Electronic Journal of Probability

Electron. J. Probab. **27** (2022), article no. 91, 1–31.
ISSN: 1083-6489 https://doi.org/10.1214/22-EJP811

Results on the contact process with dynamic edges or under renewals*

Marcelo Hilário† Daniel Ungaretti‡ § Daniel Valesin¶ ‖
Maria Eulália Vares**

Abstract

We analyze variants of the contact process that are built by modifying the percolative structure given by the graphical construction and develop a robust renormalization argument for proving extinction in such models. With this method, we obtain results on the phase diagram of two models: the Contact Process on Dynamic Edges introduced by Linker and Remenik and a generalization of the Renewal Contact Process introduced by Fontes, Marchetti, Mountford and Vares.

Keywords: contact process; percolation; random environment; renewal process.
MSC2020 subject classifications: 60K35; 60K05; 82B43.
Submitted to EJP on August 6, 2021, final version accepted on June 17, 2022.
Supersedes arXiv:2108.03219.

1 Introduction

The contact process was introduced by Harris [14] as a Markov process that models contact interactions on a lattice and has become one of the most studied interacting particle systems ever since. It may alternatively be defined in terms of a percolative structure usually referred to as a *graphical representation* [15]. In the classical setting, this is done with the aid of infinitely many independent Poisson point processes (PPP) which are suitably assigned to the sites and to the edges of the lattice. Using the common

*The research of MH was partially supported by CNPq grant 'Produtividade em Pesquisa' (312227/2020-5) and by FAPEMIG grant 'Projeto Universal' (APQ-01214-21). DU was supported by grant 2020/05555-4, São Paulo Research Foundation (FAPESP). MEV was partially supported by CNPq grant 305075/2016-0 and FAPERJ CNE grant E-26/202.636/2019.
†ICEx, Universidade Federal de Minas Gerais, Belo Horizonte, Brazil. E-mail: mhilario@mat.ufmg.com
‡IME, Universidade de São Paulo, São Paulo, Brazil.
§Instituto de Matemática, Universidade Federal do Rio de Janeiro, RJ, Brazil. E-mail: daniel@im.ufrj.br
¶Bernoulli Institute, University of Groningen, Nijenborgh 9 9747 AG Groningen.
‖Department of Statistics, University of Warwick, Coventry, CV4 7AL.
E-mail: daniel.valesin@warwick.ac.uk
**Instituto de Matemática, Universidade Federal do Rio de Janeiro, RJ, Brazil. E-mail: eulalia@im.ufrj.br

interpretation of the contact process as a model for an infection, the Poisson marks represent the space-time location where either a transmission across an edge takes place or where a site is cured. Apart from providing an appealing interpretation and allowing for a construction of the contact process, the graphical representation is also an important tool in the study of the process (and more generally, in other classes of interacting particle systems). Notably, the proof by Bezuidenhout and Grimmett [3] of the result that the critical contact process dies out makes fundamental use of it.

By replacing the PPPs in the graphical representation by other types of point processes, one is led to natural generalizations of the contact process. That is the case for the Contact Process on Dynamic Edges (CPDE) [22] and the Renewal Contact Process (RCP) [8–11]. Even though the Markov property and other useful features like the FKG inequality may no longer hold, the usual questions regarding survival or extinction still make good sense and remain interesting from various aspects, including the percolative perspective itself.

In this paper, we call a *Generalized Contact Process* (GCP) any process that is obtained from a percolative structure of recovery and transmission marks in the same way as the contact process, but where the distribution of these marks is given by some other point process. Our contributions in the study of these processes are twofold. First, by extending ideas from the existing literature and inspired by the above mentioned works (especially [10, 11]), we propose a quite robust renormalization approach for the study of survival or extinction for the GCP. Then we specialize to two types of GCP (the aforementioned CPDE and variants of the aforementioned RCP), and prove some results concerning their phase diagram. We expect that our methods may be applicable in greater generality, as for example, to allow the study of extinction for contact processes in random environments with space-time correlations.

We now provide a more detailed definition for the models under consideration and present our main results.

Generalized Contact Process. Let $\mathcal{N}_x$ and $\mathcal{N}_{x,y}$ be point processes on the line, indexed by the sites $x \in \mathbb{Z}^d$ and by the pairs of nearest neighbors $x, y \in \mathbb{Z}^d$ respectively. Given an initial configuration $\xi_0 \in \{0,1\}^{\mathbb{Z}^d}$ we define a process $(\xi_t)_{t\geq 0}$ taking values in $\{0,1\}^{\mathbb{Z}^d}$ where $\xi_t(x) = 1$ (resp. $\xi_t(x) = 0$) is interpreted as x being infected (resp. healthy) at time t. From the initial configuration the process evolves in time, with the marks in $\mathcal{N}_x$ and $\mathcal{N}_{x,y}$ playing the roles of the instants of time when x may get cured, and when the infection may be transmitted across the edge xy respectively. More precisely, if $x, y \in \mathbb{Z}^d$ and $s < t$, we define a path γ from (x, s) to (y, t) as a càdlàg function $\gamma : [s, t] \to \mathbb{Z}^d$ that fulfills the following properties:

- it does not contain any cure marks: for every $u \in [s, t]$ $(\gamma(u), u) \notin \mathcal{N}_{\gamma(u)}$;
- its discontinuities have size one and only occur at transmission times: if $\gamma(u) \neq \gamma(u-)$ then $\gamma(u)$ and $\gamma(u-)$ are nearest neighbors in $\mathbb{Z}^d$ and $u \in \mathcal{N}_{\gamma(u-),\gamma(u)}$.

The event that there is a path from (x, s) to (y, t) is denoted by $(x, s) \rightsquigarrow (y, t)$. We define $\xi_t(y)$ as

$$\xi_t(y) = 1 \text{ if and only if we have } (x, 0) \rightsquigarrow (y, t) \text{ for some } x \text{ with } \xi_0(x) = 1.$$

We may pick a set $A \subset \mathbb{Z}^d$ to be the set of initially infected sites, by taking $\xi_0^A := \mathbb{1}_A$ in which case we write ξ_t^A for the resulting process. Identifying a configuration $\xi \in \{0,1\}^{\mathbb{Z}^d}$ with its set of infected sites $\{x \in \mathbb{Z}^d;\ \xi(x) = 1\}$ allows us to write $\xi_t^A = \cup_{x\in A}\xi_t^{\{x\}}$, a property known as *additivity*. For any starting configuration A, we define the extinction time from A as

$$\tau^A := \inf\{t;\ \xi_t^A \equiv 0\}.$$

We say that the GCP dies out or that extinction occurs if $\tau^{\{0\}} < \infty$, where $\{0\}$ represents the set containing only the origin. On a vertex-transitive graph, assuming that our point processes are i.i.d., additivity implies that if $\tau^{\{0\}} < \infty$ a.s. then the same holds for any finite starting set A.

In Section 2 we present a reformulation of the renormalization approach used in [11] that can be used for proving extinction for the GCP under certain conditions. This renormalization is applied to the CPDE and the RCP models. In both models, $\mathcal{N}_x$ and $\mathcal{N}_{x,y}$ are independent renewal processes and to emphasize this we denote them by $\mathcal{R}_x$ and $\mathcal{R}_{x,y}$. However, we remark that the GCP includes more general point processes.

Contact Process on Dynamic Edges. Some of our results concern a model of contact process on a dynamic random environment that was introduced in [22]

The environment is given by a dynamic percolation [28] on the edges of the $\mathbb{Z}^d$ lattice. Initially each edge is independently declared open with probability p and closed otherwise. In the environment dynamics, each edge updates its state to open or closed with respective rates vp and $v(1-p)$. Hence the environment is in equilibrium, given by the product of Bernoulli measures of parameter p and $v > 0$ is the total rate at which edges update.

Conditional on the environment, the contact process evolves as following: infected individuals try to transmit the infection to each of its healthy neighbors at a fixed rate λ and heals at rate one. However, any attempt for transmissions is only successful when it is allowed by the environment, i.e. when it is done across an open edge. We write $\mathbb{P}_{v,p,\lambda}$ for the joint law of the process and the environment on some suitable probability space. More details about the model are given in Section 3.

Monotonicity with respect to λ allows us to define the critical parameter

$$\lambda_0(v,p) := \inf\{\lambda > 0;\ \mathbb{P}_{v,p,\lambda}(\tau^{\{0\}} = \infty) > 0\}.$$

Let $p_c(d)$ be the critical point for Bernoulli bond percolation on $\mathbb{Z}^d$. Our main result is the following:

Theorem 1.1. *Consider the CPDE on $\mathbb{Z}^d$ with $d \geq 2$.*

(i) If $p < p_c(d)$, then

$$\lim_{v \downarrow 0} \lambda_0(v,p) = \infty.$$

(ii) If $p > p_c(d)$, then

$$\sup_{v \geq 0} \lambda_0(v,p) < \infty.$$

Remark 1.2. Since $p \mapsto \lambda_0(v,p)$ is non-increasing, by monotonicity the second item implies that when $\bar{p} > p_c$

$$\sup_{p \geq \bar{p}, v \geq 0} \lambda_0(v,p) < \infty. \tag{1.1}$$

Theorem 1.1 only mentions dimension $d \geq 2$, since for $d = 1$ we have that *(i)* is covered in [22] and *(ii)* is vacuous. Several questions concerning the CPDE were investigated in [22], mostly related to the behavior of the model as a function of the parameters v, p or λ. The results obtained there hold for any infinite vertex-transitive regular graph, the only exception being Theorem 2.4 which includes item *(i)* in Theorem 1.1 for the graph $\mathbb{Z}$ (with the obvious adaptation that $p_c(1) = 1$). The proof there relies on arguments that are essentially $(1+1)$-dimensional and does not seem to extend easily to higher spatial dimensions. However, the authors conjectured that the result should still hold beyond $\mathbb{Z}$ and item *(i)* answers this conjecture affirmatively for $\mathbb{Z}^d$, $d \geq 2$.

If $\bar{\lambda}$ stands for the critical parameter of the contact process on the static lattice then a simple coupling shows that $\lambda_0(v,p) \geq \bar{\lambda}$. However, as shown in Corollary 2.8 in [22] there are choices for v and p for which $\lambda_0(v,p) = \infty$. In fact their results imply that if

$$p_1 := \sup\{p > 0;\ \text{there is } v > 0 \text{ with } \lambda_0(v,p) = \infty\},$$

then $p_1 \in (0,1)$ and Theorem 1.1(ii) complements their result with the bound $p_1 \leq p_c$, see Figure 1. A natural question that follows is if these critical points are equal or not. For CPDE on $\mathbb{Z}$ it holds $p_1 < p_c(1) = 1$, but we do not know if strict inequality holds in general. If so, then the phase diagram would be divided into three different regions, with respect to the value of p.

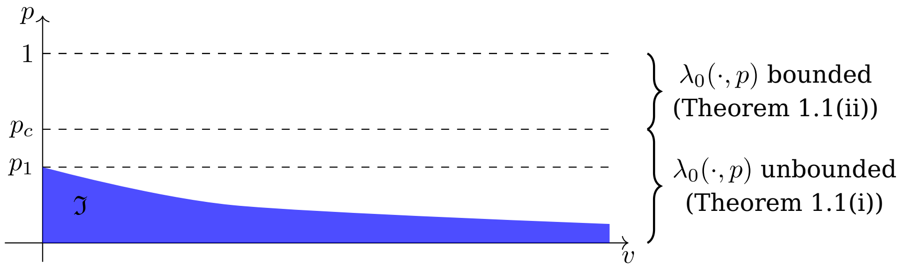

Figure 1: Phase diagram for the behavior of $\lambda_0(v,p)$, complementing Figure 1 in [22]. The immunity region $\mathfrak{I}$ in blue is the set of points with $\lambda_0(v,p) = \infty$. It falls below the line $p = p_1$, and the critical value p_1 is shown to be at most p_c by Theorem 1.1(ii).

The proof of Theorem 1.1(ii) is based on a one-step renormalization argument, while the proof of Theorem 1.1(i) relies on the multiscale renormalization construction presented in Section 2.

Renewal Contact Process. The second model for which we apply the method in Section 2 is a further generalization of the Renewal Contact Process (RCP) that appeared in the series of papers [8–11]. There, the authors consider the GCP in which the transmissions are governed by PPPs of rate λ and the cures by renewal processes whose interarrivals have a certain distribution ν. One can define the critical parameter whose value depends on the specific choice of distribution ν as:

$$\lambda_c(\nu) := \inf\{\lambda > 0;\ \mathbb{P}(\tau^{\{0\}} = \infty) > 0\}.$$

One of the goals for the investigation carried on in [8–11], was to relate the question whether $\lambda_c(\nu)$ is strictly positive to the tail decay of ν.

Analogously, if renewals with interarrival distribution μ were associated to the transmissions while the cures were given by PPP's of rate δ, then one could define the critical parameter as

$$\delta_c(\mu) := \inf\{\delta > 0;\ \mathbb{P}(\tau^{\{0\}} = \infty) = 0\}. \tag{1.2}$$

The critical parameter δ_c has not been investigated in the series of works [8–11]. The main novelty here is to allow for both the transmissions and the cures to be determined by renewal processes. Their interarrival distributions will be denoted μ and ν, respectively. Let us denote $\mathbb{P}_{\mu,\nu}$ the law of the resulting GCP on some suitable probability space.

A natural definition for the critical parameter is done by fixing one interarrival distribution while scaling the other. In fact, for any fixed $\delta > 0$ let us define the distribution ν_δ on $\mathbb{R}_+$ given by $\nu_\delta(t,\infty) := \nu(\delta t,\infty)$ for every $t \geq 0$. Then, we can define the critical parameter

$$\delta_c(\mu,\nu) := \inf\{\delta > 0;\ \mathbb{P}_{\mu,\nu_\delta}(\tau^{\{0\}} = \infty) = 0\}. \tag{1.3}$$

The definition in (1.3) includes the one in (1.2) since $\delta_c(\mu, \mathrm{Exp}(1)) = \delta_c(\mu)$. However, notice that monotonicity of $\mathbb{P}_{\mu,\nu_\delta}(\tau^{\{0\}} = \infty)$ in δ is not clear for general ν. A similar definition, considering scalings for μ, leads to the definition of $\lambda_c(\mu,\nu)$. The parameter $\lambda_c(\nu)$ investigated in [8–11] is the same as $\lambda_c(\mathrm{Exp}(1),\nu)$.

We investigate the behavior of $\delta_c(\mu,\nu)$. To emphasize that we are fixing the edge (or transmission) renewal processes and scaling the site (or cure) processes, we adopt the more explicit name Edge Renewal Contact Process (ERCP). Similarly, we write Site Renewal Contact Process (SRCP) in case we are fixing the cure processes. Also, we write ERCP(μ,ν) and SRCP(μ,ν) whenever we want to emphasize the dependence on the fixed distributions.

A first observation is that since we allow for transmissions that are given by general renewal processes, it is not clear in principle whether the infection can spread to infinitely many sites in finite time. In order to avoid such undesirable behavior, we only study $\delta_c(\mu,\nu)$ for continuous μ and ν. Such choice prevents the coincidence of transmission marks and is enough for our purposes, see Section 4.2 for the details. Another possibility would be the introduction of independent random delays to the transmission processes, using for instance i.i.d. uniform random variables in $(0,\varepsilon)$. Also, notice that if we suppress the cures in an ERCP, the resulting model can be seen as a generalization of Richardson model, which corresponds to the case where μ is an exponential distribution. We refer to a RCP without cures and transmission given by μ as a Richardson(μ) model.

Let us denote by $\mathcal{R}$ the collection of renewal marks obtained from some distribution μ. Besides continuity, there are two hypotheses on distributions that we use frequently.

The first one is a quantitative control on the renewal marks of a heavy-tailed distribution: we say that μ satisfies condition (G) if there exists $\epsilon_4 > 0$ and $t_0 > 0$ such that

$$\mathbb{P}(\mathcal{R} \cap [t, t + t^{\epsilon_4}] \neq \emptyset) \leq t^{-\epsilon_4} \quad \text{for } t \geq t_0. \tag{G}$$

Condition (G), which appeared already in [9], can be interpreted as $\mathcal{R}$ having increasingly large gaps. In [9, Proposition 7] it is shown that (G) holds whenever μ satisfies conditions A)-C) defined therein, which we reproduce here:

A) There is $1 < M_1 < \infty$, $\epsilon_1 > 0$ and $t_1 > 0$ such that

$$\text{for every } t > t_1, \qquad \epsilon_1 \int_{[0,t]} s\mu(\mathrm{d}s) < t\mu(t, M_1 t).$$

B) There is $1 < M_2 < \infty$, $\epsilon_2 > 0$ and $r_2 > 0$ such that

$$\text{for every } r > r_2, \qquad \epsilon_2 \mu[M_2^r, M_2^{r+1}] \leq \mu[M_2^{r+1}, M_2^{r+2}].$$

C) There is $M_3 < \infty$, $\epsilon_3 > 0$ such that

$$\text{for } t \geq M_3, \qquad t^{-(1-\epsilon_3)} \leq \mu(t,\infty) \leq t^{-\epsilon_3}.$$

Condition C) controls the tail decay of μ, while A) and B) concern its regularity. Thus, [9, Proposition 7] provides a straightforward way to verify if μ satisfies (G). In particular, hypothesis (G) holds when $\mu(t,\infty) = L(t)t^{-\alpha}$ with $\alpha \in (0,1)$, where L is a slowly varying function at infinity.

The second hypothesis on the interarrival distribution is a moment condition that is behind a sufficiently fast decay of correlations for events depending on $\mathcal{R}$. In [11, Theorem 1.1] it is proved that a sufficient condition for $\lambda_c(\nu) > 0$ in a SRCP($\mathrm{Exp}(1), \nu$) is the moment condition

$$\int_1^\infty x \exp\Big[\theta(\ln x)^{1/2}\Big]\nu(\mathrm{d}x) < \infty \quad \text{for some } \theta > 4\sqrt{(\ln 2)d}. \tag{M}$$

Condition (M) goes in the opposite direction of (G); it implies that it is hard to find large intervals without renewal marks, see Lemma 4.1(i). It is slightly stronger than finite first moment.

Now, we are ready to state our results about ERCP. Define

$$r_t := \max\{\|x\|_1\,;\ (0,0) \rightsquigarrow (x,t) \text{ in Richardson}(\mu) \text{ model}\}.$$

We estimate the speed of infection in Richardson(μ) model. This is based on a comparison with a toy model of iterated percolation, see Section 4.2. The estimates in Proposition 1.3 are used as a tool for proving Theorem 1.4, but we believe they are interesting by themselves.

Proposition 1.3. *Let $d \geq 1$ and μ be a continuous distribution.*

(i) For any $a > 1$ it holds $\overline{\lim}_{t\to\infty} \dfrac{r_t}{t(\ln t)^a} = 0$, almost surely.

(ii) Suppose μ satisfies (G)*. Then, the process r_t has sublinear growth: for every $\rho \in (0, \epsilon_4)$ we have*

$$\overline{\lim_{t\to\infty}} \frac{r_t}{t^{1-\rho}} = 0 \qquad \textit{almost surely.} \tag{1.4}$$

Our main contribution in the investigation of ERCP is a set of conditions on μ and ν under which we can show whether $\delta_c(\mu,\nu)$ is trivial or not. Heuristically, when μ is a heavy-tailed distribution, Proposition 1.3 shows that the speed of the infection in an environment without cures is slow. Any rate $\delta > 0$ of cure is sufficient for it to die out, as shown in Theorem 1.4(i) below. On the other hand, when μ and ν have a fast tail decay one expects a non-trivial phase transition, similar to what is observed in the standard Contact Process.

Theorem 1.4. *Let μ, ν be continuous interarrival distributions and consider a ERCP(μ, ν) in $\mathbb{Z}^d$.*

(i) If $d \geq 1$, μ satisfies A)-C) and ν satisfies $\int x^n \nu(\mathrm{d}x) < \infty$ for all $n \geq 1$, then the ERCP(μ, ν_δ) dies out almost surely, for each $\delta > 0$, i.e. , $\delta_c(\mu,\nu) = 0$.

(ii) If $d \geq 2$ and μ has finite first moment then $\delta_c(\mu,\nu) > 0$.

(iii) If $d \geq 1$ and μ and ν satisfy (M)*, then $\mathbb{P}_{\mu,\nu_\delta}(\tau^{\{0\}} = \infty) = 0$ for sufficiently large δ, i.e. , $\delta_c(\mu,\nu) < \infty$.*

Remark 1.5. The restriction to $d \geq 2$ in (ii) allows a simpler argument by using at most once the transmission process at each given edge, therefore avoiding the dependencies between various residual times. A suitable extension to $d = 1$ is expected.

Related works. Let us now comment on some related works, apart from [8–11] and [22], which were already mentioned. There is a substantial literature on the contact process on static random environments, that is, versions of the contact process in which the recovery and transmission rates may vary spatially, and are sampled from some environment distribution, but the dynamics is still driven by Poisson point processes. Due to this last point, these models are fundamentally different from the ones we consider, but there are similarities in the line of investigation and the analysis. Klein [17] considers an environment obtained from i.i.d. recovery and transmission rates, and gives a condition on the environment distribution that guarantees almost sure extinction; his method is a multi-scale construction that has similarities to the one we employ. Newman and Volchan [23] consider a one-dimensional recovery environment (and transmissions with constant rate λ), and give a condition on the environment distribution that guarantees survival regardless of λ. See also [1, 4, 12, 19].

Summary of the paper. This paper is organized as follows. In Section 2 we develop the renormalization argument for a GCP. Section 3 contains the results for CPDE and Section 4 contains the results for ERCP.

2 Renormalization scheme for generalized contact process

As mentioned in the introduction, in order to prove extinction for GCP we develop a version of the renormalization in [11]. The construction in this section is very close to the one in [11], but is presented here with some details to make it more self-contained and ease applications in different GCP. For instance, Lemmas 2.2 and 2.4 are straightforward adaptations of [11, Lemmas 2.5 and 2.6]. The main differences between the construction here and the one in [11] are:

- The previous construction was developed in the context of Renewal Contact Process and the construction here can be applied to GCP in general.
- Instead of considering a fixed sequence of boxes $B_n = [0, 2^n]^d \times [0, h_n]$, now we allow spatial dimensions to grow faster, which can be useful to decouple variations of the Contact Process with space correlations.
- In [11], the renormalization related crossing events on different scales by considering spatial and temporal crossings separately. Here, the same idea is present but we combine these crossings in a single event we call a *half-crossing*. Moreover, we now introduce the notion of a *hierarchy* of boxes, reminiscent of the arguments from [25, 29]. Whenever one has a half-crossing of a large scale box, the hierarchy encodes the structure of smaller scale boxes that are also half-crossed. This provides an alternative way of using this renormalization to prove extinction of the infection, and we apply it to CPDE in Section 3.1.

The choice of scales depends on some parameters that need to be tuned in order for the argument to work. This tuning depends on the specific point processes we choose for the model.

2.1 Main events

We begin recalling the definition of a general space-time crossing.

Definition 2.1 (Crossing)**.** *Given space-time regions $C, D, H \subset \mathbb{Z}^d \times \mathbb{R}$ we say there is a crossing from C to D in H if there is a path $\gamma : [s, t] \to \mathbb{Z}^d$ such that $\gamma(s) \in C$, $\gamma(t) \in D$ and for every $u \in [s, t]$ we have $(\gamma(u), u) \in H$.*

Given $a = (a_1, \ldots, a_d)$, and $b = (b_1, \ldots, b_d)$ with $a_i < b_i$ for every i, let $[a, b] = \prod_{i=1}^{d} [a_i, b_i]$ and consider the space-time box $B := [a, b] \times [s, t]$ whose projection into the spatial coordinates is $[a, b]$. For each $1 \leq j \leq d$ we denote by

$$\partial_j^- B := \{(x, u) \in B;\ x_j = a_j\} \quad \text{and} \quad \partial_j^+ B := \{(x, u) \in B;\ x_j = b_j\}$$

the face of B that is perpendicular to direction j. The hyperplane $\{(x, u) \in \mathbb{Z}^d \times \mathbb{R};\ x_j = \frac{a_j + b_j}{2}\}$ divides B into two half-boxes, B_j^- and B_j^+, that contain faces $\partial_j^- B$ and $\partial_j^+ B$, respectively. Using this notation, four crossing events of the box $B = [a, b] \times [s, t]$ will be important in our investigation.

Temporal crossing. Event $T(B)$ in which there is a path from $[a, b] \times \{s\}$ to $[a, b] \times \{t\}$ in B.

Temporal half-crossing. Event $\tilde{T}(B) := T([a, b] \times [s, \frac{t+s}{2}])$. In words, we have a temporal crossing from the bottom of B to the middle of its time interval.

Spatial crossing. For some fixed direction $j \in \{1, \dots, d\}$ we define $S_j(B)$ as the event that there is a crossing from $\partial_j^- B$ to $\partial_j^+ B$ in B, i.e., there is a crossing connecting the opposite faces of B that are perpendicular to direction j.

Spatial half-crossing. For some fixed direction $j \in \{1, \dots, d\}$, we define the events $\tilde{S}_{j,+}(B) := S_j(B_j^+)$ and $\tilde{S}_{j,-}(B) := S_j(B_j^-)$, in which we have a spatial crossing in B of a half-box connecting the opposite faces of direction j. To ease notation, we write $\tilde{S}_{j,+} = \tilde{S}_j$ and $\tilde{S}_{j,-} = \tilde{S}_{j+d}$, allowing indices $1 \leq j \leq 2d$.

Given a box B, consider the event

$$H(B) := \tilde{T}(B) \cup \bigcup_{j=1}^{2d} \tilde{S}_j(B) \tag{2.1}$$

that we refer to as *half-crossing* of B; this event will play a central role in the renormalization approach to be developed in the next sections. Our first aim is to show that $H(B)$ satisfies the so-called cascading property, meaning that its occurrence implies the existence of two well-positioned smaller boxes inside B which are also half-crossed.

2.2 Cascading property for half-crossing events; hierarchies

In what follows we will analyze half-crossing events inside boxes of type $B_k := [-l_k, l_k]^d \times [0, h_k]$ where $(l_k) \subset \mathbb{N}$ and $(h_k) \subset \mathbb{R}$ are increasing sequences to be determined later. They must be interpreted as sequences of spatial and temporal scales along which we analyze occurrence of half-crossing events. In fact, if the origin starts infected, i.e. if $\xi_0(\mathbf{0}) > 0$ and the resulting infection from that point survives till time h_k then either $T(B_k)$ occurs or the infection must leave box B_k through some of its faces $\partial_j^+ B_k$ or $\partial_j^- B_k$ for $1 \leq j \leq d$. Thus, one can write

$$\mathbb{P}(\tau^{\{0\}} = \infty) \leq \mathbb{P}\Big(\tilde{T}(B_k) \cup \bigcup_{j=1}^{2d} \tilde{S}_j(B_k)\Big) = \mathbb{P}(H(B_k)). \tag{2.2}$$

Any space-time translation of the box B_k is called a *scale-k* box. This section is devoted to proving a deterministic lemma that relates half-crossings of boxes at two successive scales. In Lemma 2.5 we prove that for this sequence of boxes the event $H(B_k)$ is *cascading*, meaning that its occurrence implies the occurrences of two similar events inside disjoint boxes from the previous scale. Moreover, we are able to

- find an upper bound (uniform in k) for the amount of pairs of boxes that we need to look at in order to find these two half-crossings;
- control the positions of such pairs of boxes, obtaining that they might be taken well-separated in space and time. For a class of examples, this allows to decouple the corresponding half-crossing events.

This will ultimately allow us to control the right-hand side in (2.2) which is useful for proving existence of regimes when the infection dies out.

Let us now describe the rate of growth for the sequences of scales (l_k) and (h_k) to be considered. Given the initial scales $l_0 \in \mathbb{N}$ and $h_0 > 0$ and two constants $\alpha, \beta \in \mathbb{N}$ we define recursively

$$l_{k+1} = \alpha l_k \quad \text{and} \quad h_{k+1} = \beta h_k, \qquad \text{for } k \geq 0. \tag{2.3}$$

Note that l_k and h_k grow exponentially fast. For other possibilities of scale progression, see Remark 2.6.

We are now ready to state and prove the main results in this section. Temporal and spatial half-crossing will be treated separately. Both cases follow the same pattern: given a scale-n box B_n that has been half-crossed, we can find two suitable disjoint regions inside B_n that were traversed by the half-crossing. After covering these regions with scale-$(n-1)$ boxes we conclude that in each region one of such boxes must have been half-crossed too. Let us begin with the temporal ones.

Lemma 2.2 (Temporal half-crossings)**.** *Fix $n \geq 1$ and $\beta > 4$. There are collections $\mathcal{B}_0 = \mathcal{B}_{0,n}$ and $\mathcal{B}'_0 = \mathcal{B}'_{0,n}$ of scale-$(n-1)$ boxes such that*

$$\tilde{T}(B_n) \subset \bigcup_{(B,B') \in \mathcal{B}_0 \times \mathcal{B}'_0} H(B) \cap H(B').$$

Moreover, we may assume that $\mathcal{B}_0$ and $\mathcal{B}'_0$ have $(2\alpha - 1)^d$ elements each, and that the vertical distance between any pair of boxes $B \in \mathcal{B}_0$ and $B' \in \mathcal{B}'_0$ is $(\beta/2 - 2)h_{n-1}$.

Remark 2.3. It is not needed to take $\alpha, \beta \in \mathbb{N}$, but we do so for simplicity since our model is discrete in space. The arguments would still follow by making a slightly different choice of boxes B_n.

Proof. By construction we have $h_n = \beta h_{n-1}$. The event $\tilde{T}(B_n)$ entails the two following temporal crossings

$$T([-l_n, l_n]^d \times [0, h_{n-1}]) \quad \text{and} \quad T([-l_n, l_n]^d \times [(\beta/2 - 1)h_{n-1}, (\beta/2)h_{n-1}]).$$

For $z \in [-\alpha, \alpha - 1] \cap \mathbb{Z} =: Z_\alpha$ let us define

$$I_z := l_{n-1}z + [0, l_{n-1}]$$

that forms a covering of $[-l_n, l_n]$ by 2α intervals of length l_{n-1}. On $T([-l_n, l_n]^d \times [0, h_{n-1}])$ we can find a path $\gamma : [0, h_{n-1}] \to [-l_n, l_n]^d$ spanning the box in the temporal direction. Let us consider its range $\mathcal{I} = \gamma([0, h_{n-1}])$. Projecting $\mathcal{I}$ into each one of the coordinate directions j yields discrete intervals $\mathcal{I}_j \subset [-l_n, l_n]$. Define the *box count* of $\mathcal{I}_j$ as

$$c_j := \min\{|I|;\ I \subset Z_\alpha,\ \mathcal{I}_j \subset \cup_{z \in I} I_z\}. \tag{2.4}$$

We decompose $T([-l_n, l_n]^d \times [0, h_{n-1}])$ according to the values assumed by each c_j.

If for every $1 \leq j \leq d$ we have $c_j \leq 2$ then the whole path γ is contained inside a d-dimensional box with side length $2l_{n-1}$, yielding a temporal crossing. In this case, we can choose some $z \in (Z_\alpha \setminus \{\alpha - 1\})^d$ such that

$$\mathcal{I} \subset l_{n-1}z + [0, 2l_{n-1}]^d,$$

and the number of possible z is given by $(2\alpha - 1)^d$.

Now, let us consider the case in which some $c_j \geq 3$ and thus $\mathcal{I}$ is not contained in some of the boxes with side length $2l_{n-1}$ described above. In this case, we refine the argument by considering time. For any time $t \in [0, h_{n-1}]$ we define $\mathcal{I}(t) := \gamma([0, t])$ and for any fixed direction j we consider its projection $\mathcal{I}_j(t)$ and its box count $c_j(t)$. Define

$$t_1 := \inf\{t \in [0, h_{n-1}];\ \exists 1 \leq j \leq d \text{ such that } c_j(t) \geq 3\}.$$

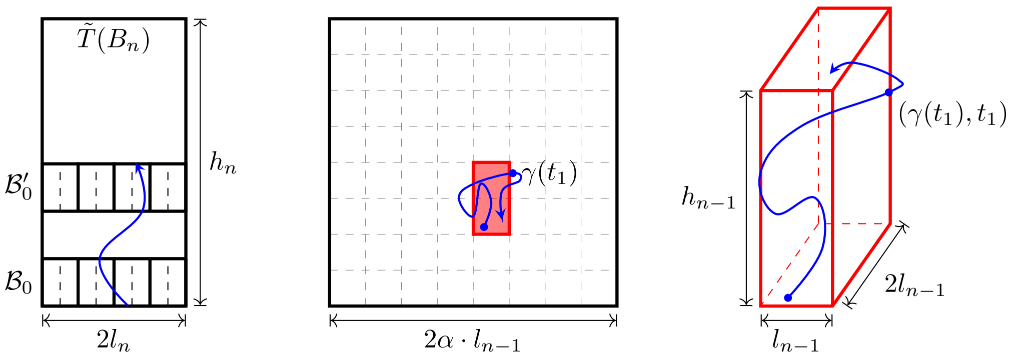

Figure 2: An illustration for the argument in Lemma 2.2 when $d = 2$. On the event $\tilde{T}(B_n)$, $H(B) \cap H(B')$ occurs for a box $B \in \mathcal{B}_0$ and another $B' \in \mathcal{B}_0'$. For $\mathcal{B}_0$, when the projection of the temporal crossing into space coordinates is not contained in one of the $(2\alpha - 1)^d$ sub-boxes of side length $2l_{n-1}$ a spatial crossing of a half-box of scale $n-1$ must occur.

Since γ can only change value when there is transmission to a neighboring site, at time t_1 we have $c_{j_0}(t_1-) = 2$ and $c_{j_0}(t_1) = 3$ for some special direction j_0 and $c_j(t_1) \leq 2$ for every other direction. Thus, there is $z \in (Z_\alpha \setminus \{\alpha - 1\})^d$ such that

$$\mathcal{I}(t_1-) \subset l_{n-1}z + [0, 2l_{n-1}]^d \quad \text{but} \quad \mathcal{I}_{j_0}(t_1) \not\subseteq l_{n-1}z + [0, 2l_{n-1}]^d \quad \text{and} \quad c_{j_0}(t_1) = 3.$$

Notice that this means path γ must have crossed a half-box of $l_{n-1}z + [0, 2l_{n-1}]^d$ on direction j_0 during time interval $[0, t_1] \subset [0, h_{n-1}]$, see Figure 2. In any case, we have that $H(B)$ happens for some box B in

$$\mathcal{B}_0 := \{(l_{n-1}z + [0, 2l_{n-1}]^d) \times [0, h_{n-1}];\ z \in (Z_\alpha \setminus \{\alpha - 1\})^d\}.$$

Applying the same argument for event $T([-l_n, l_n]^d \times [(\beta/2 - 1)h_{n-1}, (\beta/2)h_{n-1}])$, we conclude that we can take $\mathcal{B}_0'$ as the vertical translation of boxes of $\mathcal{B}_0$ by $(\beta/2-2)h_{n-1}$. □

We now turn our attention to spatial half-crossings for which a similar result also holds.

Lemma 2.4 (Spatial half-crossing)**.** *Assume that $\alpha \geq 5$. Let $n \geq 1$ and $1 \leq j \leq 2d$. There are collections $\mathcal{B}_j = \mathcal{B}_{j,n}$ and $\mathcal{B}_j' = \mathcal{B}_{j,n}'$ of scale-$(n-1)$ boxes such that*

$$\tilde{S}_j(B_n) \subset \bigcup_{(B,B') \in \mathcal{B}_j \times \mathcal{B}_j'} H(B) \cap H(B').$$

Moreover, we may assume that $\mathcal{B}_j$ and $\mathcal{B}_j'$ have $(2\beta - 1) \cdot (2\alpha - 1)^{d-1}$ elements and that any pair of boxes $B \in \mathcal{B}_j$ and $B' \in \mathcal{B}_j'$ have spatial distance at least $(\alpha/2 - 2)2l_{n-1}$.

Proof. By symmetry, we can assume $j = 1$. On the event $S_1([0, l_n] \times [-l_n, l_n]^{d-1} \times [0, h_n])$ we have a half-crossing of box B_n, that entails the crossing of two smaller boxes:

$$S_1([0, 2l_{n-1}] \times [-l_n, l_n]^{d-1} \times [0, h_n]) \quad \text{and} \quad S_1([l_n - 2l_{n-1}, l_n] \times [-l_n, l_n]^{d-1} \times [0, h_n]).$$

Similarly to what was done in the proof of Lemma 2.2, we will build a collection $\mathcal{B}_1$ inside the first box and take $\mathcal{B}_1'$ as a translation of $\mathcal{B}_1$. Hence the spatial distance of boxes in $\mathcal{B}_1$ and $\mathcal{B}_1'$ is at least $(\alpha/2 - 2)2l_{n-1}$. Consider the collection of boxes

$$\mathcal{C} := \{(l_{n-1}z + [0, 2l_{n-1}]^d) \times [0, h_n];\ z \in \{0\} \times (Z_\alpha \setminus \{\alpha - 1\})^{d-1}\}.$$

Consider a path $\gamma : [s_1, t_1] \to \mathbb{Z}^d$ that realizes $S_1([0, 2l_{n-1}] \times [-l_n, l_n]^{d-1} \times [0, h_n])$ and let $\mathcal{I}_j$ be the projection of $\gamma([s_1, t_1])$ on direction j and c_j be its box count, i.e.,

$$c_j := \min\{|I|;\ I \subset Z_\alpha,\ \mathcal{I}_j \subset \cup_{z \in I} I_z\}.$$

Like in the previous lemma, if $c_j \leq 2$ for every $2 \leq j \leq d$, we can ensure that γ is contained in some box $B \in \mathcal{C}$ and $S_1(B)$ happens. On the other hand, if some $c_j \geq 3$ then $\tilde{S}_j(B)$ happens for some box $B \in \mathcal{C}$. In both cases, the crossing of our smaller box implies the occurrence of some half-crossing of a box $B \in \mathcal{C}$ inside it, of the form $[0, 2l_{n-1}]^d \times [0, h_n]$. Finally, we adjust the time dimension of B with a similar argument. Denote by $\pi(B)$ the space-projection of a space-time box B. Define

$$\mathcal{B}_1 := \{\pi(B) \times [ih_{n-1}, (i+1)h_{n-1}];\ B \in \mathcal{C}, 0 \leq i \leq \beta - 1, i \in 1/2 + \mathbb{Z}\}.$$

Our path γ ensures that either we have $\tilde{S}_j(B)$ for some direction $1 \leq j \leq d$ and box $B \in \mathcal{B}_1$ or we have $\tilde{T}(B)$ for some $B \in \mathcal{B}_1$. It is easy to check that $\mathcal{B}_1$ has $(2\beta - 1) \cdot (2\alpha - 1)^{d-1}$ elements. □

Putting together Lemmas 2.2 and 2.4 we readily obtain a result that relates the occurrence of half-crossings at successive scales:

Lemma 2.5 (Cascading half-crossings)**.** *For any $n \geq 1$, $\alpha \geq 5$ and $\beta \geq 6$ it holds*

$$H(B_n) \subset \bigcup_{j=0}^{2d} \Big(\bigcup_{(B,B') \in \mathcal{B}_j \times \mathcal{B}'_j} H(B) \cap H(B') \Big).$$

Hence, whenever we have a half-crossing of a scale-n box we can find half-crossings of two scale-$(n-1)$ boxes B and B' that either have vertical distance at least $(\beta/2-2)h_{n-1}$ or have spatial distance at least $(\alpha/2-2)2l_{n-1}$. Moreover, we can find such pair considering at most

$$C(d, \alpha, \beta) := \big((2\alpha - 1)^d\big)^2 + (2d) \cdot \big((2\beta - 1) \cdot (2\alpha - 1)^{d-1}\big)^2 \tag{2.5}$$

pairs of boxes (B, B').

Lemma 2.5 provides explicit control on the amount and on the position of the boxes where the half-crossings are found when moving from one scale to the previous one. This allows us to derive upper bounds for the probability of half-crossings at large scales as we explain next. We will present two possible approaches for obtaining such upper bounds. One of them is to derive a contracting inequality relating the probability of the crossing events at two successive scales. The other one is to move all the way down to the bottom scale obtaining what we call an hierarchical structure.

Recurrence inequality. Let $u_n := \sup_{(x,t)} P(H((x, t) + B_n))$. Lemma 2.5 implies

$$u_n \leq C(\alpha, \beta, d) u_{n-1}^2 + C(\alpha, \beta, d) \max_{(B,B')} \operatorname{Cov}(\mathbb{1}_{H(B)}, \mathbb{1}_{H(B')}), \tag{2.6}$$

where the maximum runs over pairs of scale-$(n-1)$ boxes (B, B') ranging in $\mathcal{B}_j \times \mathcal{B}'_j$ with $0 \leq j \leq 2d$ and $C(\alpha, \beta, d)$ is given by (2.5). Provided that one is able to obtain good upper bounds on the covariance of the events $H(B)$ and $H(B')$, then (2.6) becomes a contraction, and therefore $u_n \downarrow 0$. This is a very common strategy in renormalization.

Remark 2.6. Notice that although we have worked with sequences l_k, h_k that grow at an exponential rate, Lemma 2.5 allows us to consider more general scale progressions. In fact we may consider the more general relations

$$l_{k+1} = \alpha_k l_k \quad \text{and} \quad h_{k+1} = \beta_k h_k, \qquad \text{for } k \geq 1,$$

and $H(B_n)$ is contained in a union of at most $C(d,\alpha_{n-1},\beta_{n-1})$ pairs of scale-$(n-1)$ boxes. For instance, one can use sequences that grow faster than exponentially, in order to obtain better decoupling inequalities, i.e. better bounds on the covariance. This approach has been applied in [11, Theorem 1.1], and it will be used in the proof of Theorem 1.4(iii). The main difference is that [11] does not focus on events $H(B_n)$, but instead considers spatial and temporal crossings separately as in [11, Lemmas 2.5, 2.6]. In the construction therein $\alpha = 2$, which means that pairs of boxes (B, B') with $B = B'$ are considered. Here, we have opted to take $\alpha \geq 5$ to ensure boxes B and B' are actually separated.

Hierarchy. Lemma 2.5 also allows for the definition of a *hierarchy* of boxes. The idea is simple: fix $k \geq 1$ and consider box $B_k = [-l_k, l_k]^d \times [0, h_k]$. Using Lemma 2.5, the original scale-k box gives birth to a pair of scale-$(k-1)$ boxes. After iterating the use of the same lemma we end up with a collection of 2^k scale-0 boxes, all of which have been half-crossed. This structure of boxes can be encoded via a binary tree, in a construction that is similar to the one in [25] regarding random interlacements.

Let us introduce some notation. We use words $a \in \{0,1\}^n$ to encode the leaves of a binary tree of depth n. Consider that $\varnothing$ is the root vertex, and 0 and 1 denote the left and right children of $\varnothing$, respectively. We append digits to the right of a word $a \in \{0,1\}^n$ in order to create longer words, e.g., $a1 \in \{0,1\}^{n+1}$ is the word that encodes the right child of a. Morever, for $a, b \in \{0,1\}^n$ we define the depth of a by $|a| := n$ and define $a \wedge b \in \cup_{i=0}^{n}\{0,1\}^i$ to be the common ancestor of a and b of highest depth. In other words, a and b are descendants of $a \wedge b$ but there is no c child of $a \wedge b$ that has both a and b as descendants.

For a fixed $k \geq 1$ we consider box B_k as the root of the hierarchy, and encode its descendants obtained via Lemma 2.5 by a binary tree. More precisely, a collection of boxes

$$\mathcal{H}_k = \{B_k(a);\ a \in \cup_{i=0}^{k}\{0,1\}^i\}$$

is called a *hierarchy* of B_k if $B_k(\varnothing) = B_k$, and for every $a \in \cup_{i=0}^{k-1}\{0,1\}^i$ the boxes $B_k(a0)$ and $B_k(a1)$ are disjoint scale-$(k-|a|-1)$ boxes contained in $B_k(a)$. A hierarchy is said to be *achievable* if for all boxes $B_k(a)$ its children are a pair of boxes (B, B') from the choice of pairs given in Lemma 2.5. Finally, let us define the collection of all achievable hierarchies

$$\mathcal{X}_k := \{\mathcal{H}_k;\ \mathcal{H}_k \text{ is an achievable hierarchy}\},$$

which is a deterministic set. It is clear from Lemma 2.5 that $\#\mathcal{X}_k \leq C \cdot C^2 \cdot \ldots \cdot C^{2^{k-1}} \leq C^{2^k}$ for $C = C(d,\alpha,\beta)$ given by (2.5).

Define the set of *leaves* of $\mathcal{H}_k$ as

$$L(\mathcal{H}_k) := \{B \in \mathcal{H}_k;\ B = B_k(a),\ a \in \{0,1\}^k\},$$

that is, the set of scale-0 boxes of $\mathcal{H}_k$. Then, for any probability measure given by a GCP we have

$$\mathbb{P}(H(B_k)) \leq \sum_{\mathcal{H}_k \in \mathcal{X}_k} \mathbb{P}\Big(\bigcap_{B \in L(\mathcal{H}_k)} H(B)\Big) \leq C^{2^k} \max_{\mathcal{H}_k} \mathbb{P}\Big(\bigcap_{B \in L(\mathcal{H}_k)} H(B)\Big). \tag{2.7}$$

Recall that we want to prove that the infection dies out almost surely. Since $\mathbb{P}(\tau^{\{0\}} = \infty) \leq \mathbb{P}(H(B_k))$, the estimate in (2.7) shows it is sufficient to prove that $\mathbb{P}\big(\cap_{B \in L(\mathcal{H}_k)} H(B)\big) \leq \varepsilon^{2^k}$ for ε sufficiently small, uniformly over $\mathcal{H}_k \in \mathcal{X}_k$.

Fix any achievable hierarchy $\mathcal{H}_k \in \mathcal{X}_k$. By construction, it contains 2^k scale-0 boxes (its leaves) and any two of them are either separated by a spatial distance of at least $(\alpha/2-2)2l_0$ or by a temporal distance of at least $(\beta/2-2)h_0$. Indeed, we have for $a \neq b$ with $|a| = |b| = k$ that $B_k(a)$ and $B_k(b)$ are both contained in $B_k(a \wedge b)$, but are in different children of $B_k(a \wedge b)$. Hence, by definition of achievable hierarchy boxes $B_k((a \wedge b)0)$ and $B_k((a \wedge b)1)$ are well-separated, implying that $B_k(a)$ and $B_k(b)$ enjoy the same property.

Fix some ordering $\{L_1, L_2, \dots, L_{2^k}\}$ for $L(\mathcal{H}_k)$ such that L_j is always either above or at the same height of every previous leaf. Since

$$\mathbb{P}\Big(\bigcap_{j=1}^{2^k} H(L_j)\Big) = \prod_{j=1}^{2^k} \mathbb{P}\big(H(L_j) \mid H(L_1) \cap \ldots \cap H(L_{j-1})\big), \tag{2.8}$$

we focus on estimating the conditional probabilities above. This task depends on the point processes chosen for the GCP.

3 Contact process on dynamic edges

The authors in [22] define the CPDE on $\mathbb{Z}$, but also remark that the definition extends easily to any connected graph with bounded degree. Here we will focus on the case of $\mathbb{Z}^d$. The idea is to start with an underlying dynamic *environment* process $\zeta_t \in \{0,1\}^{E(\mathbb{Z}^d)}$ and, conditional on the realization of this environment to define an *infection* process $\eta_t \in \{0,1\}^{\mathbb{Z}^d}$ similar to the classical contact process, with the main difference that its evolution depends on the changing environment. More precisely, let us fix two parameters $v > 0$ and $p \in (0,1)$. Independently of everything else, each edge e in the environment is assigned an initial state $\zeta_0(e)$ with the Bernoulli distribution with parameter p, $\mathrm{Ber}(p)$, and independently updates its state as follows:

$$\begin{aligned} 0 &\longrightarrow 1 \quad \text{at rate} \quad vp, \\ 1 &\longrightarrow 0 \quad \text{at rate} \quad v(1-p). \end{aligned}$$

We say that the edge e is open at time t if $\zeta_t(e) = 1$ and that it is closed at time t otherwise. We may think of the state of each edge e independently alternating between open and closed at the given rates. This defines the Markov process $\{\zeta_t\}_{t \geq 0}$ taking values in $\{0,1\}^{E(\mathbb{Z}^d)}$ usually called dynamic bond percolation on $\mathbb{Z}^d$ with density parameter p and rate v. In fact, the choice for the initial distribution ζ_0 as being the product of $\mathrm{Ber}(p)$ implies that ζ_t is stationary. Hence, for each fixed time t, the configuration $\{\zeta_t(e)\}_{e \in E(\mathbb{Z}^d)}$ is distributed as an independent bond percolation process on $\mathbb{Z}^d$.

We will now define the contact process η_t whose evolution will depend on a parameter $\lambda > 0$ and on the underlying environment ζ_t. At each site x, the state $\eta_t(x)$ evolves as follows:

$$\begin{aligned} 1 &\longrightarrow 0 \quad \text{at rate} \quad 1, \\ 0 &\longrightarrow 1 \quad \text{at rate} \quad \lambda \sum_{y \sim x} \zeta_t(xy)\eta_t(y). \end{aligned} \tag{3.1}$$

In words, each site y attempts to infect a neighboring site y at rate λ through the edge $e = xy$ as in the usual contact process on $\mathbb{Z}^d$. However, it will only succeed in case that edge is found open at the time of the attempt.

In [22, Section 3.1] the authors define this process via a standard graphical construction, employing a collection of independent Poisson point processes on $(0, \infty)$:

- $\{\mathcal{O}_e\}$ of rate vp, whose marks provide the opening times of edge e.
- $\{\mathcal{C}_e\}$ of rate $v(1-p)$, whose marks provide the closing times of edge e.
- $\{\mathcal{I}_e\}$ of rate λ, whose marks provide the times of potential transmissions along edge e.
- $\{\mathcal{R}_x\}$ of rate 1, whose marks provide the cure (recovery) times of site x.

This graphical construction serves as a tool in our methods. The reader may consult [22, Section 3.1] for more details.

Remark 3.1. It is worth noticing that actually the CPDE can be seen as a Renewal Contact Process with renewals on the edges and a delay. However, using the results available from previous works on RCP is not straightforward. The interarrival distribution μ of the model depends on parameters λ, v, p, and is therefore affected by changes on any of these parameters.

Using the above notation, the critical parameter is given by

$$\lambda_0(v,p) = \inf\{\lambda > 0;\ \mathbb{P}_{v,p,\lambda}(\eta_t^{\{0\}} \not\equiv 0,\ \forall t > 0) > 0\}.$$

3.1 Proof of Theorem 1.1(i)

In this section we apply the hierarchical approach to renormalization. On the course of the proof we will need to use a straightforward estimate on how long the usual contact process restricted to a finite (static) cluster can survive. We state it as a lemma and include its proof for the reader's convenience.

Lemma 3.2. *Let G be any connected subgraph of $\mathbb{Z}^d$ with at most n vertices and $\tau = \tau(G)$ the extinction time of the contact process on the subgraph G started from the configuration η_0^G. There exists $\kappa = \kappa(\lambda, d) > 0$ such that for every $\nu > 1/n$,*

$$\mathbb{P}(\tau(G) \geq e^{\nu n}) \leq \exp[-e^{(\nu-\kappa)n}/2]. \tag{3.2}$$

Proof. We fix G throughout the proof and provide bounds that are uniform in G. Let T_j be the event in which for every vertex in G there is a cure mark before the first transmission in $[j, j+1]$ (or before $j+1$, if there are no transmissions). It is clear that if T_j occurs for some $j \leq k-1$, then $\tau < k$. Hence $\{\tau \geq k\} \subset \cap_{j=0}^{k-1} T_j^{\mathsf{c}}$, which implies for $k = \lfloor e^{\nu n} \rfloor$,

$$\mathbb{P}(\tau \geq e^{\nu n}) \leq \prod_{j=0}^{k-1} \mathbb{P}(T_j^{\mathsf{c}}) = (1 - \mathbb{P}(T_0))^k \leq \exp[-k \cdot \mathbb{P}(T_0)] \leq \exp[-\mathbb{P}(T_0)\, e^{\nu n}/2],$$

where we have used the fact that the events T_j are independent and have the same probability together with the fact that $\lfloor e^{\nu n} \rfloor > e^{\nu n}/2$ if $\nu > 1/n$. Since G has at most $2dn$ edges and n vertices, it is clear that

$$\begin{aligned}\mathbb{P}(T_0) &\geq \mathbb{P}\Big(\bigcap_{e \in E(G)} \{\mathcal{I}_e \cap [0,1] = \emptyset\} \cap \bigcap_{v \in V(G)} \{\mathcal{R}_v \cap [0,1] \neq \emptyset\} \Big) \geq e^{-2d\lambda n} \cdot (1 - e^{-1})^n \\ &\geq \exp\big[-\big(2d\lambda + 1\big)n\big],\end{aligned}$$

using that $1 - e^{-1} > e^{-1}$. Taking $\kappa(\lambda, d) := 2d\lambda + 1$, the inequality (3.2) follows. □

Proof of Theorem 1.1(i). Let $p < p_c(d)$ and $\lambda > 0$ be fixed. Our goal is to show that for v smaller than some $v_0(p, \lambda, d) > 0$, the CPDE dies out almost surely. In order to apply our renormalization approach, we need to define the sequence of scales l_k and h_k as in (2.3). Recall that they become fully determined once we choose the values for α, β, l_0 and h_0. We start by fixing $\alpha = 5$. The other values will be determined next depending on p, λ and d.

Let us write $\delta = \delta(p) := \frac{1}{4}(p_c(d) - p)$ and fix $\beta = \beta(p, d) \geq 6$ sufficiently large so that

$$e^{-(\beta/2-2)\delta} < \delta. \tag{3.3}$$

Having fixed α and β it only remains to chose l_0 and h_0 suitably. In the following, we take

$$v := \delta / h_0 \tag{3.4}$$

so that v will be determined once h_0 has been chosen.

For a scale-k box $B_k = [-l_k, l_k]^d \times [0, h_k]$ and a hierarchy $\mathcal{H}_k \in \mathcal{X}_k$, label the leaves in $\mathcal{H}_k$ as $L_1, \ldots, L_{2^k}$ in such a way that leaves located higher in time are assigned greater indices. Fix some leaf L_j of the form $\pi(L_j) \times [s_j, s_j + h_0]$ (recall the notation $\pi(L_j)$ for its space projection). Also recall that $\mathcal{O}_e$ and $\mathcal{C}_e$ are PPPs whose arrivals represent the times at which the edge e opens and closes, respectively. We say that an edge e with both endvertices in $\pi(L_j)$ is L_j-*available* if at least one of the following conditions is satisfied:

(i) e opens during the time interval associated to L_j: $\mathcal{O}_e \cap [s_j, s_j + h_0] \neq \emptyset$;

(ii) e does not update in the time interval of length $(\beta/2-2)h_0$ prior to the time interval of L_j:
$(\mathcal{O}_e \cup \mathcal{C}_e) \cap [s_j - (\beta/2-2)h_0, s_j] = \emptyset$;

(iii) e updates during $[s_j - (\beta/2-2)h_0, s_j]$, and e is open at time s_j.

Hence we have

$$\begin{aligned}
\mathbb{P}(e \text{ is } L_j\text{-available}) &\leq (1 - e^{-pvh_0}) + (e^{-v(\beta/2-2)h_0}) + \mathbb{P}(\zeta_{s_j}(e) = 1) \\
&= (1 - e^{-p\delta}) + (e^{-(\beta/2-2)\delta}) + p \\
&\leq p\delta + e^{-(\beta/2-2)\delta} + p \\
&< \frac{1}{2}(p + p_c(d)),
\end{aligned} \tag{3.5}$$

where we used (3.4) in the second line and the last inequality is due to our choices of β and δ in (3.3).

Consider the graph whose vertices are sites in $\pi(L_j)$ and whose edges are those that are L_j available. Let us call $C_j(x)$ the cluster containing the vertex $x \in \pi(L_j)$ in this graph. Notice that $C_j(x)$ is either equal to $\{x\}$ or is an open cluster of a Bernoulli bond percolation process in $\pi(L_j)$ with parameter at most $(p + p_c(d))/2$, hence subcritical. By the exponential decay of the cluster size distribution (cf. [13, Theorem (6.75)])

$$\exists\, \psi = \psi(p,d) > 0 \quad \text{s.t.} \quad \mathbb{P}(|C_j(x)| \geq m) \leq e^{-\psi m} \quad \forall m \in \mathbb{N}. \tag{3.6}$$

For each $e = xy$ with both endvertices $x, y \in \pi(L_j)$ and $t \in [s_j, s_j + h_0]$ let us define

$$\hat{\zeta}_{j,t}(e) := \mathbb{1}_{\{e \text{ is } L_j\text{-available}\}}. \tag{3.7}$$

Using the graphical construction in terms of the point processes $\mathcal{O}$, $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{R}$ we can define inside L_j the process $\hat{\eta}_{j,t}$ where, the initial configuration is given by $\hat{\eta}_{j,s_j}(x) = 1$ for every $x \in \pi(L_j)$ and instead of $\zeta_t(xy)$ one uses $\hat{\zeta}_{j,t}(xy)$ in (3.1). Roughly speaking, replacing ζ by $\hat{\zeta}_j$ amounts to enlarging the open clusters at the basis of L_j and then to keep them frozen for time h_0. Indeed, their external boundary is made of edges that are not L_j-available and by (i) they remain closed. Of course the initial clusters evolve dynamically with time according to processes $\mathcal{O}$ and $\mathcal{C}$, but notice that whenever an edge is open for ζ if is also open for $\hat{\zeta}_j$. As a consequence, every infection path for η is also an infection path for $\hat{\eta}_j$.

We say that the leaf L_j is good if the half-crossing event occurs inside L_j for the process $\hat{\eta}_j$. Let us denote this event by $\hat{H}(L_j)$. Notice that for different leaves, these events depend on disjoint regions of the Poisson point processes $\mathcal{O}$, $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{R}$ so they are independent.

The infection spreads better inside L_j when using $\hat{\eta}_j$, as discussed above. Therefore, it follows that $H(L_j) \subset \hat{H}(L_j)$ and

$$\mathbb{P}\Big(\bigcap_{j=1}^{2^k} H(L_j)\Big) \leq \mathbb{P}\Big(\bigcap_{j=1}^{2^k} \hat{H}(L_j)\Big) = \mathbb{P}\big(\hat{H}(L_1)\big)^{2^k},$$

where we have used independence and translation invariance. In view of (2.7), it suffices to prove that $\mathbb{P}(\hat{H}(L_1)) < 1/(2C(d,\alpha,\beta))$ where $C(d,\alpha,\beta)$ has been fixed in Lemma 2.5. This will be done by suitably choosing l_0 and h_0.

By (3.6) the probability of the event

$$U_j := \left\{\exists x \in \pi(L_j);\ |C_j(x)| \geq (2d/\psi)\ln l_0\right\}$$

is bounded by

$$\mathbb{P}(U_j) \leq c(d) l_0^d \cdot e^{-\psi(2d/\psi)\ln l_0} = c(d) l_0^{-d}. \tag{3.8}$$

Since every infection path inside L_j must only jump through L_j-available edges, each of these paths is contained in a cluster $C_j(x)$, that is typically much smaller than $\pi(L_j)$.

Let us now fix l_0 sufficiently large so that

$$\frac{2d}{\psi}\ln l_0 < l_0 \quad \text{and} \quad l_0 \geq [4c(d)C(d,\alpha,\beta)]^{1/d}. \tag{3.9}$$

Then, estimate (3.8) implies

$$\mathbb{P}(U_j) \leq \frac{1}{4C(d,\alpha,\beta)}. \tag{3.10}$$

Moreover, any spatial half-crossing for $\hat{\eta}_j$ inside L_j has to traverse at least l_0 edges. Therefore, the occurrence of such half-crossings implies the occurrence of U_j.

On U_j^{c} we know that all of the available clusters in $\pi(L_j)$ are small, that is, each $C_j(x)$ contains at most $(2d/\psi)\ln l_0$ sites. In order for a temporal half-crossing to occur the process $\hat{\eta}_j$ must survive for time at least $h_0/2$ in one of these small clusters.

Let $\nu = \nu(p,\lambda,d,l_0) = \max\{\psi+\kappa, (2d/\psi)\ln l_0\}$ where $\kappa = \kappa(\lambda,d)$ is given in Lemma 3.2 and $\psi(p,d)$ is given in (3.6). Define

$$V_j := \left\{\exists x \in \pi(L_j);\ \hat{\eta}_j \text{ survives longer than } e^{\nu(2d/\psi)\ln l_0} \text{ inside } C_j(x)\right\}.$$

By Lemma 3.2,

$$\mathbb{P}(U^{\mathsf{c}} \cap V_j) \leq c(d) l_0^d \cdot e^{-(\nu-\kappa)(2d/\psi)\ln l_0} = c(d) l_0^{d-(\nu-\kappa)(2d/\psi)} \leq c(d) l_0^{-d} \leq \frac{1}{4C(d,\alpha,\beta)}. \tag{3.11}$$

Therefore, uniformly over

$$h_0 \geq 2 \cdot e^{\nu(2d/\psi)\ln l_0} = 2 \cdot l_0^{\nu(2d/\psi)} \tag{3.12}$$

the following bound holds

$$\mathbb{P}(\hat{H}(L_j)) \leq \mathbb{P}(U) + \mathbb{P}(U^{\mathsf{c}} \cap V) \leq \frac{1}{2C(d,\alpha,\beta)}, \tag{3.13}$$

as it can be seen by just plugging (3.8) and (3.11). Thanks to (3.4) this implies that any choice of $v \in (0, \delta l_0^{-\nu(2d/\psi)}/2)$ is sufficient to establish (3.13). This finishes the proof with $v_0 = l_0^{-\nu(2d/\psi)}/2$. □

3.2 Proof of Theorem 1.1(ii)

Let us introduce some notation that will be used throughout this section. For each $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, define

$$B_n(x) := x + \{-n,\dots,n\}^d.$$

For a bond percolation configuration ζ in $\mathbb{Z}^d$ and any connected subgraph B of $\mathbb{Z}^d$, we denote by $G_\zeta(B)$ the random subgraph of B induced by the open bonds in ζ. We denote by $G_\zeta^*(B)$ the connected component with largest cardinality of $G_\zeta(B)$ (we can adopt some arbitrary procedure to decide between components in the case of a tie).

The following result follows from Proposition 3.2 in [27] (which in turn is proved using results from [6] and [24]).

Proposition 3.3. *Assume that $d \geq 2$, $p > p_c(\mathbb{Z}^d)$ and ζ is sampled from the product Bernoulli(p) distribution. Then, there exists $\delta > 0$ such that the following holds for n sufficiently large. With probability higher than $1 - \exp\{-(\log n)^{1+\delta}\}$, the component $G^*_\zeta(B_n(o))$ has cardinality larger than $n^{d-\frac{1}{4}}$, and all other components of $G_\zeta(B_n(o))$ have cardinality smaller than $n^{d-\frac{1}{2}}$.*

For the rest of this section we fix $d \geq 2$ and $\bar{p} > p_c(\mathbb{Z}^d)$. We will prove that

$$\sup\{\lambda_0(v, \bar{p}) : v \geq 0\} < \infty. \tag{3.14}$$

By Theorem 2.3 in [22] we have that $\lambda_0(v, \bar{p})$ converges to a finite limit as $v \to \infty$. Hence, (3.14) will follow from showing that

$$\sup\{\lambda_0(v, \bar{p}) : 0 \leq v \leq \bar{v}\} < \infty \qquad \text{for all } \bar{v} > 0. \tag{3.15}$$

Therefore, for the rest of this section we fix $\bar{v} > 0$ and we will prove that (3.15) holds. The dynamic environment $\{\zeta_t\}_{t\geq 0}$ will have edge density parameter equal to $\bar{p}$ and edge update speed $v \in [0, \bar{v}]$ which will be clear from the context or irrelevant.

For $x \in \mathbb{Z}^d$, $t \geq 0$ and $n \in \mathbb{N}$, define the event

$$A_n(x,t) := \left\{ \begin{array}{c} G^*_{\zeta_t}(B_n(x)) \text{ has more than } n^{d-\frac{1}{4}} \text{ vertices; all other} \\ \text{components of } G_{\zeta_t}(B_n(x)) \text{ have fewer than } n^{d-\frac{1}{2}} \text{ vertices} \end{array} \right\}$$

Then let

$$E_n(x,t) := \Big(\bigcap_{y \in B_n(x)} A_n(y,t) \Big) \cap A_{2n}(x,t).$$

Proposition 3.3 and a union bound give the following.

Lemma 3.4. *For n sufficiently large we have, for any $x \in \mathbb{Z}^d$ and $t \geq 0$,*

$$\mathbb{P}(E_n(x,t)) > 1 - (2n+1)^d \cdot \exp\{-(\log n)^{1+\delta}\} - \exp\{-(\log 2n)^{1+\delta}\},$$

where δ is given in Proposition 3.3.

For future reference, we make a few observations about these events. First, we have

$$E_n(x,t) \subset \{\text{for every } y \in B_n(x) \text{ we have } G^*_{\zeta_t}(B_n(y)) \subset G^*_{\zeta_t}(B_{2n}(x))\}.$$

Indeed, if $E_n(x,t)$ occurs and $y \in B_n(x)$, then $G^*_{\zeta_t}(B_n(y))$ has more than $n^{d-\frac{1}{4}}$ vertices and is contained in some cluster $\mathcal{C}$ of $G_{\zeta_t}(B_{2n}(x))$; moreover, all clusters of $G_{\zeta_t}(B_{2n}(x))$ except for $G^*_{\zeta_t}(B_{2n}(x))$ have fewer than $(2n)^{d-\frac{1}{2}} \ll n^{d-\frac{1}{4}}$ vertices, so it follows that $\mathcal{C} = G^*_{\zeta_t}(B_{2n}(x))$.

Second, as an immediate consequence of the previous observation, we have that

$$\begin{aligned} &\text{if } x, x' \text{ have } B_n(x) \cap B_n(x') \neq \varnothing, \text{ then} \\ &E_n(x,t) \cap E_n(x',t) \subset \{G^*_{\zeta_t}(B_{2n}(x)) \cap G^*_{\zeta_t}(B_{2n}(x')) \neq \varnothing\}. \end{aligned} \tag{3.16}$$

Third and finally, if $E_n(x,t)$ occurs and $\tilde{\zeta}$ is a bond percolation configuration obtained from ζ_t by modifying the state of a single bond, then $G^*_{\zeta_t}(B_{2n}(x))$ and $G^*_{\tilde{\zeta}}(B_{2n}(x))$ have at least one vertex in common. This is easy to check from the definition of $E_n(x,t)$ and inspection of a few cases, so we leave the details to the reader.

We now define the event

$$E'_n(x,t) = \bigcap_{s \in [t,t+1]} E_n(x,s), \qquad n \in \mathbb{N},\ x \in \mathbb{Z}^d,\ t \geq 0.$$

We then have

Lemma 3.5. *For any $\varepsilon > 0$ there exists n_0 such that if $n \geq n_0$ we have $\mathbb{P}(E'_n(x,t)) > 1-\varepsilon$ for all $v \in [0,\bar{v}]$, $x \in \mathbb{Z}^d$ and $t \geq 0$.*

Proof. Fix v, x, t as in the statement. Let $\mathcal{T} \subset [0,\infty)$ denote the set of update times of the edges of $B_{2n}(x)$. Then, $\mathcal{T}$ is a Poisson point process whose intensity is smaller than $C_d v n^d$, for some constant $C_d > 0$. In particular, for any $s \geq 0$ we have

$$\mathbb{P}(\mathcal{T} \cap [s, s+(C_d v n^d)^{-1}] = \varnothing) \geq \mathbb{P}(\text{Poisson}(1) = 0) = e^{-1}.$$

On the event $(E'_n(x,t))^c$, let τ denote the smallest $s \in [t,t+1]$ such that $(E_n(x,s))^c$ occurs. Letting $A = (E'_n(x,t))^c \cap \{\mathcal{T} \cap (\tau, \tau+(C_d v n^d)^{-1}] = \varnothing\}$, we have, by the strong Markov property,

$$\mathbb{P}(A) \geq \mathbb{P}((E'_n(x,t))^c) \cdot e^{-1}.$$

Now, noting that

$$(C_d v n^d)^{-1} \cdot \mathbb{1}_A \leq \int_t^{t+2} \mathbb{1}_{(E_n(x,s))^c} \, \mathrm{d}s$$

and taking expectations, we obtain

$$\begin{aligned}(C_d v n^d)^{-1} \cdot e^{-1} \cdot \mathbb{P}((E'_n(x,t))^c) &\leq \mathbb{E}\left[\int_t^{t+2} \mathbb{1}_{(E_n(x,s))^c} \, \mathrm{d}s\right] \\ &\leq 2 \cdot \left[(2n+1)^d \cdot \exp\{-(\log n)^{1+\delta}\} + \exp\{-(\log 2n)^{1+\delta}\}\right],\end{aligned}$$

where the last inequality follows from Lemma 3.4 and Fubini's theorem. We thus have

$$\mathbb{P}((E'_n(x,t))^c) \leq 2eC_d v n^d \left[(2n+1)^d \cdot \exp\{-(\log n)^{1+\delta}\} + \exp\{-(\log 2n)^{1+\delta}\}\right],$$

and the right-hand side can be made as small as desired by taking n sufficiently large, uniformly for $v \in [0,\bar{v}]$. □

We now give some further definitions. A finite sequence $\gamma = (x_0, \ldots, x_m)$ of vertices of $\mathbb{Z}^d$ is called a self-avoiding path if $x_0, \ldots, x_m$ are all distinct and $x_i \sim x_{i+1}$ for each i. For such a sequence γ and $t > s \geq 0$, we let $\Phi(\gamma, s, t)$ denote the indicator function of the event that, in the graphical construction of the process, there exist times $s < t_1 < \ldots < t_m < t$ such that, for each i, there is a transmission mark in the edge $\{x_{i-1}, x_i\}$ at time t_i. We emphasize that this definition does not involve the recovery marks or the edge percolation environment, but only the transmission marks.

For a finite connected subgraph B of $\mathbb{Z}^d$, let Γ_B denote the (finite) set of all self-avoiding paths contained in B.

Lemma 3.6. *Let $t > 0$ and B be a finite connected subgraph of $\mathbb{Z}^d$. Then, for any $\varepsilon > 0$, we can take λ large enough so that*

$$\mathbb{P}(\Phi(\gamma, s, s+t) = 1 \text{ for all } \gamma \in \Gamma_B) > 1 - \varepsilon \quad \text{for all } s \geq 0.$$

Proof. By translation invariance, it suffices to treat $s = 0$. We divide $[0,t]$ into $|\Gamma_B|$ sub-intervals of equal lengths and disjoint interiors. The event in question is achieved if each edge of B has a transmission mark in the interior of each of these sub-intervals. This has probability as high as desired when $\lambda \to \infty$. □

We now define a further event $F_n(x,t)$ for $n \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $t \geq 0$, as follows. Let $t_1 < t_2 < \cdots < t_N$ denote, in increasing order, the (random) times within the time interval $[t,t+1]$ at which there is either an edge update or a recovery mark inside $B_{2n}(x)$. Also let $t_0 = t$ and $t_{N+1} = t+1$. Then, $F_n(x,t)$ is defined as the event that

$$\Phi(\gamma, t_i, t_i+1) = 1 \text{ for all } \gamma \in \Gamma_{B_{2n}(x)} \text{ and all } i \in \{0, \ldots, N\}.$$

In words, this is the event that, between two successive times t_i, t_{i+1} (each of which can correspond to a recovery mark or an edge update inside $B_{2n}(x)$), every self-avoiding path inside $B_{2n}(x)$ can be traversed by following transmissions. This guarantees that, if at time t_i one of the vertices of $G^*_{\zeta_{t_i}}(B_{2n}(x))$ is infected, then immediately before time t_{i+1} *all* vertices of this cluster will be infected.

Lemma 3.7. *For any $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\lambda' > 0$ such that*

$$\mathbb{P}(F_n(x,t)) > 1 - \varepsilon \qquad \text{for any } \lambda > \lambda',\ v \in [0, \bar{v}],\ x \in \mathbb{Z}^d,\ \text{and } t \geq 0.$$

Proof. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. By translation invariance, it is sufficient to prove that there exists $\lambda' > 0$ such that $\mathbb{P}(F_n(0,0)) > 1 - \varepsilon$ for any $\lambda > \lambda'$ and $v \in [0, \bar{v}]$. Let $t_1 < \cdots < t_N$ denote the times in $[0,1]$ at which there is either an edge update or recovery mark inside $B_{2n}(0)$, and let $t_0 = 0$ and $t_{N+1} = 1$. Let

$$X := \inf\{|t_{i+1} - t_i| : \ i \in \{0, \dots, N\}\}.$$

It is easy to see that there exists $\delta > 0$ such that

$$\mathbb{P}(X > \delta,\ N < 1/\delta) > 1 - \frac{\varepsilon}{2} \quad \text{for any } v \in [0, \bar{v}].$$

Now, by Lemma 3.6 and a union bound, we can obtain $\lambda' > 0$ such that, for any $\lambda > \lambda'$ and any $v \in [0, \bar{v}]$,

$$\mathbb{P}(F_n(0,0) \mid X > \delta,\ N < 1/\delta) > 1 - \frac{\varepsilon}{2},$$

completing the proof. □

Finally, define

$$v_n(z,k) := (nz, nk, \underbrace{0, \dots, 0}_{d-2}) \in \mathbb{Z}^d, \quad (z,k) \in \mathbb{Z}^2,\ k \geq 0$$

and

$$\eta_k(z) := \mathbb{1}\{E'_n(v_n(z,k),k) \cap F_n(v_n(z,k),k)\}, \qquad (z,k) \in \mathbb{Z}^2,\ k \geq 0.$$

It will be useful to note the following:

Claim 3.8. *If $\eta_k(z) = 1$ and at least one site of $G^*_{\zeta_k}(B_{2n}(v_n(z,k)))$ is infected at time k, then all sites of $G^*_{\zeta_{k+1}}(B_{2n}(v_n(z,k)))$ are infected at time $k+1$.*

Proof. Let $t_1 < t_2 < \cdots < t_N$ denote the times within $[k, k+1]$ at which there is either an edge update or a recovery mark inside $B_{2n}(v_n(z,k))$. The definition of $F_n(z,k)$ guarantees that, in $[k, t_1)$, the component $G^*_{\zeta_k}(B_{2n}(v_n(z,k)))$ becomes fully infected. As observed after Lemma 3.4, the definition of $E'_n(z,k)$ guarantees that the components $G^*_{\zeta_{t_1-}}(B_{2n}(v_n(z,k)))$ and $G^*_{\zeta_{t_1}}(B_{2n}(v_n(z,k)))$ have at least one vertex in common. In particular, at least one infection remains in $G^*_{\zeta_{t_1}}(B_{2n}(v_n(z,k)))$. Proceeding recursively, we obtain the result. □

Proof of Theorem 1.1(ii). Fix $\varepsilon > 0$. Assume that n is large enough, as required by Lemma 3.5, and then assume that λ is large enough, as required by Lemma 3.7. These choices guarantee that $\mathbb{P}(\eta_k(z) = 1) > 1 - 2\varepsilon$ for all $\lambda > \lambda'$, $v \in [0, \bar{v}]$, $k \in \mathbb{N}$ and $z \in \mathbb{Z}$. We also have that if (k,z) and (k',z') have either $|k - k'| > 4$ or $|z - z'| > 4$, then $\eta_k(z)$ and $\eta_{k'}(z')$ are independent. Hence, $(\eta_k(z))$ dominates a Bernoulli field with range of dependence 4 (in ℓ_∞-norm) and density above $1 - 2\varepsilon$. If ε is sufficiently small, then with positive probability there is an infinite sequence $0 = z_0, z_1, \ldots \in \mathbb{Z}$ such that $|z_k - z_{k+1}| \leq 1$ and $\eta_k(z_k) = 1$ for every k. By combining Claim 3.8 and (3.16), we obtain that there is an infinite infection path contained in the space-time set $\cup_k (B_{2n}(v_n(z_k,k)) \times [k,k+1])$. This proves that $\mathbb{P}(B_{2n}(0) \rightsquigarrow \infty) > 0$; since

$$\mathbb{P}(B_{2n}(0) \rightsquigarrow \infty) \le |B_{2n}(0)| \cdot \mathbb{P}((0,0) \rightsquigarrow \infty),$$

it follows that $\mathbb{P}((0,0) \rightsquigarrow \infty) > 0$. □

4 Edge renewal contact process

4.1 Uniform control for renewals

Our study of ERCP is based on a uniform control for the probability of having renewal marks in an interval of fixed length. The next lemma summarizes inequalities that achieve this goal. These estimates are in the core of all subsequent computations and justify our hypotheses on μ and ν.

Lemma 4.1 (Uniform estimates)**.** *Let μ be any probability distribution on $\mathbb{R}_+$ and let $\mathcal{R}$ be a renewal process with interarrival μ started from some $\tau \le 0$.*

(i) If $f : [0,\infty) \to [0,\infty)$ is non-decreasing, $\lim_{x\to\infty} f(x) = \infty$, and $\int x f(x)\,\mu(\mathrm{d}x) < \infty$, then uniformly on τ we have

$$\sup_{t\ge 0} \mathbb{P}(\mathcal{R} \cap [t,t+h] = \emptyset) \le \frac{C}{f(h)}, \tag{4.1}$$

for some positive constant $C = C(\mu,f)$ whenever $f(h) > 0$. Moreover, if $\int x\,\mu(\mathrm{d}x) < \infty$ then given any $\varepsilon > 0$ there is $h_0 = h_0(\varepsilon) > 0$ such that uniformly on τ, we have

$$\sup_{t\ge 0} \mathbb{P}(\mathcal{R} \cap [t,t+h_0] = \emptyset) \le \varepsilon. \tag{4.2}$$

(ii) If μ is continuous, then given $\varepsilon > 0$ there is $w_0 = w_0(\varepsilon) > 0$ such that uniformly on τ we have

$$\sup_{t\ge 0} \mathbb{P}(\mathcal{R} \cap [t,t+w_0] \ne \emptyset) \le \varepsilon. \tag{4.3}$$

Proof. We can assume that $\tau = 0$. Indeed, if $\mathcal{R}$ is a renewal process started from 0 and $\tau < 0$ then the process $\mathcal{R}' = \tau + \mathcal{R}$ is a renewal process started from τ and we have

$$\begin{aligned} \sup_{t\ge 0} \mathbb{P}(\mathcal{R}' \cap [t,t+h] = \emptyset) &= \sup_{t\ge 0} \mathbb{P}((\tau+\mathcal{R}) \cap [t,t+h] = \emptyset) = \sup_{s\ge -\tau} \mathbb{P}(\mathcal{R} \cap [s,s+h] = \emptyset) \\ &\le \sup_{t\ge 0} \mathbb{P}(\mathcal{R} \cap [t,t+h] = \emptyset). \end{aligned}$$

The first statement in (i) is exactly Lemma 2.3 of [11]. We also notice that the inequality (4.2) is a straightforward consequence of (4.1). Indeed, it suffices to show that when $\int x\,\mu(\mathrm{d}x) < \infty$ one can find a function f satisfying the requirements for (4.1) and then take h sufficiently large. Finding such function f is a standard analysis exercise and we omit the proof.

The proof of (ii) is based on the fact that when μ is continuous its renewal function $U(t)$, i.e. the expected number of renewals up to time t, is uniformly continuous on $\mathbb{R}_+$. The continuity of $U(\cdot)$ follows at once from that of μ. To ensure uniform continuity, we have to control the behavior of $U(t)$ as $t \to \infty$. This follows from the classical renewal theorem, which implies $\lim_{t\to\infty}(U(t+h) - U(t)) = \frac{h}{\int x\,\mu(\mathrm{d}x)}$ (understood as zero if the integral diverges). Hence, given $\varepsilon > 0$ there is $w_0 = w_0(\varepsilon)$ such that

$$\sup_{t\ge 0} \mathbb{P}(\mathcal{R} \cap (t,t+w_0] \ne \emptyset) \le \sup_{t\ge 0}\big(U(t+w_0) - U(t)\big) \le \varepsilon. \qquad \square$$

4.2 Growth of ERCP

We start this section by showing that if μ is continuous then a.s. the infection cannot reach infinitely many sites in finite time. A first observation is that conditions for finite speed of infection already exist for First Passage Percolation in $\mathbb{Z}^d$. This classical model considers that every edge e of the graph has a random passage time $\tau_e \geq 0$, with (τ_e) being i.i.d. with distribution μ, representing the time it takes for an infection to traverse edge e. In First Passage Percolation, a known condition in μ for having a finite speed of infection is $\mu(\{0\}) < p_c = p_c(\mathbb{Z}^d)$, cf. [16]. In words, one has to prevent the existence of an infinite cluster of edges that can be crossed instantaneously by the infection.

Although the infection spreads in a different way in a RCP, a similar phenomenon can occur. For instance, if μ has an atom at $t \geq 0$ with $\mu(\{t\}) > p_c$ then the cluster at time t is a.s. infinite. Moreover, the same behavior can be obtained by combining atoms at different times: e.g., if $\mu(1)^2 + \mu(2) > p_c$ then we have the same problem, since it implies $\mathbb{P}(\mathcal{R} \ni 2) > p_c$.

We estimate the speed of growth in an ERCP without cures, which is a Richardson(μ) model. Our strategy is to make a comparison with a toy model of iterated percolation. The idea is the following. Fix $p < p_c(\mathbb{Z}^d)$ and let $\mathcal{P}_i$ be a family of independent Bernoulli bond percolation models on $\mathbb{Z}^d$. Moreover, for $V \subset \mathbb{Z}^d$ let us denote by $\mathcal{C}_i(V)$ the connected component of V in $\mathcal{P}_i$ by open edges. Given an initial finite non-empty set C_0, we define an increasing sequence of sets by

$$C_n := \mathcal{C}_n(C_{n-1}), \quad \text{for every } n \geq 1.$$

Coupling. Fix $d \geq 1$ and a continuous distribution μ for the transmissions. We compare iterated percolation with Richardon(μ) model. Assume that only the origin is infected at time 0. By Lemma 4.1(ii) we can choose an increasing sequence of times $(s_n)_{n\geq 0}$ with $s_0 := 0$ and $\lim s_n = \infty$ satisfying

$$\mathbb{P}(\mathcal{R} \cap [s_n, s_{n+1}] \neq \emptyset) < \frac{1}{2} p_c(\mathbb{Z}^d). \tag{4.4}$$

Indeed, we can fix $\varepsilon = \frac{1}{4} p_c(\mathbb{Z}^d)$ and define $s_n := n w_0(\varepsilon)$. The sequence of times (s_n) is important for the coupling we describe next. Define $\mathcal{I}_0$ as the set containing only the origin of $\mathbb{Z}^d$. Notice that if a site v is infected at time s_1 there must be a sequence of sites $0 = x_0, x_1, \ldots, x_k = v$ such that $\mathcal{R}_{x_{i-1}x_i} \cap [s_0, s_1] \neq \emptyset$ for every $1 \leq i \leq k$. Hence, we can find all infected sites at time s_1 by exploring the connected component of the origin in a canonical way: order the set of edges and always explore the smallest edge that has not been explored yet but has some extremity in the current infected cluster. Due to (4.4), this exploration produces a finite (random) set $\mathcal{E}_1$ of explored edges and finds all sites that have been infected till time s_1.

Let us define $\mathcal{I}_1$ as the set of all sites that are an extremity of some edge in $\mathcal{E}_1$. Notice that sites in $\mathcal{I}_1$ may not be actually infected (since we may have $e \in \mathcal{E}_1$ with $\mathcal{R}_e \cap [0, s_1] = \emptyset$), but we consider them infected all the same.

We define sets $\mathcal{E}_{n+1}$ and $\mathcal{I}_{n+1}$ inductively. Given $\mathcal{I}_n$, consider an exploration process on edges of $\mathcal{E}_n^{\mathsf{c}}$ to find the infected cluster at time s_{n+1}, starting from $\mathcal{I}_n$ infected at time s_n. This consists of checking the processes $\{\mathcal{R}_e \cap [s_n, s_{n+1}];\ e \in \mathcal{E}_n^{\mathsf{c}}\}$ until we determine the cluster. We define $\mathcal{E}_{n+1}$ as the union of all new explored edges with $\mathcal{E}_n$ and define $\mathcal{I}_{n+1}$ as the set of all sites that are the extremity of some $e \in \mathcal{E}_{n+1}$.

Since we only look at each edge at most once and different edges have independent renewal processes, this construction is a minor modification of the iterated percolation described above. Indeed, the set of explored edges in step n, $\mathcal{E}_n \setminus \mathcal{E}_{n-1}$, is contained in the union of $\mathcal{C}_n(\mathcal{I}_{n-1})$ with its external boundary of edges, a set we denote $\bar{\mathcal{C}}_n(\mathcal{I}_{n-1})$. We conclude that it holds

$$\mathcal{I}_n \subset \bar{\mathcal{C}}_n(\mathcal{I}_{n-1}), \quad \text{for every } n \geq 1$$

and, since in each step we have $\mathcal{I}_n \setminus \mathcal{I}_{n-1}$ is finite, the infection cannot reach infinitely many sites in finite time.

Iterated percolation growth. We have just described a coupling in which the growth of an iterated percolation model dominates the growth of ERCP. We can actually use the coupling to estimate its rate of growth. We consider the variation of iterated percolation that is relevant for us: given $C_0 \subset \mathbb{Z}^d$ finite, define $C_n := \bar{\mathcal{C}}_n(C_{n-1})$, for every $n \geq 1$. The main quantity for us is

$$R_n := \max\{\|x\|_1 \,;\ x \in C_n\}.$$

Having control on R_n, we are able to control C_n since $C_n \subset B(R_n)$. We are able to prove that the growth of R_n is very close to linear.

Proposition 4.2. *For any fixed $a > 1$ we have that almost surely, as $n \to \infty$*

$$(p/2) \leq \varliminf_n \frac{R_n}{n} \quad \textit{and} \quad \varlimsup_n \frac{R_n}{n(\ln n)^a} = 0. \tag{4.5}$$

Proof. The first step of our proof is to show that R_n must grow at least linearly. This is quite straightforward, since in any step of the growth process we must have some $x \in C_n$ that achieves $\|x\| = R_n$ and an edge with extremity on x such that if it is open on $\mathcal{P}_{n+1}$ then $R_{n+1} \geq R_n + 1$. This shows R_n dominates stochastically $R_0 + \mathrm{Bin}(n, p)$. Hence, using Chernoff bounds we can write

$$\mathbb{P}\big(R_n \leq (p/2)n\big) \leq \mathbb{P}\big(\mathrm{Bin}(n,p) \leq (p/2)n\big) \leq e^{-\frac{((p/2)n)^2}{2n}} = e^{-(p^2/8)n}.$$

Since $\sum_n \mathbb{P}\big(R_n \leq (p/2)n\big)$ converges, using the Borel-Cantelli lemma we conclude that $R_n > (p/2)n$ for every sufficiently large n and the lower bound in (4.5) is proved. For the upper bound, we define events

$$A_{n+1} := \{R_{n+1} \geq R_n + \eta \ln R_n\}.$$

for some constant $\eta(p,d) > 0$ that is chosen below. Consider the filtration $\mathcal{F}_n := \sigma(\mathcal{P}_i;\ i \leq n)$ and notice that $A_n \in \mathcal{F}_n$. Given $\mathcal{F}_n$ we have that on event A_{n+1} there must be some point $x \in \partial B(R_n)$ (notice that it does not need to belong to C_n) that satisfies $x \leftrightarrow x + \partial B(\eta \ln R_n)$ in percolation $\mathcal{P}_{n+1}$. Hence, exponential decay of cluster size, see e.g. [13, Theorem (6.75)], gives the estimate

$$\mathbb{P}(A_{n+1} \mid \mathcal{F}_n) \leq \sum_{x \in \partial B(R_n)} \mathbb{P}(x \leftrightarrow x + \partial B(\eta \ln R_n) \mid \mathcal{F}_n) \leq c R_n^{d-1} e^{-\psi(p,d)\eta \ln R_n}.$$

Choose $\eta(p,d) := \frac{d+1}{\psi(p,d)}$, which leads to $\mathbb{P}(A_{n+1} \mid \mathcal{F}_n) \leq c R_n^{-2}$. The linear growth estimate says there is n_1 (random) such that $R_n > (p/2)n$ for $n \geq n_1$, and we notice that $x \mapsto cx^{-2}$ is decreasing for $n \geq n_1$. This means that

$$\sum_{n \geq n_1} \mathbb{P}(A_{n+1} \mid \mathcal{F}_n) \leq c(p,d) \sum_{n \geq n_1} n^{-2} < \infty.$$

Using a conditional Borel-Cantelli lemma, see [7, Theorem 5.3.2], we have that $\mathbb{P}(\varlimsup A_n) = 0$ implying that there is a random n_2 such that $R_{n+1} \leq R_n + \eta \ln R_n$ for $n \geq n_2$. This implies estimates on the growth of R_n. Indeed, fix $a > 1$ and define function $f : [1, \infty) \to \mathbb{R}$ given by $f(x) := x + \eta \ln x$. Notice that if we take x of the form $y(\ln y)^a$ we can write

$$\begin{aligned} f(y(\ln y)^a) &= y(\ln y)^a + \eta \ln[y(\ln y)^a] = y(\ln y)^a + \eta \ln y + \eta a \cdot \ln \ln y \\ &\leq y(\ln y)^a + (\ln y)^a \leq (y+1)(\ln(y+1))^a \end{aligned} \tag{4.6}$$

for any $y \geq y_0(a,\eta)$. Using that R_n eventually grows at least linearly, we can find a random $n_3 \geq n_2$ sufficiently large so that $R_{n_3} = y(\ln y)^a$ for some $y \geq y_0$. Now, by (4.6) we have

$$R_{n_3+1} \leq f(R_{n_3}) = f(y(\ln y)^a) \leq (y+1)(\ln(y+1))^a$$

and repeated application of (4.6) leads to $R_{n+n_3} \leq (y+n)(\ln(y+n))^a$ for every $n \geq 0$. Hence, we have asymptotically that $\overline{\lim} \frac{R_n}{n(\ln n)^a} \leq 1$. Since any choice of $a > 1$ works, the result follows. $\square$

Growth of heavy-tailed ERCP. The coupling between ERCP and iterated percolation we have just described works for any sequence of times (s_n) increasing to infinity and satisfying (4.4). Recall the definition

$$r_t := \max\{\|x\|_1\,;\ (0,0) \rightsquigarrow (x,t) \text{ in Richardson}(\mu) \text{ model}\}.$$

Proof of Proposition 1.3(i). Let $w_0 = w_0(p_c(\mathbb{Z}^d)/2)$ as given in Lemma 4.1(ii) and consider the sequence $s_n := nw_0$. We just have to combine the linear growth of s_n with the estimates given by Proposition 4.2. For any fixed time t, define $n(t) := \lceil \frac{t}{w_0} \rceil$. Clearly, we have $r_t \leq R_{n(t)}$ and since $\overline{\lim} \frac{R_n}{n(\ln n)^a} = 0$ the result holds. $\square$

Notice that in Proposition 1.3 the rate of growth of s_n is essential in the final estimate. When μ is heavy-tailed, our estimate on r_t can be greatly improved by considering a sequence of times that grows faster. We assume that μ satisfies (G). Then, for n_0 sufficiently large and the sequence of times $t_0 = 2^{n_0}$ and $t_{n+1} = t_n + t_n^{\epsilon_4}$ we can ensure

$$\mathbb{P}(\mathcal{R} \cap [t_n, t_{n+1}] \neq \emptyset) \leq p < p_c(\mathbb{Z}^d) \tag{4.7}$$

and let $\mathcal{P}_n$ be independent Bernoulli bond percolation models with parameter p. In other words, we start the coupling only at time t_0, when we have a (random) finite infected set $\mathcal{I}_0$ and then

$$\{x;\ (0,0) \rightsquigarrow (x,t_n) \text{ in Richardson}(\mu) \text{ model}\} \subset \mathcal{I}_n \subset \bar{\mathcal{C}}_n(\mathcal{I}_{n-1}) \quad \text{for every } n \geq 1. \tag{4.8}$$

Proof of Proposition 1.3(ii). The growth of R_n is estimated in Proposition 4.2. Fixing $a > 1$, we notice that $r_{t_n} \leq R_n \leq n^a$ for n large. It is also clear that r_t is non-decreasing. Thus, for some $s > 0$ sufficiently large, if we define $n = n(s)$ as the unique integer satisfying $t_{n-1} < s \leq t_n$, then $r_s \leq n^a$. We just have to estimate $n(s)$.

Consider intervals $I_i := [2^{n_0+i-1}, 2^{n_0+i}]$. Each of them cannot have too many points of sequence (t_j). Indeed, since for $t_j \in I_i$ we have $t_{j+1} - t_j = t_j^{\epsilon_4} \geq 2^{(n_0+i-1)\epsilon_4}$, we have

$$\#\{j;\ t_j \in I_i\} \leq \frac{2^{n_0+i-1}}{2^{(n_0+i-1)\epsilon_4}} = 2^{(n_0+i-1)(1-\epsilon_4)}.$$

Since $s \geq 2^{\lfloor \log_2 s \rfloor}$, we conclude that

$$n(s) \leq \sum_{i=1}^{\lfloor \log_2 s \rfloor - n_0} 2^{(n_0+i-1)(1-\epsilon_4)} \leq c(n_0,\epsilon_4) 2^{\lfloor \log_2 s \rfloor (1-\epsilon_4)} \leq c(n_0,\epsilon_4) s^{1-\epsilon_4}.$$

Notice that since we can take any $a > 1$, the result in (1.4) follows. $\square$

Corollary 4.3. *If $\mu(t,\infty) = L(t)t^{-\alpha}$ with $\alpha \in (0,1)$, $L(t)$ slowly varying, and μ satisfies the Strong Renewal Theorem (cf. [5]), then for all $\eta > 0$ we have*

$$\overline{\lim_{s\to\infty}} \frac{r_s}{s^{\alpha+\eta}} \leq 1. \tag{4.9}$$

Proof. The estimate in (4.9) follows in the same manner, by noticing that if we have the Strong Renewal Theorem then

$$\mathbb{P}(\mathcal{R}\cap[t,t+t^{\epsilon}]\neq\emptyset)\leq\sum_{s=1}^{t^{\epsilon}}\mathbb{P}(\mathcal{R}\cap[t+s-1,t+s]\neq\emptyset)\sim t^{\epsilon}c_{\alpha}\frac{L(t)}{t^{1-\alpha}}$$

for some slowly varying function L. Hence, this probability goes to zero whenever $\epsilon<1-\alpha$. This implies that we can take ϵ_4 arbitrarily close to $1-\alpha$ and the result follows. □

4.3 Extinction in heavy-tailed ERCP

Proposition 1.3(ii) gives a bound on how fast the infection can spread without any cures: for large t, the infection is contained inside $\{(x,t);\|x\|_1\leq t^{\rho}\}$ for some $\rho<1$.

Fix $\beta>0$. The next step in our investigation is to show that eventually we are able to cure all the infection in a region of the form $\mathsf{B}(2^{\beta n})\times[2^n,2^n+2^{n\epsilon_4}]$. Choosing β sufficiently large, this will imply that the process dies almost surely for any renewal process ν_δ for the cures, given that ν has moments of all orders. The following lemmas introduce some bad events that would represent obstacles for stopping the infection in one of such regions all at once. We show that each of these events cannot happen infinitely often.

Our first lemma estimates the probability of having large clusters of transmissions inside $\mathsf{B}(2^{\beta n})$. For that, we recall that a finite connected subgraph of $\mathbb{Z}^d$ that contains the origin is said to be a *lattice animal*. Let us denote by A_m the set of lattice animals with m edges. By Equation (4.24) of [13] we have that $\#A_m\leq 7^{d(m+1)}\leq C^m$ for some positive constant $C(d)$. The probability finding a cluster of m adjacent transmissions in region $\mathsf{B}(2^{\beta n})\times[2^n,2^n+2^{n\epsilon_4}]$ decays quickly with n.

Lemma 4.4. *Let μ satisfy* (G). *Consider the event $U_n=U_n(m,\beta)$ defined by*

$$U_n:=\bigcup_{x\in\mathsf{B}(2^{\beta n})}\bigcup_{M\in x+A_m}\{\mathcal{R}_e\cap[2^n,2^n+2^{n\epsilon_4}]\neq\emptyset,\ \textit{for every } e \textit{ edge of } M\}\tag{4.10}$$

There is $m(\epsilon_4,\beta,d)\in\mathbb{N}$ such that $\mathbb{P}\big(\overline{\lim}_n U_n\big)=0$.

Proof. Using the union bound and the estimate from (G), we can write

$$\mathbb{P}(U_n)\leq c(d)2^{d\beta n}\cdot C^m\cdot 2^{-n\epsilon_4 m}=cC^m\cdot 2^{(d\beta-\epsilon_4 m)n}$$

and it suffices to choose $m>\frac{d\beta}{\epsilon_4}$ to make $\sum_n\mathbb{P}(U_n)$ summable. □

A second estimate that is useful is a consequence of [9, Lemma 3]. It says that even when there are transmissions in an interval $[2^n,2^n+2^{n\epsilon_4}]$ for an edge of $\mathsf{B}(2^{\beta n})$, the probability of having too many transmissions in this edge decays fast with n.

Lemma 4.5. *Let μ satisfy C). Let $V_n(\epsilon_4,\beta,\eta)$ be the event*

$$V_n:=\{\exists e \textit{ edge of } \mathsf{B}(2^{\beta n});\ |\mathcal{R}_e\cap[2^n,2^n+2^{\epsilon_4 n}]|\geq 2^{n\epsilon_4\eta}\}.\tag{4.11}$$

There is $\eta=\eta(\mu)$ with $\eta\in(0,1)$ such that it holds $\mathbb{P}(\overline{\lim}_n V_n)=0$.

Proof. Taking $I=[2^n,2^n+2^{n\epsilon_4}]$ and denoting by $l=2^{n\epsilon_4}$ its length, Lemma 3 of [9] shows that

$$\mathbb{P}(|\mathcal{R}\cap I|\geq l^{1-\epsilon_3}\ln^2 l)\leq 2\cdot e^{-\ln^2 l}\leq 2^{-c\epsilon_4^2n^2}\qquad\text{for large } n \text{ and some } c>0,$$

where constant $\epsilon_3 > 0$ satisfies $\mu(t,\infty) \geq t^{-(1-\epsilon_3)}$ for large t (the proof of Lemma 3 of [9] only uses the lower bound of condition C)). Taking $\eta \in (1-\epsilon_3, 1)$ we have that

$$l^{1-\epsilon_3} \ln^2 l = 2^{(1-\epsilon_3)\epsilon_4 n} \ln^2 2^{\epsilon_4 n} \ll 2^{\eta \epsilon_4 n}$$

for large n. The union bound implies

$$\mathbb{P}(V_n) \leq K(d) 2^{d\beta n} \mathbb{P}(|\mathcal{R} \cap I| \geq 2^{n\epsilon_4 \eta}) \leq K(d) 2^{d\beta n} 2^{-c\epsilon_4^2 n^2},$$

for some constant $K(d) > 0$. Then, $\sum_n \mathbb{P}(V_n)$ is summable and the result follows. □

The last event we consider is the only one related to cures. Notice that on the event $U_n^{\mathsf{c}} \cap V_n^{\mathsf{c}}$ the region $\mathsf{B}(2^{\beta n}) \times [2^n, 2^n + 2^{n\epsilon_4}]$ only has clusters of transmissions with at most m edges and each of these edges do not have many transmissions. Hence, for each cluster M we can find an interval $I_M \subset [2^n, 2^n + 2^{\epsilon_4 n}]$ with length at least $2^{\epsilon_4 n}/(m 2^{\eta \epsilon_4 n}) = 2^{(1-\eta)\epsilon_4 n}/m$ satisfying that $\mathcal{R}_e \cap I_M = \emptyset$ for every edge of M.

Let us denote by $\mathcal{H}_x$ the renewal process with interarrival ν_δ that is associated to site $x \in \mathbb{Z}^d$.

Lemma 4.6. *Let $\beta, \epsilon_4 > 0$ and $\eta \in (0,1)$. Consider the event $W_n = W_n(\beta, \epsilon_4, \eta)$ defined by*

$$W_n := \bigcup_{x \in \mathsf{B}(2^{\beta n})} \bigcup_{I} \{\mathcal{H}_x \cap I = \emptyset\}, \tag{4.12}$$

where the second union is over all intervals $I \subset [2^n, 2^n + 2^{n\epsilon_4}]$ of length $|I| = 2^{(1-\eta)\epsilon_4 n}/m$. If ν has finite moments of all orders then it holds $\mathbb{P}(\overline{\lim}_n W_n) = 0$.

Proof. Let $t_j = 2^n + j \cdot 2^{(1-\eta)\epsilon_4 n}/(2m)$ and notice that intervals $I_j = [t_j, t_{j+1}]$ for $0 \leq j \leq \lceil 2m \cdot 2^{\eta \epsilon_4 n} \rceil$ cover $[2^n, 2^n + 2^{n\epsilon_4}]$. If we have an interval I of length $|I| = 2^{(1-\eta)\epsilon_4 n}/m$ that has no cure marks for every $x \in \mathsf{B}(2^{\beta n})$, then this interval must contain some I_j. Hence,

$$\mathbb{P}(W_n) \leq K(d) 2^{d\beta n} \sum_{j=0}^{\lceil 2m \cdot 2^{\eta \epsilon_4 n} \rceil} \mathbb{P}(\mathcal{H} \cap I_j = \emptyset).$$

By (4.1) in Lemma 4.1, we can translate moments of ν_δ into estimates for $\mathbb{P}(\mathcal{H} \cap I_j = \emptyset)$. Consider the function

$$f(x) := x^a, \qquad \text{with } a > \frac{d\beta + \eta\epsilon_4}{(1-\eta)\epsilon_4}.$$

Since $E_{\nu_\delta}[X f(X)] = E_\nu[(X/\delta)^{1+a}] < \infty$ and every I_j has length $2^{(1-\eta)\epsilon_4 n}/(2m)$, it follows that

$$\mathbb{P}(W_n) \leq c(d, m, a, \delta) \cdot 2^{(d\beta + \eta\epsilon_4 - a(1-\eta)\epsilon_4)n}.$$

Our choice of a makes the coeficient multiplying n in the exponent negative, and we conclude that $\sum_n \mathbb{P}(W_n)$ converges. □

From the estimates above, we have

Proof of Theorem 1.4(i). By Proposition 1.3(ii), there is $\rho(\mu) < 1$ such that $r_t \leq t^\rho$ for every large t. Fix $\beta > \rho$ and consider the bad events $U_n(m,\beta), V_n(\epsilon_4, \beta, \eta), W_n(\epsilon_4, \beta, \eta)$ described in Lemmas 4.4, 4.5 and 4.6 with m and η chosen so that all the bounds in these lemmas hold.

Since $\beta > \rho$, for large n we have that $\mathsf{B}(2^{\beta n}) \times [2^n, 2^n + 2^{\epsilon_4 n}]$ will contain all infected sites of time interval $[2^n, 2^n + 2^{\epsilon_4 n}]$. Moreover, on $U_n^{\mathsf{c}} \cap V_n^{\mathsf{c}} \cap W_n^{\mathsf{c}}$ every cluster M of transmissions cannot have more than m edges and must have an interval I_M of length at least $2^{(1-\eta)\epsilon_4 n}/m$ without transmissions. Since we also have that I_M must have a point of $\mathcal{H}_x$ for every $x \in \mathsf{B}(2^{\beta n})$, the result follows. □

4.4 Phase transition

Lemma 4.1 is at the core of the proof of Theorem 1.4(ii) and (iii).

Proof of Theorem 1.4(ii). We make a straightforward comparison with planar oriented percolation. It is sufficient to prove the statement for $d = 2$. We can actually prove there is a positive probability of survival in the quadrant $\mathbb{Z}_+^2 \times \mathbb{R}_+$. In $\mathbb{Z}_+^2$, consider the graph with oriented edges $z + (1,0)$ and $z + (0,1)$. Given any $\varepsilon > 0$, notice that by Lemma 4.1 we can find h such that

$$\mathbb{P}(\mathcal{R}^\mu \cap [t, t+h] = \emptyset) \le \varepsilon, \qquad \text{for any } t \ge 0.$$

For each site $x \in \mathbb{Z}_+^2$ we associate a point $g(x) \in \mathbb{Z}_+^2 \times \mathbb{R}_+$ defined by $g(x) := (x, h \|x\|_1)$, the only point in the intersection of the vertical line from $(x, 0)$ with the plane through $(1,0,h)$, $(0,1,h)$ and the origin. We denote by G the graph with vertex set $\{g(x);\ x \in \mathbb{Z}_+^2\}$ and oriented edges from $g(x)$ to $g(y)$ if and only if there is one from x to y in the oriented graph $\mathbb{Z}_+^2$. Clearly, G is isomorphic to $\mathbb{Z}_+^2$ as shown in Figure 3. Define a site bond percolation model in G by stating that

- A site $g(x)$ is open if and only if $\{x\} \times [h(\|x\|_1 - 1), h(\|x\|_1 + 1)]$ has no cure marks.
- An edge from $g(x)$ to $g(y)$ is open if and only if $\mathcal{R}_{x,y} \cap [h(\|x\|_1 - 1), h\|x\|_1] \neq \emptyset$.

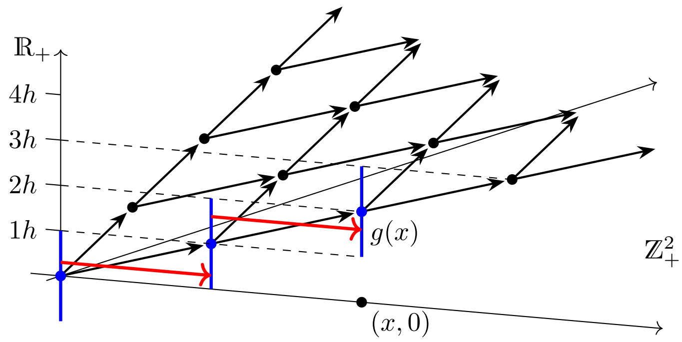

Figure 3: Coupling with oriented site bond percolation when $d \ge 2$. A path of open sites and bonds connecting 0 to $g(x)$ in G implies the infection reaches $g(x)$ in the original model.

Notice that the state of every site and bond is independent and the probability of an edge being open is at least $1 - \varepsilon$. Now, we use Lemma 4.1(ii) to obtain a similar estimate for the probability of a site being open. There is $w_0(\varepsilon) > 0$ such that it holds

$$\inf_{t \ge 0} \mathbb{P}(\mathcal{R}^\nu \cap [t, t+w] = \emptyset) \ge 1 - \varepsilon, \qquad \text{for any } w \in (0, w_0).$$

Notice that $\mathbb{P}(\mathcal{R}^\nu \cap [t, t+w] = \emptyset) = \mathbb{P}(\mathcal{R}^{\nu_\delta} \cap [\frac{1}{\delta}t, \frac{1}{\delta}t + \frac{1}{\delta}w] = \emptyset)$ for every $t \ge 0$. Taking $\delta_0 := \frac{w_0}{2h}$, it follows that for $\delta \in (0, \delta_0)$ the probability of a site being open is at least $1 - \varepsilon$. This independent site-bond model can be compared with a finite range dependent bond model: say that an edge $e = (x, y)$ is open if e, x and y are all open in G leads to a bond model in which edges that do not share extremities are independent. By the classical stochastic domination results of Liggett, Schonmann and Stacey [21], if we choose $\varepsilon > 0$ small enough in the beginning it follows that ERCP(μ, ν_δ) with $\delta \in (0, \delta_0)$ survives with positive probability. $\square$

Proof of Theorem 1.4(iii). Here we use the recurrence inequality approach. The initial part of the argument is essentially the same as in the proof of [11, Theorem 1.1], with the only difference that now transmissions and cures are given by renewal processes with interarrival distributions μ and ν_δ, respectively. Analogous to [11, Definition 2.2], we consider the uniform quantities

$$\tilde{s}_n := \sup \hat{\mathbb{P}}(\tilde{S}_j((x,t)+B_n)) \quad \text{and} \quad \tilde{t}_n := \sup \hat{\mathbb{P}}(\tilde{T}((x,t)+B_n)), \tag{4.13}$$

where the suprema above are over all $(x,t) \in \mathbb{Z}^d \times \mathbb{R}_+$ and all product probability measures $\hat{\mathbb{P}}$ describing independent renewal processes with interarrival distributions μ and ν_δ and renewal points starting at (possibly different) time points strictly less than zero. Define

$$u_n := \tilde{s}_n + \tilde{t}_n. \tag{4.14}$$

Most of the reasoning in the proof of [11, Theorem 1.1] still holds, up to the choice of box sequence. More precisely, consider boxes $B_n = [0,2^n]^d \times [0,h_n]$. Using (4.1) in Lemma 4.1(i) we estimate the probability of decoupling both transmissions and cures. For some fixed non-decreasing function f to be precised later, we estimate the probability that some site or edge in $[0,2^n]^d$ does not have a mark in the interval $[t,t+h]$ by

$$\sup_{t\geq 0} \hat{\mathbb{P}}\Big(\bigcup_e \{\mathcal{R}_e^\mu \cap [t,t+h] = \emptyset\} \cup \bigcup_x \{\mathcal{R}_x^{\nu_\delta} \cap [t,t+h] = \emptyset\}\Big) \leq \frac{d2^{dn}C(f,\mu)}{f(h)} + \frac{2^{dn}C(f,\nu_\delta)}{f(h)}.$$

More than that, the upper bound above can be taken uniform for $\delta > 1$ since

$$\mathbb{P}(\mathcal{R}_x^{\nu_\delta} \cap [t,t+h] = \emptyset) = \mathbb{P}(\mathcal{R}_x^{\nu} \cap [\delta t, \delta t + \delta h] = \emptyset) \leq \frac{C(f,\nu)}{f(\delta h)} \leq \frac{C(f,\nu)}{f(h)}, \tag{4.15}$$

and f is non-decreasing. Hence, the same line of reasoning in the proof of [11, Lemmas 2.5, 2.6] shows there are constants $c = c(d)$ and $C = C(d,\mu,\nu,f)$ such that

$$u_n \leq c \cdot (h_n/h_{n-1})^2 \cdot u_{n-1}^2 + \frac{C2^{dn}}{f(h_{n-1})}. \tag{4.16}$$

Remark 4.7. The derivation of (4.16) uses essentially the same geometric features of the paths that are used while deriving (2.6) and we omit full details.

Since μ and ν satisfy (M), if we choose a sequence $h_n = e^{(\alpha/\theta)^2 n^2}$ with an appropriate choice of α like in the proof of [11, Lemma 2.7], and

$$f(x) := e^{\theta(\ln x)^{1/2}} \cdot \mathbb{1}\{x \geq 1\}, \tag{4.17}$$

it follows that there is $n_0(\mu,\nu,\theta,d)$ such that if $u_{n_0} \leq 2^{-dn_0}$ then $u_n \leq 2^{-dn}$ for every $n \geq n_0$. For sake of completeness, we include this proof in an Appendix. To finish the proof, we only need to choose $\delta > 1$ sufficiently high so that $u_{n_0} \leq 2^{-dn_0}$.

Recall that $n_0(d,\mu,\nu,\theta)$ is fixed, and so are the dimensions l_{n_0} and h_{n_0} of box B_{n_0}. To control the probability of $\hat{\mathbb{P}}(\tilde{S}_1((x,s)+B_{n_0}))$ and $\hat{\mathbb{P}}(\tilde{T}((x,s)+B_{n_0}))$ uniformly in (x,s) and $\hat{\mathbb{P}}$ we observe the following. Firstly, we control the probability of some edge in $\pi(B_{n_0})$ having too many renewal marks. Let N denote the number of edges in $\pi(B_{n_0})$. For a single edge, we can find $k_0(n_0,\mu)$ sufficiently large so that

$$\mathbb{P}(\#\mathcal{R}_e \cap [s,s+h_{n_0}] \geq k_0) \leq \frac{1}{4N}2^{-dn_0},$$

implying that

$$\mathbb{P}\Big(\bigcup_{e\in\pi(B_{n_0})} \{\#\mathcal{R}_e \cap [s,s+h_{n_0}] \geq k_0\}\Big) \leq N \cdot \mathbb{P}(\#\mathcal{R} \cap [s,s+h_{n_0}] \geq k_0) \leq \frac{1}{4}2^{-dn_0}. \tag{4.18}$$

Estimate (4.18) controls the probability that some edge has more than k_0 renewals. When every edge of $\pi(B_{n_0})$ has less than k_0 renewals, we show that after every transmission in B_{n_0} there is a high probability that every site gets cured before the next transmission if δ is sufficiently large.

Let Z_t denote the overshoot at time t, i.e., $Z_t := \inf \mathcal{R} \cap [t, \infty) - t$. We denote by $Z_t^{\mu,e}$ and $Z_t^{\nu_\delta,x}$ the overshoots of the renewal processes of edge e and site x. By Lemma 4.1(ii), given $\varepsilon > 0$ there is $w_0(\varepsilon, \mu) > 0$ such that

$$\inf_{t\geq 0} \hat{\mathbb{P}}(\min_{e\in\pi(B_{n_0})} Z_t^{\mu,e} > w_0) \geq (1-\varepsilon)^N. \tag{4.19}$$

On the other hand, denoting by M the number of sites in $\pi(B_{n_0})$ we have that

$$\begin{aligned}\inf_{t\geq 0} \hat{\mathbb{P}}(\max_{x\in\pi(B_{n_0})} Z_t^{\nu_\delta,x} \leq w_0) &= \big(1 - \sup_{t\geq 0} \hat{\mathbb{P}}(Z_t^{\nu_\delta,x} > w_0)\big)^M \\ &\geq \big(1 - \sup_{t\geq 0} \mathbb{P}(\mathcal{R}^{\nu_\delta} \cap [t, t+w_0] = \emptyset)\big)^M \\ &\geq \Big(1 - \frac{C(\nu,f)}{f(\delta w_0)}\Big)^M,\end{aligned}$$

where the last inequality is a consequence of (4.15).

Hence, we can choose $\delta_0(\nu, w_0, f, \varepsilon) > 0$ so that for $\delta \geq \delta_0$ we have

$$\inf_{t\geq 0} \hat{\mathbb{P}}(\max_{x\in\pi(B_{n_0})} Z_t^{\nu_\delta,x} \leq w_0) \geq (1-\varepsilon)^M. \tag{4.20}$$

Combining (4.19) and (4.20) we can write that

$$\inf_{t\geq 0} \hat{\mathbb{P}}(\min_e Z_t^{\mu,e} > \max_x Z_t^{\nu_\delta,x}) \geq \inf_{t\geq 0} \hat{\mathbb{P}}(\min_e Z_t^{\mu,e} > w_0 \geq \max_x Z_t^{\nu_\delta,x}) \geq (1-\varepsilon)^{N+M}. \tag{4.21}$$

Let us estimate $\hat{\mathbb{P}}(\tilde{S}_1((x,s)+B_{n_0}))$. Define $T_0 := s$ and $T_{j+1} := T_j + \min_{e\in\pi(B_{n_0})} Z_{T_j}^e$. Then, T_j is an increasing sequence of stopping times for the filtration $\mathcal{F}_t$ that makes $\mathcal{R}_e \cap [0,t]$ and $\mathcal{R}_x \cap [0,t]$ measurable for every edge e and site x in $\pi(B_{n_0})$. By (4.18), we know that

$$\hat{\mathbb{P}}(T_{Nk_0} \leq s + h_{n_0}) \leq \frac{1}{4} 2^{-dn_0}.$$

Given $\mathcal{F}_{T_j}$, notice that at T_j there has been unique transmission in $(x,s)+B_{n_0}$ and by (4.21) all sites will cure before the next transmission with probability

$$\hat{\mathbb{P}}(\min_e Z_{T_j}^{\mu,e} > \max_x Z_{T_j}^{\nu_\delta,x} \mid \mathcal{F}_{T_j}) \geq (1-\varepsilon)^{N+M},$$

implying that

$$\hat{\mathbb{P}}\Big(\bigcap_{j=1}^{Nk_0} \{\min_e Z_{T_j}^{\mu,e} > \max_x Z_{T_j}^{\nu_\delta,x}\}\Big) \geq (1-\varepsilon)^{N(N+M)k_0}.$$

Notice that on the event

$$\{T_{Nk_0} > s + h_{n_0}\} \cap \Big(\bigcap_{j=1}^{Nk_0} \{\min_e Z_{T_{j+1}}^{\mu,e} > \max_x Z_{T_{j+1}}^{\nu_\delta,x}\}\Big)$$

there are no spatial nor temporal crossings. Hence, it follows

$$\hat{\mathbb{P}}(\tilde{S}_1((x,s)+B_{n_0})) \leq \frac{1}{4} 2^{-dn_0} + \big[1 - (1-\varepsilon)^{N(N+M)k_0}\big] \leq \frac{1}{2} 2^{-dn_0} \tag{4.22}$$

if we choose $\varepsilon(n_0, k_0, d) > 0$ sufficiently small. We emphasize that the choice of parameters above is not circular: we can choose in order $k_0(n_0, \mu)$, $\varepsilon(n_0, k_0, d)$, $w_0(\varepsilon, \mu)$ and then $\delta_0(\nu, w_0, f, \varepsilon)$, so that (4.22) holds for $\delta \geq \delta_0$, uniformly on $\hat{\mathbb{P}}$ and (x,s). We can estimate $\hat{\mathbb{P}}(\tilde{T}((x,s)+B_{n_0}))$ in a similar way, which leads to $u_{n_0} \leq 2^{-dn_0}$ and the result follows. □

5 Appendix

Lemma 5.1. (A version of [11, Lemma 2.7]) Let μ, ν be probability distributions on $\mathbb{R}_+$ satisfying condition (M). Let (u_n) satisfy a recursion of the form (4.16) with f given by (4.17). A parameter $\alpha > 0$ can be chosen so that for $h_n = e^{(\alpha/\theta)^2 n^2}$ there exists a natural number $n_0 = n_0(\mu, \nu, \theta, d)$ such that if $u_{n_0} \leq 2^{-dn_0}$, then for every $n \geq n_0$ we have $u_n \leq 2^{-dn}$.

Proof. For the above choices we have $f(h_{n-1}) := e^{\alpha(n-1)}$ for $\alpha > 0$, which is the parameter to be chosen so that $2^{nd}/f(h_{n-1})$ tends to zero sufficiently fast. The recurrence relation (4.16) becomes

$$u_n \leq c(d)\Big(\frac{h_n}{h_{n-1}} \cdot u_{n-1}\Big)^2 + C(\mu, \nu, \theta)\exp[(d\ln 2)n - \alpha(n-1)],$$

from where we see that the decay of u_n cannot be faster than $e^{-\alpha(n-1)}$. Based on this, we suppose $u_{n-1} \leq e^{-\beta(n-1)}$ for some parameter $\alpha > \beta > 0$. Under this assumption we have

$$\Big(\frac{h_n}{h_{n-1}} \cdot u_{n-1}\Big)^2 = \Big(e^{(\alpha/\theta)^2(2n-1)} \cdot u_{n-1}\Big)^2 \leq e^{2(\alpha/\theta)^2(2n-1) - 2\beta(n-1)},$$

which leads to

$$\begin{aligned} u_n &\leq c(d)e^{2(\alpha/\theta)^2(2n-1)-2\beta(n-1)} + C(\mu,\nu,\theta)e^{(d\ln 2-\alpha)n+\alpha} \\ &\leq c(d,\alpha,\beta,\theta)e^{[4(\alpha/\theta)^2-\beta]n} \cdot e^{-\beta n} + C(\mu,\nu,\theta,\alpha)e^{(\beta+d\ln 2-\alpha)n} \cdot e^{-\beta n}. \end{aligned} \tag{5.1}$$

The induction will follow once we ensure

$$4(\alpha/\theta)^2 - \beta < 0 \text{ and } \beta + d\ln 2 - \alpha < 0$$

or, equivalently,

$$\theta^2 > \frac{4\alpha^2}{\beta} \text{ and } \alpha > \beta + d\ln 2.$$

We choose α and β that would allow to take θ as small as possible while still being able to perform the induction. Combining the two inequalities above we have

$$\theta^2 > 4\Big(\sqrt{\beta} + \frac{d\ln 2}{\sqrt{\beta}}\Big)^2 \geq 16d\ln 2,$$

by AM-GM inequality, with equality when $\beta = d\ln 2$. This brings us to hypothesis (M) as the best we can hope for with this argument. We then fix $\beta = d\ln 2$ and take

$$2d\ln 2 < \alpha < \frac{\theta\sqrt{d\ln 2}}{2}.$$

Since $\frac{\theta}{2}\sqrt{d\ln 2} > 2d\ln 2$, we can take for instance $\alpha(d,\theta) := \frac{1}{2}\Big(2d\ln 2 + \frac{\theta}{2}\sqrt{d\ln 2}\Big)$. Take $n_0 = n_0(\mu,\nu,d,\theta)$ sufficiently large so that

$$c(d,\alpha,\beta,\theta)e^{[4(\alpha/\theta)^2-\beta]n} \leq \frac{1}{4} \quad \text{and} \quad C(\mu,\nu,\theta,\alpha)e^{(\beta+d\ln 2-\alpha)n} \leq \frac{1}{4} \quad \text{for all } n \geq n_0. \tag{5.2}$$

This is possible since both left hand sides tend to zero as $n \to \infty$. Suppose that $u_{n_0} \leq e^{-\beta n_0} = 2^{-dn_0}$, recalling that $\beta = d\ln 2$. Then, we have by (5.1) that

$$u_n \leq \frac{1}{4}e^{-\beta n} + \frac{1}{4}e^{-\beta n} \leq e^{-\beta n} \quad \text{for every } n \geq n_0.$$

completing the proof. $\square$

References

[1] E. D. Andjel (1992). Survival of multidimensional contact process in random environments. *Bulletin of the Brazilian Mathematical Society*, **23**(1), 109–119. MR1203176

[2] A. Auffinger, M. Damron, J. Hanson (2017). 50 years of first-passage percolation (Vol. 68). American Mathematical Soc. MR3729447

[3] C. Bezuidenhout, G. Grimmett (1990). The critical contact process dies out. *The Annals of Probability*, **18**(4), 1462–1482. MR1071804

[4] M. Bramson, R. Durrett, R. H. Schonmann (1991). The contact processes in a random environment. *The Annals of Probability*, **19**(3), 960–983. MR1112403

[5] F. Caravenna, R. Doney (2019). Local large deviations and the strong renewal theorem. *Electron. J. Probab.* **24**. MR3978222

[6] O. Couronné, R.J. Messikh (2004). Surface order large deviations for 2D FK-percolation and Potts models. *Stochastic processes and their applications*, **113**(1), 81–99. MR2078538

[7] R. Durrett (2010). Probability: theory and examples. Cambridge University press. MR2722836

[8] L. R. Fontes, P. A. Gomes, R. Sanchis (2020). Contact process under heavy-tailed renewals on finite graphs. *Bernoulli*, **27**(3), 1745–1763. MR4260508

[9] L. R. Fontes, T. S. Mountford, D. H. U. Marchetti, M. E. Vares (2019). Contact process under renewals I. *Stoch. Proc. Appl.*, **129**(8), 2903–2911. MR3980148

[10] L. R. Fontes, T. S. Mountford, M. E. Vares (2020). Contact process under renewals II. *Stoch. Proc. Appl.*, **130**(2), 1103–1118. MR4046531

[11] L. R. Fontes, T. S. Mountford, D. Ungaretti, M. E. Vares (2021). Renewal Contact Processes: phase transition and survival. arXiv preprint arXiv:2101.06207

[12] O. Garet, R. Marchand (2012). Asymptotic shape for the contact process in random environment. *The Annals of Applied Probability*, **22**(4), 1362–1410. MR2985164

[13] G. Grimmett (1999). Percolation. Springer. MR1707339

[14] T. E. Harris (1974). Contact interactions on a lattice. *The Annals of Probability*, **2**(6), 969–988. MR0356292

[15] T. E. Harris (1978). Additive Set-Valued Markov Processes and Graphical Methods. *The Annals of Probability*, **6**(3), 355–378. MR0488377

[16] H. Kesten (1986). Aspects of first passage percolation. *École d'été de probabilités de Saint Flour XIV-1984*, 125–264. MR0876084

[17] A. Klein (1994). Extinction of contact and percolation processes in a random environment. *The Annals of Probability*, 1227–1251. MR1303643

[18] T. M. Liggett (1985). Interacting Particle Systems. *Grundlehren der Mathematischen Wissenschaften* **276**, New York: Springer. MR0776231

[19] T. M. Liggett (1992). The survival of one-dimensional contact processes in random environments. *The Annals of Probability* **20** 696–723. MR1159569

[20] T. M. Liggett (2013). Stochastic interacting systems: contact, voter and exclusion processes. Vol. 324, Springer science & Business Media. MR1717346

[21] T. Liggett, R. Schonmann, A. Stacey (1997). Domination by product measures. *The Annals of Probability*, **25**(1), 71–95. MR1428500

[22] A. Linker, D. Remenik (2020). The contact process with dynamic edges on $\mathbb{Z}$. *Electron. J. Probab.* **25**. MR4125785

[23] C. M. Newman, S. B. Volchan (1996). Persistent survival of one-dimensional contact processes in random environments. *The Annals of Probability*, **24**(1), 411–421. MR1387642

[24] A. Pisztora (1996). Surface order large deviations for Ising, Potts and percolation models. *Probability Theory and Related Fields*, **104**(4), 427–466. MR1384040

[25] B. Ráth (2015). A short proof of the phase transition for the vacant set of random interlacements. *Electronic Communications in Probability*, **20**, 1–11. MR3304409

[26] D. Remenik (2008). The contact process in a dynamic random environment. *The Annals of Applied Probability*, **18**(6), 2392–2420. See also the arXiv update: arXiv:0901.2480v2. MR2474541

[27] B. Schapira, D. Valesin (2021). Exponential rate for the contact process extinction time, to appear in *Annales de la Faculté des Sciences de Toulouse (Mathématiques)*. MR4331308

[28] J. E. Steif (2009). A survey of dynamical percolation. In *Fractal geometry and stochastics IV*, 145–174. Birkhäuser Basel. MR2762676

[29] A.-S. Sznitman (2012). Decoupling inequalities and interlacement percolation on $G \times \mathbb{Z}$. *Inventiones mathematicae*, **187**(3), 645–706. MR2891880

Acknowledgments. We would like to thank an anonymous referee for carefully reading our work. Also, we would like to thank Marco Seiler for pointing out a mistake in our original proof of Theorem 1.1(ii).

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS[1])
- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4] , ProjectEuclid[5]
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1] LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2] EJMS: Electronic Journal Management System http://www.vtex.lt/en/ejms.html
[3] IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4] BS: Bernoulli Society http://www.bernoulli-society.org/
[5] Project Euclid: https://projecteuclid.org/
[6] IMS Open Access Fund: http://www.imstat.org/publications/open.htm