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SYMPLECTIC BLOWING DOWN IN DIMENSION SIX

TIAN-JUN LI & YONGBIN RUAN & WEIYI ZHANG

ABSTRACT. We establish a blowing down criterion in the context of birational symplectic geometry in dimension 6.

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1. INTRODUCTION

Symplectic blowing down is a fundamental operation in birational symplectic geometry ([6], [8]). Two symplectic manifolds are birationally cobordant equivalent if one can be obtained from the other via a sequence of symplectic blow ups, symplectic blow downs and integral deformations.

Symplectic blowing up can always be performed along any symplectic submanifold to get a new symplectic manifold with an exceptional divisor ([5], [6], [21]). On the other hand, except in dimension 4 ([18]), it is poorly understood when blowing down can be performed. In this note we will attempt to find blowing down criteria in dimension 6. Blowing up in dimension 6 gives rise to either a symplectic \mathbb{P}^2 with normal degree -1 or a symplectic \mathbb{P}^1 -bundle with normal degree -1

along the \mathbb{P}^1 -fibers. The geometry of these symplectic 4-manifolds are well understood ([18], [11], [23], [16]). Our focus is whether such a symplectic divisor always arises from a symplectic blowing up.

The case of \mathbb{P}^2 is simpler. We observe that the uniqueness of the symplectic structures by Gromov and Taubes, together with the Weinstein neighborhood theorem, implies that a symplectic \mathbb{P}^2 divisor with normal degree -1 in a symplectic 6-manifold can always be blown down just as in the case of \mathbb{P}^1 with self-intersection -1 in a symplectic 4-manifold.

The case of a \mathbb{P}^1 -bundle over a Riemann surface Σ is subtler. Topologically, blowing down can always be performed since it is the same as topologically fiber summing with the triple of a linear \mathbb{P}^2 -bundle with a \mathbb{P}^1 -subbundle over Σ with opposite normal bundle and a complementary section. However, unlike the \mathbb{P}^2 case, it is not clear that blowing down can always be performed symplectically in this case. Symplectic blowing up generally involves a small neighborhood of Σ which means that the resulting \mathbb{P}^1 -bundle divisor has a small fiber area.

We investigate this problem in the context of birational symplectic geometry in dimension 6 (which has been studied in [25], [26], [9], [12], [14], [28], [31] etc). Specifically, we study whether we can symplectically blow down up to an integral deformation to obtain a simpler birationally equivalent symplectic manifold. A nice feature is that cohomologous symplectic forms on such 4-manifolds are also isotopic in this case. Moreover, as every symplectic structure on such a 4-manifold is Kähler, we can apply algebro-geometric techniques to solve this problem.

In Section 2 we investigate this problem in a general situation. Topologically the blowing up construction gives rise to a fibred codimension 2 submanifold as in the following definition.

Definition 1.1. *Let $D^{2n} \subset (M^{2n+2}, \omega)$ be a codimension 2 symplectic submanifold which admits a linear \mathbb{P}^k -bundle structure $\pi : D^{2n} \rightarrow Y^{2n-2k}$ over an oriented $(2n - 2k)$ -manifold Y . (D, π) is called a topological exceptional divisor if (i) the normal line bundle N_D is the tautological line bundle when restricted to the projective space fibers of π , and (ii) $\omega|_D$ is almost standard.*

Here a linear \mathbb{P}^k -bundle refers to a projective bundle from a complex vector bundle of dimension $k + 1$. So the structure group is the linear group $GL(k + 1, \mathbb{C})$. And the fibers of such a bundle come with a homotopy class of almost complex structures and hence have a natural complex orientation and a line class l in H_2 . We use l to specify the

forward cone

$$\mathcal{C}_\pi(D) = \{u \in H^2(D, \mathbb{R}) \mid u^n > 0, \langle u, l \rangle > 0\}.$$

A fibred symplectic form on a linear projective bundle is called standard if it arises from the Sternberg-Weinstein universal construction as described in Section 2.1. In particular, a standard form restrict to a multiple of the Fubini-Study form on each fiber. A fibred symplectic form is said to be almost standard if it is deformation to a standard form via fibred forms.

We describe the blowing up construction in several ways, via the $U(k)$ universal construction, birational cobordism and symplectic cut. From Definition 2.2 we see that symplectic exceptional divisor from blowing up is a topological exceptional divisor.

The symplectic cut description is useful to study whether a topological exceptional divisor can be blown down up to deformation. Specifically it introduces an auxiliary S^1 -equivariant linear symplectic projective space bundle triple (see Lemma 2.8), which lies behind the following definition.

Definition 1.2. *Given a topological exceptional divisor $\pi : D^{2n} \rightarrow Y^{2n-2k}$ of (M^{2n+2}, ω) , we define a matching triple $(X, D', S; \Omega)$ to be a linear $(\mathbb{P}^{k+1}, \mathbb{P}^k, \mathbb{P}^0)$ bundle triple (X, D', S) over Y with Ω a symplectic form on X , satisfying the following conditions.*

- (1) D' is diffeomorphic to D ;
- (2) $e(N_{D'}) = -e(N_D)$;
- (3) $\Omega|_{D'}$ is almost standard and matches with $\omega|_D$;
- (4) Ω is S^1 -invariant with respect to a semi-free S^1 -action with D' and S as the only fixed point sets.

The triple is called weak if we assume the weaker condition

$$(3') \quad \Omega|_{D'} \text{ is almost standard and } [\Omega|_{D'}] = [\omega|_D].$$

Here is the main result in Section 2.

Proposition 1.3. *Suppose (M, ω) has a topological exceptional divisor D . Up to integral deformation, we can symplectically blow down (M, ω) along D if there is a matching triple $(X, D', S; \Omega)$.*

Thus we are led to search for such projective space bundle triples which match with (M, D, ω) . For blowing up a surface in arbitrary dimension, the auxiliary linear symplectic projective space bundle triple also leads to a ratio constraint.

Theorem 1.4. *Suppose a symplectic divisor $\pi : D^{2n} \rightarrow \Sigma$ of (M^{2n+2}, ω) arises from symplectically blowing up a symplectic surface Σ in a symplectic manifold. Then*

(i) $c_1(N_D)$ satisfies

$$\int_D (-1)^n c_1(N_D)^n = -\deg(N_\Sigma),$$

(ii) the symplectic form $\omega|_D$ satisfies the ratio bound

$$(1) \quad \rho_\pi([\omega|_D]) > \begin{cases} -\deg(N_\Sigma), & \text{if } g(\Sigma) > 0, \\ \max\{-\deg(N_\Sigma), t_n(-\deg(N_\Sigma))\}, & \text{if } g(\Sigma) = 0. \end{cases}$$

What is essentially new is the ratio inequality (1). We now explain the terms in this theorem. Here t_n is the function

$$t_n : \mathbb{Z} \rightarrow \{0, 1, \dots, n-1\}$$

by requiring $t_n(w) = w \pmod{n}$. The degree function $\deg(E)$ of a symplectic vector bundle E over Σ is $\deg(E) = \int_\Sigma c_1(E)$. The ratio function ρ_π on the forward cone $\mathcal{C}_\pi(D)$ for a linear projective space bundle $\pi : D^{2n} \rightarrow \Sigma$ is defined by

$$(2) \quad \rho_\pi(u) = \frac{\int_D u^n}{\langle u, l \rangle^n} \in \mathbb{R}^+.$$

Note that the ratio $\rho_\pi(u) \in \mathbb{R}^+$ determines the ray of the class u since $\rho_\pi(u)$ is scale invariant and the rank of $H^2(D; \mathbb{R})$ is 2 in this case.

We introduce the following definition.

Definition 1.5. A topological exceptional divisor $\pi : D \rightarrow \Sigma$ of (M, ω) is called admissible if the symplectic form $\omega|_D$ on D satisfies the ratio bound

$$(3) \quad \rho_\pi([\omega|_D]) > \begin{cases} \alpha_{D,M}, & \text{if } g(\Sigma) > 0, \\ \max\{\alpha_{D,M}, t_n(\alpha_{D,M})\}, & \text{if } g(\Sigma) = 0. \end{cases}$$

where $\alpha_{D,M} = \int_D (-1)^n c_1(N_D)^n$.

A symplectic exceptional divisor from blowing up a surface is an admissible topological exceptional divisor by Theorem 1.4.

Theorem 1.6. Suppose $\pi : D \rightarrow \Sigma$ is an admissible topological exceptional divisor of (M, ω) . Then there exists a weak matching S^1 -equivariant triple.

To prove this result we need to understand the symplectic cone of a linear $(\mathbb{P}^{k+1}, \mathbb{P}^k, \mathbb{P}^0)$ bundle triple over Σ . This is possible in the Kähler setting by the Kleiman's criterion. We are able to determine the restricted Kähler cone for various complex structures coming from holomorphic bundles of the form $\mathcal{V} \oplus \mathcal{O}$ over Σ with \mathcal{V} either semi-stable or decomposable. Another nice feature in the Kähler setting is that we always have the required S^1 -symmetry.

Along the way we determine the symplectic cone of an arbitrary linear projective bundle over a surface.

Proposition 1.7. *For a linear \mathbb{P}^n bundle over a surface,*

- (i) *every almost standard symplectic form is cohomologous to a Kähler form,*
- (ii) *the symplectic cone is equal to the positive cone when n is odd and the surface is of positive genus.*

Remark 1.8. *This proposition generalizes the result of [19] in the case of \mathbb{P}^1 -bundles. Cascini-Panov [3] noted that the generic Kähler cone is smaller than the symplectic cone for one point blowup when the genus is 1.*

Finally, we state our blowing-down criterion in dimension 6. In dimension 6 a symplectic exceptional divisor arising from blowing up is a topological exceptional divisor $\pi : D \rightarrow Y$ as in Definition 1.1, where Y is either a point or a surface Σ .

When Y is a point, D is a symplectic \mathbb{P}^2 embedded in (M, ω) whose normal bundle N_D has degree -1 . As already mentioned, since cohomologous symplectic forms on \mathbb{P}^2 are diffeomorphic ([30]), $\omega|_D$ is standard and a neighborhood of (D, ω) is the same as the standard symplectic ball near the boundary. So it can be symplectically blown down. When $Y = \Sigma$, D is a ruled surface. Since cohomologous symplectic forms are also diffeomorphic for ruled surfaces ([16]), we have by Proposition 1.3 and Theorem 1.6,

Proposition 1.9. *Let (M, ω) be a 6-dimensional symplectic manifold and $\pi : D^4 \rightarrow Y$ a topological exceptional divisor of (M, ω) . When Y is a point, (M, ω) can be blown down along D^4 to (M', ω') .*

When $Y = \Sigma$ and $\pi : D^4 \rightarrow \Sigma$ is admissible then a weak matching triple is a matching triple and hence, up to integral deformation, (M, ω) can be blown down along D^4 to (M', ω') .

It would be interesting to investigate, when $Y = \Sigma$, whether integral deformation is not really needed. However, note that the integral deformation is a birational equivalence so the symplectic manifolds (M, ω) and (M', ω') are birational. So this blowing-down criterion in dimension 6 is suitable in the context of symplectic birational geometry.

In the case of $Y = \Sigma$, since any \mathbb{P}^1 -bundle over Σ is linear and any symplectic form is fibred (Section 6.2 in [21]), we in fact have the following more explicit and stronger formulation.

Theorem 1.10. *Let (M, ω) be a 6-dimensional symplectic manifold and D a codimension 2 symplectic submanifold. Suppose D admits a*

\mathbb{P}^1 -bundle structure $\pi : D \rightarrow \Sigma$ over a surface Σ with $\langle c_1(N_D), l \rangle = -1$. Let $\alpha_{D,M} = c_1(N_D) \cdot c_1(N_D)$ and $\rho = \rho_\pi([\omega|_D])$.

Suppose $(D, \omega|_D)$ arises from blowing up a surface. Then $\rho \neq 2$ if $D = S^2 \times S^2$ with $\alpha_{D,M} = 2$, and $\rho > \alpha_{D,M}$ otherwise.

Conversely, if $\rho > \alpha_{D,M}$, (M, ω) can be blown down along D up to deformation. In particular, this is the case if $\alpha_{D,M} \leq 0$. Moreover, when $D = S^2 \times S^2$ with $\alpha_{D,M} = 2$, (M, ω) can be blown down along D up to deformation as long as $\rho \neq 2$.

When $D = S^2 \times S^2$ and $\alpha_{D,M} = 2$, there are two rulings and the normal bundle N_D has degree -1 along each ruling. The condition $\rho \neq 2$, just means the symplectic areas of the two rulings are not the same. In this case, up to deformation, (M, ω) can be blown down along the ruling with smaller area. This picture is consistent with the flop operation for projective 3-folds. Let $D = S^2 \times S^2$ be a divisor with normal $c_1 = (-1, -1)$. If the fibers of the two rulings $S^2 \times \{pt\}$ and $\{pt\} \times S^2$ have the same symplectic area and are not cohomologous, we can perturb the symplectic form such that the areas of two S^2 factors are different. A generic perturbation would work as the symplectic cone of the ambient manifold is open and we can perturb along any direction. Hence, in this situation, the divisor D could also be blown down up to deformation. Simplest such example might be the toric blowup of the projective cone over $S^2 \times S^2$ at the conic point.

In Section 2, we review the blow up process, establish the criterion Proposition 1.3. In Section 3 we study the curve cone and the Kähler cone of holomorphic projective bundles over a Riemann surface, and also prove Proposition 1.7. In Section 4 we prove bound (1), Theorem 1.6 and Theorem 1.10.

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2. BLOWING UP/DOWN, SYMPLECTIC CUT AND FIBER SUM

2.1. Blowing up via the $U(k)$ universal construction. We first recall the Sternberg-Weinstein universal construction to obtain a canonical symplectic structures on the normal bundle of a symplectic submanifold.

2.1.1. The universal construction. Let us first review the Sternberg-Weinstein universal construction. Let $\pi : P \rightarrow X$ be a principal bundle

with structure group G over a symplectic manifold (X, ω) . If \mathfrak{g} denote the Lie algebra of G and \mathfrak{g}^* denotes the dual, the Sternberg-Weinstein universal construction produces a G -invariant symplectic form on a neighborhood of $P \times \{0\}$ in $P \times \mathfrak{g}^*$.

The construction starts with a connection on P , which is a G -invariant \mathfrak{g} -valued 1-form A on P corresponding to a G -invariant projection onto the vertical tangent bundle VP . Equivalently, it is given by a G -invariant complementary subbundle and, dually, it induces an embedding of $P \times \mathfrak{g}^*$ into T^*P . Consider the 1-form on $P \times \mathfrak{g}^*$ given by $\gamma \cdot A$ at $(p, \gamma) \in P \times \mathfrak{g}^*$, where we use \cdot to denote the pairing between \mathfrak{g} and \mathfrak{g}^* . Denote this 1-form by $\gamma \cdot A$ as well. Notice that $d(\gamma \cdot A)$ is the restriction of the canonical 2-form on T^*P . Therefore $d(\gamma \cdot A)$ is non-degenerate on the fibers of $P \times \mathfrak{g}^*$. The 2-form

$$\omega_A = \pi^*\omega + d(\gamma \cdot A)$$

is called the coupling form of A . The G -action on $P \times \mathfrak{g}^*$ given by

$$g(p, \zeta) = (g^{-1}p, \text{Ad}(g)^*\zeta),$$

preserves $\gamma \cdot A$ and hence ω_A . Notice that at any point in $P \times \{0\}$, $\gamma = 0$ and ω_A is equal to $\pi^*\omega + d\gamma \wedge A$, hence it is symplectic there.

Lemma 2.1. *The form ω_A is a symplectic form on $P \times \mathcal{W}_A$ for some G -invariant neighborhood \mathcal{W}_A of $0 \in \mathfrak{g}^*$. The projection onto \mathcal{W}_A is a moment map on $P \times \mathcal{W}_A$.*

This lemma is well-known. Notice that the vertical bundle VP is the bundle of the null vectors of $\pi^*\omega$. So as explained in [6], the construction is just a special case of the coisotropic embedding theorem. It follows from the uniqueness part of the coisotropic embedding theorem that the symplectic structure ω_A on $P \times \mathcal{W}_A$ near $P \times 0$ is independent of A up to symplectomorphisms.

More generally, if (F, ω_F) is a symplectic manifold with a Hamiltonian G action, we can form the associated bundle $P_F = P \times_G F$. Let $\mu_F : F \rightarrow \mathfrak{g}^*$ be a moment map. Furthermore, assume that

$$(4) \quad \mu_F(F) \subset \mathcal{W}_A.$$

Then there is a symplectic structure $\omega_{F,A}$ on P_F which restricts to ω_F on each fiber.

To construct $\omega_{F,A}$ consider the 2-form $\omega_A + \omega_F$ on $P \times \mathfrak{g}^* \times F$. It is invariant under the diagonal G -action and is symplectic on $P \times \mathcal{W}_A \times F$. The G -action is Hamiltonian with

$$\Gamma_{\mathcal{W}_A} = \pi_{\mathfrak{g}^*} + \mu_F : P \times \mathcal{W}_A \times F \rightarrow \mathfrak{g}^*$$

as a moment map. Furthermore, by (4), for any $f \in F$, we have $\mu_F(f) \in \mathcal{W}_A$, thus

$$\Gamma_{\mathcal{W}_A}^{-1}(0) = \{(p, -\mu_F(f), f)\}.$$

In particular, $\Gamma_{\mathcal{W}_A}^{-1}(0)$ is G -equivariantly diffeomorphic to $P \times F$, and the symplectic reduction at 0 yields the desired symplectic form $\omega_{F,A}$ on P_F .

In fact when $(F, \omega_F) = (TG^*, \omega_{can})$, then $P \times \mathfrak{g}^* = P \times_G TG^*$.

2.1.2. The local geometry of a submanifold. We apply this construction to obtain canonical symplectic structures on small normal disk bundles of a symplectic submanifold. Suppose that X is a closed symplectic manifold of dimension $2n$ and $Y \subset X$ is a symplectic submanifold of codimension $2k$. The normal bundle N_Y is a symplectic vector bundle, *i.e.* a bundle with fiber (\mathbb{C}^k, τ) . Here τ denotes the standard symplectic form on \mathbb{C}^k . Note that τ is $U(k)$ invariant. Picking a compatible almost complex structure on N_Y , we then have an Hermitian bundle. Let P be the principal $U(k)$ bundle over Y .

Now pick a unitary connection A for P , and let $\mathcal{W}_A \subset u(k)^*$ be as in Lemma 2.1. Let $D_{\epsilon_0} \subset \mathbb{C}^k$ be the closed ϵ_0 -ball such that its image under the moment map lies inside \mathcal{W}_A . Apply the universal construction to P and $D_{\epsilon_0} \subset \mathbb{C}^k$, we get a $U(k)$ -invariant symplectic form $\omega_{\epsilon_0, A}$ on the disc bundle $N_Y(\epsilon_0)$ which restricts to τ on each fiber and restricts to $\omega|_Y$ on the zero section.

By the symplectic neighborhood theorem, and by possibly taking a smaller ϵ_0 , a tubular neighborhood $\mathcal{N}_{\epsilon_0}(Y)$ of Y in X is symplectomorphic to the disc bundle $N_Y(\epsilon_0)$ with the symplectic form $\omega_{\epsilon_0, A}$. Let $\phi : (\mathcal{N}_{\epsilon_0}(Y), \omega) \rightarrow (N_Y(\epsilon_0), \omega_{\epsilon_0, A})$ be such a symplectomorphism.

2.1.3. Blowing up via the universal construction. Let $i : BL \subset \mathbb{P}^{k-1} \times \mathbb{C}^k$ be the incidence relation. Then the projection $\alpha : BL \rightarrow \mathbb{P}^{k-1}$ makes BL into the (holomorphic) tautological line bundle. The map $\beta : BL \rightarrow \mathbb{C}^k$ sending each fiber of α into the corresponding one dimensional subspace of \mathbb{C}^k is a (holomorphic) bijection of the complement of the zero section BL_0 of BL with the complement of the origin in \mathbb{C}^k . In other words, BL is the complex blowup of \mathbb{C}^k at the origin.

Let Ω be the standard $U(k)$ invariant symplectic form on \mathbb{P}^{k-1} and recall that τ is the standard symplectic form on \mathbb{C}^k . By Theorem 5.1 in [6], for $\epsilon > 0$, the form $\omega_\epsilon = i^*(\epsilon \text{pr}_1^* \Omega + \text{pr}_2^* \tau)$ defines a $U(k)$ -invariant symplectic structure on BL . (BL, ω_ϵ) is the ϵ -symplectic blowup of (\mathbb{C}^k, τ) .

Given $\epsilon_0 > 0$, let BL_{ϵ_0} be the inverse image of the ϵ_0 disk in \mathbb{C}^k under α . Note that $(BL_{\epsilon_0}, \omega_\epsilon)$ is a $U(k)$ symplectic manifold. Pick $\epsilon < \epsilon_0$ and

apply the universal construction to the principal $U(k)$ bundle P over Y and $(F, \omega_F) = (BL_{\epsilon_0}, \omega_\epsilon)$, the resulting symplectic manifold $\tilde{N}_Y^\epsilon(\epsilon_0)$ is the local ϵ -symplectic blowup along Y . Note that the zero section BL_0 of $\alpha : BL \rightarrow \mathbb{P}^{k-1}$ is $U(k)$ -invariant and so defines a codimension 2 submanifold $\tilde{N}_Y^\epsilon(0) \subset \tilde{N}_Y^\epsilon(\epsilon_0)$. By Theorem 5.4 in [6], outside the zero section BL_0 of BL , the symplectic form ω_ϵ is equivalent to the standard symplectic form τ on $\mathbb{C}^k - D_\epsilon$. In particular, $\tilde{N}_Y^\epsilon(\epsilon_0) \setminus \tilde{N}_Y^\epsilon(0)$ and $N_Y(\epsilon_0) \setminus Y$ are symplectomorphic.

Therefore we can extend the local ϵ -symplectic blowup via $\phi : (\mathcal{N}_{\epsilon_0}(Y), \omega) \rightarrow (N_Y(\epsilon_0), \omega_{\epsilon_0, A})$ to obtain the (global) ϵ -symplectic blowup of X along Y .

Definition 2.2. *Given ϵ, A, ϕ , we call*

$$\tilde{X}_{\epsilon, A, \phi} = (X - \mathcal{N}_\epsilon(Y)) \cup_\phi \tilde{N}_Y^\epsilon(\epsilon_0)$$

the (ϵ, A, ϕ) -blowup of X along Y .

The codimension 2 divisor $\tilde{N}_Y^\epsilon(0)$ is called the (symplectic) exceptional divisor. The exceptional divisor is a linear projective space bundle and the symplectic form on it is called a standard form.

Remark 2.3. *Notice that the construction depends on ϵ , the connection A and the symplectomorphism $\phi : \mathcal{N}_{\epsilon_0}(Y) \rightarrow N_Y(\epsilon_0)$. However, as remarked on p. 250 in [21], given two different choices A, ϕ and A', ϕ' , for sufficiently small ϵ , the resulting symplectic forms are isotopic. We will often simply denote the blowup by \tilde{X} ignoring various choices.*

Observe that we can define a map

$$(5) \quad p : \tilde{X} \rightarrow X$$

which is identity away from $\mathcal{N}_{\epsilon_0}(Y)$. Such a map can be constructed by identifying $\mathcal{N}_{\epsilon_0}(Y) - \mathcal{N}_\epsilon(Y)$ with the deleted neighborhood $\mathcal{N}_{\epsilon_0}(Y) - Y$ using a diffeomorphism from (ϵ, ϵ_0) to $(0, \epsilon_0)$. Such a p is not unique, but the induced maps p_* and p^* on homology and cohomology are the same for different choices.

2.2. Up to deformation via a simple birational cobordism. Note that the symplectic blowing up construction is a local $U(k)$ equivariant construction. It is shown in [6] that there is an S^1 equivariant description.

We first recall the notion of symplectic birational cobordism. This notion is based on symplectic cobordism introduced in [6] (also see [8]).

Definition 2.4. *Two symplectic manifolds (X, ω) and (X', ω') are birational cobordant if and only if there are finite number of symplectic manifolds (X_i, ω_i) , $0 \leq i \leq k$, with $(X_0, \omega_0) = (X, \omega)$ and*

$(X_{k+1}, \omega_{k+1}) = (X', \omega')$, and for each i , (X_i, ω_i) and (X_{i+1}, ω_{i+1}) are symplectomorphic to symplectic reductions of a semi-free Hamiltonian S^1 symplectic manifold W_i (of two dimensions higher) at different regular levels or simple critical levels.

Given a semi-free Hamiltonian S^1 manifold with moment map Φ , a critical level of Φ is called simple if the corresponding set W of critical points is connected and the signature of Φ is either $(2p, 2)$ or $(2, 2q)$ for some positive values of p and q . It follows from Theorem 10.2 in [6] that the reduction at a simple critical value is still a smooth symplectic manifold. The change of the symplecto-diffeotype of the reductions when passing through a simple critical value is called a simple symplectic birational cobordism. By Theorem 11.2 in [6], the change of the symplecto-diffeotype of the reductions when passing through a critical value of signature $(2p, 2q)$ is the composition of two simple symplectic birational cobordisms, one of signature $(2p, 2)$ and the other of signature $(2, 2q)$.

In Section 12 of [6], for a symplectic submanifold $Y \subset (X, \omega)$, a semi-free Hamiltonian S^1 manifold X_I with proper moment map $\Phi : X_I \rightarrow I$ is constructed, where 0 is a simple critical value of Φ with reduction (X, ω) and the reduction at small $\epsilon \geq 0$ is the ϵ -symplectic blowup of (X, ω) along Y (including the case $\epsilon = 0$). Further, Theorem 13.1 of [6] establishes the uniqueness of X_I for a sufficiently small interval I about 0. In particular, for small ϵ , symplectic blowing up/down is unique up to symplectomorphism.

Since the change of the symplecto-diffeotype of the reductions when not passing through a critical value is an integral deformation, we have the following conclusion:

Theorem 2.5. [6] *Up to integral deformation, a simple birational cobordism is the same as a symplectic blowing up/down.*

This motivates us to also describe a symplectic blowup up to integral deformation via symplectic cut.

2.3. Up to deformation from symplectic cut. Now we apply the local S^1 equivariant symplectic cut construction by Lerman to construct the blowup along Y . This observation was already mentioned in Remark 1.3 in [10] when (X, ω) is globally Hamiltonian with Y as the maximal submanifold.

We provide some details in the general case to show that symplectic cutting a small neighborhood of Y gives rise to a simple birational cobordism. As pointed out in the introduction, this description introduces a linear $(\mathbb{P}^k, \mathbb{P}^{k-1}, \mathbb{P}^0)$ -bundle triple, which leads to a ratio

constraint. Secondly, it leads to a construction of the symplectic blowing down up to deformation via normal connected sum.

2.3.1. *Symplectic cut.* Suppose that $X_0 \subset X$ is an open codimension zero submanifold with a Hamiltonian S^1 -action. Let $H : X_0 \rightarrow \mathbb{R}$ be a Hamiltonian function with a regular value ϵ . If $H^{-1}(\epsilon)$ is a separating hypersurface of X_0 , then we obtain two connected manifolds X_0^\pm with boundary $\partial X_0^\pm = H^{-1}(\epsilon)$, where the $+$ side corresponds to $H < \epsilon$. Suppose further that S^1 acts freely on $H^{-1}(\epsilon)$. Then the symplectic reduction $Z = H^{-1}(\epsilon)/S^1$ is canonically a symplectic manifold of dimension 2 less. Collapsing the S^1 -action on $\partial X^\pm = H^{-1}(\epsilon)$, we obtain closed smooth manifolds \bar{X}^\pm containing respectively real codimension 2 submanifolds $Z^\pm = Z$ with opposite normal bundles. Furthermore \bar{X}^\pm admits a symplectic structure $\bar{\omega}^\pm$ which agrees with the restriction of ω away from Z , and whose restriction to Z^\pm agrees with the canonical symplectic structure ω_Z on Z from symplectic reduction. The pair of symplectic manifolds $(\bar{X}^\pm, \bar{\omega}^\pm)$ is called the symplectic cut of X along $H^{-1}(\epsilon)$ ([10]).

This is neatly shown by considering the standard symplectic structure $\sqrt{-1}dw \wedge d\bar{w}$ on \mathbb{C} and two Hamiltonian actions of S^1 on $X_0 \times \mathbb{C}$ with the product symplectic structure, where S^1 acts on \mathbb{C} by complex multiplication by z and z^{-1} respectively. The extended Hamiltonian is therefore $H_\pm = H \pm |w|^2$ and $(\bar{X}^\pm, \bar{\omega}^\pm)$ is the reduction at ϵ with respect to H_\pm .

Lemma 2.6. *Suppose the moment map image of the S^1 -action on X_0 is (c, d) , $c < a < b < d$ and there are no critical values in the interval $I = [a, b]$. Let (K^t, ω_t) be the symplectic cut on the $-$ side at $t \in [a, b]$. Then (K^a, ω_a) and (K^b, ω_b) are integral deformation equivalent.*

Proof. proof In the special case that (X, ω) is a global S^1 -manifold, for $\Phi = H - |w|^2$, $\Phi^{-1}(I)/S^1$ is a cobordism with no critical levels. So we have an integral deformation.

In the general local case, $H^{-1}([a-\epsilon, b])$ is equivariantly $P \times [a-\epsilon, b]$ by the equivariant version of the coisotropic embedding theorem, where P is a principal S^1 -bundle. Consider a smooth family of diffeomorphisms $\xi_t : [a-\epsilon, a] \rightarrow [a-\epsilon, t]$ for $t \in [a, b]$, which is identity near $a-\epsilon$.

We use ξ_t to define a family of S^1 -equivariant diffeomorphisms $\Theta_t : X - H^{-1}((a, d)) \rightarrow X - H^{-1}((t, d))$ such that (i) Θ_t is the identity on $X - H^{-1}([a-\epsilon, d])$, and (ii) $\Theta_t : (s, p) \mapsto (\xi_t(s), p)$ on $H^{-1}([a-\epsilon, a])$. Each Θ_t descends to a diffeomorphism from the a -cut to the t -cut, still denoted by Θ_t . The family of symplectic forms $\Theta_t^* \omega_t$ is then the desired deformation. \square

2.3.2. *Symplectic cutting a standard neighborhood of Y .* The ϵ_0 neighborhood $\mathcal{N}_{\epsilon_0}(Y)$ of Y carries a $U(k)$ action from the identification $\phi : \mathcal{N}_{\epsilon_0}(Y) \rightarrow N_Y(\epsilon_0)$. Consider the Hamiltonian S^1 -action on

$$X_0 = \mathcal{N}_{\epsilon_0}(Y)$$

which corresponds to the complex multiplication on $N_Y(\epsilon_0)$. The moment map $H(u) = |\phi(u)|^2$, $u \in \mathcal{N}_{\epsilon_0}(Y)$, where $|\phi(u)|$ is the norm of $\phi(u)$ considered as a vector in a fiber of the Hermitian bundle N_Y . Here $X_0 \times \mathbb{C} = \mathcal{N}_{\epsilon_0}(Y) \times \mathbb{C}$ is identified via ϕ with

$$N_Y(\epsilon_0) \oplus \mathbb{C}.$$

Fix ϵ with $0 < \epsilon < \epsilon_0$ and consider the hypersurface $Q = H^{-1}(\epsilon)$ in X_0 corresponding to the sphere bundle of N_Y with radius ϵ . We cut X along Q to obtain two closed symplectic manifolds $(\overline{X}^+, \omega^+)$ and $(\overline{X}^-, \omega^-)$, each containing a codimension 2 symplectic submanifold $Z^\pm = Q/S^1$.

Z^\pm is diffeomorphic to the projectivization $\mathbb{P}_s(N_Y)$ of N_Y . It is important to remember that

$$(Z^+, \omega^+|_{Z^+}) = (Z^-, \omega^-|_{Z^-}).$$

2.3.3. *The \overline{X}^- side as blowup.* We apply the birational cobordism in [6] to show that \overline{X}^- , which is constructed via a local S^1 -symmetry, gives an alternative construction of the blowup of X along Y (which uses a local $U(k)$ -symmetry).

First of all, $\overline{X}_0^- = H_-^{-1}(\epsilon)/S^1$ is the reduction at ϵ with respect to the Hamiltonian

$$H_-(u, w) = |\phi(u)|^2 - |w|^2.$$

Since H_- has only one simple critical value 0 whose signature is $(2, 2k)$ which is connected when Y is, we know \overline{X}_0^- is the ϵ -blowup of $X_0 = \mathcal{N}_{\epsilon_0}(Y)$ along Y by Theorem 13.1 of [6] (while the reduction at a small negative value is $\mathcal{N}_{\epsilon_0}(Y)$ with a deformed form). The following observation is essentially pointed out in [10].

Lemma 2.7. *The symplectic manifold $(\overline{X}^-, \omega^-)$ via symplectic cut is the ϵ_0 -blowup of X along Y , where Z^- is the exceptional divisor.*

Proof. We apply the construction in Section 12 in [6] to construct a global cobordism (X_I, Φ) with a simple critical value 0. We divide X as two open sets $X - \mathcal{N}_{\epsilon'}(Y)$ and X_0 with $\epsilon' < \epsilon_0$, construct birational cobordisms on each piece and glue them together. For the part $X - \mathcal{N}_{\epsilon'}(Y)$, we choose the trivial cobordism $I \times S^1 \times (X - \mathcal{N}_{\epsilon'}(Y))$ with

the circle action the translation on the S^1 factor. For the X_0 piece, we use the simple cobordism as in the paragraph above.¹

By applying the “real blowing up” trick in [6] to our local symplectic cut cobordism over X_0 , we can glue two birational cobordisms together along $I \times S^1 \times (\mathcal{N}_{\epsilon_0}(Y) - \mathcal{N}_{\epsilon'}(Y))$ to get a global cobordism (X_I, Φ) with a simple critical value 0, where reductions at negative values are X and positive values are \overline{X}^- . Now the lemma follows from Theorem 13.1 of [6]. \square

2.3.4. *The \overline{X}^+ side.* Let us examine the \overline{X}^+ side. First of all, we have $\overline{X}^+ = \overline{X}_0^+ = H_+^{-1}(\epsilon)/S^1$ where $H_+(u, w) = |\phi(u)|^2 + |w|^2$. Notice $H_-|_{\mathcal{N}_{\epsilon_0}(Y) \oplus 0} = H_+|_{\mathcal{N}_{\epsilon_0}(Y) \oplus 0} = H$.

Lemma 2.8. *\overline{X}^+ is diffeomorphic to the projectivization $\mathbb{P}_s(N_Y \oplus \mathbb{C})$ of $N_Y \oplus \mathbb{C}$. It has two symplectic submanifolds, the codimension 2 submanifold $Z^+ = \mathbb{P}_s(N_Y \oplus 0)$ and a copy of Y which is the infinity section $\mathbb{P}_s(0 \oplus \mathbb{C})$.*

- Via the trivial \mathbb{C} -summand, there is an embedding

$$N_Y \rightarrow \mathbb{P}_s(N_Y \oplus \mathbb{C}), \quad v \rightarrow l(v, 1).$$

Under this embedding, the zero section Y of N_Y is mapped to $\mathbb{P}_s(0 \oplus \mathbb{C}) = \{l(0, 1)\}$. Thus the normal bundle to the section $\mathbb{P}_s(\mathbb{C})$ is N_Y .

- \overline{X}^+ inherits a semi-free S^1 -action from X with Z^+ and Y as the fixed loci. This action lifts to the bundle $N_Y \oplus \mathbb{C}$, complex multiplication on N_Y and trivial action on \mathbb{C} .

- The symplectic forms on \overline{X}^+ and the restriction to Z^+ are standard.

All the statements are clear.

The symplectic section $\mathbb{P}_s(\mathbb{C}) = \mathbb{P}_s(0 \oplus \mathbb{C})$ imposes constraint on the possible symplectic structures on the \mathbb{P}^{n-1} -bundle Z^- . In the next section we search for constraints in the case that Y is a 2-manifold.

2.3.5. *Symplectic cutting a general neighborhood.* Suppose W is a neighborhood of Y containing the $U(k)$ neighborhood $\mathcal{N}_{\epsilon_0}(Y)$ and W has a Hamiltonian S^1 action with the Hamiltonian function H extending $|\phi(u)|^2$.

Lemma 2.9. *If $\alpha \geq \epsilon_0 > \epsilon$ is in the interval $H(W)$ and we cut the α neighborhood of Y , then*

- $(\overline{X}^-, \omega^-)$ is deformation to the ϵ -blowup.

¹If X_0 can be taken to be X then we are done. This is the case in Remark 1.3 in [10].

- *The symplectic form on Z^\pm is almost standard.*
- $(\overline{X}^+, Z^+, \mathbb{P}_s(\mathbb{C}); \omega^+)$ is a matching triple of (\overline{X}^-, Z^-) .

Proof. It follows from Lemma 2.6 that \overline{X}^- is deformation to the ϵ -blowup.

Recall from Definition 1.2 that a matching triple of (\overline{X}^-, Z^-) is a linear $(\mathbb{P}^k, \mathbb{P}^{k-1}, \mathbb{P}^0)$ -bundle triple $(K, D', S; \Omega)$ over Y satisfying

- (1) D' is diffeomorphic to Z^- ;
- (2) $e(N_{D'}) = -e(N_{Z^-})$;
- (3) $\Omega|_{D'}$ is almost standard and matches with $\omega^-|_{Z^-}$.
- (4) Ω is S^1 -invariant with respect to a semi-free S^1 -action, where D' and S are exactly the fixed loci.

By Lemma 2.8 the triple $(\overline{X}^+, Z^+, \mathbb{P}_s(\mathbb{C}); \omega^+)$ clearly satisfies these conditions. \square

2.4. Blowing down via the S^1 -equivariant fiber sum. Blowing down is the inverse operation of blowing up. Suppose (M, ω) has a topological exceptional divisor $\pi : D \rightarrow Y$. We will state a criterion to blow down D symplectically up to deformation using normal connected sum.

Given two symplectic manifolds containing symplectomorphic codimension 2 symplectic submanifolds with opposite normal bundles, the normal connected sum operation in [4] and [17] produces a new symplectic manifold by identifying the tubular neighborhoods. As pointed out in [10], the normal connected sum operation or the fiber sum operation is the inverse operation of the symplectic cut.

Notice that we can apply the normal connected sum operation to the pairs $(\overline{X}^\pm, \omega^\pm|_{Z^\pm})$ to recover

$$(X, \omega) = (\overline{X}^+, \omega^+) \#_{Z^+ = Z^-} (\overline{X}^-, \omega^-).$$

A matching triple of (\overline{X}^-, Z^-) also satisfy the conditions to perform a symplectic sum with (\overline{X}^+, Z^+) . Topologically, the new manifold obtained is the blow down of M . Moreover, we have the following counterpart of Lemma 2.9.

Proposition 2.10. *Suppose (M, ω) has a topological exceptional divisor D . We can symplectically blow down (M, ω) along D up to integral deformation if there is a matching triple $(K, D', S; \Omega)$.*

Proof. Let $X = M \#_{D=D'} K$ be the normal connected sum of (M, ω) and (K, Ω) along $D = D'$.

By the universal construction we can choose a tubular neighborhood $W = \mathcal{N}_\epsilon(D)$ of the symplectic submanifold D in (M, ω) with a semi-free

Hamiltonian S^1 -action, which is complex multiplication when identifying W with a normal disk bundle over D . In particular, D is fixed by this S^1 -action. By the equivariant Darboux-Weinstein Theorem, we can glue the two Hamiltonian S^1 -actions on K and W to get an S^1 -action on the open piece $X_0 = W \#_{D=D'} K$ of X . Let H be the moment map on K with value 0 at S on K and $\tau = H(D')$. We extend H to X_0 .

Pick a standard ϵ_0 -neighborhood of S in K , which of course lies in X . For some $\epsilon < \epsilon_0$, perform the symplectic cut of X along $H^{-1}(\epsilon)$ to get $(\tilde{X}_1, \tilde{D}_1)$. Then $(\tilde{X}_1, \tilde{D}_1)$ is the symplectic ϵ -blowup of (X, S) by Lemma 2.7. Note that M is the cut at τ and \tilde{X}_1 is the cut at ϵ . By Lemma 2.6, M and \tilde{X}_1 are (integral) deformation from the S^1 -action on X_0 . \square

When Y is a 2-manifold, we will show that a necessary condition for the existence of a matching triple is that D being admissible. And when D is admissible, we will construct a weak matching triple using Kähler geometry. Moreover, in dimension 6, weak matching triples are actually matching triples.

3. SYMPLECTIC GEOMETRY OF PROJECTIVE BUNDLES OVER A SURFACE

In this section let Σ be a closed, oriented 2-manifold.

3.1. Topology of linear projective bundles $\mathbb{P}_s(V)$ over a surface.

We introduce the topological type and the normal type for a linear \mathbb{P}^{n-1} -bundle over Σ , which is the projectivization of a rank n bundle V . Over Σ the 1st Chern class of a complex vector bundle V can be identified as an integer, the degree $\deg(V) = \int_{\Sigma} c_1(V)$. Another feature is that V can be decomposed as the direct sum of line bundles. Especially, every bundle admits a holomorphic structure.

We use $\mathbb{C}(i)$ to denote the topological complex line bundle with degree i . Up to twisting by a line bundle, every rank n complex vector bundle is of the form $V = \mathbb{C}^k \oplus \mathbb{C}(-1)^{n-k}$, $1 \leq k \leq n$.

3.1.1. *Topological and normal types.* First observe that

$$\pi_1(PGL(n, \mathbb{C})) = \pi_1(PU(n)) = \mathbb{Z}_n$$

since $SU(n)$ is n -fold cover of $PU(n)$. So there are n topologically distinct linear \mathbb{P}^{n-1} -bundles over Σ . Since tensoring a line bundle does not change the projective bundle, we see topological linear \mathbb{P}^{n-1} -bundles are classified by $-\deg(V) \pmod{n}$.

Definition 3.1. Suppose $D_V = \mathbb{P}_s(V)$ is a linear \mathbb{P}^{n-1} -bundle modeled on a rank n complex vector bundle V . The topological type of D_V is

$$t_n(D_V) = t_n(-\deg V) \in \{0, \dots, n-1\}$$

i.e. the smallest non-negative integer satisfying

$$t_n(D_V) \equiv -\deg(V) \pmod{n}.$$

We also have the integer valued ‘normal type’ of D_V ,

$$N_n(D_V) = -\deg(V) \in \mathbb{Z}.$$

3.1.2. *Cohomology and homology of D_V .* Suppose D_V is a linear \mathbb{P}^{n-1} -bundle modeled on V . Consider the tautological line subbundle $\Xi \subset \pi^*V$ over D_V and its dual bundle Ξ^* . Here are a few properties of Ξ .

Lemma 3.2. Ξ depends on V . If we change V by a line bundle L over Σ , then Ξ is changed to $\Xi \otimes \pi^*L$.

- (1) If V is a line bundle, then $D_V = \Sigma$ and $\Xi = V$.
- (2) For a subbundle $T \subset V$, $\Xi|_{\mathbb{P}_s(T)} \subset \pi^*T|_{\mathbb{P}_s(T)}$ is the tautological line subbundle over $\mathbb{P}_s(T)$.
- (3) Ξ^* is the normal bundle of $D_V = \mathbb{P}_s(V)$ in $\mathbb{P}_s(V \oplus \mathbb{C})$. In fact, $\mathbb{P}_s(V \oplus \mathbb{C}) \setminus \mathbb{P}_s(\mathbb{C})$ is the total space of Ξ^* . Its fiberwise picture is just $\mathbb{P}^n - 0$ is biholomorphic to the total space of $\mathcal{O}(1)$ over \mathbb{P}^{n-1} .

The dual line bundle Ξ^* is called the hyperplane line bundle. Set

$$\tau = c_1(\Xi^*).$$

Since τ restricts to a ring generator of the de Rham cohomology of the fiber, by the Leray-Hirsch principle, the de Rham cohomology group $H^*(D_V; \mathbb{R})$ is the tensor product of $H^*(\mathbb{P}^{n-1}; \mathbb{R})$ and $H^*(\Sigma; \mathbb{R})$. As an $H^*(\Sigma; \mathbb{R})$ -algebra, $H^*(D_V; \mathbb{R})$ is generated by τ subject to the relation

$$(6) \quad \sum \pi^*c_j(V) \tau^{n-j} = 0.$$

In fact, this is the defining relation of Chern classes (see e.g. [2]).

Let F denote the homology class of the fiber, as well as its Poincaré dual in $H^2(D_V; \mathbb{Z})$. Let l be the homology class of a line in the fiber. Then clearly

$$\langle F, l \rangle = 0, \quad \langle \tau, l \rangle = 1.$$

Lemma 3.3. Suppose V is a rank $n \geq 2$ vector bundle over Σ with degree d . Then $H^2(D_V; \mathbb{R})$ is 2 dimensional with $\{F, \tau\}$ as a basis. And the even cohomology ring structure is described by

$$\tau^{n-1} \cdot F = (\tau|_F)^{n-1} = 1, \quad F \cdot F = 0, \quad \tau^n = - \int_{\Sigma} c_1(V) = -d.$$

There exists a unique degree 2 integral homology class η such that $\langle \tau, \eta \rangle = 0$ and $\langle F, \eta \rangle = 1$. In particular, l and η are an integral basis of $H_2(D_V; \mathbb{Z})$.

If s is a section of a line subbundle L , then

$$\langle \tau, s \rangle = -\deg(L).$$

Proof. The relation $\tau^n = -\int_Y c_1(V) = -d$ follows from (6) and the vanishing of $c_j(V)$ for $j > 1$.

To construct the class η observe that F is primitive since $\tau^{n-1} \cdot F = 1$. So there exists a class η' whose pairing with F is 1. Since $\langle F, l \rangle = 0$ and $\langle \tau, l \rangle = 1$, we get the desired η from adjusting η' by multiple of l to achieve trivial pairing with τ .

The pairing $\langle \tau, s \rangle = -\deg(L)$ follows from the observation that, under the natural identification of the section s with Σ via π , Ξ restricted to s is identified with L . A consequence is that η is geometrically represented by a nowhere zero section of a trivial line bundle, which always exists. \square

Lemma 3.4. *For a bundle V with rank n and degree d , a class $u = x\tau + yF$ is in the forward cone $\{q \in H^2(D_V; \mathbb{R}) \mid q^n > 0, \langle q, l \rangle > 0\}$ if and only if $x > 0$ and $\frac{y}{x} > \frac{d}{n}$.*

The ratio of u is given by $\rho_\pi(u) = -d + n\frac{y}{x}$. If $\langle u, \eta \rangle > 0$, then $\rho_n(u) > -d$.

Proof. The description of the forward cone follows from

$$\langle u, l \rangle = \langle x\tau + yF, l \rangle = x > 0,$$

$$u^n = (x\tau + yF)^n = x^n \tau^n + nx^{n-1}y(\tau^{n-1} \cdot F) = x^{n-1}(-dx + ny) > 0.$$

It follows that the ratio of u is given by

$$\rho_\pi(u) = \frac{u^n}{\langle u, l \rangle^n} = \frac{x^{n-1}(-dx + ny)}{x^n} = -d + n\frac{y}{x}.$$

Note that $\langle u, \eta \rangle = y$. If $\langle u, \eta \rangle > 0$, then $\rho_\pi(u) = -d + ny/x > -d$. \square

We end this subsection with a remark on smooth, linear, symplectic and holomorphic \mathbb{P}^n -bundles.

Remark 3.5. *Every complex bundle over a Riemann surface splits smoothly into a direct sum of complex line bundles and hence can be made holomorphic. Therefore any linear \mathbb{P}^n -bundle over a Riemann surface is of the form $\mathbb{P}_s(\mathcal{E})$ of a holomorphic vector bundle \mathcal{E} . We will study the Kähler geometry of such bundles in the next subsection.*

Over an algebraic variety, a holomorphic \mathbb{P}^n -bundle arises from a holomorphic vector bundle (cf. Exercise 7.10 in Hartshorne's Algebraic

Geometry). Therefore if a smooth \mathbb{P}^n bundle over surface can be made a holomorphic \mathbb{P}^n bundle, then it is linear.

A symplectic fibration with fibers (F, σ) is a fibration with structure group $\text{Symp}(F, \sigma)$. Since the fibers of D are compact, a closed form ω on D which is symplectic along fibers gives rise to a symplectic fibration structure ([21]).

When $n = 2$, if a smooth \mathbb{P}^n bundle can be made symplectic, then the structure group is $\text{Symp}(\mathbb{P}^2, \omega_{FS})$. Since $\text{Symp}(\mathbb{P}^2, \omega_{FS})$ is homotopic to $PU(3)$, such a bundle is linear.

When $n = 1$, the classification of symplectic ruled surface shows that all smooth \mathbb{P}^1 bundles are linear.

3.2. Holomorphic projective bundles $\mathbb{P}(\mathcal{E})$.

3.2.1. *The projective bundle $\mathbb{P}(\mathcal{E})$ of quotient line bundles.* We switch to the quotient bundle convention in algebraic geometry. Let us begin with a general setting. Let X be an algebraic variety, and \mathcal{E} a holomorphic vector bundle of rank r on X . We use the Grothendick convention of projectivization and denote by $\pi : \mathbb{P}(\mathcal{E}) \rightarrow X$ the projective bundle of one-dimensional quotients of \mathcal{E} . More algebraically, $\mathbb{P}(\mathcal{E}) = \text{Proj}_{\mathcal{O}_X}(\text{Sym}(\mathcal{E}))$, where $\text{Sym}(\mathcal{E}) = \bigoplus_{m>0} s^m(\mathcal{E})$ is the symmetric algebra of \mathcal{E} .

The projective bundle $\mathbb{P}(\mathcal{E})$ carries the Serre line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, which is the tautological quotient of $\pi^*\mathcal{E}$:

$$\pi^*\mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \rightarrow 0.$$

When restricted to each fiber, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the hyperplane bundle $\mathcal{O}(1)$ on \mathbb{P}^{n-1} . $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the universal quotient bundle in the following sense. Let $p : Z \rightarrow X$ be any morphism. Then, a morphism $f : Z \rightarrow \mathbb{P}(\mathcal{E})$ over X is equivalent to a quotient line bundle $p^*\mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$. Under this correspondence, $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Since X is projective, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is represented by a divisor and we write ξ for its first Chern class. As an $H^*(X; \mathbb{R})$ -algebra, $H^*(\mathbb{P}(\mathcal{E}); \mathbb{R})$ is generated by ξ subject to the Grothendick relation

$$(7) \quad \sum (-1)^j \pi^*c_j(\mathcal{E}) \xi^{n-j} = 0.$$

We call \mathcal{E} *ample* (resp. *nef*) if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is so.

We now assume $X = \Sigma$ is a Riemann surface. We need the following result of Hartshorne in [7] later.

Theorem 3.6. *For a vector bundle \mathcal{E} on a Riemann surface Σ , $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef (resp. ample) if and only if \mathcal{E} and every quotient bundle of \mathcal{E} has non-negative (resp. positive) degree. Especially, suppose \mathcal{E} is*

semistable, then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef (resp. ample) if it has non-negative (resp. positive) degree.

3.2.2. *Cohomology and homology of $\mathbb{P}(\mathcal{E})$.* We still use F, l, η for $\mathbb{P}(\mathcal{E})$ and summarize the results on the real cohomology and integral homology of $\mathbb{P}(\mathcal{E})$, $H^*(\mathbb{P}(\mathcal{E}))$ and $H_2(\mathbb{P}(\mathcal{E}); \mathbb{Z})$.

Lemma 3.7. *Suppose \mathcal{E} is a holomorphic bundle over a Riemann surface X with rank n and degree $d = \deg(\mathcal{E})$. Then*

- (1) $\mathbb{P}(\mathcal{E}) = \mathbb{P}_s(\mathcal{E}^*)$ and $\deg(\mathcal{E}) = -\deg(\mathcal{E}^*)$. Under the identification $\mathbb{P}(\mathcal{E}) = \mathbb{P}_s(\mathcal{E}^*)$, the ξ class corresponds to the τ class of $\mathbb{P}_s(\mathcal{E}^*)$.
- (2) $H^2(\mathbb{P}(\mathcal{E}); \mathbb{Z})$ is generated by ξ and the (Poincaré dual of) the fiber class F .
- (3) The even cohomology ring structure of $\mathbb{P}(\mathcal{E})$ is described by

$$\xi^{n-1} \cdot F = (\xi_F)^{n-1} = 1, \quad F \cdot F = 0, \quad \xi^n = d.$$

- (4) $H_2(\mathbb{P}(\mathcal{E}); \mathbb{Z})$ is generated by l and η , with the pairing with $H^2(\mathbb{P}(\mathcal{E}); \mathbb{Z})$ given by

$$\langle F, l \rangle = 0, \quad \langle \xi, l \rangle = 1, \quad \langle F, \eta \rangle = 1, \quad \langle \xi, \eta \rangle = 0.$$

- (5) For a class $u = x\xi + yF \in H^2(\mathbb{P}(\mathcal{E}); \mathbb{R})$,

$$u^n = x^{n-1}(dx + ny) \quad \text{and} \quad \langle u, l \rangle = \langle x\xi + yF, l \rangle = x.$$

Hence u is in the forward cone $\{q \in H^2(\mathbb{P}(\mathcal{E}); \mathbb{R}) \mid q^n > 0, \langle q, l \rangle > 0\}$ if and only if

$$x > 0 \quad \text{and} \quad \frac{y}{x} > -\frac{d}{n}.$$

- (6) The ratio of $u = x\xi + yF$ in the forward cone is given by

$$\rho_\pi(u) = d + n\frac{y}{x}.$$

- (7) If u is in the forward cone and $\langle u, \eta \rangle > 0$, then $\rho_n(u) > d$.

Proof. For (1) just notice that a line subbundle \mathcal{L} of \mathcal{E}^* determines a hyperplane $\hat{\mathcal{L}}$ of \mathcal{E} , which in turn gives rise to a quotient line bundle \mathcal{L}^* of \mathcal{E} .

(2), (3) and (4) follow from (1) and Lemma 3.3.

(5), (6) and (7) follow from (1) and Lemma 3.4. \square

3.2.3. *The curve cone, multi-sections and symmetric powers.* The curve cone $NE(\mathbb{P}(\mathcal{E})) \subset H_2(\mathbb{P}(\mathcal{E}))$ is spanned by the classes of all effective 1-cycles. An irreducible curve is either contained in a fiber, or it intersects the fiber divisor at zero-cycles. For the first case, the classes are generated by l which is the generator of the second homology of a fiber.

For the latter case, it is a multi-section Z . Suppose Z is an m -section. Then $\langle F, [Z] \rangle = m$, and the restriction of the projection $\pi|_Z : Z \rightarrow X$ is a degree m ramified covering of X . Let $\iota : Z \rightarrow \mathbb{P}(\mathcal{E})$ denote the inclusion. By the universal property of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, we have the quotient line bundle over Z ,

$$\pi|_Z^* \mathcal{E} \rightarrow \iota^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \rightarrow 0.$$

Lemma 3.8. *Let $p : Z \rightarrow X$ be a ramified covering of algebraic curves of degree m , and \mathcal{E} a vector bundle over X . Then each quotient line bundle $p^* \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$ over Z gives rise to a quotient line bundle $\mathcal{M}_{\mathcal{L}}$ of $s^m \mathcal{E}$ over X with degree $\deg \mathcal{L}$.*

Proof. First notice that any fiber $\mathcal{L}|_y$ could be viewed as a quotient subspace of $\mathcal{E}|_{p(y)}$. For any $x \in X$, we could choose m 1-dimensional vector quotient spaces of \mathcal{E}_x by taking $\mathcal{L}|_{y_i}$ where $y_i \in p^{-1}(x)$. This may count with multiplicities if it is a ramified point. Then the line $\text{Sym}(\mathcal{L}_{y_1} \otimes \dots \otimes \mathcal{L}_{y_m})$ is a 1-dimensional quotient space in the symmetric power $s^m(\mathcal{E})|_x$. Hence we obtain a quotient line bundle of $s^m \mathcal{E}$, which is holomorphic since the transition functions are polynomials of that of \mathcal{L} .

It is clear from the construction that a section s of \mathcal{L} would give rise to a section s' of $\mathcal{M}_{\mathcal{L}}$. Moreover, the zero locus of s counted with multiplicity would exactly correspond to that of s' . Hence the degree of the quotient bundle is preserved. \square

Remark 3.9. *Note that a quotient line bundle of $s^m(\mathcal{E})$ may not correspond to a m -section of $\mathbb{P}(\mathcal{E})$. This is true if the rank of \mathcal{E} is 2. The simple algebraic fact is a symmetric tensor is not always decomposable, i.e. of the form $\text{sym}(v_1 \otimes \dots \otimes v_m)$.*

By Lemma 3.8 an m -section Z induces a quotient line bundle of $s^m \mathcal{E}$, which we denote by \mathcal{M}_Z . The homology class of Z is determined by $\deg \mathcal{M}_Z$ as follows.

Lemma 3.10. *Suppose $\iota : Z \rightarrow \mathbb{P}(\mathcal{E})$ is an m -section with the associated quotient line bundle \mathcal{M}_Z of $s^m \mathcal{E}$. Then $\langle \xi, [Z] \rangle = \deg \mathcal{M}_Z$ and hence*

$$(8) \quad [Z] = \deg(\mathcal{M}_Z)l + m\eta.$$

Proof. By Lemma 3.8,

$$\deg \mathcal{M}_Z = \deg \iota^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \int_Z c_1(\iota^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \int_Z \iota^* \xi = \langle \xi, \iota_*[Z] \rangle.$$

The last statement follows from the first and $\langle F, [Z] \rangle = m$. \square

3.2.4. Semi-stable bundles and direct sums. Recall that the slope of a vector bundle \mathcal{E} is the ratio $\mu(\mathcal{E}) = \deg(\mathcal{E})/\text{rank}(\mathcal{E})$. \mathcal{E} is called semistable if every subbundle $\mathcal{F} \subset \mathcal{E}$ satisfies $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. Equivalently, \mathcal{E} is semistable if every quotient bundle $\mathcal{G} = \mathcal{E}/\mathcal{F}$ satisfies $\mu(\mathcal{G}) \geq \mu(\mathcal{E})$.

Lemma 3.11. *If \mathcal{V} is a rank r semi-stable bundle, then $s^m \mathcal{V}$ is also semi-stable with $\mu(s^m \mathcal{V}) = m \mu(\mathcal{V})$.*

Proof. A basic fact about semi-stable bundles over curves is due to Hartshorne [7]: The symmetric powers of a semi-stable bundle over a smooth curve are semi-stable.

Since both bundles are semi-stable, the slope is determined by the rank and the degree of the bundles themselves. For $m \geq 0$ one has that $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{V})}(m) = s^m \mathcal{V}$. We note that

$$\text{rank } s^m \mathcal{V} = \binom{m+r-1}{m},$$

which is the number of r -variable monomials of degree m , and

$$c_1(s^m \mathcal{V}) = \binom{m+r-1}{m-1} c_1(\mathcal{V}),$$

which can be verified by the splitting principle. By the formulae above,

$$\mu(s^m \mathcal{V}) = \frac{\deg(s^m \mathcal{V})}{\text{rank } s^m(\mathcal{V})} = m \frac{\deg(\mathcal{V})}{\text{rank } \mathcal{V}} = m \mu(\mathcal{V}).$$

\square

Lemma 3.12. *Let $\mathcal{E} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k$ with $\mathcal{V}_1, \dots, \mathcal{V}_k$ semi-stable bundles and $\mu(\mathcal{V}_1) \geq \cdots \geq \mu(\mathcal{V}_k)$. Then*

- (1) *If \mathcal{L} is a line subbundle of \mathcal{E} , then $\deg \mathcal{L} \leq \mu(\mathcal{V}_1)$.*
- (2) *If \mathcal{L} is a quotient line bundle, then $\deg \mathcal{L} \geq \mu(\mathcal{V}_k)$.*

Proof. If \mathcal{L} is a line subbundle, then \mathcal{O} is a line subbundle of

$$\mathcal{E} \otimes \mathcal{L}^{-1} = (\mathcal{V}_1 \otimes \mathcal{L}^{-1}) \oplus \cdots \oplus (\mathcal{V}_k \otimes \mathcal{L}^{-1}).$$

As

$$\Gamma((\mathcal{V}_1 \otimes \mathcal{L}^{-1}) \oplus \cdots \oplus (\mathcal{V}_k \otimes \mathcal{L}^{-1})) = \Gamma(\mathcal{V}_1 \otimes \mathcal{L}^{-1}) \oplus \cdots \oplus \Gamma(\mathcal{V}_k \otimes \mathcal{L}^{-1}),$$

we know \mathcal{O} is a subbundle of at least one of $\mathcal{V}_i \otimes \mathcal{L}^{-1}$. As each $\mathcal{V}_i \otimes \mathcal{L}^{-1}$ is semi-stable, we have $\deg(\mathcal{V}_i \otimes \mathcal{L}^{-1}) \geq 0$ for at least one \mathcal{V}_i , *i.e.*,

$$\deg \mathcal{L} \leq \mu(\mathcal{V}_i) \leq \mu(\mathcal{V}_1).$$

If \mathcal{L} is a quotient line bundle, then the dual bundle \mathcal{L}^* is a line subbundle of $\mathcal{E}^* = \mathcal{V}_1^* \oplus \cdots \oplus \mathcal{V}_k^*$. Each \mathcal{V}_i^* is a semi-stable bundle of slope $-\mu(\mathcal{V}_i)$ and $\mu(\mathcal{V}_1^*) \leq \cdots \leq \mu(\mathcal{V}_k^*)$. By the line subbundle case,

$$\deg \mathcal{L}^* \leq \mu(\mathcal{V}_k^*) = -\mu(\mathcal{V}_k),$$

which is $\deg \mathcal{L} \geq \mu(\mathcal{V}_k)$. \square

When the base is \mathbb{P}^1 , we will need the following lemma.

Lemma 3.13. *Let $\mathcal{V} = \mathcal{O}(1)^k \oplus \mathcal{O}(2)^{n-k}$ over \mathbb{P}^1 and s^* the section from the quotient line bundle $\mathcal{O}(1)$. Then the GW invariant of $[s^*]$ is nonzero.*

Proof. Observe that $\mathbb{P}(\mathcal{V})$ is Fano so we can apply the calculation in Sections 2 and 5 of [24] here. By the paragraph above Lemma 2.2 in [24], $[s^*]$ is the extremal class $A_2 = \xi^{n-1} + (1 - c_1)F\xi^{n-2}$ and the unparametrized moduli space $\mathcal{M}([s^*], 0)$ is compact. Since

$$c_1 = \int_X c_1(\mathcal{V}) = k + 2(n - k) = 2n - k,$$

$\xi^n = c_1$ and $-K_{\mathbb{P}(\mathcal{V})} = (2 - c_1)F + n\xi$, we have

$$\langle -K_{\mathbb{P}(\mathcal{V})}, s^* \rangle = (2 - c_1) + (1 - c_1)n + n\xi^n = 2 - c_1 + n = 2 - (n - k).$$

Thus, by (3.4) in [24], the GW dimension of unparametrized moduli space is $2 - (n - k) + (n - 3)(1 - 0) = k - 1$. In fact, from the proof of Lemma 2.3 (ii) and (5.6) in [24] the moduli space $\mathcal{M}([s^*], 0)$ is \mathbb{P}^{k-1} , which is identified with the space of trivial quotient line bundles of $\mathcal{O}(1)^k$. And the obstruction bundle is trivial by (5.7) in [24]. \square

3.3. Kähler cone and restricted Kähler cone.

3.3.1. *The Kähler cone of $\mathbb{P}(\mathcal{E})$.* Let \mathcal{E} still be a holomorphic vector bundle over Σ . We determine the Kähler cone of $\mathbb{P}(\mathcal{E})$ for various types of \mathcal{E} . In particular, we establish a criterion for the Kähler cone to be the forward cone. As our objects are projective bundle $\mathbb{P}(\mathcal{E})$, and in particular they are projective, so the Kähler cone is the real extension of ample cone. Since $H^{2,0}$ vanishes for $\mathbb{P}(\mathcal{E})$, it follows from Kleiman's criterion that a class in $H^2(\mathbb{P}(\mathcal{E}); \mathbb{R})$ is Kähler if and only if it is positive on the closure of the cone of curves.

A Kähler class is in the forward cone since the line class l is in the curve cone. Let Z be an effective curve which is an m -section. Then

Z corresponds to a quotient line bundle \mathcal{M}_Z of $s^m\mathcal{E}$, and by Lemma 3.10, $[Z] = al + m\eta$ with $a = \deg \mathcal{M}_Z$.

Lemma 3.14. *Suppose $u = x\xi + yF$ is in the forward cone. For an m -section Z with $[Z] = al + m\eta$, $\langle u, [Z] \rangle > 0$ if and only if*

$$\frac{y}{x} > -\frac{a}{m}.$$

Consequently, u is in the Kähler cone if there is $\alpha > 0$ such that every m -section Z satisfies $\frac{a}{m} \geq \alpha$ and $\rho_\pi(u) > \deg(\mathcal{E}) - \text{rank}(\mathcal{E})\alpha$.

In particular, the Kähler cone of $\mathbb{P}(\mathcal{E})$ is the forward cone if

$$(9) \quad \frac{a}{m} \geq \frac{\deg(\mathcal{E})}{\text{rank}(\mathcal{E})}.$$

Proof. Since the class $u = x\xi + yF$ is in the forward cone we have $x > 0$ and $dx + ny > 0$ by Lemma 3.7 (5). The first claim follows from

$$\langle u, [Z] \rangle = \langle x\xi + yF, al + m\eta \rangle = ax + my = mx\left(\frac{a}{m} + \frac{y}{x}\right).$$

Set $d = \deg(\mathcal{E})$ and $n = \text{rank}(\mathcal{E})$. By Lemma 3.7 (6), the ratio of $u = x\xi + yF$ is given by

$$\rho_\pi(u) = d + n\frac{y}{x}.$$

Therefore, for an m -section Z with $[Z] = al + m\eta$,

$$\langle u, [Z] \rangle = mx\left(\frac{a}{m} + \frac{y}{x}\right) > 0$$

if $\frac{a}{m} \geq \alpha$ and $\rho_\pi(u) > d - n\alpha$.

The last statement is clear by taking $\alpha = d/n$. \square

Proposition 3.15. *The Kähler cone of $\mathbb{P}(\mathcal{V})$ is the forward cone if \mathcal{V} is a semi-stable rank n bundle.*

In the rank 2 case, the converse is also true.

Proof. Since \mathcal{V} is semi-stable, so are all symmetric powers $s^m(\mathcal{V})$. Hence, by Lemma 3.11, for any quotient line bundle \mathcal{M} of $s^m\mathcal{V}$, we have

$$\deg(\mathcal{M}) \geq \mu(s^m\mathcal{V}) = m\mu(\mathcal{V}) = m\frac{\deg(\mathcal{V})}{n}.$$

Therefore if we set $d = \deg(\mathcal{V})$ and apply it to \mathcal{M}_Z with $[Z] = al + m\eta$, then

$$a \geq \frac{md}{n}.$$

The first statement now follows from Lemma 3.14.

In the rank 2 case, if \mathcal{V} is not semi-stable, then there is a quotient line bundle with degree strictly less than $\frac{\deg(\mathcal{V})}{2}$. Since degree is an integer, for the corresponding section Z and $u = x\xi + yF$, we have

$$\langle u, [Z] \rangle = \langle x\xi + yF, al + \eta \rangle = ax + y \leq \frac{\deg(\mathcal{V}) - 1}{2}x + y.$$

If we choose $u = 4\xi + (1 - 2d)F$, then u satisfies $u^2 > 0$ but pairs negatively with $[Z]$. \square

It is easy to see that for a rank r bundle \mathcal{V} , if the Kähler cone of $\mathbb{P}(\mathcal{V})$ is the forward cone then there is no quotient line bundle with degree strictly less than $\frac{\deg(\mathcal{V})}{r}$.

Lemma 3.16. *Suppose $\mathcal{V} = \bigoplus_j \mathcal{L}_j$ with degrees a_j and $a_1 = \min a_j$. If we view a line bundle summand also as a quotient line bundle and let C_i be the corresponding section, then the curve cone of $\mathbb{P}(\mathcal{V})$ is bounded by $[C_1] = a_1l + \eta$ and l .*

The ratio of the Kähler cone is $\sum(a_j - a_1)$. If $a_1 = 0$, then the ratio of the Kähler cone is $\deg(\mathcal{V})$.

Proof. If $Z \rightarrow X$ is of degree m , then Z corresponds to a quotient line bundle \mathcal{M}_Z of $s^m\mathcal{V}$. Observe that $s^m\mathcal{V}$ is a direct sum of line bundles and the minimal degree of these line bundles is ma_1 . Since line bundles are stable, we get by Lemma 3.12 that $\deg \mathcal{M}_Z \geq ma_1$. This means that $\langle \xi, [Z] \rangle \geq ma_1$ if $\langle F, \xi \rangle = m$. On the other hand, C_1 is an effective curve with $\langle F, [C_1] \rangle = 1$ and $\langle \xi, [C_1] \rangle = a_1$. Hence the two extremal rays are $[C_1]$ and l .

Recall η is the class with $\langle \xi, \eta \rangle = 0$ and $\langle F, \eta \rangle = 1$. Then $[C_1] = a_1l + \eta$. By the Kleiman criterion, $u = x\xi + yF$ is in the Kähler cone of $\mathbb{P}(\mathcal{V})$ if and only if

$$(10) \quad \langle x\xi + yF, l \rangle = x > 0, \quad \langle x\xi + yF, [C_1] \rangle = a_1x + y > 0.$$

Write

$$\rho(u) = \frac{x^{n-1}((\sum a_j)x + ny)}{x^n} = \sum(a_j - a_1) + n(a_1x + y)/x.$$

By (10), the ratio of the Kähler cone is $\sum(a_j - a_1)$. In particular, when $a_1 = 0$, the Kähler cone has ratio $\deg(\mathcal{V})$. \square

The above discussion follows more clearly from Hartshorne's theorem by looking at the twisting $\mathcal{E} \otimes \mathcal{L}_1^{-1}$ and notice $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{E} \otimes \mathcal{L}_1^{-1})$. Abusing the notation, we denote the new bundle to be \mathcal{E} as well. By Theorem 3.6, \mathcal{E} is nef. When we take C corresponding to the line bundle quotient \mathcal{O} , $\langle \xi, [C] \rangle = 0$. Hence the boundary of curve cone is spanned by $[C]$ and l .

3.3.2. *The ratio of the restricted Kähler cone.* Let \mathcal{V} be a rank n quotient of \mathcal{E} . We introduce the ratio of the restricted Kähler cone $\rho(\mathcal{V}, \mathcal{E})$,

$$\rho(\mathcal{V}, \mathcal{E}) = \inf\{\rho(u|_{\mathbb{P}(\mathcal{V})}) \mid u \text{ is the class of a Kähler form on } \mathbb{P}(\mathcal{E})\}.$$

Recall that $\mathbb{P}_s(\mathcal{V}) = \mathbb{P}(\mathcal{V}^*)$ so the normal type of $\mathbb{P}(\mathcal{V})$ is actually $\deg(\mathcal{V})$ rather than $-\deg(\mathcal{V})$. We have seen that the Kähler cone of a semi-stable bundle is maximal. For the direct sum of a semi-stable bundle and a line bundle we have

Lemma 3.17. *Let $\mathcal{E} = \mathcal{V} \oplus \mathcal{L}$, where \mathcal{V} is a semi-stable bundle with $\mu(\mathcal{L}) \geq \mu(\mathcal{V})$. Then the restricted Kähler cone of $\mathbb{P}(\mathcal{V})$ is the forward cone.*

Proof. Let Z be a multisection with $\langle F, [Z] \rangle = m$. Then Z corresponds to a quotient line bundle \mathcal{M}_Z of $s^m(\mathcal{E})$. In our case,

$$(11) \quad s^m(\mathcal{E}) = s^m(\mathcal{V} \oplus \mathcal{L}) = s^m(\mathcal{V}) \oplus s^{m-1}(\mathcal{V})\mathcal{L} \oplus \cdots \oplus s^{m-i}(\mathcal{V})\mathcal{L}^i \oplus \cdots \oplus \mathcal{L}^m.$$

Since the symmetric powers of a semi-stable bundle over a curve are semi-stable, each summand in (11) is semi-stable. By Lemma 3.11, we have

$$\mu(s^{m-i}(\mathcal{V})\mathcal{L}^i) = \mu(s^{m-i}(\mathcal{V})) + \mu(\mathcal{L}^i) = (m-i)\mu(\mathcal{V}) + i\mu(\mathcal{L}) \geq m\mu(\mathcal{V})$$

since $\mu(\mathcal{L}) \geq \mu(\mathcal{V})$. So the minimal slope of the summands in (11) is achieved by $s^m(\mathcal{V})$. Since each summand is semi-stable, by Lemma 3.12 the quotient line bundle \mathcal{M}_Z of $s^m(\mathcal{E})$ satisfies

$$\deg \mathcal{M}_Z \geq \mu(s^m \mathcal{V}) = m\mu(\mathcal{V}) = m \frac{\deg(\mathcal{V})}{\text{rank}(\mathcal{V})}.$$

By Lemma 3.14, $u = x\xi_{\mathcal{E}} + yF_{\mathcal{E}}$ is in the Kähler cone of $\mathbb{P}(\mathcal{E})$ if $y/x > -\deg(\mathcal{V})/\text{rank}(\mathcal{V})$. Since $u|_{\mathbb{P}(\mathcal{V})} = x\xi_{\mathcal{V}} + yF_{\mathcal{V}}$, it follows that the restricted Kähler cone of $\mathbb{P}(\mathcal{V})$ is the forward cone. \square

Lemma 3.18. *Given the normal type $qn + t < 0$ with $0 \leq t \leq n - 1$, there exists \mathcal{V} with $\deg \mathcal{V} = qn + t$ such that*

- (i) $\rho(\mathcal{V}, \mathcal{V} \oplus \mathcal{O}) = 0$ if $g(\Sigma) > 0$;
- (ii) $\rho(\mathcal{V}, \mathcal{V} \oplus \mathcal{O}) = t$ if $g(\Sigma) = 0$.

Proof. Suppose $g(\Sigma) > 0$. Then there are semi-stable bundles of arbitrary rank $r \geq 2$ and degree d over Σ ([27] for $g \geq 2$ and [1] for $g = 1$). Just pick \mathcal{V} to be a semi-stable bundle with degree $qn + t \leq 0$ and apply Lemma 3.17.

Now we assume $g(\Sigma) = 0$ and apply Lemma 3.16. Consider the bundle $\mathcal{V} = \mathcal{O}(q)^{n-t} \oplus \mathcal{O}(q+1)^t$. By Lemma 3.16, the extremal ray of $\mathbb{P}(\mathcal{V})$ is given by $\mathcal{O}(q)$ and $\rho(\mathcal{V}) = t$. Note that $q \leq -1$, so the extremal

ray of $\mathbb{P}(\mathcal{V} \oplus \mathcal{O})$ is also given by $\mathcal{O}(q)$. Hence the restricted Kähler cone is just the Kähler cone. \square

Lemma 3.19. *Given the normal type $(p-1)n+t \geq 0$ with $0 \leq t \leq n-1$, there is a completely decomposable \mathcal{V} such that $\rho(\mathcal{V}) = t$ and*

$$\rho(\mathcal{V}, \mathcal{V} \oplus \mathcal{O}) = \deg(\mathcal{V}) = (p-1)n + t.$$

Proof. Consider a decomposable rank n bundle $\mathcal{V} = \bigoplus_{i=1}^n \mathcal{L}_i \otimes \mathcal{R}$, where \mathcal{R} has degree $(p-1)$, $\mathcal{L}_i = \mathcal{O}$ or has degree 1. The number of \mathcal{O} factors is $n-t$. By Lemma 3.16, $\rho(\mathcal{V}) = t$.

For the completely decomposable bundle $\mathcal{E} = \mathcal{V} \oplus \mathcal{O}$, the section $s_{\mathcal{O}}$ is extremal. Note this is just the η class of \mathcal{E} . Since the degree of $\mathcal{L}_i \otimes \mathcal{R}$ is ≥ 0 , by Lemma 3.16, the ratio $\rho(\mathcal{V} \oplus \mathcal{O})$ is $\deg(\mathcal{V} \oplus \mathcal{O}) = \deg(\mathcal{V})$. Since the η class of \mathcal{V} is the restriction of the η class of $\mathcal{V} \oplus \mathcal{O}$, we apply Lemma 3.4 to get the ratio $\rho(\mathcal{V}, \mathcal{V} \oplus \mathcal{O})$ to be $\deg(\mathcal{V})$. \square

3.4. Almost standard symplectic forms. Suppose $\pi : D \rightarrow \Sigma$ is a linear \mathbb{P}^{n-1} bundle over a surface Σ . Recall that a fibred symplectic form on D is called standard if it arises from the Sternberg-Weinstein universal construction as described in Section 2.1. In particular, a standard form restrict to a multiple of the Fubini-Study form on each fiber. And a fibred symplectic form on D is said to be almost standard if it is deformation to a standard form via fibred forms. The linear \mathbb{P}^{n-1} bundle $\pi : D \rightarrow \Sigma$ always has a holomorphic realization of the form $\mathbb{P}(\mathcal{E})$. A Kähler form on D refers to a Kähler form on any holomorphic realization of this sort.

Lemma 3.20. *Suppose $\pi : D \rightarrow \Sigma$ is a linear \mathbb{P}^{n-1} bundle over a surface Σ . Then the space of almost standard forms on D is path connected and contains the Kähler forms.*

Proof. Since the base Σ has dimension 2, by Proposition 4.4 in [20], the space of standard forms which restricts to the same multiple of the Fubini-Study form is path-connected. Therefore the space of all standard forms is also path connected by scaling. It follows that the space of almost standard forms is path connected since any almost standard form is connected to a standard form via a path of almost standard forms.

Fix a holomorphic realization $\mathbb{P}(\mathcal{E})$ of D . Since the fibers of $\mathbb{P}(\mathcal{E})$ are holomorphic, the Kähler forms on $\mathbb{P}(\mathcal{E})$ are fibred. Moreover, there exists standard Kähler forms on $\mathbb{P}(\mathcal{E})$ (*c.f.* Proposition 3.18 in [29]). Since the space of Kähler forms on $\mathbb{P}(\mathcal{E})$ is path connected, any Kähler form on $\mathbb{P}(\mathcal{E})$ is almost standard. \square

Remark 3.21. *There is a unique deformation class of Kähler structures for each topological type since the moduli space of holomorphic bundles over a curve with the fixed rank and degree is connected ([22]). This implies that the subspace of Kähler forms is also path connected.*

Lemma 3.22. *Suppose D is a linear \mathbb{P}^{n-1} bundle over \mathbb{P}^1 . Then $\rho_\pi([\omega]) > t_n(D)$ for an almost standard form ω on D .*

Proof. Suppose $t_n(D) = n - k$. Model D on

$$V = \mathbb{C}^k \oplus \mathbb{C}(-1)^{n-k}, 1 \leq k \leq n.$$

Let S be the section $\mathbb{P}_s(\mathbb{C})$. We will show that a GW invariant of the curve class $\eta = [S]$ is nonzero. Then the inequality $\rho_\pi([\omega]) > n - k = t_n(D)$ is a consequence of Lemma 3.4.

Tensoring V^* by $\mathbb{C}(1)$, we have $V^* \otimes \mathbb{C}(1) = \mathbb{C}(1)^k \oplus \mathbb{C}(2)^{n-k}$. Now S corresponds to S^* , a quotient line bundle $\mathbb{C}(1)$ of $V^* \otimes \mathbb{C}(1)$. If ω is Kähler, the GW invariant of $[S]$ is nonzero by Lemma 3.13. By Lemma 3.20 the space of almost standard forms on D is path connected and contains Kähler forms, so ω is deformation to a Kähler form and has the same GW invariant. \square

Definition 3.23. *Suppose D is a linear \mathbb{P}^{n-1} -bundle over Σ . We define the ratio $\rho_\pi(D)$ of the (almost standard) symplectic cone by*

$$\rho_\pi(D) = \inf\{\rho_\pi(u) | u \text{ an almost standard symplectic class}\}.$$

If D is a codimension 2 submanifold of M and $S \subset M$ is a submanifold disjoint from D , we define the relative ratio

$$\rho_\pi(D; M, S)$$

to be the infimum of $\rho_\pi(u|_D)$ for u a class of a symplectic form on M that is almost standard on D and symplectic on S .

Clearly, $\rho_\pi(D; M, S) \geq \rho_\pi(D) \geq 0$.

Proposition 3.24. *Suppose D is a linear \mathbb{P}^{n-1} -bundle over Σ . Then*

$$(12) \quad \rho_\pi(D) = \begin{cases} 0 & \text{if } g(\Sigma) > 0, \\ t_n(D) & \text{if } g(\Sigma) = 0. \end{cases}$$

Moreover, any ratio can be realized by a Kähler form.

Proof. When $g(\Sigma) > 0$, by Proposition 3.15, for any complex structure arising from a semi-stable bundle, the ratio of the Kähler cone is 0.

When the base is \mathbb{P}^1 , $\rho_\pi(D) \geq t_n(D)$ by Lemma 3.22. The reverse inequality relies on the Kähler cone computation in Lemma 3.16. D can be modeled on a holomorphic bundle \mathcal{V} as in Lemma 3.16 with $a_1 = 0$. The ratio of the Kähler cone is then $\deg(\mathcal{V})$ by Lemma 3.16. So

$\rho_\pi(D) \leq \deg(\mathcal{V})$. Notice that $\mathbb{P}(\mathcal{V}) = \mathbb{P}_s(\mathcal{V}^*)$ and $\deg(\mathcal{V}) = -\deg(\mathcal{V}^*)$. Recall the topological type for $\mathbb{P}_s(E)$ is the smallest non-negative integer congruent to $-\deg(E)$ modulo n . Therefore $\rho_\pi(D) = t_n(D)$ when the base is \mathbb{P}^1 . \square

We are ready to confirm Proposition 1.7.

Proof of Proposition 1.7. Claim (i) follows directly from Lemma 3.20 and Proposition 3.24. For Claim (ii) on the symplectic cone when n is odd and $g(\Sigma) > 0$, it follows from Proposition 3.24 once we note that $-\omega$ is also a symplectic form compatible with the orientation if ω is. \square

Proposition 3.25. *Suppose V has rank n and degree d . Let $K = \mathbb{P}_s(V \oplus \mathbb{C})$, $D = \mathbb{P}_s(V)$ and $S = \mathbb{P}_s(\mathbb{C})$. Then*

$$(13) \quad \rho_\pi(D; K, S) = \begin{cases} \max\{0, -\deg(V)\} & \text{if } g(\Sigma) > 0, \\ \max\{t_n(-\deg V), -\deg(V)\} & \text{if } g(\Sigma) = 0. \end{cases}$$

Moreover, the symplectic form on K could be chosen to be S^1 -invariant with respect to the natural semi-free S^1 -action from the splitting $V \oplus \mathbb{C}$.

Proof. Let ω be a symplectic form on K that is almost standard on D and symplectic on S . Since the η class η_D of the linear \mathbb{P}^{n-1} bundle $D = \mathbb{P}_s(V)$ is sent to the η class η_K of the \mathbb{P}^n bundle $K = \mathbb{P}_s(V \oplus \mathbb{C})$ under the inclusion $D \rightarrow K$ and η_K is represented by the symplectic section S , $\omega|_D$ is positive on η_D .

Therefore we have the inequality

$$\rho_\pi(D; K, S) \geq -\deg(V)$$

by Lemma 3.4. The equality (13) then follows from the Kähler constructions on $\mathbb{P}(\mathcal{V} \oplus \mathcal{O})$ in Lemmas 3.18 and 3.19.

It remains to verify the last statement. The holomorphic bundle $\mathbb{P}(\mathcal{V} \oplus \mathcal{O})$ has a natural semi-free holomorphic S^1 -action induced from the splitting $\mathcal{V} \oplus \mathcal{O}$ that fixes both D and S . For each element g of this S^1 automorphism of $\mathbb{P}(\mathcal{V} \oplus \mathcal{O})$, $g^*\Omega$ is still a Kähler form in the class $[\Omega]$. As the space of Kähler forms is convex, take average of $g^*\Omega$ with respect to the S^1 -action, we have a Kähler form Ω' in class $[\Omega]$ which is invariant under this S^1 -action. \square

4. PROOF OF THEOREMS

Finally, we prove Theorem 1.4, Theorem 1.6 and Theorem 1.10.

Proof of Theorem 1.4. We have a symplectic divisor $\pi : D^{2n} \rightarrow \Sigma$ of (M^{2n+2}, ω) arising from symplectically blowing up a symplectic surface Σ in a symplectic manifold. Observe that from the symplectic cut

description of symplectic blowing up, $(M, \omega) = (\overline{X}^-, \omega^-)$ and $D = Z^-$. On the \overline{X}^+ side, we have a projective bundle triple by Lemma 2.8, the \mathbb{P}^k -bundle $\overline{X}^+ = \mathbb{P}_s(N_\Sigma \oplus \mathbb{C})$, the \mathbb{P}^{k-1} -bundle $Z^+ = \mathbb{P}_s(N_\Sigma \oplus 0)$ and a copy of Σ which is the infinity section $\mathbb{P}_s(0 \oplus \mathbb{C})$. Moreover, $c_1(N_D) = -c_1(N_{Z^+})$. By Lemma 3.2(3) and Lemma 3.3, $\int_D c_1(N_{Z^+})^n = -\deg(N_\Sigma)$. Therefore $c_1(N_D)$ satisfies

$$\int_D (-1)^n c_1(N_D)^n = -\deg(N_\Sigma).$$

It remains to prove the ratio inequality

$$(14) \quad \rho_\pi([\omega|_D]) > \begin{cases} -\deg(N_\Sigma), & \text{if } g(\Sigma) > 0, \\ \max\{t_n(-\deg N_\Sigma), -\deg(N_\Sigma)\}, & \text{if } g(\Sigma) = 0. \end{cases}$$

Again we can resort to the \overline{X}^+ side since

$$(15) \quad \rho_\pi([\omega|_D]) = \rho_{\pi^-}([\omega^-|_{Z^-}]) = \rho_{\pi^+}([\omega^+|_{Z^+}]),$$

where $\pi^\pm : Z^\pm \rightarrow Y$ is the projection. Note that $[\omega^+]$ pairs positively with the η class of the \mathbb{P}^k -bundle X^+ , which is represented by the symplectic section $\mathbb{P}_s(0 \oplus \mathbb{C})$. Since the η class of the \mathbb{P}^k -bundle X^+ is also the η class of the \mathbb{P}^{k-1} -subbundle Z^+ , $[\omega^+|_D]$ pairs positively with the η class of the \mathbb{P}^{k-1} -bundle Z^+ .

Note that the \mathbb{P}^{k-1} -bundles Z^+ is modeled on N_Σ . By (15), the ratio inequality (14) follows from Lemma 3.4 when $g(\Sigma) > 0$ and Lemmas 3.4 and 3.22 when $g(\Sigma) = 0$. \square

Proof of Theorem 1.6. Here we have a topological exceptional divisor $\pi : (D, \omega|_D) \rightarrow \Sigma$ of (M, ω) satisfying the ratio bound

$$(16) \quad \rho_\pi([\omega|_D]) > \begin{cases} \alpha_{D,M}, & \text{if } g(\Sigma) > 0, \\ \max\{\alpha_{D,M}, t_n(\alpha_{D,M})\}, & \text{if } g(\Sigma) = 0. \end{cases}$$

where $\alpha_{D,M} = \int_D (-1)^n c_1(N_D)^n$.

We model the linear \mathbb{P}^{k-1} -bundle D by a complex rank k vector bundle V over Σ with $\deg V = -\alpha_{D,M}$. Let

$$K = \mathbb{P}_s(V \oplus \mathbb{C}), \quad D' = \mathbb{P}_s(V), \quad S = \mathbb{P}_s(\mathbb{C})$$

as in Proposition 3.25. Since $[\omega|_D]$ satisfies the ratio inequality (16), there exists an S^1 -equivariant symplectic form Ω on K such that $[\Omega|_{D'}] = [\omega|_D]$ by Proposition 3.25. Clearly, the triple $(K, D', S; \Omega)$ is a desired weak matching triple. \square

Since Theorem 1.10 has several statements we restate it here and deduce it from Theorem 1.4 and Proposition 1.9.

Theorem 4.1. *Let (M, ω) be a 6-dimensional symplectic manifold and D a codimension 2 symplectic submanifold. Suppose D admits a \mathbb{P}^1 -bundle structure $\pi : D \rightarrow \Sigma$ over a surface Σ with $\langle c_1(N_D), l \rangle = -1$. Let*

$$\alpha_{D,M} = c_1(N_D) \cdot c_1(N_D) \quad \text{and} \quad \rho = \rho_\pi([\omega|_D]).$$

Suppose $(D, \omega|_D)$ arises from blowing up a surface. Then $\rho \neq 2$ if $D = S^2 \times S^2$ with $\alpha_{D,M} = 2$, and $\rho > \alpha_{D,M}$ otherwise.

Conversely, if $\rho > \alpha_{D,M}$, (M, ω) can be blown down along D up to deformation. In particular, this is the case if $\alpha_{D,M} \leq 0$. Moreover, when $D = S^2 \times S^2$ with $\alpha_{D,M} = 2$, (M, ω) can be blown down along D up to deformation as long as $\rho \neq 2$.

Proof. First of all, $(D, \omega|_D)$ is a topological exceptional divisor since any \mathbb{P}^1 -bundle over Σ is linear and any symplectic form is fibred and almost standard.

We first assume $g(\Sigma) > 0$. If $(D, \pi, \omega|_D)$ is a (symplectic) exceptional divisor, then $\rho > \alpha_{D,M}$ by Theorem 1.4. Conversely, (D, ω) is admissible if $\rho > \alpha_{D,M}$. By Proposition 1.9, up to integral deformation, D can be symplectically blow down.

Suppose $g(\Sigma) = 0$. There are two cases, $\alpha_{D,M}$ is even or $\alpha_{D,M}$ is odd. When $\alpha_{D,M}$ is odd, D is the unique non-trivial \mathbb{P}^1 -bundle over \mathbb{P}^1 , which is also the 1 point blowup of \mathbb{P}^2 . In this case, $t_n(\alpha_{D,M}) = 1$ and $\rho > 1$ for any symplectic form on D . So, as in the case $g(\Sigma) > 0$, $\rho > \alpha_{D,M}$ if $(D, \pi, \omega|_D)$ is a (symplectic) exceptional divisor, and conversely, (D, ω) is admissible and hence can be symplectically blow down up to integral deformation if $\rho > \alpha_{D,M}$.

When $\alpha_{D,M}$ is even, $D = S^2 \times S^2$ and $t_n(\alpha_{D,M}) = 0$. There are two \mathbb{P}^1 bundle structures on D . Accordingly we further divide into two cases, $\alpha_{D,M} \neq 2$ and $\alpha_{D,M} = 2$. When $\alpha_{D,M} \neq 2$, there is only one fibration with the appropriate normal bundle condition and the admissible condition is again $\rho > \alpha_{D,M}$. So the conclusion is exactly the same as the $g(\Sigma) > 0$ case.

When $\alpha_{D,M} = 2$, the evaluation of $c_1(N_D)$ is -1 on both fiber classes. Suppose the symplectic areas of the two rulings are x and y . Then the symplectic volume is $2xy$. The ratio of $\omega|_D$ with respect to the first ruling is $2xy/x^2 = 2y/x$, which is bigger than $\alpha_{D,M} = 2$ if and only if $y > x$. Similarly, the ratio of $\omega|_D$ with respect to the second ruling is bigger than $\alpha_{D,M} = 2$ if and only if $x > y$. Therefore $\rho \neq 2$ if D is a symplectic exceptional divisor. Conversely, if $\rho \neq 2$, up to integral deformation we could blow down (M, ω) along D with respect to the fibers with smaller area. \square

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455

Email address: tjli@math.umn.edu

INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, EAST No.7 BUILDING, ZHEJIANG UNIVERSITY ZIJINGANG CAMPUS, HANGZHOU, CHINA

Email address: ruanyb@zju.edu.cn

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK

Email address: weiyi.zhang@warwick.ac.uk