Geometric growth on translation surfaces

by

Paul Colognese

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Warwick Mathematics Institute

September 2021

Contents

Acknowledgments

I would like to express my gratitude to my supervisor, Mark Pollicott, for his guidance, encouragement, and humor-filled discussions. Working with Mark over the last few years was an invaluable experience that taught me how to approach research problems, helped grow my confidence, and showed me how rewarding collaborating with others can be.

I would also like to thank my friends and family who brought me much joy throughout the last few years; especially my parents for their constant love and support.

Declarations

The work in Chapters 3, 4 and 5 were obtained in collaboration with my supervisor, Mark Pollicott. In particular, Chapters 3 and 4 of this thesis appear in [7], and the work in Chapter 5 has recently become available as a preprint on arXiv.

Except as noted above, I declare that the material in this thesis is my own except where otherwise indicated or cited in the text. This material has not been submitted for any other degree or qualification.

Abstract

In this thesis we study geometric growth on translation surfaces. We obtain asymptotic formulae for the growth of various geometric objects on translation surfaces such as volumes of balls and circumferences of large circles. Using these asymptotic formulae, we then prove a distribution result for large circles on translation surfaces. Finally, we explore the entropy minimization problem for translation surfaces and prove a special case. These results generalize well-known results that hold for negatively curved surfaces.

Chapter 1

Introduction

The purpose of this thesis is to explore the idea that the growth of geodesics[1] and certain geometric objects on translation surfaces is similar to the growth of such objects on surfaces of negative curvature.

Translation surfaces are examples of closed surfaces that admit a locally Euclidean metric except at finitely many points which are known as singularities (see Figure 1.1). When the genus of a closed surface is greater than one, the Gauss-Bonnet theorem states that the average curvature of the surface must be negative. Because translation surfaces of genus greater than one are locally flat except at a finite number of singular points, as a consequence of the Gauss-Bonnet theorem, we can think of these singular points as points of concentrated negative curvature. This idea suggests that, at least in some respects, translation surfaces should behave like negatively curved surfaces.

We will explore the idea that translation surfaces behave like negatively curved surfaces when it comes to the growth of geometric objects. To formalize what we mean by a geometric object[2], we provide the following definition.

Definition 1.0.1. *Let M be a metric space. A geometric object on M refers to any of the following:*

[1]We emphasize that in most of the translation surfaces literature, geodesics are defined to be locally distance minimising paths that do *not* pass through singularities. This is because geodesics are often referred to in the context of the straight-line/geodesic flow which is not defined at singularities. We are interested in comparing translation surfaces to negatively curved surfaces from a geometric, not dynamical point of view. Hence, throughout this thesis we include geodesics that pass through singularities (see Chapter 2 Section 7 for a more detailed discussion).

[2]Note that although saddle connections and cylinders on translation surfaces are indeed geometric, for the purpose of this thesis we do not include them in the term *geometric object*.

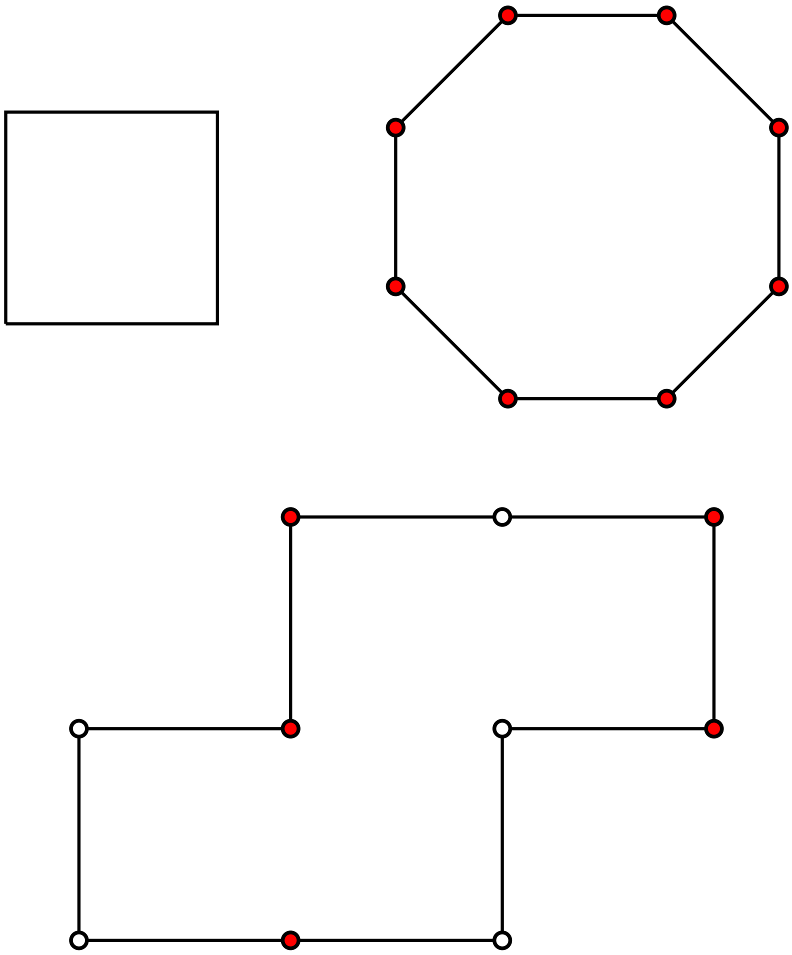


Figure 1.1: Three translation surfaces obtained by identifying opposite edges of polygons. Top left: A torus with no singularities. Top right: A genus two translation surface with one singular point corresponding to the identified vertices. Bottom: A genus two translation surface with two singularities (the filled and non-filled circles).

- *a geodesic or closed geodesic*[1]*;*
- *a ball in the universal cover of M; or*
- *a circle which is defined to be the set of end points of geodesics of length R (the radius) based at some centre point $x \in M$.*

We emphasize that the growth of other types of objects on translation surfaces, which we will refer to as *dynamical* objects, have been well studied in the literature. These dynamical objects include cylinders (which correspond to geodesics that are not allowed to pass through singularities, up to homotopy) and saddle connections on the surface (see Section 2.6 for a summary of growth results for these dynamical objects on translation surfaces). We will see that the behavior of the growth of geometric objects on translation surfaces differs from the behavior of the growth of dynamical objects on translation surfaces.

The work in this thesis generalizes several results related to growth on negatively curved surfaces to translation surfaces.

The study of the growth of such geometric objects on translation surfaces seems to have been initiated by Klaus Dankwart in [8]. An asymptotic result for the growth of closed geodesics on translation surfaces, due to Alex Eskin and Kasra Rafi, was announced at a talk in June 2018 [14].

The main results of this thesis are as follows:

- we prove asymptotic formulae for the growth of various geometric objects on translation surfaces;
- we prove a distribution result for appropriately defined circles on translation surfaces; and
- we explore how the rate of growth of these geometric objects changes over an appropriate moduli space for translation surfaces.

We conclude this introduction by summarizing the chapters of this thesis.

In Chapter 2: Background, we elaborate on the above discussion by providing a more detailed summary of the context of this work, as well as providing definitions and notation which will be used throughout the later chapters.

In Chapter 3: Asymptotic growth on infinite graphs, we develop a method for proving asymptotic formulae on infinite graphs.

In Chapter 4: Asymptotic growth on translation surfaces, we use the method developed in Chapter 3 to prove asymptotic results for translation surfaces analogous to results that hold for negatively curved surfaces.

In Chapter 5: Distribution of large circles on translation surfaces, we prove that large circles on translation surfaces distribute with respect to some measure, as the radii of these circles tend to infinity.

In Chapter 6: Minimizing entropy and equilateral surfaces, we explore which translation surfaces of a fixed area minimize the growth rate of their geometric objects. In particular, we state and discuss some conjectures and give partial evidence towards these conjectures.

In Chapter 7: Conclusion, we discuss directions for further research.

Chapter 2

Background

The purpose of this chapter is to provide context for the work in this thesis and to introduce important definitions and concepts used in later chapters.

We begin by looking at geometric growth for negatively curved surfaces and finite metric graphs. In particular, we will introduce the definition of volume entropy which quantifies the growth rate of geometric objects for various spaces, asymptotic formulae for the growth of geometric objects, distribution results, and entropy minimization problems.

We will then turn our attention to translation surfaces. First, we will define and look at the basic properties of translation surfaces. Then we will study the geometry of translation surfaces. Finally, we will look at prior work for geometric growth on translation surfaces.

2.1 Volume entropy

We begin by introducing the notion of *volume entropy* which quantifies the growth rate of geometric objects on particular spaces. More specifically, volume entropy is a non-negative number that can be defined for certain metric spaces that measures the exponential growth rate of various geometric objects defined on the space; for example, the number of geodesic paths of bounded length starting at some base point or the number of closed geodesics of a bounded length.

In this section, we will introduce the definition of volume entropy in the setting of Riemannian manifolds and finite metric graphs.

2.1.1 Volume entropy for Riemannian manifolds

We begin by introducing the definition of volume entropy in the context of Riemannian manifolds.

Let M be a closed and connected manifold with Riemannian metric ρ and universal cover $\widetilde{M}$ equipped with the lifted metric $\widetilde{\rho}$. Fix a point $c \in \widetilde{M}$ and consider a ball $\mathcal{B}(c, R)$ of radius $R > 0$ centred at c.

Definition 2.1.1. *The volume entropy of M is defined by*

$$h(M, \rho) = \lim_{R \to \infty} \frac{1}{R} \log \mathrm{Vol}_{\widetilde{\rho}}(\mathcal{B}(c, R)),$$

where $\mathrm{Vol}_{\widetilde{\rho}}(\mathcal{B}(c, R))$ denotes the Riemannian volume of $\mathcal{B}(c, R)$ in $(\widetilde{M}, \widetilde{\rho})$.

We note that the definition is well defined; Manning [31] showed that the limit always exists and $h(M, \rho)$ is independent of the base point $c \in \widetilde{M}$ (see 2.1).

One useful property of volume entropy is that it serves as an invariant for Riemannian surfaces; because isometries preserve the volume of balls, two isometric Riemannian manifolds will have the same volume entropy.

In the following example, we calculate the volume entropy of a hyperbolic surface explicitly.

Example 2.1.2. *Consider a hyperbolic surface S_g of genus $g \geq 2$ with metric ρ, which endows S_g with constant curvature equal to -1.*

The universal cover of S_g is the hyperbolic plane, $\mathbb{H}$. Let $c \in \mathbb{H}$. Using hyperbolic geometry (see [1]), one can check that

$$\mathrm{Vol}_{\widetilde{\rho}}(\mathcal{B}(c, R)) = 2\pi(\cosh(R) - 1).$$

By applying the expression $\cosh(x) = (e^x + e^{-x})/2$, it is easy to see that

$$h(S_g, \rho) = \lim_{R \to \infty} \frac{1}{R} \log \mathrm{Vol}_{\widetilde{\rho}}(B(c, R)) = 1.$$

Note that for Riemannian manifolds, at least some negative curvature is needed for positive volume entropy (see Example 2.1.3). Informally, this is because geodesics emanating from a shared point, but at different angles, grow apart exponentially

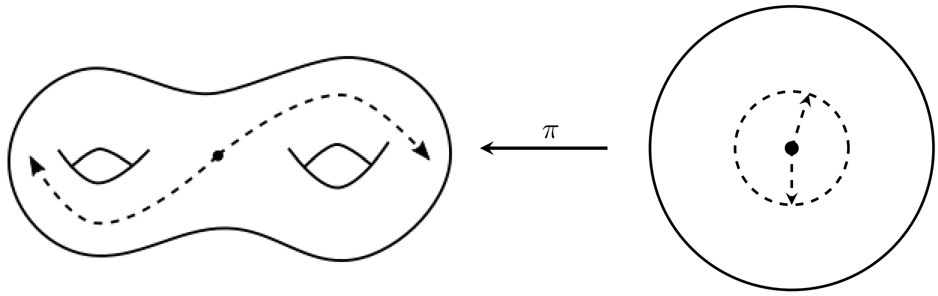


Figure 2.1: On the left is S_2, a genus 2 surface of constant negative curvature. Two geodesic paths of length R based at some point $x \in S_g$ are represented by the dashed lines. On the right we have $\mathbb{H}$, the lift of S_2, together with two lifts of the geodesic paths based at a lift $\widetilde{x}$ of x. The dashed circle represents the boundary of $\mathcal{B}(\widetilde{x}, R)$.

fast in negative curvature. The following example further highlights the need for negative curvature.

Example 2.1.3. *Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a torus equipped with a flat metric. A ball of radius $R > 0$ in the universal cover of T, the Euclidean plane $\mathbb{E}$, has area πR^2. Hence, the volume entropy of the flat torus T is*

$$h(T) = \lim_{R\to\infty} \frac{1}{R} \log(\pi R^2) = 0.$$

As mentioned at the beginning of this section, volume entropy captures the exponential growth rate of various geometric objects on a given space. For Riemannian manifolds, this means that we could replace the function $V(\mathcal{B}(c, R)) := \mathrm{Vol}_{\widetilde{\rho}}(\mathcal{B}(c, R))$ in the definition of volume entropy by any of the following counting functions associated to (M, ρ) and get the same limit (see work by [32] and [30]):

- $\ell(\mathcal{C}(c, R))$, the circumference of a ball of radius R centered at some point c in $(\widetilde{M}, \widetilde{\rho})$;
- $C(R)$, the number of closed geodesics on M, considered up to homotopy, of length less than or equal to R; and
- $N_{x,y}(R)$, the number of geodesics connecting points x and y on M of length less than or equal to R.

For Riemannian manifolds, volume entropy also captures dynamical growth on the manifold since geodesic paths on Riemannian manifolds correspond to orbits of the geodesic flow. In fact, in [31], Manning showed that for Riemannian manifolds

(M, ρ) of non-positive curvature, the volume entropy coincides with the topological entropy h_{top} of the associated geodesic flow.

2.1.2 Volume entropy for finite metric graphs

Here we introduce an analogous definition of volume entropy for finite metric graphs that quantifies the exponential growth rate of paths of a certain length based at some point. This setting is simple enough, yet it forms the foundation for our analysis of volume entropy for translation surfaces.

Let $\mathcal{G}$ be a directed, finite, connected, non-cyclic graph without terminal vertices. Let $\mathcal{V}$ and $\mathcal{E}$ be the vertex and edge sets of $\mathcal{G}$ respectively. We can give $\mathcal{G}$ a metric by defining a length function $\ell : \mathcal{E} \to \mathbb{R}^+$ on $\mathcal{G}$. Let $x \in \mathcal{V}$ and define $N(x, R)$ to be the number of non-backtracking paths based at x in $\mathcal{G}$, of length less than or equal to R.

Definition 2.1.4. *The volume entropy of $(\mathcal{G}, \ell)$ is defined by*

$$h(\mathcal{G}, \ell) := \lim_{R\to\infty} \frac{1}{R} \log N(x, R).$$

Once again, $h(\mathcal{G}, \ell)$ is independent of $x \in \mathcal{G}$ and the limit exists (see [29]). Note that one could also consider the growth of volumes of balls in the universal cover of the graph, like in the case of Riemannian manifolds (see [29]).

Example 2.1.5. *Let $\mathcal{G}$ be a graph consisting of one vertex and two oriented edges, each of length 1. Each time a path reaches the vertex, it can continue in one of two directions. Hence $N(x, R) = 2^{\lfloor R \rfloor}$ and the entropy of $(\mathcal{G}, \ell)$ is*

$$h(\mathcal{G}, \ell) = \lim_{R\to\infty} \frac{1}{R} \log(2^{\lfloor R \rfloor}) = \log(2).$$

Once again, one can replace the growth function $N(x, R)$ with other geometric growth functions such as the number of closed paths of a bounded length on the graph.

Note that in the case of Riemannian manifolds, volume entropy quantified the growth rate of geometrical objects as well as dynamical objects since geodesic paths correspond to images of points under geodesic flows. For finite metric graphs, geodesic flows cannot be defined in the same way due to the presence of vertices with multiple edges. This disconnect between the geometric and dynamical point of view will also appear when we turn our attention to translation surfaces for a similar reason.

2.2 Asymptotic formulae for counting and growth functions

In this section, we will begin by looking at a couple of well-known counting problems, and then discuss the key tool, the Ikehara-Wiener Tauberian theorem, which we will use to prove the asymptotic results in the next few chapters.

In a general sense, counting problems seek to quantify the growth of some function or collection of objects. Perhaps the most famous result for a counting problem is the prime number theorem which gives an asymptotic formula the growth of the prime numbers (see [37]).

Theorem 2.2.1 (Prime number theorem)**.** *Let $\pi(x)$ denote the number of prime numbers p such that $p \leq x$, for $x \in \mathbb{R}^+$. Then*

$$\lim_{R\to\infty} \frac{\pi(x)}{x/\log x} = 1,$$

which we write as $\pi(x) \sim x/\log(x)$.

The prime number theorem tells us that, in the limit, the number of primes below a certain number x grows like the function $x/\log(x)$. The Riemann hypothesis, if true, would give a stronger result, i.e., an asymptotic with an error term.

In the setting of closed Riemannian manifolds, we can consider the growth of volumes of balls, lengths of circles, and closed geodesics as functions for which we wish to establish asymptotic formulae.

In the case of manifolds with negative sectional curvature, work by Margulis gives asymptotic formulae for the above growth functions (see [32]). In particular, he proved that, given a Riemannian manifold (M, ρ), there exist constants $D, E, F, G > 0$ such that

$$\lim_{R\to\infty} \frac{V(\mathcal{B}(c,R))}{e^{hR}} = D, \lim_{R\to\infty} \frac{\ell(\mathcal{C}(c,R))}{e^{hR}} = E,$$

$$\lim_{R\to\infty} \frac{C(R)}{e^{hR}/hR} = F, \text{ and } \lim_{R\to\infty} \frac{N_{x,y}(R)}{e^{hR}} = G,$$

where the growth functions $V(\mathcal{B}(c,R)), \ell(\mathcal{C}(c,R)), C(R)$ and $N_{x,y}(R)$ were defined in the previous section.

The asymptotic results stated above imply the existence of the limits in the definition of volume entropy for Riemannian manifolds.

One of the goals of this thesis is to prove the analogous asymptotic formulae for translation surfaces, to those proved by Margulis for negatively curved surfaces.

We conclude this section by introducing the Ikehara-Wiener Tauberian theorem and discuss how it can be used to deduce asymptotic formulae for certain counting/growth functions. The asymptotic formulae that we prove for translation surfaces will be proved using this theorem.

The Ikehara-Wiener Tauberian theorem ([50] and [24]) tells us that the asymptotic behaviour of certain functions can be deduced from the pole structure of a meromorphic extension of the Laplace transform of the function. It can be used to prove the prime number theorem [11].

We will use the the following formulation[1] of the Ikehara-Wiener Tauberian theorem.

Theorem 2.2.2 (Ikehara–Wiener Tauberian theorem)**.** *Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing and right-continuous function. Formally denote $\eta(z) := \int_0^\infty e^{-zR} d\rho(R)$, for $z \in \mathbb{C}$. Then suppose that $\eta(z)$ has the following properties:*

1. *there exists some $a > 0$ such that $\eta(z)$ converges absolutely and is analytic on $Re(z) > a$;*
2. *$\eta(z)$ has a meromorphic extension to a neighbourhood of the half-plane $Re(z) \geq a$;*
3. *a is a simple pole for $\eta(z)$, i.e., $C = \lim_{z \to a}(z-a)\eta(z)$ exists and is positive; and*
4. *the extension of $\eta(z)$ has no poles on the line $Re(z) = a$ other than a.*

Then $\rho(R) \sim (C/a)e^{aR}$ as $R \to \infty$.

[1]This formulation comes from the typical formulation (for example in [11]) by using a change of variable.

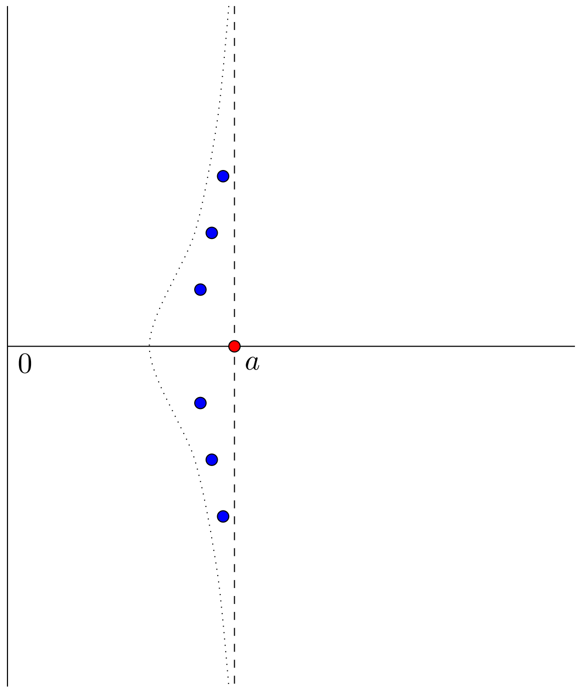


Figure 2.2: A portion of $\{z \in \mathbb{C} : Re(z) > 0\}$ is drawn along with an illustration of the conditions for the Ikehara-Wiener Tauberian theorem. In particular: $\eta(z)$ must converge absolutely and be analytic on the right side of the dashed line $Re(z) = a$; a must be a simple pole for $\eta(z)$; $\eta(z)$ must be the only pole on the line $Re(z) = a$; and there must exist a meromorphic extension of $\eta(z)$ to some neighbourhood of the line $Re(z) = a$ (represented by the dotted line). The dots represent some poles of the extension.

2.3 Distribution problems

Distribution problems are concerned with how some sequence of objects are distributed on a given space. A classic example is given by the following theorem, due to Weyl [49], which states that the images of a point on a circle under iterations of an irrational rotation are evenly distributed on the circle.

Theorem 2.3.1. *Let* $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. *Then the sequence* $\{\alpha n \textit{ (mod 1)}\}_{n\in\mathbb{N}}$ *is equidistributed on* $[0, 1)$, *i.e. for all* $0 \leq a < b < 1$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : \alpha n \textit{ (mod 1)} \in [a, b]\} = b - a.$$

We can rephrase this theorem in terms of measures. We have a sequence of probability measures μ_N on $[0, 1)$ defined by taking μ_N to be the uniform average of the

Dirac measures on the points $n\alpha \bmod 1$ for $1 \leq n \leq N$, then $\mu_N \to \mu$ as $N \to \infty$ in the sense of weak-* convergence.

In the case of Riemannian surfaces, we can ask how the geometric objects introduced earlier distribute on the surfaces. For example, given a Riemannian surface (S, ρ) of genus $g \geq 2$, we can define the circle $\mathcal{C}(x, R)$ to be the set of points on S which are joined by a geodesic of length $R > 0$ to the center point x. Equivalently, we could consider projections of circles from the surface's universal cover.

For constant negative curvature surfaces of genus $g \geq 2$, as the radius tends to infinity, these circles become equidistributed with respect to normalized volume measure (see Randol [45]).

We can state the distribution result more precisely.

Definition 2.3.2. *Let (S, ρ) be a genus $g \geq 2$ surface of constant negative curvature. Define a family of probability measures μ_R $(R > 0)$ on S by*

$$\int f d\mu_R = \frac{1}{\ell(\mathcal{C}(x, R))} \int_{\mathcal{C}(x,R)} f(x) d\lambda(x), \ \textit{for } f \in C(S, \mathbb{R}),$$

where λ is the natural length parameterization on $\mathcal{C}(x, R)$.

The result states that the measures μ_R converge to the normalized volume measure in the weak-* topology.

In Chapter 5, we will prove an analogous result for appropriately defined circles on translation surfaces.

2.4 Minimizing volume entropy

In this section, we consider volume entropy as a function defined over the moduli space of certain spaces (i.e. the space of certain metrics on a given topological space) and consider which spaces minimize volume entropy over the moduli space. We will first introduce this function in the context of negatively curved surfaces and later for finite metric graphs.

Recall that the volume entropy of a negatively curved surface (S, ρ) is defined by

$$h(S, \rho) = \lim_{R \to \infty} \frac{1}{R} \log \mathrm{Vol}_{\widetilde{\rho}}(\mathcal{B}(c, R)),$$

where $\mathrm{Vol}_{\widetilde{\rho}}(\mathcal{B}(c, R))$ denotes the Riemannian volume of $\mathcal{B}(c, R)$ in $(\widetilde{S}, \widetilde{\rho})$.

Let $\mathcal{M}$ denote the moduli space of negatively curved metrics on a surface, S, of genus $g \geq 2$. We can extend the definition of volume entropy to a function over $\mathcal{M}$, $h : \mathcal{M} \to \mathbb{R}_{\geq 0}$ where $h(\rho)$ is the volume entropy of (S, ρ).

If we consider volume entropy as quantifying the complexity of a given metric on S, it is natural to ask questions such as:

- What are the smallest and largest values that a metric on S can take?
- Which metrics, if any, minimize entropy?
- What is the regularity of the volume entropy function $h(\rho)$? Is it continuous? Smooth?

Note from the definition of volume entropy, that volume entropy scales with respect to homothety of the metric. In particular, if $h(t\rho)$ denotes the volume entropy of some metric ρ scaled by a factor of $t > 0$, then $h(t\rho) = h(\rho)/\sqrt{t}$. For this reason, the appropriate notion of volume for the metrics in question is generally fixed when asking the above questions.

The question of which surfaces minimize entropy for negatively curved surfaces was solved by Katok [25].

Theorem 2.4.1. *Let S be a surface of genus $g \geq 2$. Let ρ_0 be a metric of constant negative curvature on S and let ρ_1 be any other metric of negative curvature on S. Then*

$$h(S, \rho_1)\mathrm{Vol}_{\rho_1}(S) \geq h(S, \rho_0)\mathrm{Vol}_{\rho_0}(S),$$

with equality if and only if ρ_1 is a constant curvature metric.

In particular, if we fix the area of the metrics, we see that constant curvature metrics minimize entropy on S. Intuitively, this is because volume entropy can be thought of as a measure of complexity of a geometric space, and constant curvature metrics are the most symmetric and hence the least complex.

This work was generalized by Besson, Courtois and Gallot in [2], in which the authors showed that an analogous result holds for manifolds in higher dimensions. In particular, they show that locally symmetric spaces minimize entropy when volume is fixed.

The entropy minimization problem has been studied in other contexts, such as for finite metric graphs (see [29]). For the types of graphs defined in Section 2.1.2, the metrics which minimize entropy when the sum of the edge lengths are fixed, are those which are the most symmetric. For example, in the case of a graph $\mathcal{G}$ with one vertex and n edges, the metric which minimizes entropy is the metric which sets all the edge lengths equal to one another. In the general case, the metric that minimizes entropy is the one for which the length of any edge is proportional to the sum of the logs of the valencies of its vertices. In [29], Lim proved the following result.

Theorem 2.4.2. *Let $\mathcal{G}$ be a finite oriented connected graph with vertex set $\mathcal{V}$ and oriented edge set $\mathcal{E}$, such that the valency at each vertex x, which we denote by $k(x)+1$, is at least 3. For $e \in \mathcal{E}$, let $i(e)$ and $t(e)$ denote the initial and terminal vertex of e, respectively. Then there is a unique normalized length function which minimizes volume entropy, given by*

$$h_{min} = \frac{1}{2} \sum_{x \in V} (k(x)+1) \log(k(x)),$$

and the entropy minimizing length function is given by

$$\ell(e) = \frac{\log(k(i(e))k(t(e)))}{\sum_{x \in V} (k(x)+1) \log(k(x))},$$

for all $e \in \mathcal{E}$.

We conclude by commenting on the regularity of the entropy function. Katok, Knieper, Pollicott and Weiss [26] showed that the entropy function h has a C^∞ dependence on the metric.

2.5 Translation surfaces background

The previous sections gave an overview of some aspects of geometric growth in the context of negatively curved surfaces and finite metric graphs. The aim of this thesis is to show that some of these results generalize to translation surfaces which gives credence to the idea that, at least in some aspects, translation surfaces behave like surfaces of negative curvature.

In this section, we will give a brief introduction to translation surfaces. A good reference for this material (and more background) can be found in the surveys [54] and [51].

The study of translation surfaces has received much attention in the last couple of decades. Translation surfaces themselves arise naturally in the study of various other dynamical systems, and their study employs the tools from various fields such as: algebraic geometry; number theory; and of course, dynamical systems theory and geometry [51].

We begin by providing an informal definition of a translation surface which highlights their key features. A translation surface is a closed surface endowed with a flat metric except at, possibly, a finite number of singular points. Furthermore, there is a well-defined notion of north at every non-singular point.

Singularities on translation surfaces are cone-points. To see what this means, consider the following construction: let $k \in \mathbb{N}$ and take $(k+1)$ copies of the upper half-plane with the usual metric and $(k+1)$ copies of the lower half-plane. Then glue them together along the half infinite rays $[0, \infty)$ and $(-\infty, 0]$ in cyclic order (Figure 2.3).

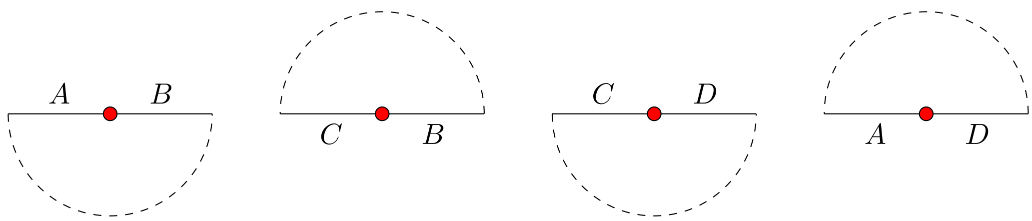


Figure 2.3: Four half-disks glued together cyclically. A singularity/cone-point of angle 4π on a translation surface has a neighbourhood isometric to a neighbourhood of the the origin in the picture (the red dot).

There are a few equivalent definitions of translation surfaces that appear in the literature. We will present the one which is most suited to our needs.

Definition 2.5.1. *A translation surface is a closed and connected topological surface, X, together with a finite set of points Σ and an atlas of charts to $\mathbb{C}$ on $X \backslash \Sigma$,* *whose transition maps are translations. Furthermore, we require that for each point $x \in \Sigma$, there exists some $k \in \mathbb{N}$ and a homeomorphism of a neighbourhood of x to a neighbourhood of the origin in the $2k+2$ half-plane construction that is an isometry away from x.*

It is easy to see that the above definition gives a finite volume and locally Euclidean metric on $X \backslash \Sigma$. The set $\Sigma = \{x_1, \ldots, x_n\}$ is the set of singularities or cone-points on the surface, where the singularity x_i has a cone-angle of $2\pi(k(x_i)+1)$ with $k(x_i) \in \mathbb{N}$.

An interesting consequence of the almost-flat metric of translation surfaces is that by cutting the surface along straight lines, one can decompose the surface into a finite collection of Euclidean polygons. Hence, any translation surface has an infinite number of representations, where each representation is a finite collection of polygons in the plane along with certain edge identifications taking an edge of one polygon to a parallel edge of (possibly) another polygon. This also gives us a convenient way of constructing examples of translation surfaces (Figure 2.4).

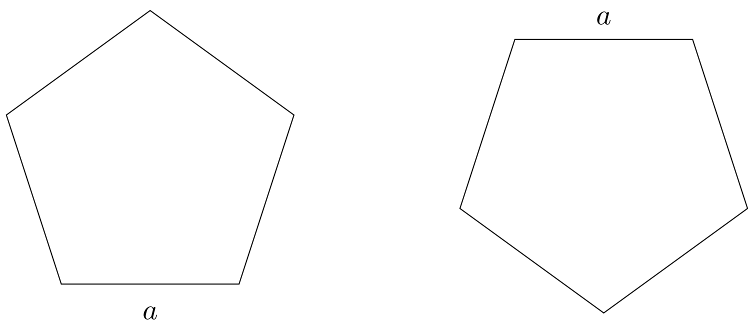


Figure 2.4: A translation surface obtained from gluing edges from one polygon to the opposite parallel edge on the other polygon. One such edge identification is denoted by “a”.

Note that two distinct polygonal representations may represent the same translation surface (Figure 2.5). In particular, two polygonal representations are equivalent if one can be subdivided into new polygons which can then be rearranged via translations to form the other polygonal representation. In general, it is difficult to tell whether two polygonal representations represent the same surface.

We conclude this section with the definition of *saddle connections* on translation surfaces.

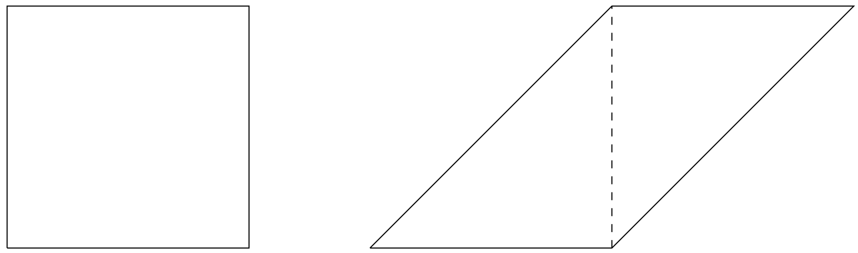


Figure 2.5: Above are two parallelograms whose opposite sides are glued together to form tori. The parallelogram on the right can be cut along the vertical dashed line two form two triangles which can be translated and glued back together to form the square on the left with opposite sides identified. Hence the two polygonal representations represent the same torus.

Definition 2.5.2. *A saddle connection, s, on a translation surface is a geodesic segment connecting one singularity to another (possibly the same) singularity such that no singularity lies in the interior of the segment.*

Saddle connections play an important role in the study of translation surfaces and continue to do so in the later chapters of this thesis.

2.6 Dynamics on translation surfaces

In this section, we briefly introduce the dynamics of the geodesic (or straight-line) flow on translation surfaces. We begin by providing some motivation for studying dynamics on translation surfaces which comes from the study of billiards (see [16] for a more detailed discussion). We then describe some dynamical growth results for translation surfaces and see how the growth differs from dynamical growth on negatively curved surfaces.

Consider a rational polygon (a Euclidean polygon whose interior angles are rational multiples of π). One can think of this polygon as a friction-less billiard table upon which we place a particle. We can then push the particle in a given direction and follow its trajectory as it moves around the table (Figure 2.6). When a particle hits the edge of a table, we assume an elastic collision where the angle of incidence is equal to the angle of reflection. Note that we ignore trajectories which eventually hit a corner because the direction of reflection is not well-defined.

It is natural to ask questions about the long term behaviour of trajectories on the table such as:

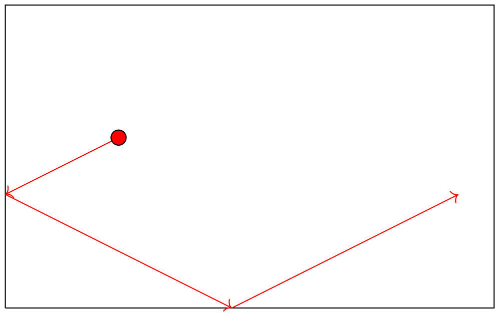


Figure 2.6: A rectangular billiard table with a particular trajectory drawn in red.

Q1) Does there exist a periodic orbit on the table?

Q2) What is the growth rate of periodic orbits of a bounded length, supposing they exist?

Q3) Do dense orbits exists on the table?

These problems are difficult for general billiard tables; in the case of non-rational polygons, it is not currently known if every triangle has a periodic orbit [20].

Studying these orbits proves to be difficult, in part due to the fact that the trajectory keeps changing direction. A fruitful method of answering these questions for rational polygonal billiards is to "unfold" them and their trajectories, so that the trajectories become straight-line flows on translation surfaces [53]. We will look at a particular example for illustrative purposes (Figure 2.7).

The unfolded tables along with the gluing described form a translation surface X which may have singularities around the vertices of the reflected tables. In this example, the translation surface has one singularity of cone-angle 6π (Figure 2.8). The straight-line flow on the unfolded tables is simply the geodesic flow on $X \setminus \Sigma$ with respect to the flat metric on the surface. This amounts to tracing out a straight-line in the direction of the flow. Note that we discard flows which hit a singularity in forwards/backwards time.

This unfolding process allows us to transfer some statements about the geodesic /straight-line flow on translation surfaces to statements about the billiard flow for rational billiard tables. This is because billiard trajectories can be unfolded and studied as geodesic/straight-line flows (and vice-versa).

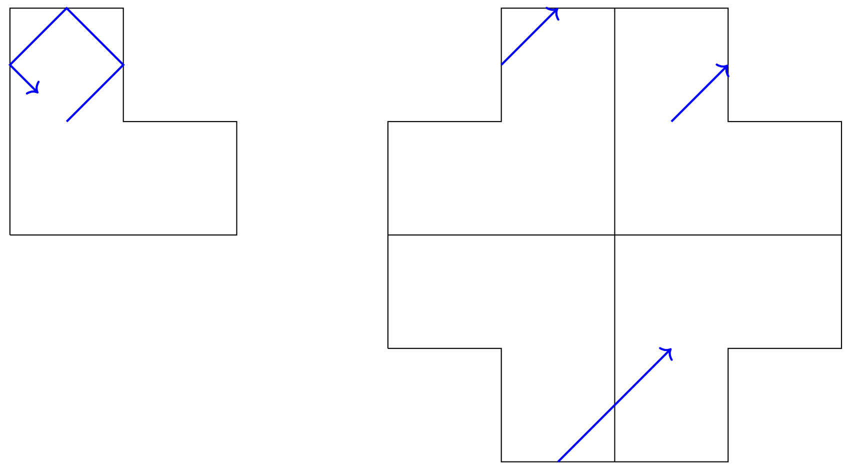


Figure 2.7: On the left we have an “L” shaped billiard table with a trajectory drawn in blue. On the right we have four unfolded copies of the original billiard table, with the unfolded trajectory in blue. The four tables are glued along the touching edges and the remaining edges are glued to their opposite edges.

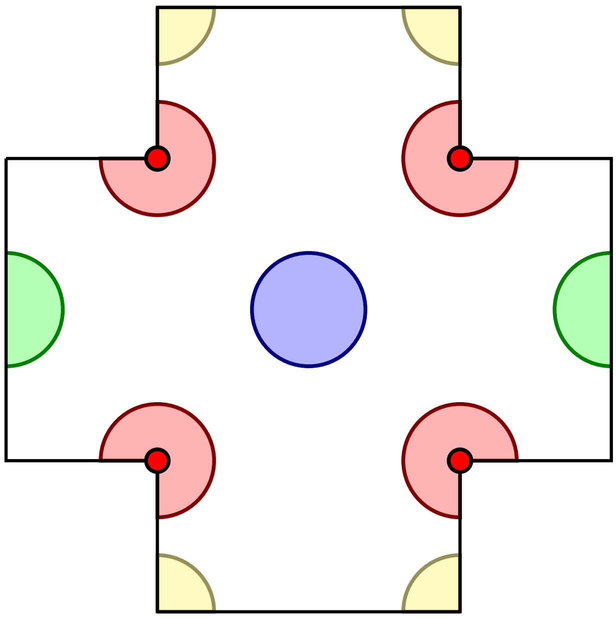


Figure 2.8: The unfolded billiard tables glue together to form a translation surface of genus 2 with one singularity (formed from the four central vertices marked in red). Every point on the surface, except at the singularity, has a neighbourhood which is isomorphic to a Euclidean ball. The singularity has a cone-angle of $4 \times 3\pi/2 = 6\pi$.

Q1 and Q2 above for billiard dynamics can be answered using this unfolding method and some results for counting cylinders on translation surfaces.

Definition 2.6.1. *A cylinder on a translation surface is an embedding of a Euclidean cylinder ($S^1 \times I$ for some interval I and S^1 is obtained from identifying the endpoints of some interval J). We insist that the boundary components of such a cylinder are composed of parallel saddle connections.*

Note that a periodic orbit on a rational billiard table unfolds to a periodic orbit on a translation surface. This periodic orbit will be contained within a cylinder and hence by translating the orbit perpendicular to the direction of the orbit, we obtain an uncountable family of orbits of the same length as the original. Hence, if we wish to count periodic orbits on the surface, it is natural to group such orbits together and count cylinders instead. Note that the length of the periodic orbits corresponding to a given cylinder will be the circumference of the cylinder.

The following result by Masur [36] proves that every translation surface contains a cylinder.

Theorem 2.6.2. *Let X be a translation surface. Then X contains an embedded cylinder.*

We can use the above result to show that there exists a periodic orbit on every rational billiard table (Q1 above): If we have a rational billiard table, we can unfold it to obtain a translation surface that contains an embedded cylinder, and so it has a periodic orbit that corresponds to a periodic orbit on the billiard table.

Similarly, Q2 is answered by the following theorem regarding the growth of cylinders on translation surfaces ([34] and [33]).

Theorem 2.6.3. *Let X be a translation surface of genus $g \geq 2$ and let $Cy(X, L)$ denote the number of cylinders on X of circumference less than or equal to L. Then there exist constants $0 < a < b < \infty$, only depending on g, such that*

$$aL^2 \leq Cy(X, L) \leq bL^2,$$

for L sufficiently large.

We will present another important growth result on translation surfaces which will be of use to us later. The growth result characterizes the growth of saddle connections on translation surfaces.

We denote the number of saddle connections (Definition 2.5.2) on a surface of length less than L by $\mathcal{S}(L)$. Again, from [34] and [33] we get a similar growth result to the one for cylinders.

Theorem 2.6.4. *Let X be a translation surface of genus g. Then there exist constants $0 < c < d < \infty$, only depending on g, such that*

$$cL^2 \leq \mathcal{S}(L) \leq dL^2,$$

for L sufficiently large.

The above growth results indicate that dynamical trajectories grow *quadratically* on translation surfaces. In the first few sections of this chapter, we saw that geometric objects (such as closed geodesics) grow exponentially on surfaces of negative curvature. For such surfaces, geodesic paths and trajectories of the geodesic flow coincide and hence dynamical trajectories on negatively curved surfaces grow exponentially.

The aim of this thesis is to explore the idea that *geometric* growth (see Definition 1.0.1) on translation surfaces behaves like geometric growth on surfaces of negative curvature.

2.7 Geometry of translation surfaces

The aim of this section is to describe the geometry of translation surfaces. We begin by looking at geodesics on translation surfaces and end by elaborating on the idea that translation surfaces with singularities are "negatively curved" from a geometrical point of view.

We begin by introducing the natural metric for translation surfaces. First note that any general path on a translation surface may pass through singularities. Hence any finite path p may be written as $p = p_1 \ldots p_n$, where the p_i are sub-paths whose interiors are contained in $X \backslash \Sigma$ and whose endpoints may be contained in Σ. The length of p, $\ell(p)$, is then the sum of the lengths of its sub-paths, i.e. $\ell(p) = \sum_{i=1}^{n} \ell(p_i)$. The length of each sub-path is obtained from the Euclidean structure of $X \backslash \Sigma$. We can then define a path metric on X by

$$d(x, y) = \inf_{\gamma} \ell(\gamma),$$

where $x, y \in X$ and the infimum is taken over all paths γ starting at x and ending at y (see [19]).

Geodesics on translation surfaces have a combinatorial structure which differs from the smooth structure of geodesics on Riemannian surfaces. We claim that geodesics on translation surfaces have the following structure (Figure 2.9).

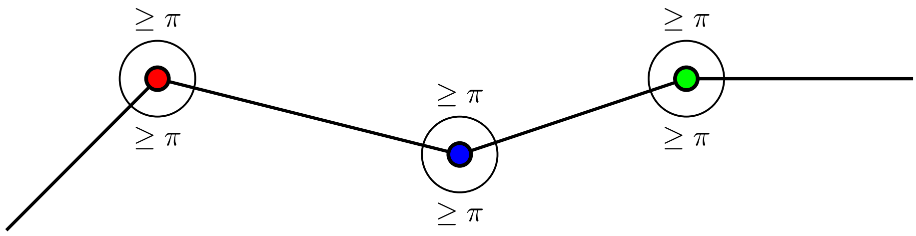


Figure 2.9: A representation of a geodesic on a translation surface.

The following result can be found in [5].

Proposition 2.7.1. *Geodesics on translation surfaces are concatenations of straight-line segments which may meet at singularities, such that the smallest angle between any two adjacent segments (as measured about the singularity) is greater than or equal to* π*.*

See [47] for a more detailed discussion of geodesics on translation surfaces. We will try to provide some intuition as to why geodesics have this structure.

To see that any geodesic will be a concatenation of straight-line segments (possibly including saddle connections), recall that any finite path can be decomposed into sub-paths contained in the Euclidean part of X, $X \backslash \Sigma$. Hence each of these sub-paths is locally distance minimising if and only if it is a straight-line segment.

The next thing to note is that not all straight-line segments meeting at a singularity form a geodesic. If the two straight-line segments form an angle of less than π at the singularity, the path formed from concatenating the two segments will not be locally distance minimising at the singularity in the same way that the path formed from two edges of a Euclidean triangle is not a geodesic in the Euclidean plane (Figure 2.10).

Next suppose that two segments which meet at a singularity x do not form an angle of less than π between them (see Figure 2.11). We claim that these two segments

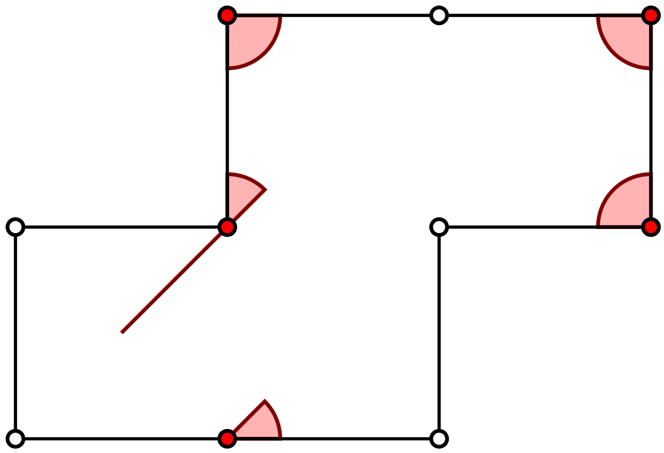


Figure 2.10: A translation surface obtained from gluing opposite sides of a polygon. This surface has two singularities represented by the filled and non-filled dots. The red line represents a straight-line segment meeting the filled singularity and the red sectors represent the directions in which a straight-line segment could be joined with the red segment to form a geodesic.

form a geodesic. To see this, suppose that they do not. Then because of the path metric, we could find a path γ connecting a point a which lies on one segment and is close to the singularity to a point b which lies on the other segment and is close to the singularity, such that $\ell(\gamma) < d(a,x)+d(x,b)$. Then because γ must be a straight-line segment which does not pass through x, it must leave some neighbourhood of the singularity via a straight-line. However, if we chose x and y to be sufficiently close to the singularity, $\ell(\gamma) > d(a,x) + d(x,b)$ which gives a contradiction.

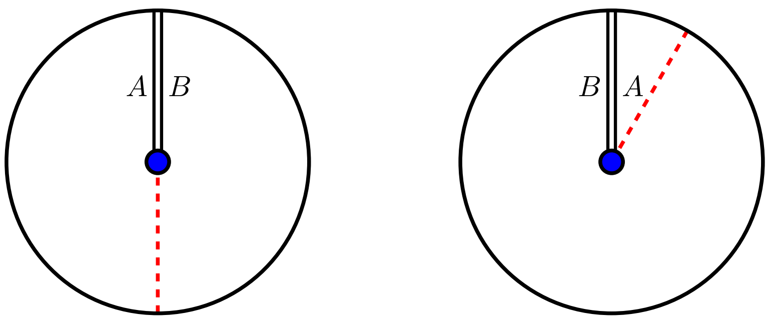


Figure 2.11: A local picture of a singularity of cone-angle 4π formed by gluing two disks along a slit. Two segments (dashed lines) meet the singularity at an angle of greater than or equal to π on both sides and hence form a geodesic.

We conclude by noting that geodesics on translation surfaces have the property that if they do pass through a singularity, then they are unique in their homotopy class (see [8]). This property is shared by negatively curved surfaces. Note that as we have seen in the previous section, closed geodesics/orbits which do not pass through singularities may have an infinite number of closed geodesics in their homotopy class (i.e. the other closed geodesics along the same cylinder). For this reason, when it comes to counting geodesics in later sections, we count geodesics up to homotopy.

2.7.1 Translation surfaces are "negatively curved"

The purpose of this section is to discuss the idea that, from a geometrical point of view, translation surfaces are in some sense similar to surfaces of non-positive curvature.

We begin by looking at the Gauss-Bonnet theorem (see [9]) which relates the curvature of a closed Riemannian surface to its underlying topology:

Theorem 2.7.2. *Let (S, g) be a closed Riemannian surface of genus $g \geq 0$ with Gaussian curvature denoted by K. Then*

$$-2\pi(2g-2) = \int_S K dA.$$

Essentially, the Gauss-Bonnet theorem states that the average curvature of the surface is constrained by the topology of the surface; in particular, the higher the genus, the more negative the average curvature will be.

A similar theorem holds for translation surfaces which can be obtained by considering a triangulation of the surface.

Theorem 2.7.3. *Fix $g \geq 2$ and let X be a translation surface with singularities $\Sigma = \{x_1, \ldots, x_n\}$, where x_i has cone-angle $2\pi(d_i + 1)$. Then*

$$-2\pi(2g-2) = \sum_{x_i \in \Sigma} -2\pi(d_i).$$

By considering these theorems side-by-side, they suggest that, heuristically, one can think of the singularities on translation surfaces as behaving like points of concentrated negative curvature, where the larger the cone-angle of a singularity, the higher the concentration of negative curvature at that particular singularity.

As mentioned in the introduction, the intuitive observation that singularities may be thought of as points of concentrated negative curvature inspires the following question: do translation surfaces behave like surfaces of negative curvature? In the next two sections we will look at existing work which explores the idea that translation surfaces behave like negatively curved surfaces when it comes to geometric growth.

2.8 Volume entropy for translation surfaces

Recall from Section 2.1, that for a negatively curved surface, volume entropy is a positive number which quantifies the exponential growth rate of various geometric objects on the manifold such as the growth of the volume of a ball as its radius goes to infinity.

The aim of this section is to explore an analogous notion of volume entropy for translation surfaces. This work has already been partially carried out by Dankwart (see [8]) and so we will briefly discuss the definition and some of its properties.

The original definition of volume entropy that Dankwart used focused on orbital counting rather than the types of geometric objects we discussed for Riemannian manifolds. We will define volume entropy in terms of the growth of geometric objects on the surface.

Let X be a translation surface with universal cover $\widetilde{X}$ that inherits a metric from the flat metric defined on X. We define the following counting functions for X inspired by those introduced for Riemannian surfaces:

- $V(\mathcal{B}(\widetilde{x}, R))$, the volume of a ball of radius R centered at $\widetilde{x} \in \widetilde{X}$;
- $\ell(\mathcal{C}(\widetilde{x}, R))$, the circumference of a ball of radius R centered at some point $\widetilde{x} \in \widetilde{X}$;
- $C(R)$, the number of closed geodesics on X of length less than or equal to R; and
- $N_{x,y}(R)$, the number of geodesics from x to y on X of length less than or equal to R.

By analogy with the definition of volume entropy for Riemannian manifolds (Definition 2.1.1) we can consider the rate of growth of balls in the universal cover $\widetilde{X}$.

Definition 2.8.1. *Let X be a translation surface. Fix $\widetilde{x} \in \widetilde{X}$. We define the volume entropy of X to be*

$$h = h(X) := \limsup_{R \to \infty} \frac{1}{R} \log V(\mathcal{B}(\widetilde{x}, R)).$$

As in the case of the definitions of volume entropy for Riemannian manifolds and finite metric graphs, h is independent of the choice of $\widetilde{x}$. This follows from the fact that any two points on X can be joined by a geodesic of length less than or equal to the diameter of X, the triangle inequality and the definition of volume entropy. For convenience, we will often assume that $\widetilde{x}$ is the lift of a singularity $x \in \Sigma$. The asymptotic formulae deduced in Chapter 4 imply that the limsup in the definition is in fact a limit, but perhaps it is also simple to deduce using a subadditivity argument.

The structure of geodesics on X and the structure of $\widetilde{X}$ itself, allow us to interpret the volume of a ball in $\widetilde{X}$ in terms of X. This interpretation will prove fruitful when proving results on the growth of objects on translation surfaces.

First we introduce some notation and definitions relating geodesics on translation surfaces to saddle connections.

Let $\mathcal{S}$ denote the set of oriented saddle connections on a surface X, partially-ordered by non-decreasing length.

Definition 2.8.2. *We define a saddle connection path $p = s_1, \ldots, s_n$ to be a finite string of oriented saddle collections $s_1, \ldots, s_n$ which form a geodesic path.*

We denote by $\ell(p) = \ell(s_1) + \ell(s_2) + \cdots + \ell(s_n)$ the sum of the lengths of the constituent saddle connections. Let $i(p), t(p) \in \Sigma$ denote the initial and terminal singularities, respectively, of the saddle connection path p.

Let $x \in \Sigma$ be a singularity, then we define

$$\mathcal{P}(x, R) := \{p : i(p) = x \text{ and } l(p) \leq R\}$$

to be the number of saddle connection paths starting at x of length less than or equal to R. We define $\mathcal{P}(x) = \bigcup_{R>0} \mathcal{P}(x, R)$ to be the set of all saddle connection paths starting at x.

Note that we obtain an additional counting function $N(x, R) := \#\mathcal{P}(x, R)$ similar the one we focused on for finite metric graphs.

We can now give an explicit formula for the volume of a ball on $\widetilde{X}$ in terms of saddle connection paths on X. See Figure 2.12 for an illustration.

Lemma 2.8.3. *Let X be a translation surface and fix a singularity $x \in \Sigma$ with lift $\widetilde{x} \in \widetilde{X}$ and let $2\pi(k(x)+1)$ be the cone angle of x. Then for $R > 0$,*

$$V(\mathcal{B}(\widetilde{x}, R)) = (k(x)+1)\pi R^2 + \sum_{p \in \mathcal{P}(x,R)} k(t(p))\pi(R - \ell(p))^2,$$

where the singularity at the end of path p has cone angle $2\pi(k(t(p))+1)$.

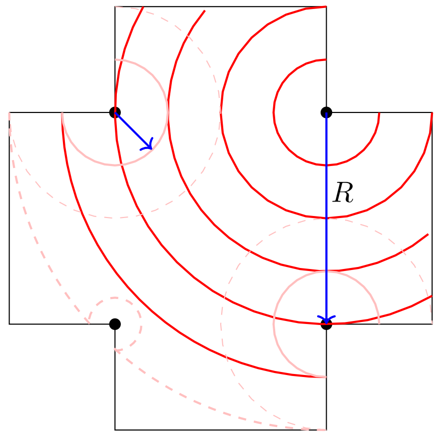
Proof. The ball $\mathcal{B}(\widetilde{x}, R)$ consists of end points of geodesics which begin at $\widetilde{x}$ and are of length less than or equal to R. The set of such geodesics is in one-to-one correspondence with the set of geodesics on X which begin at $x \in \Sigma$ of length less than or equal to R. The volume contributed by the geodesics starting from x (or equivalently $\widetilde{x}$) which do not pass through a singularity is given by $(k(x)+1)\pi R^2$, where $2\pi(k(x)+1)$ is the cone angle at x. A geodesic that begins at x and passes through a singularity will be a concatenation of a path $p \in \mathcal{P}(x, R)$ and a straight-line segment that satisfies the geodesic angle condition with p (see Proposition 2.7.1). Hence the contribution of volume from such geodesics corresponding to a saddle connection path $p \in \mathcal{P}(x, R)$, is given by $k(t(p))\pi(R - (\ell(p))^2$. $\square$

We conclude this section by noting an alternative characterization of volume entropy for translation surfaces which uses generating functions. We first note that we can replace $V(\mathcal{B}(\widetilde{x}, R))$ in the definition of volume entropy for translation surfaces with any other geometric counting function defined at the beginning of this section and obtain the constant h. In particular, we can replace it with $N(x, R) := \#\mathcal{P}(x, R)$.

Proposition 2.8.4. *Let X be a translation surface and fix $x \in X$. Then the volume entropy of X is given by*

$$h(X) = \inf_{t>0} \left\{ t : \sum_{p \in \mathcal{P}(x)} e^{-t\ell(p)} < \infty \right\}.$$

A proof for the equivalence for the two notions of volume entropy follows from the proof of a similar statement in [27] for the case of finite metric graphs. We will provide a sketch of the proof for completeness.



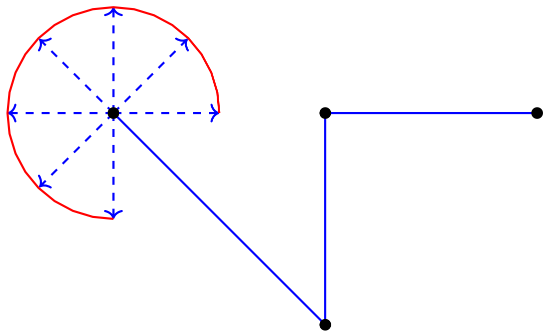


Figure 2.12: (i) The radii of $\mathcal{B}(\widetilde{x}, R)$ are concatenations of saddle connections followed by a radial line segment from a singularity; (ii) A heuristic figure illustrating that the boundary of $\mathcal{B}(\widetilde{x}, R)$ will consist of the union of circular arcs centred on singularities reached via concatenations of saddle connections

Proof. Let h' be the infimal value of t for which $\sum_{p\in\mathcal{P}(x)} e^{-t\ell(p)}$ converges. Then for $t > h$,

$$\sum_{p\in\mathcal{P}(x)} e^{-t\ell(p)} = \sum_{n=0}^{\infty} \sum_{p\in\mathcal{P}(x):n\leq\ell(p)\leq n+1} e^{-t\ell(p)} \leq \sum_{n=0}^{\infty} N(x, n+1)e^{-tn} = \sum_{n=0}^{\infty} e^{(h-t+o(1))n}.$$

It follows that $\sum_{p\in\mathcal{P}(x)} e^{-t\ell(p)}$ converges when $t > h$, hence $h \geq h'$.

Suppose that $h > h'$. Then because the set of u for which

$$\sum_{p\in\mathcal{P}(x)} e^{-u\ell(p)} < \infty$$

is an interval, we can choose some $t \in (h', h)$ such that

$$\sum_{p \in \mathcal{P}(x)} e^{-t\ell(p)} < \infty.$$

Then for $R > 0$,

$$e^{tR} \sum_{p \in \mathcal{P}(x)} e^{-t\ell(p)} = \sum_{p \in \mathcal{P}(x)} e^{t(R-\ell(p))} \geq \sum_{p \in \mathcal{P}(x,R)} e^{t(R-\ell(p))}.$$

For $\ell(p) < R$, we have $e^{t(R-\ell(p))} > 1$, hence $e^{tR} \sum_{p \in \mathcal{P}(x)} e^{-t\ell(p)} \geq N(x, R)$. Taking the logarithm of both sides and letting R tend to infinity, we obtain $t \geq h$, which gives a contradiction. Hence $h = h'$. □

2.9 Exponential growth of geometric objects on translation surfaces

We will now look at existing work for geometric growth on translation surfaces.

The study of geometric growth on translation surfaces seems to have been initiated by Dankwart in [8]. Part of his work involved studying volume entropy for translation surfaces. A result of his that forms the foundation for our later work is that translation surfaces do have positive volume entropy and so behave like negatively curved surfaces in this respect.

We will briefly sketch an argument for why volume entropy for translation surfaces is positive, then give a more intuitive picture afterwards.

We first need to introduce the following result from Dankwart [8].

Lemma 2.9.1. *Let X be a translation surface. If $s, s' \in \mathcal{S}$ are oriented saddle connections then there exists a saddle connection path which starts with s and ends with s'.*

Proposition 2.9.2. *Let X be a translation surface. Then $h(X) > 0$.*

Proof. Let s be any oriented saddle connection on X. By Lemma 2.9.1 and the fact that there exist at least two other saddle connections on X, we can find two distinct closed saddle connection paths p_1 and p_2 which pass through s.

Because p_1 and p_2 both pass through s, by concatenation we can generate a family of geodesic paths which correspond to words in $\langle p_1, p_2 \rangle$.

Let $\ell = \max\{\ell(p_1), \ell(p_2)\}$. Then $\mathcal{P}(i(s), R) \geq 2^{\lfloor R/\ell \rfloor}$. Hence

$$h(X) = \limsup_{R\to\infty} \frac{\log \mathcal{P}(i(s), R)}{R} \geq \limsup_{R\to\infty} \frac{\log 2^{\lfloor R/\ell \rfloor}}{R} = \frac{\log 2}{\ell} > 0.$$

□

Perhaps a more intuitive way to get a sense for why the volume entropy is positive for translation surfaces is to consider what happens to the set of geodesic paths based at a point $x \in \Sigma$, of length less than or equal to R, as R goes to infinity (Figure 2.12).

As the boundary of a ball increases, it passes through singularities, creating sectors of shorter radii to start growing about the singularity which correspond to geodesics which pass through that singularity. Then as the radius of the ball further increases, each of these sectors pass through more singularities, forming new sectors. This compounding effect of sectors creating new sectors is where the exponential growth comes from.

Chapter 3

Asymptotic growth on infinite graphs

3.1 Introduction

In this chapter, we prove asymptotic formulae for the growth of paths on infinite metric graphs which satisfy some appropriate conditions. We first will show that for such a graph $\mathcal{G}$, given a vertex $x \in \mathcal{V}(\mathcal{G})$, the number of non-backtracking paths starting at x of length less R, is asymptotic to $(C/h(\mathcal{G}))e^{h(\mathcal{G})R}$ as R goes to infinity, where $h(\mathcal{G})$ is the volume entropy of $\mathcal{G}$ and $C > 0$. We will then prove a similar asymptotic formula for the growth of closed paths on $\mathcal{G}$.

The method of proof we develop in this chapter will later be used to prove asymptotic results for translation surfaces. We develop the method in the context of infinite graphs, partly because infinite graphs are less complex than translation surfaces and partly because infinite graphs are interesting in their own right.

The proof follows the lines of the classical proof of the prime number theorem. In particular, the proof is based on the use of the Ikehara-Wiener Tauberian theorem (see Section 2.2.2). The theorem states that if a particular complex function $\eta_{\mathcal{G}}(z)$ associated to an infinite graph $\mathcal{G}$ satisfies certain properties, then the asymptotic formula for the growth of paths holds. We show that these properties hold for $\eta_{\mathcal{G}}(z)$ by rewriting the function in terms of infinite matrices which contain information about the poles of $\eta_{\mathcal{G}}(z)$. In the special case of finite graphs, the asymptotic result could be easily deduced using ideas in [42] for finite matrices.

3.2 Background

In this section, we will introduce the types of infinite graphs we will be working with, along with some basic definitions.

Let $\mathcal{G}$ be a non-empty strongly connected oriented graph (i.e., every point can be reached by any other point by following a path that respects the orientation on the paths edges). Let $\mathcal{V} = \mathcal{V}(\mathcal{G})$ and $\mathcal{E} = \mathcal{E}(\mathcal{G})$ be the vertex and oriented edge sets of $\mathcal{G}$, respectively. We insist that the cardinality of $\mathcal{V}$ be finite and that the cardinality of $\mathcal{E}$ be countably infinite. For every edge e, let $i(e)$ and $t(e)$ denote the initial and the terminal vertex of e, respectively. We introduce a length function $\ell : \mathcal{E} \to \mathbb{R}$ which assigns a positive real number $\ell(e)$ to each edge $e \in \mathcal{E}$. Furthermore, we will need to partially-order $\mathcal{E}$ by non-decreasing lengths with respect to ℓ.

Example 3.2.1 (Infinite Rose)**.** *A graph $\mathcal{G}$ formed from one vertex and a countably infinite number of edges (Figure 3.2.1).*

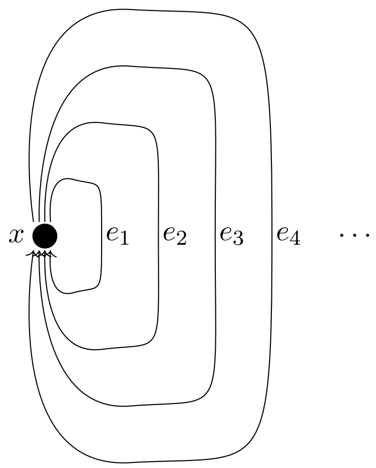


Figure 3.1: A single vertex $\mathcal{V} = \{x\}$ and infinitely many edges $\mathcal{E} = \{e_n\}_{n=1}^{\infty}$.

A *path* in $\mathcal{G}$ corresponds to a sequence of edges $p = e_1 \dots e_n$ for which $t(e_j) = i(e_{j+1})$, for $1 \leq j < n$. We denote its length by $\ell(p) = \sum_{j=1}^{n} \ell(e_j)$.

Let $\mathcal{P}_{\mathcal{G}}(x, R) := \{p = e_1 \dots e_n : i(e_1) = x, \ell(p) \leq R\}$ denote the set of paths on $\mathcal{G}$ of length at most R starting at $x \in \mathcal{V}(\mathcal{G})$. Let $\mathcal{P}_{\mathcal{G}}(x) = \bigcup_{R>0} \mathcal{P}_{\mathcal{G}}(x, R)$ denote the set of all paths in $\mathcal{G}$ starting at x. We denote the cardinality of $\mathcal{P}_{\mathcal{G}}(x, R)$ by $N_{\mathcal{G}}(x, R) = \#\mathcal{P}_{\mathcal{G}}(x, R)$.

Definition 3.2.2. *We define the volume entropy of* $(\mathcal{G}, \ell, x)$ *to be*

$$h = h(\mathcal{G}, \ell) = \limsup_{R\to\infty} \frac{1}{R} \log N_{\mathcal{G}}(x, R).$$

Note that because $\mathcal{G}$ is connected, h is independent of the base-point x.

We will often make use of the following characterization of volume entropy in terms of generating functions (see [27] or Proposition 2.8.4).

Proposition 3.2.3. *The volume entropy of* $(\mathcal{G}, \ell, x)$ *is given by*

$$h = \inf_{\sigma>0} \left\{ \sigma : \sum_{p\in\mathcal{P}_{\mathcal{G}}(x,R)} e^{-\sigma\ell(p)} < \infty \right\}.$$

We will impose further constraints on the infinite graphs $(\mathcal{G}, \ell)$. The first is that the length function $\ell : \mathcal{E} \to \mathbb{R}$ must grow sufficiently fast; this is required in order for $h(\mathcal{G}, \ell, x)$ to be finite. To see why we need this condition, consider the graph $\mathcal{G}$ in Example 6.4.1 which has a single vertex and an infinite number of edges. If the lengths of the edges of $\mathcal{G}$ have a finite upper bound then for R sufficiently large, $N_{\mathcal{G}}(x, R) = \infty$ and thus $h(\mathcal{G}, \ell, x) = \infty$.

Other constraints are required for the asymptotic result to hold. We summarize the graph properties we require under the following hypotheses.

Graph Hypotheses. Henceforth, we shall consider graphs with finite vertex set $\mathcal{V}$ and a countable oriented edge set $\mathcal{E}$. Furthermore, we require that $\mathcal{E}$ and the associated length function satisfy the following properties:

(H1) For all $\sigma > 0$ we have $\sum_{e\in\mathcal{E}} e^{-\sigma\ell(e)} < \infty$;

(H2) There exists some $L > 0$ such that for all pairs of directed edges $e, e' \in \mathcal{E}$, there exists a path in $\mathcal{G}$ which starts with e and ends with e' and is of length less than or equal to $\ell(e) + L + \ell(e')$; and

(H3) There does not exist a $d > 0$ such that

$$\{\ell(c) : c \text{ is a closed path}\} \subset d\mathbb{N}.$$

Under the above hypotheses, the volume entropy $h = h(\mathcal{G}, \ell, x)$ does not depend on the choice of base point x.

Lemma 3.2.4. *If the graph $(\mathcal{G}, \ell)$ satisfies (H1) and (H2) then $0 < h < \infty$.*

Proof. By assumption (H2) and the pigeonhole principle applied to $\mathcal{V}$, there exists a path p connecting the base point x to some vertex v and two closed paths, c_1 and c_2, which both pass through v (Figure 3.2). By considering all possible concatenations of these closed paths it is clear that $N_{\mathcal{G}}(x, R + \ell(p)) \geq 2^{\lfloor R/b \rfloor}$ for all $R > 0$, where $b = \max\{\ell(c_1), \ell(c_2)\}$. Hence $h \geq \frac{\log 2}{b} > 0$.

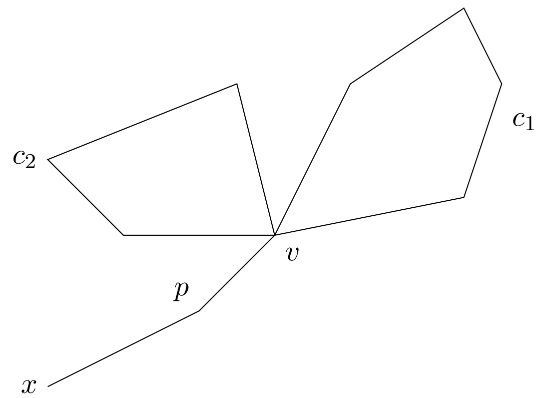


Figure 3.2: A path p connecting vertex x to v and two closed loops c_1 and c_2 which meet v. Note that the edges in c_1 and c_2 do not need to have distinct edges.

To see that h is finite, recall that we can characterize h as follows (see Proposition 3.2.3)

$$h = \inf_{\sigma > 0}\{\sigma : \sum_{p \in \mathcal{P}_{\mathcal{G}}(x)} e^{-\sigma \ell(p)} < \infty\}.$$

We can formally write

$$\sum_{p \in \mathcal{P}_{\mathcal{G}}(x,R)} e^{-\sigma \ell(p)} \leq \sum_{n=1}^{\infty} \left(\sum_{e \in \mathcal{E}} e^{-\sigma \ell(e)} \right)^n, \tag{1}$$

for $\sigma > 0$, where the right-hand side of the above equation involves all possible finite sums of edge lengths. Using (H1) one can see that for $\sigma = \sigma_0$ sufficiently large $\sum_{e \in \mathcal{E}} e^{-\sigma \ell(e)} < 1$ and thus the sums in (1) converge for $\sigma = \sigma_0$. Hence $h \leq \sigma_0 < \infty$. $\square$

The main results of this chapter are the following asymptotic formulae. First we have an asymptotic formula for $N_{\mathcal{G}}(x, R)$.

Theorem 3.2.5. *If the graph $(\mathcal{G}, \ell)$ satisfies (H1),(H2) and (H3), then there exists a constant $C > 0$ such that $N_{\mathcal{G}}(x, R) \sim (C/h)e^{hR}$, i.e.,*

$$\lim_{R\to\infty} \frac{N_{\mathcal{G}}(x, R)}{e^{hR}} = C/h.$$

Remark 3.2.6. Without hypothesis (H3) this theorem may not hold. For example, in the case of finite graphs, if we consider a metric graph $(\mathcal{G}, \ell)$ with a single vertex and two edges of length 1, then $N_{\mathcal{G}}(x, R) = 2^{\lfloor R \rfloor}$ for all $R > 0$. In this case, the limit in Theorem 2.4 does not converge.

Next, we have an asymptotic formula for the growth of closed paths on $\mathcal{G}$. A closed path on $\mathcal{G}$ is a path p such that $t(p) = i(p)$. We say a closed path is primitive if it is not multiple concatenations of another closed path. Let $C_{\mathcal{G}}(R)$ denote the number of primitive closed paths of length less than or equal to R.

Theorem 3.2.7. *If the graph $(\mathcal{G}, \ell)$ satisfies (H1),(H2) and (H3), then*

$$C_{\mathcal{G}}(R) \sim \frac{e^{hR}}{hR}.$$

The proofs will be based on the application of the Ikehara-Wiener Tauberian theorem which we rewrite here for the reader's convenience.

Ikehara-Wiener Tauberian Theorem. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing and right-continuous function. Formally denote $\eta(z) := \int_0^\infty e^{-zR} d\rho(R)$, for $z \in \mathbb{C}$. Then suppose that $\eta(z)$ has the following properties:

1. there exists some $a > 0$ such that $\eta(z)$ converges absolutely and is analytic on $Re(z) > a$;
2. $\eta(z)$ has a meromorphic extension to a neighbourhood of the half-plane $Re(z) \geq a$;
3. a is a simple pole for $\eta(z)$, i.e., $C = \lim_{z\to a}(z - a)\eta(z)$ exists and is positive; and
4. the extension of $\eta(z)$ has no poles on the line $Re(z) = a$ other than a.

Then $\rho(R) \sim (C/a)e^{aR}$ as $R \to \infty$.

In order to apply the Tauberian theorem to the non-negative, right-continuous and non-decreasing counting function $N_{\mathcal{G}}(x, R)$, we will need to study the complex function $\eta_{\mathcal{G}}(z)$ obtained from taking the Laplace transform of $N_{\mathcal{G}}(x, R)$. In the next section we will see how this function can be written as a generating function involving exponential weightings on the paths in $\mathcal{P}_{\mathcal{G}}(x,R)$. We will then rewrite the aforementioned complex function in terms of countable matrices which allows us to construct a meromorphic extension to $\eta_{\mathcal{G}}(z)$. Finally, we will use the hypotheses (H1)–(H3) to deduce that $\eta_{\mathcal{G}}(z)$ has the required properties that allow us to apply the Tauberian theorem.

3.3 The Laplace transform of $N_{\mathcal{G}}(x,R)$, $\eta_{\mathcal{G}}(z)$

In this section, we introduce a complex function whose analytic properties will allow us to derive the asymptotic formula for $N_{\mathcal{G}}(x,R)$ using the Tauberian theorem.

Definition 3.3.1. *We can formally define the following complex function using a Riemann-Stieltjes integral*

$$\eta_{\mathcal{G}}(z) = \eta_{\mathcal{G},x}(z) = \int_0^\infty e^{-zR} dN_{\mathcal{G}}(x,R), \quad z \in \mathbb{C}.$$

For $Re(z) > h$ we can rewrite $\eta_{\mathcal{G}}(z)$ in terms of the path lengths of $\mathcal{G}$ by using integration-by-parts

$$\begin{aligned}
\eta_{\mathcal{G}}(z) &= \int_0^\infty e^{-zR} dN_{\mathcal{G}}(x,R) \\
&= [e^{-zR} N_{\mathcal{G}}(x,R)]_0^\infty + z \int_0^\infty e^{-zR} N_{\mathcal{G}}(x,R) dR \\
&= \lim_{T\to\infty} e^{-zT} N_{\mathcal{G}}(x,T) + z \int_0^\infty \left(e^{-zR} \sum_{p \in \mathcal{P}_{\mathcal{G}}(x,R)} 1 \right) dR \\
&= z \sum_{p\in\mathcal{P}_{\mathcal{G}}(x)} \int_{\ell(p)}^\infty e^{-zR} dR \\
&= \sum_{p\in\mathcal{P}_{\mathcal{G}}(x)} e^{-z\ell(p)},
\end{aligned}$$

where the limit tends to 0 for $Re(z) > h$.

From the above derivation and Proposition 3.2.3, we observe that $\eta_{\mathcal{G}}(z)$ converges to an analytic function for $Re(z) > h$.

In order to deduce information about the poles of $\eta_{\mathcal{G}}(z)$ it will prove useful to rewrite $\sum_{p\in\mathcal{P}_{\mathcal{G}}(x)} e^{-z\ell(p)}$ in terms of countable matrices which contain the weighted path length data associated to $(\mathcal{G},\ell)$.

3.4 Countable Matrices

In this section, we will introduce families of countable matrices which contain information about the edge lengths of $\mathcal{E} = \mathcal{E}(\mathcal{G})$, as well as information about which pairs of edges form paths in $\mathcal{G}$. This allows us to construct the sum of exponentially weighted path lengths associated to $\mathcal{G}$ by considering powers of these matrices.

Let us order the oriented edge set $\mathcal{E} = (e_a)_{a\in\mathbb{N}}$ by non-decreasing length and write $\ell(a) := \ell(e_a)$, $a \in \mathbb{N}$.

Definition 3.4.1. *For $z \in \mathbb{C}$, we can associate to $\mathcal{G}$ the infinite matrix $M_z(a, b)$ defined by*

$$M_z(a,b) = \begin{cases} e^{-z\ell(b)} & \text{if } t(a) = i(b), \\ 0 & \text{otherwise.} \end{cases}$$

where the rows and columns are indexed by the oriented edges, partially ordered by their lengths.

Note that the path length data for $\mathcal{G}$ can be retrieved from these matrices in the following way. Let $\mathcal{P}_{\mathcal{G}}(n, a, b)$ denote the set of oriented paths p in $\mathcal{G}$ consisting of n edges, whose final edge is b, and form a path ap with a. It then follows from formal matrix multiplication that for any $n \geq 1$, we can write the $(a, b)^{th}$ entry of the n^{th} power of the matrix M_z as:

$$M_z^n(a,b) = \sum_{p\in\mathcal{P}_{\mathcal{G}}(n,a,b)} e^{-z\ell(p)}, \tag{2}$$

which will be finite for $Re(z) > 0$ by hypothesis (H1).

3.5 Countable matrices to operators

In order to deduce properties of $\eta_{\mathcal{G}}(z)$ using M_z, we will need to consider M_z as a bounded linear operator acting on $\ell^\infty(\mathbb{C})$. More generally, given an infinite matrix $L = (L(a, b))_{a,b=1}^\infty$ with $\sup_a \sum_b |L(a, b)| < \infty$, we can associate to L a bounded linear operator $\widehat{L} : \ell^\infty(\mathbb{C}) \to \ell^\infty(\mathbb{C})$ by

$$\widehat{L}(\underline{u}) = \Big(\sum_{b=1}^{\infty} L(a,b)u_b\Big)_{a=1}^{\infty} \text{ where } \underline{u} = (u_b)_{b=1}^\infty \in \ell^\infty(\mathbb{C}).$$

We denote the operator norm on the Banach space of bounded linear operators on $\ell^\infty(\mathbb{C})$ by $\|\cdot\|$.

By hypothesis (H1), when $Re(z) > 0$ we can associate to M_z a bounded linear operator $\widehat{M_z} : \ell^\infty(\mathbb{C}) \to \ell^\infty(\mathbb{C})$ defined by

$$\widehat{M_z}(\underline{u}) = \Big(\sum_{b=1}^{\infty} M_z(a,b)u_b\Big)_{a=1}^{\infty}.$$

We can now proceed to formally rewrite $\eta_{\mathcal{G}}(z)$ in terms of the operators $\widehat{M_z}$.

For $Re(z) > 0$, we define:

(a) $\underline{w}(z) = (\chi_{\mathcal{E}_x}(e_j)e^{-z\ell(j)})_{j=1}^{\infty} \in \ell^1(\mathbb{C})$ where $\chi_{\mathcal{E}_x}$ denotes the characteristic function of the set $\mathcal{E}_x = \{e \in \mathcal{E} : i(e) = x\}$ of edges whose initial vertex is x; and

(b) $\underline{1} = (1)_{j=1}^{\infty} \in \ell^\infty(\mathbb{C})$ is the vector all of whose entries are equal to 1.

Observe that for $Re(z) > 0$ we have $\underline{w}(z) \in \ell^1(\mathbb{C})$ by (H1).

Using Equation (2), we can formally rewrite $\eta_{\mathcal{G}}(z)$ as

$$\eta_{\mathcal{G}}(z) = \sum_{p \in \mathcal{P}_{\mathcal{G}}(x)} e^{-z\ell(p)} = \underline{w}(z) \cdot \Big(\sum_{n=0}^{\infty} \widehat{M_z}^n\Big)\underline{1}, \tag{3}$$

where $w \cdot v = \sum_{j=1}^{\infty} w_j v_j$ for $w \in \ell^1(\mathbb{C})$ and $v \in \ell^\infty(\mathbb{C})$.

3.6 $\eta_{\mathcal{G}}(z)$ converges absolutely and is analytic on $Re(z) > h$

In order to proceed, we would like to study the holomorphicity of the function $z \mapsto \widehat{M_z}$. We begin by recalling some basic definitions and properties of holomorphic functions from the complex plane to Banach spaces. Let $\mathcal{B}$ denote a Banach space with norm $\|\cdot\|$.

Definition 3.6.1. *A function $f : U \subset \mathbb{C} \to \mathcal{B}$ is holomorphic of for every $z_0 \in U$, there exists a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n,$$

with $\alpha_n \in \mathcal{B}$ for all $n \geq 0$, which is convergent in the sense that

$$\sum_{n=0} \|\alpha_n\| |z - z_0|^n < \infty$$

for $|z - z_0|$ sufficiently small.

We recall some standard facts about holomorphic functions from the complex plane to Banach spaces (see [41] for a reference).

Proposition 3.6.2. *Let $U \subset \mathbb{C}$ be open.*

1. *Suppose that $f_n : U \to \mathcal{B}$ is holomorphic for $n \in \mathbb{N}$. Furthermore, suppose that $f_n \to f$ to some $f : U \to \mathcal{B}$ in the sense of uniform convergence on U. Then f is also holomorphic. As a corollary, if $g_n : U \to \mathcal{B}$ are holomorphic and $\sum_{n=1}^{\infty} \sup_{z \in U} \|g_n(z)\| < \infty$, then $\sum_{n=1}^{\infty} g_n$ is holomorphic on U.*

2. *If $f : U \to \mathcal{B}$ is a holomorphic and $\ell : \mathcal{B} \to \mathbb{C}$ is a continuous linear functional, then $\ell \circ f : U \to \mathbb{C}$ is holomorphic in the usual sense.*

3. *Let X_1, X_2 be Banach spaces and define $L(X_1, X_2)$ to be the Banach space of bounded linear operators from X_1 to X_2, equipped with the operator norm topology. Let X, Y, Z be Banach spaces. If $f : U \to L(X,Y)$ and $g : U \to L(Y,Z)$ are holomorphic, then the function $z \mapsto f(z) \circ g(z)$ is also a holomorphic function from U to $L(X,Z)$.*

Lemma 3.6.3. *The function $z \mapsto \widehat{M_z}$ is holomorphic from the half-plane $Re(z) > 0$ to the Banach space of bounded linear operators on $\ell^\infty(\mathbb{C})$.*

Proof. Fix $z_0 \in \mathbb{C}$ such that $Re(z_0) > 0$. Let U denote the open ball of radius $Re(z_0)$ centred at z_0. For $n \geq 0$ we define the countable matrices

$$M_{z_0}^{(n)}(a,b) = \begin{cases} \frac{(-\ell(b))^n}{n!} e^{-z_0 \ell(b)} & \text{if } t(a) = i(b), \\ 0 & \text{otherwise.} \end{cases}$$

Let $a, b \in \mathcal{S}$. By considering the power series expansion of $e^{-z\ell(b)}$ at z_0, for $z \in \mathbb{C}$, we can write

$$M_z(a,b) = \sum_{n=0}^{\infty} M_{z_0}^{(n)}(a,b)(z - z_0)^n.$$

We will show that for $z \in U$,

$$\sum_{n=0}^{\infty} \|\widehat{M_{z_0}^{(n)}}\| |z - z_0|^n < \infty.$$

Because $z_0 \in \mathbb{C}$ with $Re(z_0) > 0$ was arbitrary, by Definition 3.6.1, the result will follow.

Let $z \in U$. Then

$$\sum_{n=0}^{\infty} \|\widehat{M_{z_0}^{(n)}}\| |z - z_0|^n \leq \sum_{n=0}^{\infty} \sum_{s \in \mathcal{S}} \left| \frac{(-\ell(s))^n}{n!} e^{-z_0 \ell(s)} \right| |z - z_0|^n.$$

Because the terms in the sum on the right-hand side are non-negative, the above sum converges if and only if the following sum converges

$$\begin{aligned}
\sum_{s \in \mathcal{S}} \sum_{n=0}^{\infty} \left| \frac{(-\ell(s))^n}{n!} e^{-z_0 \ell(s)} \right| |z - z_0|^n &= \sum_{s \in \mathcal{S}} e^{-Re(z_0)\ell(s)} \sum_{n=0}^{\infty} \left| \frac{(-\ell(s)(z - z_0))^n}{n!} \right| \\
&= \sum_{s \in \mathcal{S}} e^{-Re(z_0)\ell(s)} e^{|\ell(s)(z - z_0)|} \\
&= \sum_{s \in \mathcal{S}} e^{(-Re(z_0) + |(z - z_0)|)\ell(s)}.
\end{aligned}$$

By hypothesis (H1), the above sum converges if $|z - z_0| < Re(z_0)$ and so we are done. $\square$

Lemma 3.6.4. *The function $z \mapsto \sum_{n=0}^{\infty} \widehat{M_z}^n = (I - \widehat{M_z})^{-1}$ exists and is holomorphic on the half-plane $Re(z) > h$ to the Banach space of bounded linear operators on $\ell^\infty(\mathbb{C})$.*

Proof. Recall that hypothesis (H2) states that there exists some $L > 0$ such that for all pairs of directed edges $e, e' \in \mathcal{E}$, there exists a path p in $\mathcal{G}$ of length $\ell(p) \leq L$ such that epe' is a path in $\mathcal{G}$. Let $N := L / \min_{e \in \mathcal{E}} \ell(e)$ denote the maximum number of edges in such a path p. Fix some $a_0 \in \mathcal{E}$ satisfying $i(a_0) = x$ (recall that we are considering the growth of paths that begin at $x \in \mathcal{V}$). Our aim is to prove that the following inequality holds for all $\sigma > h$

$$\sum_{p \in \mathcal{P}_{\mathcal{G}}(x)} e^{-\sigma \ell(p)} \geq \frac{e^{-\sigma(\ell(a_0) + L)}}{N + 1} \sum_{n=0}^{\infty} \|\widehat{M_\sigma}^n\|.$$

By Proposition 3.2.3, the left-hand side of the above equation is bounded and hence for all $\epsilon > 0$, $\sum_{n=0}^{\infty} \|\widehat{M_{h+\epsilon}}^n\| \geq \sum_{n=0}^{\infty} \|\widehat{M_z}^n\|$ converges uniformly on the domain $Re(z) > h + \epsilon$. By Proposition 3.6.2 part 1, this will imply that $z \mapsto \sum_{n=0}^{\infty} \widehat{M_z}^n$ is holomorphic on $Re(z) > h$. It is then easy to check that $\sum_{n=0}^{\infty} \widehat{M_z}^n = (I - \widehat{M_z})^{-1}$.

Fix $\sigma > h$. Using Equation (3) and Equation (2), we can write

$$\begin{aligned}
(N+1)\sum_{p\in\mathcal{P}_{\mathcal{G}}(x)} e^{-\sigma\ell(p)} &= (N+1)\underline{w}(\sigma)\cdot\Big(\sum_{n=0}^{\infty}\widehat{M_\sigma}^{n}\Big)\underline{1} \\
&\geq \underline{w}(\sigma)\cdot\Big(\sum_{n=0}^{\infty}\sum_{i=0}^{N}\widehat{M_\sigma}^{n+i}\Big)\underline{1} \\
&\geq (\underline{w}(\sigma))_{a_0}\sum_{n=0}^{\infty}\sum_{i=0}^{N}\sum_{b\in\mathcal{E}}\widehat{M_\sigma}^{n+i}(a_0,b) \\
&= e^{-\sigma\ell(a_0)}\sum_{n=0}^{\infty}\sum_{i=0}^{N}\sum_{b\in\mathcal{E}}\sum_{p\in\mathcal{P}_{\mathcal{G}}(n+i,a_0,b)} e^{-\sigma\ell(p)}.
\end{aligned}$$

Let $a' \in \mathcal{E}$. Then for any $q \in \mathcal{P}_{\mathcal{G}}(n,a',b)$, by (H2), we can form a path $a'p_q q$ where $\ell(p_q) \leq L$ and the number of edges, i, in $p_q q$, satisfies $n \leq i \leq N+n$. Hence

$$\begin{aligned}
(N+1)\sum_{p\in\mathcal{P}_{\mathcal{G}}(x)} e^{-\sigma\ell(p)} &\geq e^{-\sigma\ell(a_0)}\sum_{n=0}^{\infty}\sum_{i=0}^{N}\sum_{b\in\mathcal{E}}\sum_{p\in\mathcal{P}_{\mathcal{G}}(n+i,a_0,b)} e^{-\sigma\ell(p)} \\
&\geq e^{-\sigma\ell(a_0)}\sum_{n=0}^{\infty}\sum_{a,b\in\mathcal{E}}\sum_{q\in\mathcal{P}_{\mathcal{G}}(n,a,b)} e^{-\sigma(\ell(q)+\ell(p_q))} \\
&\geq e^{-\sigma(\ell(a_0)+L)}\sum_{n=0}^{\infty}\sum_{a,b\in\mathcal{E}}\sum_{q\in\mathcal{P}_{\mathcal{G}}(n,a,b)} e^{-\sigma(\ell(q))} \\
&\geq e^{-\sigma(\ell(a_0)+L)}\sum_{n=0}^{\infty}\sup_{a\in\mathcal{E}}\left\{\sum_{b\in\mathcal{E}}\sum_{q\in\mathcal{P}_{\mathcal{G}}(n,a,b)} e^{-\sigma(\ell(q))}\right\} \\
&= e^{-\sigma(\ell(a_0)+L)}\sum_{n=0}^{\infty}\left\|\widehat{M_\sigma}^{n}\right\|.
\end{aligned}$$

□

Using Lemma 3.6.4 and Equation (3), on $Re(z) > h$, we can write

$$\eta_{\mathcal{G}}(z) = \sum_{p\in\mathcal{P}_{\mathcal{G}}(x)} e^{-z\ell(p)} = \underline{w}(z)\cdot\Big(I-\widehat{M_z}\Big)^{-1}\underline{1}, \tag{4}$$

and conclude that $\eta_{\mathcal{G}}(z)$ is analytic and bounded on $Re(z) > h$ and therefore $\eta_{\mathcal{G}}(z)$ satisfies part (1) of the Ikehara-Wiener Tauberian theorem.

We will proceed by using Equation (4) to give a meromorphic extension of $\eta_{\mathcal{G}}(z)$ to $Re(z) > 0$. In particular, we will see that understanding when $(I-\widehat{M_z})$ is invertible for $Re(z) > 0$ will allow us to understand the pole structure of the extension of $\eta_{\mathcal{G}}(z)$ to $Re(z) > 0$.

3.7 $\eta_{\mathcal{G}}(z)$ is meromorphic on $Re(z) > 0$

From the definition of entropy it followed that $\eta_{\mathcal{G}}(z)$ is analytic on $Re(z) > h$ and hence part (1) of the Tauberian theorem holds.

In this section, we will study the properties of the holomorphic operator-values function $(I - \widehat{M_z}) : \ell^\infty(\mathbb{C}) \to \ell^\infty(\mathbb{C})$, to show that $\eta_{\mathcal{G}}(z)$ is meromorphic on $Re(z) > 0$. To this end, we shall make use of an idea by Hofbauer and Keller in [21], where they observe that the invertibility of certain operators of the above form depends only on the determinant of a finite matrix associated to M_z.

Fix $\epsilon > 0$ and, for convenience, also assume that $h > \epsilon$. Given $k \geq 1$, we can truncate the matrix M_z to the $k \times k$ matrix $A_{z,k} = (M_z(i,j))_{i,j=1}^k$. We can then write

$$M_z = \begin{pmatrix} A_{z,k} & B_{z,k} \\ C_{z,k} & D_{z,k} \end{pmatrix},$$

where $B_{z,k} = (M_z(i, j+k))_{i,j=1}^\infty$, $C_{z,k} = (M_z(i+k, j))_{i,j=1}^\infty$ and $D_{z,k} = (M_z(i+k, j+k))_{i,j=1}^\infty$. Note that $A_{z,k}$, $B_{z,k}$, $C_{z,k}$ and $D_{z,k}$ are all holomorphic operator-valued functions because each is a composition of M_z with a linear map from the space of bounded linear operators on $\ell^\infty(\mathbb{C})$ to certain relevant Banach spaces of operators.

Fix $\epsilon > 0$. Then by taking k sufficiently large, by (H1) we have

$$\|\widehat{D_{z,k}}\| = \sup_{n \in \mathbb{N}} \sum_{m=1}^\infty |D_{z,k}(n,m)| \leq \sum_{m=1}^\infty e^{-Re(z)\ell(m+k)} \leq \sum_{m=1}^\infty e^{-\epsilon \ell(m+k)} < 1, \quad (5)$$

for $Re(z) \geq \epsilon$. Hence, $z \mapsto \sum_{n=0}^\infty \widehat{D_{z,k}} = (I - \widehat{D_{z,k}})^{-1}$ is a holomorphic operator-valued function on $Re(z) > \epsilon$ for k sufficiently large.

By writing $\ell^\infty(\mathbb{C})$ as the direct sum of two subspaces, we can then verify that $I - \widehat{M_z}$ can be written as

$$\begin{pmatrix} I - \widehat{A_{z,k}} - \widehat{B_{z,k}}(I - \widehat{D_{z,k}})^{-1}\widehat{C_{z,k}} & -\widehat{B_{z,k}}(I - \widehat{D_{z,k}})^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\widehat{C_{z,k}} & I - \widehat{D_{z,k}} \end{pmatrix}. \quad (6)$$

Definition 3.7.1. *Let us denote the $k \times k$ matrix*

$$W_{z,k} := A_{z,k} + B_{z,k}(I - D_{z,k})^{-1}C_{z,k}.$$

By (5), it follows that for $Re(z) > \epsilon$, whenever k is sufficiently large and $\det(I - W_{z,k}) \neq 0$, then $I - \widehat{M}_z$ is invertible with inverse

$$\begin{pmatrix} I & 0 \\ (I - \widehat{D_{z,k}})^{-1}\widehat{C_{z,k}} & (I - \widehat{D_{z,k}})^{-1} \end{pmatrix} \begin{pmatrix} (I - \widehat{W_{z,k}})^{-1} & (I - \widehat{W_{z,k}})^{-1}\widehat{B_{z,k}}(I - \widehat{D_{z,k}})^{-1} \\ 0 & I \end{pmatrix}. \tag{7}$$

Next observe that when $\det(I - W_{z,k}) \neq 0$, we can write $(I - W_{z,k})^{-1} = \mathrm{Adj}(I - W_{z,k})/\det(I - W_{z,k})$, where $\mathrm{Adj}(I - W_{z,k})$ denotes the adjugate of $(I - W_{z,k})$. Because $\mathrm{Adj}(I - W_{z,k})$ and the other entries of the decomposition of $(I - \widehat{M}_z)^{-1}$ above are holomorphic operator-valued functions, after factoring out $1/\det(I - W_{z,k})$ from the right-hand side of Equation (6), for $Re(z) \geq \epsilon$, we obtain a holomorphic operator-valued function $E_{z,k}$ (see Proposition 3.6.2).

Hence, for z satisfying $Re(z) > \epsilon$ and $\det(I - W_{z,k}) \neq 0$, we can write

$$\underline{w}(z) \cdot (I - \widehat{M}_z)^{-1}\underline{1} = \frac{\underline{w}(z) \cdot E_{z,k}\underline{1}}{\det(I - W_{z,k})}, \tag{8.}$$

Note that $\det(I - W_{z,k})$ is bounded and the sum of a countable number of holomorphic functions and so is itself a holomorphic function on $Re(z) \geq \epsilon$. Hence the function in equation (7) is meromorphic on $Re(z) \geq \epsilon$.

Equations (3) and (7) show us that $\eta_{\mathcal{G}}(z)$ has a meromorphic extension to $Re(z) \geq \epsilon$ where the poles may occur at z such that $\det(I - W_{z,k}) = 0$, for k sufficiently large. Furthermore, because $\epsilon > 0$ was arbitrary, $\eta_{\mathcal{G}}(z)$ has an extension to $Re(z) > 0$. Hence we have proven that part (2) of the Ikehara-Wiener Tauberian theorem holds for $\eta_{\mathcal{G}}(z)$, i.e. the following proposition holds

Proposition 3.7.2. *$\eta_{\mathcal{G}}(z)$ has a meromorphic extension to $Re(z) > 0$ where the poles occur at z such that $\det(I - W_{z,k}) = 0$, for k is sufficiently large..*

3.8 Poles on the line $Re(z) = h + it$

In this section, we prove that $\eta_{\mathcal{G}}(z)$ satisfies the last two requirements of the Ikehara-Wiener Tauberian theorem. In particular, we still need to show that $\eta_{\mathcal{G}}(z)$ satisfies:

3. h is a simple pole for $\eta_{\mathcal{G}}(z)$, i.e., there exists a $C > 0$ such that $C = \lim_{z \to a}(z - a)\eta_{\mathcal{G}}(z)$ exists and is positive; and

4. the extension of $\eta_{\mathcal{G}}(z)$ has no poles on the line $Re(z) = h$ other than h.

In the last section we saw that the poles of $\eta_{\mathcal{G}}(z)$ must satisfy $\det(I - W_{z,k}) = 0$ for some sufficiently large k. The graph hypotheses (H2) and (H3) will allow us to deduce that for $\sigma > 0$, the $W_{\sigma,k}$ are non-negative irreducible matrices for $\sigma > h$ and consequently we can show that the poles on the line $Re(z) = h + it$ have the properties required.

Let us begin by fixing ϵ such that $0 < \epsilon < h$. Choose k large enough so that the extension of $\eta_{\mathcal{G}}(z)$ is meromorphic on $Re(z) > \epsilon$, with poles at z such that $\det(I - W_{z,k}) = 0$. Recall that a non-negative $n \times n$ matrix M is *irreducible* if for all i, j satisfying $1 \leq i, j \leq n$ there exists a natural number m such that $(M^m)_{i,j} > 0$. In the next lemma, we require the following version of the Perron-Frobenius theorem (see [17]).

Theorem 3.8.1. *(Perron-Frobenius theorem) Let A be an irreducible non-negative $n \times n$ matrix with spectral radius ρ. Then the following statements hold*

- *ρ is a positive real number and it is an eigenvalue of A,*
- *ρ is a simple eigenvalue, and*
- *A has a left and a right eigenvector associated to ρ whose components are all positive.*

Lemma 3.8.2. *Let $\sigma > 0$. Then $W_{\sigma,k}$ is a non-negative irreducible matrix. Furthermore, there exists a neighborhood U containing σ and a function $\rho : U \to \mathbb{C}$ such that $W_{z,k}$ has a simple maximal positive eigenvalue $\rho(z) = \rho(W_{z,k})$, that depends analytically on z and satisfies $\rho'(\sigma) < 0$.*

Proof. The matrix $W_{\sigma,k} = A_{\sigma,k} + B_{\sigma,k}(I - D_{\sigma,k})^{-1}C_{\sigma,k}$ is non-negative when $\sigma > h$ because the entries in M_σ are non-negative. We will now show that the matrix $W_{\sigma,k}$ is irreducible. To see this, note that by assumption (H2), for all $1 \leq i, j \leq k$, there exists a $m > 0$ and some path p consisting of m edges, starting with edge e_i and ending with edge e_j. Such a path can be broken up into sub-paths of two types. The first type consists of those paths that stay completely within $\{e_1, \ldots, e_k\}$ and the second type which consists of those paths that initially enter the complement $\mathcal{E} - \{e_1, \ldots, e_k\}$ and finally leave at their end. Note that $W^m_{\sigma,k}$ is a binomial sum of matrices whose terms are products of powers of $A_{\sigma,k}$ (corresponding to sub-paths of the first type) and powers of $B_{\sigma,k}(I - D_{\sigma,k})^{-1}C_{\sigma,k}$ (corresponding to sub-paths of the second type), where the powers that appear in each term sum to m. By considering the entries of the matrices $W^m_{\sigma,k}$ formed from these terms (which correspond to sums of exponential weightings of paths) and noting the decomposition of p into the aforementioned sub-paths, it follows that $W^m_{\sigma,k}(i,j) \geq e^{-\sigma\ell(p)} > 0$.

We can now apply the Perron-Frobenius theorem (see Theorem 3.8.1) to deduce that the simple maximal positive eigenvalue $\rho(\sigma) > 0$ for $W_{\sigma,k}$ exists.

It is a standard fact from perturbation theory that there exists some neighborhood U of σ in $\mathbb{C}$ and a holomorphic function $\rho : U \to \mathbb{C}$ such that $\rho(z)$ is an eigenvalue of W_z for every $z \in U$, and admits a choice of left and right eigenvectors depending holomorphically on $z \in U$.

We will show that $\rho'(\sigma) < 0$. By the previous paragraph, we see that the entries in W^n_σ are of the form $\sum_p e^{-\sigma\ell(p)}$, where the sum is over some paths consisting of n edges. Let $x, y > 0$ be real numbers such that $x, x+y \in U$. Then by Gelfand's formula,

$$\rho(x+y) = \lim_{n\to\infty} \left\|W^n_{x+y}\right\|^{1/n} \leq \lim_{n\to\infty} e^{-y\min_{e\in\mathcal{E}}\ell(e)} \left\|W^n_x\right\|^{1/n} \leq e^{-y\cdot\min_{e\in\mathcal{E}}\ell(e)}\rho(x).$$

It follows directly that the derivative of ρ at x is negative. □

We can now proceed to prove that part (3) of the Tauberian theorem holds for $\eta_{\mathcal{G}}(z)$.

Proposition 3.8.3. *h is a simple pole of $\eta_{\mathcal{G}}(z)$ and the residue of this pole is positive.*

Proof. Choose k large enough so that the poles of $\eta_{\mathcal{G}}(z)$ occur when $\det(I - W_{z,k}) = 0$. We begin by noting that h is a pole for $\eta_{\mathcal{G}}(z)$ by definition of h. For z in a neighbourhood of h, we denote by $\rho(z)$ the perturbed eigenvalue of $W_{z,k}$ that corresponds to $\rho(W_{h,k})$. We can write $\det(I - W_{z,k}) = (1 - \rho(z))\Pi^k_{i=2}(1 - \lambda_i(z))$, where the $\lambda_i(z)$ denote the other eigenvalues of $W_{z,k}$. By the Perron-Frobenius theorem and standard perturbation theory, the $\lambda_i(z)$ are bounded away from 1 for z near h. Let $\psi(z) := \det(I - W_z)/(1 - \rho(z)) = \Pi^k_{i=2}(1 - \lambda_i(z))$. Because $\det(I - W_z)$ and $(1 - \rho(z))$ have a simple zero at h, $\psi(z)$ is holomorphic at, and near $z = h$. By Lemma 3.8.2, $\rho'(h) \neq 0$ and $\rho(z)$ is analytic near h. Using this and a meromorphic extension for $\eta_{\mathcal{G}}(z)$ (see Equation (7)), the residue of $\eta_{\mathcal{G}}(z)$ at h is given by

$$\begin{aligned} C := & \lim_{z \to h} (z-h)\eta_{\mathcal{G}}(z) \\ = & \lim_{z \to h} (z-h) \frac{\underline{w}(z) \cdot E_{z,k}\underline{1}}{\det(I - W_{z,k})} \\ = & \lim_{z \to h} \frac{(z-h)(\underline{w}(z) \cdot E_{z,k}\underline{1})}{(\rho'(h)(z-h) + \rho''(h)(z-h)^2/2 + \ldots)\psi(z)} \\ = & \frac{\underline{w}(h) \cdot E_{h,k}\underline{1}}{\rho'(h)\psi(h)}. \end{aligned}$$

Because the $\lambda_i(h)$ are bounded away from 1, h is either a simple pole or a removable singularity. By the characterization of entropy in terms of generating functions (Proposition 3.2.3), $z = h$ is not a removable singularity for $\eta_{\mathcal{G}}(z)$ (therefore $\underline{w}(h) \cdot E_{h,k}\underline{1} \neq 0$). One can see that $C > 0$ because $\eta_{\mathcal{G}}(\sigma) > 0$ for $\sigma > h$. □

We conclude by proving the final part of the Tauberian theorem. We will use Wielandt's theorem (see [17]), which we state here for convenience.

Theorem 3.8.4. *(Wielandt's theorem) Let the $n \times n$ matrix A satisfy the conditions of the Perron-Frobenius theorem. Suppose the $n \times n$ matrix B satisfies*

$$A(i,j) \geq |B(i,j)|$$

for $1 \leq i,j \leq n$. Then for any eigenvalue λ of B, $|\lambda| \leq \rho$, where ρ is the spectral radius of A, with equality if and only if there exists a diagonal $n \times n$ matrix D satisfying

$$B = DAD^{-1}.$$

Proposition 3.8.5. *$\eta_{\mathcal{G}}(z)$ has no poles other than h on the line $Re(z) = h$.*

Proof. Choose k large enough so that the poles of $\eta_{\mathcal{G}}(z)$ occur when $\det(I - W_{z,k}) = 0$. Suppose for a contradiction that there exists another pole at $h + it$ ($t \neq 0$). Let c be any closed path and choose an integer $k_c > k$ such that the edges of c have index smaller than k_c. Then construct the $k_c \times k_c$ matrices W_{z,k_c} and consider the new extension of $\eta_{\mathcal{G}}(z)$ to ϵ' where $0 < \epsilon' \leq \epsilon$. From Proposition 3.7.2 we see that $\det(I - W_{h,k_c}) = 0 = \det(I - W_{h+it,k_c})$ and thus 1 is an eigenvalue for W_{h+it,k_c} and W_{h,k_c} Furthermore, $\rho(W_{h,k_c}) = 1$ because otherwise $\eta_{\mathcal{G}}(z)$ would have a pole at some $c > h$, contradicting Definition 3.2.2.

Next observe that $|W_{h+it,k_c}(a,b)| \leq W_{h,k_c}(a,b)$ for all $1 \leq a,b \leq k$ and $\rho(W_{h+it,k_c}) \geq 1 = \rho(W_{h,k_c})$. Consequently, we can apply Wielandt's theorem to conclude that

$\rho(W_{h+it,k_c}) = \rho(W_{h,k_c}) = 1$ and that there exists a diagonal matrix D, whose non-zero entries have unit modulus, such that $W_{h+it,k_c} = DW_{h,k_c}D^{-1}$. Thus for all $n > 1$, we have $W^n_{h+it,k_c} = DW^n_{h,k_c}D^{-1}$.

Suppose that the closed path c contains some edge $a \in \mathcal{E}$ and consists of n edges. Note that $W^n_{h+it,k_c}(a,a)$ is a sum that includes $e^{(h+it)\ell(c)}$ as one of its terms (see the proof of Lemma 3.8.2). Since $W^n_{h+it,k_c} = DW^n_{h,k_c}D^{-1}$, it follows that $W^n_{h+it,k_c}(a,a) = W^n_{h,k_c}(a,a)$. Note that $W^n_{z,k_c}(a,a) = \sum_{p \in \mathcal{P}_G(a,k_c)} e^{-z\ell(p)}$ for some subset of paths $\mathcal{P}_\mathcal{G}(a,k_c) \subset \mathcal{P}_\mathcal{G}(x)$ that includes the closed path c. Because the summands in $W^n_{h,k_c}(a,a)$ are real and positive, the only way for $W^n_{h+it,k_c}(a,a) = W^n_{h,k_c}(a,a)$ to hold is if the summands in $W^n_{h+it,k_c}(a,a)$ are real and positive; otherwise the real component of $W^n_{h+it,k_c}(a,a)$ would be strictly less than $W^n_{h,k_c}(a,a)$. This then implies that t must satisfy $\ell(c)t = 2\pi m_c$ for some non-zero integer m_c.

Because c was arbitrary, the above construction implies that for all closed paths c, $\ell(c) \in d\mathbb{N}$ with $d = 2\pi/t$ which contradicts (H3). $\square$

Finally, we have shown that $\eta_\mathcal{G}(z)$ satisfies the hyptheses of the Ikehara-Wiener Tauberian theorem and hence

$$N_\mathcal{G}(R) \sim (C/h)e^{hR}$$

for some $C > 0$. To be more specific, C is given by

$$\lim_{z \to h}(z-h)\underline{w}(z) \cdot (I - \widehat{M}_z)^{-1}\underline{1}.$$

3.9 Asymptotic formula for the growth of closed paths

In this section, we obtain an asymptotic formula for the growth of closed paths on infinite graphs (see Theorem 3.2.7).

Definition 3.9.1. *An oriented closed path on an infinite graph $\mathcal{G}$ is an oriented path $q = (e_1, \ldots, e_n)$, of length $|q| = n$, considered up to cyclic permutation, with the additional requirement that $t(e_n) = i(e_1)$. We say that q is primitive if it is not a multiple concatenation of a shorter closed path.*

Note the asymptotic formula will hold whether we count oriented closed paths or closed paths however, in order to use the method of proof developed in the previous section, we will work with oriented closed paths.

Let $\mathcal{Q}(T)$ denote the set of oriented primitive closed paths[1] on $\mathcal{G}$ of length less than or equal to T. Let $\mathcal{Q} := \bigcup_{T>0} \mathcal{Q}(T)$ denote the set of all oriented primitive closed paths on $\mathcal{G}$. We want to count the number the number of oriented primitive closed paths $C_{\mathcal{G}}(T) := \#\mathcal{Q}(T)$.

It follows from the strong connectedness of $\mathcal{G}$ and (H2), that the exponential growth rate of $C_{\mathcal{G}}(T)$ is equal to the volume entropy of the graph, i.e.

$$h = \lim_{T\to\infty} \frac{1}{T} \log C_{\mathcal{G}}(T). \tag{9}$$

The proof of Theorem 3.2.7 follows a similar strategy to the proof presented in the previous section. In particular, it requires an application of the Ikehara-Wiener Tauberian theorem to a "zeta function" defined in terms of closed paths on the graph. A similar approach can be used to deduce the prime number theorem ([37]).

3.9.1 Zeta functions

We now present the definition of the zeta functions that will be used in the proof of Theorem 3.2.7.

Definition 3.9.2. *We can formally define the zeta function by the Euler product*

$$\zeta(z) = \prod_{q\in\mathcal{Q}} \left(1 - e^{-z\ell(q)}\right)^{-1}, z \in \mathbb{C}$$

where the product is over all oriented primitive closed paths.

This converges to a non-zero analytic function for $Re(z) > h$ by the definition of volume entropy (see Equation (8)).

The proof of Theorem 3.2.7 requires us to work with a different presentation of the zeta function. Let $\mathcal{E}_n$ denote the set of oriented edge strings $\underline{e} = (e_1, \ldots, e_n)$ of length n corresponding to general oriented (not necessarily primitive) closed paths q. Each element $q \in \mathcal{Q}$ consisting of n edges will give rise to n elements of $\mathcal{E}_n$ corresponding to cyclic permutations. For $\underline{e} \in \mathcal{E}_n$ let $\ell(\underline{e}) := \sum_{i=1}^{n} \ell(e_i)$.

Lemma 3.9.3. *For $Re(z) > h$, we can write*

[1] Formally we will count oriented primitive paths, which does not include repeated paths, but for the purposes of asymptotic counting there is no difference.

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e} \in \mathcal{E}_n} e^{-z\ell(\underline{e})}\right). \tag{10}$$

Furthermore, $\zeta(z)$ is holomorphic on $Re(z) > h$.

Proof. We begin by noting that

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e}' \in \mathcal{E}_n} e^{-z\ell(\underline{e}')} \leq \sum_{x \in \mathcal{V}} \sum_{p \in \mathcal{P}_{\mathcal{G}}(x)} e^{-z\ell(p)}$$

where the right-hand side converges for $Re(z) > h$ by Proposition 3.2.3 and the fact that the entropy is independent of $x \in \mathcal{V}$.

Hence for $\epsilon > 0$, on $Re(z) > h + \epsilon$,

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e}' \in \mathcal{E}_n} e^{-z\ell(\underline{e}')}\right)$$

converges uniformly and so this function is holomorphic on $Re(z) > h$.

Given $k \geq 1$, let $\mathcal{E}_k^{prim} \subset \mathcal{E}_k$ denote the set of (allowed) oriented edge strings $\underline{e} = (e_1, \ldots, e_k)$ corresponding to oriented primitive closed paths q which consist of k edges. In particular, each q contributes k strings in $\mathcal{E}_k^{prim}$ (due to cyclic permutations).

For each $m \geq 1$ we can write

$$\sum_{q \in \mathcal{Q}} e^{-zm\ell(q)} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\underline{e} \in \mathcal{E}_k^{prim}} e^{-zm\ell(\underline{e})}.$$

Using the above equation, we can write

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e}' \in \mathcal{E}_n} e^{-z\ell(\underline{e}')}\right) = \exp\left(\sum_{k=1}^{\infty} \sum_{\underline{e} \in \mathcal{E}_k^{prim}} \sum_{m=1}^{\infty} \frac{e^{-zm\ell(\underline{e})}}{km}\right),$$

where we have set replaced $\underline{e}' \in \mathcal{E}_n$ with $\underline{e} \in \mathcal{E}_k^{prim}$ and n with $n = km$ for $m \geq 1$.

We then rewrite $\zeta(z)$ using the Taylor expansion for $\log(1-z)$

$$\begin{aligned}\zeta(z) &= \exp\left(-\sum_{q\in\mathcal{Q}} \log(1-e^{-z\ell(q)})\right) \\ &= \exp\left(\sum_{q\in\mathcal{Q}}\sum_{m=1}^{\infty} \frac{e^{-zm\ell(q)}}{m}\right).\end{aligned} \tag{11}$$

□

3.9.2 Extending the zeta function

We want to now consider $z \in \mathbb{C}$ with $Re(z) > 0$. To extend the zeta function we will rewrite the zeta function in terms of infinite matrices introduced in the previous section.

Recall definition 3.4.1.

Definition 3.9.4. *Let us order the oriented edge set $\mathcal{E}$ by non-decreasing length. For $z \in \mathbb{C}$, we can associate to $\mathcal{G}$ the infinite matrix $M_z(e, e')$ defined by*

$$M_z(e,e') = \begin{cases} e^{-z\ell(e')} & \text{if } t(e) = i(e'), \\ 0 & \text{otherwise.} \end{cases}$$

where the rows and columns are indexed by the oriented edges, partially ordered by their lengths.

As before, we can write M_z as follows

$$M_z = \begin{pmatrix} A_{z,k} & B_{z,k} \\ C_{z,k} & D_{z,k} \end{pmatrix},$$

where $A_{z,k}$ is the $k \times k$ finite sub-matrix of M_z corresponding to the first $k \in \mathbb{N}$, say, oriented edges and the other sub-matrices $B_{z,k}, C_{z,k}, D_{z,k}$ are infinite. We define $W_{z,k} := A_{z,k} + B_{z,k}(I - D_{z,k})^{-1}C_{z,k}$.

Recall that for the proof in the previous section, for any $\epsilon > 0$, we obtained a meromorphic extension of $\eta_{\mathcal{G}}(z)$ to the half-plane $Re(z) > \epsilon$, of the form

$$\frac{\underline{w}(z) \cdot E_{z,k}\underline{1}}{\det(I - W_{z,k})},$$

whose poles occur at z for which $\det(I - W_{z,k}) = 0$. We will pursue a similar strategy here, although the details are different.

Rather than rewrite $\zeta(z)$ in terms of the operator $(I - \widehat{M_z})^{-1}$ and then apply a factorization, we will write $\zeta(z)$ in terms of two formally defined auxiliary functions formally defined on $Re(z) > 0$. The two functions are $\det(I - W_{z,k})$ and

$$g_k(z) = \exp\left(- \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e} \in \mathcal{E}_n(k)} e^{-z\ell(\underline{e})} \right),$$

where $\mathcal{E}_n(k) \subset \mathcal{E}_n$ denotes the set of oriented edge strings $\underline{e} = (e_1, \ldots, e_n)$ of length n corresponding to oriented closed paths for which all of the e_j $(1 \le j \le n)$ are disjoint from the first k edges in the ordering on $\mathcal{E}$.

We can rewrite $\zeta(z)$ in terms of $\det(I - W_{z,k})$ and $g_k(z)$ by following Hofbauer-Keller [21].

Lemma 3.9.5. *On $Re(z) > h$, we can write*

$$\zeta(z) = \frac{1}{g_k(z) \det(I - W_{z,k})}.$$

Proof. By Equation (9), on $Re(z) > h$ we can rewrite $\zeta(z)$ in terms of M_z as follows

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{tr}(M_z^n)}{n} \right),$$

where given a countable matrix A, we define the formal sum $\mathrm{tr(A)} = \sum_{i=1}^{\infty} A(i,i)$. Similarly, on $Re(z) > h$ we can write

$$g_k(z) = \exp\left(- \sum_{n=1}^{\infty} \frac{\mathrm{tr}(D_{z,k}^n)}{n} \right).$$

Next, by applying the formula $\det(I - B) = \exp(-\sum_{n=1}^{\infty} \mathrm{tr}(B^n)/n)$, for a finite matrix B, to $W_{z,k}$, we obtain the following expression for $\det(I - W_{z,k})$ on $Re(z) > h$

$$\det(I - W_{z,k}) = \exp\left(- \sum_{n=1}^{\infty} \frac{\mathrm{tr}(W_{z,k}^n)}{n} \right).$$

Note that $\det(I - W_{z,k}) \neq 0$ on $Re(z) > h$.

We claim that $\mathrm{tr}(M_z^n) = \mathrm{tr}(D_{z,k}^n) + \mathrm{tr}(W_{z,k}^n)$ on $Re(z) > h$. To see this, first note that $\mathrm{tr}(M_z^n)$ and $\mathrm{tr}(D_{z,k}^n)$ are the sums of exponentially weighted oriented edge strings in $\mathcal{E}_n$ and $\mathcal{E}_n(k)$, respectively. Similarly, $\mathrm{tr}(W_{z,k}^n)$ is the sum of exponentially weighted oriented edge strings with at least one edge in the first k saddle connections (see the proof of 3.8.2) and so corresponds to an exponentially weighted sum over $\mathcal{E}_n \backslash \mathcal{E}_n(k)$.

By applying the above observations together, it follows that on $Re(z) > h$

$$\begin{aligned}
\zeta(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{tr}(M_z^n)}{n}\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{tr}(D_{z,k}^n)}{n}\right) \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{tr}(W_{z,k}^n)}{n}\right) \\
&= \frac{1}{g_k(z)\det(I - W_{z,k})}.
\end{aligned}$$

□

We will now use the above expression to extend $\zeta(z)$. First we will study at the analyticity of $\det(I - W_{z,k})$ and $g_k(z)$.

Lemma 3.9.6. *Fix $\epsilon > 0$. Provided k (the size of $W_{z,k}$) is sufficiently large, the functions $g_k(z)$ and $\det(I - W_{z,k})$ are analytic on $Re(z) > \epsilon$. Furthermore, $g_k(z)$ is non-zero on $Re(z) > \epsilon$.*

Proof. Let $\mathcal{E}(k) \subset \mathcal{E}$ consist of the oriented edges e that are *not* in the first k in the partial ordering. Then for $Re(z) > \epsilon$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e} \in \mathcal{E}_n(k)} \left| e^{-z\ell(\underline{e})} \right| &\leq \sum_{n=1}^{\infty} \left(\sum_{e \in \mathcal{E}(k)} \left| e^{-z\ell(e)} \right| \right)^n \\
&\leq \sum_{n=1}^{\infty} \left(\sum_{e \in \mathcal{E}(k)} e^{-\epsilon\ell(e)} \right)^n .
\end{aligned}$$

Consequently, $g_k(z)$ is analytic for $Re(z) > \epsilon$ if $\sum_{e \in \mathcal{E}(k)} e^{-\epsilon\ell(e)} < 1$, which holds for k sufficiently large. Note that $g_k(z)$ is non-zero on this domain because e^z is non-zero on $\mathbb{C}$.

For $\det(I - W_{z,k})$ to be analytic on $Re(z) > \epsilon$, it suffices to show that $(I - D_{z,k})$ is invertible for such z, which holds provided $||D_{z,k}|| < 1$. To this end, we note that

$$||D_{z,k}|| \leq \sum_{e \in \mathcal{E}(k)} e^{-\epsilon \ell(e)}$$

and hence $||D_{z,k}|| < 1$ for k sufficiently large. $\square$

Fix $\epsilon < h$ and let k be sufficiently large so that $1/\zeta(z) = g_k(z)\det(I - W_{z,k})$ is analytic on $Re(z) > \epsilon$. To proceed, we need to understand the location of the poles of the extension of $\zeta(z)$ on $Re(z) > \epsilon$. Note that $g_k(z)$ is non-zero and hence poles of the extension of $\zeta(z)$ correspond to the zeros of $\det(I - W_{z,k})$ in $Re(z) > \epsilon$. Hence, we can apply the analysis for $\eta_{\mathcal{G}}(z)$ in the previous section to deduce the following Lemma.

Lemma 3.9.7. *The meromorphic extension of $\zeta(z)$ is analytic for $Re(z) > h$, with a simple pole at $z = h$ which has positive residue, and there are no other poles on the line $Re(z) = h$.*

3.9.3 Proof of Theorem 3.2.7

Having established the properties of the complex function $\zeta(z)$, the derivation of the asymptotic formula follows a classical route (see [43]). Using Equation (10), for $Re(z) > h$, we can write

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{n=1}^{\infty} \sum_{q \in \mathcal{Q}} \ell(q) e^{-zn\ell(q)} = \int_0^{\infty} e^{-zT} dF(T) \tag{12}$$

where

$$F(T) := \sum_{n\ell(q) \leq T} \ell(q) = \sum_{n\ell(q) \leq T} \ell(q) \left[\frac{T}{\ell(q)}\right] \leq C_{\mathcal{G}}(T)T, \tag{13}$$

with the summation over pairs $(n, q) \in \mathbb{N} \times \mathcal{Q}$ provided $n\ell(q) \leq T$, and $C_{\mathcal{G}}(T) = \mathrm{Card}\{q \in \mathcal{Q} : \ell(q) \leq T\} = \#\mathcal{Q}(T)$.

By Lemma 3.9.7, we can write $\zeta(z) = \psi(z)/(z - h)$ where $\psi(z)$ is analytic in a neighbourhood of $Re(z) \geq h$ and non-zero at h. Thus

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{-1}{z - h} + \frac{\psi'(z)}{\psi(z)}. \tag{14}$$

Comparing (11) and (13) we can apply the Ikehara-Wiener Tauberian theorem to deduce that $F(T) \sim e^{hT}/h$ as $T \to \infty$. Using (12), it follows that $\liminf_{T\to\infty} \frac{C_{\mathcal{G}}(T)}{e^{hT}/hT} \geq 1$.

For any $\sigma > h$ and sufficiently large $T > 0$ we can sum the geometric series in (11) to bound

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} \geq \sum_{\ell(q)\leq T} \frac{1}{e^{\sigma\ell(q)}} \frac{\ell(q)}{\left(1-e^{-\sigma\ell(q)}\right)} \geq \sum_{\ell(q)\leq T} \frac{\ell(q)}{\sigma\ell(q)} \frac{1}{e^{\sigma T}} = \frac{1}{\sigma} \frac{C_{\mathcal{G}}(T)}{e^{\sigma T}},$$

using $e^x - 1 \leq xe^x$ for $x \geq 0$.

Thus for any $\sigma' > \sigma$ we have

$$\frac{C_{\mathcal{G}}(T)}{e^{\sigma' T}} \leq e^{(\sigma-\sigma')T}\sigma\left(-\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) \to 0 \text{ as } T \to \infty.$$

Since $\sigma > h$ is arbitrary, we deduce that $C_{\mathcal{G}}(T)/e^{\sigma T} \to 0$ as $T \to \infty$.

Given T sufficiently large, we choose $y < T$ such that $e^{\sigma y} = e^{hT}/T$ and write

$$C_{\mathcal{G}}(T) - C_{\mathcal{G}}(y) = \sum_{y<\ell(q)\leq T} 1 \leq \sum_{\ell(q)\leq T} \frac{\ell(q)}{y} \leq \frac{F(T)}{y}.$$

By rearranging this inequality, we can write

$$C_{\mathcal{G}}(T)\frac{hT}{e^{hT}} \leq C_{\mathcal{G}}(y)\frac{hT}{e^{hT}} + F(T)\frac{hT}{e^{hT}y} = C_{\mathcal{G}}(y)\frac{h}{e^{\sigma y}} + F(T)\frac{h}{e^{hT}}\left(\frac{\sigma}{h-\frac{\log T}{T}}\right).$$

By using the asymptotic formula for $F(T)$ and the limit for $C_{\mathcal{G}}(T)/e^{\sigma T}$ above, it then follows that $\limsup_{T\to\infty} C_{\mathcal{G}}(T)\frac{hT}{e^{hT}} \leq \frac{\sigma}{h}$. Since $\sigma > h$ can be chosen arbitrarily, we deduce that $C_{\mathcal{G}}(T) \sim \frac{e^{hT}}{hT}$, as required.

Chapter 4

Asymptotic growth on translation surfaces

4.1 Introduction

In this chapter, we will use the method developed in the previous chapter for proving the asymptotic growth formulae for paths on infinite graphs, to prove some asymptotic results for translation surfaces.

The key insight here is that the growth of geodesics on translation surfaces depend on the growth of saddle connection paths on the surfaces (concatenations of saddle connections which form geodesics). To understand how saddle connection paths grow on the surface we observe that we can form an infinite graph (with a finite vertex set and a countable edge set) from a translation surface by taking the singularities as vertices and the saddle connections as edges. Then saddle connection paths on the surface *almost* correspond to the paths on the associated infinite graph. We say almost because any two edges which meet at a vertex on a graph form a path but due to the angle condition for saddle connection paths, two saddle connections which meet at a singularity may not form a saddle connection path. This detail will turn out not to be an issue after we make some minor changes to the countable matrices we introduced for infinite graphs.

We proceed by describing the counting and growth functions for which we wish to establish asymptotic formulae, namely functions which count saddle connection paths, volume of balls in universal covers, circumferences of geodesic circles on translation surfaces and closed geodesics of a bounded length. We then introduce countable matrices analogous to the ones used in the previous chapter and deduce that the countable matrices for translation surfaces have the same spectral properties as the ones we introduce for infinite graphs. The asymptotics for the aforementioned counting and growth functions will then follow from the Ikehara-Wiener Tauberian theorem.

4.2 Background

We begin by recalling some key notation we introduced in the background.

Let X be a translation surface with singularity set Σ. Let $h = h(X)$ denote the volume entropy of X. We denote the set of oriented saddle connections on X by $\mathcal{S}$, which we order by non-decreasing length. Recall that a saddle connection path p is a sequence of saddle connections which form a geodesic on X. We denote the length of such a path with respect to the metric on X by $\ell(p)$. Let $i(p)$ and $t(p)$ denote the initial and terminal singularity of an oriented saddle connection path p, respectively. Let $\mathcal{P}(x) := \{p : i(p) = x\}$ denote the set of saddle connection paths starting at x and $\mathcal{P}(x, R) := \{p : i(p) = x, \ell(p) \leq R\}$ denote the set of saddle connection paths starting at x of length less than or equal to $R > 0$.

We will now introduce the growth and counting functions that we will be proving asymptotic formulae for.

Let Vol_X denote the volume measure on X and $\mathrm{Vol}_{\widetilde{X}}$ denote the corresponding measure on the universal cover of X, $\widetilde{X}$. We define $\mathcal{B}(\widetilde{x}, R)$ to be the ball of radius $R > 0$ based at $\widetilde{x} \in \widetilde{X}$. Recall from the the background chapter that this ball in the universal cover corresponds to the set of end-points of geodesics based at $\widetilde{x}$ of length less than or equal to $R > 0$. Next, we define $\mathcal{C}(x, R)$ to be the "circle" based at x which corresponds to the set of end-points of geodesic paths based at x and of length equal to $R > 0$ and so this set is a one-dimensional curve. Note that this curve will have the same length as the boundary of $\mathcal{B}(\widetilde{x}, R)$.

Definition 4.2.1. *Let X be a translation surface with singularity set Σ and universal cover $\widetilde{X}$. After fixing $x \in \Sigma$ with lift $\widetilde{x}$, We define*

1. *$N(x, R) = \#\mathcal{P}(x, R)$ to be the number of saddle connection paths on X, based at x, and of length less than or equal to R;*
2. *$V(\mathcal{B}(\widetilde{x}, R)) = Vol_{\widetilde{X}}(\mathcal{B}(\widetilde{x}, R))$ to be the volume of $\mathcal{B}(\widetilde{x}, R)$ with respect to $Vol_{\widetilde{X}}$;*

3. $\ell(\mathcal{C}(x,R))$ *to be the length of the the circle* $\mathcal{C}(x,R)$ *with respect to the metric on* X*;*

4. $N_{x,y}(R)$ *to be the number of geodesics paths starting at* x*, ending at* y *and of length less than or equal to* R*, for* $x, y \in \Sigma$*; and let*

5. $\pi(R)$ *to be the number of closed geodesics on* X *up to homotopy and of length less than or equal to* R*.*

For simplicity we are considering (lifts of) base points in Σ for now however, we will see how we can generalize the method of proof to arbitrary points $z \in X$ in Section 4.5.

Our aim is to show that the functions above have an asymptotic formulae.

Theorem 4.2.2. *Let* X *be a translation surface with entropy* h*. Consider the counting and growth functions defined above. Then there exist constants* $A, C, D > 0$ *such that*

$$N(x,R) \sim Ae^{hR}, V(\mathcal{B}(\widetilde{x},R)) \sim (C/h)e^{hR},$$

$$\ell(\mathcal{C}(x,R)) \sim Ce^{hR} \text{ and } N_{x,y}(R) \sim De^{hR}.$$

The following asymptotic formula for the growth of closed geodesic was announced by Eskin based on joint work with Rafi (see [14]. They have not published a proof and so we will provide a proof for completeness in this chapter (see Section 4.6).

Theorem 4.2.3. *Let* X *be a translation surface with volume entropy* h*. Then*

$$\pi(R) \sim \frac{e^{hR}}{hR}.$$

We will proceed by introducing countable matrices associated to a given translation surface. We will then prove that these matrices satisfy the same spectral properties as the matrices in the previous chapter which allows us to deduce the asymptotic formulae.

4.3 Countable matrices for translation surface

Recall that the countable matrices in the previous chapter contained data about a graph's paths. In particular, the matrices kept track of which pairs of edges could form a path as well as exponential weightings of the lengths of the graph's edges. We would like to define an analogous family of matrices which do the same for saddle connection paths on a given translation surface.

Definition 4.3.1. *For $z \in \mathbb{C}$, we can associate to X the infinite matrices $M_z(s, s')$ defined by*

$$M_z(s,s') = \begin{cases} e^{-z\ell(s')} & \textit{if } ss' \textit{ form a saddle connection path,} \\ 0 & \textit{otherwise.} \end{cases}$$

where the rows and columns are indexed by $s \in \mathcal{S}$, partially ordered by their lengths.

For infinite graphs, we restricted our attention to graphs which satisfied certain hypotheses which ensured that the matrices had the necessary spectral properties which guaranteed that we could apply the Tauberian theorem.

We will now show that the saddle connection paths on any translation surface satisfy analogous properties that ensure that the matrices defined above have the same spectral properties as the matrices defined for infinite graphs.

Translation surface properties. We claim that the following three properties hold for all translation surfaces X.

(T1) For all $\sigma > 0$ we have $\sum_{s\in\mathcal{S}} e^{-\sigma\ell(s)} < \infty$;

(T2) There exists some $L > 0$ such that for all pairs of directed saddle connections $s, s' \in \mathcal{S}$, there exists a saddle connection path beginning with s and ending with s' of length less than or equal to $\ell(s) + L + \ell(e')$; and

(T3) There does not exist a $d > 0$ such that

$$\{\ell(c) : c \text{ is a closed saddle connection path}\} \subset d\mathbb{N}.$$

The first two properties follow from existing work for translation surfaces.

Property (T1) follows from the lower bound in the following result for the growth of saddle connections on translation surfaces (see [33] and [34]).

Proposition 4.3.2. *Let X be a translation surface and let $N(X, L)$ denote the number of saddle connections on X of length less than or equal to L. Then there exists constants $0 < c_1 < c_2 < \infty$ such that*

$$c_1 L^2 \leq N(X, L) \leq c_2 L^2,$$

for L sufficiently large.

The following result from [8] (which we restate for our purposes here) allows us to conclude that (T2) holds for all translation surfaces.

Proposition 4.3.3. *Let X be a translation surface. There exists a constant $L > 0$ such that for all pairs of saddle connections $s, s' \in \mathcal{S}$, there exists a saddle connection path p which starts with s and ends with s' of length $\ell(p) \leq \ell(s) + L + \ell(s')$.*

We will provide a proof that Property (T3) holds for all translation surfaces.

Proposition 4.3.4 (T3)**.** *Let X be a translation surface. Then there does not exist a $d > 0$ such that*

$$\{\ell(c) : c \textit{ is a closed saddle connection path}\} \subset d\mathbb{N}.$$

Proof. We proceed by contradiction. First note that if the lengths of all closed saddle connection paths were an integer multiple of some constant d, then the length of every saddle connection would be an integer multiple of $d/2$. To see this, let s be any saddle connection on X. If $i(s) = t(s)$ then s is a closed saddle connection path and so we are done. If $i(s) \neq t(s)$ then by Proposition 4.3.3, there exists a closed saddle connection path c_i such that c_i passes through $i(s)$ and $\bar{s}c_i s$ forms a saddle connection path (where $\bar{s}$ is the saddle connection s with reversed orientation). Similarly, there exists a closed saddle connection path c_t which starts and ends at $t(s)$, such that $sc_t\bar{s}$ forms a saddle connection path (Figure 4.1). The concatenation $sc_t\bar{s}c_i$ is also a closed saddle connection path of length $2\ell(s)+\ell(c_t)+\ell(c_i)$ and so by our assumption, $\ell(s) \in (d/2)\mathbb{N}$. We shall now show that this is impossible.

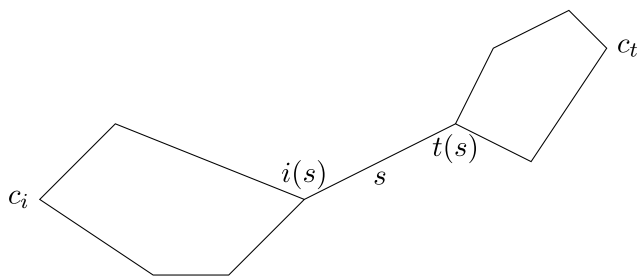


Figure 4.1: Given a saddle connection s we can find two closed saddle connection paths c_i and c_t which start and end at $i(s)$ and $t(s)$ respectively.

It follows from [36] that X contains an embedded cylinder C, i.e., the product of a circle with an interval $I = [0, L]$ for some $L > 0$, whose boundaries consist of a single saddle connection or multiple parallel saddle connections. We will now construct a countable family of triangles whose edges correspond to unions of saddle connections using this cylinder (Figure 4.2) and whose edge length spectrum cannot lie in $(d/2)\mathbb{N}$.

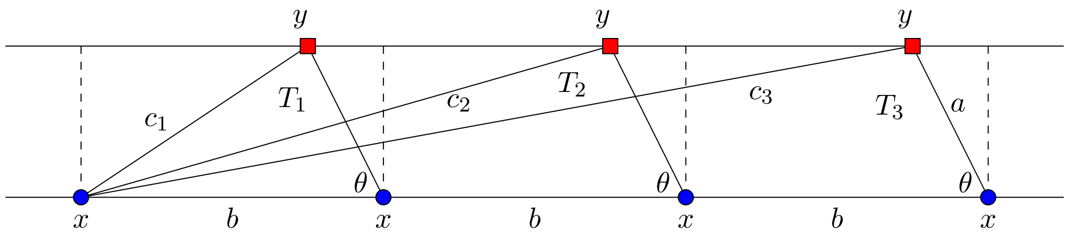


Figure 4.2: Three copies of a cylinder on X with two singularities on separate boundaries represented by circles and squares (labelled x and y, respectively). The corresponding triangles T_1, T_2 and T_3 are also drawn. The edges of T_n are given by a, n copies of b and c_n which are all saddle connections or unions of saddle connections. The angle between a and b is given by $\theta \notin \pi\mathbb{Z}$.

Fix two singularities x and y, one from each boundary of the cylinder. Let b denote the union of saddle connections that form the boundary of the cylinder connecting x to itself. Let a be the shortest saddle connection connecting x to y across the cylinder. Then consider the unique saddle connection c_n connecting x to y which is defined to be the third side in a triangle T_n whose other edges are b concatenated with itself n times and a (Figure 4.2). By our assumption, the length of each edge is an integer multiple of $d/2$ because the edges are formed from saddle connections. To see that this leads to a contradiction, we first observe that by scaling all of the T_n by a factor of $2/d$, T_n's edges will be of integer length.

Observe that the angle θ between a and the edge formed by n copies of b in T_n is independent of n. By the cosine formula

$$\ell(c_n) = \sqrt{\ell(a)^2 + (n\ell(b))^2 - 2\ell(a)n\ell(b)\cos(\theta)} \tag{1}$$

for all $n \in \mathbb{N}$.

Note if $\cos(\theta)$ were irrational then $\ell(c_n)$ would not be an integer for any $n \in \mathbb{N}$. Hence we can write $\cos(\theta) = p/q$ where p and q are coprime integers.

By squaring both sides of Equation (1) and multiplying through by q^2, we see that

$$q^2\ell(c_n)^2 = (q\ell(a))^2 + (qn\ell(b))^2 - 2\ell(a)n\ell(b)pq.$$

By replacing n with a variable $x \in \mathbb{R}$, we obtain a polynomial

$$P(x) := (q\ell(b))^2x^2 - 2\ell(a)\ell(b)pqx + (q\ell(a))^2 \in \mathbb{Z}[x]$$

whose values at the integers are squares.

By a classical result (see for instance [46]), it follows that there exists some $Q(x) \in \mathbb{Z}[x]$ such that $P(x) = Q(x)^2$.

Let $Q(x) = \sum_{i=0}^{M} d_i x^i$ for some $M \in \mathbb{N}$ and $d_i \in \mathbb{Z}$. By equating the coefficients in the equation $P(x) = Q(x)^2$, we see that $d_i = 0$ for $i \geq 2$ and obtain the following equations for the remaining coefficients:

$$d_1^2 = (q\ell(b))^2,\ d_0^2 = (q\ell(a))^2, \text{ and } d_0 d_1 = -\ell(a)\ell(b)pq.$$

By comparing the above equations, it follows that $p = \pm q$. Hence, $\cos(\theta) = \pm 1$, i.e., $\theta \in \pi\mathbb{Z}$, which implies that the width of the cylinder is zero. Hence we obtain a contradiction. $\square$

We have seen that Properties $(T1)$, $(T2)$ and $(T3)$ hold for all translation surfaces X. We are now in a position to follow the same approach we used for infinite graphs to deduce the asymptotics in Theorem 4.2.2.

4.4 Asymptotics for $N(x, R)$, $V(\mathcal{B}(\widetilde{x}, R))$, $\ell(\mathcal{C}(x, R))$ and $N_{x,y}(R)$

In this section, we will prove Theorem 4.2.2. We will follow the method used in the previous chapter. We will leave the proof of the asymptotic for closed geodesics to the final section of this chapter (Section 4.6).

As in the case of infinite graphs, in order to deduce asymptotic results for translation surfaces we will be using the Ikehara-Wiener Tauberian theorem which we recall now for the reader's convenience.

Ikehara-Wiener Tauberian Theorem. Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing and right-continuous function. Formally denote $\eta(z) := \int_0^\infty e^{-zR} d\rho(R)$, for $z \in \mathbb{C}$. Then suppose that $\eta(z)$ has the following properties:

1. there exists some $a > 0$ such that $\eta(z)$ converges absolutely and is analytic on $Re(z) > a$;
2. $\eta(z)$ has a meromorphic extension to a neighbourhood of the half-plane $Re(z) \geq a$;
3. a is a simple pole for $\eta(z)$, i.e., $C = \lim_{z \to a}(z-a)\eta(z)$ exists and is positive; and
4. the extension of $\eta(z)$ has no poles on the line $Re(z) = a$ other than a.

Then $\rho(R) \sim (C/a)e^{aR}$ as $R \to \infty$.

We begin by noting that the functions $N(x, R))$, $V(\mathcal{B}(\widetilde{x}, R))$, $\ell((\widetilde{x}, R))$, $N_{x,y}(R)$ and $\pi(R)$ that we wish to establish asymptotic formulae for are clearly non-negative, right-continuous and non-decreasing. We will now define four complex functions $\eta_N(z)$, $\eta_V(z)$, $\eta_\ell(z)$ and $\eta_{x,y}(z)$ which correspond to $N(x, R)$, $V(\mathcal{B}(\widetilde{x}, R))$, $\ell(\mathcal{C}(x, R))$ and $N_{x,y}(R)$ respectively, by

$$\begin{aligned}
\eta_N(z) &= \int_0^\infty e^{-zR} dN(x, R)), \\
\eta_V(z) &= \int_0^\infty e^{-zR} dV(\mathcal{B}(\widetilde{x}, R)), \\
\eta_\ell(z) &= \int_0^\infty e^{-zR} d\ell(\mathcal{C}(x, R)) \text{ and} \\
\eta_{x,y}(z) &= \int_0^\infty e^{-zR} dN_{x,y}(R).
\end{aligned}$$

As before, we will rewrite these complex functions in terms of the matrices M_z, more specifically, their associated operators $\widehat{M_z}$. By Lemma 3.6.4, for $Re(z) > h$ we can write

$$\begin{aligned}
\eta_N(z) &= \sum_{p \in \mathcal{P}(x)} e^{-z\ell(p)} \\
&= \underline{v}(z) \cdot \Big(\sum_{n=0}^\infty \widehat{M_z}^n \Big) \underline{1} \\
&= \underline{v}(z) \cdot \Big(I - \widehat{M_z} \Big)^{-1} \underline{1},
\end{aligned}$$

where $\underline{v}(z) = (\chi_{\mathcal{E}_x}(s) e^{-z\ell(s)})_{s \in \mathcal{S}}$, where $\chi_{\mathcal{E}_x}$ denotes the characteristic function of the set $\mathcal{E}_x = \{s \in \mathcal{S} : i(s) = x\}$ of saddle connections starting from the singularity

$x \in \Sigma$, and $\underline{1} = (1)_{s \in \mathcal{S}} \in \ell^\infty(\mathbb{C})$. Note that by (T1), $\underline{v}(z) \in \ell^1(\mathbb{C})$. By Lemma 3.6.4, $\eta_N(z)$ is analytic and bounded on $Re(z) > h$.

Let $\mathcal{P}(x, y)$ denote the set of saddle connection paths starting at a and ending at $y \in \Sigma$. We can rewrite $\eta_{x,y}(z)$ as

$$\eta_{x,y}(z) = \sum_{p \in \mathcal{P}(x,y)} e^{-z\ell(p)} = \underline{v}(z) \cdot \left(I - \widehat{M_z}\right)^{-1} \underline{u}_y,$$

where $\underline{u}_y \in \ell^\infty(\mathbb{C})$ is given by $(\underline{u}_y)_s = \chi_y(s)$ where $\chi_y(s) = 1$ if $t(s) = y$ and 0 otherwise.

In order to rewrite $\eta_V(z)$ and $\eta_\ell(z)$ in terms of $\widehat{M_z}$ we recall Lemma 2.8.3 for the reader's convenience, which relates $\mathcal{B}(\widetilde{x}, R)$ to the saddle connection paths on X.

Lemma 4.4.1. *Let X be a translation surface and fix a singularity $x \in \Sigma$. Let $2\pi(k(x) + 1)$ be the cone angle of x. Then for $R > 0$,*

$$V(\mathcal{B}(\widetilde{x}, R)) = (k(x) + 1)\pi R^2 + \sum_{p \in \mathcal{P}(x,R)} k(t(p))\pi(R - \ell(p))^2,$$

where the singularity at the end of path p has cone angle $2\pi(k(t(p)) + 1)$.

A similar expression for the circumference $\ell(\mathcal{C}(x, R))$ holds:

$$\ell(\mathcal{C}(x, R)) = 2(k(x) + 1)\pi R + \sum_{p \in \mathcal{P}(x,R)} 2k(t(p))\pi(R - \ell(p)).$$

Recall that for standard Euclidean circles, the circumference of the circle is given by $\frac{\partial}{\partial r}(\pi r^2) = 2\pi r$. Similar reasoning allows us to deduce that

$$\ell(\mathcal{C}(\widetilde{x}, R)) = \frac{d}{dR} V(\mathcal{B}(\widetilde{x}, R)). \tag{2}$$

Note that the expressions for $V(\mathcal{B}(\widetilde{x}, R))$ and $\ell(\mathcal{C}(\widetilde{x}, R))$ imply that we can write the entropy of X as $h = \lim_{R\to\infty} \frac{1}{R} \log(\ell(\mathcal{C}(\widetilde{x}, R)))$ or $h = \lim_{R\to\infty} \frac{1}{R} \log(N(x, R))$.

Using the above expression for $\ell(\mathcal{C}(x, R))$, we can use integration-by-parts to rewrite $\eta_\ell(z)$ in terms of saddle connection paths on X.

Lemma 4.4.2. *For $Re(z) > h$, we have that*

$$\eta_\ell(z) = \frac{2\pi}{z}(k(x) + 1) + \frac{2\pi}{z} k(t(p)) \sum_{p \in \mathcal{P}(x,R)} e^{-z\ell(p)}.$$

Furthermore, $\eta_\ell(z)$ is analytic on $Re(z) > h$.

Proof. We begin by showing that for all z such that $Re(z) > h$,

$$\lim_{R\to\infty} \ell(\mathcal{C}(x,R))e^{zR} = 0.$$

To see this, note that because $h = \lim_{R\to\infty} \frac{1}{R}\log(\ell(\mathcal{C}(\widetilde{x},R)))$, given $\epsilon > 0$, there exists some R_ϵ such that for $R > R_\epsilon$, $\ell(\mathcal{C}(\widetilde{x},R)) < e^{(h+\epsilon)R}$. Hence if we let $\epsilon = (Re(z)-h)/2$

$$\lim_{R\to\infty} |\ell(\mathcal{C}(x,R))e^{zR}| \leq \lim_{R\to\infty} e^{(h+\epsilon)R}e^{Re(z)R} = \lim_{R\to\infty} e^{-\epsilon R} = 0.$$

For $Re(z) > h$ we use integration-by-parts to write to rewrite $\eta_\ell(z)$ as follows

$$\begin{aligned}
\eta_\ell(z) &= \int_0^\infty e^{-zR} d\ell(\mathcal{C}(x,R)) \\
&= [e^{-zR}\ell(\mathcal{C}(x,R))]_0^\infty + z\int_0^\infty e^{-zR}\ell(\mathcal{C}(x,R))dR \\
&= z\left(2\pi(k(x)+1)\int_0^\infty Re^{-zR}dR + \sum_{p\in\mathcal{P}(x)} \int_{\ell(p)}^\infty 2k(t(p))\pi(R-\ell(p))e^{-zR}dR\right) \\
&= z\left(2\pi(k(x)+1)\int_0^\infty Re^{-zR}dR + \sum_{p\in\mathcal{P}(x)} 2\pi k(t(p))e^{-z\ell(p)}\int_0^\infty Te^{-zT}dT\right) \\
&= \frac{2\pi}{z}(k(x)+1) + \frac{2\pi}{z}k(t(p)) \sum_{p\in\mathcal{P}(x,R)} e^{-z\ell(p)}.
\end{aligned}$$

Note that we used the fact that the limit disappears as $Re(z) > h$ as well as the change of variable $T = R - \ell(p)$ for each saddle connection path $p \in \mathcal{P}(x)$.

By Proposition **??**, it follows that the summation over all of these contributions is uniformly convergent for $Re(z) > h$ which gives the required result. $\square$

We can rewrite $\eta_V(z)$ using a similar approach or by using the relationship between $V(\mathcal{B}(\widetilde{x},R))$ and $\ell(\mathcal{C}(x,R))$ (Equation (1)). In particular, for $Re(z) > h$, we can rewrite $\eta_V(z)$ using Equation (1) and the proof of the above Lemma:

$$\begin{aligned}
\eta_V(z) &= \int_0^\infty e^{-zR} dV(\mathcal{B}(\widetilde{x}, R)) \\
&= \int_0^\infty e^{-zR} \frac{d}{dR} V(\mathcal{B}(\widetilde{x}, R)) dR \\
&= \int_0^\infty e^{-zR} \ell(\mathcal{C}(x, R)) dR \\
&= \frac{\eta_\ell(z)}{z}.
\end{aligned}$$

Hence

$$\eta_V(z) = \frac{2\pi}{z^2}(k(x)+1) + \frac{2\pi}{z^2} k(t(p)) \sum_{p \in \mathcal{P}(x,R)} e^{-z\ell(p)}.$$

We can now write $\eta_\ell(z)$ and $\eta_V(z)$ in terms of $\widehat{M_z}$. Let $\underline{u} = (k(t(s)))_{s \in \mathcal{S}} \in \ell^\infty(\mathbb{C})$. Then using Lemma 4.4.2 and Lemma 3.6.4, we can write

$$\eta_\ell(z) = \frac{2\pi}{z}\left((k(x)+1) + \underline{v}(z) \cdot \left(I - \widehat{M_z}\right)^{-1} \underline{u}\right) = z\eta_V(z),$$

on $Re(z) > h$.

Note that because of the above formula, the residue of $\eta_\ell(z)$ at $z = h$ is given by h times the residue of $\eta_V(z)$ at h.

We can now deduce the asymptotic results for $N(R)$, $V(\mathcal{B}(\widetilde{x}, R))$, $\ell(\mathcal{C}(x, R))$ and $N_{x,y}(R)$. In particular, because M_z satisfies $(T1)-(T3)$, we can apply the proofs for Lemma 3.6.4, Proposition 3.7.2, Proposition 3.8.3 and Proposition 3.8.5 to deduce that $\eta_N(z)$, $\eta_V(z)$, $\eta_\ell(z)$ and $\eta_{x,y}(z)$ have meromorphic extensions to $Re(z) > 0$ and that these extensions all satisfy the assumptions of the Tauberian theorem. Hence there exist constants $A > 0$, $C > 0$ and $D > 0$ such that

$$N(x, R) \sim Ae^{hR},\ V(\mathcal{B}(\widetilde{x}, R)) \sim (C/h)e^{hR},$$

$$\ell(\mathcal{C}(x, R)) \sim Ce^{hR} \text{ and } N_{x,y}(R) \sim De^{hR}.$$

Note that the constant in each asymptotic formula is the residue given by

$$\lim_{z \to h} (z-h)\eta(z)$$

for the respective complex function.

As a consequence we see that if we change singularity $x \in \Sigma$ we are using as the initial point for our geodesics, we will typically get different coefficients for volume and circumference growth, although the growth rate, h, remains the same.

4.5 Basepoints in $X \backslash \Sigma$

In this section, we consider a slight generalization which allows us to consider geodesic paths that start at an arbitrary point $x \in X$ rather than a singularity.

Let $y \in X \backslash \Sigma$ and let G be the set of oriented geodesics g, from y to a singularity, such that g has length $\ell(g)$. Order G by non-decreasing lengths. We define a family of matrices $(P_z)_{z \in \mathbb{C}}$ where

1. the rows are indexed by such geodesics g and the columns are indexed by the oriented saddle connections s;
2. the non-zero entries correspond to pairs g, s such that the geodesic g and saddle connection s form a geodesic on X; and
3. The non-zero entries are given by $P_z(g, s) = e^{-z\ell(s)}$.

One can then modify the complex functions using these new matrices. For example, if we want to prove an asymptotic result for $\ell(\mathcal{C}(y, R))$, the length of the circle of radius R based at $y \in X \backslash \Sigma$, we can consider

$$\eta_{V,y}(z) = \frac{2\pi}{z}(k(y) + 1) + \frac{2\pi}{z}\underline{v}_y(z) \cdot \widehat{P_z}(I - \widehat{M_z})^{-1}\underline{u},$$

where $\underline{v}_y(z) = (e^{-z\ell(g)})_{g \in G}$. Note that it follows from the triangle inequality, Property $(T1)$, and the fact that X has finite diameter, that $\underline{v}_y(z) \in \ell^1(\mathbb{C})$, i.e. the number of geodesics based at y which end at a singularity of length less than or equal to R grows quadratically. Finally, the asymptotic result follows by applying the same method as before.

4.6 Closed geodesics

We conclude this chapter by proving the asymptotic result for closed geodesics on translation surfaces (Theorem 4.2.3). We note that this result, due to Eskin and Rafi, was announced by Eskin in the talk [14]. Their proof has not been published at the time of writing so we provide a proof for completeness.

The asymptotic result for closed geodesics on translation surfaces follows from the asymptotic result for closed paths on infinite graphs (Chapter 3 Section 3.9) and the work in this chapter.

We begin by noting that geodesics on translation surfaces will either be saddle connection paths or will be part of a family of closed geodesics which span the circumference of a cylinder and so do not pass through a singularity. We adopt the convention that we do not count closed geodesics which do not pass through a singularity, thus avoiding the complication of having uncountably many closed geodesics of the same length.[1]

Definition 4.6.1. *A closed geodesic on a translation surface is a saddle connection path corresponding to an (allowed) finite string of oriented saddle connections $q = (s_1, \dots, s_n)$, of length $|q| = n$ and up to cyclic permutation, with the additional requirement that $s_n s_1$ is a saddle connection path. We say that q is primitive if it is not a multiple concatenation of a shorter closed geodesic.*

Let X be a translation surface with singularity set Σ and volume entropy $h = h(X)$. Let $\mathcal{Q}(T)$ denote the set of such oriented closed geodesics on X of length less than or equal to T, and let $\mathcal{Q} := \bigcup_{T>0} \mathcal{Q}(T)$.

It follows from (T2) and the fact that there are finitely many singularities on X, that the exponential growth rate of $\#\mathcal{Q}(T)$ is equal to the volume entropy of X, i.e.

$$h = \lim_{T\to\infty} \frac{1}{T} \log \#\mathcal{Q}(T). \tag{3}$$

We define the following zeta function analogous to the zeta function in Definition 3.9.2.

Definition 4.6.2. *We can formally define the zeta function by the Euler product*

$$\zeta_X(z) = \prod_{q\in\mathcal{Q}} \left(1 - e^{-z\ell(q)}\right)^{-1}, z \in \mathbb{C}$$

where the product is over all oriented primitive closed paths.

[1] Alternatively, we could count one such geodesic from each family but then their growth would only be polynomial and this would not affect the asymptotic.

By Lemma 3.9.3, it follows that $\zeta_X(z)$ is a non-zero analytic function on $Re(z) > h$.

Once again, we will rewrite $\zeta_X(z)$ in terms of the following family of infinite matrices associated to X.

Definition 4.6.3. *For $z \in \mathbb{C}$, we can associate to X the infinite matrices $M_z(s, s')$ defined by*

$$M_z(s, s') = \begin{cases} e^{-z\ell(s')} & \textit{if } ss' \textit{ form a saddle connection path,} \\ 0 & \textit{otherwise.} \end{cases}$$

where the rows and columns are indexed by $s \in \mathcal{S}$, partially ordered by their lengths.

Let $k \in \mathbb{N}$. We can write M_z as follows

$$M_z = \begin{pmatrix} A_{z,k} & B_{z,k} \\ C_{z,k} & D_{z,k} \end{pmatrix},$$

where $A_{z,k}$ is the $k \times k$ finite sub-matrix of M_z corresponding to the first $k \in \mathbb{N}$, say, oriented saddle connections and the other sub-matrices $B_{z,k}, C_{z,k}, D_{z,k}$ are infinite. We formally define $W_{z,k} := A_{z,k} + B_{z,k}(I - D_{z,k})^{-1}C_{z,k}$.

Next we define the following auxiliary function

$$g_k(z) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{s} \in \mathcal{S}_n(k)} e^{-z\ell(\underline{s})}\right),$$

where $\mathcal{S}_n(k) \subset \mathcal{S}_n$ denotes the set of oriented saddle connection strings $\underline{s} = (s_1, \ldots, s_n)$ of length n corresponding to oriented closed geodesics for which all of the s_j $(1 \leq j \leq n)$ are disjoint from the first k edges in the ordering on $\mathcal{S}$.

By the proof of Lemma 3.9.5, on $Re(z) > h$, we can write

$$\zeta_X(z) = \frac{1}{g_k(z)\det(I - W_{z,k})}.$$

Using the fact that all translation surfaces satisfy properties (T1)–(T3), we can apply the proofs of Lemma 3.9.6 and Lemma 3.9.7 to deduce that $\zeta_X(z)$ satisfies the assumptions of the Ikehara-Wiener Tauberian theorem (see Theorem 2.2.2).

The fact that $\zeta_X(z)$ satisfies the assumptions of the Tauberian theorem, allows us to apply the proof in Section 3.9.3 to $\zeta_X(z)$, to deduce the asymptotic for closed geodesics on translation surfaces, Theorem 4.2.3.

Chapter 5

Distribution of large circles on translation surfaces

5.1 Introduction

In this chapter, we will study how large circles, $\mathcal{C}(x,R)$, on X distribute on the surface as R tends to infinity. We will show that as R goes to infinity, the circles distribute with respect to some measure on the surface. In the final section, we will briefly look at a distribution result for closed geodesics on translation surfaces.

We begin by stating the main theorem. Let X be a translation surface of entropy $h = h(X)$ and fix a singularity $x \in \Sigma$ (the singularity set of X). Let $\mathcal{C}(x,R)$ be the curve (or "circle") defined by the set of endpoints of geodesics on X of length R which begin at x. Note that if $\widetilde{x}$ is a lift of x, then $\ell(\mathcal{C}(x,R)) = \ell(\mathcal{C}(\widetilde{x},R))$ where $\ell(\mathcal{C}(\widetilde{x},R))$ denotes the length of the circle based at $\widetilde{x} \in \widetilde{X}$.

We define a family of natural probability measures μ_R supported on the sets $\mathcal{C}(x,R)$ for $R > 0$. These correspond to the normalized arc length measure on the curve $\mathcal{C}(x,R)$.

Definition 5.1.1. *We can define a family of probability measures μ_R ($R > 0$) on X by*

$$\mu_R(A) = \frac{\ell(\mathcal{C}(x,R)\cap A)}{\ell(\mathcal{C}(x,R))}, \text{ for Borel sets } A \subset X,$$

where $\ell(\mathcal{C}(x,R)\cap A)$ denotes the length of the part of $\mathcal{C}(x,R)$ which lies in A.

The next result describes the convergence of the probability measures μ_R as the radius R tends to infinity.

Theorem 5.1.2 (Circle distribution)**.** *The sequence of measures $(\mu_R)_{R>0}$ converge in the weak-* topology to a measure μ, which is equivalent to the volume measure Vol_X on X, i.e. $\lim_{R\to\infty}\int f d\mu_R = \int f d\mu$ for any $f \in C(X)$.*

We note that this is not quite a traditional equidistribution result in the sense that although μ is equivalent to the volume measure Vol_X, it seems likely that the Radon-Nikodym derivative is not constant.

Example 5.1.3 (L-shaped surface)**.** *We can consider an L-shaped surface where the identification of opposite sides gives a surface of genus 2 and only one singularity. The singularity comes from the identification of the corners and has a cone-angle of 6π. The circle drawn represents $\mathcal{C}(x, R)$.*

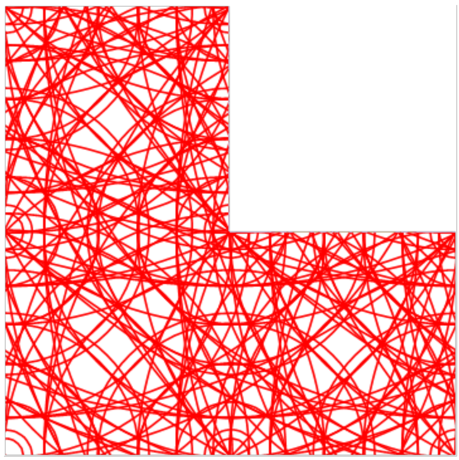


Figure 5.1: A large circle projected onto a L-shaped domain.

We briefly describe the strategy for the proof of Theorem 5.1.2. We begin by noting that because the set of all Borel probability measures on X is compact in the weak-* topology, it follows that the set of measures μ_R accumulate on some set of limit measures as $R \to \infty$. To show that the set of limit measures contains only one measure, it suffices to show that $\lim_{R\to\infty}\mu_R(B)$ converges for a set of sufficiently small closed balls $B \subset X\backslash\Sigma$ that generate the Borel σ-algebra.

To deduce the above limit, we first prove asymptotic results for $\ell(\mathcal{C}(x, R) \cap B)$ for small closed balls $B \subset X\backslash\Sigma$. In particular, for such balls B, we will show that there exist some constants $C(B) > 0$ such that

$$\ell(\mathcal{C}(x, R) \cap B) \sim C(B)e^{hR} \text{ as } R \to \infty.$$

Combining this asymptotic with the asymptotic $\ell(\mathcal{C}(x,R)) \sim Ce^{hR}$ proved in the previous chapter, we have

$$\lim_{R\to\infty} \mu_R(B) = \lim_{R\to\infty} \frac{\ell(\mathcal{C}(x,R)\cap B)}{\ell(\mathcal{C}(x,R))} = \lim_{R\to\infty} \frac{C(B)e^{hR}}{Ce^{hR}} = \frac{C(B)}{C} =: \mu(B).$$

This then proves the theorem.

In order to deduce the asymptotic for the functions $\ell(\mathcal{C}(x,R)\cap B)$, it might seem natural to consider an appropriate complex function for $\ell(\mathcal{C}(x,R)\cap B)$ and then apply the Tauberian theorem, as in the proof of the asymptotic formulae for the circumference of circles. However, due to the fact that $\ell(\mathcal{C}(x,R)\cap B)$ may not be monotonic, we cannot directly apply the Tauberian theorem. Therefore, we will instead prove an asymptotic result for the non-decreasing continuous function $V_A(R) := \mathrm{Vol}_{\widetilde{X}}(\mathcal{B}(\tilde{x},R)\cap \widetilde{A})$, for Borel sets $A \subset X$ i.e., the area of the intersection of a ball of radius R in the universal cover of X intersected with the lifts of A. We will then be able to use these asymptotics to indirectly deduce the corresponding asymptotics for $\ell(\mathcal{C}(x,R)\cap B)$ for small closed balls[1] in $X\backslash\Sigma$.

5.1.1 Notation

We begin by introducing some useful notation. Once again, we fix a translation surface X with singularity set Σ and choice of base point for the saddle connection paths, $x \in \Sigma$. Let $\mathcal{P}(x)$ denote the the set of saddle connection paths based at x and $\mathcal{P}(x,R)$ denote the set of saddle connection paths based at x of length less than or equal to $R > 0$. Let $\pi : \widetilde{X} \to X$ denote the canonical projection from the universal cover $\widetilde{X}$ to X. Let $\widetilde{\Sigma}$ and $\widetilde{\mathcal{S}}$ denote the lifts of the singularity set Σ and oriented saddle connections $\mathcal{S}$, respectively. Throughout this section, we fix a lift of x, $\widetilde{x} \in \widetilde{\Sigma}$. Let $\widetilde{p}$ denote a lift of $p \in \mathcal{P}(x)$. Then we let $\ell(\widetilde{p})$ denote the length of $\widetilde{p}$ and $i(\widetilde{p}), t(\widetilde{p}) \in \widetilde{\Sigma}$ denote the singularities at the beginning and end of $\widetilde{p}$, respectively.

Definition 5.1.4. *Let $z \in \Sigma$ with a choice of lift $\widetilde{z} \in \widetilde{\Sigma}$ (i.e., $\pi(\widetilde{z}) = z$) and let $R > 0$.*

1. *We denote a Euclidean disk $\mathcal{D}(\widetilde{z},R)$ in $\widetilde{X}$ (with centre $\widetilde{z}$ and radius $R > 0$) by*

$$\mathcal{D}(\widetilde{z},R) \subset \widetilde{X}$$

[1]The proof of the asymptotic result for $V_A(R)$ works for Borel sets A, however we need to restrict our attention to open balls B to deduce the asymptotic for $\ell(\mathcal{C}(x,R)\cap B)$.

consisting of the set of those points $\widetilde{y} \in \widetilde{X}$ which are joined to $\widetilde{z}$ by a straight line segment of length at most $R > 0$, which does not have a singularity in its interior.

2. *Let $p \in \mathcal{P}(x)$ be a saddle connection path with unique lift $\widetilde{p}$ based at $\widetilde{x} \in \widetilde{\Sigma}$. Let $\widetilde{w} := t(\widetilde{p}) \in \widetilde{\Sigma}$. We define a Euclidean sector*

$$\mathcal{E}(p, R) \subset \mathcal{D}(\widetilde{w}, R) \subset \widetilde{X}$$

associated to p (with centre $\widetilde{w}$ and radius $R > 0$) by the set of points $y \in \widetilde{X}$ which are joined to $\widetilde{w}$ by a straight line segment of length at most $R > 0$, which does not have a singularity in its interior, and which additionally forms a geodesic in $\widetilde{X}$ with $\widetilde{p}$.

On occasion it will be convenient to consider sectors on X, which we define in an analogous way and denote by $\mathcal{E}_X(p, R)$.

Given a radius $R > 0$ we can write the ball $\mathcal{B}(\widetilde{x}, R)$ in $\widetilde{X}$ as

$$\mathcal{B}(\widetilde{x}, R) = \mathcal{D}(\widetilde{x}, R) + \bigcup_{p \in \mathcal{P}(x,R)} \mathcal{E}(p, R - \ell(p)).$$

Fix a Borel set $A \subset X$. As mentioned at the beginning of this section, we are interested in an asymptotic formula for $V_A(R) := \mathrm{Vol}_{\widetilde{X}}(\mathcal{B}(\widetilde{x}, R) \cap \widetilde{A})$. We will proceed with a similar approach to the one we used for the asymptotic results in the previous section by making the following observation, analogous to Lemma 2.8.3.

Lemma 5.1.5. *For $R > 0$ we can write,*

$$V_A(R) = Vol_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A}) + \sum_{p \in \mathcal{P}(x,R)} Vol_{\widetilde{X}}(\mathcal{E}(p, R - \ell(p)) \cap \widetilde{A}). \tag{1}$$

where the first term is the volume of the Euclidean disk and the second term is expressed in terms of the volumes of Euclidean sectors.

The heuristic of the basic identity (1) is illustrated in Figure 2.

Example 5.1.6. *In the particular case that $A = X$, the identity reduces to*

$$V_X(R) = (k(x) + 1)\pi R^2 + \sum_{p \in \mathcal{P}(x,R)} k(t(p))\pi(R - \ell(p))^2.$$

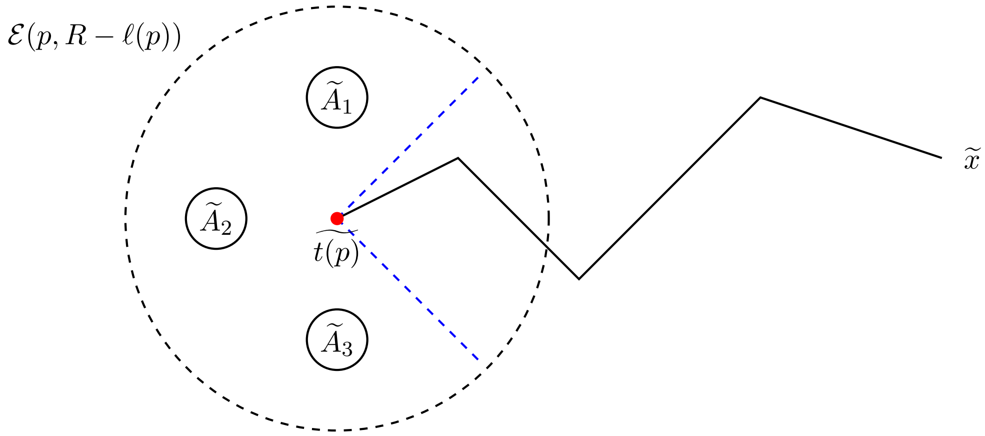


Figure 5.2: The ball $\mathcal{B}(\widetilde{x}, R) \subset \widetilde{X}$ is a union of appropriate Euclidean sectors $\mathcal{E}(\cdot, \cdot)$ centred at lifts of singularities. We are interested in the volume of the lifts of A, represented by $\widetilde{A}_1$, $\widetilde{A}_2$ and $\widetilde{A}_3$, which intersect $\mathcal{E}(p, R-\ell(p))$.

5.1.2 Asymptotic formula for $V_A(R)$

In order to derive an asymptotic formula for $V_A(R)$, we can use the same strategy developed for proving the asymptotic results in the previous two chapters. In particular, we aim to show that the following complex function satisfies the hypotheses of the Ikehara-Wiener Tauberian theorem (Theorem 2.2.2).

Definition 5.1.7. *For Borel sets $A \subset X$ with $Vol_X(A) > 0$, we can formally define a complex function by the Riemann-Stieltjes integral*

$$\eta_A(z) = \int_0^\infty e^{-zR} dV_A(R), \text{ for } z \in \mathbb{C}. \tag{2}$$

First we need to show that the growth rate of $V_A(R)$ is positive. Note that if $\mathrm{Vol}_X(A) = 0$ then $V_A(R) = 0$ for all $R > 0$. Before we proceed, we require the following lemma (a similar result can be found in [8]).

Lemma 5.1.8. *Let $A \subset X$ be a Borel set such that $\mathrm{Vol}_X(A) > 0$. Let $\mathrm{diam}(X)$ denote the finite diameter of X. Then there exists a saddle connection $s' \in \mathcal{S}$ such that*

$$\mathrm{Vol}_X(\mathcal{E}_X(s', \mathrm{diam}(X)) \cap A) > 0.$$

Proof. We require two simple preliminary results.

Claim 1. For any $x \in X$ there is a straight line segment g_x joining x to some singularity $y := t(g_x) \in \Sigma$ of length at most $\mathrm{diam}(X)$.

Proof of claim 1. A translation surface is a geodesic space of finite diameter because it has finite volume. In particular, we can connect x to a singularity in X by a geodesic which necessarily takes the form $p_x = g_x s_1 \ldots s_n$ or $p_x = g_x$ of length $\ell(p_x) \leq \mathrm{diam}(X)$, where the s_i are oriented saddle connections and g_x is an oriented straight line segment from x to some some singularity $t(g_x) \in \Sigma$. In either case, g_x is the required straight line segment. $\square$

Claim 2. Let $a \in A$. By claim 1, there exists an oriented straight line segment g_a connecting a to some singularity $t(g_a)$. The sector $\mathcal{E}_X(g_a, 2\mathrm{diam}(X))$ must contain a singularity.

Proof of claim 2. Assume for a contradiction that $\mathcal{E}_X(g_a, 2\mathrm{diam}(X)) \cap \Sigma = \emptyset$. Since the angle of the sector is greater than or equal to 2π and by assumption $\mathcal{E}_X(g_a, 2\mathrm{diam}(X))$ is Euclidean, one can choose a ball $\mathcal{B}(c, \mathrm{diam}(X)) \subset \mathcal{E}_X(g_a, 2\mathrm{diam}(X))$ centred at $c \in \mathcal{E}_X(g_a, 2\mathrm{diam}(X))$ of diameter $\mathrm{diam}(X)$ (see Figure 5). However, by claim 1, there exists a straight line segment g_c of length $\ell(g_c) \leq \mathrm{diam}(X)$, connecting c to some singularity $z \in \Sigma$. This implies that $z \in \mathcal{B}(c, \mathrm{diam}(X)) \subset \mathcal{E}_X(g_a, 2\mathrm{diam}(X))$ which gives a contradiction. $\square$

We can now complete the proof of the lemma. For any $a \in A$, claim 2 implies we can choose $z \in \mathcal{E}_X(g_a, 2\mathrm{diam}(X)) \cap \Sigma$. Thus we can choose an oriented saddle connection s_a of length $\ell(s_a) \leq 2\mathrm{diam}(X)$, from z to $y = t(g_a)$, such that $s_a g_a^{-1}$ is an allowed geodesic. Because $\ell(g_a^{-1}) = \ell(g_a) \leq \mathrm{diam}(X)$, it follows that $a \in \mathcal{E}_X(s_a, \mathrm{diam(X)})$.

Finally, because $\{s_a\}_{a \in A}$ is countable, $\mathrm{Vol}_X(A) > 0$ and $\bigcup_{\{s_a : a \in A\}} \mathcal{E}_X(s_a, \mathrm{diam}(X) \cap A) = A$, at least one of the finite number of sectors, $\mathcal{E}_X(s_a, \mathrm{diam}(X))$, must satisfy $\mathrm{Vol}_X(\mathcal{E}_X(s_a, \mathrm{diam}(X) \cap A) > 0$. $\square$

Lemma 5.1.9. *Let $A \subset X$ be a Borel set such that $Vol_X(A) > 0$. Then*

$$\lim_{R \to \infty} \frac{1}{R} \log(V_A(R)) = h.$$

Proof. We will prove the result by considering upper and lower bounds for $V_A(R)$ and their logarithmic limits. For the upper bound, it suffices to use $V_A(R) \leq V_X(R) = \mathrm{Vol}_{\widetilde{X}}(\mathcal{B}(\widetilde{x}, R))$ and the definition of h. For the lower bound, observe that

$$V_A(R) \geq \mathrm{Vol}_X(\mathcal{E}(s', \mathrm{diam}(X)) \cap A) \cdot N(x, s', R - \mathrm{diam}(X)),$$

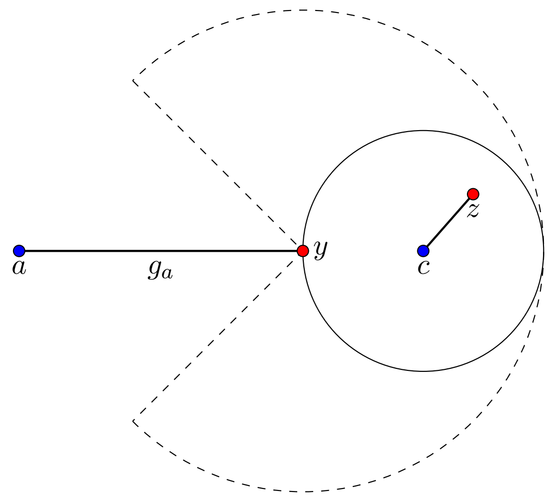


Figure 5.3: The point $a \in A$ is connected to $y \in \Sigma$ via segment g_a. The dotted sector represents $\mathcal{E}_X(g_a, 2\text{diam}(X))$. The ball $\mathcal{B}(c, \text{diam}(X)) \subset \mathcal{E}_X(g_a, 2\text{diam}(X))$ contains a singularity $z \in \Sigma$.

where $N(x, s', R)$ denotes the number of saddle connection paths starting at x ending with saddle connection s' and of length less than or equal to R.

Next we recall a result from Dankwart (see [8] or Proposition 4.3.3) which states that any two oriented saddle connections s_1, s_2 can be connected by a third oriented saddle connection s of length smaller than a given $L > 0$, such that the path $s_1 s s_2$ form an allowed saddle connection path. Using this result, we see that $N(x, s', R) \geq N(x, R - (L + \ell(s'))$ and hence

$$V_A(R) \geq \text{Vol}_X(\mathcal{E}(s', \text{diam}(X)) \cap A) \cdot N(x, R - (\text{diam}(X) + L + \ell(s')),$$

where $N(x, R) := \#\mathcal{P}(x, R)$. Finally, the definition of volume entropy and the asymptotic formula for $N(x, R)$ derived in the previous chapter allows us to conclude that

$$\lim_{R \to \infty} \frac{1}{R} \log(V_X(R)) = \lim_{R \to \infty} \frac{1}{R} \log(N(x, R)) = h$$

as required. $\square$

We next show that Lemma 5.1.9 can be improved to an asymptotic formula by following the method developed in the previous chapter.

Proposition 5.1.10. *If* $\mathrm{Vol}_X(A) > 0$ *then there exists* $C(A) > 0$ *such that*

$$V_A(R) \sim (C(A)/h)e^{hR} \text{ as } R \to \infty.$$

Proof. By Lemma 5.1.9 and the assumption that $\mathrm{Vol}_X(A) > 0$, the complex function $\eta_A(z)$ has a pole at $z = h$ and converges to an analytic function on $Re(z) > h$. In particular, for $Re(z) > h$, we can can use (1) to write

$$\begin{aligned}
\eta_A(z) &= \int_0^\infty e^{-zR} dV_A(R) \\
&= z\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})e^{-zR} dR \\
&+ z\sum_{p\in\mathcal{P}(x)} \int_{\ell(p)}^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{E}(p, R - \ell(p)) \cap \widetilde{A})e^{-zR} dR \\
&= z\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})e^{-zR} dR \\
&+ z\sum_{p\in\mathcal{P}(x)} e^{-z\ell(p)} \int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{E}(p, r) \cap \widetilde{A})e^{-zr} dr
\end{aligned}$$

using the change of variables $r = R - \ell(p)$ for each of the terms in the final summation.

By using the matrices M_z for translation surfaces (Definition 4.3.1), we can write $\eta_A(z)$ as

$$\begin{aligned}
\eta_A(z) &= z\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})e^{-zR} dR + z\underline{v}(z) \cdot \left(\sum_{n=0}^\infty \widehat{M}_z^n\right) \underline{u}_A(z) \\
&= z\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})e^{-zR} dR + z\underline{v}(z) \cdot \left(I - \widehat{M}_z\right)^{-1} \underline{u}_A(z)
\end{aligned}$$

where

a) $\underline{u}_A(z) = \left(\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{E}(s, r) \cap \widetilde{A})e^{-zr} dr\right)_{s\in\mathcal{S}} \in \ell^\infty$; and

b) $\underline{v}(z) = (\chi_{\mathcal{E}_x}(s)e^{-z\ell(s)})_{s\in\mathcal{S}} \in \ell^1$, where $\chi_{\mathcal{E}_x}$ denotes the characteristic function of the set $\mathcal{E}_x = \{s \in \mathcal{S} : i(s) = x\}$.

The quadratic growth of the volume function $R \mapsto \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})$ gives that the term

$$z\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})e^{-zR} dR$$

is analytic for $Re(z) > 0$. Moreover, the sequences $\underline{v}(z)$ and $\underline{u}_A(z)$ are analytic on $Re(z) > 0$ and by Lemma 5.1.8, $\underline{u}_A(h)$ is non-zero. It follows from the work in Sections 3.7 and 3.8 in Chapter 3, that the complex function $\eta_A(z)$ has the following properties:

1. $\eta_A(z)$ converges absolutely to a non-zero analytic function for $Re(z) > h$;
2. $\eta_A(z)$ extends to a simple pole at $z = h$ with residue $C(A) > 0$; and
3. $\eta_A(z)$ has an analytic extension to a neighbourhood of

$$\{z \in \mathbb{C} : Re(z) > h\} - \{h\}.$$

Finally, we can apply the Ikehara-Wiener Tauberian theorem to the monotone continuous function $V_A(R)$ to deduce the asymptotic formula

$$V_A(R) \sim (C(A)/h)e^{hR} \text{ as } R \to \infty, \tag{3}$$

where $C(A) > 0$ is the residue of $\eta_A(z)$ at $z = h$. This completes the proof of the proposition. $\square$

5.2 Proof of Theorem 5.1.2

We now have all the ingredients to complete the proof of Theorem 5.1.2. Recall from the previous chapter, that $\ell(\mathcal{C}(x, R)) \sim Ce^{hR}$ and in the previous section we showed that if $\mathrm{Vol}_X(A) > 0$ then $V_A(R) \sim (C(A)/h)e^{hR}$ for some $C(A) > 0$, and if $\mathrm{Vol}_X(A) = 0$ then for all $R > 0$, $V_A(R) = 0$ (and so we can formally write $V_A(R) \sim 0e^{hR}$). We use this to define the measure μ as follows: for all Borel sets $A \subset X$ we define

$$\mu(A) = \begin{cases} \frac{C(A)}{C} & \text{if } \mathrm{Vol}_X(A) > 0, \\ 0 & \text{if } \mathrm{Vol}_X(A) = 0. \end{cases}$$

It follows from the definition of $C(A)$ as the residue of $\eta_A(z)$ at h and the expression

$$\eta_A(z) = z\int_0^\infty \mathrm{Vol}_{\widetilde{X}}(\mathcal{D}(\widetilde{x}, R) \cap \widetilde{A})e^{-zR}dR + z\underline{v}(z) \cdot \left(I - \widehat{M_z}\right)^{-1} \underline{u}_A(z),$$

that $\mu(A)$ defines a probability measure on X.

Furthermore, it is easy to see that μ is equivalent to the volume measure on X, Vol_X, (i.e. $\mu(A) = 0$ if and only if $\mathrm{Vol}_X(A) = 0$ for all Borel sets A). In particular, if $\mathrm{Vol}_X(A) = 0$ then $V_A(R) = 0$ for all $R > 0$ and $\mu(A) = 0$. For the case where $\mu(A) = 0$, we can consider the contrapositive statement and observe that if $\mathrm{Vol}_X(A) > 0$ then we have shown that $\mu(A) = C(A)/C > 0$.

It remains to show that $\mu_R \to \mu$ in the weak-* topology. To this end it suffices to consider $\mu_R(B)$ for appropriately small balls $B \subset X\backslash\Sigma$ (see [48]).

The proof of Theorem 5.1.2 now comes in two steps. The first step is to deduce an asymptotic result for annuli. The second step is to let the thickness of the annuli tend to zero.

To achieve the first step, given $\epsilon > 0$ we denote by

$$\mathcal{A}(\widetilde{x}, R-\epsilon, R) := \mathcal{B}(\widetilde{x}, R) - \mathcal{B}(\widetilde{x}, R-\epsilon), \quad \text{for } \widetilde{x} \in \widetilde{\Sigma} \text{ and } R > \epsilon,$$

the corresponding annulus. We can then use (3) twice to deduce an asymptotic expression for $\mathrm{Vol}_{\widetilde{X}}(\mathcal{A}(\widetilde{x}, R-\epsilon, R) \cap \widetilde{B})$ of the form

$$\mathrm{Vol}_{\widetilde{X}}(\mathcal{A}(\widetilde{x}, R-\epsilon, R) \cap \widetilde{B}) \sim (C(B)/h)e^{hR}\left(1 - e^{-h\epsilon}\right) \text{ as } R \to \infty. \tag{4}$$

For the second step, we require an approximation argument. Let $B \subset X\backslash\Sigma$ denote a closed ball with centre $c \in X\backslash\Sigma$ and radius $t > 0$. Let $d = \|B - \Sigma\|$ denote the Hausdorff distance of B from Σ. For sufficiently small $\delta > 0$ (with $\delta \ll d$), let B_δ and $B_{-\delta}$ denote concentric balls also centred at c, with radii $t + \delta$ and $t - \delta$, respectively. Fix $R > 0$ and ϵ such that $\epsilon \ll \delta$.

Let $\mathcal{L}(R)$ denote the set of connected components of $\mathcal{C}(x, R) \cap B$. Similarly, let $\mathcal{A}_\delta(R)$ and $\mathcal{A}_{-\delta}(R)$ denote the set of connected components of $\mathcal{A}(\widetilde{x}, R, R-\epsilon) \cap \widetilde{B_\delta}$ and $\mathcal{A}(\widetilde{x}, R, R-\epsilon) \cap \widetilde{B_{-\delta}}$, respectively (see Figure 5.4).

Note that to each connected component $A_{-\delta} \in \mathcal{A}_{-\delta}(R)$, we can associate a segment $L \in \mathcal{L}(R)$, namely the segment which corresponds to the boundary component of $A_{-\delta}$ furthest away from the associated singularity. Similarly, for each $L \in \mathcal{L}(R)$ we can associate a connected component $A_\delta \in \mathcal{A}_\delta(R)$ (see Figure 5.4). Hence we have the following inclusions:

$$\mathcal{A}_{-\delta}(R) \hookrightarrow \mathcal{L}(R) \hookrightarrow \mathcal{A}_\delta(R).$$

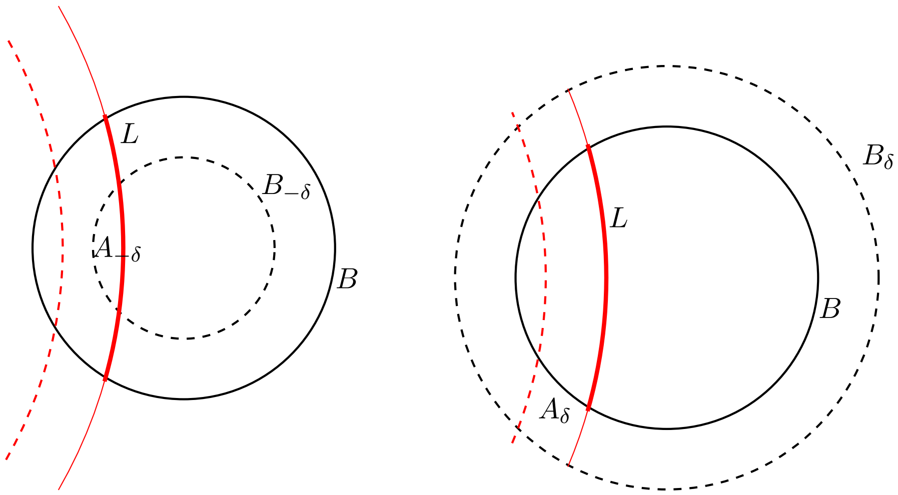


Figure 5.4: (a) We can associate to a connected component $A_{-\delta} \in \mathcal{A}_{-\delta}(R)$, a segment $L \in \mathcal{L}(R)$; and (b) We can associate to L the connected component $A_\delta \in \mathcal{A}_\delta(R)$

Note that the reverse inclusions do not necessarily hold.

For $L \in \mathcal{L}(R)$, we will compare $\ell(L)\epsilon$ to the volume $\mathrm{Vol}_{\widetilde{X}}(A_\delta)$ of the associated connected component $A_\delta \in \mathcal{A}_\delta(R)$. Using the assumption that $\epsilon \ll \delta$ and a little Euclidean geometry, it follows that

$$\ell(L)\epsilon \leq \frac{\mathrm{Vol}_{\widetilde{X}}(A_\delta)}{(1 - \frac{\epsilon}{2d})}.$$

Similarly, for $A_{-\delta} \in \mathcal{A}_{-\delta}(R)$, we can compare $\mathrm{Vol}_{\widetilde{X}}(A_{-\delta})$ to $\ell(L)\epsilon$ for the associated $L \in \mathcal{L}(R)$ and deduce that

$$\mathrm{Vol}_{\widetilde{X}}(A_{-\delta}) \leq L\epsilon.$$

By summing up the contributions from the aforementioned connected components and using the bounds above, it follows that

$$\mathrm{Vol}_{\widetilde{X}}(\mathcal{A}(\widetilde{x}, R, R-\epsilon) \cap \widetilde{B_{-\delta}}) \leq \ell(\mathcal{C}(x, R) \cap \widetilde{B})\epsilon \leq \frac{\mathrm{Vol}_{\widetilde{X}}(\mathcal{A}(\widetilde{x}, R, R-\epsilon) \cap \widetilde{B_\delta})}{\left(1 - \frac{\epsilon}{2d}\right)}.$$

Using the asymptotic formulae for annuli (4) and the above bounds, we can deduce that

$$\begin{aligned}\frac{C(B_{-\delta})}{C(B)}\frac{(1-e^{-h\epsilon})}{\epsilon} &\leq \liminf_{R\to\infty}\frac{\ell(\mathcal{C}(x,R)\cap B)}{(C(B)/h)e^{hR}}\\ &\leq \limsup_{R\to\infty}\frac{\ell(\mathcal{C}(x,R)\cap B)}{(C(B)/h)e^{hR}}\\ &\leq \frac{C(B_{\delta})}{C(B)}\frac{(1-e^{-h\epsilon})}{\epsilon}\frac{1}{\left(1-\frac{\epsilon}{2d}\right)}.\end{aligned}$$

Since $\left(1-e^{-h\epsilon}\right)/\epsilon = h+O(\epsilon)$ independently of R, letting $\epsilon\to 0$ we can deduce that

$$\frac{C(B_{-\delta})}{C(B)}h \leq \liminf_{R\to\infty}\frac{\ell(\mathcal{C}(x,R)\cap B)}{(C(B)/h)e^{hR}} \leq \limsup_{R\to\infty}\frac{\ell(\mathcal{C}(x,R)\cap B)}{(C(B)/h)e^{hR}} \leq \frac{C(B_{\delta})}{C(B)}h.$$

We can deduce an asymptotic formula for $\ell(\mathcal{C}(x,R)\cap B)$ by letting $\delta\to 0$ and using the absolute continuity of the measure $\mu(A) := C(A)/C$ to conclude that

$$\ell(\mathcal{C}(x,R)\cap B) \sim C(B)e^{hR} \text{ as } R\to\infty.$$

Finally, we can prove Theorem 5.1.2 by considering the above asymptotic formula and the asymptotic formula for $\ell(\mathcal{C}(x,R))$,

$$\lim_{R\to\infty}\mu_R(B) = \lim_{R\to\infty}\frac{\ell(\mathcal{C}(x,R)\cap B)}{\ell(\mathcal{C}(x,R))} = \lim_{R\to\infty}\frac{C(B)e^{hR}}{Ce^{hR}} = \frac{C(B)}{C} =: \mu(B),$$

for all balls $X\backslash\Sigma$. In particular, this implies that μ_R converges to μ in the weak-* topology (see [48]).

Remark 5.2.1. Although the probability measures μ and Vol_X are equivalent they (probably) are not equal. A proof of this statement can probably be deduced by examining how $C(B)$ varies as we translate some ball B around $X\backslash\Sigma$.

5.3 Distribution result for closed geodesics

We conclude this chapter by noting that one can obtain a distribution result for the growth of closed geodesics on translation surfaces. A more general result can be found in [6] which uses different techniques. We provide a sketch of the proof based on the techniques developed in the previous chapters.

Recall the following definition of closed geodesics on translation surfaces (Definition 4.6.1).

Definition 5.3.1. *A closed geodesic on a translation surface is a saddle connection* *path corresponding to an (allowed) finite string of oriented saddle connections $q = (s_1, \ldots, s_n)$, of length $|q| = n$ and up to cyclic permutation, with the additional requirement that $s_n s_1$ is a saddle connection path. We say that q is primitive if it is not a multiple concatenation of a shorter closed geodesic.*

Let $\mathcal{Q}(T)$ denote the set of oriented primitive closed geodesics on X of length less than or equal to T. Let $\mathcal{Q} := \bigcup_{T>0} \mathcal{Q}(T)$ denote the set of all oriented primitive closed geodesics on X. Let $C(T) := \#\mathcal{Q}(T)$ denote the number of of oriented primitive closed geodesics of length at most T.

We adopt the convention that we do not count closed geodesics that do not pass through a singularity, thus avoiding the complication of having uncountably many closed geodesics of the same length.

Given a Borel set $A \subset X$, we define $C_A(T) := \sum_{q \in \mathcal{Q}(T)} \frac{\ell_A(q)}{\ell(q)}$, where $\ell_A(q)$ denotes the length of the part of q which lies in A.

We define the following modified zeta function.

Definition 5.3.2. *We can formally define a modified zeta function for a given Borel set $A \subset X$, by*

$$\zeta_A(z, t) = \prod_{q \in \mathcal{Q}} \left(1 - e^{-z\ell(q) + t\ell_A(q)}\right)^{-1}, z \in \mathbb{C} \text{ and } t \in \mathbb{R}.$$

By applying a modified proof to the one presented in Section 3.9 for the asymptotic formula for closed geodesics on infinite graphs, we can obtain the following result.

Theorem 5.3.3. *Given a Borel set $A \subset X$, there exists a constant $0 \leq \nu(A) \leq 1$ such that*

$$C_A(T) \sim \nu(A) \frac{e^{hT}}{hT}.$$

We can then use this asymptotic result to obtain a distribution result for closed geodesics on translation surfaces.

Theorem 5.3.4. *Let X be a translation surface. Then for all Borel subsets $A \subset X$, there exists some $0 \leq \nu(A) \leq 1$ such that*

$$\lim_{T \to \infty} \frac{C_A(T)}{C(T)} = \nu(A).$$

Furthermore, ν defines a probability measure on X which is singular with respect to the volume measure on X

Proof. It follows from the previous theorem and the asymptotic formula for closed geodesics that

$$\lim_{T\to\infty} \frac{C_A(T)}{C(T)} = \nu(A).$$

Furthermore, one can check that $\nu(A)$ defines a probability measure on X.

The measure ν obtained is singular with respect to the volume measure $(\mathrm{Vol})_X$ (and thus with respect to μ), since one can show that the Borel set $Y = \cup_{s\in\mathcal{S}} s$ corresponding to the union of the saddle connections has $\nu(Y) = 1$, but $(\mathrm{Vol})_X(Y) = 0$. $\square$

Chapter 6

Minimizing entropy and equilateral surfaces

6.1 Introduction

The aim of this chapter is to introduce and study the entropy minimization problem for translation surfaces.

In Chapter 2, we saw that one can associate an entropy function to the moduli space of negatively curved metrics on certain closed surfaces or metrics on certain finite graphs. Because entropy scales inversely with respect to area, it is natural to restrict our attention to metrics of constant area. One can then ask various questions about this entropy function such as:

- How smooth is it?
- What are its minimum values?
- Which metrics which achieve the minima? (entropy minimization problem).

We will attempt a similar study in the context of translation surfaces, focusing on the entropy minimization problem.

Taking inspiration from the solution to the entropy minimization problem in the setting of finite metric graphs and Riemannian manifolds, we provide a conjecture for the analogous problem for translation surfaces and give evidence in favor of this conjecture. Our conjecture says for any $n, k \in \mathbb{N}$, the volume entropy function defined over the moduli space of unit area translation surfaces with n singularities, each of cone-angle $2\pi(k+1)$, has global minima at the surfaces that can be tiled by equilateral triangles whose vertices coincide with the singularities of the surface.

We will begin by introducing the natural definition of moduli spaces for translation surfaces. We will then highlight the difficulty of studying the entropy function for translation surfaces and state our conjecture properly. Afterwards, we will introduce special subsets of the moduli space for which the entropy function is better behaved. Finally, we will prove that equilaterally triangulated translation surfaces are the entropy function's global minima when restricted to these subsets.

We will state the above result in a rough form before proceeding. Let X be any translation surface and let $A \in SL(2, \mathbb{R})$. We can obtain a new translation surface $A(X)$ by cutting X into polygons, applying A to the polygons, and then gluing the polygons back together. The resultant surface will be a translation surface with the same area and singularity data as X. In particular, we obtain a space of translation surfaces by considering $SL(2, \mathbb{R})(X)$, i.e. the $SL(2, \mathbb{R})$-orbit of X.

Theorem 6.1.1. *Let E be a translation surface that can be tiled by equilateral triangles such that the vertices of the triangles occur at the singularities of E. Furthermore, assume that each of E's singularities have the same cone-angle, $2\pi(k+1)$ for a given $k \in \mathbb{N}$. Then $h : SL(2, \mathbb{R})(E) \to \mathbb{R}^+$ has a minimum value of $h(E) > 0$ and the only surfaces in $SL(2, \mathbb{R})(E)$ that obtain this minimum are equilaterally tiled surfaces.*

6.2 Background

In this section, we will introduce a couple of notions of moduli spaces of translation surfaces and review what is known about the entropy function defined on these spaces. We use [54] and [51] as references.

6.2.1 Strata

Let S be a closed topological surface of genus $g \geq 2$. We denote the set of all translation surface structures on S by $\mathcal{Q}(g)$.

This set can be partitioned into sets of translation surface structures that share the same singularity data. Recall from the background chapter that if X is a translation surface of genus $g \geq 2$ with singularity set $\Sigma \neq \emptyset$, then

$$\sum_{x \in \Sigma} k(x) = 2g - 2, \tag{1}$$

where $2\pi(k(x) + 1)$ is the cone-angle of singularity x.

Definition 6.2.1. *Let $\mathcal{H}(k_1, \ldots, k_n)$ denote the set of translation surfaces with n singularities of cone-angles $2\pi(k_1 + 1), \ldots, 2\pi(k_n + 1)$, such that $\underline{k} = (k_1, \ldots, k_n)$ is a partition of $2g - 2$, i.e. a non-increasing list of positive integers whose sum is $2g - 2$. Then by Equation (1), we see that there is a natural partition of $\mathcal{Q}(g)$ into a finite number of $\mathcal{H}(\underline{k})$. We call the sets $\mathcal{H}(\underline{k})$ strata.*

Strata are known to be complex orbifolds of dimension $2g + n - 1$ which consist of at most three connected components. Furthermore, away from the orbifold points, each stratum has an atlas of charts to $\mathbb{C}^n$ with transition functions in $GL(2, \mathbb{Z})$. We let $\mathcal{H}_A(\underline{k})$ denote the set of area $A > 0$ surfaces in $\mathcal{H}(\underline{k})$.

Another interesting property of strata, is that their connected components are never compact with respect to the analytic topology. Masur's criterion (see [35]) states that a closed subset of $\mathcal{H}_A(\underline{k})$ is compact if and only if there is a positive lower bound for the lengths of the saddle connections on all the surfaces in the subset.

We will now look at the moduli space of unit area flat tori which shares some properties with general strata. Furthermore, it highlights the $SL(2, \mathbb{R})$ action on strata which we will study later. We will consider the tori up to rotation.

Example 6.2.2. *The moduli space which we denote by $\mathcal{M}$ can be identified with $SO(2)\backslash SL(2, \mathbb{R})/SL(2, \mathbb{Z})$, where $SL(2, \mathbb{Z})$ is the space of 2×2 matrices with entries in $\mathbb{Z}$ and determinant 1, and $SO(2) \subset SL(2, \mathbb{R})$ is the space of rotation matrices. To see this, note that any unit area torus can be represented by a parallelogram in the plane with opposite sides identified. Hence the action of $SL(2, \mathbb{R})$ on the plane defines an action on $\mathcal{M}$. This action is easily seen to be transitive as any parallelogram can be sent to any other by an element of $SL(2, \mathbb{R})$. Furthermore, it is easily seen that the stabilizer subgroup of the unit torus is $SL(2, \mathbb{Z})$. Hence by the Orbit-Stabilizer theorem and the fact that we are interested in tori up to rotation, $\mathcal{M} \cong SO(2)\backslash SL(2, \mathbb{R})/SL(2, \mathbb{Z})$.*

$\mathcal{M}$ shares properties with general strata. In particular, $\mathcal{M}$ is an orbifold; the square torus (corresponding to the identity matrix) and the torus that can be tiled by two

equilateral triangles are the orbifold points of $\mathcal{M}$. Furthermore, $\mathcal{M}$ has a cusp corresponding to tori with a short closed geodesic.

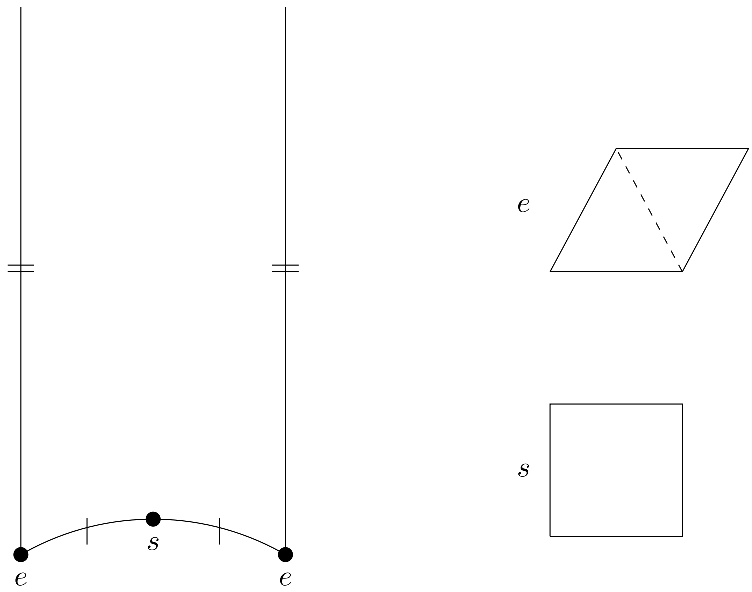


Figure 6.1: Left: The fundamental domain for $\mathcal{M}$. The vertical lines are glued together as well as the two line segments joining s to the corners labelled e. The point labelled s corresponds to the square torus and the points labelled e are both identified and correspond to the torus triangulated by two equilateral triangles. The cusp (north) corresponds to tori which have a short closed geodesic.

6.2.2 Entropy function for strata

In this section, we will introduce the entropy function for strata and examine what is already known about it.

We begin by noting that from the definition of volume entropy, it follows that volume entropy scales inversely and quadratically with respect to area i.e. if we scale our surface by doubling its area, we scale its entropy by a factor of $1/\sqrt{2}$. For this reason we will consider the subset of strata consisting of constant area surfaces to be the moduli spaces of translation surfaces. We will refer to these constant area subsets as just strata for convenience.

By analogy with the case of metric graphs and negatively curved surfaces, we can naturally extend the definition of volume entropy for individual translation surfaces to a function on strata. In particular, given a stratum $\mathcal{H}_A(\underline{k})$, we define the entropy function $h : \mathcal{H}_A(\underline{k}) \to \mathbb{R}^+$, where $h(X)$ is the volume entropy of $X \in \mathcal{H}_A(\underline{k})$.

The study of volume entropy for translation surfaces seems to have been initiated by Dankwart in his PhD thesis [8]. Dankwart defined volume entropy in terms of orbital counting which is equivalent to the definition we use. He explored some of the questions we posed in the introduction and obtained some partial results.

We will summarize and present simplified statements of some of his results that are of interest to us.

Theorem 6.2.3 (Dankwart)**.** *Fix some stratum $\mathcal{H}_A(\underline{k})$ such that $\underline{k}$ is a partition of $2g-2$, for $g \geq 2$. Then:*

1. *The entropy function $h : \mathcal{H}_A(\underline{k}) \to \mathbb{R}^+$ is continuous;*
2. *There exists a constant $c(g)$ such that $h(X) \geq c(g)$ for all $X \in \mathcal{H}_A(\underline{k})$; and*
3. *We can find surfaces of arbitrarily high entropy in $\mathcal{H}_A(\underline{k})$. These correspond to surfaces with a short closed geodesic.*

For some intuition as to why part (3) of the above results holds, see the proof of Lemma 3.2.4.

The methods used by Dankwart often rely on taking advantage of the fact that there exists a quasi-isomorphism between the universal covers of hyperbolic surfaces and translation surfaces, as well as some results regarding the fundamental group of genus $g \geq 2$ surfaces. These methods do not seem to generalize to studying the differentiability of the entropy function or finding local and global minima. We do not study the smoothness of the entropy function, except when we restrict to certain subsets (see Proposition 6.5.4).

6.2.3 Difficulties of studying the entropy function

In this section, we will look at the difficulties of studying the entropy function for translation surfaces. To do this, we will compare this setting with the case of finite metric graphs (see [29]).

Recall from Chapter 2, that for finite connected graphs, one can define a notion of volume entropy as follows. Let $(\mathcal{G}, \ell)$ be a finite connected graph with a length function on its edges $\ell : \mathcal{E}(\mathcal{G}) \to \mathbb{R}^+$. Let $x \in \mathcal{G}$ and let $N(x, R)$ denote the number of non-backtracking paths that start at x and are of length less than or equal to R. Then the volume entropy of $(\mathcal{G}, \ell)$ is defined to be

$$h(\mathcal{G}, \ell) = \lim_{R \to \infty} \frac{1}{R} \log(N(x, R)).$$

We can define the moduli space of metric graphs to be the set of length functions $\ell : \mathcal{E}(\mathcal{G}) \to \mathbb{R}^+$ such that the sum of the edge lengths with respect to this length function is equal to 1, i.e. $\sum_{e \in \mathcal{E}(\mathcal{G})} \ell(e) = 1$.

In this setting, when one varies the length function on the graph $\mathcal{G}$, the underlying path structure of the $\mathcal{G}$ remains constant and only the lengths of the paths vary. Because volume entropy for finite metric graphs depends on the growth of paths, this means that when comparing the entropy of two length functions, it suffices to compare lengths of the paths on $\mathcal{G}$ with respect to the two length functions.

Recall from Chapter 4, that the volume entropy of a translation surface is similar to the volume entropy of certain infinite graphs. In contrast to the finite graph case, one can perturb a translation surface in such a way that at least one of its saddle connection paths (for example saddle connection paths which come from boundaries of cylinders) will no longer satisfy the geodesic angle condition under this perturbation (see Figure 6.2). This means that as we vary the metric on the translation surface, not only do the lengths of the saddle connection paths change, but a path that once was a saddle connection path may no longer be a saddle connection path and vice versa.

In order to make progress on the entropy minimization problem, we will restrict the domain of the entropy function to certain natural subsets of strata, namely $SL(2, \mathbb{R})$-orbits of translation surfaces. $SL(2, \mathbb{R})$-orbits have the property that the saddle connection path structure of surfaces in these spaces remain constant as in the case of the moduli spaces of metric graphs.

In the next subsection, we will introduce these $SL(2, \mathbb{R})$-orbits before returning to the topic of entropy on strata in the next section, where we will motivate and state a conjecture for the entropy minimization problem.

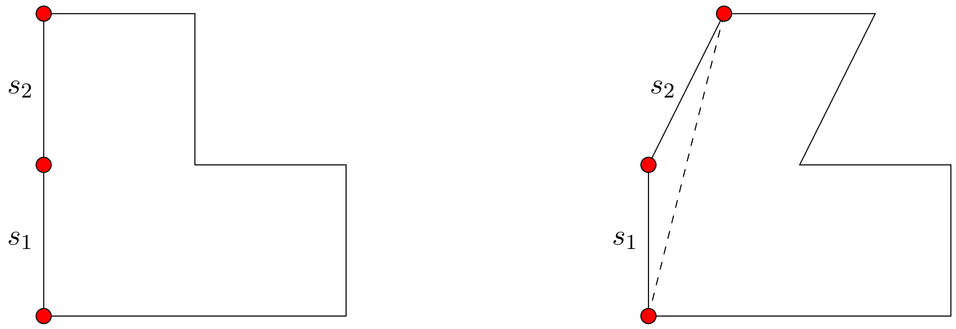


Figure 6.2: Two translation surfaces obtained from identifying opposite sides of two polygons. Both translation surfaces belong to $\mathcal{H}_A(2)$ and the translation surface on the right is obtained by continuously modifying the surface on the left. Two saddle connections s_1 and s_2 on the left surface form a saddle connection path. The corresponding saddle connections on the right surface no longer form a saddle connection path due to the geodesic angle condition (the shortest path between their endpoints is now the dashed line).

6.2.4 $SL(2,\mathbb{R})$-orbits of translation surfaces

Recall from Chapter 2 (or [51]), that any translation surface X admits a polygonal decomposition P. Given $A \in SL(2,\mathbb{R})$, A acts linearly on $\mathbb{R}^2$ and hence we obtain obtain a new polygonal decomposition $A(P)$. Note that A sends parallel lines in P to parallel lines in $A(P)$. We can obtain a new translation surface $A(X)$ from $A(P)$ by identifying edges in $A(P)$ as follows: if edge e was identified to e' in P to obtain X, then we identify edge $A(e)$ to edge $A(e')$ (we note that this construction is well-defined (see [52] or see [38] for a more rigorous construction). Because $SL(2,\mathbb{R})$ preserves the area of the polygons, and the angle around any vertex under the edge identifications, we see that $SL(2,\mathbb{R})$ produces a natural action on strata.

The $SL(2,\mathbb{R})$-action on strata is a powerful tool for studying dynamics on translation surfaces (see [52] for a good introduction to this topic). We note that the closure of any $SL(2,\mathbb{R})$-orbit will be an orbifold and there are at most countably orbits many in a given stratum [13].

The $SL(2,\mathbb{R})$-action is relevant to our purposes because it preserves the saddle connection path structure of translation surfaces. To see this, fix $A \in SL(2,\mathbb{R})$ and take two saddle connections on a translation surface X joined at a singularity which also form an angle of less than π. Because A acts as a linear map on polygons (preserving parallel lines) the images of the saddle connections in $A(X)$ will also join at a singularity and will have angle less than π (Figure 6.2.4). Hence under the action of $SL(2,\mathbb{R})$, saddle connection paths are sent to saddle connection paths. For this reason, working with entropy functions becomes more tractable when restricting to $SL(2,\mathbb{R})$-orbits of translation surfaces, at least when approaching the problem from the perspective developed in this thesis.

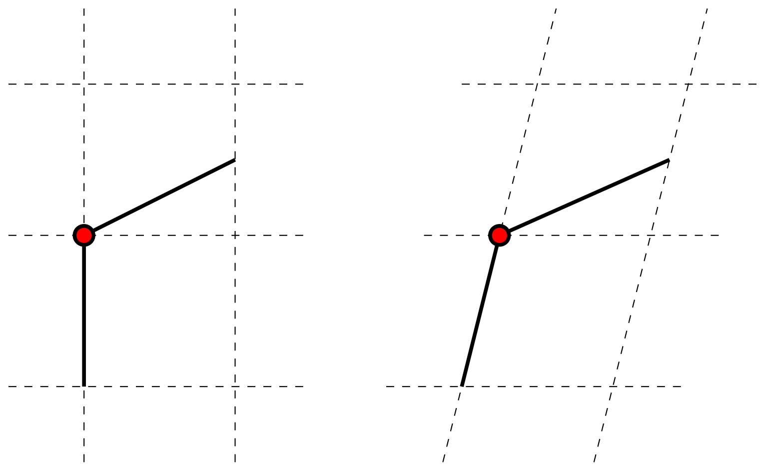


Figure 6.3: The action of some $A \in SL(2,\mathbb{R})$ on two saddle connections which meet at a singularity. If the original angle between the saddle connections is less than π then it will remain less than π under the action of A.

Although the saddle connection path preserving property of the $SL(2,\mathbb{R})$-action simplifies the study of the entropy function, it is still difficult to track how the lengths of the different saddle connection paths vary under the $SL(2,\mathbb{R})$-action of an arbitrary surface.

Remark 6.2.4. It follows from the definition of a translation surface (see Definition 2.5.1), that each point on a translation surface outside of the singularity set has a well-defined notion of north. This is because outside of the singularity set, the surface has an atlas of charts to $\mathbb{C}$ whose transition maps are translations. Because of this, translation surfaces that differ by a rotation are generally considered to be distinct. For studying volume entropy, this distinction does not matter since the volume entropy of two translation surfaces that differ by a rotation is the same. For presentation's sake we assume two surfaces are distinct if they differ by a rotation, unless otherwise specified.

6.3 Entropy minimization conjecture

In this section, we state a conjecture for the entropy minimization problem for strata. We motivate this conjecture by looking at the results of the entropy minimization problems for finite metric graphs and negatively curved surfaces.

6.3.1 Initial observations

In Chapter 2, we saw that in the case of surfaces of negative curvature, the metrics which minimise entropy are those which are the most symmetric (i.e. hyperbolic metrics). A similar result holds when considering metrics on finite graphs. For example, a k-regular graph (a graph where each vertex has the same valence $k \in \mathbb{N}$, see Figure 6.4) will have its entropy minimized by the length function which sets all edges to have the same length. In general, the metrics which minimise entropy for finite metric graphs are those which cause vertices of high valence to be far apart. The following result (see [29]) gives the precise edge weighting.

Theorem 6.3.1 (Lim). *Let $\mathcal{G}$ be a finite connected graph such that the valency at each vertex x, which we denote by $k(x)+1$, is at least 3. Let $\mathcal{V}(\mathcal{G})$ denote the vertex set of $\mathcal{G}$. Then there is a unique normalized length distance that minimizes volume entropy, given by*

$$h_{min} = \frac{1}{2} \sum_{x \in \mathcal{V}(\mathcal{G})} (k(x)+1) \log(k(x)),$$

and the entropy minimizing length function is given by

$$\ell(e) = \frac{\log(k(i(e))k(t(e)))}{\sum_{x \in \mathcal{V}(\mathcal{G})} (k(x)+1) \log(k(x))},$$

for all $e \in \mathcal{E}(\mathcal{G})$.

Vertices of high valency can informally be thought of as points of high expansion or concentrated negative curvature. This is because paths that meet a vertex of high valency can be continued in a large number of distinct ways.

Recall from Chapter 3, that the notion of entropy for translation surfaces is closely related to the notion of entropy for infinite graphs. For translation surfaces, singularities with large cone-angles can be thought of as points of high negative curvature or expansion. Using this observation and the result of the entropy minimization problem for finite metric graphs, we might expect translation surfaces with the lowest entropy in their stratum, are those that have their singularities of large cone-angles

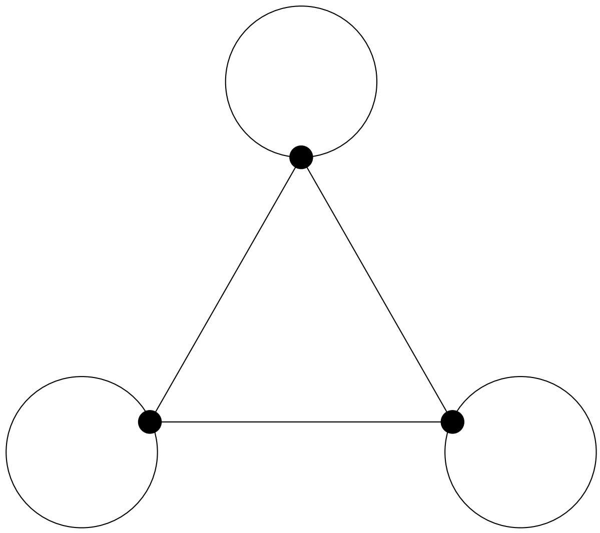


Figure 6.4: A 3-regular graph with 3 vertices.

far away from one another.

If we restrict our attention to strata $\mathcal{H}_A(\underline{k})$ that have the property that all singularities have the same cone-angle, then the problem becomes more tractable (but less ambitious) due to the additional symmetry on the singularity set. We denote such a stratum by $\mathcal{H}_A(\underline{k}_n)$ where $\underline{k}_n$ denotes the tuple consisting of n copies of $k \in \mathbb{N}$. Taking inspiration from k-regular graphs, it is natural to conjecture that the entropy of surfaces of the aforementioned type is minimized when the singularities on the surface are spread out as far as possible from one another (or themselves) in a symmetric way. In particular, the surfaces that achieve this would be tiled by equilateral triangles whose vertices correspond to the singularities of the surface. A priori, it is not evident that such surfaces exist in a given stratum $\mathcal{H}_A(\underline{k}_n)$.

In the next section, we will show that any connected component of a stratum of type $\mathcal{H}_A(\underline{k}_n)$ contains a surface described in the previous paragraph.

6.3.2 Square-tiled surfaces and equilateral surfaces

In this section, we will define square-tiled surfaces and equilateral surfaces. We will show that every component of a stratum $\mathcal{H}_A(\underline{k})$ contains an equilateral surface (and a square-tiled surface).

We begin by looking at square-tiled surfaces which are special types of translation surfaces (see [23] for a more detailed introduction to square-tiled surfaces).

Definition 6.3.2. *A square-tiled surface is a translation surface obtained from a finite collection of equal area squares in $\mathbb{R}^2$ after identifying pairs of parallel sides via translations. We add the additional requirement that each corner of the squares must correspond to a singularity of the surface.*[1]

Examples of square-tiled surfaces include the left translation surface in Figure 6.2 and the surface in Figure 2.10.

Given a square-tiled surface X_S, there is a natural map $\pi_X : X_S \to T$ where T is the standard square torus. In particular, given some $x \in X_S$ corresponding to a point in some square tile, $\pi_X(x)$ is the corresponding point on the square torus obtained after identifying opposite sides of the square tile. This map is well-defined due to the edge identifications of the square tiles. It is easy to see that $\pi_X(x)$ is a finite cover of the standard square torus except at the singularities which are ramification points. At a singularity of cone-angle $2\pi(k+1)$, this covering map looks like $w = z^{k+1}$ in local complex coordinates where $z = 0$ corresponds to the singularity.

We will now turn our attention to surfaces that can be tiled by equilateral triangles (see [4]).

Definition 6.3.3. *We define an equilateral surface to be a translation surface that admits a triangulation into equilateral triangles, such that the vertices of the equilateral triangles are singularities of X.*

See Figure 6.5 for an example of an equilateral surface.

Observe that square-tiled surfaces and equilateral surfaces are related in the sense that any equilateral surface can be obtained by applying the matrix

$$\Delta = \begin{pmatrix} \sqrt{\frac{2}{\sqrt{3}}} & \sqrt{\frac{1}{2\sqrt{3}}} \\ 0 & \sqrt{\frac{\sqrt{3}}{2}} \end{pmatrix}$$

to a square-tiled surface. Similarly, any equilateral surface gives rise to a square-tiled surface.

[1]Typically one allows square-tiled surfaces to include square tiles whose vertices do not necessarily correspond to singularities.

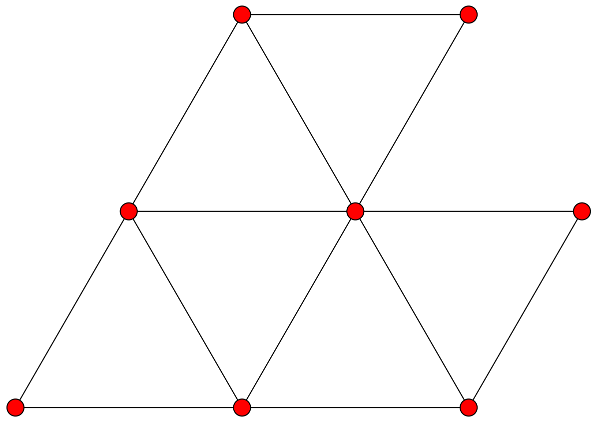


Figure 6.5: An equilateral surface obtained from gluing opposite sides of a polygon. This surface has one singularity of cone-angle 6π. This surface corresponds to a shearing of the L-shaped surface (see the left surface in Figure 6.2).

Equilateral surfaces have been studied in [4]. In this paper, the authors study the length of the shortest saddle connection on a translation surfaces, called the systole of the surface, and calculate the maximum systole over any stratum. In particular, they show that the surfaces that realize the maximum systole over any given stratum are equilateral surfaces.

The following result (Lemma 3.2 in [4]) guarantees the existence of square-tiled and equilateral surfaces in any connected component of any stratum.

Proposition 6.3.4. *Let $\mathcal{C} \subset \mathcal{H}_A(\underline{k})$ be a connected component of a stratum. Then $\mathcal{C}$ contains an equilateral surface and a square-tiled surface.*

Proof. The result follows a result Lemma 4 in [28] which states that there exists a translation surface in $\mathcal{C}$ which can be decomposed into a single horizontal cylinder. Such a surface will be a rectangle whose two vertical sides correspond to the same single saddle connection and its two horizontal sides will both decompose into horizontal saddle connections, where each saddle connection appears on both the top and the bottom. By changing the lengths of the saddle connections appropriately, we can obtain a square-tiled surface.

We then obtain an equilateral surface by rotating the vertical side to form an angle of $\pi/3$ with the horizontal sides (by applying Δ as defined above). $\square$

6.3.3 Entropy minimization conjecture

In this section, we state a conjecture regarding entropy minimization and equilateral surfaces.

Using the discussion in Section 6.3.1, in particular, taking inspiration from the entropy minimization result for k-regular graphs, we make the following conjecture.

Conjecture 6.3.5. *Let $\mathcal{H}_A(\underline{k}_n)$ denote a stratum with n singularities, each of cone-angle $2\pi(k+1)$. Then the entropy function $h : \mathcal{H}_A(\underline{k}_n) \to \mathbb{R}^+$ has global minima at the equilateral surfaces in $\mathcal{H}_A(\underline{k}_n)$.*

As discussed in Section 6.2.3, tackling this conjecture looks difficult due to the fact that the saddle connection path structure may change as we compare two different surfaces in a given connected component of a stratum.

We note that for strata corresponding to singularities with varying cone-angles, we do not expect that equilateral surfaces will minimize entropy (compare to the general minimization result for finite metric graphs - Theorem 6.3.1).

In order to make progress in studying this conjecture, we restrict our attention to minimizing the volume entropy function over particularly nice subsets of connected components of strata that contain equilateral surfaces.

6.4 Entropy minimization for $SL(2,\mathbb{R})$-orbits of equilateral surface in $\mathcal{H}_A(\underline{k}_n)$

In this section, we will study the entropy function on the $SL(2,\mathbb{R})$-orbit of equilateral surfaces whose singularities all have the same cone-angle (i.e., are in strata of the form $\mathcal{H}_A(\underline{k}_n)$.

We will begin by looking at $SL(2,\mathbb{R})$-orbit(s) of equilateral surfaces in $\mathcal{H}_A(2)$. We will see that there is only one such orbit which contains all the equilateral surfaces in $\mathcal{H}_A(2)$.

We will then prove that equilateral surfaces whose singularities have the same cone-angle minimize entropy in their $SL(2,\mathbb{R})$-orbits (Theorem 6.1.1). The proof involves first understanding the structure of saddle connection paths on surfaces in the aforementioned $SL(2,\mathbb{R})$-orbits. We then use this structure to obtain a simplified equation for the entropy of such surfaces and then apply a result that can be found in [3] related to minimizing functions defined on lattices to deduce our result.

Using the simplified equation for entropy, we will also show that the entropy function is smooth on these orbits (Proposition 6.5.4).

Then in subsection 6.6, we will develop a method for calculating the entropy of surfaces in such orbits.

6.4.1 Equilateral surfaces in $\mathcal{H}_A(2)$

In this section, we will see that there is one equilateral surfaces, up to isometry (including rotations), in $\mathcal{H}_A(2)$.

We begin by briefly noting that the $SL(2, \mathbb{R})$-orbit of an equilateral surface in $\mathcal{H}_A(2)$ is the unit tangent bundle of a genus 0 orbifold with two cusps (see Appendix A in [22]). Rotating the surface corresponds to rotating the associated vector in the unit tangent bundle.

Our approach is to construct a set of candidate equilateral surfaces combinatorially and then show that one of the resulting surfaces is a torus (and hence not in $\mathcal{H}_A(2)$) and that the others are isometric.

Let E denote an equilateral surface in $\mathcal{H}_A(2)$. It follows from the Gauss-Bonnet theorem, that E will be triangulated by 6 equilateral triangles, under the condition that the vertex of every triangle meets a singularity (see [4] for details).

Starting with 6 equilateral triangles in the plane, in order to form a translation surface with these triangles, we are required to glue every edge to another edge via a translation. Hence we are forced to arrange the triangles in the shape shown in Figure 6.6.

Despite being forced to arrange the triangles into the polygon in Figure 6.6, we still have freedom in deciding the identification of the edges on the boundary of this polygon. Given the restriction that edges must be glued together via translations, it is easy to see that there are two ways to glue the two top edges to the two bottom edges and two ways to glue the left edges to the right edges. In particular, we see that there are four ways of gluing 6 equilateral triangles to give a translation surface

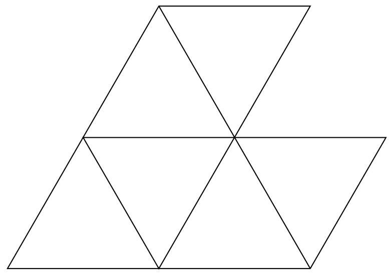


Figure 6.6: In order for the triangles to form a translation surface, they must be configured in the above shape (up to translations of the triangles), where two edges overlapping signify a gluing. We do not specify a gluing on the outer edges of the shape but note that they must be glued to other outer edges via translations.

(Figure 6.7).

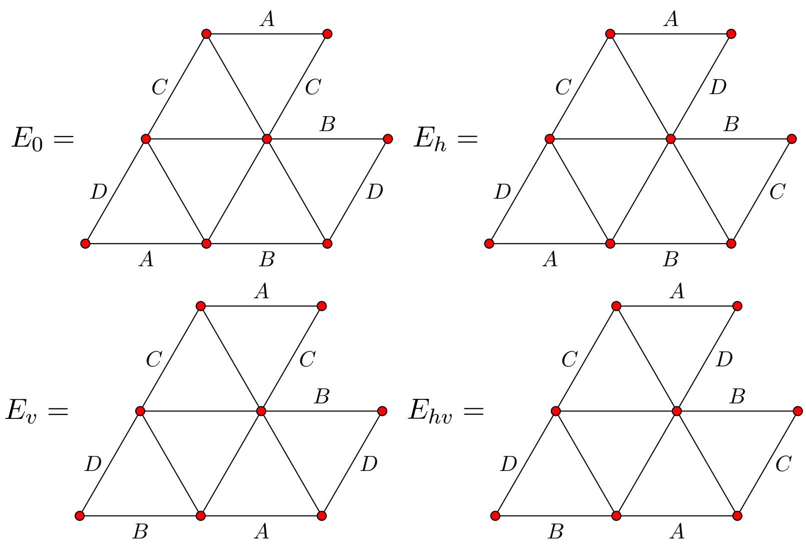


Figure 6.7: There are four ways to glue the outer edges to each other via translations.

Out of these four surfaces there is only one distinct translation surfaces in $\mathcal{H}_A(2)$ (up to rotation) because three of the configurations yield isometric surfaces and one of them is a torus. One can check that E_{hv} is a torus (and hence not in $\mathcal{H}_A(2)$) by observing that the cone-angle around the any of the red points is equal to 2π. The remaining three surface can be obtained from one another by applying rotations of $\pi/3$ or $2\pi/3$. One can check that this is so by applying the rotations and then using cut-and-paste operations on the triangles.

6.4.2 Saddle connection paths on the $SL(2,\mathbb{R})$-orbits of equilateral surfaces

The aim of this section is to give a comprehensive description of the saddle connection paths on surfaces in the $SL(2,\mathbb{R})$-orbits of equilateral surfaces/square-tiled surfaces whose singularities all have the same cone-angle. We will see that the length spectrum of saddle connection paths of these surfaces depend on the length spectrum of the torus formed by the tiles of these surfaces. In this section, we fix a square-tiled surface $X_S \in \mathcal{H}_A(\underline{k}_n)$ with singularity set Σ.

For reasons that will become apparent later, we will introduce the following generalisation of a saddle connection.

Definition 6.4.1. *An oriented singular connection, e, is a finite sequence of oriented saddle connections, i.e. $e = s_1 \dots s_n$ where the s_i are saddle connections, such that for $1 \leq i < n$, $t(s_i) = i(s_{i+1})$ and the angle formed by starting at s_i and moving clockwise about $t(s_i)$ (with respect to some fixed orientation) to s_{i+1} is equal to π (Figure 6.8).*

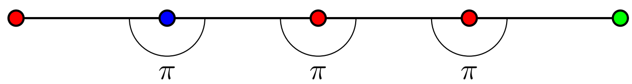


Figure 6.8: Four saddle connections that form a single singular connection.

Note that a single singular connection can still be a saddle connection.

It will prove useful to write saddle connection paths in terms of oriented singular connections instead of saddle connections. In particular, we can write any saddle connection as $p = e_1 \dots e_n$, where the e_i are singular connections such that $t(e_i) = i(e_{i+1})$ and the angle condition for geodesics on translation surfaces holds for $e_i e_{i+1}$. Furthermore, we make the additional requirement that the singular connections in the decomposition of p are "maximal with respect to p" i.e. for $1 \leq i < n$, $e_i e_{i+1}$ is not a singular connection. Note that the set of saddle connection paths on X correspond to sequences of singular connections with the above restrictions.

We will proceed by studying saddle connection paths on a square-tiled surface X_S, and note that saddle connection paths on general surfaces in $SL(2,\mathbb{R})(X_S)$ have a similar structure (see Figure 6.2.4).

Recall from Section 6.3.2, that X_S is a cover of the square torus $T(X_S)$ with covering map $\pi_{X_S} : X_S \to T(X_S)$. Because oriented singular connections begin and end at singularities, under the projection map oriented singular connections are sent to oriented closed geodesics on $T(X_S)$. Hence, an oriented singular connection has its holonomy (the vector associated to the singular connection) given by some $(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Furthermore, there are exactly $k+1$ oriented singular connections based at a given singularity x with holonomy $(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ because at the singularities of cone-angle $2\pi(k+1)$ the projection map $\pi_{X_S}(x)$ can be written in the form $w = z^{k+1}$ using local complex coordinates (see Section 6.3.2). In particular, the set of oriented singular connections based at x corresponds to $k+1$ copies of the lattice $\mathbb{Z}^2 \backslash \{(0,0)\}$.

Now we will show that given an oriented singular connection e_1 such that $t(e_1) = y \in \Sigma$, the set of saddle connection paths of the form $e_1 e_2$ corresponds to k copies of the lattice $\mathbb{Z}^2 \backslash \{(0,0)\}$. Recall that we insisted that in order for $e_1 e_2$ to form a saddle connection path, $e_1 e_2$ cannot itself be a singular connection. It follows that the clockwise angle formed by e_1 and e_2 must be greater than π (otherwise it would not be a geodesic or the concatenation itself would be a singular connection) and the anticlockwise angle must be greater than or equal to π. One can check that this eliminates exactly 1 out of the $k+1$ oriented singular connections of a given holonomy $(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ (see Figure 6.9).

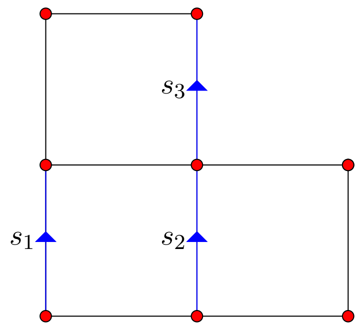


Figure 6.9: Three oriented saddle/singular connections s_1, s_2 and s_3 of holonomy $(0, 1)$ on a square-tiled surface with one singularity of cone-angle 6π. The singular connection s_1 can be followed by the other two singular connections to form a saddle connection path such that the singular connections will be maximal with respect to this path. However, when it is concatenated with itself, the concatenation will also be a singular connection as s_1 forms a clockwise angle of π with itself. Hence, we do not consider this concatenation to be a saddle connection path. In particular, we see that the s_1 can be concatenated with two out of three singular connections with holonomy $(0, 1)$ to form a singular connection.

Hence we have shown that any saddle connection path on X_S can be written as a sequence of n singular connections, where the first singular connection that starts at x can be chosen to correspond to any of the points in the $k+1$ copies of the lattice $\mathbb{Z}^2\backslash\{(0,0)\}$, the next singular connection can be chosen to be any of the allowed singular connections corresponding to k copies of the lattice $\mathbb{Z}^2\backslash\{(0,0)\}$, and so on. The length of the path will then be the sum of the lengths of the corresponding vectors associated to the lattice points. Similarly, if we take any point in the $k+1$ copies of the $\mathbb{Z}^2\backslash\{(0,0)\}$ lattice, followed by any finite sequence of points in the k copies of the lattice, this sequence will determine a unique saddle connection path on X_S.

If we consider $A(X_S) \in SL(2, \mathbb{R})(X_S)$, its saddle connection paths will also have the aforementioned correspondence, except that we replace $\mathbb{Z}^2\backslash\{(0,0)\}$ with $A(\mathbb{Z}^2\backslash\{(0,0)\})$.

6.5 Minimizing entropy over $SL(2,\mathbb{R})$-orbits of equilateral surfaces

The aim of this section is to prove Theorem 6.1.1, i.e. solve the entropy minimization problem when we restrict our attention to certain $SL(2,\mathbb{R})$-orbits of equilateral surfaces. We will restate Theorem 6.1.1 here for the reader's convenience.

Theorem 6.1.1. *Let $\mathcal{H}_A(\underline{k}_n)$ denote the stratum of area A translation surfaces, consisting of surfaces with n singularities, each of cone-angle $2\pi(k+1)$. Let X_S be an square-tiled surface. Then $h : SL(2,\mathbb{R})(X_S) \to \mathbb{R}^+$ is uniquely minimized by the equilateral surfaces in $SL(2,\mathbb{R})(X_S)$.*

The proof of the above theorem uses the saddle connection path structure of surfaces in the $SL(2,\mathbb{R})$-orbit of square-tiled surfaces (described in the previous section) to reduce the entropy minimization problem to a lattice minimization problem that can be solved using the method outlined in [3].

Before we continue, we introduce some notation. We fix a stratum $\mathcal{H}_A(\underline{k}_n)$, a square-tiled surface $X_S \in \mathcal{H}_A(\underline{k}_n)$, and a singularity $x \in \Sigma$. For convenience we assume that the squares tiling X_S each have area 1. We will denote a surface in $SL(2,\mathbb{R})(X_S)$ by $A(X_S)$ where $A \in SL(2,\mathbb{R})$. Let $\mathcal{P}_A$ denote the set of saddle connection paths on $A(X_S)$, based at the singularity $A(x)$ and we denote the length of a path $p \in \mathcal{P}_A$ by $\ell_A(p)$. We will consider the entropy function defined on $SL(2,\mathbb{R})(X_S)$, i.e. $h : SL(2,\mathbb{R})(X_S) \to \mathbb{R}^+$ and write $h(A) := h(A(X_S))$.

Recall that a surface $A(X_S)$ is a cover of some torus $T(A(X_S))$ (see the previous section). The holonomy vectors corresponding to the oriented closed geodesics on $T(A(X_S))$ can be identified with the lattice $A(\mathbb{Z}^2\backslash\{(0,0)\})$ which we denote by $A(\Lambda)$. We denote the length of a vector $\underline{v} \in A(\Lambda) \subset \mathbb{R}^2$ by $\ell(\underline{v}) := ||\underline{v}||_2$. We note that the lattice corresponding to any equilateral surface is given by $\Delta(\mathbb{Z}^2\backslash\{(0,0)\})$, where

$$\Delta = \begin{pmatrix} \sqrt{\frac{2}{\sqrt{3}}} & \sqrt{\frac{1}{2\sqrt{3}}} \\ 0 & \sqrt{\frac{\sqrt{3}}{2}} \end{pmatrix}.$$

We begin by defining the the following family of functions $f_t : SL(2,\mathbb{R}) \to \mathbb{R}^+$, for $t > 0$, by

$$f_t(A) = \sum_{\underline{v} \in A(\Lambda)} e^{-t\ell(\underline{v})}.$$

We will now see how the $f_t(A)$ relate to the entropy function $h(A)$. Recall from Definition 2.8.4, that the entropy of $A(X_S)$ is given by

$$h(A) = \inf_{t>0}\{t : \sum_{p\in\mathcal{P}_A} e^{-t\ell_A(p)} < \infty\}.$$

Using the saddle connection path structure for $A(X_S)$ (see Section 6.4.2), we can formally rewrite the exponentially weighted sum of the lengths of saddle connection paths on $A(X_S)$ using the $f_t(A)$ as follows

$$\begin{aligned}\sum_{p\in\mathcal{P}_A} e^{-t\ell_A(p)} &= (k+1)\sum_{\underline{v}\in A(\Lambda)} e^{-t\ell(\underline{v})}\sum_{n=0}^{\infty}\left(k\sum_{\underline{v}\in A(\Lambda)} e^{-t\ell(\underline{v})}\right)^n \\ &= (k+1)f_t(A)\sum_{n=0}^{\infty}\left(kf_t(A)\right)^n.\end{aligned}$$

Provided that t is sufficiently large, $f_t(A) < 1$ and we can rewrite the right-hand side of the equation above as follows

$$\sum_{p\in\mathcal{P}_A} e^{-t\ell_A(p)} = (k+1)f_t(A)\sum_{n=0}^{\infty}(kf_t(A))^n \quad = \frac{(k+1)f_t(A)}{1-kf_t(A)} < \infty. \tag{2}$$

We can then use (2) to obtain an expression for $h(A)$ in terms of the $f_t(A)$.

Lemma 6.5.1. *Let $A \in SL(2,\mathbb{R})$. Then $h(A)$ is the unique $t > 0$ that satisfies*

$$f_t(A) = 1/k.$$

Proof. The result follows from rewriting the exponentially weighted sum of the saddle connection paths using (2), the definition of $h(A)$ in terms of generating functions and the fact that $f_t(A)$ is a decreasing with respect to t. □

We note that it follows from Lemma 6.5.1, that the entropy of a surface $A(X_S) \in SL(2,\mathbb{R})(X_S)$ depends only on the corresponding lattice $A(\Lambda)$ and the singularity data of the surface. It follows that all equilateral surfaces in a given stratum have the same entropy.

In order to complete the proof of Theorem 6.1.1, we require the following result that can be found in [3] (Proposition 3.1 and Example 2.8).

Proposition 6.5.2. *For any $t > 0$, $f_t(A)$ is uniquely minimized (up to rotation) by Δ, where $\Delta(\mathbb{Z}^2\backslash\{(0,0)\})$ corresponds to the the equilateral lattice.*

Remark 6.5.3. Bétermin in [3], showed that the equilateral lattice minimizes a larger class of functions of a similar form to $f_t(A)$ by building on work by Montgomery [40].

We can now prove Theorem 6.1.1.

Proof. Let X_S be a square-tiled surface in $\mathcal{H}_A(\underline{k}_n)$. Let $B, \Delta \in SL(2,\mathbb{R})$, such that $\Delta(X_S)$ is an equilateral surface with entropy $h(\Delta)$. By Lemma 6.5.1 and Proposition 6.5.2,

$$f_{h(B)}(B) = 1/k = f_{h(\Delta)}(\Delta) \leq f_{h(\Delta)}(B),$$

with equality if and only if $B(X_S)$ is an equilateral surface.

Because $f_t(A)$ is decreasing with respect to $t > 0$, it follows from the above inequality that $h(\Delta) \leq h(B)$ with equality if and only if $B(X_S)$ is an equilateral surface. □

We can use $f_t(A)$ to deduce the regularity of $h : SL(2,\mathbb{R})(X_S) \to \mathbb{R}^+$.

Proposition 6.5.4. *Let $X_S \in \mathcal{H}_A(\underline{k}_n)$ be a square-tiled surface. Then the entropy function $h : SL(2,\mathbb{R})(X_S) \to \mathbb{R}^+$ is C^∞.*

Proof. We begin by noting that the lengths of singular connections on X_S have a C^∞ dependence on the $SL(2,\mathbb{R})$-orbit. The result then follows from applying the Implicit Function Theorem to $f_t(A) = 1/k$ and Lemma 6.5.1. □

We conclude this section by providing a plot for $f_t(A)$ over the modular surface $\mathcal{M}$[2] using approximations to $f_t(A)$ for $t = 2.511$ (Figure 6.10).

[2]Here we use the observation that $f_t(A)$ is well-defined on the space of unit area tori considered up to rotation (the modular surface).

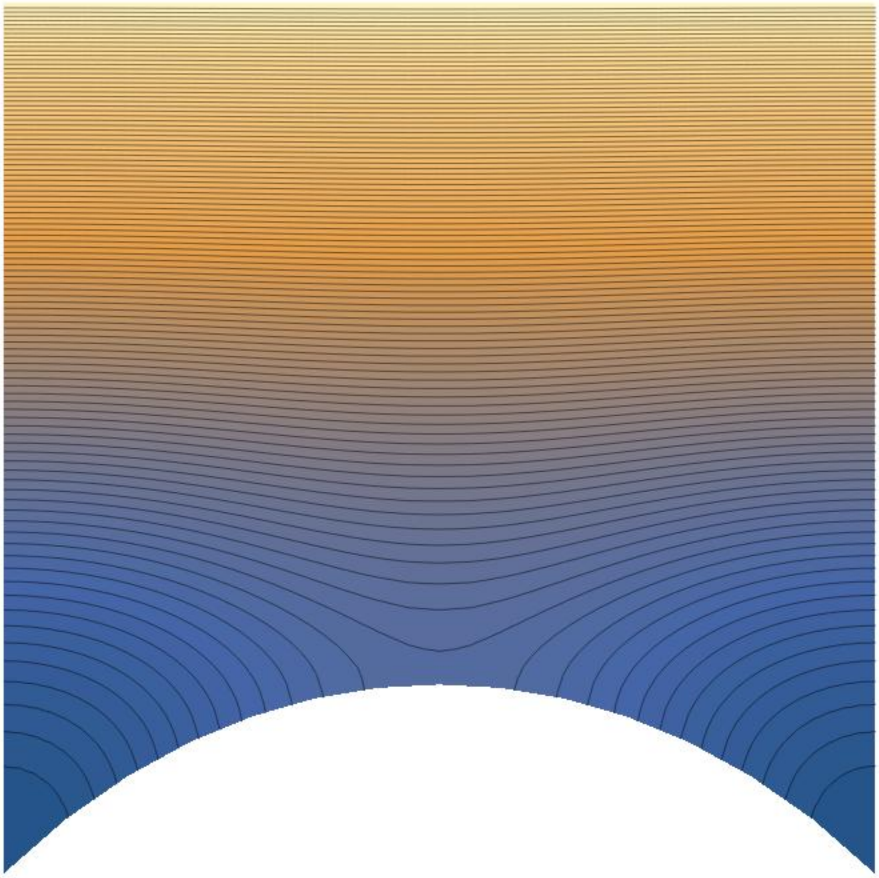


Figure 6.10: A contour plot of an approximation to $f_t(A) : \mathcal{M} \to \mathbb{R}^+$, $t = 2.511$. The darker regions represents the smaller values of $f_t(A)$ (the value decreases towards the bottom corners of the region) and the corners correspond the the equilaterally tiled torus.

6.6 Calculating the entropy of equilateral surfaces

In the previous section, we described an approach to finding minimal points for the entropy function defined over the $SL(2,\mathbb{R})$-orbits of equilateral surfaces in strata of the form $\mathcal{H}_A(\underline{k}_n)$. Now we will present a method of calculating the values of surfaces in the orbit.

To be more specific, let X_S denote a square-tiled surface in $\mathcal{H}_A(\underline{k}_n)$. We develop a method for finding arbitrarily good approximations to $h(A)$, for a given $A \in SL(2,\mathbb{R})$, and calculate the error terms of these approximations. We will then use this method to approximate $h(\Delta)$, the entropy of the equilateral surface in $\mathcal{H}_3(2)$ (see Section 6.4.1).

Let $N \in \mathbb{N}$ and define $\mathbb{Z}_N = \{n \in \mathbb{Z} : |n| \leq N\}$. We can then define the finite square lattice $\Lambda_N := \mathbb{Z}_N^2 \backslash \{(0,0)\}$ and the sheared finite square lattices $A(\Lambda_N)$ for $A \in SL(2,\mathbb{R})$.

We can then define an N^{th} approximation to $f_t(A)$ by considering a truncation of the infinite series in the definition of $f_t(A)$:

$$f_t^{(N)}(A) = \sum_{\underline{v} \in A(\Lambda_N)} e^{-t\ell(\underline{v})}.$$

Note that the derivatives of $f_t^{(N)}(A)$ give approximations to the respective derivatives of $f_t(A)$.

The following lemma allows us to bound the error of the approximations.

Lemma 6.6.1. *Fix $A \in SL(2,\mathbb{R})$. We define*

$$d(A) := \inf_{\underline{x} \in \mathbb{R}^2 : ||\underline{x}||_2 = 1} ||A(\underline{x})||_2.$$

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-negative integrable non-increasing function such that $\int_0^\infty g(x)dx < \infty$. Then the following inequalities hold:

1. $$\sum_{\underline{v} \in A(\Lambda)} g(\ell(\underline{v})) \leq 2\pi \int_0^\infty Rg(R)dR + 4\int_0^\infty g(R)dR,$$

2. *For $N \in \mathbb{N}$,*

$$\sum_{\underline{v} \in A(\Lambda)} g(\ell(\underline{v})) - \sum_{\underline{v} \in A(\Lambda_N)} g(\ell(\underline{v})) \leq 2\pi \int_{d(A)N}^\infty Rg(R)dR + 4\int_{d(A)N}^\infty g(R)dR,$$

Proof. We begin by defining $Q = \{(m,n) \in \mathbb{Z}^2 : m \neq 0 \text{ and } n \neq 0\}$. We can partition $A(\Lambda)$ into $A(Q)$ and $A(\Lambda \backslash Q)$. We begin by noting that $\sum_{\underline{v} \in A(Q)} g(\ell(\underline{v}))$ has a natural geometric interpretation which we describe in what follows. First observe that we can tile $\mathbb{R}^2$ with parallelograms $p(m,n)$, where $p(m,n)$ has its vertices at $A(m,n), A(m-1,n), A(m,n-1)$ and $A(m-1,n-1)$. We can extend each parallelogram $p(m,n)$ into a parallelepiped $P(m,n)$ in $\mathbb{R}^3$ by defining $P(m,n) =$

$p(m,n) \times [0, g(\ell(\underline{v}(m,n))]$, where $\underline{v}(m,n)$ is the point in $p(m,n)$ furthest away from the origin. The total volume of the $P(m,n)$ is equal to

$$\sum_{\underline{v} \in A(Q)} g(\ell(\underline{v})).$$

We can consider $g(x)$ as a function on $\mathbb{R}^2$ by defining $g(\underline{x}) := g(\ell(\underline{x}))$ for $\underline{x} \in \mathbb{R}^2$. Note that because $g(x)$ is a non-increasing function, for $\underline{x} \in p(m,n)$ $g(\underline{x}) \geq g(v(m,n))$. It follows that the integral of $g(\underline{x})$ over $\mathbb{R}^2$ is the volume of a region in $\mathbb{R}^3$ which contains the region defined by the parallelepipeds $P(m,n)$ and hence is greater than $\sum_{\underline{v} \in Q} g(\ell(\underline{v}))$. By considering the integral in terms of polar coordinates we obtain

$$\sum_{\underline{v} \in A(Q)} g(\ell(\underline{v})) \leq 2\pi \int_0^\infty Rg(R)dR.$$

To obtain the full inequality in (1) we need to bound the contribution to the sum from points in $A(\Lambda \backslash Q)$, i.e.

$$\sum_{\underline{v} \in A(m,0): m \in \mathbb{Z} \backslash \{\underline{0}\}} g(\ell(\underline{v})) + \sum_{\underline{v} \in A(0,n): n \in \mathbb{Z} \backslash \{\underline{0}\}} g(\ell(\underline{v})).$$

The bound follows from a similar geometric interpretation that we just used, except we observe that the sums correspond to areas of rectangles rather than volumes of parallelepipeds.

For the second inequality, we begin by defining $Q_N := \{(m,n) \in \mathbb{Z}_N^2 \backslash \{\underline{0}\} : m \neq 0 \text{ and } n \neq 0\}$, where $\mathbb{Z}_N := \{m \in \mathbb{Z} : m \leq N\}$. We will first consider contributions to the sum on the left-hand side of the second inequality from $\underline{v} \in A(Q \backslash Q_N)$. Once again, this sum is the volume of parallelepipeds, however this time the parallelepipeds correspond to parallelograms $p(m,n)$ such that $(m,n) \in Q \backslash Q_N$.

Observe that by definition of $d(A)$, a ball of radius $d(A)$ centered at any of the parallelograms $p(m,n)$, lies completely inside the parallelogram (Figure 6.11).
By scaling the parallelogram it is easy to check that that a ball of radius $2d(A)N$ centered at the origin will be contained in the set of parallelograms $p(m,n)$ such that $(m,n) \in Q_N$. Hence the integral

$$2\pi \int_{d(A)N}^\infty Rg(R)dR$$

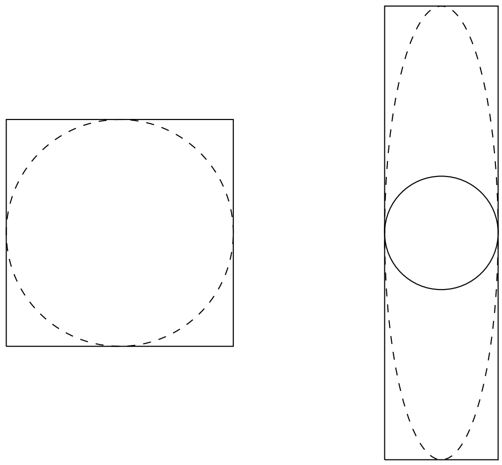


Figure 6.11: The parallelogram on the right is obtained from applying some matrix $A \in SL(2,\mathbb{R})$ to the square on the left. The ball of radius $d(A)$ (drawn as a black circle) centered at the center of the parallelogram is contained within the parallelogram.

corresponds to a volume greater than the volume of the region corresponding to volume of the parallelepipeds, i.e. the sum

$$\sum_{\underline{v}\in A(Q\setminus Q_N)} g(\ell(\underline{v})).$$

Similar reasoning gives the bound for the remaining points. $\square$

We will now use the $f_t^N(A)$ and the above lemma to approximate $h(\Delta)$, where $h(\Delta)$ denotes the entropy of the equilateral surfaces in $\mathcal{H}_3(2)$.

Recall from Lemma 6.5.1 that $h(\Delta)$ is the unique $t > 0$ such that $f_t(\Delta) = 1/2$.

By applying Lemma 6.6.1 to $f_t(\Delta) - f_t^{(N)}(\Delta)$, we obtain an upper bound for $f_t(\Delta) - f_t^{(N)}(\Delta)$ which we denote by $E_t^{(N)}(\Delta)$. Next observe the following inequalities:

$$f_t^{(N)}(\Delta) \le f_t(\Delta) \le f_t^{(N)}(\Delta) + E_t^{(N)}(\Delta),$$

where each of the terms are decreasing in t.

Let $h_L^{(N)}(\Delta)$ denote the unique $t > 0$ such that $f_t^{(N)}(\Delta) = 1/2$ and let $h_U^{(N)}(\Delta)$ denote the unique $t > 0$ such that $f_t^{(N)}(\Delta) + E_t^{(N)}(\Delta) = 1/2$.

It follows from the previous inequality that for all $N \in \mathbb{N}$, $h_L^{(N)}(\Delta) \leq h(\Delta) \leq h_U^{(N)}(\Delta)$. Because $f_t^N(\Delta)$ converges to $f_t(\Delta)$ and $E_t^{(N)}(\Delta)$ converges to 0 as $N \to \infty$, we obtain arbitrarily close bounds to $h(\Delta)$ by taking N sufficiently large.

We will first calculate $E_t^{(N)}(\Delta)$ using Lemma 6.6.1 and then compute the bounds for N sufficiently large. Note that $d(\Delta) := \inf_{\underline{x}\in\mathbb{R}^2:||\underline{x}||_2=1} ||\Delta(\underline{x})||_2$ is the smallest singular value of Δ, i.e. the square root of the smallest eigenvalue of $\Delta^*\Delta$, where Δ^* denotes the adjoint of Δ (see [18]). By a standard calculation, one can show that $d(\Delta) = 0.75836...$

Next we apply Lemma 6.6.1 to get

$$E_t^{(N)}(\Delta) = 2\pi \int_{d(\Delta)N}^{\infty} Re^{-tR} dR + 4 \int_{d(\Delta)N} e^{-tR} dR$$

$$= \frac{2\pi}{t^2} e^{-d(\Delta)Nt}(d(\Delta)Nt + 1) + \frac{4}{t} e^{-d(\Delta)Nt}.$$

Using Mathematica's NSolve with working precision equal to 30, we solve $f_t^{(N)}(\Delta) = 1/2$ for t, with $N = 100$ to obtain $h_L(\Delta) = 2.51109553318836192072366801885....$

Again, using NSolve, we numerically solve $f_t^{(N)}(\Delta) + E_t^{(N)}(\Delta) = 1/2$ for t, using the expression for $E_t^{(N)}(\Delta)$ with $N = 100$ to also get
$h_U(\Delta) = 2.51109553318836192072366801885...$

Hence we see that $h(\Delta) = 2.51109553318836192072366801885...$ (up to 29 decimal places).

Remark 6.6.2. We conclude by noting that the method of approximating $f_t(A)$ by the functions $f_t^{(N)}(A)$ can serve as an alternate method for approaching the minimization problem tackled in the previous section. By studying partial derivatives of the $f_t^{(N)}(A)$ and using the bounds in Lemma 6.6.1, we were able to show that the equilateral surfaces *locally* minimize entropy over their $SL(2, \mathbb{R})$-orbits.

Chapter 7

Conclusion

In this concluding chapter we will briefly look at possible directions for further research.

7.1 Geometric growth for flat surfaces

In this section we discuss the possibility of generalizing the work in this thesis from translation surfaces to general flat surfaces with singularities.

The purpose of this thesis was to show that when it comes to geometric growth, translation surfaces behave like surfaces of negative curvature. The key idea was that the singularities on translation surfaces behave like points of concentrated negative curvature. It is natural to ask: are there other surfaces that have this property and do they also exhibit similar geometric growth behaviour?

We begin by defining a generalization of translation surfaces, flat surfaces, which have singularities that will be "negatively curved" by the same reasoning used in Section 2.7.1.

Definition 7.1.1. *A flat surface is a topological surface S of genus $g \geq 2$ together with a finite set of points $\Sigma \subset S$ and a metric on $S \backslash \Sigma$ which is locally Euclidean. The set Σ is the singularity set of S.*

Clearly translation surfaces are flat surfaces. By the Gauss-Bonnet theorem, at least one of the singularities in Σ will have cone-angle greater than 2π and so the surface can be thought of having a point of high negative curvature. Note that a path passing through a singularity of cone-angle less than 2π cannot be a geodesic due to the geodesic angle condition described for translation surfaces.

One could ask whether these flat surfaces, or at least certain examples of flat surfaces (for example half-translation surfaces [15]), exhibit the geometric growth behaviour we saw for negatively curved and translation surfaces. It is possible that the approach we developed for infinite graphs and translation surfaces may extend to the general flat surface setting provided that translation surface hypotheses (T1)-(T3) hold for flat surfaces.

7.2 Error bounds for asymptotic formulae

A stronger result than an asymptotic formula would involve error bounds that quantify how quickly the ratio of the counting/growth function to the exponential function converge.

We expect polynomial error bounds if a Diophantine condition would hold for the closed geodesic length spectrum of translation surfaces (see [44] for an analogous result for hyperbolic flows on closed manifolds). In particular, if there exist closed geodesics c_1, c_2 on X such that $\ell(c_1)/\ell(c_2)$ is diophantine, i.e., there exists some $\tau > 0$ such that $|\frac{\ell(c_1)}{\ell(c_2)} - \frac{p}{q}| \geq \frac{1}{q^\tau}$ has only finitely many rational solutions, then we would obtain the following error bounds for volume growth on translation surfaces

$$V(\mathcal{B}(\widetilde{x}, R)) = e^{hR}(1 + O(R^{-\beta})),$$

as $R \to \infty$ for some $\beta > 0$. Similar error terms would hold for the other growth functions for translation surfaces. However, we wouldn't expect exponential error bounds, i.e. $V(\mathcal{B}(\widetilde{x}, R)) = e^{hR}(1 + O(e^{-\epsilon R}))$ as $R \to \infty$ for some $\epsilon > 0$.

7.3 Further directions for entropy functions on strata

Perhaps the most interesting direction of further research on the entropy functions for strata would be to further study the entropy minimization problem for strata. Recall that we gave the following conjecture (Conjecture 6.3.5):

Let $\mathcal{H}_A(\underline{k}_n)$ denote a stratum with n singularities, each of cone-angle $2\pi(k+1)$. Then the entropy function $h : \mathcal{H}_A(\underline{k}_n) \to \mathbb{R}^+$ has global minima at the equilateral surfaces in $\mathcal{H}_A(\underline{k}_n)$.

A natural easier problem may be to find the surfaces that minimize entropy over $SL(2,\mathbb{R})$-orbits of general square-tiled surfaces (where the corners of the squares do not have to meet at singularities) or just $SL(2,\mathbb{R})$-orbits in general. It is quite possible that the "infinite graph" perspective that we have taken in this thesis may be the wrong approach for the minimization problem.

Another direction of research would be to consider the regularity of the entropy function on strata/$SL(2,\mathbb{R})$-orbits. We suspect that the entropy function restricted to $SL(2,\mathbb{R})$-orbits are smooth.

It would also be interesting to calculate the entropy of various equilateral surfaces of genus $g \geq 2$ and compare them to the entropy of hyperbolic surfaces.

7.4 Simple closed geodesics on translation surfaces

Another interesting direction to pursue regarding geometric growth on translation surfaces is to examine the growth of simple closed geodesics on translation surfaces. Previous work on simple closed geodesics for translation surfaces includes [10] and [12].

We start by recalling the definition of a simple closed geodesic for general metric spaces.

Definition 7.4.1. *Let M be a metric space. A simple closed geodesic α on M is the image of an embedding $f : S^1 \to M$, such that α is a locally distance minimising curve.*

Let $\mathcal{M}_{g,n}$ denote the moduli space of complete hyperbolic Riemann surfaces of genus g with n cusps. In [39], Mirzakhani proved the following theorem (which we have written in a simplified form).

Theorem 7.4.2. *Fix $X \in \mathcal{M}_{g,n}$. Let $S_X(L)$ denote the number of simple closed geodesics of length less than L on X. Then*

$$\lim_{L\to\infty} \frac{S_X(L)}{L^{6g-6+2n}} = n(X),$$

where $n : \mathcal{M}_{g,n} \to \mathbb{R}_+$ is a continuous proper function.

Remark 7.4.3. Note that the growth rate is independent of the hyperbolic metric and depends only on the topology of the underlying surface. However, the constant does vary over the moduli space.

We will see that the above theorem cannot hold for all translation surfaces for a simple reason.

Recall that a closed geodesic on a translation surface is a sequence of saddle connections $s_1, \dots, s_n$ such that $t(s_j) = i(s_{j+1})$ for $i = 1, \dots n-1$, $t(s_n) = i(s_1)$ and the angle between any two consecutive saddle connections at least π. From this we easily see that simple closed geodesics on translation surfaces are either:

1. closed geodesic that pass through any singularity at most once and no two of its saddle connections intersect one another, except possibly at singularities; or

2. closed geodesics that live on the interior of some embedded cylinder on X.

We define S_X to be the set of simple closed geodesics on X (up to homotopy) and $S_X(L) := \{\alpha \in S_X : \ell_X(\alpha) \leq L\}$.

If we fix a translation surface with one singularity, then $S_X(L)$ will grow quadratically independently of the genus chosen. To see this note that because simple closed geodesics cannot pass through a singularity more than once, simple closed geodesics on translation surfaces with one singularity will correspond to cylinders on the surface (up to homotopy). It immediately follows from Theorem 2.6.3, that for translation surfaces X with a single singularity, $S_X(L)$ has quadratic growth. This immediately contrasts with the growth of simple closed geodesics on hyperbolic surfaces as surfaces of arbitrarily large genus admit a translation surface structure with a single singularity.

A key reason why Mirzakhani's result does not generalize to translation surfaces is due to the difference in behaviour between simple closed geodesics on hyperbolic surfaces and translation surfaces. On hyperbolic surfaces, simple closed curves have a unique geodesic representative which is also a simple closed curve. On translation surfaces, a simple closed curve can have a non-simple geodesic representative (Figure 7.1). Another way to think of this behaviour, is that closed geodesics on translation surfaces will need to pass through the singularities on the surface which increases their chance of self-intersecting themselves.

The above discussion highlights a possible difference between translation surfaces and negatively curved surfaces when it comes to geometric growth.

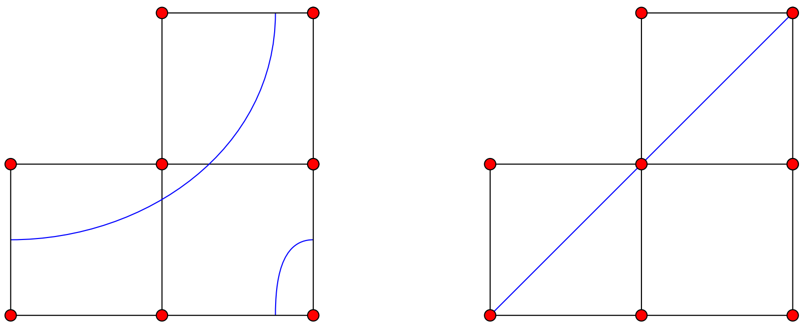


Figure 7.1: Left: A simple closed curve α on a translation surface with opposite edges identified. Right: The unique closed geodesic β homotopic to α. Note that β passes through the singularity (red) twice.

Bibliography

[1] Alan F. Beardon. An introduction to hyperbolic geometry. In *Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989)*, Oxford Sci. Publ., pages 1–33. Oxford Univ. Press, New York, 1991.

[2] G. Besson, G. Courtois, and S. Gallot. Entropies et rigidités des espaces localement symétriques de courbure strictement négative. *Geom. Funct. Anal.*, 5(5):731–799, 1995.

[3] Laurent Bétermin. Two-dimensional theta functions and crystallization among Bravais lattices. *SIAM J. Math. Anal.*, 48(5):3236–3269, 2016.

[4] Corentin Boissy and Slavyana Geninska. Systoles in translation surfaces (preprint). July 2017.

[5] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[6] B. Call, D. Constantine, A. Erchenko, N. Sawyer, and G. Work. Unique equilibrium states states for geodesic flows on flat surfaces with singularities. *preprint (ArXiv: 2101.11806v2).*

[7] P. Colognese and M. Pollicott. Volume growth for infinite graphs and translation surfaces. In *Dynamics: topology and numbers*, volume 744 of *Contemp. Math.*, pages 109–123. Amer. Math. Soc., [Providence], RI, [2020] ©2020.

[8] Klaus Dankwart. Typical geodesics on flat surfaces. *Conform. Geom. Dyn.*, 15:188–209, 2011.

[9] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[10] Moon Duchin, Christopher J. Leininger, and Kasra Rafi. Length spectra and degeneration of flat metrics. *Invent. Math.*, 182(2):231–277, 2010.

[11] William Ellison and Fern Ellison. *Prime numbers.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York; Hermann, Paris, 1985.

[12] Viveka Erlandsson and Caglar Uyanik. Length functions on currents and applications to dynamics and counting. In *In the tradition of Thurston*, pages 423–458. Springer, Cham, [2020] ©2020.

[13] Alex Eskin and Maryam Mirzakhani. Invariant and stationary measures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. *Publ. Math. Inst. Hautes Études Sci.*, 127:95–324, 2018.

[14] Alex Eskin and Kasra Rafi. A. Eskin - Counting closed geodesics on translation surfaces. `https://www.youtube.com/watch?v=hP2y-wrMGwY`, 2015.

[15] Max Forester, Robert Tang, and Jing Tao. Veech surfaces and simple closed curves. *Israel J. Math.*, 223(1):323–342, 2018.

[16] Giovanni Forni and Carlos Matheus. Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. *J. Mod. Dyn.*, 8(3-4):271–436, 2014.

[17] F. R. Gantmacher. *The theory of matrices. Vol. 1.* AMS Chelsea Publishing, Providence, RI, 1998. Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation.

[18] Gene H. Golub and Charles F. Van Loan. *Matrix computations.* Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, fourth edition, 2013.

[19] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[20] Eugene Gutkin. Billiard dynamics: an updated survey with the emphasis on open problems. *Chaos*, 22(2):026116, 13, 2012.

[21] Franz Hofbauer and Gerhard Keller. Zeta-functions and transfer-operators for piecewise linear transformations. *J. Reine Angew. Math.*, 352:100–113, 1984.

[22] Pascal Hubert and Samuel Lelièvre. Prime arithmetic Teichmüller discs in $H(2)$. *Israel J. Math.*, 151:281–321, 2006.

[23] Pascal Hubert and Thomas A. Schmidt. An introduction to Veech surfaces. In *Handbook of dynamical systems. Vol. 1B*, pages 501–526. Elsevier B. V., Amsterdam, 2006.

[24] Shikao Ikehara. *On a theorem of Landau.* ProQuest LLC, Ann Arbor, MI, 1930. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[25] A. Katok. Entropy and closed geodesics. *Ergodic Theory Dynam. Systems*, 2(3-4):339–365 (1983), 1982.

[26] A. Katok, G. Knieper, M. Pollicott, and H. Weiss. Differentiability and analyticity of topological entropy for Anosov and geodesic flows. *Invent. Math.*, 98(3):581–597, 1989.

[27] Wooyeon Kim and Seonhee Lim. Notes on the values of volume entropy of graphs. *Discrete Contin. Dyn. Syst.*, 40(9):5117–5129, 2020.

[28] Maxim Kontsevich and Anton Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Invent. Math.*, 153(3):631–678, 2003.

[29] Seonhee Lim. Minimal volume entropy for graphs. *Trans. Amer. Math. Soc.*, 360(10):5089–5100, 2008.

[30] Ricardo Mañé. On the topological entropy of geodesic flows. *J. Differential Geom.*, 45(1):74–93, 1997.

[31] Anthony Manning. Topological entropy for geodesic flows. *Ann. of Math. (2)*, 110(3):567–573, 1979.

[32] G. A. Margulis. Certain applications of ergodic theory to the investigation of manifolds of negative curvature. *Funkcional. Anal. i Priložen.*, 3(4):89–90, 1969.

[33] Howard Masur. Lower bounds for the number of saddle connections and closed trajectories of a quadratic differential. In *Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986)*, volume 10 of *Math. Sci. Res. Inst. Publ.*, pages 215–228. Springer, New York, 1988.

[34] Howard Masur. The growth rate of trajectories of a quadratic differential. *Ergodic Theory Dynam. Systems*, 10(1):151–176, 1990.

[35] Howard Masur. Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. *Duke Math. J.*, 66(3):387–442, 1992.

[36] Howard Masur and John Smillie. Hausdorff dimension of sets of nonergodic measured foliations. *Ann. of Math. (2)*, 134(3):455–543, 1991.

[37] Barry Mazur and William Stein. *Prime numbers and the Riemann hypothesis*. Cambridge University Press, Cambridge, 2016.

[38] Curtis T. McMullen. Dynamics of $\mathrm{SL}_2(\mathbb{R})$ over moduli space in genus two. *Ann. of Math. (2)*, 165(2):397–456, 2007.

[39] Maryam Mirzakhani. Growth of the number of simple closed geodesics on hyperbolic surfaces. *Ann. of Math. (2)*, 168(1):97–125, 2008.

[40] Hugh L. Montgomery. Minimal theta functions. *Glasgow Math. J.*, 30(1):75–85, 1988.

[41] Jorge Mujica. *Complex analysis in Banach spaces*, volume 120 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1986. Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática [Mathematical Notes], 107.

[42] William Parry. An analogue of the prime number theorem for closed orbits of shifts of finite type and their suspensions. *Israel J. Math.*, 45(1):41–52, 1983.

[43] William Parry. Bowen's equidistribution theory and the Dirichlet density theorem. *Ergodic Theory Dynam. Systems*, 4(1):117–134, 1984.

[44] Mark Pollicott and Richard Sharp. Error terms for closed orbits of hyperbolic flows. *Ergodic Theory Dynam. Systems*, 21(2):545–562, 2001.

[45] Burton Randol. The behavior under projection of dilating sets in a covering space. *Trans. Amer. Math. Soc.*, 285(2):855–859, 1984.

[46] Harold S. Shapiro. The range of an integer-valued polynomial. *Amer. Math. Monthly*, 64:424–425, 1957.

[47] Kurt Strebel. *Quadratic differentials*, volume 5 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[48] Peter Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[49] Hermann Weyl. Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.*, 77(3):313–352, 1916.

[50] Norbert Wiener. Tauberian theorems. *Ann. of Math. (2)*, 33(1):1–100, 1932.

[51] Alex Wright. Translation surfaces and their orbit closures: an introduction for a broad audience. *EMS Surv. Math. Sci.*, 2(1):63–108, 2015.

[52] Alex Wright. From rational billiards to dynamics on moduli spaces. *Bull. Amer. Math. Soc. (N.S.)*, 53(1):41–56, 2016.

[53] A. N. Zemljakov and A. B. Katok. Topological transitivity of billiards in polygons. *Mat. Zametki*, 18(2):291–300, 1975.

[54] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.