^p_s(\mathbb{R}^d)} \, \mathrm{d}s < \infty.$$

If $p = \infty$ we require $\int_{0^+} \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{s^{d+2}} \|f\|_{L^p_s(\mathbb{R}^d)} \, \mathrm{d}s < \infty$.

Then $u_* = u_*(f)$ constructed in Theorem 5.1 satisfies $u_* \in L^q_0(\mathbb{R}^d)$ for any $1 \leq p \leq q < \infty$ such that $q < \frac{pd}{(d-2p)_+}$. If $p > \frac{d}{2}$ then we can take $q = \infty$ as well.

If moreover

$$\int_{0^+} \frac{\mathrm{e}^{-\frac{\lambda}{4\gamma s}}}{s^{\frac{3}{2}}} \|f\|_{L^p_s(\mathbb{R}^d)} \, \mathrm{d}s < \infty$$

for some $\gamma > 1$ then $\nabla u_* \in L^r_0(\mathbb{R}^d)$ for any $r < \frac{pd}{(d-p)_+}$. If $p > d$ then we can take $r = \infty$ as well.

In particular if $p > \frac{d}{2}$ then $u_* \in C(\mathbb{R}^d)$.

Proof Observe that from (5.4) and the estimates in Proposition 2.7, if $1 \leq p \leq q < \infty$,

$$\|u_*\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq \int_0^\infty \mathrm{e}^{-\lambda t} \phi_{p,q}(t) \|f\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)} \, \mathrm{d}t, \quad \delta_p(t) = \frac{p\delta}{1 + 4p\delta t} \tag{5.17}$$

with $\phi_{p,q}(t) = c\left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$, see (2.21), which is integrable at $t = 0$ if $\frac{1}{p} - \frac{1}{q} < \frac{2}{d}$, that is $q < \frac{pd}{(d-2p)_+}$. Changing variables as $s = \delta_p(t)$, $t = \frac{1}{4p\delta}\left(\frac{p\delta - s}{s}\right)$, $\mathrm{d}t = \frac{-1}{4s^2}\mathrm{d}s$, $\frac{1+4p\delta t}{4p\delta t} = \frac{p\delta}{p\delta - s}$ and

$$\|u_*\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq C_{\delta,p} \int_0^{p\delta} \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{(p\delta - s)^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} s^2} \|f\|_{L^p_s(\mathbb{R}^d)} \, \mathrm{d}s < \infty. \tag{5.18}$$

For $q = \infty$, by (2.22)

$$\|u_*\|_{L^\infty_\delta(\mathbb{R}^d)} \le \int_0^\infty \mathrm{e}^{-\lambda t}\phi_{p,\infty}(t)\|f\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}\,\mathrm{d}t, \qquad \delta_p(t) = \frac{p\delta}{1+4p\delta t}$$

with $\phi_{p,\infty}(t) = c\big(\frac{1+4p\delta t}{4p\delta t}\big)^{\frac{d}{2p}}$ which is integrable at $t = 0$ if $p > \frac{d}{2}$. Changing variables as above leads to

$$\|u_*\|_{L^\infty_\delta(\mathbb{R}^d)} \le C_{\delta,p}\int_0^{p\delta} \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{(p\delta - s)^{\frac{d}{2p}} s^2}\|f\|_{L^p_s(\mathbb{R}^d)}\,\mathrm{d}s < \infty. \tag{5.19}$$

The case $p = \infty$ follows as above using (2.23)

$$\|u_*\|_{L^\infty_\delta(\mathbb{R}^d)} \le \int_0^\infty \mathrm{e}^{-\lambda t}\phi_{\infty,\infty}(t)\|f\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)}\,\mathrm{d}t, \qquad \delta(t) = \frac{\delta}{1+4\delta t}$$

with $\phi_{\infty,\infty}(t) = (1+4\delta t)^d$. Changing variables as $s = \delta(t)$, $t = \frac{1}{4\delta}\big(\frac{\delta - s}{s}\big)$, $\mathrm{d}t = \frac{-1}{4s^2}\mathrm{d}s$, and then

$$\|u_*\|_{L^\infty_\delta(\mathbb{R}^d)} \le C_\delta \int_0^\delta \frac{\mathrm{e}^{-\frac{\lambda}{4s}}}{s^{d+2}}\|f\|_{L^\infty_s(\mathbb{R}^d)}\,\mathrm{d}s < \infty. \tag{5.20}$$

For the estimate on the gradients, note that for $1 \le p \le r < \infty$, from (2.24) in Proposition 2.7, we have

$$\|\nabla u_*\|_{L^r_{r\delta}(\mathbb{R}^d)} \le \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{t^{\frac{1}{2}}}\Phi_{p,r}(t)\|f\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}\,\mathrm{d}t, \qquad \tilde{\delta}_p(t) = \frac{p\delta}{1+4p\gamma\delta t} \tag{5.21}$$

with $\gamma > 1$ and $\Phi_{p,r}(t) = c\big(\frac{1+4p\gamma\delta t}{t}\big)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}$ which is integrable at $t = 0$ provided that $\frac{1}{p} - \frac{1}{r} < \frac{1}{d}$, that is $r < \frac{pd}{(d-p)_+}$. Changing variables as $s = \tilde{\delta}_p(t)$, $t = \frac{1}{4p\gamma\delta}\big(\frac{p\delta - s}{s}\big)$, $\mathrm{d}t = \frac{-1}{4\gamma s^2}\mathrm{d}s$, $\frac{1+4p\delta t}{t} = \frac{c}{p\delta - s}$ and we get

$$\|\nabla u_*\|_{L^r_{r\delta}(\mathbb{R}^d)} \le C_{\delta,p,\gamma}\int_0^{p\delta} \frac{\mathrm{e}^{-\frac{\lambda}{4\gamma s}}}{(p\delta - s)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{r})} s^{\frac{3}{2}}}\|f\|_{L^p_s(\mathbb{R}^d)}\,\mathrm{d}s < \infty. \tag{5.22}$$

Analogously, for $r = \infty$, we get

$$\|\nabla u_*\|_{L^\infty_\delta(\mathbb{R}^d)} \le \int_0^\infty \frac{\mathrm{e}^{-\lambda t}}{t^{\frac{1}{2}}}\Phi_{p,\infty}(t)\|f\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}\,\mathrm{d}t, \qquad \tilde{\delta}_p(t) = \frac{p\delta}{1+4p\gamma\delta t} \tag{5.23}$$

with $\Phi_{p,\infty}(t) = c\big(\frac{1+4p\gamma\delta t}{t}\big)^{\frac{d}{2p}}$, which is integrable at $t = 0$ provided that $p > d$. Changing variables as above we conclude.

Finally, if $p > \frac{d}{2}$ then we can take $r > d$ and then $\nabla u_* \in L^r_{loc}(\mathbb{R}^d)$ which implies $u_* \in C(\mathbb{R}^d)$. □

Further properties of the solutions of (5.1) constructed in Theorem 5.1 can be obtained if, in a similar way to (2.19) and (2.20) we define $L^p_{0,B}(\mathbb{R}^d)$ as the subset of $L^p_0(\mathbb{R}^d)$ made of functions such that

$$|||u_0|||_{L^p_{0,B}(\mathbb{R}^d)} := \sup_{\varepsilon > 0}\|u_0\|_{L^p_\varepsilon(\mathbb{R}^d)} < \infty. \tag{5.24}$$

Notice that we have for $1 \leq p \leq q \leq \infty$

$$L^q_{0,B}(\mathbb{R}^d) \subset L^p_{0,B}(\mathbb{R}^d) \subset L^1_{0,B}(\mathbb{R}^d) \subset \mathcal{M}_{0,B}(\mathbb{R}^d).$$

Then from Theorem 5.1 and Corollary 5.12 we get at once the following result.

Corollary 5.13 *Assume $\lambda > 0$. If $f \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ as in (2.19) then u_* constructed in Theorem 5.1 satisfies*

$$\|u_*\|_{L^1_{0,B}(\mathbb{R}^d)} + \|\nabla u_*\|_{L^1_{0,B}(\mathbb{R}^d)} \leq \frac{c}{\lambda}\|f\|_{\mathcal{M}_{0,B}(\mathbb{R}^d)} \tag{5.25}$$

and

$$\|u_*\|_{L^q_{0,B}(\mathbb{R}^d)} \leq c(\lambda)\|f\|_{\mathcal{M}_{0,B}(\mathbb{R}^d)}, \qquad \|\nabla u_*\|_{L^r_{0,B}(\mathbb{R}^d)} \leq c(\lambda)\|f\|_{\mathcal{M}_{0,B}(\mathbb{R}^d)} \tag{5.26}$$

with $1 \leq q < \frac{d}{(d-2)_+}$ and $1 \leq r < \frac{d}{(d-1)_+}$.

If moreover, $f \in L^p_{0,B}(\mathbb{R}^d)$ as in (5.24) then

$$\|u_*\|_{L^q_{0,B}(\mathbb{R}^d)} \leq c(\lambda)\|f\|_{L^p_{0,B}(\mathbb{R}^d)}, \qquad \|\nabla u_*\|_{L^r_{0,B}(\mathbb{R}^d)} \leq c(\lambda)\|f\|_{L^p_{0,B}(\mathbb{R}^d)} \tag{5.27}$$

with $q < \frac{pd}{(d-2p)_+}$ and $r < \frac{pd}{(d-p)_+}$. If $p > \frac{d}{2}$ or $p > d$ then we can take $q = \infty$ or $r = \infty$ as well, respectively.

Proof If $f \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ then (5.5) and (5.13) and (5.7) and (5.15) are finite for every $\lambda > 0$ and (5.25) and (5.26) follow. The rest is immediate. □

We also get the following.

Corollary 5.14 *The analytic semigroup $S(t)$ of contractions in $L^1_{0,B}(\mathbb{R}^d)$ in Proposition 2.6 has $-\Delta$ as its infinitesimal generator with a domain*

$$D(-\Delta) = \{u_0 \in L^1_{0,B}(\mathbb{R}^d),\ -\Delta u_0 \in L^1_{0,B}(\mathbb{R}^d)\}$$

and if $u_0 \in D(-\Delta)$ then $u_0 \in L^q_0(\mathbb{R}^d)$ for any $q < \frac{d}{(d-2)_+}$ and $\nabla u_0 \in L^r_0(\mathbb{R}^d)$ for any $r < \frac{d}{(d-1)_+}$.

Proof A careful revision of the proof of the proof of Proposition 3.5 in [5] shows that if $u_0 \in \mathcal{M}_{0,B}(\mathbb{R}^d)$ then for $t > 0$, $\frac{S(t+h)u_0 - S(t)u_0}{h} + \Delta S(t)u_0$ goes to zero in $L^1_{0,B}(\mathbb{R}^N)$, as $h \to 0$. Hence the generator has to coincide with $-\Delta$ on its domain. Hence the description of $D(-\Delta)$ in the statement follows.

Finally, from Corollary 5.13, we obtain the regularity of the functions in the domain. □

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