



Bubbles in discrete-time models

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Abstract

We introduce a new definition of bubbles in discrete-time models based on the discounted stock price losing mass under an equivalent martingale measure at some finite drawdown. We provide equivalent probabilistic characterisations of this definition and give examples of discrete-time martingales that are bubbles and others that are not. In the Markovian case, we provide sufficient analytic conditions for the presence of bubbles. We also show that the existence of bubbles is directly linked to the existence of a non-trivial solution to a linear Volterra integral equation of the second kind involving the Markov kernel. Finally, we show that our definition of bubbles in discrete time is consistent with the strict local martingale definition of bubbles in continuous time in the sense that a properly discretised strict local martingale in continuous time is a bubble in discrete time.

Keywords Bubble · Strict local martingale · Markov martingale · Volterra integral equation

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JEL Classification G10 · C60

1 Introduction

Over the last 20 years, a mathematical theory of bubbles for continuous-time models has been developed based on the concept of strict local martingales; see the seminal papers by Loewenstein and Willard [12], Cox and Hobson [4], Jarrow et al. [9, 10] as well as the survey article by Protter [17] and the references therein. In economic

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terms, an asset price bubble exists if the fundamental value of the asset deviates from its current price. If the fundamental value is understood to be the expectation of the (discounted) price process $S = (S_t)_{t \geq 0}$ under the equivalent local martingale measure \mathbb{P} , then the asset S has a \mathbb{P} -bubble if $\mathbb{E}^{\mathbb{P}}[S_T] < \mathbb{E}^{\mathbb{P}}[S_0]$ for some fixed time $T > 0$, i.e., if S is a strict local \mathbb{P} -martingale, that is, a local \mathbb{P} -martingale that fails to be a \mathbb{P} -martingale.

While many implications and extensions of this definition have been discussed in the literature (see e.g. Ekström and Tysk [6], Bayraktar et al. [1], Biagini et al. [2], Herdegen and Schweizer [7]), the strict local martingale definition of bubbles has no direct analogue in discrete-time models. The reason for this is that a nonnegative local martingale $S = (S_k)_{k \in \mathbb{N}_0}$ in discrete time with $S_0 \in L^1$ is automatically a (true) martingale. Hence, a definition of bubbles based on strict local martingales is void. Also, trying to define a bubble in discrete time as a martingale $S = (S_k)_{k \in \mathbb{N}_0}$ that is not uniformly integrable does not lead to a meaningful concept as this would imply that virtually all relevant models such as the standard binomial model (considered on an unbounded time horizon) are bubbles, which seems absurd.

Despite the above negative results, the goal of this paper is to introduce a new definition of bubbles for discrete-time models on an unbounded time horizon – keeping the standard assumption that the discounted stock price is a martingale. This definition has to satisfy at least two conditions.

(I) It should *split* martingales that are not uniformly integrable into two sufficiently rich classes: those that are bubbles and those that are not. In particular, standard discrete-time models with i.i.d. returns like the binomial model should not be bubbles.

(II) It should be *consistent* with the strict local martingale definition in continuous time in the sense that a continuous local martingale in continuous time is a strict local martingale if and only if all appropriate discretisations thereof are bubbles in discrete time.

To the best of our knowledge, there has been no attempt in the extant literature to extend the martingale theory of bubbles to discrete time. The only slight exception is Roch [18] who introduced the notion of asymptotic asset price bubbles using the concept of weakly convergent discrete-time models (“large financial market”). More precisely, he showed that even if the price process is a martingale in a sequence of weakly convergent discrete-time models, it can have properties similar to a bubble in that the fundamental value in the asymptotic market can be lower than the current price in the asymptotic market. In contrast to [18], our approach is non-asymptotic.

To motivate our definition of a bubble in a discrete-time model, consider a non-sophisticated investor who follows a simple buy-and-hold strategy to invest into an asset with (discounted) price process $S = (S_k)_{k \in \mathbb{N}_0}$. The investor buys the asset at time $k = 0$ and hopes that it will rise and rise. When the (discounted) asset price drops for the first time, the investor fears to lose money and sells the asset. Denoting by $\tau_1 := \inf\{j > 0 : S_j < S_{j-1}\}$ the time of the first drawdown of the asset, the fundamental value under \mathbb{P} of S at time 0 (viewed with regard to the first drawdown) is $\mathbb{E}^{\mathbb{P}}[S_{\tau_1}]$, where \mathbb{P} denotes an equivalent martingale measure. As the process S is a nonnegative supermartingale, we always have $\mathbb{E}^{\mathbb{P}}[S_{\tau_1}] \leq S_0$. If $\mathbb{E}^{\mathbb{P}}[S_{\tau_1}] < S_0$, the fundamental value of the asset (viewed with regard to the first drawdown) is lower than its initial price and hence S might be considered a \mathbb{P} -bubble. Indeed, if the market

is complete, the predictable representation theorem implies that a sophisticated investor might choose a dynamic trading strategy ϑ whose (discounted) value process $V(\vartheta) = (V_k(\vartheta))_{k \in \mathbb{N}_0}$ satisfies $V_0(\vartheta) = \mathbb{E}^{\mathbb{P}}[S_{\tau_1}] < S_0$ and $V_{\tau}(\vartheta) = S_{\tau}$.

Of course, the requirement that S loses mass at the *first* drawdown is somewhat arbitrary and unrealistic. For this reason, our precise definition of a bubble in discrete time is more general and only requires S to lose mass at the k th drawdown for some $k \in \mathbb{N}$. While this definition is very simple, it leads to a rich theory.

In Sect. 2, we provide several equivalent probabilistic characterisations for a non-negative discrete-time martingale to be a bubble. We also provide necessary and sufficient characterisations for a discrete-time martingale with independent increments to have a bubble. In particular, we show that i.i.d. returns models such as the standard binomial model do not have a bubble, which implies that Condition (I) above is satisfied.

In Sect. 3, we look at the special case that S is a Markov martingale. We provide characterisations for the presence or absence of bubbles, depending on the *probability* $a(x) = \mathbb{P}_x[S_1 < x]$ of going down, and the *relative recovery* $b(x) = \mathbb{E}_x[\frac{S_1}{x} \mathbf{1}_{\{S_1 < x\}}]$ when going down. Loosely speaking, it turns out that S is a bubble if and only if $b(x)$ converges to 0 fast enough as $x \rightarrow \infty$. To make this precise, however, is quite involved. While we are able to give sufficient conditions in the general case, we provide necessary and sufficient conditions in the case of complete markets.

In Sect. 4, we continue our study of Markov martingales by looking more closely at the underlying Markov kernel. We show that the existence of bubbles for S is directly linked to the existence of a non-trivial nonnegative solution to a linear Volterra integral equation of the second kind involving the Markov kernel. Among other things, this allows us to give some additional sufficient conditions for the existence of bubbles that cannot be covered with the results from Sect. 3.

Finally, in Sect. 5, we discuss how our definition of a bubble in discrete time relates to the strict local martingale definition in continuous time. We show that when discretising a positive continuous strict local martingale along sequences of stopping times in a certain somewhat canonical class, one obtains a bubble in discrete time. Conversely, we show that a positive continuous local martingale is a strict local martingale if for all localising sequences in the same class, the corresponding discretised martingales are bubbles. This shows that Condition (II) above is also satisfied.¹ To prove these discretisation results, we rely on the deep change of measure techniques first employed by Delbaen and Schachermayer [5] and further developed by Pal and Protter [15], Kardaras et al. [11] and Perkowski and Ruf [16] that allow turning the inverse of a nonnegative strict local martingale into a true martingale under a locally dominating probability measure. Some technical proofs of this section are shifted to the [Appendix](#).

¹As pointed out by one referee, our concept of a discrete-time bubble has a clear economic foundation which the strict local martingale definition of bubbles in continuous time maybe lacks – at least at a first glance. Hence, by showing that appropriately discretised strict local martingales yield discrete-time bubbles, we provide additional and new support to modelling continuous-time bubbles by strict local martingales.

2 Definition and characterisation of bubbles

In this section, we introduce our definition of a bubble in discrete time and provide equivalent probabilistic characterisations of this concept.

Definition 2.1 Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}, \mathbb{P})$ be a filtered probability space. A non-negative (\mathbb{P}, \mathbb{F}) -martingale $S = (S_k)_{k \in \mathbb{N}_0}$ is called a *bubble* if

$$\mathbb{E}[S_{\tau_k}] < \mathbb{E}[S_0] \quad (2.1)$$

for some $k \in \mathbb{N}$, where $\tau_0 := 0$ and $\tau_k := \inf\{j > \tau_{k-1} : S_j < S_{j-1}\}$, $k \in \mathbb{N}$, denotes the k th *drawdown* of S . We also call \mathbb{P} a *bubble measure* for S .

Some comments on the above definition are in order.

Remark 2.2 (a) We include the possibility that $\mathbb{P}[\tau_k = \infty] > 0$. Since S is a nonnegative martingale, it converges \mathbb{P} -a.s. to some integrable random variable S_∞ by Doob's supermartingale convergence theorem. Hence S_{τ_k} is well defined in any case.

(b) If S is a Markov process, S will be a bubble if and only if $\mathbb{E}[S_{\tau_1}] < \mathbb{E}[S_0]$; cf. Sect. 3 below. In general, however, the above definition does not contain any redundancy; one may think for example of dynamics with a change point.

(c) By the stopping theorem for uniformly integrable martingales, S can only be a bubble if it is *not* uniformly integrable. Example 2.9, however, shows that not every martingale that fails to be uniformly integrable is a bubble. Also, note that the precise definition of the stopping time τ_k in (2.1) is important and cannot naively be replaced by an arbitrary stopping time.

(d) Our bubbles are strictly speaking \mathbb{P} -*bubbles* since the definition depends on the choice of the (equivalent) martingale measure \mathbb{P} . In incomplete markets, it is possible to have a \mathbb{P} -*bubble* under some equivalent martingale measure (EMM) \mathbb{P} but not a $\tilde{\mathbb{P}}$ -*bubble* under a different EMM $\tilde{\mathbb{P}}$. A simple Markovian trinomial-type example for this can be constructed by using the results in Sect. 3. In this sense, our definition is not robust with respect to the choice of EMM. Note that exactly the same issue arises in the strict local martingale definition of bubbles in continuous time. Using similar ideas as in Herdegen and Schweizer [7], one could introduce the notion of a *strong bubble* in discrete time. We leave the details of this to future work.

(e) Definition 2.1 can be reformulated in the following way: For some $k \in \mathbb{N}$, the stopped processes S^{τ_k} fails to be uniformly integrable. This reformulation allows applying results from the literature that are proved for general càdlàg (local) martingales. For example, Hulley and Ruf [8] provide necessary and sufficient characterisations for a càdlàg (local) martingale to be uniformly integrable. However, the conditions given in [8] require calculating the distribution of the supremum of S^{τ_k} and the distribution of the jumps of S^{τ_k} at certain hitting times, both of which are difficult to get hold of. For this reason, they are not very useful in our discrete-time setup.

We proceed to give a first simple example of a bubble in a complete market model.

Example 2.3 Define the process $S = (S_k)_{k \in \mathbb{N}_0}$ and the probability measure \mathbb{P} recursively by $S_0 = s_0 > \frac{1}{2}$ and for $k \in \mathbb{N}$,

$$\mathbb{P} \left[S_k = 2S_{k-1} - \frac{1}{2} \mid S_{k-1} > \frac{1}{2} \right] = \mathbb{P} \left[S_k = \frac{1}{2} \mid S_{k-1} > \frac{1}{2} \right] = \frac{1}{2},$$

$$\mathbb{P} \left[S_k = \frac{1}{2} \mid S_{k-1} = \frac{1}{2} \right] = 1.$$

Then S is a \mathbb{P} -martingale for its natural filtration, we have $\tau_1 < \infty$ \mathbb{P} -a.s., and $\mathbb{E}[S_{\tau_1}] = \frac{1}{2} < s_0 = \mathbb{E}[S_0]$, i.e., S is a bubble.

The following result provides two equivalent characterisations of bubbles. The first shows that a nonnegative martingale S is a bubble if and only if there exists a deterministic time $k \in \mathbb{N}_0$ such that S loses mass at the first drawdown after k . The second provides a limit characterisation. The latter characterisation is particularly useful for checking whether or not a martingale S is a bubble.

Theorem 2.4 Let $S = (S_k)_{k \in \mathbb{N}_0}$ be a nonnegative martingale. For $k \in \mathbb{N}_0$, define the stopping time $\tilde{\tau}_k$ by

$$\tilde{\tau}_k := \inf\{j > k : S_j < S_{j-1}\}.$$

Then the following are equivalent:

- (a) S is a bubble.
- (b) There exists $k \geq 0$ such that $\mathbb{E}[S_{\tilde{\tau}_k}] < \mathbb{E}[S_k]$.
- (c) There exists $k \geq 0$ such that $\lim_{n \rightarrow \infty} \mathbb{E}[(S_n - S_\infty)\mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots \leq S_n\}}] > 0$.

Proof (a) \Rightarrow (b) Suppose that (b) is not true, i.e., $\mathbb{E}[S_{\tilde{\tau}_k}] = \mathbb{E}[S_k]$ for all $k \geq 0$. Then for all $k \geq 0$, the stopped process $S^{\tilde{\tau}_k}$ is a right-closed supermartingale which does not lose mass at ∞ and hence is a uniformly integrable martingale. We proceed by induction to show that $\mathbb{E}[S_{\tau_\ell}] = \mathbb{E}[S_0]$ for all $\ell \geq 0$, whence (a) is not true. The induction basis $\ell = 0$ is trivial. For the induction step, suppose that $\mathbb{E}[S_{\tau_{\ell-1}}] = \mathbb{E}[S_0]$ for some $\ell \geq 1$. Then using the stopping theorem for the uniformly integrable martingale $S^{\tilde{\tau}_k}$ in the third equality, we obtain

$$\begin{aligned} \mathbb{E}[S_{\tau_\ell}] &= \sum_{k=\ell-1}^{\infty} \mathbb{E}[S_{\tau_\ell} \mathbf{1}_{\{\tau_{\ell-1}=k\}}] + \mathbb{E}[S_\infty \mathbf{1}_{\{\tau_{\ell-1}=\infty\}}] \\ &= \sum_{k=\ell-1}^{\infty} \mathbb{E}[S_{\tilde{\tau}_k} \mathbf{1}_{\{\tau_{\ell-1}=k\}}] + \mathbb{E}[S_{\tau_{\ell-1}} \mathbf{1}_{\{\tau_{\ell-1}=\infty\}}] \\ &= \sum_{k=\ell-1}^{\infty} \mathbb{E}[S_{\tau_{\ell-1}} \mathbf{1}_{\{\tau_{\ell-1}=k\}}] + \mathbb{E}[S_{\tau_{\ell-1}} \mathbf{1}_{\{\tau_{\ell-1}=\infty\}}] = \mathbb{E}[S_{\tau_{\ell-1}}] = \mathbb{E}[S_0]. \end{aligned}$$

(b) \Rightarrow (a) This follows from the fact that $\mathbb{E}[S_k] = \mathbb{E}[S_0]$ by the martingale property of S together with the fact that $\tilde{\tau}_k \leq \tau_{k+1}$.

(b) \Leftrightarrow (c) The equivalence follows from the following calculation, which uses the martingale property of S in the third equality:

$$\begin{aligned}
 \mathbb{E}[S_{\tilde{\tau}_k}] &= \lim_{n \rightarrow \infty} \sum_{\ell=k+1}^n \mathbb{E}[S_\ell \mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots \leq S_{\ell-1} > S_\ell\}}] + \mathbb{E}[S_\infty \mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots\}}] \\
 &= \lim_{n \rightarrow \infty} \sum_{\ell=k+1}^n (\mathbb{E}[S_\ell \mathbf{1}_{\{S_k \leq \dots \leq S_{\ell-1}\}}] - \mathbb{E}[S_\ell \mathbf{1}_{\{S_k \leq \dots \leq S_\ell\}}]) \\
 &\quad + \mathbb{E}[S_\infty \mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots\}}] \\
 &= \lim_{n \rightarrow \infty} \sum_{\ell=k+1}^n (\mathbb{E}[S_{\ell-1} \mathbf{1}_{\{S_k \leq \dots \leq S_{\ell-1}\}}] - \mathbb{E}[S_\ell \mathbf{1}_{\{S_k \leq \dots \leq S_\ell\}}]) \\
 &\quad + \mathbb{E}[S_\infty \mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots\}}] \\
 &= \mathbb{E}[S_k] + \lim_{n \rightarrow \infty} \mathbb{E}[(S_\infty - S_n) \mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots \leq S_n\}}]. \tag{2.2}
 \end{aligned}$$

This finishes the proof. □

The characterisation (c) in Theorem 2.4 is generally the most useful to decide whether or not S is a bubble. The following corollary strengthens this characterisation. It shows directly that bounded martingales fail to be bubbles. (Of course, this follows directly from the fact that a bounded martingale is uniformly integrable.)

Corollary 2.5 *Let $S = (S_k)_{k \in \mathbb{N}_0}$ be a nonnegative martingale. Then the following are equivalent:*

- (a) S is a bubble.
- (b) There exist $x \geq 0$ and $k \geq 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[(S_n - S_\infty) \mathbf{1}_{\{x \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] > 0.$$

- (c) For all $x \geq 0$, there exists $k \geq 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[(S_n - S_\infty) \mathbf{1}_{\{x \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] > 0.$$

Proof It is clear that (c) implies (b), and (b) a fortiori implies condition (c) of Theorem 2.4, which by Theorem 2.4 gives (a). It remains to prove (a) \Rightarrow (c). So fix $x \geq 0$. Since S is a bubble, by Theorem 2.4 (b) and the martingale property of S , there exists $\ell \geq 0$ such that $\mathbb{E}[S_{\tilde{\tau}_\ell}] < \mathbb{E}[S_\ell] = \mathbb{E}[S_0]$. Define the stopping times $\sigma_{\ell,x}$ and $\tau_1^{\sigma_{\ell,x}}$ by

$$\sigma_{\ell,x} := \inf\{k \geq \ell : S_k \geq x\}, \quad \tau_1^{\sigma_{\ell,x}} := \inf\{j > \sigma_{\ell,x} : S_j < S_{j-1}\}.$$

Then $\sigma_{\ell,x}$ is the first hitting time of $[x, \infty)$ after ℓ , and $\tau_1^{\sigma_{\ell,x}}$ is the first drawdown of S after $\sigma_{\ell,x}$. Since $\sigma_{\ell,x} \geq \ell$, it follows that $\tau_1^{\sigma_{\ell,x}} \geq \tilde{\tau}_\ell$.

By the definition of $\sigma_{\ell,x}$, the stopped process $S^{\sigma_{\ell,x}}$ is uniformly integrable because

$$S_k^{\sigma_{\ell,x}} \leq \max(S_0, \dots, S_\ell, S_{\sigma_{\ell,x}}, x) \in L^1.$$

This implies that $\mathbb{E}[S_{\sigma_{\ell,x}}] = \mathbb{E}[S_0]$. Since $\tau_1^{\sigma_{\ell,x}} \geq \tilde{\tau}_\ell$, this in turn implies that

$$\mathbb{E}[S_{\tau_1^{\sigma_{\ell,x}}}] \leq \mathbb{E}[S_{\tilde{\tau}_\ell}] < \mathbb{E}[S_0] = \mathbb{E}[S_{\sigma_{\ell,x}}].$$

A similar calculation as in (2.2) shows that

$$\mathbb{E}[S_{\tau_1^{\sigma_{\ell,x}}}] = \mathbb{E}[S_{\sigma_{\ell,x}}] + \lim_{n \rightarrow \infty} \mathbb{E}[(S_\infty - S_{\sigma_{\ell,x}+n})\mathbf{1}_{\{S_{\sigma_{\ell,x}} \leq \dots \leq S_{\sigma_{\ell,x}+n}\}}\mathbf{1}_{\{\sigma_{\ell,x} < \infty\}}].$$

This together with the tower property of conditional expectations and dominated convergence yields

$$\mathbb{E}\left[\lim_{n \rightarrow \infty} \mathbb{E}[(S_{\sigma_{\ell,x}+n} - S_\infty)\mathbf{1}_{\{S_{\sigma_{\ell,x}} \leq S_{\sigma_{\ell,x}+1} \leq \dots \leq S_{\sigma_{\ell,x}+n}\}}|\mathcal{F}_{\sigma_{\ell,x}}]\mathbf{1}_{\{\sigma_{\ell,x} < \infty\}}\right] > 0.$$

We may deduce that there is $k \geq \ell$ such that

$$\mathbb{E}\left[\lim_{n \rightarrow \infty} \mathbb{E}[(S_{k+n} - S_\infty)\mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots \leq S_{k+n}\}}\mathbf{1}_{\{\sigma_{\ell,x} = k\}}|\mathcal{F}_k]\right] > 0.$$

Using that $S_k \geq x$ on $\{\sigma_{\ell,x} = k\}$, this implies that

$$\mathbb{E}\left[\lim_{n \rightarrow \infty} \mathbb{E}[(S_{k+n} - S_\infty)\mathbf{1}_{\{x \leq S_k \leq S_{k+1} \leq \dots \leq S_{k+n}\}}|\mathcal{F}_k]\right] > 0.$$

Dominated convergence and the tower property of conditional expectations give (c). □

While characterisations (b) and (c) in Corollary 2.5 are an improvement of Theorem 2.4 (c), they still depend on S_∞ , of which we generally do not have a good knowledge. The following corollary provides a mild condition on S under which the bubble behaviour of S can be characterised without involving S_∞ .

Proposition 2.6 *Let $S = (S_k)_{k \in \mathbb{N}_0}$ be a nonnegative martingale and $x > 0$. Suppose that*

$$\sum_{k=0}^{\infty} \mathbb{P}[S_k < x \text{ or } S_k > S_{k+1}|\mathcal{F}_k] = \infty \quad \mathbb{P}\text{-a.s.}$$

Then for each $k \in \mathbb{N}_0$,

$$\lim_{n \rightarrow \infty} \mathbb{E}[S_\infty \mathbf{1}_{\{x \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] = 0. \tag{2.3}$$

In particular, S is a bubble if and only if there exists $k \in \mathbb{N}_0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[S_n \mathbf{1}_{\{x \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] > 0.$$

Proof The conditional Borel–Cantelli lemma implies that

$$\mathbb{P}\left[\liminf_{k \rightarrow \infty} \{x \leq S_k \leq S_{k+1}\}\right] = 1 - \mathbb{P}\left[\limsup_{k \rightarrow \infty} (\{S_k < x\} \cup \{S_k > S_{k+1}\})\right] = 0.$$

This implies a fortiori that $\{x \leq S_k \leq S_{k+1} \leq \dots\}$ is a \mathbb{P} -null set for each $k \in \mathbb{N}_0$. This gives (2.3). The final claim now follows from Corollary 2.5. \square

The following result gives simple necessary and sufficient conditions for a non-negative martingale with independent increments to be a bubble.

Theorem 2.7 *Let $(X_k)_{k \in \mathbb{N}}$ be an independent sequence of nonnegative random variables with $\mathbb{E}[X_k] = 1$ for $k \in \mathbb{N}$. Define the process $S = (S_k)_{k \in \mathbb{N}_0}$ by $S_k = \prod_{\ell=1}^k X_\ell$ and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$ by $\mathcal{F}_k := \sigma(S_0, \dots, S_k)$. Moreover, for $k \in \mathbb{N}$, set*

$$a_k := \mathbb{P}[X_k < 1] \in [0, 1), \quad b_k := \mathbb{E}[X_k \mathbf{1}_{\{X_k < 1\}}] \in [0, a_k].$$

Then S is a positive (\mathbb{P}, \mathbb{F}) -martingale. It is a bubble if and only if

$$\sum_{k=1}^{\infty} a_k = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} b_k < \infty.$$

Proof It is clear by construction that S is a nonnegative (\mathbb{P}, \mathbb{F}) -martingale.

First, we argue that if $\sum_{k=1}^{\infty} a_k < \infty$, then S is uniformly integrable and hence cannot be a bubble. To this end, note that

$$\infty > \sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} \mathbb{P}[X_k < 1] \geq \sum_{k=0}^{\infty} \mathbb{E}[(1 - \sqrt{X_k}) \mathbf{1}_{\{X_k < 1\}}] \geq \sum_{k=0}^{\infty} \mathbb{E}[(1 - \sqrt{X_k})].$$

By Kakutani’s theorem (see e.g. Williams [19, Theorem 14.12 (v)]), this implies that S is uniformly integrable.

Next, if $\sum_{k=1}^{\infty} a_k = \infty$, by independence of the X_k ,

$$\sum_{k=0}^{\infty} \mathbb{P}[S_k < 1 \text{ or } S_k > S_{k+1} | \mathcal{F}_k] \geq \sum_{k=0}^{\infty} \mathbb{P}[X_{k+1} < 1 | \mathcal{F}_k] = \sum_{k=1}^{\infty} a_k = \infty.$$

This together with Corollary 2.5 implies that S is a bubble if and only if there exists $k \geq 0$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}[S_n \mathbf{1}_{\{1 \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] &= \mathbb{P}[S_k \geq 1] \prod_{\ell=k+1}^{\infty} \mathbb{E}[X_\ell \mathbf{1}_{\{X_\ell \geq 1\}}] \\ &= P[S_k \geq 1] \prod_{\ell=k+1}^{\infty} (1 - b_\ell) > 0. \end{aligned}$$

Since $\mathbb{P}[S_k \geq 1] > 0$ and $1 - b_\ell > 0$ for all $\ell \in \mathbb{N}$, this is equivalent to

$$\prod_{\ell=1}^{\infty} (1 - b_\ell) > 0,$$

which in turn is equivalent to $\sum_{k=1}^{\infty} b_k < \infty$. □

We illustrate the above theorem by two examples. The first gives a bubble in a time-dependent binomial-type model, where the downward jumps get more and more severe.

Example 2.8 Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of independent random variables satisfying $P[X_k = \frac{1}{k}] = \frac{1}{k}$ and $P[X_k = 1 + \frac{1}{k}] = 1 - \frac{1}{k}$. Define the process $S = (S_k)_{k \in \mathbb{N}_0}$ by $S_k := \prod_{\ell=1}^k X_\ell$ and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$ by $\mathcal{F}_k := \sigma(S_0, \dots, S_k)$. Then $a_k = \frac{1}{k}$ and $b_k = \frac{1}{k^2}$. Hence $\sum_{k=1}^{\infty} a_k = \infty$ and $\sum_{k=1}^{\infty} b_k < \infty$, whence S is a bubble.

The second example shows that a martingale with i.i.d. returns is never a bubble. In particular, a standard binomial model is never a bubble, which agrees with our intuition.

Example 2.9 Let $(X_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. random variables that are nonnegative and satisfy $\mathbb{E}[X_k] = 1$ and $\mathbb{P}[X_k \neq 1] > 0$. Define the process $S = (S_k)_{k \in \mathbb{N}_0}$ by $S_k := \prod_{\ell=1}^k X_\ell$ and the filtration $\mathbb{F} = (\mathcal{F}_k)_{k \in \mathbb{N}_0}$ by $\mathcal{F}_k := \sigma(S_0, \dots, S_k)$. Then S is a martingale but fails to be uniformly integrable because $\mathbb{E}[\log X_k] < 1$ implies that $S_n = \exp(\sum_{k=1}^n \log X_k) \xrightarrow{\text{a.s.}} 0$ by the strong law of large numbers. However, setting $b_k := \mathbb{E}[X_k \mathbf{1}_{\{X_k < 1\}}] = b > 0$, we obtain

$$\sum_{k=1}^{\infty} b_k = \infty.$$

Thus S fails to be a bubble.

3 Characterisation of bubble measures for Markov chains

Throughout this section, we suppose that $S = (S_k)_{k \in \mathbb{N}_0}$ is a positive martingale which is a Markov process with transition kernel $K : (0, \infty) \times \mathcal{B}(0, \infty) \rightarrow [0, \infty)$ and starting from $S_0 = x > 0$. Our goal is to determine under which conditions on the kernel $K(x, dy)$ the measure \mathbb{P}_x is a bubble measure. To this end, we define the functions $a, b : (0, \infty) \rightarrow [0, 1)$ by

$$a(x) := \mathbb{P}_x[S_1 < x] = \int_{[0,x)} K(x, dy), \tag{3.1}$$

$$b(x) := \mathbb{E}_x \left[\frac{S_1}{x} \mathbf{1}_{\{S_1 < x\}} \right] = \int_{[0,x)} \frac{y}{x} K(x, dy). \tag{3.2}$$

Hence $a(x)$ denotes the *probability* of a downward jump and $b(x)$ the *relative recovery* in case of a downward jump.

First, we show we show that S cannot be a bubble unless the relative recovery function b converges to zero at infinity.

Proposition 3.1 *Assume that $\liminf_{x \rightarrow \infty} b(x) > 0$. Then S fails to be a bubble under \mathbb{P}_x for any $x \in (0, \infty)$.*

Proof There exist $x_0 > 0$ and $\varepsilon \in (0, 1)$ such that $b(x) \geq \varepsilon$ for all $x \geq x_0$. Pick $x \in (0, \infty)$. By Corollary 2.5 and using that $S_\infty \geq 0$, it suffices to show that for each $k \in \mathbb{N}_0$,

$$\lim_{n \rightarrow \infty} \mathbb{E}_x [S_n \mathbf{1}_{\{x_0 \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] = 0.$$

So let $k < n$. Then by the Markov property and the definition of b ,

$$\begin{aligned} & \mathbb{E}_x [S_n \mathbf{1}_{\{x_0 \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] \\ &= \mathbb{E}_x \left[\mathbb{E}_x \left[\frac{S_n}{S_{n-1}} \mathbf{1}_{\{S_n \geq S_{n-1}\}} \middle| \mathcal{F}_{n-1} \right] S_{n-1} \mathbf{1}_{\{x_0 \leq S_k \leq S_{k+1} \leq \dots \leq S_{n-1}\}} \right] \\ &= \mathbb{E}_x \left[(1 - b(S_{n-1})) S_{n-1} \mathbf{1}_{\{x_0 \leq S_k \leq S_{k+1} \leq \dots \leq S_{n-1}\}} \right] \\ &\leq (1 - \varepsilon) \mathbb{E}_x [S_{n-1} \mathbf{1}_{\{x_0 \leq S_k \leq S_{k+1} \leq \dots \leq S_{n-1}\}}] \\ &\leq (1 - \varepsilon)^{n-k} \mathbb{E}_x [S_k \mathbf{1}_{\{x_0 \leq S_k\}}] \leq (1 - \varepsilon)^{n-k} x. \end{aligned}$$

Now the claim follows by letting $n \rightarrow \infty$. □

We proceed to formulate a mild condition on the function a which allows us to characterise the bubble behaviour of S without involving S_∞ .

Assumption 3.2 There exists $x_a > 0$ such that $\inf_{x \in [x_a, y]} a(x) > 0$ for all $y > x_a$.

Remark 3.3 Assumption 3.2 is in particular fulfilled if a is positive and lower semi-continuous.

Proposition 3.4 *Suppose Assumption 3.2 is satisfied for some $x_a > 0$. Let $x' \geq x_a$. Then for each $k \in \mathbb{N}_0$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}_x [S_\infty \mathbf{1}_{\{x' \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] = 0. \tag{3.3}$$

Moreover, S is a bubble under \mathbb{P}_x if and only if there exists $k \in \mathbb{N}_0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}_x [S_n \mathbf{1}_{\{x' \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] > 0.$$

Proof By Proposition 2.6, it suffices to check that \mathbb{P}_x -a.s.,

$$\sum_{k=0}^{\infty} \mathbb{P}_x [S_k < x' \text{ or } S_k > S_{k+1} | \mathcal{F}_k] = \sum_{k=0}^{\infty} (\mathbf{1}_{\{S_k < x'\}} + a(S_k) \mathbf{1}_{\{S_k \geq x'\}}) = \infty.$$

We now distinguish two cases. If $\omega \in \limsup_{k \rightarrow \infty} \{S_k < x'\}$, it follows that we have $\sum_{k=0}^{\infty} \mathbf{1}_{\{S_k(\omega) < x'\}} = \infty$. If $\omega \in \liminf_{k \rightarrow \infty} \{S_k \geq x'\} \cap \{\sup_{k \geq 0} S_k < \infty\}$, then

$\sum_{k=0}^{\infty} a(S_k(\omega)) \mathbf{1}_{\{S_k(\omega) \geq x\}} = \infty$ by the assumption on a . Since $\sup_{k \geq 0} S_k < \infty$ \mathbb{P}_x -a.s. by Doob’s martingale convergence theorem, the claim follows. \square

We next aim to give a sufficient condition for S to be a bubble. To this end, we need to slightly relax the definition of the function b from (3.2). For $\varepsilon > 0$, define the function $b_\varepsilon : (0, \infty) \rightarrow (0, 1]$ by

$$b_\varepsilon(x) := \mathbb{E}_x \left[\frac{S_1}{x} \mathbf{1}_{\{S_1 < x(1+\varepsilon)\}} \right] = \int_{[0, x(1+\varepsilon))} \frac{y}{x} K(x, dy).$$

Theorem 3.5 *Suppose that Assumption 3.2 is satisfied and there exist $\varepsilon > 0$ and $x_b > 0$ with the property that the function b_ε is nonincreasing for $x \geq x_b$ and satisfies $\int_{\log x_b}^{\infty} b_\varepsilon(\exp(x)) dx < \infty$. Then S is a bubble under each \mathbb{P}_x for which S is not \mathbb{P}_x -a.s. bounded.*

Proof Suppose that S is not \mathbb{P}_x -a.s. bounded. Let x_a be the constant in Assumption 3.2. We may assume without loss of generality that $x_b \geq x_a$. By Proposition 3.4, it suffices to check that there is $k \in \mathbb{N}_0$ such that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}_x [S_n \mathbf{1}_{\{x_b \leq S_k \leq S_{k+1} \leq \dots \leq S_{k+n}\}}] \\ & \geq \lim_{n \rightarrow \infty} \mathbb{E}_x \left[S_k \mathbf{1}_{\{S_k \geq x_b\}} \prod_{j=1}^n \frac{S_{k+j}}{S_{k+j-1}} \mathbf{1}_{\{S_{k+j} \geq S_{k+j-1}(1+\varepsilon)\}} \right] > 0. \end{aligned}$$

Since S is not \mathbb{P}_x -a.s. bounded, there exists $k \in \mathbb{N}_0$ with $\mathbb{E}_x[S_k \mathbf{1}_{\{S_k \geq x_b\}}] > 0$. Fix $n \in \mathbb{N}$. By the definition of b_ε , we obtain for $j \in \{1, \dots, n\}$ that

$$\mathbb{E}_x \left[\frac{S_{k+j}}{S_{k+j-1}} \mathbf{1}_{\{S_{j+k-1}(1+\varepsilon) \leq S_{j+k}\}} \middle| \mathcal{F}_{k+j-1} \right] = (1 - b_\varepsilon(S_{k+j-1})) \quad \mathbb{P}_x\text{-a.s.}$$

This together with the tower property of conditional expectations and the fact that b_ε is nonincreasing for $x \geq x_b$ gives

$$\begin{aligned} & \mathbb{E}_x \left[S_k \mathbf{1}_{\{S_k \geq x_b\}} \prod_{j=1}^n \frac{S_{k+j}}{S_{k+j-1}} \mathbf{1}_{\{S_{k+j} \geq S_{k+j-1}(1+\varepsilon)\}} \right] \\ & = \mathbb{E}_x \left[S_k \mathbf{1}_{\{S_k \geq x_b\}} \left(\prod_{j=1}^{n-1} \frac{S_{k+j}}{S_{k+j-1}} \mathbf{1}_{\{S_{k+j} \geq S_{k+j-1}(1+\varepsilon)\}} \right) (1 - b_\varepsilon(S_{k+n-1})) \right] \\ & \geq \mathbb{E}_x \left[S_k \mathbf{1}_{\{S_k \geq x_b\}} \left(\prod_{j=1}^{n-1} \frac{S_{k+j}}{S_{k+j-1}} \mathbf{1}_{\{S_{k+j} \geq S_{k+j-1}(1+\varepsilon)\}} \right) (1 - b_\varepsilon(x_b(1 + \varepsilon)^{n-1})) \right] \\ & \geq \mathbb{E}_x \left[S_k \mathbf{1}_{\{S_k \geq x_b\}} \prod_{j=0}^{n-1} (1 - b_\varepsilon(x_b(1 + \varepsilon)^j)) \right] \\ & = \mathbb{E}_x [S_k \mathbf{1}_{\{S_k \geq x_b\}}] \prod_{j=0}^{n-1} (1 - b_\varepsilon(x_b(1 + \varepsilon)^j)). \end{aligned}$$

Thus it remains to show that $\prod_{j=0}^{\infty}(1 - b_{\varepsilon}(x_b(1 + \varepsilon)^j)) > 0$. The latter condition is equivalent to $\sum_{j=0}^{\infty} b_{\varepsilon}(\exp(\log x_b + \log(1 + \varepsilon)j)) < \infty$, which is equivalent to $\int_{\log x_b}^{\infty} b_{\varepsilon}(\exp(x)) dx < \infty$. □

We illustrate the above result by two examples. The first is a “smooth” version of Example 2.3; the second is an example of a “discrete diffusion” for the log price.

Example 3.6 Assume that the Markov kernel is given by

$$K(x, dy) = \begin{cases} \frac{1}{2}\mathbf{1}_{(0,1)}(y) dy + \frac{1}{2}\mathbf{1}_{(2x-1,2x)}(y) dy & \text{if } x > 1, \\ \frac{1}{2x}\mathbf{1}_{(0,2x)}(y) dy & \text{if } x \leq 1. \end{cases}$$

Then $a(x) = \frac{1}{2}$ and for $\varepsilon \in (0, 1)$,

$$b_{\varepsilon}(x) = \frac{(1 + \varepsilon)^2}{4} \mathbf{1}_{\{x \leq 1\}} + \left(1 - x \left(1 - \frac{(1 + \varepsilon)^2}{4}\right)\right) \mathbf{1}_{\{1 \leq x \leq \frac{1}{1-\varepsilon}\}} + \frac{1}{4x} \mathbf{1}_{\{x > \frac{1}{1-\varepsilon}\}}.$$

By Theorem 3.5, S is a bubble under \mathbb{P}_x for all $x > 0$.

Example 3.7 Let $(Z_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. standard normal random variables and $\sigma : \mathbb{R} \rightarrow (0, \infty)$ a measurable function such that $\sigma(x)$ is nondecreasing for large values of x . Define the process $(X_k)_{k \in \mathbb{N}_0}$ recursively by $X_0 := 0$ and

$$X_{k+1} := X_k + \sigma(X_k)Z_{k+1} - \frac{\sigma^2(X_k)}{2}, \quad k \in \mathbb{N}_0.$$

Then the process $S = (S_k)_{k \in \mathbb{N}_0}$ defined by $S_k := \exp(X_k)$ for $k \in \mathbb{N}_0$ is a Markov martingale.

Denoting by Φ the distribution function of a standard normal random variable, it is not difficult to check that for $x > 0$ and $\varepsilon \geq 0$,

$$a(x) = \Phi\left(\frac{\sigma(\log x)}{2}\right), \quad b_{\varepsilon}(x) = \Phi\left(\frac{\log(1 + \varepsilon)}{\sigma(\log x)} - \frac{\sigma(\log x)}{2}\right),$$

where $b_0 = b$. Thus by Proposition 3.1, for S to be a bubble, it is necessary that $\sigma(x) \rightarrow \infty$ as $x \rightarrow \infty$. Moreover, by Theorem 3.5, a sufficient condition for S to be a bubble is given by

$$\int_{x_0}^{\infty} \Phi\left(\frac{\log 2}{\sigma(x)} - \frac{\sigma(x)}{2}\right) dx < \infty \quad \text{for some } x_0 \in \mathbb{R}. \tag{3.4}$$

Denoting the density function of a standard normal random variable by φ , using Mills’ ratio and the fact that $|\left(\frac{\log 2}{\sigma(x)} - \frac{\sigma(x)}{2}\right)^2 - \left(\frac{\sigma(x)}{2}\right)^2| \leq \left(\frac{\log 2}{\sigma(x)}\right)^2 + \log 2$ is uniformly bounded for all sufficiently large x as σ is nondecreasing, it is not difficult to check that (3.4) is equivalent to

$$\int_{x_0}^{\infty} \frac{1}{\sigma(x)} \varphi\left(-\frac{\sigma(x)}{2}\right) dx < \infty \quad \text{for some } x_0 \in \mathbb{R}. \tag{3.5}$$

Remark 3.8 It is insightful to compare Example 3.7 to the continuous-time theory of bubbles. To this end, recall that in continuous time, the process $S = (S_t)_{t \geq 0}$ given by $S_t := \exp(X_t)$, where $X = (X_t)_{t \geq 0}$ solves the SDE

$$dX_t = \sigma(X_t)dW_t - \frac{1}{2}\sigma^2(X_t)dt,$$

is a strict local martingale and hence a bubble if and only if for some $x_0 \in \mathbb{R}$,

$$\int_{x_0}^{\infty} \frac{dx}{\sigma^2(x)} < \infty; \tag{3.6}$$

cf. Mijatović and Urusov [14, Corollary 4.3]. While (3.5) and (3.6) both say that S is a bubble if and only if $\sigma(x) \rightarrow \infty$ fast enough as $x \rightarrow \infty$, the exact rate of increase of σ required for a bubble is quite different as (3.6) is a much stronger requirement on the growth of σ than (3.5). The reason for this is that the discretisation of the diffusion model in continuous time should not be done along a deterministic time grid, but along certain sequences of stopping times; see Sect. 5 below.

While Proposition 3.1 and Theorem 3.5 give useful general sufficient conditions for the absence or presence of a bubble, respectively, these conditions are not necessary. In the complete Markov case, we can give a necessary and sufficient characterisation of bubbles under mild assumptions on the functions a and b .

Theorem 3.9 *Suppose the Markov kernel is given by*

$$K(x, dy) = a(x)\delta_{\frac{b(x)x}{a(x)}}(dy) + (1 - a(x))\delta_{\frac{(1-b(x))x}{1-a(x)}}(dy),$$

where $0 \leq b(x) < a(x) < 1$ and $0 < \liminf_{x \rightarrow \infty} a(x) \leq \limsup_{x \rightarrow \infty} a(x) < 1$. Moreover, suppose that there exists $x_b > 0$ such that the function b is nonincreasing for $x \geq x_b$. Then S is a bubble under \mathbb{P}_x if and only if S is not \mathbb{P}_x -a.s. bounded and $\int_{\log x_b}^{\infty} b(\exp(x)) dx < \infty$.

The above process corresponds to a binomial-type model where the probability of downward jumps is bounded away from 0 and 1 and the relative recovery in case of a downward jump decreases for large values.

Proof of Theorem 3.9 Suppose that S is not \mathbb{P}_x -a.s. bounded. We may focus on the case where $\lim_{x \rightarrow \infty} b(x) = 0$. Indeed, otherwise it follows that $\lim_{x \rightarrow \infty} b(x) > 0$ and hence $\int_{\log x_b}^{\infty} b(\exp(x)) dx = \infty$, and S fails to be a bubble by Proposition 3.1.

Since $\lim_{x \rightarrow \infty} b(x) = 0$ and $0 < \liminf_{x \rightarrow \infty} a(x) \leq \limsup_{x \rightarrow \infty} a(x) < 1$, after potentially enlarging x_b , we may assume that there exists $0 < c < C$ such that $1 + c \leq \frac{1-b(x)}{1-a(x)} \leq 1 + C$ for all $x \geq x_b$. By Proposition 3.4, S is a bubble under \mathbb{P}_x if and only if there exists $k \geq 0$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}_x [S_n \mathbf{1}_{\{x_b \leq S_k \leq S_{k+1} \leq \dots \leq S_{k+n}\}}] > 0.$$

Using that $\{S_j \leq S_{j+1}\} = \{S_{j+1} = \frac{1-b(S_j)}{1-a(S_j)}S_j\}$ for $j \in \mathbb{N}_0$ and arguing as in the proof of Theorem 3.5 gives for each $0 \leq k \leq n$ that

$$\begin{aligned} \mathbb{E}_x[S_k \mathbf{1}_{\{S_k \geq x_b\}}] \prod_{j=0}^{n-1} \left(1 - b(x_b(1 + C)^j)\right) &\geq \mathbb{E}_x[S_n \mathbf{1}_{\{x_b \leq S_k \leq S_{k+1} \leq \dots \leq S_{k+n}\}}] \\ &\geq \mathbb{E}_x[S_k \mathbf{1}_{\{S_k \geq x_b\}}] \prod_{j=0}^{n-1} \left(1 - b(x_b(1 + c)^j)\right). \end{aligned}$$

Now using that for $\gamma \in \{c, C\}$, $\prod_{j=0}^{\infty} (1 - b(x_b(1 + \gamma)^j)) > 0$ if and only if

$$\sum_{j=0}^{\infty} b\left(\exp(\log x_b + \log(1 + \gamma)j)\right) < \infty,$$

which in turn is equivalent to $\int_{\log x_b}^{\infty} b(\exp(x)) dx < \infty$, the claim follows. □

4 A fixed point equation associated to a Markovian bubble

In this section, we continue our study of Markov martingales, taking a more analytic perspective. We assume throughout that $S = (S_k)_{k \in \mathbb{N}_0}$ is a positive Markov martingale with kernel $K : (0, \infty) \times \mathcal{B}(0, \infty) \rightarrow [0, \infty)$, starting from $S_0 = x > 0$. The key object of this section is the *default function* of S .

Definition 4.1 The Borel-measurable function $M_S : (0, \infty) \rightarrow [0, \infty)$ defined by

$$M_S(x) := \lim_{n \rightarrow \infty} \mathbb{E}_x[(S_n - S_\infty) \mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq x\}}]$$

is called the *default function* of S .

It follows from (2.2) (with $k = 0$) that $M_S(x) = \mathbb{E}_x[S_{\tau_1}]$, so that M_S measures the loss of mass at the first drawdown of S . It is clear that \mathbb{P}_x is a bubble measure for S if $M_S(x) > 0$. The following two results show that M essentially fully characterises the bubble behaviour of S under \mathbb{P}_x for all $x > 0$.

Proposition 4.2 *Suppose $M_S(x) = 0$ for all $x \geq x' > 0$. Then $M_S(x) = 0$ for all $x > 0$, and \mathbb{P}_x fails to be a bubble measure for S for any $x > 0$.*

Proof Fix $x > 0$. Then for all $k \in \mathbb{N}_0$, the Markov property of S , the choice of x' and dominated convergence give

$$\begin{aligned} &\lim_{n \rightarrow \infty} \mathbb{E}_x[(S_n - S_\infty) \mathbf{1}_{\{x' \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] \\ &= \mathbb{E}_x \left[\lim_{n \rightarrow \infty} \mathbb{E}_{S_k}[(S_n - S_\infty) \mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots \leq S_n\}}] \mathbf{1}_{\{S_k \geq x'\}} \right] \\ &= \mathbb{E}_x[M_S(S_k) \mathbf{1}_{\{S_k \geq x'\}}] = 0. \end{aligned}$$

Thus Corollary 2.5 implies that S is not a bubble under \mathbb{P}_x , and so $M_S(x) = 0$. □

Proposition 4.3 *Suppose that $\liminf_{x \rightarrow \infty} M_S(x) > 0$. Then S is a bubble under \mathbb{P}_x for all $x > 0$ for which S is not \mathbb{P}_x -a.s. bounded.*

Proof Fix $x > 0$ and suppose that S is not \mathbb{P}_x -a.s. bounded. By hypothesis, there exists $x' \geq x$ such that $M_S(y) > 0$ for all $y \geq x'$. Since S is not \mathbb{P}_x -a.s. bounded, there is $k \geq 0$ such that $\mathbb{P}_x[S_k \geq x'] > 0$. Then by the Markov property of S , the choice of x' and dominated convergence,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}_x[(S_n - S_\infty)\mathbf{1}_{\{x' \leq S_k \leq S_{k+1} \leq \dots \leq S_n\}}] \\ &= \mathbb{E}_x \left[\lim_{n \rightarrow \infty} \mathbb{E}_{S_k}[(S_n - S_\infty)\mathbf{1}_{\{S_k \leq S_{k+1} \leq \dots \leq S_n\}}] \mathbf{1}_{\{S_k \geq x'\}} \right] \\ &= \mathbb{E}_x[M_S(S_k)\mathbf{1}_{\{S_k \geq x'\}}] > 0. \end{aligned}$$

Thus S is a bubble under \mathbb{P}_x by Corollary 2.5. □

In the remainder of this section, we seek to characterise the function M_S in an analytic way and provide conditions for it to be non-zero.

First, we show that M_S solves a fixed point equation, more precisely a homogeneous *Volterra integral equation of the second kind*; cf. Brunner [3, Chap. 1.2] for a textbook treatment.

Lemma 4.4 *The default function M_S is a solution to the Volterra integral equation*

$$M_S(x) = \int_{[x, \infty)} M_S(y)K(x, dy), \quad x > 0. \tag{4.1}$$

Proof Fix $x > 0$. Using the definition of M_S , dominated convergence and the Markov property of S , we obtain

$$\begin{aligned} \int_{[x, \infty)} M_S(y)K(x, dy) &= \mathbb{E}_x[M_S(S_1)\mathbf{1}_{\{S_1 \geq x\}}] \\ &= \mathbb{E}_x \left[\lim_{n \rightarrow \infty} \mathbb{E}_{S_1}[(S_n - S_\infty)\mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq x\}}] \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_x[(S_n - S_\infty)\mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq x\}}] = M_S(x). \quad \square \end{aligned}$$

Note that (4.1) is non-standard in that the domain is non-compact. Therefore, we cannot apply standard existence and uniqueness results for Volterra integral equations, cf. [3, Chap. 8]. In fact, existence is anyway not an issue since the zero function always solves (4.1). Since the bubble case corresponds to (4.1) having a non-zero (nonnegative) solution, we are actually interested in *non-uniqueness*, i.e., the case that (4.1) has multiple nonnegative solutions. By homogeneity of (4.1), we then always have infinitely many solutions, and so it is clear that we need an additional condition to pin down the default function M_S .

It follows from the definition of M_S that $M_S(x) \leq x$ for all $x > 0$. So we consider nonnegative solutions to (4.1) that are dominated by the identity. To this end, denote

by \mathcal{I} all Borel-measurable functions $M : (0, \infty) \rightarrow [0, \infty)$ satisfying $M(x) \leq x$ for all $x > 0$. Using that

$$0 \leq \int_{[x, \infty)} M(y)K(x, dy) \leq \int_{(0, \infty)} M(y)K(x, dy) \leq \int_{(0, \infty)} yK(x, dy) = x$$

for all $M \in \mathcal{I}$ and $x > 0$, we can define the map $\mathcal{K} : \mathcal{I} \rightarrow \mathcal{I}$ by

$$\mathcal{K}(M)(x) = \int_{[x, \infty)} M(y)K(x, dy), \quad x > 0.$$

Then the nonnegative solutions to (4.1) dominated by the identity are precisely given by fixed points of \mathcal{K} .

While the map \mathcal{K} is in general not a contraction (and therefore (4.1) may have multiple solutions on \mathcal{I}), it is *monotone*, and this property will prove crucial for our subsequent analysis.

Proposition 4.5 *The map \mathcal{K} is monotone on \mathcal{I} .*

Proof Let $M_1, M_2 \in \mathcal{I}$ with $M_1 \leq M_2$. Then monotonicity of the integral gives for $x > 0$ that

$$\mathcal{K}(M_1)(x) = \int_{[x, \infty)} M_1(y)K(x, dy) \leq \int_{[x, \infty)} M_2(y)K(x, dy) = \mathcal{K}(M_2)(x). \quad \square$$

Due to monotonicity of \mathcal{K} , it is very useful to consider *subsolutions* and *supersolutions* to (4.1) on \mathcal{I} .

Definition 4.6 A function $M \in \mathcal{I}$ is called a *subsolution* to (4.1) if

$$M(x) \leq \int_{[x, \infty)} M(y)K(x, dy), \quad x > 0.$$

It is called a *supersolution* to (4.1) if

$$M(x) \geq \int_{[x, \infty)} M(y)K(x, dy), \quad x > 0.$$

The following result shows that we can construct from each sub- or supersolution a solution to (4.1) by Picard iteration. To this end, for $n \in \mathbb{N}_0$, define $\mathcal{K}^n(M)$ recursively by $\mathcal{K}^0(M) := M$ and $\mathcal{K}^n(M) := \mathcal{K}(\mathcal{K}^{n-1}(M))$ for $n \geq 1$.

Proposition 4.7 *Let $M \in \mathcal{I}$ be a sub- or supersolution to (4.1). Then the limit $\mathcal{K}^\infty(M) = \lim_{n \rightarrow \infty} \mathcal{K}^n(M)$ exists and is a solution to (4.1). Moreover,*

- if M is a subsolution, then the sequence $(\mathcal{K}^n(M))_{n \in \mathbb{N}}$ is nondecreasing and $\mathcal{K}^\infty(M)$ is the smallest solution dominating M ;
- if M is a supersolution, then the sequence $(\mathcal{K}^n(M))_{n \in \mathbb{N}}$ is nonincreasing and $\mathcal{K}^\infty(M)$ is the largest solution dominated by M .

Proof We only consider the case that M is a subsolution; the proof for the case that M is a supersolution is analogous. If M is a subsolution, the sequence $(\mathcal{K}^n(M))_{n \in \mathbb{N}_0}$ is nondecreasing in M by monotonicity of \mathcal{K} . Hence the limit $\lim_{n \rightarrow \infty} \mathcal{K}^n(M)$ exists and is in \mathcal{I} since each $\mathcal{K}^n(M)$ is in \mathcal{I} . Moreover, it follows from monotone convergence that

$$\begin{aligned} \mathcal{K}^\infty(M)(x) &= \lim_{n \rightarrow \infty} \mathcal{K}^n(M)(x) = \lim_{n \rightarrow \infty} \int_{[x, \infty)} \mathcal{K}^{n-1}(M)(y)K(x, dy) \\ &= \int_{[x, \infty)} \mathcal{K}^\infty(M)(y)K(x, dy), \quad x > 0, \end{aligned}$$

whence $\mathcal{K}^\infty(M)$ is a solution to (4.1).

Now let $\tilde{M} \in \mathcal{I}$ be any solution to (4.1) dominating M . It suffices to show that $\tilde{M} \geq \mathcal{K}^n(M)$ for all $n \in \mathbb{N}_0$. We argue by induction. The induction basis is trivial. For the induction step, suppose that $n \geq 1$ and $\tilde{M} \geq \mathcal{K}^{n-1}(M)$. Then by the induction hypothesis and the definition of $\mathcal{K}^n(M)$, for $x > 0$,

$$\tilde{M}(x) = \int_{[x, \infty)} \tilde{M}(y)K(x, dy) \geq \int_{[x, \infty)} \mathcal{K}^{n-1}(M)(y)K(x, dy) = \mathcal{K}^n(M)(x). \quad \square$$

We note the following important corollary.

Corollary 4.8 *The largest solution to (4.1) on \mathcal{I} is given by $\mathcal{K}^\infty(\text{id})$, where id denotes the identity function.*

It follows from Lemma 4.4 and Corollary 4.8 that the default function M_S is dominated by $\mathcal{K}^\infty(\text{id})$. Under a mild assumption on the kernel K , we can assert that M_S coincides with $\mathcal{K}^\infty(\text{id})$. Thus in this case, we can characterise the default function M_S as the maximal solution to (4.1) dominated by the identity.

Theorem 4.9 *Suppose that Assumption 3.2 is satisfied for any $x_a > 0$. Then M_S is the maximal solution to (4.1) dominated by the identity. It is given by $M_S = \mathcal{K}^\infty(\text{id})$. Moreover, $M_S(x) < x$ for all $x > 0$ and if $M_S \not\equiv 0$, then*

$$\limsup_{x \rightarrow \infty} \frac{M_S(x)}{x} = 1. \tag{4.2}$$

Proof We first show by induction that for each $n \in \mathbb{N}_0$, we have

$$\mathcal{K}^n(\text{id})(x) = \mathbb{E}_x[S_n \mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq S_0\}}], \quad x > 0.$$

The induction basis $n = 0$ follows from the martingale property of S . For the induction step, let $n \geq 1$. By the definition of $\mathcal{K}^n(\text{id})$ and the Markov property of S , we

obtain

$$\begin{aligned} \mathcal{K}^n(\text{id})(x) &= \int_{[x, \infty)} \mathcal{K}^{n-1}(\text{id})(y)K(x, dy) = \mathbb{E}_x[\mathcal{K}^{n-1}(\text{id})(S_1)\mathbf{1}_{\{S_1 \geq x\}}] \\ &= \mathbb{E}_x[\mathbb{E}_{S_1}[S_n \mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1\}}]\mathbf{1}_{\{S_1 \geq x\}}] \\ &= \mathbb{E}_x[S_n \mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq S_0\}}]. \end{aligned}$$

Hence the definitions of $\mathcal{K}^\infty(\text{id})$ and M_S together with (3.3) give

$$\begin{aligned} \mathcal{K}^\infty(\text{id})(x) &= \lim_{n \rightarrow \infty} \mathbb{E}_x[S_n \mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq S_0\}}] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}_x[(S_n - S_\infty)\mathbf{1}_{\{S_n \geq S_{n-1} \geq \dots \geq S_1 \geq S_0\}}] = M_S(x), \quad x > 0. \end{aligned}$$

It follows from Corollary 4.8 that M_S is the maximal solution to (4.1) dominated by the identity. Moreover, $M_S \leq \text{id}$, the fact that $\mathbb{P}_x[S_1 < x] = a(x) > 0$ for all $x > 0$ and the martingale property of S give

$$M_S(x) = \mathbb{E}_x[M_S(S_1)\mathbf{1}_{\{S_1 \geq x\}}] \leq \mathbb{E}_x[S_1 \mathbf{1}_{\{S_1 \geq x\}}] < \mathbb{E}_x[S_1] = x.$$

Finally, to establish (4.2), let us define the function $H_S : (0, \infty) \rightarrow [0, 1]$ by $H_S(x) = \sup_{y \geq x} \frac{M_S(x)}{x}$. If $M_S \not\equiv 0$, it follows from Proposition 4.2 that $H_S(x) > 0$ for all $x > 0$. It suffices to show that $H_S(x) = 1$ for all $x > 0$. Seeking a contradiction, suppose there exists $x' > 0$ such that $H_S(x') < 1$. Define the function $M : (0, \infty) \rightarrow [0, \infty)$ by

$$M(y) = \begin{cases} 0 & \text{if } x < x', \\ \frac{M_S(x)}{H_S(x')} & \text{if } x \geq x'. \end{cases}$$

Then $M \in \mathcal{I}$ by the fact that $H_S(x') \geq \frac{M_S(x)}{x}$ for $x \geq x'$. Moreover, M is a subsolution to (4.1) and $M(x) > M_S(x)$ for $x \geq x'$. This together with Proposition 4.7 implies that $\mathcal{K}^\infty(M)$ is a solution to (4.1) dominated by the identity. Since $\mathcal{K}^\infty(M)(x) \geq M(x) > M_S(x)$ for $x > x'$, this is in contradiction to M_S being the maximal solution to (4.1) dominated by the identity. \square

The following corollary shows that if we can find a non-trivial subsolution to (4.1), then S is a bubble.

Corollary 4.10 *Suppose Assumption 3.2 is satisfied for any $x_a > 0$. If $M \in \mathcal{I}$ is a subsolution to (4.1), then $M \leq M_S$. If in addition $\liminf_{x \rightarrow \infty} M(x) > 0$, then S is a bubble under \mathbb{P}_x for all $x > 0$ for which S is not \mathbb{P}_x -a.s. bounded.*

Proof Proposition 4.7 and Theorem 4.9 give $M \leq \mathcal{K}^\infty(M) \leq M_S$. The additional claim then follows from Proposition 4.3. \square

A typical candidate for a subsolution in Corollary 4.10 is given by the call function $M(x) = (x - L)^+$ for some $L > 0$. This is illustrated by the following example. Note that this example cannot be addressed with the results from Sect. 3.

Example 4.11 Suppose that $K(x, dy) = k(x, y) dy$ for all $x > 0$, where the density k satisfies

$$k(x, y) = \frac{2}{3(x + 1)} \mathbf{1}_{[x, 2x]}(y) \quad \text{for } y \geq x > 0$$

and $k(x, y)$ on $\{(x, y) : 0 < y < x\}$ is chosen such that K is a martingale kernel. Then the function a from (3.1) satisfies $a(x) = 1 - \int_{[x, \infty)} k(x, y) dy = \frac{3+x}{3+3x} \geq \frac{1}{3}$ so that Assumption 3.2 is satisfied for any $x_a > 0$. Consider $M(x) := (x - 3)^+$. Then trivially $\int_x^\infty M(y)k(x, y) dy \geq 0 = M(x)$ for $x \leq 3$ and

$$\int_x^\infty M(y)k(x, y) dy = x - 3 \frac{x}{x + 1} \geq x - 3 = M(x) \quad \text{for } x > 3.$$

It follows that M is subsolution to (4.1), and we deduce that S has a bubble under \mathbb{P}_x for all $x > 0$ by Corollary 4.10 since S is not \mathbb{P}_x -a.s. bounded for any $x > 0$.

While Theorem 4.9 provides a characterisation of the default function M , it does not provide a criterion to decide whether (4.1) has a non-trivial, i.e., a non-zero non-negative solution dominated by the identity. Moreover, it does not provide a criterion to decide whether a given candidate solution M to (4.1) is indeed maximal. Under a stronger assumption on the kernel K , we can provide a sufficient criterion for the existence of non-trivial solutions to (4.1) dominated by the identity. Moreover, we obtain a local uniqueness result in this case. To this end, recall the definitions of the functions a and b from (3.1) and (3.2), respectively. Moreover, denote by $\|\cdot\|_{\text{sup}}$ the supremum norm.

Theorem 4.12 *Suppose $\inf_{x>0} a(x) > 0$. Then the following are equivalent:*

- (a) $\sup_{x>0} xb(x) < \infty$.
- (b) *For all $L > 0$ sufficiently large, the call function $M(x) = (x - L)^+$ is a subsolution to (4.1).*
- (c) *The default function M_S is non-trivial and satisfies $\|\text{id} - M_S\|_{\text{sup}} < \infty$.*

Moreover, if one of the above conditions is satisfied, M_S is the unique solution to (4.1) among all solutions $M \in \mathcal{I}$ satisfying $\|\text{id} - M\|_{\text{sup}} < \infty$.

Proof (a) \Rightarrow (b) Set $\alpha := \inf_{x>0} a(x) > 0$, $\beta := \sup_{x>0} xb(x)$ and $L \geq \frac{\beta}{\alpha}$. Then the call function $M(x) := (x - L)^+$ satisfies $\int_{[x, \infty)} M(y)K(x, dy) \geq 0 = M(x)$ for $x \leq L$ and

$$\begin{aligned} \int_{[x,\infty)} M(y)K(x, dy) &= x(1 - b(x)) - L(1 - a(x)) \\ &= M(x) - \beta(x)x + \alpha(x)L \\ &\geq M(x) - \beta + \alpha L \geq M(x), \quad x > L. \end{aligned}$$

Hence M is a subsolution to (4.1).

(b) \Rightarrow (c) Let $L > 0$ be such that $M(x) = (x - L)^+$ is a subsolution to (4.1). Then we obtain $M_S \geq M$ by Corollary 4.10, whence M_S is non-trivial and satisfies $\| \text{id} - M_S \|_{\text{sup}} < \| \text{id} - M \|_{\text{sup}} = L$.

(c) \Rightarrow (a) Since $\mathcal{K}(\text{id}) \geq \mathcal{K}^\infty(\text{id}) = M_S$ by Proposition 4.7 and Theorem 4.9, it follows that $\| \text{id} - \mathcal{K}(\text{id}) \|_{\text{sup}} \leq \| \text{id} - M_S \|_{\text{sup}} < \infty$. Now the claim follows from the fact that $xb(x) = \int_{(0,x)} yK(x, dy) = x - \mathcal{K}(\text{id})(x) = \text{id}(x) - \mathcal{K}(\text{id})(x)$ for $x > 0$.

For the additional claim, set $\mathcal{I}_{\text{sup}} := \{M \in \mathcal{I} : \| \text{id} - M \|_{\text{sup}} < \infty\}$. Then \mathcal{I}_{sup} is a complete metric space for the metric generated by the supremum norm. Moreover, \mathcal{K} maps \mathcal{I}_{sup} to itself. Indeed, for each $M \in \mathcal{I}_{\text{sup}}$, there exists by (b) some $L \geq \| \text{id} - M \|_{\text{sup}}$ such that $\tilde{M}(x) := (x - L)^+$ is a subsolution to (4.1). Hence by monotonicity of \mathcal{K} and the fact that $\tilde{M} \leq M$ is a subsolution, we get

$$\| \text{id} - \mathcal{K}(M) \|_{\text{sup}} \leq \| \text{id} - \mathcal{K}(\tilde{M}) \|_{\text{sup}} \leq \| \text{id} - \tilde{M} \|_{\text{sup}} = L < \infty.$$

Finally, we show that \mathcal{K} is a contraction on \mathcal{I}_{sup} . Let $M_1, M_2 \in \mathcal{I}_{\text{sup}}$. Then

$$\begin{aligned} | \mathcal{K}(M_1)(x) - \mathcal{K}(M_2)(x) | &= \left| \int_{[x,\infty)} (M_1(y) - M_2(y))K(x, dy) \right| \\ &\leq \| M_1 - M_2 \|_{\text{sup}} \int_{[x,\infty)} K(x, dy) \\ &\leq (1 - \alpha) \| M_1 - M_2 \|_{\text{sup}}. \end{aligned}$$

Taking the supremum over x shows that \mathcal{K} is indeed a contraction since $\alpha > 0$. Now Banach’s fixed point theorem implies that \mathcal{K} has a unique fixed point, and by (c), this fixed point is M_S . □

We proceed to illustrate Theorem 4.12 by an example.

Example 4.13 Suppose that $K(x, dy) = k(x, y) dy$ for all $x > 0$, where the density satisfies

$$k(x, y) = \frac{e}{2} \frac{1 - e^{-x}}{1 - e^{-y}} \frac{1}{x} e^{-y/x} \quad \text{for } y \geq x > 0$$

and $k(x, y)$ on $\{(x, y) : 0 < y < x\}$ is chosen such that K is a martingale kernel. Note that

$$\int_x^\infty k(x, y) dy < \frac{e}{2} \int_x^\infty \frac{1}{x} e^{-y/x} dy = \frac{1}{2}.$$

This implies in particular that $a(x) \geq 1/2$ for all $x > 0$. In this case, the fixed point equation (4.1) is given by

$$M(x) = \int_x^\infty M(y) \frac{e}{2} \frac{1 - e^{-x}}{1 - e^{-y}} \frac{1}{x} e^{-y/x} dy,$$

which is equivalent to

$$\frac{M(x)x}{1 - e^{-x}} = \frac{e}{2} \int_x^\infty \frac{M(y)}{1 - e^{-y}} e^{-y/x} dy.$$

As one easily checks, the function $M_\lambda(x) := \lambda x(1 - e^{-x})$, $x > 0$, is a solution in \mathcal{I} to (4.1) for any $\lambda \in [0, 1]$. As M_S is the largest solution to (4.1) dominated by the identity, this yields the candidate $M_1(x) = x(1 - e^{-x})$. But $\|id - M_1\|_{\text{sup}} = e^{-1} < \infty$, and so it follows from Theorem 4.12 that $M_S = M_1$.

Combining Theorem 4.12 with Proposition 4.3, we get the following existence results for bubbles. Note that this result covers cases that cannot be treated with the theory of Sect. 3.

Corollary 4.14 *Suppose that $\inf_{x>0} a(x) > 0$ and $\sup_{x>0} xb(x) < \infty$. Then S is a bubble under \mathbb{P}_x for all $x > 0$ for which S is not \mathbb{P}_x -a.s. bounded.*

5 Relation to the strict local martingale definition of asset price bubbles in continuous-time models

In this final section, we discuss how our definition of bubbles in discrete time relates to the strict local martingale definition of bubbles in continuous time. To approach this question, one first has to discretise a positive continuous local martingale $X = (X_t)_{t \geq 0}$ in continuous time in such a way that it becomes a discrete-time martingale. Of course, there are many ways to do this, and we choose a somewhat canonical construction. More precisely, we consider localising sequences $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\tau_n \rightarrow \infty$ \mathbb{P} -a.s. such that for each n , both τ_n and the stopped process X^{τ_n} are uniformly bounded. We then define the discrete-time process $S = (S_n)_{n \in \mathbb{N}}$ by $S_n := X_{\tau_n}$. Then S is a martingale by the stopping theorem and satisfies $S_\infty = X_\infty$ \mathbb{P} -a.s., which implies that S is uniformly integrable if and only if X is uniformly integrable.

The simplest way to get localising sequences as above is to choose two increasing sequences of positive real numbers $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ converging to infinity and to define the sequence $(\tau_n^{a,b})_{n \in \mathbb{N}}$ of stopping times by $\tau_0^{a,b} := 0$ and

$$\tau_n^{a,b} := \inf\{t \geq 0 : X_t \geq b_n\} \wedge a_n, \quad n \in \mathbb{N}. \tag{5.1}$$

Then $(\tau_n^{a,b})_{n \in \mathbb{N}}$ is a localising sequence of stopping times for X with $\tau_n^{a,b} \leq a_n$ and $\sup_{t \geq 0} X_t^{\tau_n^{a,b}} \leq b_n$, by continuity of X .

In the special case that X is a Markov process, we should like to stop in such a way that the discrete-time process S is again a Markov process. In this case, the simplest way to get localising sequences as above is to choose two constants $\alpha, \beta > 0$ and to define the sequence of stopping times $(\tau_n^{\alpha,\beta})_{n \in \mathbb{N}}$ by $\tau_0^{\alpha,\beta} := 0$ and

$$\tau_n^{\alpha,\beta} := \inf \left\{ t \geq \tau_{n-1}^{\alpha,\beta} : X_t \geq (1 + \beta) X_{\tau_{n-1}^{\alpha,\beta}} \right\} \wedge (\tau_{n-1}^{\alpha,\beta} + \alpha), \quad n \in \mathbb{N}. \tag{5.2}$$

In this case, it is still true that $(\tau_n^{\alpha,\beta})_{n \in \mathbb{N}}$ is a localising sequence of stopping times for X and that $\tau_n^{\alpha,\beta}$ and $X_{\tau_n^{\alpha,\beta}}$ are uniformly bounded.

Our first goal in this section is to show that if X is a continuous positive strict local martingale, then the discrete-time process S is a bubble for either choice of stopping times above. The proof of this result relies on the following deep characterisation of strict local martingales in continuous time; cf. Meyer [13], Delbaen and Schachermayer [5] and Kardaras et al. [11]. Let $X = (X_t)_{t \geq 0}$ be a positive càdlàg local \mathbb{P} -martingale with $X_0 = x$. Then under some technical assumptions on the probability space and the underlying filtration, there exists a probability measure \mathbb{Q} with $\mathbb{Q}|_{\mathcal{F}_t} \gg \mathbb{P}|_{\mathcal{F}_t}$ for all $t \geq 0$ such that $Y := 1/X$ is a nonnegative true \mathbb{Q} -martingale, and for all bounded stopping times τ and all $A \in \mathcal{F}_\tau$,

$$\mathbb{P}[A] = x \mathbb{E}^{\mathbb{Q}} [Y_\tau \mathbf{1}_A].$$

Especially, we have the identity

$$\mathbb{E}^{\mathbb{P}} [X_0 - X_t] = x \mathbb{Q}[Y_t = 0], \quad t \geq 0,$$

i.e., X is a strict local martingale on $[0, t]$ if and only if $\mathbb{Q}[Y_t = 0] > 0$.

With this, we have the following two results.

Proposition 5.1 *Let $X = (X_t)_{t \geq 0}$ be a continuous positive strict local \mathbb{P} -martingale. Let $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ be increasing sequences of positive real numbers converging to ∞ . Define the sequence of stopping times $(\tau_n^{a,b})_{n \in \mathbb{N}}$ by (5.1) and set $S_n^{a,b} = X_{\tau_n^{a,b}}$ for $n \in \mathbb{N}_0$. Then the measure \mathbb{P} is a bubble measure for the discrete-time martingale $S^{a,b} = (S_n^{a,b})_{n \in \mathbb{N}_0}$.*

Proposition 5.2 *Let $X = (X_t)_{t \geq 0}$ be a continuous positive strict local Markov martingale under the measure \mathbb{P}_x . Let $\alpha, \beta > 0$, define the sequence of stopping times $(\tau_n^{\alpha,\beta})_{n \in \mathbb{N}}$ by (5.2) and set $S_n^{\alpha,\beta} = X_{\tau_n^{\alpha,\beta}}$ for $n \in \mathbb{N}_0$. Then the measure \mathbb{P}_x is a bubble measure for the discrete-time Markov martingale $S^{\alpha,\beta} = (S_n^{\alpha,\beta})_{n \in \mathbb{N}_0}$.*

We only establish the proof of Proposition 5.1. The proof of Proposition 5.2 is similar and left to the reader.

Proof of Proposition 5.1 Since X is a local martingale with respect to its natural filtration and $(\tau_n^{a,b})_{n \in \mathbb{N}}$ is adapted to this filtration, we may assume without loss of generality that X is the canonical process on $C([0, \infty); (0, \infty])$ with $X_0 = x > 0$. Set $\tau_\infty := \inf\{t \geq 0 : X_t = \infty\}$. Then $\tau_n^{a,b} < \tau_\infty$ for all $n \in \mathbb{N}$.

By the above results, there exists a measure \mathbb{Q} on $C([0, \infty); (0, \infty])$ which has $\mathbb{Q}|_{\mathcal{F}_t} \gg \mathbb{P}|_{\mathcal{F}_t}$ for all $t \geq 0$ and such that $Y := 1/X$ is a nonnegative \mathbb{Q} -martingale and $\mathbb{E}^{\mathbb{P}}[X_\tau \mathbf{1}_A] = x\mathbb{Q}[A]$ for each bounded stopping time $\tau < \tau_\infty$ and each $A \in \mathcal{F}_\tau$. Let $k := \min\{n \in \mathbb{N} : \mathbb{E}[X_0 - X_{a_n}] > 0\}$. Then for $n \geq k$, using that $\tau_n^{a,b}$ is bounded by a_n and $\tau_n^{a,b} < \tau_\infty$, we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[S_n^{a,b} \mathbf{1}_{\{S_k^{a,b} \leq S_{k+1}^{a,b} \leq \dots \leq S_n^{a,b}\}}] &= \mathbb{E}^{\mathbb{P}}[X_{\tau_n^{a,b}} \mathbf{1}_{\{X_{\tau_k^{a,b}} \leq X_{\tau_{k+1}^{a,b}} \leq \dots \leq X_{\tau_n^{a,b}}\}}] \\ &= x\mathbb{Q}[Y_{\tau_n^{a,b}} \leq Y_{\tau_{n-1}^{a,b}} \leq \dots \leq Y_{\tau_k^{a,b}}] \\ &\geq x\mathbb{Q}\left[Y_{a_n} < \frac{1}{b_n}, \dots, Y_{a_{k+1}} < \frac{1}{b_{k+1}}\right] \\ &\geq x\mathbb{Q}[Y_{a_{k+1}} = 0] \\ &= \mathbb{E}^{\mathbb{P}}[X_0 - X_{a_{k+1}}] > 0. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ on the left-hand side, it follows that \mathbb{P} is a bubble measure for S by Theorem 2.4. □

We proceed to illustrate Proposition 5.2 by an example.

Example 5.3 Let $(X_t)_{t \geq 0}$ be the three-dimensional inverse Bessel process, i.e., X is the unique strong solution to the SDE $dX_t = -X_t^2 dB_t$, where $B = (B_t)_{t \geq 0}$ is a \mathbb{P}_x -Brownian motion. The process $Y := 1/X$ is then a $\mathbb{Q}_{1/x}$ -Brownian motion stopped when it reaches zero.

Fix $\alpha, \beta > 0$. We proceed to calculate the Markov kernel $K(x, dy)$ for $S^{\alpha, \beta}$ under \mathbb{P}_x . Denote by W a standard Brownian motion starting at zero and by Φ the distribution function of a standard normal random variable. Using the reflection principle for Brownian motion and denoting the running supremum and the running infimum of a process Z by \overline{Z} and \underline{Z} , respectively, we obtain

$$\begin{aligned} \mathbb{P}_x[S_1^{\alpha, \beta} = (1 + \beta)x] &= \mathbb{P}_x[X_{\tau_1^{\alpha, \beta}} = (1 + \beta)x] \\ &= x\mathbb{E}_{1/x}^{\mathbb{Q}}[Y_{\tau_1^{\alpha, \beta}} \mathbf{1}_{\{Y_{\tau_1^{\alpha, \beta}} = \frac{1}{(1+\beta)x}\}}] = \frac{1}{1 + \beta} \mathbb{Q}_{1/x}\left[Y_\alpha \leq \frac{1}{(1 + \beta)x}\right] \\ &= \frac{1}{1 + \beta} \mathbb{Q}\left[\overline{W}_\alpha \geq \frac{1}{x} - \frac{1}{(1 + \beta)x}\right] = \frac{2}{1 + \beta} \Phi\left(-\frac{\beta}{(1 + \beta)x\sqrt{\alpha}}\right). \end{aligned}$$

Moreover, for $z \in (0, (1 + \beta)x)$, using the joint density of Brownian motion and its running supremum, we obtain

$$\begin{aligned}
 \mathbb{P}_x[S_1^{\alpha,\beta} \leq z] &= \mathbb{P}_x[X_{\tau_1^{\alpha,\beta}} \leq z] = \mathbb{P}_x[X_\alpha \leq z, \bar{X}_\alpha < (1 + \beta)x] \\
 &= x \mathbb{E}^{\mathbb{Q}}_{1/x} \left[Y_\alpha \mathbf{1}_{\{Y_\alpha \geq \frac{1}{z}\}} \mathbf{1}_{\{Y_\alpha > \frac{1}{(1+\beta)x}\}} \right] \\
 &= x \mathbb{E}^{\mathbb{Q}} \left[\left(\frac{1}{x} - W_\alpha \right) \mathbf{1}_{\{W_\alpha \leq \frac{1}{x} - \frac{1}{z}\}} \mathbf{1}_{\{\bar{W}_\alpha < \frac{1}{x} - \frac{1}{(1+\beta)x}\}} \right] \\
 &= x \int_{-\infty}^{\frac{1}{x} - \frac{1}{z}} \left(\frac{1}{x} - y \right) \int_0^{\frac{\beta}{(1+\beta)x}} \frac{2(2u - y)}{\sqrt{2\pi\alpha^3}} \exp\left(-\frac{(2u - y)^2}{2\alpha}\right) du dy \\
 &= x \int_{-\infty}^{\frac{1}{x} - \frac{1}{z}} \left(\frac{1}{x} - y \right) \int_{\frac{-y}{\sqrt{\alpha}}}^{\frac{\frac{2\beta}{(1+\beta)x} - y}{\sqrt{\alpha}}} \frac{v}{\sqrt{2\pi\alpha}} \exp\left(-\frac{v^2}{2}\right) dv dy \\
 &= x \int_{-\infty}^{\frac{1}{x} - \frac{1}{z}} \frac{1}{\sqrt{2\pi\alpha}} \left(\frac{1}{x} - y \right) \left(\exp\left(-\frac{y^2}{2\alpha}\right) - \exp\left(-\frac{(\frac{2\beta}{(1+\beta)x} - y)^2}{2\alpha}\right) \right) dy \\
 &= x \int_0^z \frac{1}{\sqrt{2\pi\alpha}w^3} \left(\exp\left(-\frac{(\frac{1}{x} - \frac{1}{w})^2}{2\alpha}\right) - \exp\left(-\frac{(\frac{2\beta}{(1+\beta)x} - \frac{1}{x} + \frac{1}{w})^2}{2\alpha}\right) \right) dw.
 \end{aligned}$$

Thus the Markov kernel $K(x, dy)$ for $S^{\alpha,\beta}$ under \mathbb{P}_x is given by

$$\begin{aligned}
 K(x, dy) &= \frac{2}{1 + \beta} \Phi\left(-\frac{\beta}{(1 + \beta)x\sqrt{\alpha}}\right) \delta_{(1+\beta)x}(dy) \\
 &\quad + \frac{x}{\sqrt{2\pi\alpha}y^3} \left(e^{-\frac{(\frac{1}{y} - \frac{1}{x})^2}{2\alpha}} - e^{-\frac{(\frac{1}{y} + \frac{\beta - 1}{(1+\beta)x})^2}{2\alpha}} \right) \mathbf{1}_{(0, (1+\beta)x)}(y) dy.
 \end{aligned}$$

In this way, we have constructed a somewhat natural Markov bubble in discrete time.

We finish this section by providing a converse to Proposition 5.1.

Theorem 5.4 *Let $X = (X_t)_{t \geq 0}$ be a continuous local \mathbb{P} -martingale that \mathbb{P} -a.s. never becomes constant, i.e., $\mathbb{P}[X_t = X_\infty \text{ for all } t \geq s] = 0$ for all $s \geq 0$. Then X is a strict local \mathbb{P} -martingale if and only if for all sequences $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ converging to infinity, the measure \mathbb{P} is a bubble measure for the discrete-time martingale $S^{a,b} = (S_n^{a,b})_{n \in \mathbb{N}_0}$.*

Proof If X is a strict local martingale, the result follows from Proposition 5.1. Conversely, suppose X is a true \mathbb{P} -martingale. As in the proof of Proposition 5.1, we assume without loss of generality that X is the canonical process on $C([0, \infty); (0, \infty))$ with $X_0 = 1$. Then there exists a measure \mathbb{Q} on $C([0, \infty); (0, \infty))$ with $\mathbb{Q}|_{\mathcal{F}_t} \gg \mathbb{P}|_{\mathcal{F}_t}$ for all $t \geq 0$ and such that $Y := 1/X$ is a positive \mathbb{Q} -martingale that converges to 0 \mathbb{Q} -almost surely and $\mathbb{E}^{\mathbb{P}}[X_\tau \mathbf{1}_A] = \mathbb{Q}[A]$ for each bounded stopping time $\tau < \tau_\infty$ and each $A \in \mathcal{F}_\tau$. By Proposition A.1, there exists an increasing sequence $a = (a_n)_{n \in \mathbb{N}}$ converging to infinity such that for each $k \in \mathbb{N}$,

$$\mathbb{Q}[Y_{a_k} \geq Y_{a_{k+1}} \geq \dots] = 0. \tag{5.9}$$

Since Y is positive \mathbb{Q} -a.s., we can find an increasing sequence $b = (b_n)_{n \in \mathbb{N}}$ converging to infinity such that $\mathbb{Q}[Y_{a_n} \leq 1/b_n] < 2^{-n}$ for all $n \in \mathbb{N}$. By the Borel–Cantelli lemma, this implies that

$$\mathbb{Q} \left[Y_{a_k} \leq \frac{1}{b_k} \text{ for infinitely many } k \right] = 0. \tag{5.4}$$

Then for any $k \in \mathbb{N}$, using (5.3) and (5.4) and recalling that each $\tau_n^{a,b}$ is bounded by a_n , we obtain

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}} \left[S_n^{a,b} \mathbf{1}_{\{S_k^{a,b} \leq S_{k+1}^{a,b} \leq \dots \leq S_n^{a,b}\}} \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{Q} \left[Y_{\tau_n^{a,b}} \leq Y_{\tau_{n-1}^{a,b}} \leq \dots \leq Y_{\tau_k^{a,b}} \right] \\ &= \mathbb{Q} \left[Y_{\tau_k^{a,b}} \geq Y_{\tau_{k+1}^{a,b}} \geq \dots \right] \\ &\leq \mathbb{Q} \left[Y_{a_\ell} \leq \frac{1}{b_\ell} \text{ for infinitely many } \ell \right] + \mathbb{Q} [Y_{a_\ell} \leq Y_{a_{\ell-1}} \text{ eventually}] = 0. \end{aligned}$$

By Theorem 2.4, this shows that \mathbb{P} is not a bubble measure for $S^{a,b}$. □

Appendix: Auxiliary results

Proposition A.1 *Let $(M_t)_{t \geq 0}$ be a positive continuous local martingale that \mathbb{P} -a.s. never becomes constant, i.e., $\mathbb{P}[M_t = M_\infty \text{ for all } t \geq s] = 0$ for all $s \geq 0$. Then there exists an increasing sequence of nonnegative real numbers $(a_k)_{k \in \mathbb{N}}$ satisfying $\lim_{k \rightarrow \infty} a_k = \infty$ such that for each $k \in \mathbb{N}$,*

$$P[M_{a_k} \geq M_{a_{k+1}} \geq \dots] = 0.$$

Proof For $n, m \in \mathbb{N}_0$, set $D_m^n := \{n + j2^{-m} : j \in \mathbb{N}_0\}$. Then for each fixed $n \in \mathbb{N}_0$, D_m^n is increasing in m . Set $D_\infty^n := \bigcup_{m \in \mathbb{N}_0} D_m^n$, which is dense in $[n, \infty)$. Since M is a continuous local martingale that \mathbb{P} -a.s. never becomes constant, its paths \mathbb{P} -a.s. never become monotone. By continuity of the paths of M , this implies that for each $n \in \mathbb{N}_0$,

$$\mathbb{P}[M_s \geq M_t \text{ for all } s, t \in D_\infty^n \text{ with } s < t] = 0.$$

By σ -continuity of \mathbb{P} , for each $n \in \mathbb{N}_0$, there exists $m_n \in \mathbb{N}_0$ such that

$$\mathbb{P}[M_s \geq M_t \text{ for all } s, t \in D_{m_n}^n \text{ with } s < t] \leq 2^{-n}.$$

We may assume without loss of generality that the sequence $(m_n)_{n \in \mathbb{N}}$ is nondecreasing. Define the set D by

$$D := \bigcup_{n \in \mathbb{N}_0} (D_{m_n}^n \cap [n, n + 1)).$$

Then $D \cap [n, \infty) \supseteq D_{m_n}^{n'}$ for all $n' \geq n$. This implies that for each $k \in \mathbb{N}_0$,

$$\mathbb{P}[M_s \geq M_t \text{ for all } t, s \in D \cap [n, \infty) \text{ with } s < t] \leq \lim_{n' \rightarrow \infty} 2^{-n'} = 0.$$

If $(a_k)_{k \in \mathbb{N}}$ is an enumeration of D in increasing order, the result follows. □

Remark A.2 Note that in the situation of Proposition A.1, there also exist in general sequences $(a_k)_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} a_k = \infty$ such that

$$P[M_{a_1} \geq M_{a_2} \geq \dots] > 0.$$

For example, let $M_t = \exp(W_t - t/2)$, where W is a Brownian motion. Define the sequence $(a_k)_{k \in \mathbb{N}}$ by $a_1 = 1$ and $a_k = a_{k-1} + k$. Then using that Brownian motion has independent and normally distributed increments, denoting the distribution function of a standard normal random variable by Φ , we obtain

$$\mathbb{P}[M_{a_1} \geq M_{a_2} \geq \dots] = \prod_{k=1}^{\infty} \Phi\left(\frac{\sqrt{k}}{2}\right) \geq \prod_{k=1}^{\infty} \left(1 - \frac{1}{\sqrt{2\pi} \frac{\sqrt{k}}{2}} \exp\left(-\frac{1}{8}k\right)\right) > 0.$$

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