Interacting loop ensembles and Bose gases

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Abstract

We study interacting Bose gases in thermal equilibrium on a lattice. We establish convergence of the grand-canonical Gibbs states of such gases to their mean-field (classical field) and large-mass (classical particle) limits. The former is a classical field theory for a complex scalar field with quartic self-interaction. The latter is a classical theory of point particles with two-body interactions. Our analysis is based on representations in terms of ensembles of interacting random loops, the *Ginibre loop ensemble* for Bose gases and the *Symanzik loop ensemble* for classical scalar field theories. For small enough interactions, our results also hold in infinite volume. The results of this paper were previously sketched in [11].

1. Introduction and main results

1.1. Overview. In this paper we study equilibrium properties of Bose gases and of systems emerging from Bose gases in various limiting regimes, by representing these systems in terms of ensembles of interacting random loops. These include the *Ginibre loop ensemble* [13–16], which describes an interacting Bose gas in thermal equilibrium, and the *Symanzik loop ensemble* [32], which describes the equilibrium state of an interacting classical field theory. The main goals of this paper are to highlight the usefulness of such random loop representations and to develop a unified approach to analyse their relationships and their behaviour in various limiting regimes.

Among different limiting regimes, we analyse the *mean-field* and the *large-mass* limits of interacting Bose gases using their random loop representations. These representations are particularly well-suited for proving results in infinite volume, and all our results also hold in infinite volume assuming the interaction strength is small enough. Our results on the interacting Bose gas are mostly new. We also obtain a new proof of convergence to the mean-field limit on a finite lattice, which was previsouly established using other methods.

For concreteness, we focus on the Euclidean lattice $\mathbb{Z}^d$, where the random loops are defined in terms of continuous-time simple random walks. With fairly straightforward modifications, our results and proofs – with the exception of convergence to the mean-field limit in dimensions $d > 1$, treated in separate papers [10, 26] – extend to continuum gases defined on $\mathbb{R}^d$, where the random loops are defined in terms of Brownian motion. For conciseness, we shall however not pursue this direction in the present paper.

Next, we describe the main results established in this paper and the methods used to prove them. For $d \in \mathbb{N}^*$ and $L \in \mathbb{N}^*$ we define the discrete cube $\Lambda := [-L/2, L/2)^d \cap \mathbb{Z}^d$. On Λ we define the discrete Laplacian

$$\Delta f(x) := \sum_{y : |y-x|=1} (f(y) - f(x)) , \qquad f : \Lambda \to \mathbb{C} , \tag{1.1}$$

with periodic boundary conditions on the cube Λ. For $T \geqslant 0$ and $x, y \in \Lambda$ we denote by $\Omega^T_{y,x}$ the set of càdlàg[1] paths $\omega : [0,T] \to \Lambda$ satisfying $\omega(0) = x$ and $\omega(T) = y$. We also abbreviate $\Omega^T := \bigcup_{x,y\in\Lambda} \Omega^T_{y,x}$ and $\Omega := \bigcup_{T\geqslant 0} \Omega^T$. For $\omega \in \Omega^T_{y,x}$, we use the shorthands $T(\omega) := T$, $x(\omega) := x$, and $y(\omega) := y$.

For $x \in \Lambda$ and $T > 0$, let $\mathbb{P}^T_x$ denote the law on Ω^T of the continuous-time random walk starting at x, which is by definition the Markovian jump process with generator $\Delta/2$. On Ω^T we define the path measures

$$\mathbb{W}^T_{y,x}(\mathrm{d}\omega) := \mathbf{1}_{\omega(T)=y}\, \mathbb{P}^T_x(\mathrm{d}\omega)\,, \qquad \mathbb{W}^T := \int \mathrm{d}x\, \mathbb{W}^T_{x,x}\,. \tag{1.2}$$

The measure $\mathbb{W}^T$ is the path measure for *closed paths* (i.e. loops) and $\mathbb{W}^T_{y,x}$ the path measure for *open paths* from x to y. Here we use the abbreviation $\int \mathrm{d}x := \sum_{x\in\Lambda}$.

By definition, a *loop ensemble* is a random point process Φ on the polish space Ω. That is, Φ is a random locally finite collection of elements of Ω (see e.g. [19]). To describe it in more detail, we suppose that we are given a *single-loop measure* $\mathbb{L}$, which is a positive measure on Ω. To simplify the presentation, we suppose here that $\mathbb{L}$ is finite (although in our proofs we shall have to abandon this assumption). As a point process, the loop ensemble is characterized by its *p-loop correlation functions* γ_p, $p \in \mathbb{N}^*$, which are defined through

$$\int f(\omega_1,\dots,\omega_p)\, \gamma_p(\omega_1,\dots,\omega_p)\, \mathbb{L}(\mathrm{d}\omega_1)\cdots\mathbb{L}(\mathrm{d}\omega_p) = \mathbb{E} \sum_{\omega_1,\dots,\omega_p\in\Phi}^{*} f(\omega_1,\dots,\omega_p)$$

for any nonnegative symmetric test function f, where the expectation is taken over the point process Φ and the sum is taken over all pairwise distinct p-tuples of loops in Φ. The *noninteracting loop ensemble* associated with the single-loop measure $\mathbb{L}$ is the Poisson point process on Ω with intensity measure $\mathbb{L}$. More concretely, in the noninteracting loop ensemble the loop configuration $(\omega_1,\dots,\omega_n)$ carries the weight

$$\frac{1}{Z}\frac{1}{n!}\mathbb{L}(\mathrm{d}\omega_1)\cdots\mathbb{L}(\mathrm{d}\omega_n)\,, \qquad Z = \sum_{n\in\mathbb{N}} \frac{1}{n!}\int \mathbb{L}(\mathrm{d}\omega_1)\cdots\mathbb{L}(\mathrm{d}\omega_n)\,. \tag{1.3}$$

Here the factor $1/n!$ compensates the overcounting from permuting the n loops. The p-loop correlation function of the noninteracting loop ensemble is simply 1 for all $p \in \mathbb{N}^*$.

In order to define an *interacting loop ensemble*, we introduce a *two-loop interaction* $\mathcal{V}(\omega,\tilde{\omega})$, which is a real-valued function on $\Omega\times\Omega$. This determines an n-loop interaction potential through

$$V(\omega_1,\dots,\omega_n) := \frac{1}{2}\sum_{i,j=1}^{n} \mathcal{V}(\omega_i,\omega_j)\,. \tag{1.4}$$

The interacting loop ensemble is then obtained from the corresponding noninteracting loop ensemble by weighting the contribution of each loop configuration $(\omega_1,\dots,\omega_n)$ in (1.3) by the Boltzmann factor $\mathrm{e}^{-V(\omega_1,\dots,\omega_n)}$. Recalling the definition (1.3), we then easily find that the p-loop correlation function of the interacting loop ensemble is

$$\gamma_p(\omega_1,\dots,\omega_p) = \frac{Z(\omega_1,\dots,\omega_p)}{Z}\,, \tag{1.5}$$

[1] Right-continuous with left limits.

where we defined the loop partition functions

$$Z(\omega_1,\dots,\omega_p) := \sum_{n\in\mathbb{N}} \frac{1}{n!} \int \mathbb{L}(\mathrm{d}\tilde\omega_1)\cdots\mathbb{L}(\mathrm{d}\tilde\omega_n)\, \mathrm{e}^{-V(\omega_1,\dots,\omega_p,\tilde\omega_1,\dots,\tilde\omega_n)}\,, \qquad Z := Z(\emptyset)\,. \tag{1.6}$$

Next, we explain the three interacting loop ensembles used in this paper and explain how they are related to each other. The *Symanzik loop ensemble* has the single-loop measure

$$\int \mathbb{L}^{\mathrm{cl}}(\mathrm{d}\omega) := \int_0^\infty \mathrm{d}T\, \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega)\,, \tag{1.7}$$

where $\kappa > 0$ is a killing rate (a negative chemical potential in physics terminology). The factor $1/T$ has the interpretation of compensating the overcounting arising from the arbitrary choice of the origin in the time interval $[0,T]$ parametrizing the closed loop. The factor $\mathrm{e}^{-\kappa T}$ entails an exponential suppression of long loops. We note that $\mathbb{L}^{\mathrm{cl}}$ is not finite, owing to the contribution of very short-lived loops; we temporarily ignore this issue here. In our proofs, we shall regularize $\mathbb{L}^{\mathrm{cl}}$ by truncating it at small values of T, and then show that the truncation can be removed in the quotient (1.5). The two-loop interaction of the Symanzik loop ensemble is

$$\mathcal{V}^{\mathrm{cl}}(\omega,\tilde\omega) := \int_0^{T(\omega)} \mathrm{d}t \int_0^{T(\tilde\omega)} \mathrm{d}\tilde t\, v(\omega(t)-\tilde\omega(\tilde t))\,, \tag{1.8}$$

where $v : \Lambda \to \mathbb{R}$ is a *two-body interaction potential*. We denote the associated n-loop interaction potential in (1.4) by $V^{\mathrm{cl}}(\omega_1,\dots,\omega_p)$

The Symanzik loop ensemble was introduced in [32] to describe interacting Euclidean field theories. To define a field theory of a complex scalar field ϕ on Λ with interaction potential v, we define the complex Gaussian measure with mean 0 and covariance $(-\Delta/2+\kappa)^{-1}$ through

$$\mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) := \frac{1}{\pi^{|\Lambda|}\det(-\Delta/2+\kappa)^{-1}}\, \mathrm{e}^{\langle\phi,(\Delta/2-\kappa)\phi\rangle}\, \mathrm{d}\phi\,, \tag{1.9}$$

where $\mathrm{d}\phi$ denotes Lebesgue measure on $\mathbb{C}^\Lambda$. The *(relative) classical partition function* is

$$\mathcal{Z}^{\mathrm{cl}} := \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\, \mathrm{e}^{-\frac12\int_\Lambda \mathrm{d}x\int_\Lambda \mathrm{d}y\, |\phi(x)|^2\, v(x-y)\, |\phi(y)|^2}\,. \tag{1.10}$$

The *classical p-point correlation function* is

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} := \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) \prod_{i=1}^p \bar\phi(y_i) \prod_{i=1}^p \phi(x_i)\, \mathrm{e}^{-\frac12\int_\Lambda \mathrm{d}x\int_\Lambda \mathrm{d}y\, |\phi(x)|^2\, v(x-y)\, |\phi(y)|^2}\,, \tag{1.11}$$

where $\mathbf{x},\mathbf{y} \in \Lambda^p$. As observed in [32], the relation between the classical correlation function of the interacting field theory $(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}}$ from (1.11) and the correlation function of the Symanzik loop ensemble $\gamma_p^{\mathrm{cl}}(\omega_1,\dots,\omega_p)$ from (1.5) is given by

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \sum_{\pi\in S_p} \Bigg(\prod_{i=1}^p \int_0^\infty \mathrm{d}T_i\, \mathrm{e}^{-\kappa T_i}\Bigg) \Bigg(\prod_{i=1}^p \int \mathbb{W}^{T_i}_{y_{\pi(i)},x_i}(\mathrm{d}\omega_i)\Bigg)\, \gamma_p^{\mathrm{cl}}(\omega_1,\dots,\omega_p)\,. \tag{1.12}$$

See Proposition 2.1 and Appendix A.1 below.

Next, we describe the *Ginibre loop ensemble*. It depends on three parameters $\nu, \kappa, \lambda > 0$. It has the single-loop measure

$$\int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\omega) := \nu \sum_{T \in \nu\mathbb{N}^*} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega)\,. \tag{1.13}$$

Note that (1.13) is obtained from (1.7) by replacing the Lebesgue integral $\int \mathrm{d}T$ with its discrete (Riemann-sum) approximation $\nu \sum_{T \in \nu\mathbb{N}^*}$. In the Ginibre loop ensemble, the two-loop interaction is

$$\mathcal{V}^{\nu,\lambda}(\omega, \tilde{\omega}) := \frac{\lambda}{\nu^2}\, \nu \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s<T(\omega)}\, \nu \sum_{\tilde{s} \in \nu\mathbb{N}} \mathbf{1}_{\tilde{s}<T(\tilde{\omega})}\, \frac{1}{\nu} \int_0^\nu \mathrm{d}t\, v(\omega(s+t) - \tilde{\omega}(\tilde{s}+t))\,, \tag{1.14}$$

and we denote the associated n-loop interaction potential by $V^{\nu,\lambda}(\omega_1, \dots, \omega_n)$ (see (1.4)). Thus, the Ginibre loop ensemble can be regarded as a discretized version of the Symanzik loop ensemble, where the times are constrained to belong to the lattice $\nu\mathbb{Z}$. See Figure 1.1 for an illustration of the Symanzik and Ginibre ensembles.

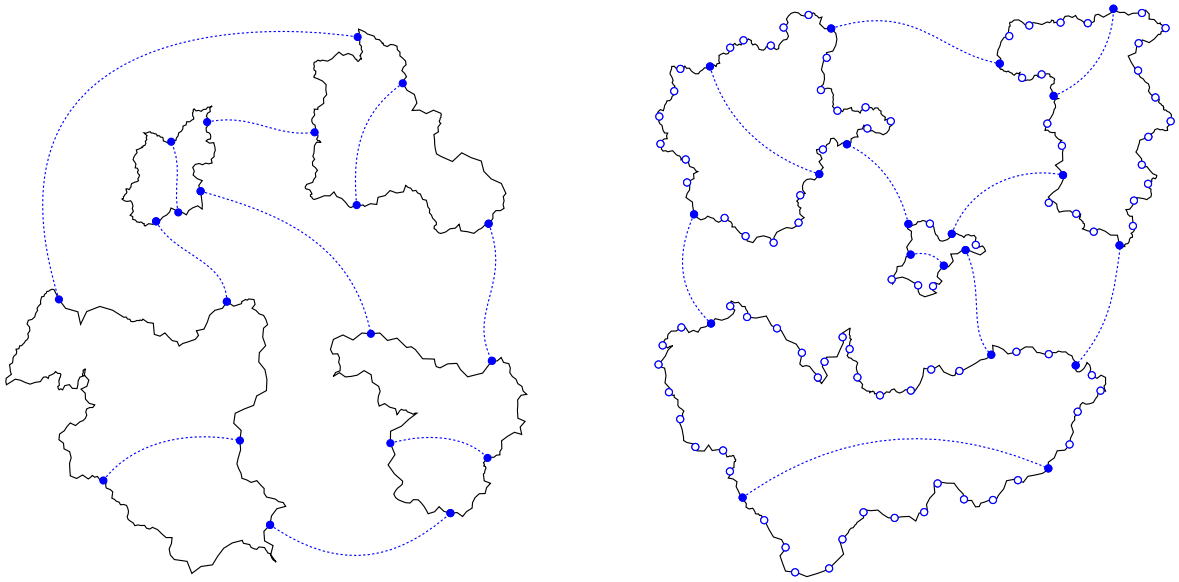

Figure 1.1. An illustration of the Symanzik (left) and Ginibre (right) loop ensembles. The random loops ω are drawn in black. An interaction $\mathcal{V}(\omega, \tilde{\omega})$ is drawn with a dotted blue line, joining the points $\omega(t)$ and $\tilde{\omega}(\tilde{t})$ that appear in the argument of the interaction potential v. Note that each loop can interact with each other loop and with itself. In the Ginibre ensemble, the duration of the loops is a multiple of ν, and the times at which the loops interact differ by integer multiples of ν, indicated using empty blue dots. In the Symanzik ensemble, all times are arbitrary real numbers.

The Ginibre loop ensemble was introduced in [13–16] to describe the statistical mechanics of interacting Bose gases at positive temperatures. A system of n spinless bosons of mass $m > 0$ confined to Λ is governed by the Hamiltonian

$$\mathbb{H}_n := -\sum_{i=1}^n \frac{\Delta_i}{2m} + \frac{\lambda}{2} \sum_{i,j=1}^n v(x_i - x_j)\,, \tag{1.15}$$

where Δ_i is the discrete Laplacian introduced in (1.1) acting on the variable x_i, $\lambda \geqslant 0$ is a coupling constant, and $v : \Lambda \to \mathbb{R}$ is a two-body interaction potential. The Hamiltonian (1.15) acts on the n-particle bosonic Hilbert space $\mathcal{H}_n := P_n^+ L^2(\Lambda^n)$, where

$$P_n^+ f(x_1, \dots, x_n) := \frac{1}{n!} \sum_{\pi \in S_n} f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

is the orthogonal projection onto the subspace of symmetric functions, and $L^2(\Lambda^n)$ is the L^2-space with respect to the counting measure on Λ^n.

We analyse the Bose gas in the *grand-canonical ensemble* at positive temperature. Its equilibrium state is described by a sequence of density matrices $(\rho_n)_{n\in\mathbb{N}}$ defined by

$$\rho_n := \frac{1}{\Xi} \mathrm{e}^{-\beta(\mathbb{H}_n - \mu n)} \quad \text{with} \quad \Xi := \sum_{n\in\mathbb{N}} \mathrm{Tr}\big(\mathrm{e}^{-\beta(\mathbb{H}_n - \mu n)}\big)\,.$$

Here $\beta > 0$ is the inverse temperature, $\mu < 0$ is the chemical potential, and Ξ is the *grand-canonical partition function*, which is a normalization constant chosen such that $\sum_n \mathrm{Tr}(\rho_n) = 1$.

Without loss of generality, we set $\beta = 1$ as it can be absorbed into the other parameters, and we replace the parameters m and μ with

$$\nu := \frac{1}{m} > 0\,, \qquad \kappa := -\mu m > 0\,. \tag{1.16}$$

Thus we find that the grand-canonical ensemble is characterized by the sequence $(\rho_n^{\nu,\kappa,\lambda})_{n\in\mathbb{N}}$, where

$$\rho_n^{\nu,\kappa,\lambda} := \frac{1}{\Xi^{\nu,\kappa,\lambda}} \mathrm{e}^{-(H_n^{\nu,\lambda} + \kappa\nu n)}\,, \qquad \Xi^{\nu,\kappa,\lambda} := \sum_{n\in\mathbb{N}} \mathrm{Tr}\big(\mathrm{e}^{-(H_n^{\nu,\lambda} + \kappa\nu n)}\big)\,, \tag{1.17}$$

and

$$H_n^{\nu,\lambda} := -\frac{\nu}{2} \sum_{i=1}^n \Delta_i + \frac{\lambda}{2} \sum_{i,j=1}^n v(x_i - x_j)\,. \tag{1.18}$$

The *reduced p-particle density matrix* of the grand-canonical ensemble is defined as

$$\Gamma_p^{\nu,\kappa,\lambda} = \sum_{n\in\mathbb{N}} \frac{(p+n)!}{n!} \mathrm{Tr}_{p+1,\dots,p+n}(\rho_{p+n}^{\nu,\kappa,\lambda})\,, \tag{1.19}$$

where $\mathrm{Tr}_{p+1,\dots,p+n}$ denotes the partial trace over the coordinates $x_{p+1}, \dots, x_{p+n}$. We denote by $(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}}$ the operator kernel of $\Gamma_p^{\nu,\kappa,\lambda}$. As observed in [13–16], the relation between the reduced density matrix $(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}}$ from (1.19) and the correlation function of the Ginibre loop ensemble $\gamma_p^{\nu,\kappa,\lambda}(\omega_1, \dots, \omega_p)$ from (1.5) is given by

$$(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}} = \frac{1}{\nu^p} \sum_{\pi\in S_p} \left(\prod_{i=1}^p \nu \sum_{T_i \in \nu\mathbb{N}^*} \mathrm{e}^{-\kappa T_i} \right) \left(\prod_{i=1}^p \int \mathbb{W}^{T_i}_{y_{\pi(i)}, x_i}(\mathrm{d}\omega_i) \right) \gamma_p^{\nu,\kappa,\lambda}(\omega_1, \dots, \omega_p)\,. \tag{1.20}$$

See Proposition 2.3 and Appendix A.2 below.

In this paper we analyse various limiting regimes of the Ginibre loop ensemble.

(a) *The mean-field (or classical field) limit:* $\nu \to 0$, $\lambda = \nu^2$, κ fixed. Recalling (1.16), we see that this amounts to a high temperature and high density limit, or, equivalently, to a large mass and large chemical potential limit. Our main result is the convergence of the Ginibre loop ensemble to the Symanzik loop ensemble,

$$\lim_{\nu \to 0} \gamma_p^{\nu,\kappa,\nu^2} = \gamma_p^{\mathrm{cl}} \,. \tag{1.21}$$

At a formal level, this convergence is plausible after comparing the single-loop measures (1.7), (1.13) and the two-loop interactions (1.8), (1.14). As a corollary, using the representations (1.12), (1.20), we deduce the convergence of the rescaled reduced density matrices of the quantum Bose gas to the correlation functions of the classical field theory:

$$\lim_{\nu \to 0} \nu^p \, \Gamma_p^{\nu,\kappa,\nu^2} = \Gamma_p^{\mathrm{cl}} \,; \tag{1.22}$$

see Theorem 1.2.

(b) *The large-mass limit:* $\nu \to 0$, $\lambda = 1$, $\kappa = \kappa_0/\nu$ for some fixed κ_0. Recalling (1.16), we see that this amounts to the limit of large m for fixed β and μ. In this limit, we show that the Ginibre loop ensemble converges to an ensemble of interacting classical particles,

$$\lim_{\nu \to 0} \gamma_p^{\nu,\kappa_0/\nu,1} = \gamma_p^{\mathrm{lm}} \,, \tag{1.23}$$

where γ_p^{lm} is the correlation function of a process of interacting weighted particles which we describe in more detail below[2]. We conclude convergence of the reduced density matrices

$$\lim_{\nu \to 0} \Gamma_p^{\nu,\kappa_0/\nu,1} = \Gamma_p^{\mathrm{lm}} \,, \tag{1.24}$$

where Γ_p^{lm} is a classical correlation function defined in terms of a process of interacting weighted particles. See Theorem 1.4.

In addition, we extend both convergence results to the thermodynamic limit $L \to \infty$, under the assumption that the two-body potential v is of short range and not too large. Indicating the L-dependence of all quantities with a superscript, we extend (1.22) to

$$\lim_{\nu \to 0} \lim_{L \to \infty} \nu^p \, \Gamma_p^{\nu,\kappa,\nu^2,L} = \lim_{L \to \infty} \Gamma_p^{\mathrm{cl},L} \tag{1.25}$$

where all limits exist; see Theorem 1.6. Similarly, we extend (1.24) to

$$\lim_{\nu \to 0} \lim_{L \to \infty} \Gamma_p^{\nu,\kappa_0/\nu,1,L} = \lim_{L \to \infty} \Gamma_p^{\mathrm{lm},L} \,, \tag{1.26}$$

where all limits exist. See Theorem 1.8.

The convergence (1.22) is not new and was previously established in [20], using different methods. The convergence of the loop ensembles (1.21), however, and the other three results (1.24), (1.25), and (1.26) appear to be new.

Finally, we describe the process of interacting weighted particles that emerges in the large-mass limit (b). It may be formulated as an ensemble of interacting stationary loops of integer time length.

[2] In the continuum $\mathbb{R}^d$ instead of the lattice $\mathbb{Z}^d$, with the correct choice of κ as a function of ν, the limit $\nu \to 0$ describes a classical gas of point particles with two-body interactions given by the potential v.

To describe this precisely, introduce the measure $\mathbb{D}_x^T$ as the atomic measure on Ω^T at the constant loop $\omega(t) = x$ for all $t \in [0,T]$. The loop ensemble in the large-mass limit has the single-loop measure

$$\int \mathbb{D}^{\mathrm{lm}}(\mathrm{d}\omega) := \sum_{k\in\mathbb{N}^*} \frac{\mathrm{e}^{-\kappa_0 k}}{k} \int_\Lambda \mathrm{d}x \int \mathbb{D}_x^k(\mathrm{d}\omega)\,,$$

and the two-loop interaction is

$$\mathcal{V}^{\mathrm{lm}}(\omega,\tilde\omega) := \sum_{0\leqslant k<T(\omega)} \sum_{0\leqslant \tilde k<T(\tilde\omega)} \int_0^1 \mathrm{d}t\, v(\omega(k+t)-\tilde\omega(\tilde k+t))\,.$$

The right-hand side of (1.23) is by definition the p-loop correlation function of this loop ensemble, and analogously the right-hand side of (1.24) is given by

$$(\Gamma_p^{\mathrm{lm}})_{\mathbf{x},\mathbf{y}} := \sum_{\pi\in S_p} \left(\prod_{i=1}^p \sum_{k_i\in\mathbb{N}^*} \mathrm{e}^{-\kappa_0 k_i}\, \delta(y_{\pi(i)}-x_i) \int \mathbb{D}_{x_i}^{k_i}(\mathrm{d}\omega_i) \right) \gamma_p^{\mathrm{lm}}(\omega_1,\dots,\omega_p)\,.$$

The interpretation is that in the large-mass limit, only loops of duration of order ν contribute, in which time the simple random walk cannot make even a single jump. Loops thus collapse to points. We can make this collapse more explicit by parametrizing a stationary loop ω with its location $x \in \Lambda$ and its duration $k\nu$, where $k \in \mathbb{N}^*$. The couple (k,x) describes a weighted particle, where k is the occupation number and x the position. Using this parametrization, the single-loop measure and the two-loop interaction become

$$\int \mathbb{D}^{\mathrm{lm}}(\mathrm{d}(k,x)) = \sum_{k\in\mathbb{N}^*} \frac{\mathrm{e}^{-\kappa_0 k}}{k} \int_\Lambda \mathrm{d}x\,, \qquad \mathcal{V}^{\mathrm{lm}}((k,x),(\tilde k,\tilde x)) = k\,\tilde k\, v(x-\tilde x)\,.$$

It is of some interest to consider also interaction potentials with a hard core repulsion, i.e. $v(0) = +\infty$. In that case the interaction energy in (1.18) is always infinite, but we can renormalize it by omitting the diagonal terms $i = j$. Then the density matrices in (1.17) and the Ginibre loop ensembles are well defined. For systems with hard core two-body potentials, only loops of duration ν contribute in the large-mass limit, i.e. all occupation numbers k are zero or one. In particular, one can see that loops of duration $k\nu$ with $k \geqslant 2$ are eliminated by a self-interaction, which is absent from loops of duration ν. The interpretation is that, due to hard core repulsion, multiple occupancies of a single site are excluded.

1.2. Statement of the main results. Let

$$\mathcal{Z}^{\nu,\kappa,\lambda} := \frac{\Xi^{\nu,\kappa,\lambda}}{\Xi^{\nu,\kappa,0}} \tag{1.27}$$

be the relative grand-canonical partition function, where $\Xi^{\nu,\kappa,\lambda}$ was defined in (1.17).

We first state our result on the mean-field regime on a finite lattice. We begin by stating precise assumptions on the interaction potential v.

Assumption 1.1. We consider an interaction potential $v : \Lambda \to \mathbb{R}$ which is pointwise nonnegative and of positive type (i.e. its Fourier transform is pointwise nonnegative).

Theorem 1.2 (Mean-field limit on a finite lattice)**.** *With v as in Assumption 1.1, the following limits hold.*

(i) $\lim_{\nu\to 0} \mathcal{Z}^{\nu,\kappa,\nu^2} = \mathcal{Z}^{\mathrm{cl}}$.

(ii) For all $p \in \mathbb{N}^*$ *and* $\mathbf{x}, \mathbf{y} \in \Lambda^p$, *we have* $\lim_{\nu\to 0} \nu^p \, (\Gamma_p^{\nu,\kappa,\nu^2})_{\mathbf{x},\mathbf{y}} = (\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}}$.

Next, we state our result on the large-mass limit on a finite lattice. In this regime, we modify Assumption 1.1 to account for interaction potentials that can have a hard core.

Assumption 1.3. We consider an interaction potential $v : \Lambda \to [0,\infty]$ for which there exists $R \in \{0,1\}$ such that (i) $v(x) \in [0,\infty)$ for $|x| \geqslant R$. (ii) $v(x) = \infty$ for $|x| < R$.

We note that the analysis can be extended to consider arbitrary $R \geqslant 0$, but we consider $R \in \{0,1\}$ for simplicity. As remarked in the overview, we take $\lambda = 1$ and $\kappa = \kappa_0/\nu$ for some fixed κ_0. In light of Assumption 1.3, we modify (1.18) and work with

$$H_n^{\nu,1} = -\frac{\nu}{2} \sum_{i=1}^n \Delta_i + \frac{1}{2} \sum_{i,j=1}^n (1 - \mathbf{1}_{i=j} \mathbf{1}_{R=1}) v(x_i - x_j) \,, \tag{1.28}$$

thereby eliminating the infinite self-interaction in the presence of a hard core interaction potential.

Given $n \in \mathbb{N}^*, \mathbf{k} \in (\mathbb{N}^*)^n, \mathbf{x} \in \Lambda^n$, we define

$$V^{\mathrm{lm}}(\mathbf{k},\mathbf{x}) := \begin{cases} \frac{1}{2} \sum_{i,j=1}^n k_i k_j v(x_i - x_j) & \text{if } R = 0 \\ \frac{1}{2} \sum_{i,j=1}^n v(x_i - x_j) \, \mathbf{1}_{i\neq j} \, \mathbf{1}_{\mathbf{k}=\mathbf{1}_n} + \infty \, \mathbf{1}_{\mathbf{k}\neq\mathbf{1}_n} & \text{if } R = 1 \,. \end{cases} \tag{1.29}$$

Here, we write

$$\mathbf{1}_n := (1, \dots, 1) \in (\mathbb{N}^*)^n \,. \tag{1.30}$$

We then consider the (relative) classical partition function with infinite mass, which is defined as

$$\mathcal{Z}^{\mathrm{lm}} := \left\{ \sum_{n=0}^\infty \frac{1}{n!} \sum_{\mathbf{k}\in(\mathbb{N}^*)^n} \prod_{i=1}^n \frac{1}{k_i} \int_{\Lambda^n} \mathrm{d}\mathbf{x} \, \mathrm{e}^{-\kappa_0|\mathbf{k}|} \exp\bigl(-V^{\mathrm{lm}}(\mathbf{k},\mathbf{x})\bigr) \right\} \Bigg/ \exp\left\{ \sum_{k=1}^\infty \frac{\mathrm{e}^{-\kappa_0 k}}{k} |\Lambda| \right\} . \tag{1.31}$$

Furthermore, for $p \in \mathbb{N}^*$ and $\mathbf{x}, \mathbf{y} \in \Lambda^p$, we consider

$$(\Gamma_p^{\mathrm{lm}})_{\mathbf{x},\mathbf{y}} := \sum_{\mathbf{k}\in(\mathbb{N}^*)^p} \sum_{\pi\in S_p} \mathrm{e}^{-\kappa_0|\mathbf{k}|} \, \delta(\pi\mathbf{y} - \mathbf{x}) \, \frac{Z^{\mathrm{lm}}(\mathbf{k},\mathbf{x})}{Z^{\mathrm{lm}}} \,, \tag{1.32}$$

where

$$Z^{\mathrm{lm}}(\mathbf{k},\mathbf{x}) = \sum_{n=0}^\infty \frac{1}{n!} \sum_{\tilde{\mathbf{k}}\in(\mathbb{N}^*)^n} \prod_{i=1}^n \frac{1}{\tilde{k}_i} \int_{\Lambda^n} \mathrm{d}\tilde{\mathbf{x}} \, \mathrm{e}^{-\kappa_0|\tilde{\mathbf{k}}|} \exp\bigl(-V^{\mathrm{lm}}(\mathbf{k}\tilde{\mathbf{k}}, \mathbf{x}\tilde{\mathbf{x}})\bigr) \,, \qquad Z^{\mathrm{lm}} = Z^{\mathrm{lm}}(\emptyset) \,. \tag{1.33}$$

Here S_p is the set of permutations of $\{1,\dots,p\}$, $\pi\mathbf{y} = (y_{\pi(1)}, \dots, y_{\pi(p)})$, and we use the notation (2.1) below.

We prove the following result.

Theorem 1.4 (Large-mass limit on a finite lattice)**.** *With* v *as in Assumption 1.3, and with notations as in* (1.31)–(1.32)*, the following holds.*

(i) $\lim_{\nu\to 0} \mathcal{Z}^{\nu,\kappa_0/\nu,1} = \mathcal{Z}^{\mathrm{lm}}$.

(ii) For all $p \in \mathbb{N}^*$ *and* $\mathbf{x}, \mathbf{y} \in \Lambda^p$, *we have that* $\lim_{\nu\to 0} (\Gamma_p^{\nu,\kappa_0/\nu,1})_{\mathbf{x},\mathbf{y}} = (\Gamma_p^{\mathrm{lm}})_{\mathbf{x},\mathbf{y}}$.

Next, we extend the results of Theorems 1.2 and 1.4 to the infinite lattice $\mathbb{Z}^d$. To that end, we explicitly include the side length L of the cube $\Lambda \equiv \Lambda_L$ in our notation.

Assumption 1.5. We consider $v : \mathbb{Z}^d \to \mathbb{R}$ which satisfies the following properties. (i) v is pointwise nonnegative. (ii) v is of positive type. (iii) $v \in \ell^1(\mathbb{Z}^d)$.

With v given as in Assumption 1.5, we work with $v^L : \Lambda_L \to \mathbb{R}$ given by

$$v^L(x) := \sum_{k \in (L\mathbb{Z})^d} v(x+k)\,. \tag{1.34}$$

Furthermore, we define the *specific (relative) Gibbs potential* of the Bose gas by

$$g^{\nu,\kappa,\lambda,L} := \frac{1}{|\Lambda_L|} \log \mathcal{Z}^{\nu,\kappa,\lambda,L} \tag{1.35}$$

and the *classical specific (relative) Gibbs potential* by

$$g^{\mathrm{cl},L} := \frac{1}{|\Lambda_L|} \log \mathcal{Z}^{\mathrm{cl},L}\,. \tag{1.36}$$

In order to study the convergence of the reduced density matrices, we need to define an appropriate norm. Given $p \in \mathbb{N}^*$ and $L_0 \in \mathbb{N}^*$, we define

$$\Pi_{L_0,p} := P_{L_0}^{\otimes p}\,(\cdot)\,P_{L_0}^{\otimes p}\,, \tag{1.37}$$

where $P_{L_0} : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda_{L_0})$ denotes the canonical orthogonal projection. With $\Pi_{L_0,p}$ given as in (1.37), and for A an operator on $\ell^2(\mathbb{Z}^d)^{\otimes p}$, we define

$$\|A\|_{L_0,p} := \|\Pi_{L_0,p}\,A\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}}\,. \tag{1.38}$$

Theorem 1.6 (Infinite volume mean-field limit)**.** *Let v be as in Assumption 1.5. If $\|v\|_{\ell^1(\mathbb{Z}^d)}$ is sufficiently small depending on κ, the following limits exist and satisfy the following relations.*

(i) We have

$$\lim_{\nu\to 0}\lim_{L\to\infty} g^{\nu,\kappa,\nu^2,L} = \lim_{L\to\infty} g^{\mathrm{cl},L}\,.$$

(ii) Fix $p \in \mathbb{N}^*$. Then for any $L_0 \in \mathbb{N}^*$ we have

$$\lim_{\nu\to 0}\lim_{L\to\infty} \nu^p\,\Gamma^{\nu,\kappa,\nu^2,L} = \lim_{L\to\infty} \Gamma_p^{\mathrm{cl},L}$$

with respect to $\|\cdot\|_{L_0,p}$, uniformly in L_0.

When studying the large-mass limit in the infinite volume, we modify Assumption 1.5 as follows.

Assumption 1.7. We consider an interaction potential $v : \mathbb{Z}^d \to [0,\infty]$ for which there exists $R \in \{0,1\}$ such that (i) $v\,\mathbf{1}_{|x|\geqslant R} \in \ell^1(\mathbb{Z}^d)$ and (ii) $v(x) = \infty$ for $|x| < R$.

We consided the Hamiltonian defined in (1.28) with interaction potential $v^L : \Lambda_L \to [0,\infty]$, which is given by (1.34) with v as in Assumption 1.7, and modify notations accordingly.

Theorem 1.8 (Infinite volume large-mass limit)**.** *Let v be as in Assumption 1.7. If $\|v\|_{\ell^1(\mathbb{Z}^d)}$ is sufficiently small in terms of κ_0, the following limits exist and satisfy the following relations.*

(i) We have

$$\lim_{\nu\to 0}\lim_{L\to\infty} g^{\nu,\kappa_0/\nu,1,L} = \lim_{L\to\infty} g^{\mathrm{lm},L},$$

where $g^{\mathrm{lm},L} := \frac{1}{|\Lambda_L|}\log \mathcal{Z}^{\mathrm{lm},L}$.

(ii) Fix $p \in \mathbb{N}^*$. *Then for any* $L_0 \in \mathbb{N}^*$ *we have*

$$\lim_{\nu\to 0}\lim_{L\to\infty} \Gamma_p^{\nu,\kappa_0/\nu,1,L} = \lim_{L\to\infty} \Gamma_p^{\mathrm{lm},L}$$

with respect to $\|\cdot\|_{L_0,p}$, *uniformly in* L_0.

Remark 1.9. We note that the results stated in Theorems 1.6 and 1.8 extend to more general boundary conditions for (1.1) by using very similar arguments. This can be seen by appropriately modifying the proofs of Propositions 5.5 and 5.20 below. We study periodic boundary conditions for concreteness.

Remark 1.10. By using subadditivity arguments [28] one can show convergence of the thermodynamic potentials (as $L \to \infty$) for all temperatures.

Remark 1.11. When studying the mean-field limit, the assumption that v is of positive type and pointwise nonnegative (see Assumption 1.1 and 1.5 above) is needed purely for mathematical reasons. More precisely, the assumption that v is of positive type is needed to apply the Hubbard-Stratonovich formula (A.3), which is the starting point of the proof of the Symanzik loop representation, stated in Proposition 2.1. Furthermore, the proof of the Ginibre loop representation, stated in Proposition 2.3, relies on the pointwise nonnegativity of v.

1.3. Related results. The methods used in this paper are inspired by representations of Bose gases and Euclidean field theories in terms of interacting random loops developed by Ginibre [13–16] and Symanzik [32], respectively. We note that the mean-field limit was studied for $d = 1, 2, 3$ in the continuum in the work of Lewin, Nam, and Rougerie [23–26], in our previous work [8,10–12], in the work of the fourth author [30], as well as in the work of Rout and the fourth author [27]. When studying the limiting regime of Bose gases, as $\nu \to 0$, in dimensions $d = 2, 3$ in the continuum, it is necessary to introduce a renormalization in the form of Wick ordering. This serves to control ultraviolet (short-distance) singularities. The latter do not occur in the study of lattice Bose gases (see also Appendix C). We note that the results in [8] have been extended to gases with singular interaction potentials in [30], and *time-dependent* correlation functions in one dimension have been constructed and studied in [9].

Cluster expansions, which we use to extend our results to the infinite lattice, are ubiquitous in statistical mechanics. For further results on cluster expansions and their applications, we refer the reader to [2, 5, 17, 18] and the references therein. Concerning the infinite-volume limit in the continuum, we note that the normalization of the classical Gibbs measure and its distributional properties in the limit $L \to \infty$ have been studied in [1].

In the recent paper [29], a construction of regularized coherent-state functional integrals for ensembles of bosons on a lattice is given. As in [10], an important tool is the Hubbard-Stratonovich formula.

2. The Symanzik and Ginibre representations

2.1. Notation. Let us first introduce some notation that we will use throughout the paper. We write $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, 3, \dots\}$. We use the notation $A_{x,y}$ for the operator kernel of an operator A with respect to the counting measure. When working on a fixed spatial domain Λ, we write $\int \mathrm{d}x$ to mean $\int_\Lambda \mathrm{d}x \equiv \sum_{x\in\Lambda}$, if there is no possibility for confusion. We denote by $\mu_{\mathcal{C}}$ a Gaussian measure with covariance $\mathcal{C}$. We state all of the properties of $\mu_{\mathcal{C}}$ that we use throughout the paper in Appendix B.

Given $a, b \geqslant 0$, we write $a \lesssim b$ if there exists $C > 0$ such that $a \leqslant Cb$. We sometimes also write this as $a = O(b)$. Furthermore, if C depends on a set of parameters $p_1, \dots, p_k$, we write this as $C = C_{p_1,\dots,p_k}$ or $a \lesssim_{p_1,\dots,p_k} b$, $a = O_{p_1,\dots,p_k}(b)$. $\mathbf{1}_X$ denotes the indicator function of a set X.

We use the notation

$$\mathbf{x}\mathbf{y} = (x_1, \dots, x_p, y_1, \dots, y_n) \tag{2.1}$$

for vectors $\mathbf{x} = (x_1, \dots, x_p), \mathbf{y} = (y_1, \dots, y_n)$. Moreover, the symmetric group S_p acts on p-component vectors $\mathbf{x} = (x_1, \dots, x_p)$ according to $\pi\mathbf{x} = (x_{\pi(1)}, \dots, x_{\pi(p)})$.

For $n \in \mathbb{N}^*$ and $\mathbf{T} \in (0,\infty)^n$, denote by $\Omega^{\mathbf{T}} := \Omega^{T_1} \times \cdots \times \Omega^{T_n}$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in \Omega^{\mathbf{T}}$ where $\omega_i \in \Omega^{T_i}$. For $\mathbf{x}, \mathbf{y} \in \Lambda^n$, we define the product measure on $\Omega^{\mathbf{T}}$ by $\mathbb{W}^{\mathbf{T}}_{\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) := \mathbb{W}^{T_1}_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \mathbb{W}^{T_n}_{y_n,x_n}(\mathrm{d}\omega_n)$. Given $\mathbf{T} \in (0,\infty)^p$, we write $|\mathbf{T}| := \sum_{i=1}^p T_i$. Analogously to (2.1), given $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n) \in \Omega^{\mathbf{T}}$ and $\tilde{\boldsymbol{\omega}} = (\tilde{\omega}_1, \dots, \tilde{\omega}_m) \in \Omega^{\tilde{\mathbf{T}}}$, we write

$$\boldsymbol{\omega}\tilde{\boldsymbol{\omega}} = (\omega_1, \dots, \omega_n, \tilde{\omega}_1, \dots, \tilde{\omega}_m) \in \Omega^{\mathbf{T}\tilde{\mathbf{T}}} .$$

2.2. The Symanzik representation. In this section, we derive the Symanzik representation for the classical partition function (1.10) and for the classical p-particle correlation functions (1.11) following [32]. The precise statement is given in Proposition 2.1 below. In Corollary 2.2, we use the methods from the proof of Proposition 2.1 to give a proof of a correlation inequality for classical p-particle correlation functions. The proofs of both Proposition 2.1 and Corollary 2.2 are given in Appendix A.1.

In this subsection, the interaction $V \equiv V^{\mathrm{cl}}$ is given by (1.4) using (1.8). For the single-loop measure, we will work with a suitable regularization of $\mathbb{L}^{\mathrm{cl}}$ given by (1.7). In particular, given $\varepsilon > 0$, we consider $\mathbb{L}^{\mathrm{cl},\varepsilon}$ given by

$$\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega) := \int_\varepsilon^\infty \mathrm{d}T \, \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega) \, . \tag{2.2}$$

We define

$$\mathcal{Z}^{\mathrm{cl},\varepsilon} := \sum_{n=0}^\infty \frac{1}{n!} \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_1) \cdots \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_n) \, \exp(-V^{\mathrm{cl}}(\boldsymbol{\omega})) \, \exp(K^\varepsilon) \, , \tag{2.3}$$

where

$$K^\varepsilon := -\int_\varepsilon^\infty \frac{\mathrm{d}T}{T} \, \mathrm{e}^{-\kappa T} \int \mathbb{W}^T(\mathrm{d}\omega) \, . \tag{2.4}$$

Note that $\lim_{\varepsilon\to 0} K^\varepsilon = -\infty$. Furthermore, we let

$$(\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}} := \sum_{\pi\in S_p} \int_{(0,\infty)^p} \mathrm{d}\mathbf{T} \, \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \, \frac{\mathcal{Z}^{\mathrm{cl},\varepsilon}(\boldsymbol{\omega})}{\mathcal{Z}^{\mathrm{cl}}} \, , \tag{2.5}$$

where

$$\mathcal{Z}^{\mathrm{cl},\varepsilon}(\boldsymbol{\omega}) := \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\tilde{\omega}_1) \cdots \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\tilde{\omega}_n) \, \exp(-V^{\mathrm{cl}}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) \, \exp(K^{\varepsilon}) \, . \tag{2.6}$$

Note that (2.6) reduces to (2.3) when $\boldsymbol{\omega} = \emptyset$.

Proposition 2.1 (Symanzik loop representation)**.** *Let v be of positive type. With notation as in (1.10), (1.11), (2.3), and (2.5), the following claims hold.*

(i) $\mathcal{Z}^{\mathrm{cl}} = \lim_{\varepsilon \to 0} \mathcal{Z}^{\mathrm{cl},\varepsilon}$.

(ii) For all $p \in \mathbb{N}^*$ and $\mathbf{x}, \mathbf{y} \in \Lambda^p$, we have $(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \lim_{\varepsilon \to 0} (\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}}$.

We now introduce a coupling constant $\lambda \geqslant 0$ in front of the interaction v. This changes the exponential weight in (1.10). Note that if v is of positive type, then so is λv. The proof of Proposition 2.1 (see Appendix A.1) allows us to deduce a correlation inequality for the (unnormalized) classical p-point correlation function

$$(\widehat{\Gamma}_p^{\mathrm{cl},\lambda})_{\mathbf{x},\mathbf{y}} := \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) \, \bar{\phi}(y_1) \cdots \bar{\phi}(y_p) \, \phi(x_1) \cdots \phi(x_p) \, \mathrm{e}^{-\frac{\lambda}{2} \int \mathrm{d}x \int \mathrm{d}y \, |\phi(x)|^2 \, v(x-y) \, |\phi(y)|^2} \, . \tag{2.7}$$

Corollary 2.2 (Correlation inequality)**.** *Let v be of positive type. Given $p \in \mathbb{N}^*$, and $\mathbf{x}, \mathbf{y} \in \Lambda^p$, with notation as in (2.7), we have*

$$0 \leqslant (\widehat{\Gamma}_p^{\mathrm{cl},\lambda})_{\mathbf{x},\mathbf{y}} \leqslant (\widehat{\Gamma}_p^{\mathrm{cl},0})_{\mathbf{x},\mathbf{y}} \, . \tag{2.8}$$

2.3. The Ginibre representation. In this section, we recall the *Ginibre loop representation* of the reduced p-particle density matrices and of the partition function. These results appeared originally in the work of Ginibre [13–16].

Proposition 2.3 (Ginibre loop representation)**.** *Let v be pointwise nonnegative. Recalling (1.5)–(1.6), the operator kernel of $\Gamma_p^{\nu,\kappa,\lambda}$ defined in (1.19) satisfies identity (1.20). Here, the single loop measure $\mathbb{L} \equiv \mathbb{L}^{\nu,\lambda}$ is given by (1.13) and the interaction $V \equiv V^{\nu,\lambda}$ is given by (1.4) using (1.14). Furthermore, we can write (1.27) as*

$$\mathcal{Z}^{\nu,\kappa,\lambda} = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\omega_1) \cdots \mathbb{L}^{\nu,\kappa}(\mathrm{d}\omega_n) \, \exp(-V^{\nu,\lambda}(\boldsymbol{\omega})) \right\} \Big/ \exp\left\{ \int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\omega) \right\} . \tag{2.9}$$

For completeness, we give the proof of Proposition 2.3 in Appendix A.2.

It is useful to rewrite the Ginibre representation of (1.19) in terms of relative quantities.

Remark 2.4. Let us define

$$\Xi^{\nu,\kappa,\lambda}(\boldsymbol{\omega}) := \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\tilde{\omega}_1) \cdots \mathbb{L}^{\nu,\kappa}(\mathrm{d}\tilde{\omega}_n) \, \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) \right\} , \quad \Xi^{\nu,\kappa,\lambda} := \Xi^{\nu,\kappa,\lambda}(\emptyset) \, , \tag{2.10}$$

and we let $\mathcal{Z}^{\nu,\kappa,\lambda}(\boldsymbol{\omega}) := \frac{\Xi^{\nu,\kappa,\lambda}(\boldsymbol{\omega})}{\Xi^{\nu,\kappa,\lambda}}$. Then, Proposition 2.3 implies that for $p \in \mathbb{N}^*$ and $\mathbf{x}, \mathbf{y} \in \Lambda^p$, we have

$$(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}} = \sum_{\pi \in S_p} \sum_{\mathbf{T} \in (\nu\mathbb{N}^*)^p} \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \, \frac{\mathcal{Z}^{\nu,\kappa,\lambda}(\boldsymbol{\omega})}{\mathcal{Z}^{\nu,\kappa,\lambda}} \, . \tag{2.11}$$

3. The mean-field limit: Convergence of the Ginibre representation to the Symanzik representation

In this section, we study the mean-field convergence on the finite lattice Λ and we prove Theorem 1.2. Let us recall that in this regime, we consider $\nu \to 0$ with $\lambda = \nu^2$ and κ fixed. Throughout the section, we assume that v satisfies Assumption 1.1. Recalling (1.8) and (1.14), by pointwise nonnegativity of v, we have that

$$\mathcal{V}^{\mathrm{cl}}(\omega, \tilde{\omega}) \geqslant 0\,, \quad \mathcal{V}^{\nu,\nu^2}(\omega, \tilde{\omega}) \geqslant 0\,. \tag{3.1}$$

Given $n \in \mathbb{N}^*$, and $\mathbf{T} \in (0,\infty)^n$, in the sequel we write

$$\mathbb{W}^{\mathbf{T}}(\mathrm{d}\boldsymbol{\omega}) := \int_{\Lambda^n} \mathrm{d}\mathbf{x}\, \mathbb{W}^T_{\mathbf{x},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega})\,. \tag{3.2}$$

We denote by

$$\psi^t(x) := (\mathrm{e}^{t\Delta/2})_{x,0} \tag{3.3}$$

the heat kernel on Λ. Let us note that for $x, y \in \Lambda$ and $t > 0$ we have

$$\int \mathbb{W}^t_{y,x}(\mathrm{d}\omega) = \psi^t(y-x)\,. \tag{3.4}$$

Further properties of the heat kernel on the lattice are given in Appendix C.

Proof of Theorem 1.2. Let us first prove (i). For $\boldsymbol{\omega} \in \Omega^{\mathbf{T}}$, we can write $V^{\nu,\nu^2}(\boldsymbol{\omega}) = \frac{1}{2}\langle f, \nu v f\rangle$, where

$$f(x) = \sum_{i=1}^n \sum_{r_i \in \nu\mathbb{N}} \mathbf{1}_{r_i<T_i} \int_0^\nu \mathrm{d}t\, \delta(x - \omega_i(t+r_i))\,. \tag{3.5}$$

We rewrite $\mathcal{Z}^{\nu,\kappa,\nu^2}$ by starting from (2.9). By using the Hubbard-Stratonovich formula (A.3) for f as in (3.5), collecting terms in the exponential, and recalling (1.13) we can write

$$\mathcal{Z}^{\nu,\kappa,\nu^2} = \int \mu_{\nu v}(\mathrm{d}\sigma)\, \exp\left\{ \int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\omega) \left[\mathrm{e}^{\mathrm{i}\sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \int_0^\nu \mathrm{d}t\, \sigma(\omega(t+r))} - 1 \right] \right\}. \tag{3.6}$$

Given $\varepsilon > 0$, we let

$$\mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon} := \int \mu_{\nu v}(\mathrm{d}\sigma)\, \exp\left\{ \nu \sum_{T\in\nu\mathbb{N}^*\cap[\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega) \left[\mathrm{e}^{\mathrm{i}\sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T} \int_0^\nu \mathrm{d}t\, \sigma(\omega(t+r))} - 1 \right] \right\}. \tag{3.7}$$

We first show that there exists $\nu_0 > 0$ sufficiently small such that

$$\lim_{\varepsilon\to 0} \mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon} = \mathcal{Z}^{\nu,\kappa,\nu^2}\,, \quad \text{uniformly in } \nu \in (0,\nu_0)\,. \tag{3.8}$$

In order to prove (3.8), we use (3.6)–(3.7) to compute

$$\begin{aligned}
&\mathcal{Z}^{\nu,\kappa,\nu^2} - \mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon} \\
&\quad = \int \mu_{\nu v}(\mathrm{d}\sigma)\, \exp\left\{ \nu \sum_{T\in\nu\mathbb{N}^*\cap[\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega) \left[\mathrm{e}^{\mathrm{i}\sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T} \int_0^\nu \mathrm{d}t\, \sigma(\omega(t+r))} - 1 \right] \right\} \\
&\qquad \times \left(\exp\left\{ \nu \sum_{T\in\nu\mathbb{N}^*\cap(0,\varepsilon)} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega) \left[\mathrm{e}^{\mathrm{i}\sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T} \int_0^\nu \mathrm{d}t\, \sigma(\omega(t+r))} - 1 \right] \right\} - 1 \right).
\end{aligned} \tag{3.9}$$

We estimate each factor of the integrand in (3.9) separately. For the first factor, we recall (1.2) and (3.4) to note that $\int \mathbb{W}^T(\mathrm{d}\omega) = \psi^T(0)|\Lambda|$. Furthermore, we use the elementary inequality $|\mathrm{e}^{\mathrm{i}a} - 1| \leqslant C|a|$ for $a \in \mathbb{R}$ and consider Riemann sums with mesh size ν for the integral

$$\int_\varepsilon^\infty \mathrm{d}T\, \mathrm{e}^{-\kappa T}\, \psi^T(0) = O_\kappa(1)\,, \tag{3.10}$$

to deduce that

$$\left|\exp\left\{\nu \sum_{T\in\nu\mathbb{N}^*\cap[\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega)\left[\mathrm{e}^{\mathrm{i}\sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T}\int_0^\nu \mathrm{d}t\,\sigma(\omega(t+r))} - 1\right]\right\}\right| \leqslant \mathrm{e}^{C_\kappa|\Lambda|\|\sigma\|_\infty}\,. \tag{3.11}$$

Note that for (3.10) we used that $\psi^T(0) \leqslant 1$, which follows from Lemma C.1 (i). Similarly, for the second factor of the integrand in (3.9), we use the elementary inequality $|\mathrm{e}^\zeta - 1| \leqslant |\zeta|\mathrm{e}^{|\zeta|}$ for $\zeta \in \mathbb{C}$ and consider Riemann sums with mesh size ν for the integral $\int_0^\varepsilon \mathrm{d}T\, \mathrm{e}^{-\kappa T}\, \psi^T(0) \leqslant \varepsilon$ to deduce that

$$\left|\exp\left\{\nu \sum_{T\in\nu\mathbb{N}^*\cap(0,\varepsilon)} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^T(\mathrm{d}\omega)\left[\mathrm{e}^{\mathrm{i}\sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T}\int_0^\nu \mathrm{d}t\,\sigma(\omega(t+r))} - 1\right]\right\} - 1\right| \leqslant C\varepsilon|\Lambda|\|\sigma\|_\infty\, \mathrm{e}^{C\varepsilon|\Lambda|\|\sigma\|_\infty}\,. \tag{3.12}$$

Combining (3.9)–(3.12), it follows that

$$|\mathcal{Z}^{\nu,\kappa,\nu^2} - \mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon}| \leqslant C\varepsilon \int \mathrm{d}\mu_{\nu v}(\sigma)\, \mathrm{e}^{C_\kappa|\Lambda|\|\sigma\|_\infty}\,. \tag{3.13}$$

Recalling (A.16), we note that (3.8) follows from (3.13) if we prove that, for fixed $x \in \Lambda$, we have

$$\int \mathrm{d}\mu_{\nu v}(\sigma)\, \mathrm{e}^{C|\sigma(x)|} = O(1)\,, \quad \text{uniformly in } \nu \in (0,\nu_0)\,, \tag{3.14}$$

for $\nu_0 > 0$ sufficiently small. We deduce (3.14) by using (A.17) (with v replaced by νv). Namely, we note the elementary inequality $\sqrt{\frac{(2i)!}{i!\,2^i}} \leqslant \frac{(2i)!}{i!\,2^i} \leqslant 2^i\, i!$ and thus obtain that the left-hand side of (3.14) is $\leqslant \sum_{i=0}^\infty \big(2Cv(0)^{1/2}\nu^{1/2}\big)^i = O(1)$, provided that $\nu \in (0,\nu_0)$ for $\nu_0 > 0$ sufficiently small.

By using (A.3) as in the proof of (3.6), we can rewrite (3.7) as

$$\mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon} = \sum_{n=0}^\infty \frac{\nu^n}{n!} \sum_{\mathbf{T}\in(\nu\mathbb{N}^*\cap[\varepsilon,\infty))^n} \prod_{i=1}^n \frac{1}{T_i}\, \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}(\mathrm{d}\boldsymbol{\omega}) \exp(-V^{\nu,\nu^2}(\boldsymbol{\omega}))\, \exp(K_\nu^\varepsilon)\,, \tag{3.15}$$

where

$$K_\nu^\varepsilon := -\nu \sum_{T\in\nu\mathbb{N}^*\cap[\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T}}{T}\psi^T(0)|\Lambda| \leqslant 0\,. \tag{3.16}$$

For the inequality above, we used Lemma C.1 (i).

Recalling (2.3), we now show that for all $\varepsilon > 0$

$$\lim_{\nu\to 0} \mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon} = \mathcal{Z}^{\mathrm{cl},\varepsilon}\,. \tag{3.17}$$

In order to prove (3.17), we show three auxiliary claims. Let us henceforth fix $\varepsilon > 0$.

(1) For $u_1, u_2 \in \Lambda$, we have

$$\nu^2 \sum_{T_1 \in \nu\mathbb{N}^* \cap [\varepsilon,\infty)} \sum_{T_2 \in \nu\mathbb{N}^* \cap [\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T_1}\mathrm{e}^{-\kappa T_2}}{T_1 T_2} \int \mathbb{W}^{T_1}_{u_1,u_1}(\mathrm{d}\omega_1) \int \mathbb{W}^{T_2}_{u_2,u_2}(\mathrm{d}\omega_2) \Big| \exp(-\mathcal{V}^{\nu,\nu^2}(\omega_1,\omega_2)/2) - \exp(-\mathcal{V}^{\mathrm{cl}}(\omega_1,\omega_2)/2) \Big| \leqslant C_\kappa \nu \|v\|_{\ell^\infty} . \quad (3.18)$$

(2) For $u \in \Lambda$, we have

$$\nu \sum_{T \in \nu\mathbb{N}^* \cap [\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T}}{T} \int \mathbb{W}^{T}_{u,u}(\mathrm{d}\omega) = O_{\varepsilon,\kappa}(|\Lambda|) . \quad (3.19)$$

(3) With K^ε_ν as in (3.16) and K^ε as in (2.4), we have

$$\lim_{\nu \to 0} K^\varepsilon_\nu = K^\varepsilon . \quad (3.20)$$

Let us assume (3.18)–(3.20) for now. We show how one can then deduce (3.17). Let us define for $\nu > 0$ the auxiliary quantity

$$\widetilde{\mathcal{Z}}^{\nu,\kappa,\nu^2,\varepsilon} := \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \sum_{\mathbf{T} \in (\nu\mathbb{N}^*) \cap [\varepsilon,\infty)^n} \prod_{i=1}^{n} \frac{1}{T_i} \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}(\mathrm{d}\boldsymbol{\omega}) \exp(-V^{\mathrm{cl}}(\boldsymbol{\omega})) \, \exp(K^\varepsilon_\nu) . \quad (3.21)$$

Using (3.1), (3.16), (3.18)–(3.19), and applying a telescoping argument in comparing (3.15) and (3.21), we deduce that

$$\mathcal{Z}^{\nu,\kappa,\nu^2,\varepsilon} - \widetilde{\mathcal{Z}}^{\nu,\kappa,\nu^2,\varepsilon} = O_{\varepsilon,v,\Lambda,\kappa}(\nu) . \quad (3.22)$$

By using (2.3), (3.20), (3.21), and by considering Riemann sums with mesh size ν we deduce that

$$\lim_{\nu \to 0} \widetilde{\mathcal{Z}}^{\nu,\kappa,\nu^2,\varepsilon} = \mathcal{Z}^{\mathrm{cl},\varepsilon} . \quad (3.23)$$

We hence obtain (3.17) from (3.22)–(3.23). Claim (i) then follows from Proposition 2.1 (i), (3.8), and (3.17).

Let us now show (3.18)–(3.20). We first show (3.18). By using (3.1) and recalling (1.8), (1.14), we have that the contribution to the left-hand side of (3.18) for fixed T_1, T_2 is

$$\leqslant \frac{1}{2} \int \mathbb{W}^{T_1}_{u_1,u_1}(\mathrm{d}\omega_1) \int \mathbb{W}^{T_2}_{u_2,u_2}(\mathrm{d}\omega_2) \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{r<T_1} \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s<T_2} \int_0^\nu \mathrm{d}t_1 \int_0^\nu \mathrm{d}t_2 \big| v(\omega_1(t_1+r) - \omega_2(t_1+s)) - v(\omega_1(t_1+r) - \omega_2(t_2+s)) \big| . \quad (3.24)$$

The contribution to (3.24) when $t_1 \leqslant t_2$ is

$$\leqslant \frac{1}{2} \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{r<T_1} \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s<T_2} \int_0^\nu \mathrm{d}t_1 \int_0^\nu \mathrm{d}t_2 \int \mathrm{d}\zeta \int \mathrm{d}\eta_1 \int \mathrm{d}\eta_2 \, \psi^{t_1+r}(\zeta - u_1) \, \psi^{T_1-t_1-r}(u_1 - \zeta) \times \psi^{t_1+s}(\eta_1 - u_2) \, \psi^{t_2-t_1}(\eta_2 - \eta_1) \, \psi^{T_2-t_2-s}(u_2 - \eta_2) \, \big| v(\zeta - \eta_1) - v(\zeta - \eta_2) \big| , \quad (3.25)$$

which by Lemma C.1 is[3]

$$\lesssim \nu^3 \, \|v\|_{\ell^\infty} \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{r<T_1} \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s<T_2} = \nu \, \|v\|_{\ell^\infty} \, T_1 T_2 . \quad (3.26)$$

[3]Note that (3.25) vanishes if $\eta_1 = \eta_2$, hence we estimate $\psi^{t_2-t_1}(\eta_2 - \eta_1)$ by Lemma C.1 (ii).

The contribution to (3.24) when $t_1 > t_2$ is estimated analogously. Using (3.26), it follows that the expression on the left-hand side of (3.18) is

$$\lesssim \nu \, \|v\|_{\ell^\infty} \sum_{T_1 \in \nu\mathbb{N}^* \cap [\varepsilon,\infty)} \sum_{T_2 \in \nu\mathbb{N}^* \cap [\varepsilon,\infty)} \nu^2 \mathrm{e}^{-\kappa T_1} \mathrm{e}^{-\kappa T_2} \leqslant C_\kappa \nu \|v\|_{\ell^\infty} \,, \tag{3.27}$$

as was claimed. In order to obtain (3.27), we considered a Riemann sum of mesh size ν for the integral $\int_\varepsilon^\infty \mathrm{d}T_1 \int_\varepsilon^\infty \mathrm{d}T_2 \, \mathrm{e}^{-\kappa T_1} \mathrm{e}^{-\kappa T_2} = O_\kappa(1)$.

We now show (3.19). By Lemma C.1 (i), it follows that the expression on the left-hand side of (3.19) is

$$\leqslant \nu \sum_{T \in \nu\mathbb{N}^* \cap [\varepsilon,\infty)} \frac{\mathrm{e}^{-\kappa T}}{T} |\Lambda| = O_{\varepsilon,\kappa}(|\Lambda|) \,, \tag{3.28}$$

and we hence obtain (3.19). In (3.28), we considered a Riemann sum of mesh size ν for the integral $\int_\varepsilon^\infty \mathrm{d}T \, \frac{\mathrm{e}^{-\kappa T}}{T} = O_{\varepsilon,\kappa}(1)$.

Finally, in order to show (3.20), we rewrite (2.4) using Lemma A.1 as

$$K^\varepsilon = -\int_\varepsilon^\infty \frac{\mathrm{d}T}{T} \, \mathrm{e}^{-\kappa T} \psi^T(0) \, |\Lambda| \,. \tag{3.29}$$

Using (3.16), (3.29), and considering Riemann sums of mesh size ν for the integral in (3.29), we deduce (3.20). Claim (i) now follows.

The proof of (ii) is similar to that of (i). We just outline the main differences. We now start from (2.11) and hence work with paths that can also be open. For fixed $\mathbf{T} \in (\nu\mathbb{N}^*)^p, \tilde{\mathbf{T}} \in (\nu\mathbb{N}^*)^n, \mathbf{x}, \mathbf{y} \in \Lambda^p, (\boldsymbol{\omega}, \tilde{\boldsymbol{\omega}}) \in \Omega^{\mathbf{T}}_{\mathbf{y},\mathbf{x}} \times \Omega^{\tilde{\mathbf{T}}}$, we can write $V^{\nu,\nu^2}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}) = \frac{1}{2}\langle f, \nu v f \rangle$, where

$$f(x) = \sum_{j=1}^p \sum_{r_j \in \nu\mathbb{N}} \mathbf{1}_{r_j < T_j} \int_0^\nu \mathrm{d}t \, \delta(x - \omega_j(t + r_j)) + \sum_{i=1}^n \sum_{s_i \in \nu\mathbb{N}} \mathbf{1}_{s_i < \tilde{T}_i} \int_0^\nu \mathrm{d}t \, \delta(x - \tilde{\omega}_i(t + s_i)) \,. \tag{3.30}$$

We use (2.11) and apply (A.3) with f as in (3.30) to obtain that

$$(\Gamma_p^{\nu,\kappa,\nu^2})_{\mathbf{x},\mathbf{y}} = \frac{1}{\mathcal{Z}^{\nu,\kappa,\nu^2}} \, (\widehat{\Gamma}_p^{\nu,\kappa,\nu^2})_{\mathbf{x},\mathbf{y}} \,, \tag{3.31}$$

where $(\widehat{\Gamma}_p^{\nu,\kappa,\nu^2})_{\mathbf{x},\mathbf{y}}$ is

$$\begin{aligned} &:= \sum_{\mathbf{T} \in (\nu\mathbb{N}^*)^p} \sum_{\pi \in S_p} \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \int \mu_v(\mathrm{d}\sigma) \, \exp\Bigg\{ \mathrm{i} \sum_{j=1}^p \sum_{r_j \in \nu\mathbb{N}} \mathbf{1}_{r_j < T_j} \int_0^\nu \mathrm{d}t \, \sigma(\omega_j(t + r_j)) \Bigg\} \\ &\times \exp\Bigg\{ \int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\tilde{\omega}) \Big[\mathrm{e}^{\mathrm{i} \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s < T(\tilde{\omega})} \int_0^\nu \mathrm{d}t \, \sigma(\tilde{\omega}(t+s))} - 1 \Big] \Bigg\} \,. \end{aligned}$$

By arguing analogously as for (3.8) we have that there exists $\nu_0 > 0$ sufficiently small such that

$$\lim_{\varepsilon \to 0} \nu^p (\widehat{\Gamma}_p^{\nu,\kappa,\nu^2,\varepsilon})_{\mathbf{x},\mathbf{y}} = \nu^p (\widehat{\Gamma}_p^{\nu,\kappa,\nu^2})_{\mathbf{x},\mathbf{y}} \,, \quad \text{uniformly in } \nu \in (0, \nu_0) \,, \tag{3.32}$$

where

$$\begin{aligned} (\widehat{\Gamma}_p^{\nu,\kappa,\nu^2,\varepsilon})_{\mathbf{x},\mathbf{y}} = \sum_{\mathbf{T} \in (\nu\mathbb{N}^*)^p} \sum_{\pi \in S_p} \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \sum_{n=0}^\infty \frac{\nu^n}{n!} \sum_{\tilde{\mathbf{T}} \in (\nu\mathbb{N}^*)^n \cap [\varepsilon,\infty)^n} \prod_{i=1}^n \frac{1}{\tilde{T}_i} \mathrm{e}^{-\kappa|\tilde{\mathbf{T}}|} \\ \times \int \mathbb{W}^{\tilde{\mathbf{T}}}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \exp(-V^{\nu,\nu^2}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) \, \exp(K_\nu^\varepsilon) \,, \end{aligned} \tag{3.33}$$

for K_ν^ε as given by (3.16). We now use (3.33) and consider Riemann sums as in the proof of (3.17) to deduce that for all $\varepsilon > 0$ we have

$$\lim_{\nu \to 0} \nu^p (\widehat{\Gamma}_p^{\nu,\kappa,\nu^2,\varepsilon})_{\mathbf{x},\mathbf{y}} = \mathcal{Z}^{\mathrm{cl}} \, (\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}} \,, \tag{3.34}$$

where $(\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}}$ is given by (2.5). Claim (ii) now follows from (3.31), (3.32), (3.34), Proposition 2.1 (ii), and from part (i) of the theorem.

□

4. The large-mass limit

In this section, we analyse the large-mass limit on the finite lattice Λ and provide the proof of Theorem 1.4. We recall that in this regime, we consider $\nu \to 0$ with $\lambda = 1$ and $\kappa = \frac{\kappa_0}{\nu}$. Throughout this section, we assume that v satisfies Assumption 1.3. In light of (1.28), we need to slightly modify (1.4) when studying the large-mass limit. Namely, recalling (1.14), we work with

$$V^{\nu,1}(\boldsymbol{\omega}) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathcal{V}^{\nu,1}(\omega_i, \omega_j) + \frac{1}{2} \sum_{i=1}^{n} \widetilde{\mathcal{V}}^{\nu,1}(\omega_i) + \frac{v(0)}{2\nu} |\mathbf{T}| \, \mathbf{1}_{R=0} \,, \tag{4.1}$$

where

$$\widetilde{\mathcal{V}}^{\nu,1}(\omega) := \frac{1}{\nu} \sum_{r,s \in \nu\mathbb{N}} \mathbf{1}_{r,s<T(\omega)} \, \mathbf{1}_{r \neq s} \int_0^\nu \mathrm{d}t \, v(\omega(t+r) - \omega(t+s)) \,. \tag{4.2}$$

Note that (4.1) differs from (1.4) only in the presence of a hard core potential. With this modification, the result of Proposition 2.3 holds. We use this without further comment below.

Before proceeding to the proof of Theorem 1.4, we first note several basic facts.

Lemma 4.1. *Let $g_1 : \Lambda \to \mathbb{C}$ and $g_2 : \Lambda \times \Lambda \to \mathbb{C}$ be given. The following estimates hold.*

(i) For $x, x' \in \Lambda, T > 0$, and $r, s, t \in [0, T]$ satisfying $r \leqslant s$ and $t + s \leqslant T$, we have

$$\int \mathbb{W}^T_{x',x}(\mathrm{d}\omega) \, |g_1(\omega(t+r) - \omega(t+s)) - g_1(0)| = O_\Lambda(T \, \|g_1\|_{\ell^\infty}) \,. \tag{4.3}$$

(ii) For $x_1, x_2, x_1', x_2' \in \Lambda, T_1, T_2 > 0, t_1 \in [0, T_1], t_2 \in [0, T_2]$ we have

$$\int \mathbb{W}^{T_1}_{x_1',x_1}(\mathrm{d}\omega_1) \int \mathbb{W}^{T_2}_{x_2',x_2}(\mathrm{d}\omega_2) \, |g_2(\omega_1(t_1), \omega_2(t_2)) - g_2(x_1, x_2)| = O_\Lambda((T_1 + T_2) \, \|g_2\|_{\ell^\infty}) \,. \tag{4.4}$$

Proof. We first prove (i). Let us first consider the case when $x = x'$. The expression on the left-hand side of (4.3) is then given by

$$\int \mathrm{d}\zeta \int \mathrm{d}\eta \, \psi^{t+r}(\zeta - x) \, \psi^{s-r}(\eta - \zeta) \, \psi^{T-s-t}(x - \eta) \, |g_1(\zeta - \eta) - g_1(0)| \,. \tag{4.5}$$

By Lemma C.1 (i)–(ii), it follows that the expression in (4.5) is

$$\leqslant \int \mathrm{d}\zeta \int \mathrm{d}\eta \, \psi^{s-r}(\eta - \zeta) \, |g_1(\zeta - \eta) - g_1(0)| = O_\Lambda(T \, \|g_1\|_{\ell^\infty}) \,. \tag{4.6}$$

Let us now consider the case when $x \neq x'$. By using (3.4) and Lemma C.1 (ii), it follows that the expression on the left-hand side of (4.3) is

$$\leqslant 2\|g_1\|_{\ell^\infty} \psi^T(x'-x) \lesssim T\|g_1\|_{\ell^\infty} . \tag{4.7}$$

Claim (i) follows from (4.6)–(4.7).

We now prove (ii). Let us first consider the case when $x_1 = x_1'$ and $x_2 = x_2'$. Similarly as in the proof of (i), we need to estimate

$$\int \mathrm{d}\zeta \int \mathrm{d}\eta\, \psi^{t_1}(\zeta - x_1)\, \psi^{T_1-t_1}(x_1-\zeta)\, \psi^{t_2}(\eta - x_2)\, \psi^{T_2-t_2}(x_2-\eta)\, |g_2(\zeta,\eta) - g_2(x_1,x_2)| ,$$

which by the triangle inequality and Lemma C.1 (i) is

$$\leqslant \int \mathrm{d}\eta \int \mathrm{d}\zeta\, \psi^{t_1}(\zeta - x_1)\, |g_2(\zeta,\eta) - g_2(x_1,\eta)| + \int \mathrm{d}\eta \int \mathrm{d}\zeta\, \psi^{t_2}(\eta - x_2)\, |g_2(x_1,\eta) - g_2(x_1,x_2)| .$$

By Lemma C.1 (ii), the above expression is

$$= O_\Lambda((T_1+T_2)\, \|g_2\|_{\ell^\infty}) . \tag{4.8}$$

Let us now consider the case when $x_1 \neq x_1'$. By (3.4) and Lemma C.1 (i)–(ii), we obtain that the expression on the left-hand side of (4.4) is

$$\leqslant 2\|g_2\|_{\ell^\infty}\, \psi^{T_1}(x_1'-x_1)\, \psi^{T_2}(x_2'-x_2) \lesssim T_1\|g_2\|_{\ell^\infty} . \tag{4.9}$$

Finally, when $x_2 \neq x_2'$, we obtain by analogous arguments that the left-hand side of (4.4) is

$$\lesssim T_2\|g_2\|_{\ell^\infty} . \tag{4.10}$$

Claim (ii) follows from (4.8)–(4.10). □

We define

$$\widehat{V}^{\nu,1}(\boldsymbol{\omega}) := \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \widehat{\mathcal{V}}^{\nu,1}(\omega_i,\omega_j) + \frac{1}{2}\sum_{i=1}^{n} \widehat{\mathcal{V}}^{\nu,1}(\omega_i,\omega_i)\, \mathbf{1}_{R=0} , \tag{4.11}$$

where we let

$$\widehat{\mathcal{V}}^{\nu,1}(\omega,\tilde{\omega}) := \frac{1}{\nu^2}\, T(\omega)\, T(\tilde{\omega}) v(x(\omega) - x(\tilde{\omega})) . \tag{4.12}$$

We note that $\widehat{V}^{\nu,1}(\boldsymbol{\omega})$ depends only on the durations $\mathbf{T} = (T(\omega_1), \dots, T(\omega_n))$ and initial points $\mathbf{x} = (x(\omega_1), \dots, x(\omega_n))$ of the paths $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$. Let us note a consequence of Lemma 4.1.

Lemma 4.2. *The following estimates hold.*

(i) Suppose that $R = 0$ in Assumption 1.3. For $x, x' \in \Lambda$ and $T \in \nu\mathbb{N}^*$, we have

$$\int \mathbb{W}^T_{x',x}(\mathrm{d}\omega) \left| \mathrm{e}^{-\mathcal{V}^{\nu,1}(\omega,\omega)} - \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1}(\omega,\omega)} \right| = O_\Lambda\left(\frac{T^3}{\nu^2}\, \|v\|_{\ell^\infty} \right).$$

(ii) With v as in Assumption 1.3, we define the function $\tilde{v} : \Lambda \to \mathbb{R}$ by

$$\tilde{v}(x) := v(x)\, \mathbf{1}_{|x| \geqslant R}\,. \tag{4.13}$$

For $x_1, x_2, x_1', x_2' \in \Lambda$ and $T_1, T_2 \in \nu\mathbb{N}^*$, we have

$$\int \mathbb{W}^{T_1}_{x_1', x_1}(\mathrm{d}\omega_1) \int \mathbb{W}^{T_2}_{x_2', x_2}(\mathrm{d}\omega_2) \left| \mathrm{e}^{-\mathcal{V}^{\nu,1}(\omega_1,\omega_2)} - \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1}(\omega_1,\omega_2)} \right| = O_\Lambda\!\left(\frac{T_1 T_2 (T_1 + T_2)}{\nu^2} \, \|\tilde{v}\|_{\ell^\infty} \right) + O(T_1^2 + T_2^2)\,. \tag{4.14}$$

Proof. In order to prove (i), we note that for $\omega \in \Omega^T_{x',x}$, we have by (4.1), (4.12), and Assumption 1.3

$$\left| \mathrm{e}^{-\mathcal{V}^{\nu,1}(\omega,\omega)} - \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1}(\omega,\omega)} \right| \leqslant \frac{1}{2\nu} \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{r<T} \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s<T} \int_0^\nu \mathrm{d}t \left| v(\omega(t+r) - \omega(t+s)) - v(0) \right|. \tag{4.15}$$

Using (4.15), Fubini's theorem, and Lemma 4.1 (i) with $g_1 = v$, we first integrate in ω for fixed $t \in [0, \nu], r, s \in \nu\mathbb{N}$ with $r, s < T$. Then, we integrate in t and sum in r, s to deduce claim (i).

We now prove (ii). We consider the cases $R = 0$ and $R = 1$ separately.

Case 1: $R = 0$. Let $\omega_1 \in \Omega^{T_1}_{x_1', x_1}$ and $\omega_2 \in \Omega^{T_2}_{x_2', x_2}$ be given. As in (4.15), we get that

$$\left| \mathrm{e}^{-\mathcal{V}^{\nu,1}(\omega_1,\omega_2)} - \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1}(\omega_1,\omega_2)} \right| \leqslant \frac{1}{2\nu} \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{r<T_1} \sum_{s \in \nu\mathbb{N}} \mathbf{1}_{s<T_2} \int_0^\nu \mathrm{d}t \left| v(\omega_1(t+r) - \omega_2(t+s)) - v(x_1 - x_2) \right|. \tag{4.16}$$

We use Fubini's theorem and first integrate in ω_1 and ω_2 in (4.16). In doing so, for fixed $t \in [0, \nu]$, and $r, s \in \nu\mathbb{N}$, with $r < T_1, s < T_2$, we use Lemma 4.1 (ii) with $t_1 = t+r, t_2 = t+s, g_2(\zeta, \eta) = v(\zeta - \eta)$. We then integrate in t and sum in r, s and recall that by (4.13) we have $\tilde{v} = v$. Claim (ii) for $R = 0$ now follows.

Case 2: $R = 1$. If $x_1 = x_2$, then by the right continuity of ω_1 and ω_2, there exists $\varepsilon \in (0, \nu)$ such that $\omega_1 = \omega_2$ on $[0, \varepsilon)$. Hence, recalling (1.14) and (4.12), the expression on the left-hand side of (4.14) is zero by Assumption 1.3 (ii).

We henceforth assume that $x_1 \neq x_2$. Let us separately consider the integral over two regions in $(\omega_1, \omega_2) \in \Omega^{T_1}_{x_1', x_1} \times \Omega^{T_2}_{x_2', x_2}$, which are defined depending on whether ω_1 and ω_2 intersect.

(i) ω_1 and ω_2 do not intersect. In other words, we consider the region

$$\mathcal{R}_1 := \left\{ (\omega_1, \omega_2) \in \Omega^{T_1}_{x_1', x_1} \times \Omega^{T_2}_{x_2', x_2}\,,\ \forall t_1 \in [0, T_1]\ \forall t_2 \in [0, T_2]\,,\ \omega_1(t_1) \neq \omega_2(t_2) \right\}. \tag{4.17}$$

Using (4.13) and (4.17), we see that in the contribution from $\mathcal{R}_1$ to the left-hand side of (4.14), we can replace v by $\tilde{v}$ in (1.14) and (4.12). Therefore, we can argue analogously as in Case 1 and obtain the same upper bound.

(ii) ω_1 and ω_2 intersect. We consider the region

$$\mathcal{R}_2 := \left\{ (\omega_1, \omega_2) \in \Omega^{T_1}_{x_1', x_1} \times \Omega^{T_2}_{x_2', x_2}\,,\ \exists t_1 \in [0, T_1]\ \exists t_2 \in [0, T_2]\,,\ \omega_1(t_1) = \omega_2(t_2) \right\}.$$

Since $x_1 \neq x_2$, we have $\mathcal{R}_2 \subset \mathcal{R}_2^{(1)} \cup \mathcal{R}_2^{(2)}$, where

$$\mathcal{R}_2^{(j)} := \left\{ (\omega_1, \omega_2) \in \Omega^{T_1}_{x_1', x_1} \times \Omega^{T_2}_{x_2', x_2} \,,\; \exists t_j \in [0, T_j] \,,\; \omega_j(t_j) \neq x_j \right\}.$$

By using $|\mathrm{e}^{-\mathcal{V}^{\nu,1}(\omega_1,\omega_2)} - \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1}(\omega_1,\omega_2)}| \leqslant 1$ and recalling (3.4), we get that the contribution from $\mathcal{R}_2^{(1)}$ to the left-hand side of (4.14) is

$$\left[\int_{\Lambda \setminus \{x_1\}} \mathrm{d}y \int_0^{T_1} \mathrm{d}t_1 \, \psi^{t_1}(y - x_1) \, \psi^{T_1 - t_1}(x_1' - y) \right] \psi^{T_2}(x_2' - x_2) = O(T_1^2) \,. \tag{4.18}$$

In the last step above, we used Lemma C.1. Similarly, the contribution from $\mathcal{R}_2^{(2)}$ to the left-hand side of (4.14) is $O(T_2^2)$ and the claim follows.

□

We now have the necessary ingredients to prove Theorem 1.4.

Proof of Theorem 1.4. We first prove (i).

Let us note that it suffices to consider unnormalized quantities. In particular, recalling (1.33) and (2.10), it suffices to show that

$$\lim_{\nu \to 0} \Xi^{\nu, \kappa_0/\nu, 1} = Z^{\mathrm{lm}} \,. \tag{4.19}$$

Namely, by (1.13), (3.4), Lemma C.1 (i)–(ii), and the dominated convergence theorem, we have that

$$\lim_{\nu \to 0} \int \mathbb{L}^{\nu, \kappa_0/\nu}(\mathrm{d}\omega) = \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-\kappa_0 k}}{k} |\Lambda| \,. \tag{4.20}$$

Therefore, (4.19) implies claim (i) by (1.31), (2.9), and (4.20).

We note that (4.19) follows if we show that for all $n \in \mathbb{N}^*, \mathbf{k} \in (\mathbb{N}^*)^n$, we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{\nu \mathbf{k}}(\mathrm{d}\boldsymbol{\omega}) \, \exp(-V^{\nu,1}(\boldsymbol{\omega})) = \int_{\Lambda^n} \mathrm{d}\mathbf{x} \, \exp(-V^{\mathrm{lm}}(\mathbf{k}, \mathbf{x})) \,. \tag{4.21}$$

Namely, since $\int \mathbb{W}^{k\nu}(\mathrm{d}\omega) \leqslant |\Lambda|$, we have

$$\sum_{k=1}^{\infty} \frac{\mathrm{e}^{-\kappa_0 k}}{k} \int \mathbb{W}^{k\nu}(\mathrm{d}\omega) = O_{\kappa_0}(|\Lambda|) \,. \tag{4.22}$$

Using (4.22), $V^{\nu,1}(\boldsymbol{\omega}) \geqslant 0$, and the dominated convergence theorem, we deduce that the claim of the theorem indeed follows from (4.21). We now show (4.21). In doing so, we consider the two cases $R = 0$ and $R = 1$ separately.

Case 1: $R = 0$. Recalling (4.1), (4.11), and using Lemma 4.2 together with a telescoping argument, it follows that

$$\int \mathbb{W}^{\nu \mathbf{k}}(\mathrm{d}\boldsymbol{\omega}) \, \exp(-V^{\nu,1}(\boldsymbol{\omega})) = \int \mathbb{W}^{\nu \mathbf{k}}(\mathrm{d}\boldsymbol{\omega}) \, \exp(-\widehat{V}^{\nu,1}(\boldsymbol{\omega})) + O_{\Lambda, v, |\mathbf{k}|, n}(\nu) \,. \tag{4.23}$$

We use (1.29), (3.4), and (4.11)–(4.12) to write

$$\int \mathbb{W}^{\nu \mathbf{k}}(\mathrm{d}\boldsymbol{\omega}) \, \exp(-\widehat{V}^{\nu,1}(\boldsymbol{\omega})) = \prod_{i=1}^{n} \psi^{k_i \nu}(0) \int_{\Lambda^n} \mathrm{d}\mathbf{x} \, \exp(-V^{\mathrm{lm}}(\mathbf{k}, \mathbf{x})) \,. \tag{4.24}$$

Note that by Lemma C.1 (ii), we have that for all $k \in \mathbb{N}^*$

$$\psi^{k\nu}(0) = 1 + O(k\nu) \,. \tag{4.25}$$

Combining (4.23)–(4.25), we deduce (4.21).

Case 2: $R = 1$. We recall (1.29) and deduce that (4.21) is the consequence of the following two claims.

(1) For all $n \in \mathbb{N}^*$ and $\mathbf{k} \in (\mathbb{N}^*)^n \setminus \{\mathbf{1}_n\}$, we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{\nu \mathbf{k}}(\mathrm{d}\boldsymbol{\omega}) \, \exp\bigl(-V^{\nu,1}(\boldsymbol{\omega})\bigr) = 0 \,. \tag{4.26}$$

(2) For all $n \in \mathbb{N}^*$, we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{\nu \mathbf{1}}(\mathrm{d}\boldsymbol{\omega}) \, \exp\bigl(-V^{\nu,1}(\boldsymbol{\omega})\bigr) = \int_{\Lambda^n} \mathrm{d}\mathbf{x} \, \exp\bigl(-V^{\mathrm{lm}}(\mathbf{1}_n, \mathbf{x})\bigr) \,. \tag{4.27}$$

Here we recall (1.30).

We first prove claim (1). We recall (4.1)–(4.2), and use the nonnegativity of v, to deduce that (4.26) follows if we show that for all $k \geqslant 2$ and $x \in \Lambda$, we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{k\nu}_{x,x}(\mathrm{d}\omega) \, \exp\bigl(-\widetilde{\mathcal{V}}^{\nu,1}(\omega)\bigr) = 0 \,. \tag{4.28}$$

Namely, we know that there exists a component of $\mathbf{k}$ which is at least 2. Let's assume without loss of generality that $k_1 \geqslant 2$. When integrating in ω_1, we use (4.28). When integrating in $\omega_j, 2 \leqslant j \leqslant n$, we use the nonnegativity of v, (3.4) and Lemma C.1 (i). Therefore, claim (i) indeed follows from (4.28).

Let us now prove (4.28). We define $\mathcal{L}^T_x := \bigl\{\omega \in \Omega^T_{x,x} \,, \exists t \in [0,T] \,, \omega(t) \neq x\bigr\}$ and $\mathcal{S}^T_x := \Omega^T_{x,x} \setminus \mathcal{L}^T_x$. With this notation, we have

$$\int \mathbb{W}^{k\nu}_{x,x}(\mathrm{d}\omega) \, \exp\bigl(-\widetilde{\mathcal{V}}^{\nu,1}(\omega)\bigr) \leqslant \int \mathbb{W}^{k\nu}_{x,x}(\mathrm{d}\omega) \, \mathbf{1}_{\mathcal{L}^{k\nu}_x}(\omega) + \int \mathbb{W}^{k\nu}_{x,x}(\mathrm{d}\omega) \, \exp\bigl(-\widetilde{\mathcal{V}}^{\nu,1}(\omega)\bigr) \, \mathbf{1}_{\mathcal{S}^{k\nu}_x}(\omega) \,. \tag{4.29}$$

By arguing analogously as for (4.18), it follows that the first term on the right-hand side of (4.29) is $O(k^2\nu^2)$. Furthermore, by (4.2) and the assumptions on v, the second term is equal to zero (here, we are crucially using the fact that $k \geqslant 2$). We therefore obtain (4.28) and claim (i) follows.

Let us now prove claim (2). We apply Lemma 4.2 (ii) with $T = \nu$ and use a telescoping argument to deduce that

$$\lim_{\nu \to 0} \biggl[\int \mathbb{W}^{\nu \mathbf{1}}(\mathrm{d}\boldsymbol{\omega}) \, \exp\bigl(-V^{\nu,1}(\boldsymbol{\omega})\bigr) - \int \mathbb{W}^{\nu \mathbf{1}}(\mathrm{d}\boldsymbol{\omega}) \, \exp\bigl(-V^{\mathrm{lm}}(\mathbf{1}_n, \mathbf{x})\bigr) \biggr] = 0 \,. \tag{4.30}$$

Here, we recall (4.1)–(4.2), (1.29), (4.12), and keep in mind that the left-hand side of (4.27) does not contain any self-interactions. Claim (ii) now follows from (4.30) by iteratively applying (3.4) and Lemma C.1 (ii) in the second term and thus obtaining the expression on the right-hand side of (4.27) in the limit. We deduce claim (i).

Let us now show claim (ii). The proof is similar to that of (i), so we will just outline the main differences. Let $p \in \mathbb{N}^*$ and $\mathbf{x}, \mathbf{y} \in \Lambda^p$ be given. By (1.32), (2.10)–(2.11), and (4.19), the claim follows if we show that

$$\lim_{\nu \to 0} \sum_{\mathbf{T} \in (\nu \mathbb{N}^*)^p} \sum_{\pi \in S_p} \mathrm{e}^{-\kappa_0 |\mathbf{T}|/\nu} \int \mathbb{W}^{\mathbf{T}}_{\pi \mathbf{y}, \mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \, \Xi^{\nu, \kappa_0/\nu, 1}(\boldsymbol{\omega}) = \sum_{\mathbf{k} \in (\mathbb{N}^*)^p} \sum_{\pi \in S_p} \mathrm{e}^{-\kappa_0 |\mathbf{k}|} \, \delta(\pi \mathbf{y} - \mathbf{x}) \, Z^{\mathrm{lm}}(\mathbf{k}, \mathbf{x}) \,. \tag{4.31}$$

Analogously as for (4.21), it suffices to show that given $\mathbf{k} \in (\mathbb{N}^*)^p, \pi \in S_p, n \in \mathbb{N}^*, \tilde{\mathbf{k}} \in (\mathbb{N}^*)^n$ we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{\nu \mathbf{k}}_{\pi \mathbf{y}, \mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \int \mathbb{W}^{\nu \tilde{\mathbf{k}}}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \, \exp(-V^{\nu,1}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) = \delta(\pi \mathbf{y} - \mathbf{x}) \int_{\Lambda^n} \mathrm{d}\tilde{\mathbf{x}} \, \exp(-V^{\mathrm{lm}}(\mathbf{k}\tilde{\mathbf{k}}, \mathbf{x}\tilde{\mathbf{x}})) \, . \quad (4.32)$$

When $R = 0$, the proof of (4.32) proceeds as that of (4.21). The only difference is that we now apply Lemma 4.2 for both open and closed paths. The p open paths give rise to delta functions by Lemma C.1 (ii). We omit the details.

We henceforth show (4.32) when $R = 1$. Arguing as for (4.26)–(4.27), it suffices to show the following two claims.

(a) For all $n \in \mathbb{N}^*$, $(\mathbf{k}, \tilde{\mathbf{k}}) \in (\mathbb{N}^*)^p \times (\mathbb{N}^*)^n \setminus \{\mathbf{1}_{p+n}\}$, and $\pi \in S_p$, we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{\nu \mathbf{k}}_{\pi \mathbf{y}, \mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \int \mathbb{W}^{\nu \tilde{\mathbf{k}}}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \, \exp(-V^{\nu,1}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) = 0 \, . \quad (4.33)$$

(b) For all $n \in \mathbb{N}^*$, we have

$$\lim_{\nu \to 0} \int \mathbb{W}^{\nu \mathbf{1}_p}_{\pi \mathbf{y}, \mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \int \mathbb{W}^{\nu \mathbf{1}_n}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \, \exp(-V^{\nu,1}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) = \delta(\pi \mathbf{y} - \mathbf{x}) \int_{\Lambda^n} \mathrm{d}\tilde{\mathbf{x}} \, \exp(-V^{\mathrm{lm}}(\mathbf{x}\tilde{\mathbf{x}})) \, . \quad (4.34)$$

We first prove claim (a). Note that if $\tilde{\mathbf{k}} \neq \mathbf{1}_n$, then (4.33) follows from (4.28) by using $V^{\nu,1}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}) \geqslant V^{\nu,1}(\boldsymbol{\omega})$, and by recalling (3.4) and Lemma C.1 (i). If $\mathbf{k} \neq \mathbf{1}_p$, then we consider two subcases. If $\pi \mathbf{y} \neq \mathbf{x}$, then (4.33) follows from (3.4) and Lemma C.1 (i)–(ii), since $V^{\nu,1}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}) \geqslant 0$. If $\pi \mathbf{y} = \mathbf{x}$, then (4.33) again follows from (4.28), since $V^{\nu,1}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})$ contains self-interactions (4.2).

Let us now prove claim (b). If $\pi \mathbf{y} \neq \mathbf{x}$, then the limit is zero by arguing as in the proof of (a). If $\pi \mathbf{y} = \mathbf{x}$, then (4.34) follows by using Lemma 4.2 (ii) analogously as in the proof of (4.27). Note that now we are integrating only over the endpoints of the closed paths. We hence obtain (b) and claim (ii) follows. □

5. The infinite-volume limit

In this section, we study the infinite-volume limit. In Section 5.1, we study the mean-field regime and prove Theorem 1.6. In Section 5.2, we study the large-mass regime and prove Theorem 1.8.

5.1. Infinite-volume limit of the specific relative Gibbs potential and reduced density matrices I: the mean-field regime. In this subsection, we work in the mean-field regime. In the sequel, we vary the size $L \in \mathbb{N}^*$ of the box Λ_L. Throughout, we assume that the interaction potential on Λ_L is given by (1.34) for v as in Assumption 1.5. As was mentioned in the introduction, we keep track of the L-dependence of all the quantities by adding a superscript L. More precisely, we write $\Omega^{L,T}$ and $\Omega^{L,T}_{y,x}$ for the appropriate space of càdlàg paths on Λ_L. Analogously, we write $\mathbb{W}^{L,T}_{y,x}(\mathrm{d}\omega), \mathbb{W}^{L,T}(\mathrm{d}\omega)$ for (1.2) respectively. Finally, we write $\psi^t \equiv \psi^{L,t}$ for the heat kernel (3.3) on Λ_L.

Let us rewrite the reduced p-particle density matrix (1.19) as a power series representation amenable to a cluster expansion. Our starting point is the Ginibre representation given in Proposition 2.3 above. Before stating the precise formula, we introduce some notation. We consider

$\mathcal{V}^{\nu,\lambda,L}(\omega,\tilde{\omega})$ given by (1.14) with interaction as in (1.34) and define positive measures on $\bigcup_{T>0}\Omega^{L,T}$ through

$$\mu^L(\mathrm{d}\omega) := \nu \sum_{T\in\nu\mathbb{N}^*} \frac{\mathrm{e}^{-\kappa T}}{T}\, \mathbb{W}^{L,T}(\mathrm{d}\omega)\, \mathrm{e}^{-\mathcal{V}^{\nu,\lambda,L}(\omega,\omega)/2}\,, \tag{5.1}$$

$$\hat{\mu}^L_{y,x}(\mathrm{d}\omega) := \sum_{T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa T}\, \mathbb{W}^{L,T}_{y,x}(\mathrm{d}\omega)\, \mathrm{e}^{-\mathcal{V}^{\nu,\lambda,L}(\omega,\omega)/2}\,. \tag{5.2}$$

Given $p\in\mathbb{N}^*$ and $\omega_1,\dots,\omega_p\in\Omega$ we define

$$X^L(\omega_1,\dots,\omega_p) := \sum_{n\geqslant p\vee 1} \frac{n!}{(n-p)!}\int \mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\varphi^L(\omega_1,\dots,\omega_n)\,,\quad X^L := X^L(\emptyset)\,, \tag{5.3}$$

with the *Ursell function* given by

$$\varphi^L(\omega_1,\dots,\omega_n) := \frac{1}{n!}\sum_{\mathcal{G}\in\mathfrak{G}_n^c}\prod_{\{i,j\}\in\mathcal{G}}\zeta^L(\omega_i,\omega_j)\,,\qquad \zeta^L(\omega,\tilde{\omega}) := \exp\bigl(-\mathcal{V}^{\nu,\lambda,L}(\omega,\tilde{\omega})/2\bigr)-1\,. \tag{5.4}$$

Here $\mathfrak{G}_n^c$ denotes the set of all connected graphs on $[n]=\{1,2,\dots,n\}$. In this section, we work in the mean-field limit and therefore set $\lambda=\nu^2$ in the definitions above [4].

Throughout the sequel, we use the identity

$$\|v^L\|_{\ell^1(\Lambda_L)} = \|v\|_{\ell^1(\mathbb{Z}^d)}\,, \tag{5.5}$$

which follows directly from (1.34). We henceforth write $\|v\|_{\ell^1}\equiv\|v\|_{\ell^1(\mathbb{Z}^d)}$.

Proposition 5.1 (The cluster expansion)**.** *With notation as in (5.3) and assuming that $\|v\|_{\ell^1}$ is sufficiently small, we have the following identities.*

(i) $\log\mathcal{Z}^{\nu,\kappa,\nu^2,L} = X^L - X^{L,0}$*, where $X^{L,0}$ is given by (5.3) with interaction potential set to zero (i.e. $\phi^L=1$).*

(ii) Let $p\in\mathbb{N}^*$ and $\mathbf{x},\mathbf{y}\in\Lambda^p$ be given. We have

$$(\Gamma_p^{\nu,\kappa,\nu^2,L})_{\mathbf{x},\mathbf{y}} = \sum_{\pi\in S_p}\int \hat{\mu}^L_{y_{\pi(1)},x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^L_{y_{\pi(p)},x_p}(\mathrm{d}\omega_p)\sum_{\Pi\in\mathfrak{P}_p}\prod_{\xi\in\Pi}X^L((\omega_i)_{i\in\xi})\,, \tag{5.6}$$

where $\mathfrak{P}_p$ is the set of partitions of $\{1,\dots,p\}$.

Proof. The identity in (i) follows from [33, Theorem 1]. In order to prove (ii), we define

$$\tilde{X}^L(\omega_1,\dots,\omega_p) := \sum_{n=p}^{\infty}\frac{1}{(n-p)!}\int\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\prod_{1\leqslant i<j\leqslant n}\exp\bigl(-\mathcal{V}^{\nu,\nu^2,L}(\omega_i,\omega_j)/2\bigr) \tag{5.7}$$

and $\tilde{X}^L := \tilde{X}^L(\emptyset)$. By Proposition 2.3, (5.1)–(5.2), and (5.7), we have

$$(\Gamma_p^{\nu,\kappa,\nu^2,L})_{\mathbf{x},\mathbf{y}} = \sum_{\pi\in S_p}\int \hat{\mu}^L_{y_{\pi(1)},x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^L_{y_{\pi(p)},x_p}(\mathrm{d}\omega_p)\frac{\tilde{X}^L(\omega_1,\dots,\omega_p)}{\tilde{X}^L}\,. \tag{5.8}$$

[4] In order to simplify notation in the sequel, we write $\mu^L,\hat{\mu}^L_{y,x},X^L$, etc. instead of $\mu^{\nu,\kappa,\lambda,L},\hat{\mu}^{\nu,\kappa,\lambda,L}_{y,x},X^{\nu,\lambda,L}$, etc. We bear in mind that all of the above quantities depend on ν,κ,λ as well. In this section we fix κ and take $\lambda=\nu^2$.

By [33, Theorem 2], we have

$$\frac{\tilde{X}^L(\omega_1,\dots,\omega_p)}{\tilde{X}^L} = \sum_{\Pi\in\mathfrak{P}_p}\prod_{\xi\in\Pi} X^L((\omega_i)_{i\in\xi}) \tag{5.9}$$

and the claim follows from (5.8)–(5.9). □

The cluster expansion from Proposition 5.1 allows us to prove the bounds on the specific relative Gibbs potential and the reduced density matrices, which are uniform in ν, for ν small enough, and in L. For the remainder of the section, we assume that

$$\nu \in (0, 1/\kappa]\,. \tag{5.10}$$

Given $p \in \mathbb{N}^*$, we consider the norm on operators acting on $\ell^2(\Lambda_L)^{\otimes p}$ given by $\|A\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} \equiv \sup_{\mathbf{x}\in\Lambda_L^p}\int_{\Lambda_L^p} \mathrm{d}\mathbf{y}\,|A_{\mathbf{x},\mathbf{y}}|$. By Schur's test, we note that for self-adjoint operators A we have $\|A\| \leqslant \|A\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}}$, where $\|\cdot\|$ denotes the operator norm on $\ell^2(\Lambda_L)^{\otimes p}$.

Proposition 5.2 (Bounds on the specific relative Gibbs potential and reduced density matrices in a finite volume)**.** *For $\|v\|_{\ell^1}$ sufficiently small depending on κ, we have the following bounds for all $L \in \mathbb{N}^*$.*

(i) The specific relative Gibbs potential (1.35) satisfies

$$g^{\nu,\kappa,\nu^2,L} = O_{\kappa,\|v\|_{\ell^1}}(1)\,. \tag{5.11}$$

(ii) For $p \in \mathbb{N}^*$, the p-particle reduced density matrix $\Gamma_p^{\nu,\kappa,\nu^2,L}$ satisfies

$$\nu^p \left\|\Gamma_p^{\nu,\kappa,\nu^2,L}\right\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} = O_{\kappa,p,\|v\|_{\ell^1}}(1)\,. \tag{5.12}$$

From Proposition 5.2, we can deduce the existence of the specific relative Gibbs potential and reduced density matrices in the infinite volume.

Corollary 5.3 (Specific relative Gibbs potential and reduced density matrices in the infinite volume)**.** *With assumptions as in Proposition 5.2, the following claims hold.*

(i) The quantity

$$\lim_{L\to\infty} g^{\nu,\kappa,\nu^2,L} =: g^{\nu,\kappa,\nu^2,\infty} \tag{5.13}$$

exists.

(ii) Let $p \in \mathbb{N}^*$ and be given. We can take $L = \infty$ in (5.6) and obtain an operator $\Gamma_p^{\nu,\kappa,\nu^2,\infty}$ on $\ell^2(\mathbb{Z}^d)^{\otimes p}$ which satisfies

$$\nu^p \left\|\Gamma_p^{\nu,\kappa,\nu^2,\infty}\right\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} = O_{\kappa,p,\|v\|_{\ell^1}}(1)\,. \tag{5.14}$$

In (5.12), we are taking the $\|\cdot\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}}$ norm on operators acting on $\ell^2(\mathbb{Z}^d)^{\otimes p}$.

Remark 5.4. We note that the results in Proposition 5.2 (i) and Corollary 5.3 (i) above are not new. Furthermore, they hold under more general assumptions that do not require the interaction to be small, see [6,28]. Below, we give a short proof under our assumptions using cluster expansions for completeness and for expository purposes. This proof allows us to deduce more specific properties of the thermodynamic limit, such as analyticity.

We can use Proposition 5.2 and Corollary 5.3 to show that specific relative Gibbs potential and reduced density matrices converge to their corresponding infinite-volume limits uniformly in ν.

Proposition 5.5 (Convergence to the infinite-volume limit)**.** *With assumptions as in Proposition 5.2, the following claims hold.*

(i) The convergence in (5.13) holds uniformly in $\nu \in (0, 1/\kappa]$.

(ii) Let $p \in \mathbb{N}^*$ and $L_0 \in \mathbb{N}^*$ be given. Let $\Gamma_p^{\nu,\kappa,\nu^2,\infty}$ be as in Corollary 5.3 (ii). Then, we have

$$\Gamma_p^{\nu,\kappa,\nu^2,\infty} = \lim_{L\to\infty} \Gamma_p^{\nu,\kappa,\nu^2,L} \,. \tag{5.15}$$

The convergence in (5.15) holds in $\|\cdot\|_{L_0,p}$ given by (1.38) and is uniform in $\nu \in (0, 1/\kappa]$ and $L_0 \in \mathbb{N}^*$.

Before proceeding to the proofs of Proposition 5.2, Corollary 5.3, and Proposition 5.5, we note several auxiliary results. The first one is a useful estimate on the Ursell function (5.4).

Lemma 5.6 (Tree bound)**.** *For $n \in \mathbb{N}$ and $\omega_1, \dots, \omega_n \in \Omega$ we have*

$$|\varphi^L(\omega_1, \dots, \omega_n)| \leqslant \frac{1}{n!} \sum_{\mathcal{T}\in\mathfrak{T}_n} \prod_{\{i,j\}\in\mathcal{T}} |\zeta^L(\omega_i, \omega_j)| \,.$$

Here $\mathfrak{T}_n$ denotes the set of all trees on $[n] = \{1, 2, \dots, n\}$.

We prove Lemma 5.6 in Appendix D.1. The proof is based on *Kruskal's algorithm* [21], which we also recall in Appendix D.1 for completeness. The second auxiliary result is the basis of an algorithm that allows us to iteratively integrate all of the paths ω_j in the representation of Proposition 5.1 (recalling (5.1)–(5.4)). We now state the integration lemma.

Lemma 5.7 (Integrating out a vertex)**.** *Let $\omega \in \Omega^{L,T(\omega)}$ with $T(\omega) \in \nu\mathbb{N}^*$, $q \in \mathbb{N}$, and $x \in \Lambda_L$ be given. Then, the following estimates hold.*

(i) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega, \tilde{\omega})| \lesssim \frac{T(\omega)}{\kappa^{q+1}}\, q!\, \|v\|_{\ell^1}$.

(ii) $\nu \int_{\Lambda_L} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega, \tilde{\omega})| \lesssim \frac{T(\omega)}{\kappa^{q+2}}\, (q+1)!\, \|v\|_{\ell^1}$.

(iii) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q \lesssim \frac{(q-1)!}{\kappa^q}\, |\Lambda_L|$ *if* $q \in \mathbb{N}^*$.

(iv) $\nu \int_{\Lambda_L} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q \lesssim \frac{q!}{\kappa^{q+1}}$.

In the proof of Lemma 5.7, we use the following lemma concerning Riemann sums, whose proof is given in Appendix D.2.

Lemma 5.8. *Given $q \in \mathbb{N}$ and assuming (5.10), we have*

$$\nu \sum_{T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa T}\, T^q \lesssim \frac{q!}{\kappa^{q+1}} \,. \tag{5.16}$$

Proof of Lemma 5.7. Throughout the proof, we use the observation that

$$|\zeta^L(\omega,\tilde\omega)| \leqslant \frac{1}{2}\,\mathcal{V}^{\nu,\nu^2,L}(\omega,\tilde\omega)\,, \tag{5.17}$$

which follows from (5.4) and (3.1). Let us first prove (i). By using (1.14), (3.1), (5.1), and (5.17), we need to estimate

$$\begin{aligned}&\int \mu^L(\mathrm{d}\tilde\omega)\,T(\tilde\omega)^q\,\mathcal{V}^{\nu,\nu^2,L}(\omega,\tilde\omega)\\ &\leqslant \nu^2 \sum_{\tilde T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde T}\,\tilde T^{q-1} \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde T} \int_0^\nu \mathrm{d}t \int_{\Lambda_L}\mathrm{d}x \int \mathbb{W}_{x,x}^{L,\tilde T}(\mathrm{d}\tilde\omega)\,v^L(\omega(t+r)-\tilde\omega(t+s))\,.\end{aligned} \tag{5.18}$$

We note that, for fixed $\tilde T\in\nu\mathbb{N}^*, y\in\Lambda_L, t\in[0,\nu], s\in\nu\mathbb{N}$ with $s<\tilde T$ we have

$$\int_{\Lambda_L}\mathrm{d}x \int \mathbb{W}_{x,x}^{L,\tilde T}(\mathrm{d}\tilde\omega)\,v^L(y-\tilde\omega(t+s)) = \int_{\Lambda_L}\mathrm{d}x\int_{\Lambda_L}\mathrm{d}z\int \mathbb{W}_{z,x}^{L,t+s}(\mathrm{d}\tilde\omega_1)\int \mathbb{W}_{x,z}^{L,\tilde T-(t+s)}(\mathrm{d}\tilde\omega_2)\,v^L(y-z)\,, \tag{5.19}$$

which is

$$\begin{aligned}&= \int_{\Lambda_L}\mathrm{d}z\left[\int_{\Lambda_L}\mathrm{d}x\int \mathbb{W}_{x,z}^{L,\tilde T-(t+s)}(\mathrm{d}\tilde\omega_2)\int \mathbb{W}_{z,x}^{L,t+s}(\mathrm{d}\tilde\omega_1)\right] v^L(y-z)\\ &= \int_{\Lambda_L}\mathrm{d}z\,v^L(y-z)\int \mathbb{W}_{z,z}^{L,\tilde T}(\mathrm{d}\tilde\omega) = \int_{\Lambda_L}\mathrm{d}z\,v^L(y-z)\,\psi^{\tilde T}(0) \leqslant \|v^L\|_{\ell^1(\Lambda_L)} = \|v\|_{\ell^1}\,.\end{aligned} \tag{5.20}$$

In order to deduce (5.20), we used Fubini's theorem, the time-reversibility of the random walk, (3.4), Lemma C.1 (i), the assumption that v is even, and (5.5).

Using (5.20) we get that the expression in (5.18) is

$$\leqslant \nu^3 \sum_{\tilde T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde T}\,\tilde T^{q-1} \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde T}\,\|v\|_{\ell^1} = \nu \sum_{\tilde T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde T}\,\tilde T^{q}\,T(\omega)\,\|v\|_{\ell^1}\,. \tag{5.21}$$

We hence deduce (i) from (5.21) and Lemma 5.8.

We now prove (ii). Similarly as in (5.18), we have

$$\begin{aligned}&\nu\int_{\Lambda_L}\mathrm{d}y\int \hat\mu_{y,x}^L(\mathrm{d}\tilde\omega)\,T(\tilde\omega)^q\,\mathcal{V}^{\nu,\nu^2,L}(\omega,\tilde\omega)\\ &\leqslant \nu^2\int_{\Lambda_L}\mathrm{d}y \sum_{\tilde T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde T}\,\tilde T^{q} \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde T} \int \mathbb{W}_{y,x}^{L,\tilde T}(\mathrm{d}\tilde\omega)\int_0^\nu \mathrm{d}t\,v^L(\omega(t+r)-\tilde\omega(t+s))\,.\end{aligned} \tag{5.22}$$

Let us consider fixed $\tilde T\in\nu\mathbb{N}^*$, $r,s\in\nu\mathbb{N}$ with $r<T(\omega), s<\tilde T$, and $t\in[0,\nu]$. We then write

$$\begin{aligned}&\int_{\Lambda_L}\mathrm{d}y\int \mathbb{W}_{y,x}^{L,\tilde T}(\mathrm{d}\tilde\omega)\,v^L(\omega(t+r)-\tilde\omega(t+s))\\ &= \int_{\Lambda_L}\mathrm{d}y\int_{\Lambda_L}\mathrm{d}z\int \mathbb{W}_{z,x}^{L,t+s}(\mathrm{d}\tilde\omega_1)\int \mathbb{W}_{y,z}^{L,\tilde T-(t+s)}(\mathrm{d}\tilde\omega_2)\,v^L(\omega(t+r)-z)\,,\end{aligned} \tag{5.23}$$

which we rewrite by Fubini's theorem as

$$\int_{\Lambda_L}\mathrm{d}z\int \mathbb{W}_{z,x}^{L,t+s}(\mathrm{d}\tilde\omega_1)\left(\int_{\Lambda_L}\mathrm{d}y\int \mathbb{W}_{y,z}^{L,\tilde T-(t+s)}(\mathrm{d}\tilde\omega_2)\right) v^L(\omega(t+r)-z) \leqslant \|v\|_{\ell^1}\,. \tag{5.24}$$

Above, we used (3.4), Lemma C.1 (i), (iii), and (5.5). Using (5.24) and Lemma 5.8, we get that the expression in (5.22) is

$$\leqslant \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}}\,\tilde{T}^{q+1}\,T(\omega)\,\|v\|_{\ell^1} \lesssim \frac{T(\omega)}{\kappa^{q+2}}\,(q+1)!\,\|v\|_{\ell^1}\,. \tag{5.25}$$

This proves (ii).

We now prove (iii). Let $q\in\mathbb{N}^*$ be given. By using (5.1), (3.1), and recalling (3.4), we have

$$\int \mu^L(\mathrm{d}\tilde{\omega})\,T(\tilde{\omega})^q \leqslant \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}}\,\tilde{T}^{q-1}\int_{\Lambda_L}\mathrm{d}x\int \mathbb{W}^{L,\tilde{T}}_{x,x}(\mathrm{d}\tilde{\omega}) = \Big(\nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}}\,\tilde{T}^{q-1}\,\psi^{L,\tilde{T}}(0)\Big)\,|\Lambda_L|\,, \tag{5.26}$$

which by Lemma C.1 (i) and Lemma 5.8 is

$$\lesssim \Big(\nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}}\,\tilde{T}^{q-1}\Big)\,|\Lambda_L| \lesssim \frac{(q-1)!}{\kappa^q}\,|\Lambda_L|\,. \tag{5.27}$$

We hence deduce (iii).

Finally, we prove (iv). By using (5.2), (3.1), followed by (3.4), Lemma C.1 (iii), and Lemma 5.8, we have

$$\nu\int_{\Lambda_L}\mathrm{d}y\int \hat{\mu}^L_{y,x}(\mathrm{d}\tilde{\omega})\,T(\tilde{\omega})^q \leqslant \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}}\,\tilde{T}^{q}\int_{\Lambda_L}\mathrm{d}y\int \mathbb{W}^{L,\tilde{T}}_{y,x}(\mathrm{d}\tilde{\omega}) = \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}}\,\tilde{T}^{q} \lesssim \frac{q!}{\kappa^{q+1}}\,. \tag{5.28}$$

We hence deduce (iv). □

We now introduce some terminology and notation. Given $n\in\mathbb{N}^*$ and $(\delta_1,\dots,\delta_n)\in\mathbb{N}^n$, we define

$$\mathfrak{T}_n^{\delta_1,\dots,\delta_n} := \{\mathcal{T}\in\mathfrak{T}_n\,,\deg(i)=\delta_i\,,i=1\,,\dots,n\}\,. \tag{5.29}$$

In (5.29), $\deg(i)$ denotes the degree of i in $\mathcal{T}$, i.e. the number of vertices in $[n]\setminus\{i\}$ with which i is connected by edges of $\mathcal{T}$. We note that $\mathfrak{T}_1^0$ consists of a single element, i.e. the tree with one vertex and no edges and that $\mathfrak{T}_1^{\delta_1}=\emptyset$ if $\delta_1\neq 0$. For $n\geqslant 2$, $\mathfrak{T}_n^{\delta_1,\dots,\delta_n}$ is nonempty only if $(\delta_1,\dots,\delta_n)\in(\mathbb{N}^*)^n$ and $\sum_{i=1}^n\delta_i = 2(n-1)$.

Let us now note how we can use Lemma 5.7 (i)–(ii) to integrate all of the paths ω_j which correspond to vertices of a tree that are not the root (which we henceforth designate to be 1).

Lemma 5.9 (Integration algorithm)**.** *Let $n\in\mathbb{N}^*$, $(\delta_1,\dots,\delta_n)\in(\mathbb{N}^*)^n$ with $\sum_{i=1}^n\delta_i=2(n-1)$ and $\mathcal{T}\in\mathfrak{T}_n^{\delta_1,\dots,\delta_n}$ be given. Furthermore, let $\mathcal{O},\mathcal{C}\subset\{2,3,\dots,n\}$ such that $\mathcal{O}\sqcup\mathcal{C}=\{2,3,\dots,n\}$ and $(x_i)_{i\in\mathcal{O}}\in\Lambda^{\mathcal{O}}$ be given. For $i\in\{2,3,\dots,n\}$ we let*

$$\hat{\delta}_i := \begin{cases}\delta_i & \text{if } i\in\mathcal{C}\\ \delta_i+1 & \text{if } i\in\mathcal{O}\end{cases} \qquad \Theta_i(\mathrm{d}\omega) := \begin{cases}\mu^L(\mathrm{d}\omega) & \text{if } i\in\mathcal{C}\\ \nu\int_{\Lambda_L}\mathrm{d}y\,\hat{\mu}^L_{y,x_i}(\mathrm{d}\omega) & \text{if } i\in\mathcal{O}\,.\end{cases}$$

With this notation, we have that for $\omega_1\in\Omega^{T_1}$

$$\int \Theta_2(\mathrm{d}\omega_2)\,\Theta_3(\mathrm{d}\omega_3)\cdots\Theta_n(\mathrm{d}\omega_n)\prod_{(i,j)\in\mathcal{T}}|\zeta^L(\omega_i,\omega_j)| \leqslant C^{n-1}\|v\|_{\ell^1}^{n-1}\prod_{i=2}^n\Big(\kappa^{-\hat{\delta}_i}\,(\hat{\delta}_i-1)!\Big)\,T(\omega_1)^{\delta_1}\,. \tag{5.30}$$

We note that, in the statement above, $\mathcal{O}, \mathcal{C}$, denotes the set of vertices $\neq 1$ which correspond to open paths and closed paths respectively.

Proof. We prove the claim by induction on n.

Base. The claim trivially holds when $n = 1$.

Step. Suppose that $n \geqslant 2$ and that (5.30) holds for all trees on at most $n-1$ vertices.

Let $k := \delta_1$. Then ω_1 is connected to $\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_k}$ for $1 < i_1 < i_2 < \cdots < i_k \leqslant n$. By deleting the edges $(1, i_1), \dots, (1, i_k)$ from $\mathcal{T}$, we get a forest $\mathcal{T}_1 \sqcup \mathcal{T}_2 \sqcup \cdots \sqcup \mathcal{T}_k$, where $i_\ell \in \mathrm{V}(\mathcal{T}_\ell)$ for $\ell = 1, \dots, k$. Here, $\mathrm{V}(\cdot)$ denotes the vertex set of a graph.

For $\ell = 1, \dots, k$, let $n_\ell := |\mathrm{V}(\mathcal{T}_\ell)|$. By the inductive assumption, we have

$$\int \prod_{i \in \mathrm{V}(\mathcal{T}_\ell) \setminus \{i_\ell\}} \Theta_i(\mathrm{d}\omega_i) \prod_{\{i,j\} \in \mathcal{T}_\ell} |\zeta^L(\omega_i, \omega_j)| \leqslant C^{n_\ell - 1} \|v\|_{\ell^1}^{n_\ell - 1} \prod_{i \in \mathrm{V}(\mathcal{T}_\ell) \setminus \{i_\ell\}} \left(\kappa^{-\hat{\delta}_i} (\hat{\delta}_i - 1)! \right) T(\omega_{i_\ell})^{\delta_{i_\ell} - 1}, \tag{5.31}$$

for all $\ell = 1, \dots, k$.

We use (5.31) to deduce that the expression on the left-hand side of (5.30) is

$$\leqslant C^{n-k-1} \|v\|_{\ell^1}^{n-k-1} \prod_{i \in [n] \setminus \{i_1, \dots, i_k\}} \left(\kappa^{-\hat{\delta}_i} (\hat{\delta}_i - 1)! \right) \prod_{\ell=1}^{k} \left(\int \Theta_{i_\ell}(\mathrm{d}\omega_{i_\ell}) \, T(\omega_{i_\ell})^{\delta_{i_\ell} - 1} \, |\zeta^L(\omega_1, \omega_{i_\ell})| \right). \tag{5.32}$$

We now apply Lemma 5.7 in each of the k factors in (5.32). More precisely, we apply Lemma 5.7 (i) if $i_\ell \in \mathcal{C}$ and Lemma 5.7 (ii) if $i_\ell \in \mathcal{O}$ and deduce (5.30). □

We record a well-known result about the cardinality of $\mathfrak{T}_n^{\delta_1, \dots, \delta_n}$, which can be obtained e.g. from [31, Theorem 5.3.4].

Lemma 5.10. *Let $n \geqslant 2$ and $(\delta_1, \dots, \delta_n) \in (\mathbb{N}^*)^n$ such that $\sum_{i=1}^n \delta_i = 2(n-1)$ be given. We then have $\left|\mathfrak{T}_n^{\delta_1, \dots, \delta_n}\right| = \frac{(n-2)!}{(\delta_1 - 1)! \cdots (\delta_n - 1)!}$.*

We now have the necessary tools to prove Proposition 5.2.

Proof of Proposition 5.2. Throughout the proof, we work with v such that $\|v\|_{\ell^1}$ is sufficiently small depending on κ in a way to be precisely determined later. We first prove (i). By using Proposition 5.1 (i), (5.3), (5.4), and Lemma 5.6, we deduce that

$$\left| \log \mathcal{Z}^{\nu, \kappa, \nu^2, L} \right| \leqslant \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} \int \mu^L(\mathrm{d}\omega_1) \cdots \mu^L(\mathrm{d}\omega_n) \prod_{\{i,j\} \in \mathcal{T}} |\zeta^L(\omega_i, \omega_j)| \,. \tag{5.33}$$

By recalling (5.29), we rewrite the right-hand side of (5.33) as

$$\sum_{n=2}^{\infty} \frac{1}{n!} \sum_{\substack{(\delta_1, \dots, \delta_n) \in \mathbb{N}^n \\ \delta_1 + \cdots + \delta_n = 2(n-1)}} \sum_{\mathcal{T} \in \mathfrak{T}_n^{\delta_1, \dots, \delta_n}} \int \mu^L(\mathrm{d}\omega_1) \cdots \mu^L(\mathrm{d}\omega_n) \prod_{\{i,j\} \in \mathcal{T}} |\zeta^L(\omega_i, \omega_j)| \,. \tag{5.34}$$

By using (5.33)–(5.34) as well as Lemma 5.9 with $\mathcal{C} = \{2, \dots, n\}$ in each term of (5.34), we deduce that

$$\left| \log \mathcal{Z}^{\nu, \kappa, \nu^2, L} \right| \lesssim \sum_{n=2}^{\infty} \frac{C^{n-1}}{n!} (n-1)! \, \|v\|_{\ell^1}^{n-1} \frac{1}{\kappa^{2n-2}} |\Lambda_L| = \sum_{n=2}^{\infty} \frac{1}{n} \left(\frac{C}{\kappa^2} \|v\|_{\ell^1} \right)^{n-1} |\Lambda_L| \,. \tag{5.35}$$

Here, we used the observation that for $(\delta_1, \dots, \delta_n)$ as above, we have

$$\prod_{i=2}^{n} \Big(\kappa^{-\delta_i} \, (\delta_i - 1)! \Big) \int \mu^L(\mathrm{d}\omega_1) \, T_1^{\delta_1} \lesssim \frac{1}{\kappa^{2n-2}} \prod_{i=1}^{n} (\delta_i - 1)! \, |\Lambda| \,, \tag{5.36}$$

which follows from Lemma 5.7 (iii). We deduce (5.35) from (5.36) by noting that for $n \geqslant 2$, there are $\binom{2n-3}{n-1} \leqslant C^{n-1}$ possible choices of $(\delta_1, \dots, \delta_n) \in (\mathbb{N}^*)^n$ such that $\delta_1 + \cdots + \delta_n = 2(n-1)$ and by using Lemma 5.10. Claim (i) now follows.

We now prove (ii). By Proposition 5.1 (ii), it suffices to show that for all $\mathbf{x} \in \Lambda^p$, we have

$$\nu^p \int_{\Lambda_L^p} \mathrm{d}\mathbf{y} \int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \hat{\mu}^L_{y_p,x_p}(\mathrm{d}\omega_p) \sum_{\Pi \in \mathfrak{P}_p} \prod_{\xi \in \Pi} |X^L((\omega_i)_{i \in \xi})| = O_{\kappa,p,\|v\|_{\ell^1}}(1) \,. \tag{5.37}$$

Let us first estimate the contribution to the left-hand side of (5.37) coming from the trivial partition $\Pi = [p]$. To this end, we define

$$(\widetilde{\Gamma}_p^{\nu,\kappa,\nu^2,L})_{\mathbf{x},\mathbf{y}} := \int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \hat{\mu}^L_{y_p,x_p}(\mathrm{d}\omega_p) \, |X^L(\omega_1, \dots, \omega_p)|$$

for $\mathbf{x}, \mathbf{y} \in \Lambda^p$. Recalling (5.3) and arguing analogously as in (5.33)–(5.34), we deduce that

$$\begin{aligned} (\widetilde{\Gamma}_p^{\nu,\kappa,\nu^2,L})_{\mathbf{x},\mathbf{y}} \leqslant \sum_{n \geqslant p} \frac{1}{(n-p)!} \sum_{\substack{(\delta_1,\dots,\delta_n) \in \mathbb{N}^n \\ \delta_1 + \cdots + \delta_n = 2(n-1)}} \sum_{\mathcal{T} \in \mathfrak{T}_n^{\delta_1,\dots,\delta_n}} \int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \hat{\mu}^L_{y_p,x_p}(\mathrm{d}\omega_p) \\ \times \mu^L(\mathrm{d}\omega_{p+1}) \cdots \mu^L(\mathrm{d}\omega_n) \prod_{\{i,j\} \in \mathcal{T}} |\zeta^L(\omega_i, \omega_j)| \,. \end{aligned} \tag{5.38}$$

We first integrate (5.38) with respect to $y_2, \dots, y_p$. By using Lemma 5.9 with $\mathcal{O} = \{2, 3, \dots, p\}, \mathcal{C} = \{p+1, \dots, n\}$ in each term of (5.38), we deduce that

$$\begin{aligned} \nu^p \int_{\Lambda_L^{p-1}} \mathrm{d}y_2 \cdots \mathrm{d}y_p \, (\widetilde{\Gamma}_p^{\nu,\kappa,\nu^2,L})_{\mathbf{x},\mathbf{y}} &\leqslant \mathbf{1}_{p=1} \int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1) \\ &+ \sum_{n \geqslant p \vee 2} \frac{1}{(n-p)!} \sum_{\substack{(\delta_1,\dots,\delta_n) \in \mathbb{N}^n \\ \delta_1 + \cdots + \delta_n = 2(n-1)}} \sum_{\mathcal{T} \in \mathfrak{T}_n^{\delta_1,\dots,\delta_n}} C^{n-1} \|v\|_{\ell^1}^{n-1} \prod_{i=2}^{n} \Big(\kappa^{-\hat{\delta}_i} \, (\hat{\delta}_i - 1)! \Big) \int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1) \, T_1^{\delta_1} \,. \end{aligned} \tag{5.39}$$

We now use (5.39), Lemma 5.7 (iv), the fact that $\delta_1, \dots, \delta_n \leqslant n-1$, and argue analogously as for (5.35) to deduce that

$$\nu^p \int_{\Lambda_L^p} \mathrm{d}\mathbf{y} \, (\widetilde{\Gamma}_p^{\nu,\kappa,\nu^2,L})_{\mathbf{x},\mathbf{y}} \leqslant \frac{C}{\kappa} + \sum_{n \geqslant p \vee 2} \bigg(\frac{(n-1)^2}{\kappa} \bigg)^p \bigg(\frac{C}{\kappa^2} \, \|v\|_{\ell^1} \bigg)^{n-1} . \tag{5.40}$$

We use (5.40) to estimate the contribution to the left-hand side of (5.37) coming from $\Pi = [p]$.

By arguing analogously as for (5.40) we deduce that for a general nonempty set $\xi \subset [p]$

$$\nu^{|\xi|} \int_{\Lambda_L^{|\xi|}} \prod_{i \in \xi} \mathrm{d}y_i \prod_{i \in \xi} \int \hat{\mu}^L_{y_i,x_i}(\mathrm{d}\omega_i) \, |X^L((\omega_i)_{i \in \xi})| \leqslant \frac{C}{\kappa} + \sum_{n \geqslant |\xi| \vee 2} \bigg(\frac{(n-1)^2}{\kappa} \bigg)^{|\xi|} \bigg(\frac{C}{\kappa^2} \, \|v\|_{\ell^1} \bigg)^{n-1} . \tag{5.41}$$

In the proof of (5.41), it is important that the last vertex over which we integrate corresponds to an open path (which in the proof of (5.40) was ω_1). This is possible to do by construction. We deduce (5.37) from (5.41) and claim (ii) follows. □

We can now deduce Corollary 5.3.

Proof of Corollary 5.3. We first prove (i). By standard arguments, it suffices to show the following two claims.

(1) For fixed $\nu > 0$, the quantity $g_{\nu,\kappa,\nu^2,L}$ is bounded in $L \in \mathbb{N}^*$.

(2) For disjoint boxes $\Lambda', \Lambda'' \subset \mathbb{Z}^d$ and $\Lambda := \Lambda' \cup \Lambda''$, we have

$$\Xi^{\nu,\kappa,\nu^2,\Lambda} \geqslant \Xi^{\nu,\kappa,\nu^2,\Lambda'} \Xi^{\nu,\kappa,\nu^2,\Lambda''}, \qquad \Xi^{\nu,\kappa,0,\Lambda} = \Xi^{\nu,\kappa,0,\Lambda'} \Xi^{\nu,\kappa,0,\Lambda''}. \tag{5.42}$$

Here, we slightly modify the notation to replace the L-dependence by dependence on the domain.

For a more detailed explanation on why these conditions are sufficient, we refer the reader to [16, Section 2.2], as well as [6, 28]. We note that (1) follows immediately from Proposition 5.2 (i). In order to prove the inequality in (2), we note that by the nonnegativity of v that for $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n), \boldsymbol{\omega}' = (\omega_1, \dots, \omega_{n'})$, we have

$$V^{\nu,\nu^2}(\boldsymbol{\omega}, \boldsymbol{\omega}') - V^{\nu,\nu^2}(\boldsymbol{\omega}) - V^{\nu,\nu^2}(\boldsymbol{\omega}') = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n'} \mathcal{V}^{\nu,\nu^2}(\omega_i, \omega'_j) \geqslant 0. \tag{5.43}$$

The arguments in [16, Section 2.2] allow us to deduce the inequality (5.42) from (5.43). The equality follows by analogous arguments, since in this case equality holds in (5.43). Hence (i) follows. Claim (ii) follows directly from the proof of Proposition 5.2 (ii). □

Before proving Proposition 5.5, we note two technical lemmas that we apply in the proof. We first introduce some notation. Throughout the sequel, we denote by $|\cdot|_L$ the (periodic) Euclidean norm on Λ_L. Furthermore, we fix $c > 0$ small and let

$$\mathcal{D}_c^L := \Big\{ \omega \in \bigcup_{T>0} \Omega^{L,T}, |\omega(s) - \omega(t)|_L \geqslant cL \text{ for some } s, t \in [0, T(\omega)] \Big\}. \tag{5.44}$$

All of the estimates below depend on c, but we do not keep explicit track of this dependence.

The first lemma is a modification of Lemma 5.7 telling us that we get small contributions if we integrate over long paths.

Lemma 5.11. *Let $\omega \in \Omega^{L,T(\omega)}$ with $T(\omega) \in \nu\mathbb{N}^*$ and $q \in \mathbb{N}$ be given. Then, the following estimates hold.*

(i) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega, \tilde{\omega})|\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \lesssim_{\kappa,q} T(\omega)\, \|v\|_{\ell^1}\, \mathrm{e}^{-CL}$.

(ii) Let $L_0 \in \mathbb{N}^*$ with

$$L_0 \leqslant L/4 \tag{5.45}$$

be given. Then, for $x \in \Lambda_{L_0}$, we have

$$\nu \int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega, \tilde{\omega})|\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \lesssim_{\kappa,q} T(\omega)\, \|v\|_{\ell^1}\, \mathrm{e}^{-CL}.$$

(iii) For L_0 as in (5.45) and $x \in \Lambda_{L_0}$, we have

$$\nu \int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \lesssim_{\kappa,q} \mathrm{e}^{-CL}. \tag{5.46}$$

The second lemma states that the contribution from two interacting paths which are far away is small. In order to state this precisely, we need to introduce some notation. With $c > 0$ as earlier, we let $\mathcal{V}_c^{\nu,\nu^2,L}(\omega,\tilde{\omega})$ denote the quantity given as in (1.14) with interaction potential given by

$$v_c^L(x) := v^L(x)\,\mathbf{1}_{|x|_L \geqslant cL}\,. \tag{5.47}$$

Note that, by (1.34) and Assumption 1.5 (iii), we have that

$$\lim_{L\to\infty} \|v_c^L\|_{\ell^1(\Lambda_L)} = 0\,. \tag{5.48}$$

In analogy with (5.4), we define

$$\zeta_c^L(\omega,\tilde{\omega}) := \exp\bigl(-\mathcal{V}_c^{\nu,\nu^2,L}(\omega,\tilde{\omega})/2\bigr) - 1\,. \tag{5.49}$$

With this notation, we have the following lemma.

Lemma 5.12. *With assumptions as in Lemma 5.11, the following estimates hold.*

(i) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta_c^L(\omega,\tilde{\omega})| \lesssim_{\kappa,q} T(\omega)\, \|v_c^L\|_{\ell^1(\Lambda_L)}$.

(ii) $\nu \int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta_c^L(\omega,\tilde{\omega})| \lesssim_{\kappa,q} T(\omega)\, \|v_c^L\|_{\ell^1(\Lambda_L)}$ *for all* $x \in \Lambda_{L_0}$.

We prove Lemmas 5.11 and 5.12 in Appendix D.3. We now have all the necessary tools to prove Proposition 5.5.

Proof of Proposition 5.5. We first show claim (ii) and then explain how the proof can be modified to obtain claim (i). By Proposition 5.2 (ii) and Corollary 5.3 (ii), we can consider fixed $n \in \mathbb{N}^*$ in (5.3) and fixed $\pi \in S_p$ in (5.6). Without loss of generality, we can assume that π is the identity and reduce to proving that

$$\begin{aligned}\lim_{L\to\infty} \nu^p \Bigl\| \int \hat{\mu}_{y_1,x_1}^L(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^L(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\varphi^L(\omega_1,\dots,\omega_n) \\ - \int \hat{\mu}_{y_1,x_1}^\infty(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^\infty(\mathrm{d}\omega_p)\,\mu^\infty(\mathrm{d}\omega_{p+1})\cdots\mu^\infty(\mathrm{d}\omega_n)\,\varphi^\infty(\omega_1,\dots,\omega_n)\Bigr\|_{L_0,p} = 0\,,\end{aligned} \tag{5.50}$$

where we recall (1.38) and also consider (5.1)–(5.2) and (5.4) on the infinite lattice $\mathbb{Z}^d$, with analogous definitions. In (5.50) and in the remainder of the proof, all of the convergence claims are interpreted as being uniform in $\nu \in (0, 1/\kappa]$.

Let us now prove (5.50). Throughout the sequel, we assume that $L \in \mathbb{N}^*$ satisfies (5.45) above. Given such an L, we take $L_1 \in \mathbb{N}^*$ to be the smallest even integer

$$L_1 \geqslant L/2\,, \tag{5.51}$$

and define

$$\mathcal{A}^L := \Bigl\{\omega \in \bigcup_{T>0} \Omega^{\infty,T}\,, \omega(t) \in \Lambda_{L_1}\ \forall t \in [0,T(\omega)]\Bigr\}\,, \quad \mathcal{B}^L := \bigcup_{T>0} \Omega^{\infty,T} \setminus \mathcal{A}^L\,. \tag{5.52}$$

We first show that for all $k \in \{1,\dots,n\}$, we have

$$\lim_{L\to\infty} \nu^p \Bigl\| \int \hat{\mu}_{y_1,x_1}^L(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^L(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\mathbf{1}_{\mathcal{B}^L}(\omega_k)\,\varphi^L(\omega_1,\dots,\omega_n)\Bigr\|_{L_0,p} = 0\,. \tag{5.53}$$

By Lemma 5.6, (5.53) follows if we show that

$$\lim_{L\to\infty} \nu^p \Big\| \int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^L_{y_p,x_p}(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\mathbf{1}_{\mathcal{B}^L}(\omega_k)\prod_{\{i,j\}\in\mathcal{T}}|\zeta^L(\omega_i,\omega_j)|\Big\|_{L_0,p} = 0\,, \tag{5.54}$$

for a fixed $\mathcal{T}\in\mathfrak{T}_n$. We note that (5.54) follows by applying the triangle inequality together with Lemmas 5.11 and 5.12. More precisely, we note that in (5.54), the nonzero contribution comes from $x_1\in\Lambda_{L_0}$. For such x_1, we recall that by (5.45) and (5.52), there exists a point z on ω_k with the property that $|z-x_1|_L \gtrsim L$. In particular, by the triangle inequality, it follows that, for the paths over which we are integrating in (5.54), at least one of the following cases occur with suitable $c\gtrsim\frac{1}{n}$.

(1) There exists $i\in\{1,\dots,n\}$ such that $\omega_i\in\mathcal{D}^{L,c}$. Here, we recall (5.44).

(2) There exists $\{i,j\}\in\mathcal{T}$ such that $\zeta^L(\omega_i,\omega_j)=\zeta^L_c(\omega_i,\omega_j)$. In other words, the two-particle interaction is given by (5.47).

We now prove (5.54) by arguing analogously as in the proof of Proposition 5.2 (ii). We just modify the proof to keep track of cases (1) and (2) above. If (1) occurs, we apply Lemma 5.11 when integrating ω_i. If (2) occurs, we apply Lemma 5.12 when integrating the path ω_i or ω_j (determined by the algorithm from Lemma 5.54). In this case, we also recall (5.48) to note that the estimates that we get from Lemma 5.12 tend to zero as $L\to\infty$. We hence deduce (5.53).

By arguing analogously as for (5.53), we get that for all $k\in\{1,\dots,n\}$,

$$\lim_{L\to\infty} \nu^p \Big\| \int \hat{\mu}^\infty_{y_1,x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^\infty_{y_p,x_p}(\mathrm{d}\omega_p)\,\mu^\infty(\mathrm{d}\omega_{p+1})\cdots\mu^\infty(\mathrm{d}\omega_n)\,\mathbf{1}_{\mathcal{B}^L}(\omega_k)\,\varphi^\infty(\omega_1,\dots,\omega_n)\Big\|_{L_0,p} = 0\,. \tag{5.55}$$

Combining (5.52), (5.53), (5.55), and recalling (1.38), we deduce that (5.50) follows if we prove that

$$\begin{aligned}\lim_{L\to\infty} \nu^p \Big\| &\int \hat{\mu}^L_{y_1,x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^L_{y_p,x_p}(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\prod_{k=1}^n \mathbf{1}_{\mathcal{A}^L}(\omega_k)\,\varphi^L(\omega_1,\dots,\omega_n)\\ &- \int \hat{\mu}^\infty_{y_1,x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^\infty_{y_p,x_p}(\mathrm{d}\omega_p)\,\mu^\infty(\mathrm{d}\omega_{p+1})\cdots\mu^\infty(\mathrm{d}\omega_n)\prod_{k=1}^n \mathbf{1}_{\mathcal{A}^L}(\omega_k)\,\varphi^\infty(\omega_1,\dots,\omega_n)\Big\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} = 0\,.\end{aligned} \tag{5.56}$$

By (5.1)–(5.2) and (5.52), we have that

$$\hat{\mu}^L_{y,x}(\mathrm{d}\omega)\,\mathbf{1}_{\mathcal{A}^L}(\omega) = \hat{\mu}^\infty_{y,x}(\mathrm{d}\omega)\,\mathbf{1}_{\mathcal{A}^L}(\omega)\,,\quad \mu^L(\mathrm{d}\omega)\,\mathbf{1}_{\mathcal{A}^L}(\omega) = \mu^\infty(\mathrm{d}\omega)\,\mathbf{1}_{\mathcal{A}^L}(\omega)\,. \tag{5.57}$$

In other words, the presence of the indicator functions $\mathbf{1}_{\mathcal{A}^L}(\omega)$ erases the boundary effects. By (5.57), we note that (5.56) follows if we prove that

$$\begin{aligned}\lim_{L\to\infty} \nu^p \Big\| &\int \hat{\mu}^\infty_{y_1,x_1}(\mathrm{d}\omega_1)\cdots\hat{\mu}^\infty_{y_p,x_p}(\mathrm{d}\omega_p)\,\mu^\infty(\mathrm{d}\omega_{p+1})\cdots\mu^\infty(\mathrm{d}\omega_n)\prod_{k=1}^n \mathbf{1}_{\mathcal{A}^L}(\omega_k)\\ &\times\,|\varphi^L(\omega_1,\dots,\omega_n)-\varphi^\infty(\omega_1,\dots,\omega_n)|\Big\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} = 0\,.\end{aligned} \tag{5.58}$$

Recalling the bound from Lemma 5.6 and the integration algorithm from the proof of Proposition 5.2, by the dominated convergence theorem, we deduce that (5.58) follows if we prove that

$$\lim_{L\to\infty} \varphi^L(\omega_1,\dots,\omega_n) = \varphi^\infty(\omega_1,\dots,\omega_n)\,, \tag{5.59}$$

for fixed paths $\omega_1,\dots,\omega_n$. We note that by (1.34) and Assumption 1.5 (iii), it follows that

$$\lim_{L\to\infty} v^L(x) = v(x) \quad \forall x\in\mathbb{Z}^d\,. \tag{5.60}$$

Using (5.60) and recalling (1.14), it follows that

$$\lim_{L\to\infty} \mathcal{V}^{\nu,\nu^2,L}(\omega,\tilde\omega) = \mathcal{V}^{\nu,\nu^2,\infty}(\omega,\tilde\omega) \tag{5.61}$$

for all paths $\omega,\tilde\omega$. Substituting (5.61) into (5.4), we deduce (5.59). By construction, the convergence is uniform in ν and L_0. Claim (ii) now follows.

We note that (i) follows by arguing as in the proof of (ii). Namely, we differentiate with respect to κ to obtain

$$\frac{\partial g^{\nu,\kappa,\nu^2,L}}{\partial\kappa} = -\frac{\nu}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}x\,(\Gamma_1^{\nu,\kappa,\nu^2,L})_{x,x} + \frac{\nu}{|\Lambda_L|}\int_{\Lambda_L}\mathrm{d}x\,(\Gamma_1^{\nu,\kappa,0,L})_{x,x}\,. \tag{5.62}$$

We analyse each of the terms in (5.62) separately. We fix $\varepsilon>0$ small and let $L_{0,\varepsilon}\in\mathbb{N}^*$ be the largest even integer

$$L_{0,\varepsilon} \leqslant L - L^\varepsilon\,. \tag{5.63}$$

By Proposition 5.2 (ii) and the observation that

$$\lim_{L\to\infty}\frac{|\Lambda_L\setminus\Lambda_{L_{0,\varepsilon}}|}{|\Lambda_L|} = 0\,,$$

we deduce that the first term in (5.62) is

$$= -\frac{\nu}{|\Lambda_L|}\int_{\Lambda_{L_{0,\varepsilon}}}\mathrm{d}x\,(\Gamma_1^{\nu,\kappa,\nu^2,L})_{x,x} + o_L(1)\,, \tag{5.64}$$

where $o_L(1)$ denotes a quantity that converges to zero as $L\to\infty$ uniformly in ν. By an analogous proof as in (i), we have that for all $x\in\Lambda_{L_{0,\varepsilon}}$

$$\nu(\Gamma_1^{\nu,\kappa,\nu^2,L})_{x,x} = \nu(\Gamma_1^{\nu,\kappa,\nu^2,\infty})_{x,x} + o_L(1)\,, \tag{5.65}$$

More precisely, we replace L_0 (as in (5.45)) with $L_{0,\varepsilon}$ as in (5.63). Furthermore, we replace L_1 as in (5.51) with $L_{1,\varepsilon}\in\mathbb{N}^*$ the smallest even integer such that $L_{1,\varepsilon}\geqslant L-\frac{1}{2}L^\varepsilon$. We then deduce (5.65) by arguing as for (i). Using (5.65) and noting that $(\Gamma_1^{\nu,\kappa,\nu^2,\infty})_{x,x} = (\Gamma_1^{\nu,\kappa,\nu^2,\infty})_{0,0}$ by translation invariance, we have

$$(5.64) = -\frac{|\Lambda_{L_{0,\varepsilon}}|}{|\Lambda_L|}\nu(\Gamma_1^{\nu,\kappa,\nu^2,\infty})_{0,0} + o_L(1) = -\nu(\Gamma_1^{\nu,\kappa,\nu^2,\infty})_{0,0} + o_L(1)\,, \tag{5.66}$$

where for the last equality we recalled the construction of $L_{0,\varepsilon}$. An identity analogous to (5.66) with $v=0$ holds for the second term in (5.62). In particular, we can rewrite (5.62) as

$$\frac{\partial g^L_{\nu,\kappa}}{\partial\kappa} = -\nu(\Gamma_1^{\nu,\kappa,\nu^2,\infty})_{0,0} + \nu(\Gamma_1^{\nu,\kappa,0,\infty})_{0,0} + o_L(1)\,. \tag{5.67}$$

We deduce (i) from (5.67) by integrating in κ. □

Let us recall a general fact about interchanging the order of limits, whose proof we omit.

Lemma 5.13. *Suppose that $\rho : \mathbb{N}^* \times (0,\infty) \to \mathbb{C}$ is a function that satisfies the following properties.*

(1) There exists $\nu_0 > 0$ such that the limit

$$\lim_{L\to\infty} \rho(L,\nu) =: \rho(\infty,\nu)\,, \tag{5.68}$$

exists uniformly in $\nu \in (0,\nu_0]$.

(2) For all $L \in \mathbb{N}^*$, there exists $\rho(L,0)$ such that

$$\lim_{\nu\to 0} \rho(L,\nu) = \rho(L,0)\,. \tag{5.69}$$

Then, the following properties hold.

(i) $\rho(\infty,0) := \lim_{L\to\infty} \rho(L,0)$ exists.

(ii) With $\rho(\infty,0)$ given as in (i), we have that $\lim_{\nu\to 0} \rho(\infty,\nu) = \rho(\infty,0)$.

We now have the necessary tools to prove Theorem 1.6.

Proof of Theorem 1.6. We first prove claim (i). This follows from Theorem 1.2 (i), Proposition 5.5 (i), and Lemma 5.13 by taking $\rho(L,\nu) = g^{\nu,\kappa,\nu^2,L}$ for $\nu \in (0,1/\kappa]$ and $\rho(L,0) = g^{\mathrm{cl},L}$ with $\nu_0 = 1/\kappa$.

In order to prove claim (ii), we also apply a suitable modification of Lemma 5.13 where all of the convergence is taken in the norm $\|\cdot\|_{L_0,p}$ given by (1.38). Namely, we first use Proposition 5.5 (ii) and Theorem 1.2 (ii) to note that assumptions (1) and (2) of Lemma 5.13 hold if we take $\rho(L,\nu) = \nu^p\,\Gamma_p^{\nu,\kappa,\nu^2,L}$, $\rho(L,0) = \Gamma_p^{\mathrm{cl},L}$ and if the convergence in (5.68)–(5.69) is interpreted with respect to $\|\cdot\|_{L_0,p}$. By Lemma 5.13 (i), we deduce that the limit $\lim_{L\to\infty} \gamma_p^L$ exists in $\|\cdot\|_{L_0,p}$. Note that, a priori this quantity depends on L_0 (since the norm $\|\cdot\|_{L_0,p}$ depends on L_0). However, by recalling (1.37)–(1.38) and by construction, it follows that this limit is independent of L_0. We conclude the result of claim (ii) from Lemma 5.13 (ii). □

Remark 5.14. In the noninteracting case $v = 0$, the measure $\mu \equiv \mu^L$ given in (5.1) is a discrete-time version of Lawler's and Werner's loop soup intensity measure [22], to which it converges as $\nu \to 0$. Indeed, for $v = 0$ the measure $\mu(\mathrm{d}\omega)$ converges as $\nu \to 0$ to $\xi(\mathrm{d}\omega) = \int_0^\infty \mathrm{d}T\, \frac{\mathrm{e}^{-\kappa T}}{T}\, \mathbb{W}^T(\mathrm{d}\omega)$, which is precisely the intensity measure of the loop soup (a Poisson process with intensity ξ).

5.2. Infinite-volume limit of the specific relative Gibbs potential and reduced density matrices II: the large-mass regime. In this subsection, we work in the large-mass regime. Throughout, we assume that the interaction potential on Λ_L is given by (1.34) for v as in Assumption 1.7. We recall that we are considering the parameters $\kappa = \kappa_0/\nu$ for fixed κ_0 and $\lambda = 1$ and that the many-body Hamiltonian is given by (1.28).

Recalling the definition of $\widetilde{\mathcal{V}}^{\nu,1}(\omega) \equiv \widetilde{\mathcal{V}}^{\nu,1,L}(\omega)$ with interaction v^L given by (4.2), we define the self-interaction

$$\widehat{\mathcal{V}}^{\nu,1,L}(\omega) := \widetilde{\mathcal{V}}^{\nu,1,L}(\omega) + \frac{1}{\nu}\, v^L(0) T(\omega)\, \mathbf{1}_{R=0} \geqslant 0\,. \tag{5.70}$$

Furthermore, we modify the definition of the measures (5.1)–(5.2) according to

$$\mu^L(\mathrm{d}\omega) := \nu \sum_{T\in\nu\mathbb{N}^*} \frac{\mathrm{e}^{-\kappa_0 T/\nu}}{T}\, \mathbb{W}^{L,T}(\mathrm{d}\omega)\, \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1,L}(\omega)/2}\,, \tag{5.71}$$

$$\hat{\mu}^L_{y,x}(\mathrm{d}\omega) := \sum_{T\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0 T/\nu}\, \mathbb{W}^{L,T}_{y,x}(\mathrm{d}\omega)\, \mathrm{e}^{-\widehat{\mathcal{V}}^{\nu,1,L}(\omega)/2}\,. \tag{5.72}$$

With v and R as in Assumption 1.7, we define $v^{(1)}, v^{(2)} : \mathbb{Z}^d \to [0,\infty]$ by

$$v^{(1)} := v\,\mathbf{1}_{|x|<R}\,, \qquad v^{(2)} := v - v^{(1)}\,. \tag{5.73}$$

Note that, by Assumption 1.7 (i), we have that

$$v^{(2)} \in \ell^1(\mathbb{Z}^d)\,. \tag{5.74}$$

With notation as in (5.73), we define $v^{L,(1)}, v^{L,(2)} : \Lambda_L \to [0,\infty]$ by

$$v^{L,(j)}(x) := \sum_{k\in(L\mathbb{Z})^d} v^{(j)}(x+k)\,, \quad j=1,2\,. \tag{5.75}$$

Throughout, we assume that

$$\kappa_0 \leqslant 1\,, \quad \nu \leqslant 1\,. \tag{5.76}$$

With notation as in (5.71)–(5.73), and assuming (5.76), we note the following analogue of Lemma 5.7.

Lemma 5.15. *Let $\omega \in \Omega^{L,T(\omega)}$ with $T(\omega) \in \nu\mathbb{N}^*$, $q \in \mathbb{N}$, and $x \in \Lambda_L$ be given. Then, the following estimates hold.*

(i) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega,\tilde{\omega})| \lesssim \frac{T(\omega)}{\kappa_0^{q+1}}\, q!\, (1+\|v^{(2)}\|_{\ell^1})\, \nu^{q-1}$.

(ii) $\int_{\Lambda_L} \mathrm{d}y \int \hat{\mu}^L_{y,x}(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega,\tilde{\omega})| \lesssim \frac{T(\omega)}{\kappa_0^{q+2}}\, (q+1)!\, (1+\|v^{(2)}\|_{\ell^1})\, \nu^{q-1}$.

(iii) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q \lesssim \frac{(q-1)!}{\kappa_0^q}\, |\Lambda_L|\, \nu^q$ *if* $q \in \mathbb{N}^*$.

(iv) $\int_{\Lambda_L} \mathrm{d}y \int \hat{\mu}^L_{y,x}(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q \lesssim \frac{q!}{\kappa_0^{q+1}}\, \nu^q$.

Here we recall the definition (5.4).

Proof. We first prove (i). We let $\mathcal{V}^{\nu,1,L,(2)}(\omega,\tilde{\omega})$ be given as in (1.14), where we replace $v \equiv v^L$ by $v^{L,(2)}$ as given in (5.75) above. We define the set $\mathcal{I}(\omega,\nu)$ as

$$\mathcal{I}(\omega,\nu) := \left\{ \tilde{\omega} \in \bigcup_{\tilde{T}>0} \Omega^{L,\tilde{T}}\,,\ \exists t \in [0,T(\omega)]\ \exists \tilde{t} \in [0,T(\tilde{\omega})]\,,\ \omega(t) = \tilde{\omega}(\tilde{t})\,,\ t-\tilde{t} \in \nu\mathbb{Z} \right\}. \tag{5.77}$$

In other words $\tilde{\omega} \in \mathcal{I}(\omega,\nu)$ if and only if ω and $\tilde{\omega}$ intersect at times which are equal modulo $\nu\mathbb{Z}$. By Assumption 1.7, the construction of $\mathcal{V}^{\nu,1,L,(2)}(\omega,\tilde{\omega})$ and (5.73), (5.77), we have

$$|\zeta^L(\omega,\tilde{\omega})| \leqslant \mathbf{1}_{\mathcal{I}(\omega,\nu)}(\tilde{\omega}) + \frac{1}{2}\, \mathcal{V}^{\nu,1,L,(2)}(\omega,\tilde{\omega})\,. \tag{5.78}$$

We now estimate the contribution to the left-hand side of (i) coming from each of the expressions in the bound (5.78). For the first term, we use (5.70)–(5.71) to write

$$\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, \mathbf{1}_{\mathcal{I}(\omega,\nu)}(\tilde{\omega}) \leqslant \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\, \tilde{T}^{q-1} \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\, \mathbf{1}_{\mathcal{I}(\omega,\nu)}(\tilde{\omega})\,,$$

which, recalling (5.77) is

$$\leqslant \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\, \tilde{T}^{q-1} \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde{T}} \int_0^\nu \mathrm{d}t \int_{\Lambda_L} \mathrm{d}x\, \psi^{L,t+s}(\omega(t+r)-x)\, \psi^{L,\tilde{T}-(t+s)}(x-\omega(t+r))\,. \quad (5.79)$$

By using Lemma C.1 (i) and (iii), we note that (5.79) is

$$\leqslant \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\, \tilde{T}^q\, T(\omega) \leqslant \nu^q\, T(\omega) \sum_{j=1}^\infty \mathrm{e}^{-\kappa_0 j}\, j^q \lesssim \frac{T(\omega)}{\kappa_0^{q+1}}\, q!\, \nu^q\,. \quad (5.80)$$

In order to obtain (5.80), we recalled (5.76) and used Lemma 5.8.

For the second term, we argue argue analogously as in (5.18)–(5.21) with v replaced by $v^{L,(2)}$ and deduce that

$$\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, \mathcal{V}^{\nu,1,L,(2)}(\omega,\tilde{\omega}) \lesssim \frac{T(\omega)}{\kappa_0^{q+1}}\, q!\, \|v\|_{\ell^1}\, \nu^{q-1}\,. \quad (5.81)$$

In order to deduce (5.81), we used the observation that

$$\|v^{L,(2)}\|_{\ell^1(\Lambda_L)} = \|v^{(2)}\|_{\ell^1(\mathbb{Z}^d)}\,, \quad (5.82)$$

which follows from (5.75). Claim (i) then follows from (5.78), (5.80)–(5.81).

We now prove (ii). Let $x\in\Lambda_L$ be fixed. As in the proof of (i), we need to estimate the two terms coming from (5.78). By (5.70), (5.72), the first term is

$$\leqslant \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\, \tilde{T}^q \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<T(\tilde{\omega})} \int_0^\nu \mathrm{d}t \int_{\Lambda_L} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\, \mathbf{1}_{\omega(t+r)=\tilde{\omega}(t+s)} \lesssim \frac{T(\omega)}{\kappa_0^{q+2}}\, (q+1)!\, \nu^q\,. \quad (5.83)$$

In (5.83), we bounded the indicator function by 1 and argued as in (5.80). By analogous arguments as in (5.22)–(5.25), and recalling (5.82), the second term coming from (5.78) is

$$\lesssim \frac{T(\omega)}{\kappa_0^{q+2}}\, (q+1)!\, \|v^{(2)}\|_{\ell^1}\, \nu^{q-1}\,. \quad (5.84)$$

Claim (ii) now follows from (5.78), (5.83)–(5.84).

We now prove (iii), we use (5.70)–(5.71) and argue analogously as in (5.26)–(5.27) to deduce that

$$\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q \lesssim \frac{(q-1)!}{\kappa_0^q}\, |\Lambda_L|\, \nu^q\,.$$

Claim (iv) follows by analogous arguments. □

We can now deduce an analogue of Proposition 5.2.

Proposition 5.16. *For $\|v\|_{\ell^1}$ sufficiently small depending on κ_0, we have the following bounds for all $L \in \mathbb{N}^*$.*

(i) The specific relative Gibbs potential (1.35) satisfies $g^{\nu,\kappa_0/\nu,1,L} = O_{\kappa_0,\|v\|_{\ell^1}}(1)$.

(ii) For $p \in \mathbb{N}^*$, we have $\|\Gamma^{\nu,\kappa_0/\nu,1,L}\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} = O_{\kappa_0,p,\|v\|_{\ell^1}}(1)$.

Proof. The proof is analogous to that of Proposition 5.2. Instead of Lemma 5.7, we apply Lemma 5.15. We only need to note that the powers of ν that we obtain by applying the estimates in Lemma 5.15 cancel out. In order to do this, we note a general fact about trees. Let $\mathcal{T} \in \mathfrak{T}_n$ with a distinguished root r be given. We recall that $\mathrm{V}(\mathcal{T})$ denotes the set of vertices of $\mathcal{T}$. For $w \in \mathrm{V}(\mathcal{T})$, we denote by $\mathcal{Q}(w)$ the set of *direct descendants* of w. By definition, this is the set of all $w' \in \mathrm{V}(\mathcal{T})$ such that the unique path in $\mathcal{T}$ connecting w' to r starts with the edge joining w' to w. By induction on the number of vertices of $\mathcal{T}$, we obtain that

$$\sum_{w \in \mathrm{V}(\mathcal{T}) \setminus \{r\}} (1 - |\mathcal{Q}(w)|) = |\mathcal{Q}(r)| \,. \tag{5.85}$$

Using (5.85), we deduce that, when applying Lemma 5.15 in the argument of the proof of Proposition 5.2, the powers of ν in the upper bound cancel out. The claim follows. □

Arguing analogously as for Corollary 5.3, we can use Proposition 5.16 to deduce the following result.

Corollary 5.17. *With assumptions as in Proposition 5.16, the following claims hold.*

(i) The quantity

$$\lim_{L\to\infty} g^{\nu,\kappa_0/\nu,1,L} =: g^{\nu,\kappa_0/\nu,1,\infty} \tag{5.86}$$

exists.

(ii) Let $p \in \mathbb{N}^*$ and be given. We can take $L = \infty$ in (5.6) and obtain an operator $\Gamma^{\nu,\kappa_0/\nu,1,\infty}$ on $\ell^2(\mathbb{Z}^d)^{\otimes p}$ which satisfies $\|\Gamma^{\nu,\kappa_0/\nu,1,\infty}\|_{\ell^\infty_{\mathbf{x}}\ell^1_{\mathbf{y}}} = O_{\kappa,p,\|v\|_{\ell^1}}(1)$.

With notation as in (5.44), we note the following analogue of Lemma 5.11 for given $c > 0$ small.

Lemma 5.18. *Let $\omega \in \Omega^{L,T(\omega)}$ with $T(\omega) \in \nu\mathbb{N}^*$ and $q \in \mathbb{N}$ be given. Then, the following estimates hold.*

(i) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega,\tilde{\omega})|\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \lesssim_{\kappa,q} T(\omega)\,(1 + \|v^{(2)}\|_{\ell^1})\,\nu^{q-1}\,\mathrm{e}^{-CL}$.

(ii) Let $L_0 \in \mathbb{N}^*$ as in (5.45) and $x \in \Lambda_{L_0}$ be given. Then, we have

$$\int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}^L_{y,x}(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta^L(\omega,\tilde{\omega})|\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \lesssim_{\kappa,q} T(\omega)\,(1 + \|v^{(2)}\|_{\ell^1})\,\nu^{q-1}\,\mathrm{e}^{-CL}\,.$$

(iii) With L_0 as in (ii) and $x \in \Lambda_{L_0}$, we have

$$\int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}^L_{y,x}(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \lesssim_{\kappa,q} T(\omega)\,\nu^q\,\mathrm{e}^{-CL}\,.$$

We also note an analogue of Lemma 5.12. Before stating the result, we need to modify some of the notation. With $c > 0$ as earlier, we let $\mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega})$ denote the quantity given as in (1.14) with interaction potential replaced by $v_c^{L,(2)}(x) := v^{L,(2)}(x)\,\mathbf{1}_{|x|_L \geqslant cL}$. Here, we recall (5.74)–(5.75) to see that, similarly as in (5.48), we have $\lim_{L\to\infty} \|v_c^{L,(2)}\|_{\ell^1(\Lambda_L)} = 0$.

We modify (5.49) and let

$$\zeta_c^{L,(2)}(\omega,\tilde{\omega}) := \exp\bigl(-\mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega})\bigr) - 1\,. \tag{5.87}$$

With this notation, we have the following lemma.

Lemma 5.19. *With assumptions and notation as in Lemma 5.18, the following estimates hold.*

(i) $\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta_c^{L,(2)}(\omega,\tilde{\omega})| \lesssim_{\kappa,q} T(\omega)\, \|v_c^{L,(2)}\|_{\ell^1(\Lambda_L)}\, \nu^{q-1}$.

(ii) $\int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, |\zeta_c^{L,(2)}(\omega,\tilde{\omega})| \lesssim_{\kappa,q} T(\omega)\, \|v_c^{L,(2)}\|_{\ell^1(\Lambda_L)}\, \nu^{q-1}$ *for all* $x \in \Lambda_{L_0}$.

We prove Lemmas 5.18 and 5.19 in Appendix D.4. Let us note an analogue of Proposition 5.5.

Proposition 5.20. *With assumptions as in Proposition 5.16, the following claims hold.*

(i) The convergence in (5.86) holds uniformly in $\nu \leqslant 1$.

(ii) Let $p \in \mathbb{N}^*$ *and* $L_0 \in \mathbb{N}^*$ *be given. Let* $\Gamma_{p,m}^{\infty}$ *be as in Corollary 5.17 (ii). Then, we have*

$$\Gamma^{\nu,\kappa_0/\nu,1,\infty} = \lim_{L\to\infty} \Gamma^{\nu,\kappa_0/\nu,1,L}\,. \tag{5.88}$$

The convergence in (5.88) holds in $\|\cdot\|_{L_0,p}$ *given by (1.38) and is uniform in* $\nu \leqslant 1$ *and* $L_0 \in \mathbb{N}^*$.

Proof of Proposition 5.20. The proof is similar to that of Proposition 5.5. We just comment on the main differences. As in Proposition 5.5, the proof of (ii) allows us to obtain claim (i). In order to prove claim (ii), we note that by Proposition 5.16 (ii), Corollary 5.17 (ii), and arguing as for (5.50), it suffices to prove

$$\begin{aligned}\lim_{L\to\infty} \Bigl\| &\int \hat{\mu}_{y_1,x_1}^L(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^L(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\varphi^L(\omega_1,\dots,\omega_n) \\ &- \int \hat{\mu}_{y_1,x_1}^{\infty}(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^{\infty}(\mathrm{d}\omega_p)\,\mu^{\infty}(\mathrm{d}\omega_{p+1})\cdots\mu^{\infty}(\mathrm{d}\omega_n)\,\varphi^{\infty}(\omega_1,\dots,\omega_n)\Bigr\|_{L_0,p} = 0\,,\end{aligned} \tag{5.89}$$

uniformly in $\nu \leqslant 1$ and L_0. Note that in (5.89), the path measures are given by (5.71)–(5.72). Recalling (5.52), and arguing analogously as in (5.53)–(5.54), we have that for all $k \in \{1,\dots,n\}$,

$$\lim_{L\to\infty} \Bigl\| \int \hat{\mu}_{y_1,x_1}^L(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^L(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\mathbf{1}_{\mathcal{B}^L}(\omega_k)\,\varphi^L(\omega_1,\dots,\omega_n)\Bigr\|_{L_0,p} = 0\,, \tag{5.90}$$

which follows from the observation that

$$\lim_{L\to\infty} \Bigl\| \int \hat{\mu}_{y_1,x_1}^L(\mathrm{d}\omega_1)\cdots\hat{\mu}_{y_p,x_p}^L(\mathrm{d}\omega_p)\,\mu^L(\mathrm{d}\omega_{p+1})\cdots\mu^L(\mathrm{d}\omega_n)\,\mathbf{1}_{\mathcal{B}^L}(\omega_k) \prod_{\{i,j\}\in\mathcal{T}} |\zeta^L(\omega_i,\omega_j)|\Bigr\|_{L_0,p} = 0\,, \tag{5.91}$$

for a fixed $\mathcal{T} \in \mathfrak{T}_n$. The proof of (5.91) is analogous to that of (5.54), except that we now use Lemmas 5.18–5.19 instead of Lemmas 5.11–5.12. Similarly, we have that for all $k \in \{1, \dots, n\}$,

$$\lim_{L\to\infty} \Big\| \int \hat{\mu}^{\infty}_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \hat{\mu}^{\infty}_{y_p,x_p}(\mathrm{d}\omega_p)\, \mu^{\infty}(\mathrm{d}\omega_{p+1}) \cdots \mu^{\infty}(\mathrm{d}\omega_n)\, \mathbf{1}_{\mathcal{B}^L}(\omega_k)\, \varphi^{\infty}(\omega_1, \dots, \omega_n) \Big\|_{L_{0,p}} = 0\,. \tag{5.92}$$

By using (5.90), (5.92), and arguing as in the proof of Proposition 5.5, the claim follows if we show

$$\begin{aligned} \lim_{L\to\infty} \Big\| & \int \hat{\mu}^{L}_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \hat{\mu}^{L}_{y_p,x_p}(\mathrm{d}\omega_p)\, \mu^{L}(\mathrm{d}\omega_{p+1}) \cdots \mu^{L}(\mathrm{d}\omega_n) \prod_{k=1}^{n} \mathbf{1}_{\mathcal{A}^L}(\omega_k)\, \varphi^{L}(\omega_1, \dots, \omega_n) \\ & - \int \hat{\mu}^{\infty}_{y_1,x_1}(\mathrm{d}\omega_1) \cdots \hat{\mu}^{\infty}_{y_p,x_p}(\mathrm{d}\omega_p)\, \mu^{\infty}(\mathrm{d}\omega_{p+1}) \cdots \mu^{\infty}(\mathrm{d}\omega_n) \prod_{k=1}^{n} \mathbf{1}_{\mathcal{A}^L}(\omega_k)\, \varphi^{\infty}(\omega_1, \dots, \omega_n) \Big\|_{\ell^{\infty}_{\mathbf{x}} \ell^{1}_{\mathbf{y}}} = 0\,. \end{aligned} \tag{5.93}$$

By arguing as in the proof of (5.56), we note that (5.93) follows from (5.60) with v as in Assumption 1.7 and v^L as in (1.34). Note that (5.60) indeed holds by Assumption 1.7 (i). □

We now have the necessary tools to prove Theorem 1.8.

Proof of Theorem 1.8. The proof is similar to that of Theorem 1.6. We combine Theorem 1.4, Proposition 5.20 and Lemma 5.13. □

A. Derivation of the Symanzik and Ginibre loop representations

In this appendix we derive the Symanzik and Ginibre loop representations of the classical field theory and the interacting Bose gas, respectively. We shall use the following standard tool.

Lemma A.1 (Feynman-Kac formula)**.** *For any $V : \Lambda \to \mathbb{C}$ and $t > 0$ we have*

$$\big(\mathrm{e}^{t(\Delta/2 - V)}\big)_{y,x} = \int \mathbb{W}^{t}_{y,x}(\mathrm{d}\omega)\, \mathrm{e}^{-\int_0^t \mathrm{d}s\, V(\omega(s))}\,.$$

A.1. The Symanzik representation: proofs of Proposition 2.1 and Corollary 2.2. In this appendix, we give give the proof of Proposition 2.1. Let us first comment on the main proof strategy. Our starting point is the observation that the weight

$$\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}x \int \mathrm{d}y\, |\phi(x)|^2\, v(x-y)\, |\phi(y)|^2}\,, \tag{A.1}$$

occurring in (1.10) and (1.11) is a function of $|\phi|^2 \equiv (|\phi(u)|^2)_{u\in\Lambda}$. We rewrite (A.1) using the Hubbard-Stratonovich formula (see (A.3) below). As a result, we obtain integrals over a field $\sigma : \Lambda \to \mathbb{R}$. After performing a Gaussian integration in the field ϕ, we can resum the σ integration to obtain the result. For similar arguments based on the Hubbard-Stratonovich formula, we refer the reader to [10, Sections 3-4]. Let us note that arguments based on rewriting (A.1) using the Fourier transform were applied in [3, Section 2], [4, Section 2], and [7, Section 5]. We now prove Proposition 2.1.

Proof of Proposition 2.1. Let us first prove claim (i). We identify $v : \Lambda \to \mathbb{R}$ with a positive quadratic form $f \mapsto \langle f, vf \rangle := \int \mathrm{d}x \int \mathrm{d}y\, f(x)v(x-y)f(y)$. Note that the positivity of the quadratic form follows since v is of positive type. Let μ_v be a Gaussian measure on $\mathbb{R}^\Lambda$ with covariance v, i.e.

$$\int \mu_v(\mathrm{d}\sigma)\, \sigma(x)\, \sigma(y) = v(x-y)\,. \tag{A.2}$$

The *Hubbard-Stratonovich formula* then states that

$$\int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\langle f,\sigma\rangle} = e^{-\frac{1}{2}\langle f,vf\rangle}\,, \tag{A.3}$$

which follows from Lemma B.1. By (A.3) with $f = |\phi|^2$ we can rewrite (1.10) as

$$\mathcal{Z}^{\mathrm{cl}} = \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\Big(\int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}x\, \sigma(x)|\phi(x)|^2}\Big)\,, \tag{A.4}$$

which by using Fubini's theorem and evaluating a Gaussian integral equals

$$\begin{aligned}&\int \mu_v(\mathrm{d}\sigma) \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}x\, \sigma(x)|\phi(x)|^2}\\ &= \int \mu_v(\mathrm{d}\sigma)\, \det(-\Delta/2+\kappa-\mathrm{i}\sigma)^{-1}\, \det(-\Delta/2+\kappa)\,.\end{aligned} \tag{A.5}$$

We note that

$$\det(-\Delta/2+\kappa-\mathrm{i}\sigma)^{-1}\, \det(-\Delta/2+\kappa) = \exp\Big\{-\operatorname{Tr}\log(-\Delta/2+\kappa-\mathrm{i}\sigma)+\operatorname{Tr}\log(-\Delta/2+\kappa)\Big\}\,, \tag{A.6}$$

since the arguments of the logarithm have strictly positive real part.

We note that for all $a, b \in \mathbb{C}$ of strictly positive real part we have

$$\log a - \log b = -\int_0^\infty \frac{\mathrm{d}t}{t}\,(\mathrm{e}^{-ta} - \mathrm{e}^{-tb})\,. \tag{A.7}$$

A direct calculation yields

$$\log a - \log b = -\int_0^\infty \mathrm{d}t\,\Big(\frac{1}{t+a} - \frac{1}{t+b}\Big)\,. \tag{A.8}$$

We deduce (A.7) from (A.8) by noting that for $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$ and $t \geqslant 0$ we have $\frac{1}{t+c} = \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s(t+c)}$, and by using Fubini's theorem. By using (A.7) followed by Lemma A.1, the fact that $\operatorname{Tr} A = \int \mathrm{d}u\, A_{u,u}$ and (1.2), we can write

$$-\operatorname{Tr}\log(-\Delta/2+\kappa-\mathrm{i}\sigma)+\operatorname{Tr}\log(-\Delta/2+\kappa) = \int_0^\infty \frac{\mathrm{d}T}{T} \int \mathbb{W}^T(\mathrm{d}\omega)\, \mathrm{e}^{-\kappa T}\,\Big(\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\, \sigma(\omega(t))} - 1\Big)\,. \tag{A.9}$$

From (A.4)–(A.9), we conclude

$$\mathcal{Z}^{\mathrm{cl}} = \int \mu_v(\mathrm{d}\sigma)\, \exp\Big\{\int_0^\infty \frac{\mathrm{d}T}{T}\, \mathrm{e}^{-\kappa T} \int \mathbb{W}^T(\mathrm{d}\omega)\,\Big(\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\, \sigma(\omega(t))} - 1\Big)\Big\}\,. \tag{A.10}$$

In what follows, we fix $\varepsilon > 0$ and rewrite for fixed σ the expression (A.9) as

$$\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\, \mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\, \sigma(\omega(t))} + K^\varepsilon + \int_0^\varepsilon \frac{\mathrm{d}T}{T}\, \mathrm{e}^{-\kappa T} \int \mathbb{W}^T(\mathrm{d}\omega)\,\Big(\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\, \sigma(\omega(t))} - 1\Big)\,, \tag{A.11}$$

where we recall (2.2) and (2.4). We now show that the third term in (A.11) is $\varepsilon|\Lambda|O(\|\sigma\|_\infty)$.

Indeed, we obtain this by noting that

$$\left|\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} - 1\right| \leqslant T\|\sigma\|_\infty$$

and using (3.4) combined with Lemma C.1 (i). In particular, we can rewrite (A.11) as

$$\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\,\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} + K^\varepsilon + \varepsilon|\Lambda|O(\|\sigma\|_\infty)\,. \tag{A.12}$$

We now need to exponentiate and integrate in σ. Before doing so, we analyse (A.12) more closely. We first note that, by (2.4) we have

$$\mathrm{Re}\left(\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\,\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))}\right) + K^\varepsilon \leqslant 0\,. \tag{A.13}$$

Hence, (A.13) implies that

$$\exp\left\{\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\,\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} + K^\varepsilon\right\} = O(1)\,, \tag{A.14}$$

uniformly in $\varepsilon, \kappa > 0$, Λ and $\sigma : \Lambda \to \mathbb{R}$.

We now analyse the third (i.e. the error) term in (A.12). Given $C > 0$ we show that

$$\int \mu_v(\mathrm{d}\sigma)\,\mathrm{e}^{C\varepsilon\|\sigma\|_\infty} \to 1 \quad \text{as} \quad \varepsilon \to 0\,. \tag{A.15}$$

We note that (A.15) follows from the dominated convergence theorem provided that we show that for $C > 0$ we have $\int \mu_v(\mathrm{d}\sigma)\,\mathrm{e}^{C\|\sigma\|_\infty} < \infty$. We prove this by writing

$$\int \mu_v(\mathrm{d}\sigma)\,\mathrm{e}^{C\|\sigma\|_\infty} \leqslant \int \mathrm{d}x \int \mu_v(\mathrm{d}\sigma)\,\mathrm{e}^{C|\sigma(x)|}\,. \tag{A.16}$$

Note that in (A.16), we used the fact that μ_v is a positive measure. We expand the exponential, and use the Cauchy-Schwarz inequality, Wick's theorem (Lemma B.1), and (A.2) to deduce

$$\int \mu_v(\mathrm{d}\sigma)\,\mathrm{e}^{C|\sigma(x)|} \leqslant \sum_{i=0}^\infty \frac{C^i}{i!}\left(\int \mu_v(\mathrm{d}\sigma)\,\sigma(x)^{2i}\right)^{1/2} = \sum_{k=0}^\infty \frac{C^i}{i!}\sqrt{\frac{(2i)!}{i!\,2^i}}\,v(0)^{i/2} < \infty\,, \tag{A.17}$$

as desired.

We now combine (A.10)–(A.11), (A.12), (A.14)–(A.15), and apply an $L^\infty(\mathrm{d}\mu_v) - L^1(\mathrm{d}\mu_v)$ Hölder's inequality in σ, to deduce $\mathcal{Z}^{\mathrm{cl}} = \lim_{\varepsilon\to 0}\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon}$, where

$$\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon} := \int \mu_v(\mathrm{d}\sigma)\,\exp\left\{\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\,\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} + K^\varepsilon\right\}. \tag{A.18}$$

What remains, therefore, is to show that

$$\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon} = \mathcal{Z}^{\mathrm{cl},\varepsilon}\,. \tag{A.19}$$

By expanding the exponential and using Fubini's theorem we have

$$\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon} = \sum_{n=0}^\infty \frac{1}{n!}\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_1)\cdots\mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_n)\left(\int \mu_v(\mathrm{d}\sigma)\,\mathrm{e}^{\mathrm{i}\sum_{i=1}^n \int_0^{T_i}\mathrm{d}t\,\sigma(\omega_i(t))}\right)\exp(K^\varepsilon)\,. \tag{A.20}$$

The σ-integration can be performed in (A.20) by using the Hubbard-Stratonovich formula (A.3), noting that $\sum_{i=1}^n \int_0^{T_i} \mathrm{d}t\, \sigma(\omega_i(t)) = \langle f, \sigma\rangle$ with $f(x) := \sum_{i=1}^n \int_0^{T_i} \mathrm{d}t\, \delta(x-\omega_i(t))$. We therefore obtain (A.19) as claimed, and we deduce (i).

The proof of claim (ii) is similar to that of (i). We just outline the main differences. We first apply (A.3) with $f = |\phi|^2$ in (1.11) followed by Fubini's theorem to deduce that

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \int \mu_v(\mathrm{d}\sigma) \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\, \mathrm{e}^{\mathrm{i}\langle\phi,\sigma\phi\rangle}\, \bar\phi(y_1)\cdots\bar\phi(y_p)\,\phi(x_1)\cdots\phi(x_p)\,. \tag{A.21}$$

By (B.1) and Lemma B.3, we can rewrite (A.21) as

$$= \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \sum_{\pi\in S_p} \int \mu_v(\mathrm{d}\sigma) \prod_{i=1}^p \Big(\frac{1}{-\Delta/2+\kappa-\mathrm{i}\sigma}\Big)_{y_{\pi(i)},x_i} \det(-\Delta/2+\kappa-\mathrm{i}\sigma)^{-1} \det(-\Delta/2+\kappa)\,. \tag{A.22}$$

Furthermore we have for all $x, y \in \Lambda$

$$\Big(\frac{1}{-\Delta/2+\kappa-\mathrm{i}\sigma}\Big)_{y,x} = \int_0^\infty \mathrm{d}T \left(\mathrm{e}^{T(\Delta/2-\kappa+\mathrm{i}\sigma)}\right)_{y,x} = \int_0^\infty \mathrm{d}T\, \mathrm{e}^{-\kappa T} \int \mathbb{W}_{y,x}^T(\mathrm{d}\omega)\, \mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))}\,, \tag{A.23}$$

where in the last equality we used Lemma A.1. We now combine (A.22) and (A.23) to deduce that

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \sum_{\pi\in S_p} \int_{(0,\infty)^p} \mathrm{d}\mathbf{T}\, \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}_{\pi\mathbf{y},\mathbf{x}}^{\mathbf{T}}(\mathrm{d}\boldsymbol{\omega}) \left(\int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\sum_{j=1}^p \int_0^{T_j} \mathrm{d}t\,\sigma(\omega_j(t))}\right)$$
$$\times \det(-\Delta/2+\kappa-\mathrm{i}\sigma)^{-1} \det(-\Delta/2+\kappa)\,. \tag{A.24}$$

We rewrite the factor of $\det(-\Delta/2+\kappa-\mathrm{i}\sigma)^{-1}\det(-\Delta/2+\kappa)$ as in (A.6)–(A.9), recall (A.12), (A.14)–(A.15), and proceed analogously as in the remainder of the proof of Proposition 2.1 (i) to rewrite (A.24) as

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \lim_{\varepsilon\to 0} (\tilde\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}}\,, \tag{A.25}$$

where for $\varepsilon > 0$, we let

$$(\tilde\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}} := \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \sum_{\pi\in S_p} \int_{(0,\infty)^p} \mathrm{d}\mathbf{T}\, \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}_{\pi\mathbf{y},\mathbf{x}}^{\mathbf{T}}(\mathrm{d}\boldsymbol{\omega}) \sum_{n=0}^\infty \frac{1}{n!} \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\tilde\omega_1)\cdots\mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\tilde\omega_n)$$
$$\times \exp(-V^{\mathrm{cl}}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) \left(\int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\sum_{j=1}^p \int_0^{T_j} \sigma(\omega_j(t))\,\mathrm{d}t + \mathrm{i}\sum_{i=1}^n \int_0^{T_i} \sigma(\tilde\omega_i(t))\,\mathrm{d}t}\right) \exp(K^\varepsilon) \tag{A.26}$$

We can now perform the σ integration in (A.26) analogously as in (A.19)–(A.20). Note that we now apply (A.3) with $f = \sum_{j=1}^p \int_0^{T_j} \mathrm{d}t\,\delta(x-\omega_j(t)) + \sum_{i=1}^n \int_0^{T_i} \mathrm{d}t\,\delta(x-\tilde\omega_i(t))$. In particular, recalling (2.5) we obtain that $(\tilde\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}} = (\Gamma_p^{\mathrm{cl},\varepsilon})_{\mathbf{x},\mathbf{y}}$ and we deduce the claim (ii) from (A.25). □

We also give the proof of Corollary 2.2.

Proof of Corollary 2.2. We recall (1.9), and argue as in (A.21)–(A.23) with v replaced by λv to rewrite (2.7) as

$$(\widehat\Gamma_p^{\mathrm{cl},\lambda})_{\mathbf{x},\mathbf{y}} = \frac{1}{\pi^{|\Lambda|}\det(-\Delta/2+\kappa)^{-1}} \sum_{\pi\in S_p} \int_{(0,\infty)^p} \mathrm{d}\mathbf{T}\, \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}_{\pi\mathbf{y},\mathbf{x}}^{\mathbf{T}}(\mathrm{d}\boldsymbol{\omega}) \int \prod_{u\in\Lambda} \mathrm{d}\phi(u)\, \mathrm{e}^{\langle\phi,(\Delta/2-\kappa)\phi\rangle}$$
$$\times \int \mu_{\lambda v}(\mathrm{d}\sigma)\, \exp\left[\left\langle \mathrm{i}|\phi|^2 + \mathrm{i}\sum_{j=1}^p \tau_{(\cdot)}(\omega_j), \sigma\right\rangle\right]. \tag{A.27}$$

In (A.27), given $u \in \Lambda$ and a path ω, $\tau_u(\omega)$ denotes the *local time*, i.e. the amount of time that ω spends at u. We use (A.3) with

$$f(x) = |\phi(x)|^2 + \sum_{j=1}^{p} \tau_x(\omega_j)$$

to obtain that $(\widehat{\Gamma}_p^{\mathrm{cl},\lambda})_{\mathbf{x},\mathbf{y}} = \sum_{\pi \in S_p} \int_{(0,\infty)^p} \mathrm{d}\mathbf{T}\, \mathrm{e}^{-\kappa|\mathbf{T}|} \int \mathbb{W}^{\mathbf{T}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\, \mathrm{e}^{-\frac{\lambda}{2}\langle f, vf\rangle}$, from where we deduce (2.8) since $\langle f, vf\rangle \geqslant 0$ because v is of positive type. □

A.2. The Ginibre representation: proof of Proposition 2.3.

Proof of Proposition 2.3. We recall (1.18) and apply the Feynman-Kac formula, Lemma A.1 to obtain that

$$(\mathrm{e}^{-H_n^{\nu,\lambda}})_{\mathbf{y},\mathbf{x}} = \int \mathbb{W}^{\nu\mathbf{1}}_{\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}))\,, \tag{A.28}$$

where we write $\mathbf{1} \equiv \mathbf{1}_n$ and recall (1.30) (in the sequel we drop the subscript n in (1.30)). Recalling (1.1), we have

$$(\mathrm{e}^{-H_n} P_n^+)_{\mathbf{y},\mathbf{x}} = \frac{1}{n!} \sum_{\pi \in S_n} (\mathrm{e}^{-H_n})_{\pi\mathbf{y},\mathbf{x}}\,. \tag{A.29}$$

We use (A.28)–(A.29) in (1.19) and obtain

$$(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}} = \frac{1}{\Xi^{\nu,\kappa,\lambda}} \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\kappa\nu(p+n)}}{n!} \sum_{\pi \in S_{p+n}} \int_{\Lambda^n} \mathrm{d}\mathbf{u} \int \mathbb{W}^{\nu\mathbf{1}}_{\pi(\mathbf{y}\mathbf{u}),\mathbf{x}\mathbf{u}}(\mathrm{d}\boldsymbol{\omega}) \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}))\,. \tag{A.30}$$

Now we perform the first step of the loop integration. We distinguish between two types of paths. The first type are *open paths* with endpoints x_j and $y_{j'}$ for $j, j' \in \{1, \dots, p\}$. The *internal points* in these open paths are of the form u_j for some $1 \leqslant j \leqslant n$. The second type are *closed paths*, all of whose vertices are of the form u_j for some $1 \leqslant j \leqslant n$. Let us denote by r the number of vertices contained in all of the closed paths. We give an example in Figure A.1 below. The first step of the loop integration consists in integrating over the internal vertices of the open paths.

In the sequel, we write k_i for the total number of edges in the open path with one endpoint x_i for $i = 1, \dots, p$. Hence, in the example given in Figure A.1 we have $k_1 = 4$. We rewrite (A.30) as

$$\begin{aligned}(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}} = \frac{1}{\Xi^{\nu,\kappa,\lambda}} \sum_{\mathbf{k} \in (\mathbb{N}^*)^p} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \delta(|\mathbf{k}| + r - n - p) \binom{n}{r} (n-r)! \\ \times \frac{1}{n!}\, \mathrm{e}^{-\kappa\nu(|\mathbf{k}|+r)} \sum_{\pi \in S_p} \int \mathbb{W}^{\nu\mathbf{k}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \int_{\Lambda^r} \mathrm{d}\mathbf{u} \sum_{\sigma \in S_r} \mathbb{W}^{\nu\mathbf{1}}_{\sigma\mathbf{u},\mathbf{u}}(\mathrm{d}\tilde{\boldsymbol{\omega}})\, \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}))\,.\end{aligned} \tag{A.31}$$

In (A.31), we chose for fixed r the $\binom{n}{r}$ elements of the form u_j that are taken as vertices of closed paths. For fixed $\mathbf{k}$, the remaining $n - r$ u's can be distributed among the open paths in $(n-r)!$ different ways. Furthermore, we used the presence of the delta function to deduce that $n+p = |\mathbf{k}|+r$ and to then perform the sum in n. In particular, we conclude that

$$\begin{aligned}(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}} = \frac{1}{\Xi^{\nu,\kappa,\lambda}} \sum_{\mathbf{T} \in (\nu\mathbb{N}^*)^p} \sum_{\pi \in S_p} \int \mathrm{e}^{-\kappa|\mathbf{T}|}\, \mathbb{W}^{\mathbf{T}}_{\pi\mathbf{y},\mathbf{x}}(\mathrm{d}\boldsymbol{\omega}) \\ \sum_{r=0}^{\infty} \frac{\mathrm{e}^{-\kappa\nu r}}{r!} \int_{\Lambda^r} \mathrm{d}\mathbf{u} \sum_{\sigma \in S_r} \int \mathbb{W}^{\nu\mathbf{1}}_{\sigma\mathbf{u},\mathbf{u}}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}))\,.\end{aligned} \tag{A.32}$$

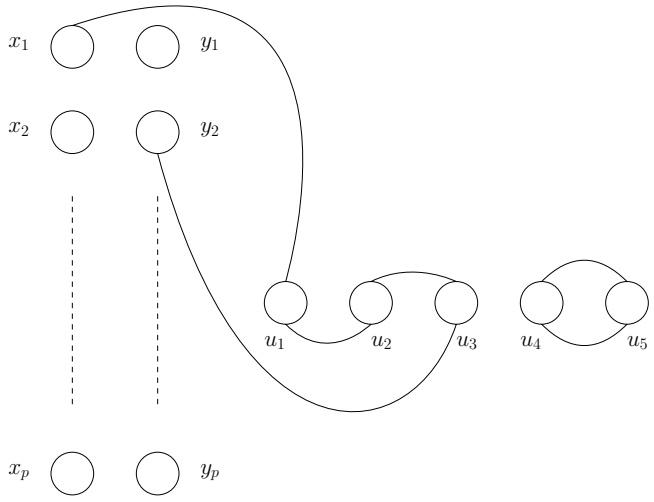

Figure A.1. In this example, we have $n = 5$ and $r = 2$. The open path with endpoint x_1 has y_2 as its other endpoint. It has three internal points u_1, u_2, u_3. The points u_4, u_5 belong to a closed path of length two.

Next, we perform the second step of the loop integration, by decomposing σ into cycles. For a permutation σ and $k \in \mathbb{N}^*$, denote by $a_k(\sigma)$ the number of cycles of length k in σ. Write $\mathbf{a}(\sigma) = (a_1(\sigma), a_2(\sigma), \dots) \in \mathbb{N}^{\mathbb{N}^*}$. For $\mathbf{a} = (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}^*}$ we define $|\mathbf{a}| := \sum_{k=1}^{\infty} a_k$ (number of cycles) and $r(\mathbf{a}) := \sum_{k=1}^{\infty} k a_k$ (number of elements in all cycles). Thus, if $\sigma \in S_r$ then $r(\mathbf{a}(\sigma)) = r$. Note that the number of permutations σ satisfying $\mathbf{a}(\sigma) = \mathbf{a}$ is equal to

$$\frac{r(\mathbf{a})!}{\prod_{k=1}^{\infty}(k^{a_k} a_k!)}\,.$$

For $\boldsymbol{\omega} \in \Omega^{\mathbf{T}}$ we let

$$\begin{aligned}\widetilde{\Xi}^{\nu,\kappa,\lambda}(\boldsymbol{\omega}) &:= \sum_{r=0}^{\infty} \frac{\mathrm{e}^{-\kappa\nu r}}{r!} \int_{\Lambda^r} \mathrm{d}\mathbf{u} \sum_{\sigma\in S_r} \int \mathbb{W}^{\nu\mathbf{1}}_{\sigma\mathbf{u},\mathbf{u}}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) \\ &= \sum_{\mathbf{a}\in\mathbb{N}^{\mathbb{N}^*}} \prod_{k=1}^{\infty} \frac{1}{k^{a_k} a_k!} \int \mathbb{W}^{\nu\boldsymbol{\ell}(\mathbf{a})}(\mathrm{d}\tilde{\boldsymbol{\omega}}) \mathrm{e}^{-\kappa\nu r(\mathbf{a})} \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}}))\,, \end{aligned} \tag{A.33}$$

where $\boldsymbol{\ell}(\mathbf{a}) \equiv (\ell_1(\mathbf{a}), \dots, \ell_{|\mathbf{a}|}(\mathbf{a})) \in (\mathbb{N}^*)^{|\mathbf{a}|}$ is an (arbitrary) family of cycle lengths corresponding to $\mathbf{a}$, i.e. satisfying $\sum_{i=1}^{|\mathbf{a}|} \mathbf{1}_{\ell_i(\mathbf{a})=k} = a_k$ for all $k \in \mathbb{N}^*$. Here, we recall the notation (3.2). Recalling (1.13) and using the multinomial identity in (A.33), we find (writing $n = |\mathbf{a}|$ for the number of cycles) that

$$\widetilde{\Xi}^{\nu,\kappa,\lambda}(\boldsymbol{\omega}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\nu,\kappa}(\mathrm{d}\tilde{\omega}_1) \cdots \mathbb{L}^{\nu,\kappa}(\mathrm{d}\tilde{\omega}_n) \, \exp(-V^{\nu,\lambda}(\boldsymbol{\omega}\tilde{\boldsymbol{\omega}})) = \Xi^{\nu,\kappa,\lambda}(\boldsymbol{\omega})\,, \tag{A.34}$$

where we recall the notation from (2.10). We recall (1.17) and use the same arguments as above to obtain

$$\widetilde{\Xi}^{\nu,\kappa,\lambda}(\emptyset) = \Xi^{\nu,\kappa,\lambda}\,. \tag{A.35}$$

The identity (1.20) now follows by substituting (A.34)–(A.35) into (A.32). The proof of (2.9) is analogous. □

B. Remarks on Gaussian integrals

We collect several standard facts about Gaussian integrals.

Lemma B.1. *Let $\mathcal{C} > 0$ be a positive real $n \times n$ matrix. We define the Gaussian probability measure on $\mathbb{R}^n$ with covariance $\mathcal{C}$ through*

$$\mu_{\mathcal{C}}(\mathrm{d}\mathbf{x}) := \frac{1}{\sqrt{(2\pi)^n \det \mathcal{C}}} \, \mathrm{e}^{-\frac{1}{2}\langle \mathbf{x}, \mathcal{C}^{-1}\mathbf{x}\rangle} \, \mathrm{d}\mathbf{x} \, .$$

It has Fourier transform given by $\int_{\mathbb{R}^n} \mu_{\mathcal{C}}(\mathrm{d}x) \, \mathrm{e}^{\mathrm{i}\langle \mathbf{a}, \mathbf{x}\rangle} = \mathrm{e}^{-\frac{1}{2}\langle \mathbf{a}, \mathcal{C}\mathbf{a}\rangle}$ for all $\mathbf{a} \in \mathbb{R}^n$.

On $\mathbb{C}^n$, we denote by $\langle z, w\rangle = \sum_i \overline{z}_i w_i$ the complex inner product and by $\mathrm{d}z$ the Lebesgue measure.

Lemma B.2. *Let $\mathcal{C}$ be a complex $n \times n$ matrix with $\operatorname{Re}\mathcal{C} = (\mathcal{C} + \mathcal{C}^*)/2 > 0$. Then we have*

$$\int_{\mathbb{C}^n} \mathrm{d}\mathbf{z} \, \mathrm{e}^{-\langle \mathbf{z}, \mathcal{C}^{-1}\mathbf{z}\rangle} = \pi^n \det \mathcal{C} \, .$$

Hence, for $\mathcal{C}$ a complex $n \times n$ matrix with $\operatorname{Re}\mathcal{C} > 0$, let us define the Gaussian probability measure on $\mathbb{C}^n$ with covariance $\mathcal{C}$ as

$$\mu_{\mathcal{C}}(\mathrm{d}\mathbf{z}) := \frac{1}{\pi^n \det \mathcal{C}} \, \mathrm{e}^{-\langle \mathbf{z}, \mathcal{C}^{-1}\mathbf{z}\rangle} \, \mathrm{d}\mathbf{z} \, . \tag{B.1}$$

Lemma B.3. *Let $\mathcal{C}$ be a complex $n \times n$ matrix with $\operatorname{Re}\mathcal{C} > 0$ and let $\mu_{\mathcal{C}}$ be given as in (B.1). For $i_1, \dots, i_p, j_1, \dots, j_p \in \{1, \dots, n\}$, we have $\int_{\mathbb{C}^n} \mu_{\mathcal{C}}(\mathrm{d}\mathbf{z}) \, \overline{z}_{j_1} \cdots \overline{z}_{j_p} \, z_{i_1} \cdots z_{i_p} = \sum_{\pi \in S_p} \prod_{k=1}^p \mathcal{C}_{i_k, j_{\pi(k)}}$.*

C. The heat kernel on the lattice

We note several useful estimates for the heat kernel $\psi^{L,t}$ on the finite lattice Λ_L and on the infinite lattice $\mathbb{Z}^d$.

Lemma C.1. *(i)* $0 \leqslant \psi^{L,t}(x) \leqslant 1$.

(ii) $\psi^{L,t}(x) = \delta(x) + O(t)$. *Here δ denotes the Kronecker delta function on Λ_L.*

(iii) $\int_{\Lambda_L} \mathrm{d}x \, \psi^{L,t}(x) = 1$.

For the following estimates, we use Fourier analysis. We denote the dual lattice by $\Lambda_L^* := \frac{2\pi}{L}\Lambda_L$. The heat kernel $\psi^{L,t}$ on Λ_L can be written as

$$\psi^{L,t}(x) = \frac{1}{L^d} \sum_{\xi \in \Lambda_L^*} \mathrm{e}^{-t\lambda_\xi} \, \mathrm{e}^{\mathrm{i}\xi \cdot x} \, , \qquad \lambda_\xi := \left(d - \sum_{j=1}^d \cos \xi_j \right) .$$

For $L = \infty$, we have

$$\psi^{\infty,t}(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi)^d} \mathrm{d}\xi \, \mathrm{e}^{-t\lambda_\xi} \, \mathrm{e}^{\mathrm{i}\xi \cdot x} \, . \tag{C.1}$$

Lemma C.2. *The following estimates hold.*

(i) There exist constants $c_1, c_2 > 0$ with c_2 depending on d such that for every $\delta \in (0,1)$, and $x, y \in \Lambda$ and $t \geqslant 0$, we have

$$\psi^{\infty,t}(x) = O\Big(\mathrm{e}^{c_1 t\delta}\, \mathrm{e}^{-c_2\delta|x|}\Big)\,. \tag{C.2}$$

(ii) Given $L \in \mathbb{N}^*$, $x \in \Lambda_L$, and $t \geqslant 0$, we have

$$\psi^{L,t}(x) = O_d\bigg(\frac{1}{\delta}\,\mathrm{e}^{c_1 t\delta}\, \mathrm{e}^{-c_2\delta|x|_L}\bigg)\,, \tag{C.3}$$

where $c_1, c_2 > 0$ and $\delta \in (0,1)$ are as in part (i).

Proof. We first prove (i). We can assume without loss of generality, that $|x_1| = \max_{1\leqslant i\leqslant d}|x_i|$. In particular, we have that $|x_1| \geqslant \frac{1}{\sqrt{d}}|x|$. We then rewrite (C.1) as

$$\psi^{\infty,t}(x) = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\mathrm{d}\xi_2\cdots\int_{-\pi}^{\pi}\mathrm{d}\xi_d\,\mathrm{e}^{-t\left[(d-1)-\sum_{j=2}^{d}\cos(\xi_j)\right]}\,\mathrm{e}^{\mathrm{i}\sum_{j=2}^{d}\xi_j x_j}\int_{-\pi}^{\pi}\mathrm{d}\xi_1\,\mathrm{e}^{-t[1-\cos(\xi_1)]}\,\mathrm{e}^{\mathrm{i}\xi_1 x_1}\,. \tag{C.4}$$

By a contour deformation, we can rewrite the ξ_1 integral in (C.4) as

$$\int_{-\pi}^{\pi}\mathrm{d}\xi_1\,\mathrm{e}^{-t[1-\cos(\xi_1\pm\mathrm{i}\delta)]}\,\mathrm{e}^{\mathrm{i}(\xi_1\pm\mathrm{i}\delta)x_1}\,, \tag{C.5}$$

where the sign is taken to be $+$ if $x_1 \geqslant 0$ and $-$ otherwise. Therefore, since $\cos(\xi_1 \pm \mathrm{i}\delta) - 1 \leqslant C\delta$, we deduce that the expression in (C.5) is

$$= O\Big(\mathrm{e}^{Ct\delta}\,\mathrm{e}^{-\delta|x_1|}\Big) = O\bigg(\mathrm{e}^{Ct\delta}\,\mathrm{e}^{-\frac{\delta|x|}{\sqrt{d}}}\bigg)\,. \tag{C.6}$$

Substituting (C.5)–(C.6) into (C.4), we deduce (i). In order to show (ii), we note that, by periodicity we have for $x \in \Lambda_L$

$$\psi^{L,t}(x) = \sum_{k\in(L\mathbb{Z})^d}\psi^{\infty,t}(x+k)\,. \tag{C.7}$$

Using (C.2) for each term on the right-hand side of (C.7), and considering Riemann sums, we deduce (C.3). □

D. Proofs of auxiliary claims from Section 5

D.1. Kruskal's algorithm and proof of Lemma 5.6. In this subsection, we give an outline of Kruskal's algorithm, which we then use to prove Lemma 5.6. Kruskal's algorithm [21] defines a map $\mathcal{K} : \mathfrak{G}_n^c \to \mathfrak{T}_n$ with the property that $\mathcal{K}(\mathcal{G}) \subset \mathcal{G}$ is a spanning tree of $\mathcal{G} \in \mathfrak{G}_n^c$. For completeness, let us briefly recall the construction of the map $\mathcal{K}$. We first order all the edges of the complete graph on n vertices according to an arbitrary (strict) linear order $<$. Given $\mathcal{G} \in \mathfrak{G}_n^c$, we define the following sequence $(\mathcal{F}_k) \equiv (\mathcal{F}_k(\mathcal{G}))$ of *forests* on n vertices.

(i) $\mathcal{F}_0 := \emptyset$.

(ii) Let $k \in \mathbb{N}$ be given. We find the smallest edge $e_{k+1} \in \mathcal{G}\backslash\mathcal{F}_k$ with the property that $\mathcal{F}_k \cup \{e_{k+1}\}$ contains no cycles, in which case we let $\mathcal{F}_{k+1} := \mathcal{F}_k \cup \{e_{k+1}\}$. If no such e_{k+1} exists, we let $\mathcal{F}_{k+1} := \mathcal{F}_k$ and we terminate the procedure.

Given $\mathcal{G} \in \mathfrak{G}_n^c$, there exists $k \in \mathbb{N}$ such that the above procedure terminates at the k-th step. We then define $\mathcal{K}(\mathcal{G}) := \mathcal{F}_k$. We note the following observation about the preimage of any tree under the Kruskal map.

Lemma D.1. *Let $\mathcal{T} \in \mathfrak{T}_n$ be given. Then there exists $M(\mathcal{T}) \in \mathfrak{G}_n^c$ containing $\mathcal{T}$ such that*

$$\mathcal{K}^{-1}(\mathcal{T}) = \{\mathcal{G} \in \mathfrak{G}_n^c \,, \mathcal{T} \subset \mathcal{G} \subset M(\mathcal{T})\} \,.$$

Proof. We let

$$M(\mathcal{T}) := \bigcup_{\mathcal{G} \in \mathfrak{G}_n^c \,, \mathcal{K}(\mathcal{G}) = \mathcal{T}} \mathcal{G} \,. \tag{D.1}$$

We obtain that (D.1) satisfies the wanted properties if we show that the following three claims hold.

(i) $\mathcal{K}(\mathcal{T}) = \mathcal{T}$.

(ii) Let $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}_n^c$ be such that $\mathcal{K}(\mathcal{G}_1) = \mathcal{K}(\mathcal{G}_2) = \mathcal{T}$. Then $\mathcal{K}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{T}$.

(iii) Let $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}_n^c$ be such that $\mathcal{T} \subset \mathcal{G}_2 \subset \mathcal{G}_1$ and $\mathcal{K}(\mathcal{G}_1) = \mathcal{T}$. Then $\mathcal{K}(\mathcal{G}_2) = \mathcal{T}$.

Claims (i) and (iii) follow immediately from the construction of the Kruskal algorithm. We now prove claim (ii). We argue by contradiction. Assume that $\mathcal{K}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{T}'$ for some $\mathcal{T}' \in \mathfrak{T}_n$ with $\mathcal{T}' \neq \mathcal{T}$. In particular, there exists $m \in \mathbb{N}$ such that

$$\mathcal{F}_m(\mathcal{G}_1 \cup \mathcal{G}_2) \neq \mathcal{F}_m(\mathcal{G}_1) = \mathcal{F}_m(\mathcal{G}_2) \,. \tag{D.2}$$

Note that $m \geqslant 2$ since $\mathcal{F}_1(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{F}_1(\mathcal{G}_1) = \mathcal{F}_1(\mathcal{G}_2)$ consists of the smallest edge in $\mathcal{T}$. In particular, we have that

$$\mathcal{F}_{m-1}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{F}_{m-1}(\mathcal{G}_1) = \mathcal{F}_{m-1}(\mathcal{G}_2) \,. \tag{D.3}$$

By construction

$$\mathcal{F}_m(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{F}_{m-1}(\mathcal{G}_1 \cup \mathcal{G}_2) \cup \{e\} \tag{D.4}$$

for some edge $e \in \mathcal{G}_1 \cup \mathcal{G}_2$. If $e \in \mathcal{G}_1$, then by the construction of the Kruskal algorithm, as well as (D.3)–(D.4), we get that $\mathcal{F}_m(\mathcal{G}_1) = \mathcal{F}_{m-1}(\mathcal{G}_1) \cup \{e\} = \mathcal{F}_m(\mathcal{G}_1 \cup \mathcal{G}_2)$, which contradicts (D.2). Analogously, we obtain a contradiction if $e \in \mathcal{G}_2$. Claim (ii) then follows. □

We now have the necessary tools to prove Lemma 5.6.

Proof of Lemma 5.6. We show the claim by applying Kruskal's algorithm in (5.4) and by resumming the contributions of edges that do not belong to the thus obtained spanning trees. Recalling (5.4), we have that

$$\begin{aligned}
\varphi^L(\omega_1, \dots, \omega_n) &= \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} \sum_{\mathcal{G} \in \mathcal{K}^{-1}(\mathcal{T})} \prod_{\{i,j\} \in \mathcal{G}} \zeta^L(\omega_i, \omega_j) \\
&= \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} \prod_{\{i,j\} \in \mathcal{T}} \zeta^L(\omega_i, \omega_j) \sum_{\mathcal{G} \in \mathcal{K}^{-1}(\mathcal{T})} \prod_{\{i,j\} \in \mathcal{G} \setminus \mathcal{T}} \left(\mathrm{e}^{-\mathcal{V}^{\nu,\lambda,L}(\omega_i,\omega_j)/2} - 1 \right) \\
&= \frac{1}{n!} \sum_{\mathcal{T} \in \mathfrak{T}_n} \prod_{\{i,j\} \in \mathcal{T}} \zeta^L(\omega_i, \omega_j) \, \mathrm{e}^{-\sum_{\{i,j\} \in M(\mathcal{T}) \setminus \mathcal{T}} \mathcal{V}^{\nu,\lambda,L}(\omega_i,\omega_j)/2} \,.
\end{aligned} \tag{D.5}$$

Note that in the last line we applied Lemma D.1. The claim follows from (D.5) by recalling (3.1). □

Remark D.2. A similar method to bound the Ursell function has been applied in a more general context in [2, Theorem 3.1].

D.2. Proof of Lemma 5.8. In this subsection, we prove Lemma 5.8.

Proof of Lemma 5.8. Let us first consider the case when $q \geqslant 1$. We estimate the expression on the left-hand side of (5.16) by analysing Riemann sums. We note that the function $f(t) = \mathrm{e}^{-\kappa t}\, t^q$ is increasing on $[0, \frac{q}{k}]$ and decreasing on $[\frac{q}{k}, \infty)$. We choose $\ell_0 \in \mathbb{N}^*$ such that $\ell_0 \nu \leqslant \frac{q}{\kappa} < (\ell_0 + 1)\nu$. By construction of ℓ_0, we have that

$$\nu \sum_{\substack{\tilde{T} \in \nu\mathbb{N}^* \\ \tilde{T} \leqslant (\ell_0 - 1)\nu}} \mathrm{e}^{-\kappa \tilde{T}}\, \tilde{T}^q + \nu \sum_{\substack{\tilde{T} \in \nu\mathbb{N}^* \\ \tilde{T} \geqslant (\ell_0 + 2)\nu}} \mathrm{e}^{-\kappa \tilde{T}}\, \tilde{T}^q \leqslant \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\kappa t}\, t^q = \frac{q!}{\kappa^{q+1}}\,. \tag{D.6}$$

We now show that the bound in (D.6) also holds for the terms with $\ell \in \{\ell_0, \ell_0 + 1\}$. We note that for $\ell \in \{\ell_0, \ell_0 + 1\}$

$$\mathrm{e}^{-\kappa\nu\ell} \lesssim \mathrm{e}^{-q}\,, \quad (\nu\ell)^q \leqslant \left(\frac{q+1}{\kappa}\right)^q \leqslant \mathrm{e}\left(\frac{q}{\kappa}\right)^q. \tag{D.7}$$

Using (D.7), applying Stirling's formula, and recalling (5.10), we deduce indeed that $\nu\, \mathrm{e}^{-\kappa\nu\ell}\, (\nu\ell)^q \lesssim \frac{q!}{\kappa^{q+1}}$. For $q = 0$, we also get the bound (5.16), but the proof is simplified since the function $f(t) = \mathrm{e}^{-\kappa t}$ is decreasing on $[0, \infty)$ and we can estimate the Riemann sum by the integral. □

D.3. Proofs of Lemmas 5.11 and 5.12. In this subsection, we prove Lemmas 5.11 and 5.12, which were used in the proof of Proposition 5.5.

Proof of Lemma 5.11. The proof is similar to that of Lemma 5.7. We just outline the main differences. We first prove (i). Recalling (5.44) and arguing analogously as in (5.18)–(5.20), we reduce to estimating

$$\int_{\Lambda_L} \mathrm{d}z\, v^L(y - z) \int \mathbb{W}_{z,z}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega})\,, \tag{D.8}$$

for fixed $\tilde{T} \in \nu\mathbb{N}^*$ and $y \in \Lambda_L$. Here we used that, with notation as in (5.19)–(5.20), we have

$$\tilde{\omega}_1 \oplus \tilde{\omega}_2 \in \mathcal{D}_c^L \iff \tilde{\omega}_2 \oplus \tilde{\omega}_1 \in \mathcal{D}_c^L\,. \tag{D.9}$$

In (D.9), given $\omega_1 \in \Omega^{L,T(\omega_1)}, \omega_2 \in \Omega^{L,T(\omega_2)}$, we define their concatenation $\omega_1 \oplus \omega_2 \in \Omega^{L,T(\omega_1)+T(\omega_2)}$ by

$$\omega_1 \oplus \omega_2(t) := \begin{cases} \omega_1(t) & \text{if } 0 \leqslant t < T(\omega_1) \\ \omega_2(t - T(\omega_1)) & \text{if } T(\omega_1) \leqslant t \leqslant T(\omega_1) + T(\omega_2)\,. \end{cases}$$

We now integrate over all possible pairs of points $(w_1, w_2) \in \Lambda_L^2$ with $|w_1 - w_2|_L \geqslant cL$ and all possible times at which the path $\tilde{\omega}$ reaches these points (assuming without loss of generality that it reaches w_1 before w_2) to obtain

$$\begin{aligned}(\text{D.8}) \leqslant \int_{\Lambda_L} \mathrm{d}z\, v^L(y - z) \int_0^{\tilde{T}} \mathrm{d}t_1 \int_{t_1}^{\tilde{T}} \mathrm{d}t_2 \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2\, \mathbf{1}_{|w_1 - w_2|_L \geqslant cL}\, \psi^{L,t_1}(w_1 - z) \\ \times \psi^{L,t_2 - t_1}(w_2 - w_1)\, \psi^{L,\tilde{T} - t_2}(z - w_2)\,.\end{aligned} \tag{D.10}$$

We estimate

$$\psi^{L,t_2 - t_1}(w_2 - w_1) \lesssim \mathrm{e}^{\kappa\tilde{T}/2}\, \mathrm{e}^{-C\kappa L}\,, \tag{D.11}$$

which follows from Lemma C.2 (ii) since $t_2 - t_1 \leqslant \tilde{T}$. We then integrate in w_1 using $\psi^{L,s}(w_1 - z)$, in w_2 using $\psi^{L,t_1}(z - w_2)$, in z using $v^L(y - z)$, and finally in t_1, t_2 to deduce that

$$(\text{D.10}) \lesssim \tilde{T}^2 \, \mathrm{e}^{\kappa \tilde{T}/2} \, \mathrm{e}^{-C\kappa L} \, \|v\|_{\ell^1} \, . \tag{D.12}$$

Here, we used (5.5) and Lemma C.1 (iii). We deduce claim (i) from (D.12) by arguing analogously as in the proof of Lemma 5.7 (i).

We now prove (ii). Let us consider fixed $\tilde{T} \in \nu\mathbb{N}^*$, $r, s \in \nu\mathbb{N}$ with $r < T(\omega), s < \tilde{T}$, and $t \in [0, \nu]$. Instead of (5.23), we need to estimate

$$\int_{\Lambda_{L_0}} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega}) \, v^L(\omega(t+r) - \tilde{\omega}(t+s)) \, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \, , \tag{D.13}$$

which is

$$\leqslant \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2 \int_{\Lambda_{L_0}} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega}) \, v^L(\omega(t+r) - \tilde{\omega}(t+s)) \, \mathbf{1}_{|w_1 - w_2|_L \geqslant cL} \, \mathbf{1}_{w_1, w_2 \in [\tilde{\omega}]} \, . \tag{D.14}$$

In (D.14), we use the shorthand $w_1, w_2 \in [\tilde{\omega}]$ to denote that there exist $t_1, t_2 \in [0, \tilde{T}]$ such that $\tilde{\omega}(t_1) = w_1, \tilde{\omega}(t_2) = w_2$. We assume without loss of generality that $t_1 \leqslant t_2$. We need to consider several cases depending on the relative size of t_1, t_2 with respect to $t + s$.

Case 1: $t_1 \leqslant t_2 \leqslant t + s$. The contribution to (D.14) from this case is

$$\begin{aligned} \int_{\Lambda_{L_0}} \mathrm{d}y \int_0^{t+s} \mathrm{d}t_1 \int_{t_1}^{t+s} \mathrm{d}t_2 \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2 \int_{\Lambda_L} \mathrm{d}z \, \mathbf{1}_{|w_1 - w_2|_L \geqslant cL} \, \psi^{L,t_1}(w_1 - x) \\ \times \psi^{L,t_2 - t_1}(w_2 - w_1) \, \psi^{L,t+s-t_2}(z - w_2) \, \psi^{L,\tilde{T}-(t+s)}(y - z) \, v^L(\omega(t+r) - z) \, . \end{aligned} \tag{D.15}$$

In (D.15), we estimate $\psi^{L,t_2-t_1}(w_2 - w_1)$ as in (D.11). We then integrate in w_1 by $\psi^{L,t_1}(w_1 - x)$, in w_2 by $\psi^{L,t+s-t_2}(z - w_2)$, in y by $\psi^{L,\tilde{T}-(t+s)}(y - z)$, and in z by $v^L(\omega(t+r) - z)$. Using (5.5), Lemma C.1 (iii), and integrating in t_1, t_2, we get that

$$(\text{D.15}) \lesssim \tilde{T}^2 \, \mathrm{e}^{\kappa \tilde{T}/2} \, \mathrm{e}^{-C\kappa L} \, \|v\|_{\ell^1} \, . \tag{D.16}$$

Case 2: $t_1 \leqslant t + s \leqslant t_2$. The contribution to (D.14) from this case is

$$\begin{aligned} \int_{\Lambda_{L_0}} \mathrm{d}y \int_0^{t+s} \mathrm{d}t_1 \int_{t+s}^{\tilde{T}} \mathrm{d}t_2 \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2 \int_{\Lambda_L} \mathrm{d}z \, \mathbf{1}_{|w_1 - w_2|_L \geqslant cL} \, \psi^{L,t_1}(w_1 - x) \\ \times \psi^{L,t+s-t_1}(z - w_1) \, \psi^{L,t_2-(t+s)}(w_2 - z) \, \psi^{L,\tilde{T}-t_2}(y - w_2) \, v^L(\omega(t+r) - z) \, . \end{aligned} \tag{D.17}$$

By using the triangle inequality, it suffices to consider two cases when estimating (D.17).

Case 2A: $|w_1 - z|_L \geqslant cL/2$. In this case, we estimate $\psi^{L,t+s-t_1}(z - w_1)$ as in (D.11). We then integrate in y by $\psi^{L,\tilde{T}-t_2}(y - w_2)$, in w_1 by $\psi^{L,t_1}(w_1 - x)$, in w_2 by $\psi^{L,t_2-(t+s)}(w_2 - z)$, in z by $v^L(\omega(t+r) - z)$ and argue as before to get the same upper bound as in (D.16). Here, we also used Lemma C.1 (iii) and (5.5).

Case 2B: $|w_2 - z|_L \geqslant cL/2$. In this case, we use Lemma C.2 (ii) to estimate

$$\psi^{L,t_2-(t+s)}(w_2 - z) \lesssim \mathrm{e}^{\kappa \tilde{T}/2} \, \mathrm{e}^{-C\kappa |w_2 - z|_L} \, . \tag{D.18}$$

We first integrate in y using $\psi^{L,\tilde{T}-t_2}(y - w_2)$, then in w_2 using $\psi^{L,t_2-(t+s)}(w_2 - z)$ together with (D.18) and the assumption that $|w_2 - z|_L \geqslant cL/2$, in w_1 using $\psi^{L,t_1}(w_1 - x) \, \psi^{L,t+s-t_1}(z - w_1) \leqslant \psi^{L,t_1}(w_1 - x)$ (the last inequality follows from Lemma C.1 (i)), and in z by $v^L(\omega(t+r) - z)$. In particular, we deduce that

$$(\text{D.17}) \lesssim \tilde{T}^2 \, \mathrm{e}^{\kappa \tilde{T}/2} \, \mathrm{e}^{-C\kappa L} \, \|v\|_{\ell^1} \, . \tag{D.19}$$

Case 3: $t+s \leqslant t_1 \leqslant t_2$. The contribution to (D.14) from this case is

$$\int_{\Lambda_{L_0}} \mathrm{d}y \int_{t+s}^{\tilde{T}} \mathrm{d}t_1 \int_{t_1}^{\tilde{T}} \mathrm{d}t_2 \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2 \int_{\Lambda_L} \mathrm{d}z \, \mathbf{1}_{|w_1-w_2|_L \geqslant cL} \, \psi^{L,t+s}(z-x) \\ \times \psi^{L,t_1-(t+s)}(w_1-z) \, \psi^{L,t_2-t_1}(w_2-w_1) \, \psi^{L,\tilde{T}-t_2}(y-w_2) \, v^L(\omega(t+r)-z) \, . \quad \text{(D.20)}$$

We use Lemma C.2 (ii) to estimate

$$\psi^{L,t_2-t_1}(w_2-w_1) \lesssim \mathrm{e}^{\kappa\tilde{T}/2} \, \mathrm{e}^{-C\kappa|w_2-w_1|_L} \, . \quad \text{(D.21)}$$

In (D.20), we first integrate in y by using $\psi^{L,\tilde{T}-t_2}(y-w_2)$, then in w_2 by using $\psi^{L,t_2-t_1}(w_2-w_1)$ together with (D.21) and the assumption that $|w_1-w_2|_L \geqslant cL$, in w_1 by using $\psi^{L,t_1-(t+s)}(w_1-z)$, and in z by using $v^L(\omega(t+r)-z) \, \psi^{L,t_1-(t+s)}(w_1-z) \leqslant v^L(\omega(t+r)-z)$. In particular, we deduce that

$$\text{(D.20)} \lesssim \tilde{T}^2 \, \mathrm{e}^{\kappa\tilde{T}/2} \, \mathrm{e}^{-C\kappa L} \, \|v\|_{\ell^1} \, . \quad \text{(D.22)}$$

Combining (D.16), (D.19), and (D.22) we deduce that

$$\text{(D.13)} \lesssim \tilde{T}^2 \, \mathrm{e}^{\kappa\tilde{T}/2} \, \mathrm{e}^{-C\kappa L} \, \|v\|_{\ell^1} \, . \quad \text{(D.23)}$$

We now deduce claim (ii) from (D.23) by arguing analogously as in the proof Lemma 5.7 (ii).

In order to prove (iii), we argue similarly as in (5.28) and note that that left-hand side of (5.46) is

$$\leqslant \nu \sum_{\tilde{T} \in \nu\mathbb{N}^*} \mathrm{e}^{-\kappa\tilde{T}} \, \tilde{T}^q \int_{\Lambda_{L_0}} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega}) \, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \, . \quad \text{(D.24)}$$

We note that, for fixed $\tilde{T} \in \nu\mathbb{N}^*$,

$$\int_{\Lambda_{L_0}} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega}) \, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \leqslant \int_{\Lambda_{L_0}} \mathrm{d}y \int_0^{\tilde{T}} \mathrm{d}t_1 \int_{t_1}^{\tilde{T}} \mathrm{d}t_2 \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2 \, \mathbf{1}_{|w_1-w_2|_L \geqslant cL} \\ \psi^{L,t_1}(w_1-x) \, \psi^{L,t_2-t_1}(w_2-w_1) \, \psi^{L,\tilde{T}-t_2}(y-w_2) \lesssim \tilde{T}^2 \, \mathrm{e}^{\kappa\tilde{T}/2} \, \mathrm{e}^{-C\kappa L} \, . \quad \text{(D.25)}$$

In order to obtain the last estimate in (D.25), we used Lemma C.1 (iii) to integrate in y by $\psi^{L,\tilde{T}-t_2}(y-w_2)$, in w_2 by $\psi^{L,t_2-t_1}(w_2-w_1)$, and in w_1 by $\psi^{L,t_1}(w_1-x)$. Substituting (D.25) into (D.24), we deduce claim (iii) as in the proof of Lemma 5.7 (iv). □

Proof of Lemma 5.12. By recalling (5.49) and arguing analogously as in the proof of Lemma 5.7 (i) and (ii), we deduce the following estimates.

(1) $\int \mu^L(\mathrm{d}\tilde{\omega}) \, T(\tilde{\omega})^q \, |\zeta_c^L(\omega,\tilde{\omega})| \lesssim \frac{T(\omega)}{\kappa^{q+1}} \, q! \, \|v_c^L\|_{\ell^1(\Lambda_L)}$.

(2) $\int_{\Lambda_{L_0}} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega}) \, T(\tilde{\omega})^q \, |\zeta_c^L(\omega,\tilde{\omega})| \lesssim \frac{T(\omega)}{\kappa^{q+2}} \, (q+1)! \, \|v_c^L\|_{\ell^1(\Lambda_L)}$ for all $x \in \Lambda_{L_0}$.

The claim now follows. □

D.4. Proofs of Lemmas 5.18 and 5.19. In this subsection, we prove Lemmas 5.18 and 5.19.

Proof of Lemma 5.18. The proof is similar to that of Lemma 5.11. The main difference is that we need to analyse the case when there is a hard core. Let us first prove (i). We estimate the two terms that come from (5.78). For the first term, we use (5.70)–(5.71) and estimate

$$\int \mu^L(\mathrm{d}\tilde{\omega})\, T(\tilde{\omega})^q\, \mathbf{1}_{\mathcal{I}(\omega,\nu)}(\tilde{\omega})\, \mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega}) \leqslant \nu \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\, \tilde{T}^{q-1} \int_0^\nu \mathrm{d}t \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde{T}} \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2\, \mathbf{1}_{|w_1-w_2|_L\geqslant cL} \int \mathbb{W}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\, \mathbf{1}_{\omega(t+r)=\tilde{\omega}(t+s)}\, \mathbf{1}_{w_1,w_2\in[\tilde{\omega}]}\,. \quad \text{(D.26)}$$

In the last line of (D.26), we use the same shorthand as in (D.14). We write $w_j = \tilde{\omega}(t_j), j = 1, 2$, where $0 \leqslant t_1 \leqslant t_2 \leqslant \tilde{T}$, without loss of generality. As in the proof of Lemma 5.11, we need to consider three cases depending on the relative sizes of $t_1, t_2, t+s$.

Case 1: $t_1 \leqslant t_2 \leqslant t+s$. In this case, the last line of (D.26) is

$$\int_{\Lambda_L} \mathrm{d}x \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2\, \mathbf{1}_{|w_1-w_2|_L\geqslant cL} \int_0^{t+s} \mathrm{d}t_1 \int_{t_1}^{t+s} \mathrm{d}t_2\, \psi^{L,t_1}(w_1-x)\, \psi^{L,t_2-t_1}(w_2-w_1) \times \psi^{L,(t+s)-t_2}(\omega(t+r)-w_2)\, \psi^{L,\tilde{T}-(t+s)}(x-\omega(t+r)) \lesssim \tilde{T}^2\, \mathrm{e}^{\frac{\kappa_0\tilde{T}}{2\nu}}\, \mathrm{e}^{-CL}\,, \quad \text{(D.27)}$$

where we use Lemma C.2 (ii) to estimate $\psi^{L,t_2-t_1}(w_2-w_1)$. We then use Lemma C.1 (iii) to integrate in w_1 by $\psi^{L,t_1}(w_1-x)$, in w_2 by $\psi^{L,(t+s)-t_2}(\omega(t+r)-w_2)$, and in x by $\psi^{L,\tilde{T}-(t+s)}(x-\omega(t+r))$.

Case 2: $t_1 \leqslant t+s \leqslant t_2$. In this case, the last line of (D.26) is

$$\int_{\Lambda_L} \mathrm{d}x \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2\, \mathbf{1}_{|w_1-w_2|_L\geqslant cL} \int_0^{t+s} \mathrm{d}t_1 \int_{t+s}^{\tilde{T}} \mathrm{d}t_2\, \psi^{L,t_1}(w_1-x)\, \psi^{L,(t+s)-t_1}(\omega(t+r)-w_1) \times \psi^{L,t_2-(t+s)}(w_2-\omega(t+r))\, \psi^{L,\tilde{T}-t_2}(x-w_2) \lesssim \tilde{T}^2\, \mathrm{e}^{\frac{\kappa_0\tilde{T}}{2\nu}}\, \mathrm{e}^{-CL}\,. \quad \text{(D.28)}$$

In order to deduce (D.28), we considered two subcases, which follow by the triangle inequality.

Case 2A: $|\omega(t+r)-w_1|_L \geqslant cL/2$. Here, we first use Lemma C.1 (i) and Lemma C.2 (ii) to integrate in w_1 by $\psi^{L,t_1}(w_1-x)\, \psi^{L,(t+s)-t_1}(\omega(t+r)-w_1) \leqslant \psi^{L,(t+s)-t_1}(\omega(t+r)-w_1)$. Then, we use Lemma C.1 (iii) to integrate in w_2 by $\psi^{L,t_2-(t+s)}(w_2-\omega(t+r))$ and in x by $\psi^{L,\tilde{T}-t_2}(x-w_2)$. The result in the second subcase is obtained analogously.

Case 2B: $|w_2-\omega(t+r)|_L \geqslant cL/2$. Here, we first use Lemma C.1 (i) and Lemma C.1 (iii) to integrate in x by $\psi^{L,t_1}(w_1-x)\, \psi^{L,\tilde{T}-t_2}(x-w_2) \leqslant \psi^{L,\tilde{T}-t_2}(x-w_2)$. We then use Lemma C.1 (iii) and Lemma C.2 (ii) to integrate in w_1 by $\psi^{L,(t+s)-t_1}(\omega(t+r)-w_1)$ and in w_2 by $\psi^{L,t_2-(t+s)}(w_2-\omega(t+r))$ respectively.

Case 3: $t+s \leqslant t_1 \leqslant t_2$. We note that the last line of (D.26) is

$$\int_{\Lambda_L} \mathrm{d}x \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2\, \mathbf{1}_{|w_1-w_2|_L\geqslant cL} \int_{t+s}^{\tilde{T}} \mathrm{d}t_1 \int_{t_1}^{\tilde{T}} \mathrm{d}t_2\, \psi^{L,t+s}(\omega(t+r)-x)\, \psi^{L,t_1-(t+s)}(w_1-\omega(t+r)) \times \psi^{L,t_2-t_1}(w_2-w_1)\, \psi^{L,\tilde{T}-t_2}(x-w_2) \lesssim \tilde{T}^2\, \mathrm{e}^{\frac{\kappa_0\tilde{T}}{2\nu}}\, \mathrm{e}^{-CL}\,. \quad \text{(D.29)}$$

In order to obtain (D.29), we estimate $\psi^{L,t+s}(\omega(t+r)-x)$ by using Lemma C.1 (i). We then integrate in x by $\psi^{L,\tilde{T}-t_2}(x-w_2)$, in w_2 by $\psi^{L,t_2-t_1}(w_2-w_1)$ which we previously estimate as in

(D.21), and in w_1 by $\psi^{L,t_1-(t+s)}(w_1-\omega(t+r))$. For the last two integrations, we use Lemma C.1 (iii).

Using (D.27)–(D.29), we deduce that

$$(\text{D.26}) \lesssim \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\frac{\kappa_0\tilde{T}}{2\nu}}\,\tilde{T}^{q+2}\,T(\omega)\,\mathrm{e}^{-CL} \lesssim \frac{T(\omega)\,\mathrm{e}^{-CL}}{\kappa_0^{q+3}}\,(q+2)!\,\nu^{q+2}\,. \tag{D.30}$$

In order to deduce (D.30), we recalled (5.76) and used Lemma 5.8.

We now estimate the second term coming from (5.78), i.e.

$$\frac{1}{2}\int \mu^L(\mathrm{d}\tilde{\omega})\,T(\tilde{\omega})^q\,\mathcal{V}^{\nu,1,L,(2)}(\omega,\tilde{\omega})\,\mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega})\,. \tag{D.31}$$

Arguing analogously as in the proof of Lemma 5.11 (i), we reduce to estimating (D.8) with v^L replaced by $v^{L,(2)}$. Arguing analogously as in (D.10)–(D.12), we get that this quantity is $\lesssim \tilde{T}^2\,\mathrm{e}^{\frac{\kappa_0\tilde{T}}{2\nu}}\,\mathrm{e}^{-CL}\,\|v^{(2)}\|_{\ell^1}$. In particular, we deduce that

$$(\text{D.31}) \lesssim \frac{1}{\nu}\sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\frac{\kappa_0\tilde{T}}{2\nu}}\,\tilde{T}^{q+2}\,T(\omega)\,\|v^{(2)}\|_{\ell^1}\,\mathrm{e}^{-CL} \lesssim \frac{T(\omega)\,\mathrm{e}^{-CL}}{\kappa_0^{q+3}}\,(q+2)!\,\nu^{q+1}\,\|v^{(2)}\|_{\ell^1}\,. \tag{D.32}$$

Claim (i) follows from (5.78), (D.30), and (D.32).

The proof of claim (ii) is similar. As in (i), we have to study the two terms that come from (5.78). By (5.70), (5.72), the first term is

$$\begin{aligned}\leqslant \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\,\tilde{T}^q \int_0^\nu \mathrm{d}t \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde{T}} \\ \int_{\Lambda_{L_0}} \mathrm{d}y \int_{\Lambda_L} \mathrm{d}w_1 \int_{\Lambda_L} \mathrm{d}w_2\,\mathbf{1}_{|w_1-w_2|_L\geqslant cL} \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\,\mathbf{1}_{\omega(t+r)=\tilde{\omega}(t+s)}\,\mathbf{1}_{w_1,w_2\in[\tilde{\omega}]}\,.\end{aligned} \tag{D.33}$$

By arguing analogously as in (D.27)–(D.29), we deduce that the last line of (D.33) is $\lesssim \tilde{T}^2\,\mathrm{e}^{\frac{\kappa_0\tilde{T}}{2\nu}}\,\mathrm{e}^{-CL}$. Here we note that, in the proof of (D.27)–(D.29), we are always integrating in x in the fourth factor of $\psi^{L,t}(\cdot)$. Substituting this bound back into (D.33), and arguing as for (D.30), we deduce that

$$(\text{D.33}) \lesssim \frac{T(\omega)\,\mathrm{e}^{-CL}}{\kappa_0^{q+4}}\,(q+3)!\,\nu^{q+2}\,. \tag{D.34}$$

The second term coming from (5.78) is

$$\begin{aligned}\leqslant \frac{2}{\nu}\sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\,\tilde{T}^q \int_0^\nu \mathrm{d}t \sum_{r\in\nu\mathbb{N}} \mathbf{1}_{r<T(\omega)} \sum_{s\in\nu\mathbb{N}} \mathbf{1}_{s<\tilde{T}} \\ \int_{\Lambda_{L_0}} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\,v^{L,(2)}\big(\omega(t+r)-\tilde{\omega}(t+s)\big)\,\mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega})\,.\end{aligned} \tag{D.35}$$

By arguing as for (D.23), the last line in (D.35) is $\lesssim \tilde{T}^2\,\mathrm{e}^{\frac{\kappa_0\tilde{T}}{2\nu}}\,\mathrm{e}^{-CL}\,\|v^{(2)}\|_{\ell^1}$. Substituting this bound back into (D.35), and arguing as for (D.30), we deduce that

$$(\text{D.35}) \lesssim \frac{T(\omega)\,\mathrm{e}^{-CL}}{\kappa_0^{q+4}}\,(q+3)!\,\|v^{(2)}\|_{\ell^1}\,\nu^{q+1}\,. \tag{D.36}$$

We deduce claim (ii) from (D.34) and (D.36).

Finally, we prove claim (iii). By (5.70), (5.72), the expression that we want to estimate is

$$\leqslant \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa_0\tilde{T}/\nu}\,\tilde{T}^q \left[\int_{\Lambda_{L_0}} \mathrm{d}y \int \mathbb{W}_{y,x}^{L,\tilde{T}}(\mathrm{d}\tilde{\omega})\,\mathbf{1}_{\mathcal{D}_c^L}(\tilde{\omega})\right] \lesssim \sum_{\tilde{T}\in\nu\mathbb{N}^*} \mathrm{e}^{-\frac{\kappa_0\tilde{T}}{2\nu}}\,\tilde{T}^{q+2}\,\mathrm{e}^{-CL}$$
$$\lesssim \frac{\mathrm{e}^{-CL}}{\kappa_0^{q+3}}\,(q+2)!\,\nu^{q+2}\,, \tag{D.37}$$

where in the second inequality in (D.37), we argued as for (D.25) to estimate the expression in square brackets. □

Proof of Lemma 5.19. We use the estimate $|\zeta_c^{L,(2)}(\omega,\tilde{\omega})| \leqslant \mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega})$, which follows from (5.87) and the nonnegativity of $\mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega})$. We obtain (i) by arguing as in (5.81) above to deduce that $\int \mu^L(\mathrm{d}\tilde{\omega})\,T(\tilde{\omega})^q\,\mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega}) \lesssim \frac{T(\omega)}{\tilde{\kappa}^{q+1}}\,q!\,\|v_c^{L,(2)}\|_{\ell^1(\Lambda)}\,\nu^{q-1}$. Note that the first term in the upper bound (5.78) does not appear by construction of $\mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega})$. Likewise, we obtain (ii) by arguing as in (5.84) to deduce that

$$\int_{\Lambda_L} \mathrm{d}y \int \hat{\mu}_{y,x}^L(\mathrm{d}\tilde{\omega})\,T(\tilde{\omega})^q\,\mathcal{V}_c^{\nu,1,L,(2)}(\omega,\tilde{\omega}) \lesssim \frac{T(\omega)}{\tilde{\kappa}^{q+2}}\,(q+1)!\,\|v_c^{L,(2)}\|_{\ell^1(\Lambda_L)}\,\nu^{q-1}\,. \qquad \square$$

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