Mean-ρ Portfolio Selection

Nazem Khan

Thesis submitted for the degree of *Doctor of Philosophy*

University of Warwick
Department of Statistics

March 2022

Contents

List of Figures v

Abstract xi

1 Introduction 1

1.1 Motivating Example . . . 3
1.2 Objective . . . 6
1.3 Literature . . . 8
1.4 Structure . . . 12
1.5 Model . . . 16

2 Mean-Expected Shortfall Portfolio Selection 19

2.1 Mean-Expected Shortfall Portfolio Optimisation . . . 19
2.1.1 Optimal Portfolios . . . 21
2.1.2 Efficient Portfolios . . . 23
2.2 ES^{α}-Arbitrage and its Dual Characterisation . . . 25
2.3 Proofs of the Main Results . . . 27

3 Mean-ρ Portfolio Selection for Coherent Risk Measures 35

3.1 Mean-ρ Portfolio Optimisation . . . 36
3.1.1 Optimal Portfolios . . . 36
3.1.2 Efficient Portfolios . . . 40
3.1.3 ρ-Arbitrage . . . 42
3.1.4 ρ-Arbitrage for Elliptical Returns . . . 46
3.2 Dual Characterisation of (Strong) ρ-Arbitrage . . . 50
3.2.1 Preliminary Considerations and Conditions . . . 51
3.2.2 Dual Characterisation of Strong ρ-Arbitrage . . . 56
3.2.3 Dual Characterisation of ρ-Arbitrage . . . 58

3.3 Examples . . . 60
3.3.1 Worst-Case Risk Measure . . . 60
3.3.2 Value at Risk and Expected Shortfall . . . 61
3.3.3 Spectral Risk Measures . . . 63
3.3.4 Coherent Risk Measures on Orlicz Spaces . . . 64

4 Mean-ρ Portfolio Selection for Convex Risk Measures 69
4.1 Sensitivity to Large Losses . . . 71
4.1.1 Weak Sensitivity to Large Losses . . . 72
4.1.2 Optimal Boundary . . . 74
4.1.3 Efficient Portfolios . . . 77
4.1.4 (Strong) ρ-Arbitrage . . . 79
4.1.5 Strong Sensitivity to Large Losses . . . 80
4.1.6 Suitability for Risk Management and Portfolio Selection 83
4.2 Dual Results . . . 86
4.2.1 Preliminary Considerations and Conditions . . . 87
4.2.2 (Strong) ρ-Arbitrage . . . 91
4.2.3 Risk Measures Suitable for Portfolio Selection . . . 94
4.3 Examples . . . 96
4.3.1 Risk Functionals Based on Loss Functions . . . 96
4.3.2 g-Adjusted Expected Shortfall . . . 102
4.3.3 Loss Sensitive Expected Shortfall . . . 104

Closing Remarks 109

A Appendix 111
A.1 Expected Shortfall Deviation . . . 111
A.2 Coherent Risk Measures on Orlicz Spaces . . . 113
A.3 Counterexamples . . . 116
A.4 Star-Shapedness, Recession Cones and Recession Functions . . . 122
A.5 Key Definitions and Results on Convex Analysis . . . 123
A.6 Additional Results . . . 126

Bibliography 142

List of Figures

1 Mean-ES portfolio selection for the market (S^0, S^1) 4

2 General shapes of the ES^α-optimal boundary 23
3 ES^α-optimal boundary (red) and ES^α-efficient frontier (green) . 25

4 General shapes of the ρ-optimal boundary when $\rho_0 = 0$ and $\rho_1 \in \mathbb{R}$ 37
5 ρ-optimal boundary (red) and ρ-efficient frontier (green) when $\Pi^\rho_\nu \neq \emptyset$ for all $\nu \geq 0$ 42
6 ρ-arbitrage for ES (blue) and VaR (orange), for Gaussian returns 49

7 The ρ-optimal boundary (solid) and ρ-efficient frontier (green solid) with the ρ^∞-optimal boundary (dashed) and ρ^∞-efficient frontier (green dashed) when ρ satisfies convexity, the Fatou property on $\mathcal{X}$, weak sensitivity to large losses on $\mathcal{X}$, $\rho_0 = 0$ and $\rho_\nu < \infty$ for all $\nu \geq 0$. The green dot in the right lower panel indicates the efficient frontier for ρ^∞. 78
8 Dependence of α^* on b/σ. 107

9 The ρ-optimal boundary corresponding to Example A.3.8 121

Acknowledgements

First, I would like to thank my supervisors Vicky Henderson and Martin Herdegen. I am extremely grateful for their help, encouragement and guidance over the past three and a half years. I consider myself very lucky to have been given the chance to work with them.

During my time in Warwick University, I have had the pleasure to exchange words on mathematics, finance and beyond with several people. In particular, I will remember with joy discussions with Gulfraz Ahmed, Bapinder Bansal, Justin Gwee, Aparajita Haldar, Mohammad Hasan, Waseem Hussain and Eunji Lee. I am also thankful to the members of the Stochastic Finance Group at Warwick – they have all, in some way, contributed to this thesis. In particular, Urvashi Behary Paray, Miha Bresar, Nikolaos Constantinou, Joseph Jerome, Ruiqi Liu and Osian Shelley. Finally, I am especially grateful for the friendship of Xin Zhi.

Last but not least, I would like to thank my parents and siblings for their unconditional love and tremendous support during all the years of my life. Without them, I would not be where I am.

March 2022

Addendum: I would like to thank my examiners Cosimo Munari and Jon Warren for taking the time to read this thesis and for their helpful comments and suggestions.

June 2022

Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. Work based on collaborative research has all been conducted with my supervisor Martin Herdegen and is declared as follows:

- Chapter 2 is based on the working paper titled "A dual characterisation of regulatory arbitrage for Expected Shortfall".
- Chapter 3 is based on the paper [54] titled "Mean-ρ portfolio selection and ρ-arbitrage for coherent risk measures" which appears in the journal *Mathematical Finance*.
- Chapter 4 is based on the paper "Sensitivity to large losses and ρ-arbitrage for convex risk measures" which has been submitted to the journal *Management Science*.

Place: University of Warwick
Date: March 29, 2022 Nazem Khan

Abstract

The three pillars of Mathematical Finance are optimal investment, pricing, and risk management. In this thesis, we intertwine all three in the context of a one-period economy by replacing the variance in mean-variance portfolio selection by a risk measure ρ. This entanglement stems from ρ-arbitrage, which is a generalisation of ordinary arbitrage where – unlike in the classical theory of Markowitz – no efficient portfolios exist.

We first assume an Expected Shortfall (ES) risk constraint and prove that the market does not admit ES-arbitrage at confidence level α if and only if there exists an equivalent martingale measure $\mathbb{Q} \approx \mathbb{P}$ such that $\left\|\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right\|_\infty < \frac{1}{\alpha}$.

We then quantify risk by a general positively homogeneous risk measure. After providing a primal characterisation of ρ-arbitrage we prove that it cannot be excluded in this setting unless ρ is as conservative as the worst-case risk measure. In the case where ρ is a coherent risk measure that admits a dual representation, we further give a necessary and sufficient dual characterisation of ρ-arbitrage. This is intimately linked to the interplay between the set of equivalent martingale measures for the discounted risky assets and the set of absolutely continuous measures in the dual representation of ρ.

We end our exploration by considering star-shaped risk measures. We introduce the new axiom of strong sensitivity to large losses and show it is key to ensure the absence of ρ-arbitrage. This leads to a new class of risk measures that are suitable for portfolio selection. Specialising to the case that ρ is convex and admits a dual representation allows us to derive equivalent dual characterisations of ρ-arbitrage as well as the property that ρ is suitable for portfolio selection. Finally, we introduce the new risk measure of Loss Sensitive Expected Shortfall, which is similar to and not more complicated to compute than Expected Shortfall, but suitable for portfolio selection – which Expected Shortfall is not.

1

Introduction

It has been said (cf. [28]) that there have been three major revolutions in Mathematical Finance. The first one was Markowitz' mean-variance analysis [68]. Markowitz is widely regarded as the father of portfolio theory. Denoting by X_π the excess return of a portfolio $\pi \in \mathbb{R}^d$, he considered the following two problems:

- Given a minimal desired expected excess return $\nu^* \geq 0$, minimise the variance $\mathrm{Var}(X_\pi)$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\mathbb{E}[X_\pi] \geq \nu^*$;
- Given a maximal variance threshold $V^* \geq 0$, maximise the return $\mathbb{E}[X_\pi]$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\mathrm{Var}(X_\pi) \leq V^*$.

This is very intuitive and reduces the portfolio choice to a two-stage process: determining a set of *efficient* portfolios, and then selecting the most appropriate portfolio from the efficient set. Moreover, it is mathematically elegant and leads to nice explicit formulas due to the simplicity of the computation of the variance. Markowitz' work popularised concepts like diversification and overall portfolio risk and return, moving away from the performance of individual stocks. It led to the CAPM of Treynor [86], Sharpe [85], Lintner [66, 65] and Mossin [71], and changed the way people invested.

The second revolution was the Black-Scholes-Merton formula [17]. This helped theorise options trading, making it seem less like speculation. It had a significant impact on how financial markets were viewed, and paved the way towards the use of more sophisticated mathematical methods. The formula is based around the crucial principle of *no-arbitrage*, which states that a model of a financial market should not admit any arbitrage opportunities. Loosely speaking, an arbitrage opportunity is a tradable payoff producing a sure gain with no chance of loss. Thus, the absence of arbitrage is a natural criterion. It took over two decades to crystallise the mathematics of arbitrage (cf. [37]), which allows us to systematically derive prices for options and other contingent claims in an economically convincing way.

Finally, the third revolution was the development of *coherent risk measures* by Artzner, Delbaen, Eber and Heath [11], which was later extended to *convex risk measures* in [44] and [47]. Risk plays a role in every decision we face and is at the heart of finance. Its quantification is a key issue, both for regulators and financial institutions. Furthermore, new ways of looking at risk yield new approaches to other problems in finance. The main contribution of the above authors was shifting the paradigm of dealing with financial risks towards the use of axioms, and introducing a mathematical notion of acceptability. They laid the foundation of a new theory of risk measurement that has turned out to be surprisingly rich and holds strong connections to many other areas in economics, statistics and mathematics, cf. [46].

In this thesis, we link the first and third revolution by substituting the variance in the theory of Markowitz by a general measure of risk ρ. Denoting by X_π the excess return of a portfolio $\pi \in \mathbb{R}^d$, we consider the following two problems:

(1) Given a minimal desired expected excess return $\nu^* \geq 0$, minimise the risk $\rho(X_\pi)$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\mathbb{E}[X_\pi] \geq \nu^*$;

(2) Given a maximal risk threshold $\rho^* \geq 0$, maximise the return $\mathbb{E}[X_\pi]$ among all portfolios $\pi \in \mathbb{R}^d$ that satisfy $\rho(X_\pi) \leq \rho^*$.

Herein, we refer to this as *mean-ρ portfolio selection*, and we refer to the solutions of (1) and (2) as *ρ-efficient portfolios*.

When computing the risk of a portfolio, the variance has some deficiencies due to its symmetry and inability to capture the risk of low probability events, as for example default. Moreover, it is well-known that mean-variance preferences are not consistent with *second-order stochastic dominance*, while mean-ρ preferences can be. However, unlike classical mean-variance portfolio selection, mean-ρ portfolio selection may be *ill-posed* in the sense that there are no ρ-efficient portfolios, or even worse, that there is a sequence of portfolios $(\pi_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}[X_{\pi_n}] \uparrow \infty$ and $\rho(X_{\pi_n}) \downarrow -\infty$. We refer to these situations as *ρ-arbitrage* and *strong ρ-arbitrage*, respectively. This terminology is motivated by the fact that they generalise the ordinary notion of arbitrage, cf. Proposition 3.1.21. As a byproduct, we extend the methods relating to the second revolution by replacing "no-arbitrage" with "no-(strong)-ρ-arbitrage".

1.1 Motivating Example

We begin with a toy example. Consider a market with two assets. Asset 0 is riskless, hence by definition we know exactly what it is worth at time $t = 1$. Assume its price at time $t = 0$ is equal to 1 and for simplicity neglect interest rates, i.e.,

$$S_0^0 = S_1^0 = 1.$$

Assume the risky asset, asset 1, has the following dynamics

$$S_0^1 = 1 \quad \text{and} \quad S_1^1 = \begin{cases} 2, & \text{with probability } \frac{1}{3}, \\ 1, & \text{with probability } \frac{1}{3}, \\ \frac{1}{2}, & \text{with probability } \frac{1}{3}. \end{cases}$$

Denote the return of asset $i \in \{0, 1\}$ by $R^i := (S_1^i - S_0^i)/S_0^i$ and fix an initial wealth $x_0 > 0$. We can parametrise trading in fractions of wealth. Let $\pi^1 \in \mathbb{R}$ and $\pi^0 = 1 - \pi^1$ denote the fractions of wealth invested in the risky asset and riskless asset, respectively. Then the space of obtainable returns is given by the set

$$\mathcal{X} := \{\pi^0 R^0 + \pi^1 R^1 : \pi^1 \in \mathbb{R} \text{ and } \pi^0 = 1 - \pi^1\} = \{\pi^1 R^1 : \pi^1 \in \mathbb{R}\},$$

which does *not* depend on π^0 nor x_0.

Mean-ES Portfolio Selection

In this market, every portfolio can be described by a real number π^1. For given desired return $\nu \geq 0$, one has to invest $\pi_\nu^1 = 6\nu$ into the risky asset to obtain an expected return of ν. As for the associated risk, we use the *Expected Shortfall*, which is the most important risk measure in banking practice and insurance regulation as of today. The Expected Shortfall (ES) of an integrable random variable X at confidence level $\alpha \in (0,1)$ is given by

$$\mathrm{ES}^{\alpha}(X) := \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}^u(X)\,\mathrm{d}u,$$

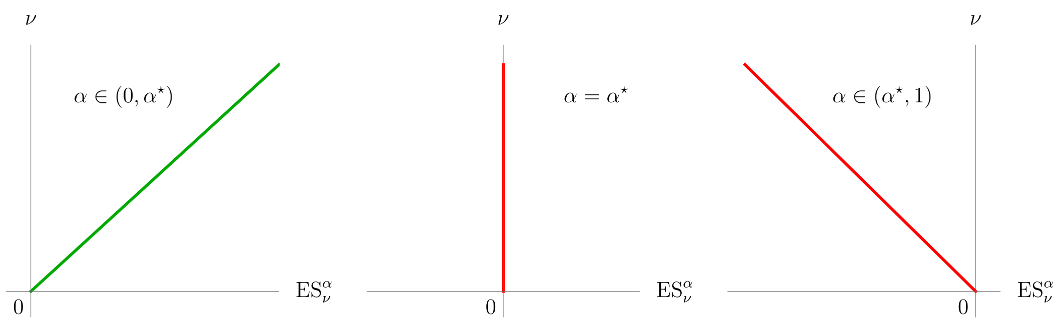
where $\mathrm{VaR}^u(X) := \inf\{m \in \mathbb{R} : \mathbb{P}[m+X<0] \leq u\}$ is the *Value at Risk* (VaR) of X at confidence level $u \in (0,1)$. Here, portfolios with an expected return of $\nu \geq 0$ have an ES at level α given by

$$\mathrm{ES}^{\alpha}_{\nu} = \begin{cases} 3\nu, & \text{if } \alpha \in (0,1/3], \\ \frac{\nu}{\alpha}, & \text{if } \alpha \in (1/3,2/3), \\ \frac{5\nu}{\alpha} - 6\nu, & \text{if } \alpha \in [2/3,1). \end{cases}$$

Fixing $\alpha^* = 5/6$, for $\nu > 0$ we obtain

$$\mathrm{ES}^{\alpha}_{\nu} \begin{cases} > 0, & \text{if } \alpha < \alpha^*, \\ = 0, & \text{if } \alpha = \alpha^*, \\ < 0, & \text{if } \alpha > \alpha^*. \end{cases}$$

If we plot expected return against ES of return, we get the following three cases:



(a) (b) (c)

Figure 1: Mean-ES portfolio selection for the market (S^0, S^1)

It is easy to check that in case (a) of Figure 1 we do not have ES^{α}-arbitrage. There is a risk-reward tradeoff. Efficient portfolios exist and they "lie" on the green branch. However, we do have ES^{α}-arbitrage for cases (b) and (c). The risk constraint becomes *counterproductive* for $\alpha \geq \alpha^*$. In the case of (c), i.e., $\alpha > \alpha^*$, we further have strong ES^{α}-arbitrage.

No-(Strong)-ES-Arbitrage Pricing

Now suppose we introduce a third asset S^2 in our model, an option on S^1 with strike price K: the buyer of the option has the right, but not the obligation, to buy one share in asset 1 at time $t = 1$ at a predetermined price K. For directness, we fix $K = 1$. The option is worthless at time $t = 1$ if $S_1^1 \leq 1$, otherwise it is worth $S_1^1 - 1$. Therefore,

$$S_1^2 = (S_1^1 - 1)^+.$$

However, the value of the option at time $t = 0$ is not clear. One approach is to find the values S_0^2 for which the augmented market (S^0, S^1, S^2) *does not admit (strong)* ES^α*-arbitrage.*

To that end, if we assume $S_0^2 = x > 0$, then $R^2 = (S_1^2 - x)/x$ and for $\nu \geq 0$

$$\mathbb{E}[\pi^1 R^1 + \pi^2 R^2] = \nu \iff \pi^1 = 6\nu - 2\pi^2((1-x)/x - 2).$$

Using this, it follows that any portfolio (in fractions of wealth) $(\pi^1, \pi^2) \in \mathbb{R}^2$ that has an expected return of ν has a corresponding ES at level α given by

$$\mathrm{ES}^\alpha(\pi^1 R^1 + \pi^2 R^2) = \begin{cases} -m_1, & \text{if } \alpha \in (0, 1/3], \\ \frac{1}{\alpha}(-\frac{1}{3}m_1 - (\alpha - \frac{1}{3})m_2), & \text{if } \alpha \in (1/3, 2/3), \\ \frac{1}{\alpha}(-\frac{1}{3}m_1 - \frac{1}{3}m_2 - (\alpha - \frac{2}{3})m_3), & \text{if } \alpha \in [2/3, 1), \end{cases}$$

where

$$\begin{aligned} m_1 &:= \min(\pi^1 R^1 + \pi^2 R^2) = \min\{6\nu + \pi^2(4 - \tfrac{1-x}{x}), -\pi^2 - 3\nu - \pi^2(3 - \tfrac{1-x}{x})\}; \\ m_2 &:= \min(\{6\nu + \pi^2(4 - \tfrac{1-x}{x}), -\pi^2, -3\nu - \pi^2(3 - \tfrac{1-x}{x})\} \setminus \{m_1\}); \\ m_3 &:= \max(\pi^1 R^1 + \pi^2 R^2) = \max\{6\nu + \pi^2(4 - \tfrac{1-x}{x}), -\pi^2, -3\nu - \pi^2(3 - \tfrac{1-x}{x})\}. \end{aligned}$$

Letting $\mathrm{ES}_\nu^\alpha := \min\{\mathrm{ES}^\alpha(\pi^1 R^1 + \pi^2 R^2) : \pi^1 = 6\nu - 2\pi^2((1-x)/x - 2)\}$, for $\nu > 0$ we obtain $\mathrm{ES}_\nu^\alpha = \nu \mathrm{ES}_1^\alpha$, where

$$\begin{aligned} \mathrm{ES}_1^\alpha > 0 &\iff \max(0, \tfrac{1}{3} - \tfrac{1}{9\alpha}) < x < \min(\tfrac{1}{3}, \tfrac{1}{6\alpha}); \\ \mathrm{ES}_1^\alpha \geq 0 &\iff \max(0, \tfrac{1}{3} - \tfrac{1}{9\alpha}) \leq x \leq \min(\tfrac{1}{3}, \tfrac{1}{6\alpha}). \end{aligned}$$

When $S_0^2 = x < 0$, we work with the equivalent market $(S^0, S^1, \tilde{S}^2)$ where

$\tilde{S}^2 := -S^2$ (so we can use fractions of wealth). By arguing similar to above, one can show $\mathrm{ES}^\alpha_\nu = \nu \mathrm{ES}^\alpha_1$ where $\mathrm{ES}^\alpha_1 < 0$. Finally when $S^2_0 = 0$, we work with the equivalent market $(S^0, S^1, \tilde{S}^2)$ where $\tilde{S}^2 := S^2 + S^0$ (so that the relative return is well-defined). Here, $\mathrm{ES}^\alpha_\nu = \nu \mathrm{ES}^\alpha_1$ where $\mathrm{ES}^\alpha_1 = 0$ if $\alpha \in (0, \alpha^*]$ and $\mathrm{ES}^\alpha_1 < 0$ otherwise.

Using similar logic to the reasoning after Figure 1 together with the analysis above reveals that the no-(strong)-ES^α-arbitrage pricing interval for this new financial instrument is given by the (closed) open interval from $\max(0, \frac{1}{3} - \frac{1}{9\alpha})$ to $\min(\frac{1}{3}, \frac{1}{6\alpha})$. (Note that this interval is empty for $\alpha \geq (>)\ 5/6 = \alpha^*$ as expected.)

1.2 Objective

The preceding example was simple enough to take a direct approach. However, we want to develop a deeper understanding. We want to construct a theory that can be applied to more general markets and to a wide variety of risk measures, allowing us to bypass complicated calculations.

Fixing a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Riesz space $L^\infty \subset L \subset L^1$ with the $\mathbb{P}$-a.s. ordering, the objective of this thesis is to study mean-ρ portfolio selection where $\rho : L \to (-\infty, \infty]$ satisfies

- Monotonicity: For $X_1, X_2 \in L$ such that $X_1 \leq X_2$ $\mathbb{P}$-a.s., $\rho(X_1) \geq \rho(X_2)$;
- Normalisation: $\rho(0) = 0$;
- Star-shapedness: For all $X \in L$ and $\lambda \geq 1$, $\rho(\lambda X) \geq \lambda \rho(X)$.

Here, monotonicity means that higher payoffs have lower risk, which is a very natural property. Normalisation encodes that no investment means no risk. Finally, star-shapedness captures the idea that a position's risk should increase at least proportionally when scaled by a factor greater than one. (Note that this framework is rich enough to include any convex risk measure.)

The way we tackle mean-ρ portfolio selection is to first study the mean-ρ problem (1) with an equality constraint, i.e., for fixed $\nu \geq 0$, find the minimal risk among the portfolios with expected excess return ν:

(1') For $\nu \geq 0$, minimise $\rho(X_\pi)$ among all portfolios π with $\mathbb{E}[X_\pi] = \nu$.

Solutions to (1') are referred to as *ρ-optimal portfolios*. The first question we seek to answer (Q1), concerns their existence:

(Q1) **Existence of optimal portfolios.** What conditions guarantee that ρ-optimal portfolios for a desired excess return $\nu \geq 0$ exist?

The next step is to plot ν against $\rho_\nu := \inf_{\pi \in \Pi_\nu} \rho(X_\pi)$ as in Figure 1, and identify the set of ρ-efficient portfolios. We want to perform this procedure only if we know this set is nonempty, otherwise the risk constraint is *void*. Thus, the next question we consider is:

(Q2) **Absence of (strong) ρ-arbitrage.** What are necessary and sufficient conditions to ensure that a market does not admit (strong) ρ-arbitrage?

This is also important in the context of pricing. Indeed, we have seen in the motivating example that obtaining no-(strong)-ES-arbitrage price bounds directly for a new asset can be tedious. This can be simplified if we can find a simple characterisation of (strong) ρ-arbitrage.

Remark 1.2.1. In classical mean-variance portfolio selection, the solution to (1') exists for all $\nu \geq 0$. It also gives the solution to (1) and provides the so-called efficient frontier, which in turn can be used to derive the solution to (2). In particular, (1) and (2) are always well-posed and equivalent problems. By contrast, in the mean-ρ setting, existence in (1') is not guaranteed. Moreover, even if (1') has a solution for all $\nu \geq 0$, (1) and (2) may both be ill-posed, or (1) may be well-posed and (2) ill-posed. This implies in particular that (1) and (2) are no longer equivalent. These issues arise exactly when the market admits *ρ-arbitrage*.

These first two questions are local, in the sense that we work on a fixed market. The final question we consider is global:

(Q3) **Suitable for risk management/portfolio selection.**

(a) When is ρ suitable for risk management, i.e., when does ρ satisfy the following universal property: *every* market that satisfies no-arbitrage does not admit strong ρ-arbitrage?

(b) When is ρ suitable for portfolio selection, i.e., when does ρ satisfy the following universal property: for *every* market that satisfies no-arbitrage and for every $\nu^* \geq 0$ and $\rho^* \geq 0$, the mean-ρ problems (1) and (2) admit at least one solution with finite risk?

The notion of suitability for risk management is crucial from a regulator's perspective. They want to avoid situations where there is a sequence of portfolios whose reward increases to ∞ and risk decreases to $-\infty$. In addition, being suitable for portfolio selection is desirable from an investor's perspective as efficient portfolios exist. Economically speaking, (Q3) is the most important question. The qualifier "suitable" should be interpreted from a purely theoretical perspective. Whether or not risk measures deemed "unsuitable" for portfolio portfolio selection/risk management are actually unsuitable in practise is left for further research, cf. Remark 2.3.2.

1.3 Literature

Mean-ρ Portfolio Selection

Mean-ρ portfolio selection for specific (classes of) risk measures has been well studied in the extant literature. Alexander and Baptista [5] solved the problem of mean-VaR portfolio selection explicitly for multivariate normal returns distributions. Rockafellar and Uryasev [77] studied mean-ES portfolio selection for continuous returns distributions and showed that the optimisation problem could be reduced to linear programming. Subsequently, the results of [77] were extended to general returns distributions by the same authors [78] and later generalised to spectral risk measures by Adam et al. [2].

A related strand of literature has looked at mean-ρ portfolio selection, where ρ is a *deviation risk measure* (a generalisation of standard deviation). This class of risk measures has been axiomatically studied by Rockafellar et al. [80]. They showed in [81] that if $\mathcal{D}$ is a deviation risk measure, then mean-$\mathcal{D}$ portfolio selection is always well posed. However, deviation risk measures quantify the *degree of uncertainty* in a random variable, while regulators are more concerned with the *overall seriousness of possible losses*. In particular, deviation risk measures are not monotone.

Beyond the aforementioned research, the theory surrounding mean-ρ portfolio selection is sparse. Notwithstanding, the minimisation of convex risk measures has been studied by Ruszczyński and Shapiro [84], and their results were later extended to *quasiconvex* risk measures by Mastrogiacomo and Gianin [69]. These two papers study the following question: Given a vector space $\mathcal{Z}$ representing the set of actions and a function $F : \mathcal{Z} \to L$ which maps each action to a payoff, when is $\min_{z \in C} \rho(F(z))$ well-posed for some some given convex subset C of $\mathcal{Z}$. While their setting is more general than ours, their assumptions on ρ are stronger (in particular a 'nice' dual representation). As an application, they consider the mean-ρ problem (1) and provide sufficient conditions that guarantee the existence of a solution to (1). In particular, these conditions imply the existence of optimal portfolios (in our sense) and thereby answer (Q1) at least partially. Nevertheless, their results do not contribute to answering (Q2). Indeed, as we have seen in the toy example, even if the mean-ρ problem (1) has a solution, there might still be ρ-arbitrage – in which case the mean-ρ problem (2) does not have a solution. Finally, neither [84] nor [69] consider (Q3) which we believe to be the most interesting and important question from the point of view of the regulator.

(Strong) ρ-Arbitrage

The existence of ρ-arbitrage is puzzling at first sight because this is a situation that does not appear in the classical mean-variance framework. Its occurrence was first recognised for VaR by [5] who gave necessary and sufficient conditions for its absence in the case of multivariate normal returns distributions. For ES, the possibility of ρ-arbitrage was first noted in a working paper by De Giorgi [34] in the case of elliptical returns distributions and later observed in a simulation study by Kondor et al. [60]. The latter paper led to a more detailed study by Ciliberti et al. [31], who concluded that there is a *phase transition*, i.e., for small values of α, mean-ES portfolio selection is well-posed, and from a certain critical value α^* onwards, mean-ES portfolio selection becomes ill-posed. The working paper [79] also recognised the occurrence of ρ-arbitrage for coherent risk measures, noting that minimising the risk subject to an inequality constraint on the expected return may fail to have a solution. They called this an "acceptably free lunch". More recently, Armstrong and Brigo [9] showed that VaR and ES constraints may be void for behavioural investors with an S-shaped utility. They proceeded to study ρ-arbitrage for general coherent risk measures [10], focusing on multivariate normal returns and looking at the issue from an empirical/statistical perspective, demonstrating that it is relevant in practise.

(Strong) ρ-arbitrage is related to the theory of *good-deal pricing*. To make this connection, let us fix some pieces of notation. Denote by $\mathcal{X}$ the set of excess returns, $F \subset \mathcal{X}$ the set of "free" (i.e., whose price is nonpositive) nonzero payoffs in the market and let $D \subset L$ be a set of "desirable claims". Then the market is said to satisfy *no-good-deals* if $F \cap D = \emptyset$; in which case the set of no-good-deal prices for a financial contract Y outside the market is given by

$$\mathrm{I}_D(Y) := \{y \in \mathbb{R} : \text{the augmented market with } Y \text{ priced at } y \\ \text{satisfies no-good-deals}\}.$$

The absence of good-deals simply translates to forbidding positions that are "too good to be true". The no-good-deals pricing technique allows us to extend the pricing rule in a "market consistent way". Often both parts are expressed, via duality, using *pricing kernels*.

When $D = L_+ := \{X \in L : X \geq 0 \ \mathbb{P}\text{-a.s.}\}$, we are in the classical setting of arbitrage pricing. (For a historical overview of arbitrage pricing, refer to [37].) While the absence of arbitrage is universally accepted, its implications for pricing are often rather *weak*, since for incomplete markets, the interval $\mathrm{NA}(Y) := \mathrm{I}_{L_+}(Y)$ is too large to provide any useful information. Sharper bounds can be obtained by incorporating *individual preferences*.

A problem that arises immediately is how to define a good-deal, which unlike arbitrage opportunities, may expose to downside risk. Cochrane and Saa-Requejo [32] and Bernardo and Ledoit [15] initiated this study. The former used Sharpe ratios to govern the set D, while the latter employed gain-loss ratios. Their results were generalised by Černý and Hodges [23] who developed an abstract theory for *closed boundedly generated sets* D and applied it, in a finite state setting, to good-deals defined via utility functions. For a multi-period and continuous time treatment of utility-based good-deal bounds see [58] and [7], respectively. An alternative, but somewhat related way to define a good-deal is through risk measures. This was first explored by Jaschke and Küchler [55], who studied the situation where $D = \{X \in L : \rho(X) < 0\}$ and ρ is a coherent risk measure; this was later extended by Cherny [28].

In all the above concepts of a good-deal, $D \not\supset L_+$, which is problematic since it may mean that $\mathrm{I}_D(Y) \not\subset \mathrm{NA}(Y)$. The only notion (aside from ρ-arbitrage) we are aware of that truly subsumes ordinary arbitrage is the *(scalable) acceptable deal* by Arduca and Munari [8]. There, the authors derive a fundamental theorem of asset pricing for pointed convex cones D that contain

L_+. Their main result, albeit in a much more complicated setting, holds a strong connection with our main result in Chapter 3, cf. Remark 4.2.10.

Our contribution to the theory of good-deal pricing is expressed through Corollaries 4.2.8 and 4.2.11. In essence we replace the term "no-good-deal" with "no-(strong)-ρ-arbitrage", and are able to derive price bounds based on convex risk measures, which is missing is in the literature. One could argue that since (strong) ρ-arbitrage relies on the expectation playing the role of a *reward measure*, it would be more fitting to use the term (strong) $(\mathbb{E}, \rho)$-arbitrage. However, it is often the case that (strong) ρ-arbitrage can be reformulated in terms of a statement involving only ρ, cf. Remarks 3.1.17 and 3.1.19. Furthermore, as we do not differ how the reward is quantified in this thesis, there is nothing to be gained by complicating our terminology.

Related Concepts

A strand of literature that is not directly related to mean-ρ portfolio selection but is close from a conceptual point of view is mean-variance portfolio selection under *ambiguity aversion*. Here, the idea is that the investor is uncertain about the probabilistic model but otherwise stays in the classical mean-variance framework. Let us just mention two key contributions: Boyle et al. [20] assume that the investor is uncertain about the mean (but not the variance) of the risky assets and hence first minimises over the expected returns they consider plausible. If the investor has less uncertainty about the returns of some "familiar" assets, they hold – compared to classical mean-variance portfolio selection – a higher proportion of "familiar" assets and a lower proportion of "unfamiliar" ones, where they have more uncertainty about the returns. Maccheroni et al. [67] consider the Bayesian framework of model uncertainty from [57], where the agent has a prior on plausible models and penalises the mean-variance utility under the so-called ambiguity neutral model by a variance term describing the model uncertainty. In a setting with a riskless and two risky assets (one with and one without model uncertainty), they show that the alpha of the ambiguous asset is the key additional statistic in this problem.

Both mean-ρ portfolio selection and mean-variance portfolio selection under ambiguity aversion can be seen as a way of making the classical Markowitz problem more *robust*. In the former, the focus is on making the risk measure more robust (and correcting for the theoretical shortcomings of the variance as a measure of risk). In the latter, the focus is on making the probabilistic model more robust by taking uncertainty on the mean/the probabilistic model into account. Both extensions are important but address different issues.

Last but not least, ρ-arbitrage is conceptually related to the notion of *regulatory arbitrage*. Here the idea is that the risk measure constraint can be interpreted as a regulatory capital requirement imposed by the regulator. If the agent can act in some way to avoid (or weaken) the regulatory constraint, they perform a regulatory arbitrage. The term 'regulatory-arbitrage' has been emphasised in the literature more intensively since 2004 as explained in [88]. However, there is no universal definition for this concept. The general consensus is that it is a notion that refers to actions performed by financial institutions to avoid unfavourable regulation.

The closest paper to our work in that direction is Wang [87], who defines regulatory arbitrage quantitatively as the level of *superadditivity* that a risk measure possesses. The larger the latter, the more the agent can weaken the regulatory constraint by splitting up their position. While this definition is somewhat different from our notion of ρ-arbitrage, it captures the same idea that risk measure constraints may be (partially) *avoided* by financial agents. In our case, for certain market environments, the regulatory constraint becomes void in portfolio optimisation, whereas in [87], the action of splitting up the position can weaken the regulatory constraint.

1.4 Structure

In addition to this introductory part and to the following section, where we describe our model of a financial market, the dissertation is divided into three chapters.

Chapter 2 is devoted to a rigorous study of mean-ES portfolio selection. It is widely argued that the financial crisis of 2007-2009 was a result of excessive risk-taking by banks; see e.g. [41, 89]. Consequently, the financial regulators have tried to impose better risk constraints on financial institutions, which for the banking sector are codified in the Basel accords. One of the key changes from Basel II to Basel III was updating the 'official' risk measure from VaR to ES in the hope of better financial regulation; cf. the discussion in [59]. Thus, we begin our exploration with mean-ES portfolio selection. More precisely, we show that optimal portfolios always exist, we give necessary and sufficient conditions for the existence of (strong) ES-arbitrage, and conclude that ES is *not* suitable for risk management/portfolio selection.

Many of the techniques in Chapter 2 can be generalised to positively homogeneous risk measures, ρ ($\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \geq 0$). This is what we do in Chapter 3, focusing particularly on answering (Q1)-(Q3) for coherent measures of risk. We begin by showing that under mild regularity assumptions, positive homogeneity *alone* (without convexity) is enough to ensure existence of optimal portfolios. We then provide a primal characterisation of (strong) ρ-arbitrage.

We then introduce necessary and sufficient dual criteria for the absence of (strong) ρ-arbitrage when ρ is a coherent risk measure that admits a dual representation $\rho(X) = \sup_{Z\in\mathcal{Q}} \mathbb{E}[-ZX]$, where $\mathcal{Q}$ describes some dual set of probability measures that are absolutely continuous with respect to $\mathbb{P}$. Our main result in this chapter Theorem 3.2.18, shows that absence of ρ-arbitrage is equivalent to $\mathcal{P}\cap\tilde{\mathcal{Q}} \neq \emptyset$, where $\mathcal{P}$ describes the set of all equivalent martingale measures (EMMs) for the discounted risky assets and $\tilde{\mathcal{Q}}$ is the "interior" of $\mathcal{Q}$. The precise definition for this "interior" of $\mathcal{Q}$ is very delicate because both topological and algebraic notions fail. For this reason, we define $\tilde{\mathcal{Q}}$ in an abstract way that also gives some additional flexibility. This is worth the effort: As a by-product of our main result, we get a refined version of the fundamental theorem of asset pricing in a one-period market: for returns in L^1, we show in Theorem 3.3.2 that standard no-arbitrage is equivalent to the existence of an EMM $\mathbb{Q}$ whose Radon-Nikodým derivative is uniformly bounded *away from* 0.

We proceed to apply our dual results to a large variety of examples. These examples also highlight an important technical feature of our analysis. In order to achieve a maximum level of generality, we do *not* assume that the set $\mathcal{Q}$ in the dual representation of ρ (which is not unique) is L^1-closed (as for example assumed by [28]), nor do we assume that it coincides with the maximal dual set. This extra flexibility allows us to get dual characterisations of ρ-arbitrage even in cases when ρ might take the value ∞, and to explicitly characterise the "interior" set $\tilde{\mathcal{Q}}$ for a large class of examples.

In this chapter, we also explain how ρ-arbitrage and strong ρ-arbitrage generalise the classical notions of arbitrage of the first and second kind, respectively, and show in Theorem 3.1.22 that (strong) ρ-arbitrage *cannot be excluded* (under standard no-arbitrage) unless ρ is as conservative as the *worst-case risk measure*. Since a worst-case approach to risk is infeasible in practise, this indicates that one should move *beyond* the class of positively homogeneous risk measures for effective risk constraints in the context of portfolio selection. This is the content of Chapter 4.

In Chapter 4 we complete our objective and answer (Q1)-(Q3) in the case where ρ is monotone, normalised and star-shaped. We first address (Q1) and show in Theorem 4.1.5 that the crucial ingredient for the existence of optimal portfolios is that ρ satisfies on the set of returns $\mathcal{X} \subset L$, the following new axiom:

- *Weak sensitivity to large losses on $\mathcal{X}$:* For any $X \in \mathcal{X}$ with $\mathbb{P}[X < 0] > 0$ and $\mathbb{E}[X] = 0$, there exists $\lambda > 0$ such that $\rho(\lambda X) > 0$.

The economic meaning of this axiom is simple and intuitive: Apart from the riskless portfolio, any portfolio that is expected to break-even has a positive risk if it is scaled by a sufficiently large amount.

We then turn our attention to (Q2). Here, it turns out that the crucial ingredient is a stronger version of the above axiom:

- *Strong sensitivity to large losses on $\mathcal{X}$*: For all $X \in \mathcal{X}$ with $\mathbb{P}[X < 0] > 0$, there exists $\lambda > 0$ such that $\rho(\lambda X) > 0$.

Again, its economic meaning is simple and intuitive: Apart from the riskless portfolio, any portfolio has a positive risk if it is scaled by a sufficiently large amount. The main issue with positively homogeneous risk measures, as mentioned in [46, page 306], is that an acceptable position X remains acceptable if it is multiplied by an arbitrarily large factor $\lambda > 0$; and this is exactly what makes it undesirable in portfolio selection since it can lead to an "undetected accumulation of risk" [11, page 233]. Strong sensitivity to large losses overcomes this pitfall. With the help of this axiom, we can provide a primal characterisation of ρ-arbitrage in Theorem 4.1.23.

We also seek to derive a *dual characterisation* for the absence of (strong) ρ-arbitrage. To this end, a key methodological tool is to consider ρ^∞, the *smallest positively homogeneous* risk functional that *dominates* ρ. Together with the fact that if ρ has a dual representation then so does ρ^∞, this allows us to lift the results from Chapter 3 on the dual characterisation of ρ-arbitrage for coherent risk measures to convex risk measures. We also provide a dual characterisation of *strong* ρ-arbitrage for convex risk measures. However, in this case the link to ρ^∞ breaks down and so the result is more involved.

Finally, we address (Q3). Here again, the key methodological tool is to consider ρ^∞. A key observation is that ρ satisfies weak/strong sensitivity to large losses if and only if ρ^∞ does. For part (a), we show in Lemma 4.1.26 that a risk measure ρ is suitable for risk management if and only if ρ^∞ is the worst-case risk measure. And for part (b), we prove in Lemma 4.1.30 that a convex risk measure ρ is suitable for portfolio selection if and only if it is real-valued and ρ^∞ is the worst-case risk measure. Combining these two results yields in Theorem 4.1.31 the unexpected result that suitability for risk management is equivalent to suitability for portfolio selection for a very wide class of risk measures.

While the above results fully answer (Q3) from a theoretical perspective, it leaves open the questions *how large* the subclass of risk measures suitable for portfolio selection is and how concrete examples look like. Perhaps surprisingly, we can describe in Theorem 4.2.13 *all* such convex risk measures in a *dual way* if L is an Orlicz heart, which includes all L^p-spaces for $p \in [1, \infty)$.

Of course, of special interest is the case $L = L^1$. We first show that an important subset of risk measures that are suitable for portfolio selection on L^1 are given by a subclass of so-called *g-adjusted Expected Shortfall* risk measures, recently studied by Burzoni et al. [21]. In particular, we introduce the new risk measure *Loss Sensitive Expected Shortfall*, which is not more complicated to compute than ES, but unlike ES, is suitable for portfolio selection on L^1. We believe that this new risk measure could become of great relevance to the regulator because it keeps many attractive features of ES, while being strongly sensitive to large losses.

The thesis ends with some closing remarks. Past that, there is the appendix and the bibliography. The appendix is split into six sections. Appendix A.1 contains some auxiliary results on the Expected Shortfall Deviation. Appendix A.2 starts with a brief overview of some key definition and results concerning Orlicz spaces before summarising existing results on the dual representation of real-valued coherent risk measures defined on Orlicz spaces. Counterexamples complementing our theory are collected in Appendix A.3. Appendix A.4 recalls some results relating star-shaped sets/functions with their recession cones/functions, whilst Appendix A.5 contains key definitions and results on convex analysis. Last but not least, some additional technical results can be found in Appendix A.6.

1.5 Model

The assumptions and notation introduced here will be adhered to throughout this thesis. We consider a one-period $(1+d)$-dimensional market

$$(S_t^0, \dots, S_t^d)_{t \in \{0,1\}}$$

on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that S^0 is riskless and satisfies $S_0^0 = 1$ and $S_1^0 = 1 + r$, where $r > -1$ denotes the riskless rate. We further assume that $S^1, \dots, S^d$ are risky assets, where $S_0^1, \dots, S_0^d > 0$ and $S_1^1, \dots, S_1^d$ are real-valued integrable random variables. We denote the relative return of asset $i \in \{0, \dots, d\}$ by

$$R^i := \frac{S_1^i - S_0^i}{S_0^i},$$

and its expectation by $\mu^i := \mathbb{E}[R^i]$. For notational convenience, we set $S := (S^1, \dots, S^d)$, $R := (R^1, \dots, R^d)$ and $\mu := (\mu^1, \dots, \mu^d) \in \mathbb{R}^d$. We may assume without loss of generality that the market is *nonredundant*, i.e., $\sum_{i=0}^d \vartheta^i S^i = 0$ $\mathbb{P}$-a.s. implies that $\vartheta^i = 0$ for all $i \in \{0, \dots, d\}$. We also assume that the risky returns are *nondegenerate* in the sense that for at least one $i \in \{1, \dots, d\}$, $\mu^i \neq r$. If this were not true, then there would be no incentive to trade in the financial market.

Portfolios

As $S_0^0, \dots, S_0^d > 0$, we can parametrise trading in *fractions of wealth*, and we assume that trading is frictionless. More precisely, we fix an initial wealth $x_0 > 0$ and describe any portfolio (for this initial wealth) by a vector $\pi = (\pi^1, \dots, \pi^d) \in \mathbb{R}^d$, where π^i denotes the fraction of wealth invested in asset $i \in \{1, \dots, d\}$. We do not impose any portfolio constraints, so in particular short selling is permitted. The fraction of wealth invested in the riskless asset is in turn given by $\pi^0 := 1 - \sum_{i=1}^d \pi^i = 1 - \pi \cdot \mathbf{1}$, where $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^d$. The *return* of a portfolio $\pi \in \mathbb{R}^d$ (which is independent of the initial wealth

x_0) can be computed by

$$R_\pi := (1 - \pi \cdot \mathbf{1})r + \pi \cdot R,$$

and the *excess return* of a portfolio $\pi \in \mathbb{R}^d$ over the riskless rate r is in turn given by

$$X_\pi := R_\pi - r = (1 - \pi \cdot \mathbf{1})r + \pi \cdot R - r = \pi \cdot (R - r\mathbf{1}). \tag{1.1}$$

It follows that $\mathcal{X} = \{X_\pi : \pi \in \mathbb{R}^d\}$ is a subspace of L^1. The *expected excess return* of a portfolio $\pi \in \mathbb{R}^d$ over the riskless rate r can be calculated as

$$\mathbb{E}[X_\pi] = \pi \cdot (\mu - r\mathbf{1}).$$

For fixed $\nu \in \mathbb{R}$, we set

$$\Pi_\nu := \{\pi \in \mathbb{R}^d : \mathbb{E}[X_\pi] = \nu\}, \tag{1.2}$$

i.e., Π_ν denotes the set of all portfolios with expected excess return ν. By nondegeneracy, Π_ν is nonempty for all $\nu \in \mathbb{R}$, and it is an affine subspace. Moreover, the definition of Π_ν in (1.2) implies that

$$\Pi_k = \begin{cases} k\Pi_1 := \{k\pi : \pi \in \Pi_1\}, & \text{if } k > 0, \\ (-k)\Pi_{-1} := \{-k\pi : \pi \in \Pi_{-1}\}, & \text{if } k < 0. \end{cases} \tag{1.3}$$

Finally, in the sequel, we will only focus on nonnegative excess returns.

2

Mean-Expected Shortfall Portfolio Selection

In this chapter, we study how the Expected Shortfall (ES) performs as a regulatory constraint imposed by the regulator on a financial agent seeking to optimise a portfolio. For an axiomatic justification of mean-ES portfolio selection, we refer to [52].

Standing Assumption. Throughout the entire chapter we consider the market (S^0, S) described in the introduction with the additional assumption that it satisfies no-arbitrage, i.e., there is no trading strategy $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}$ (in numbers of shares) such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0, \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \geq 0 \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[\vartheta^0 S_1^0 + \vartheta \cdot S_1 > 0] > 0.$$

By the Dalang-Morton-Willinger theorem [33], this means that the set

$$\mathcal{P} := \{\mathbb{Q} \approx \mathbb{P} : \mathbb{E}^{\mathbb{Q}}[S_1^i/(1+r)] = S_0^i \text{ for all } i = 1, \ldots, d\},$$

of all equivalent martingale measures for the discounted risky assets is nonempty and there is $\mathbb{Q} \in \mathcal{P}$ such that the Radon-Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is in L^∞.

2.1 Mean-Expected Shortfall Portfolio Optimisation

We start by recalling the definitions of Value at Risk and Expected Shortfall.

Definition 2.1.1. Let $\alpha \in (0,1)$ be a confidence level and X an integrable random variable.

- The *Value at Risk* (VaR) of X at confidence level α is given by

$$\mathrm{VaR}^{\alpha}(X) := \inf\{m \in \mathbb{R} : \mathbb{P}[m + X < 0] \leq \alpha\}.$$

- The *Expected Shortfall* (ES) of X at confidence level α is given by

$$\mathrm{ES}^{\alpha}(X) := \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}^{u}(X)\,\mathrm{d}u.$$

Remark 2.1.2. (a) It follows immediately from the definition that for fixed X, the function $\alpha \mapsto \mathrm{VaR}^{\alpha}(X)$ is nonincreasing, which in turn implies that the function $\alpha \mapsto \mathrm{ES}^{\alpha}(X)$ is nonincreasing and continuous. Note that $\alpha \mapsto \mathrm{VaR}^{\alpha}(X)$ might not be continuous.

(b) Basel III currently endorses ES at level 2.5%.

(c) Both VaR and ES are positively homogeneous. However, unlike VaR, ES satisfies *convexity*, which means it encourages diversification. It also admits a dual representation. To this end, for $\alpha \in (0,1)$, set

$$\mathcal{Q}_{\alpha} := \left\{ \mathbb{Q} \ll \mathbb{P} : \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \leq \frac{1}{\alpha}\ \mathbb{P}\text{-a.s.} \right\}. \tag{2.1}$$

By [45, Theorem 4.47] which extends to the case $X \in L^1$, we have the following dual characterisation of ES:

$$\mathrm{ES}^{\alpha}(X) = \max_{\mathbb{Q} \in \mathcal{Q}_{\alpha}} \mathbb{E}^{\mathbb{Q}}[-X]. \tag{2.2}$$

(d) By equation (A.2) ES is *(strictly) expectation bounded*, that is

$$\mathrm{ES}^{\alpha}(X) \geq \mathbb{E}[-X], \quad X \in L^1,$$

(where the inequality is strict if X is not $\mathbb{P}$-a.s. constant). This is an important property in what follows. Note that VaR is not expectation bounded. For further properties of VaR/ES we refer to [45, Section 4.4].

We start our discussion on mean-ES portfolio optimisation by introducing a partial preference order on the set of portfolios. This preference order formalises the idea that return is “desirable” and risk is “undesirable”.

Definition 2.1.3. Fix a confidence level $\alpha \in (0,1)$. A portfolio $\pi \in \mathbb{R}^d$ is strictly ES^α-preferred over a portfolio $\pi' \in \mathbb{R}^d$, if $\mathbb{E}[X_\pi] \geq \mathbb{E}[X_{\pi'}]$ and $\mathrm{ES}^\alpha(X_\pi) \leq \mathrm{ES}^\alpha(X_{\pi'})$, with at least one inequality being strict.

2.1.1 Optimal Portfolios

We approach the problem of mean-ES portfolio selection by first looking at the slightly simplified problem of finding the minimum risk portfolio given a fixed excess return.

Definition 2.1.4. Fix a confidence level $\alpha \in (0,1)$ and an excess return $\nu \geq 0$. A portfolio $\pi \in \mathbb{R}^d$ is called ES^α-*optimal* for ν if $\pi \in \Pi_\nu$ and $\mathrm{ES}^\alpha(X_\pi) \leq \mathrm{ES}^\alpha(X_{\pi'})$ for all $\pi' \in \Pi_\nu$. We denote the set of all ES^α-optimal portfolios for ν by Π^α_ν.

The following result gives existence and further properties of ES^α-optimal portfolios. It relies on properties of the so-called *Expected Shortfall Deviation* that are discussed in Appendix A.1.

Proposition 2.1.5. *Fix a confidence level* $\alpha \in (0,1)$ *and* $\nu \geq 0$. *Then the set* Π^α_ν *is nonempty, compact and convex. Moreover,* $\Pi^\alpha_0 = \{\mathbf{0}\}$ *and* $\Pi^\alpha_\nu = \nu\Pi^\alpha_1$.

Proof. This follows from Corollary A.1.4 in Appendix A.1. □

For $\nu > 0$, the set Π^α_ν is in general not a singleton, i.e., there may be multiple optimal portfolios. Nevertheless, for all $\pi \in \Pi^\alpha_\nu$, the associated risk $\mathrm{ES}^\alpha(X_\pi)$ is the same. Therefore, for $\alpha \in (0,1)$ and $\nu \geq 0$, we may define

$$\mathrm{ES}^\alpha_\nu := \mathrm{ES}^\alpha(X_\pi), \quad \pi \in \Pi^\alpha_\nu. \tag{2.3}$$

We proceed to study the properties of ES^α_ν. First, we consider it as a function of ν.

Lemma 2.1.6. *Fix a confidence level* $\alpha \in (0,1)$ *and* $\nu \geq 0$. *Then* $\mathrm{ES}^\alpha_\nu = \nu\mathrm{ES}^\alpha_1$ *where* $\mathrm{ES}^\alpha_1 \geq -1$.

Proof. This follows immediately from (1.1), Proposition 2.1.5, positive homogeneity of ES and (A.2). □

Next, we show that ES^α_1 as a function of α is nonincreasing and continuous. These properties will turn out to be very useful in the proof of our main results.

Proposition 2.1.7. *The function $\alpha \mapsto \mathrm{ES}_1^\alpha$ is nonincreasing and continuous.*

Proof. First, we establish monotonicity of ES_1^α. Let $0 < \alpha_1 < \alpha_2 < 1$. Let $\pi_{\alpha_1} \in \Pi_1^{\alpha_1}$ and $\pi_{\alpha_2} \in \Pi_1^{\alpha_2}$. Then by (2.3), the fact that ES^α is nonincreasing in α and the definition of ES^α-optimal portfolios, we obtain

$$\mathrm{ES}_1^{\alpha_1} = \mathrm{ES}^{\alpha_1}(X_{\pi_{\alpha_1}}) \geq \mathrm{ES}^{\alpha_2}(X_{\pi_{\alpha_1}}) \geq \mathrm{ES}^{\alpha_2}(X_{\pi_{\alpha_2}}) = \mathrm{ES}_1^{\alpha_2}.$$

Next, we establish left-continuity of $\alpha \mapsto \mathrm{ES}_1^\alpha$. So fix $\alpha \in (0,1)$ and let $\pi_\alpha \in \Pi_1^\alpha$. Let $\varepsilon > 0$. By continuity of ES^α in α, there is $\delta \in (0, \alpha)$ such that

$$\mathrm{ES}^\beta(X_{\pi_\alpha}) - \mathrm{ES}^\alpha(X_{\pi_\alpha}) < \varepsilon, \quad \text{for all } \beta \in (\alpha - \delta, \alpha].$$

Hence, by definition of ES_1^α, we obtain

$$\mathrm{ES}_1^\beta \leq \mathrm{ES}^\beta(X_{\pi_\alpha}) \leq \mathrm{ES}^\alpha(X_{\pi_\alpha}) + \varepsilon = \mathrm{ES}_1^\alpha + \varepsilon, \quad \text{for all } \beta \in (\alpha - \delta, \alpha].$$

Since $\mathrm{ES}_1^\beta \geq \mathrm{ES}_1^\alpha$ for $\beta \in (0, \alpha]$ by the first part of the proof, left-continuity of $\alpha \mapsto \mathrm{ES}_1^\alpha$ follows.

Finally, we establish right-continuity of $\alpha \mapsto \mathrm{ES}_1^\alpha$. Suppose to the contrary that there is $\alpha \in (0,1)$, a nonincreasing sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $(\alpha, 1)$ with $\lim_{n \to \infty} \alpha_n = \alpha$, and $\varepsilon > 0$ such that

$$\mathrm{ES}_1^{\alpha_n} \leq \mathrm{ES}_1^\alpha - \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, the set

$$A_n := \{\pi \in \Pi_1 : \mathrm{ES}^{\alpha_n}(X_\pi) \leq \mathrm{ES}_1^\alpha - \varepsilon\}$$

is nonempty. Each A_n is also compact by Proposition A.1.2(d), (A.4) and the fact that Π_1 is closed. Moreover, the sequence $(A_n)_{n \in \mathbb{N}}$ is nested in the sense that each A_n contains A_{n+1} by the fact that ES is nonincreasing in its confidence level. Hence $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$ by Cantor's intersection theorem, i.e., there exists $\pi \in \Pi_1$ such that

$$\mathrm{ES}^{\alpha_n}(X_\pi) \leq \mathrm{ES}_1^\alpha - \varepsilon \text{ for all } n \in \mathbb{N}.$$

By continuity of expected shortfall in its confidence level, we arrive at the contradiction

$$\mathrm{ES}^\alpha(X_\pi) \le \mathrm{ES}_1^\alpha - \varepsilon,$$

We conclude that $\alpha \mapsto \mathrm{ES}_1^\alpha$ is right-continuous. $\square$

We proceed to define the ES^α-*optimal boundary.*

Definition 2.1.8. Fix a confidence level $\alpha \in (0,1)$. Define the ES^α-*optimal boundary* by

$$\mathcal{O}_{\mathrm{ES}^\alpha} := \{(\mathrm{ES}_\nu^\alpha, \nu) : \nu \ge 0\} \subset \mathbb{R} \times \mathbb{R}_+.$$

The following result gives a full description of the shape of the ES^α-optimal boundary. It follows directly from Lemma 2.1.6.

Proposition 2.1.9. *Fix a confidence level $\alpha \in (0,1)$. Then $\mathcal{O}_{\mathrm{ES}^\alpha}$ is given by*

$$\mathcal{O}_{\mathrm{ES}^\alpha} = \{(k\mathrm{ES}_1^\alpha, k) : k \ge 0\}, \tag{2.4}$$

where $\mathrm{ES}_1^\alpha > -1$ is a constant that depends on α.

The following figure gives a graphical illustration of Proposition 2.1.9. It shows that $\mathcal{O}_{\mathrm{ES}^\alpha}$ can takes three different shapes depending on the sign of ES_1^α.

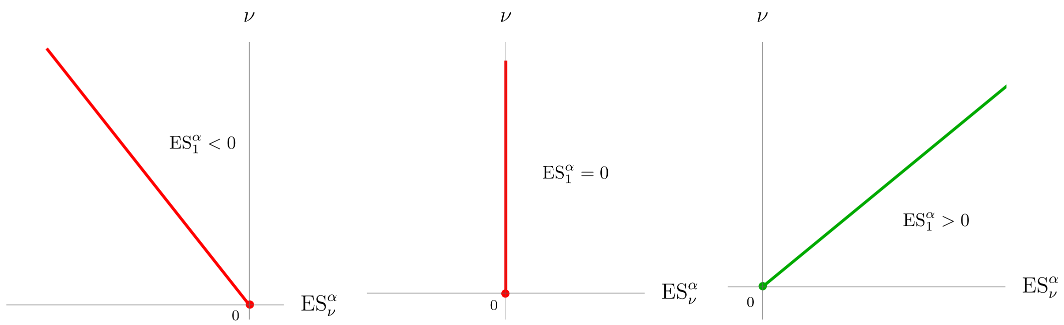


Figure 2: General shapes of the ES^α-optimal boundary

2.1.2 Efficient Portfolios

We proceed to study the notion of ES^α-efficient portfolios, which are defined in analogy to efficient portfolios in the classical mean-variance sense.

Definition 2.1.10. Fix a confidence level $\alpha \in (0,1)$. A portfolio $\pi \in \mathbb{R}^d$ is called ES^α-*efficient* if there does not exist another portfolio $\pi' \in \mathbb{R}^d$ that is strictly ES^α-preferred over π. We denote the ES^α-*efficient frontier* by

$$\mathcal{E}_{\mathrm{ES}^\alpha} := \{(\mathrm{ES}^\alpha(X_\pi), \mathbb{E}[X_\pi]) : \pi \text{ is } \mathrm{ES}^\alpha\text{-efficient}\}.$$

Remark 2.1.11. Note that a portfolio $\pi \in \mathbb{R}^d$ can only be ES^α-efficient if it is ES^α-optimal for some $\nu := \mathbb{E}[X_\pi] \geq 0$. Indeed if $\pi \notin \Pi_\nu^\alpha$, then any $\pi^* \in \Pi_\nu^\alpha$ is strictly ES^α-preferred over π because $\mathrm{ES}^\alpha(X_{\pi^*}) < \mathrm{ES}^\alpha(X_\pi)$ and $\mathbb{E}[X_{\pi^*}] = \mathbb{E}[X_\pi] = \nu$. Likewise if $\nu < 0$, then $\mathrm{ES}^\alpha(X_\pi) \geq \mathbb{E}[-X_\pi] = -\nu > 0$ by expectation boundedness of ES, and so the riskless portfolio $\mathbf{0}$ is strictly ES^α-preferred over π.

As we have seen in the introduction, it can happen that there are no ES^α-efficient portfolios – despite the fact that ES^α-optimal portfolios exists for all $\nu \geq 0$. The following result shows that the existence of the ES^α-efficient frontier depends only on the sign of ES_1^α.

Theorem 2.1.12. *Fix a sensitivity level $\alpha \in (0,1)$. Then the following are equivalent:*

(a) $\mathrm{ES}_1^\alpha > 0$.

(b) $\mathcal{E}_{\mathrm{ES}^\alpha} \neq \emptyset$.

Moreover, if $\mathrm{ES}_1^\alpha > 0$, then the efficient frontier is given by

$$\mathcal{E}_{\mathrm{ES}^\alpha} = \{(k\mathrm{ES}_1^\alpha, k) : k \geq 0\} \tag{2.5}$$

Proof. First assume that $\mathrm{ES}_1^\alpha > 0$. We proceed to show that any ES^α-optimal portfolio for $\nu \geq 0$ is efficient, and so (2.5) follows from Remark 2.1.11 and Proposition 2.1.9. Seeking a contradiction, let $\pi \in \Pi_\nu^\alpha$ for some $\nu \geq 0$ and assume that there is $\pi' \in \mathbb{R}^d$ such that $\mathbb{E}[X_{\pi'}] \geq \mathbb{E}[X_\pi] = \nu$ and $\mathrm{ES}^\alpha(X_{\pi'}) \leq \mathrm{ES}^\alpha(X_\pi) = \nu\mathrm{ES}_1^\alpha$, with one inequality being strict. Set $\nu' := \mathbb{E}[X_{\pi'}]$. If $\nu' = \nu$, then $\mathrm{ES}^\alpha(X_{\pi'}) < \mathrm{ES}^\alpha(X_\pi)$ and we arrive at a contradiction as $\pi \in \Pi_\nu^\alpha$. Otherwise, if $\nu' > \nu$, let $\pi^* \in \Pi_{\nu'}^\alpha$. Then $\mathbb{E}[X_{\pi^*}] = \nu' > \mathbb{E}[X_\pi] = \nu$ and $\mathrm{ES}^\alpha(X_{\pi^*}) = \nu'\mathrm{ES}_1^\alpha \leq \mathrm{ES}^\alpha(R_{\pi'} - r) \leq \mathrm{ES}^\alpha(R_\pi - r) = \nu\mathrm{ES}_1^\alpha$. Since $\mathrm{ES}_1^\alpha > 0$, we arrive at the contradiction that $\nu' > \nu$ and $\nu' \leq \nu$.

Finally, assume that $\mathrm{ES}_1^\alpha \leq 0$. We proceed to show that there does not exist any ES^α-efficient portfolio. Seeking a contradiction, suppose that $\pi \in \mathbb{R}^d$ is ES^α-efficient. Then $\pi \in \Pi_\nu^\alpha$ for some $\nu \geq 0$ by the first step. Pick $\nu' > \nu$ and let $\pi' \in \Pi^{\alpha}_{\nu'}$. It follows from Proposition 2.1.9 that $\mathbb{E}[X_{\pi'}] = \nu' > \mathbb{E}[X_{\pi}] = \nu$ and $\mathrm{ES}^{\alpha}(R_{\pi'} - r) = \nu' \mathrm{ES}^{\alpha}_1 \leq \nu \mathrm{ES}^{\alpha}_1$. Hence, π' is ES^{α}-preferred over π and we arrive at a contradiction. □

The following figure gives a graphical illustration of Theorem 2.1.12.

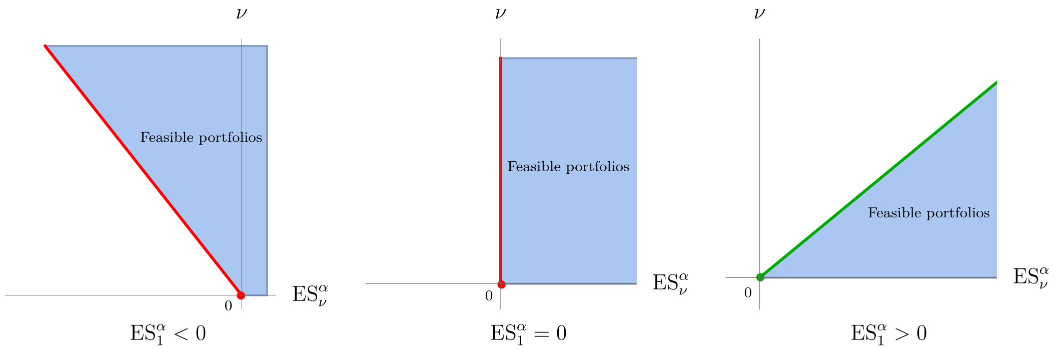


Figure 3: ES^{α}-optimal boundary (red) and ES^{α}-efficient frontier (green)

2.2 ES^{α}-Arbitrage and its Dual Characterisation

As we have seen in the previous section, mean-ES^{α} portfolio optimisation is not always well defined as it can happen that there are no ES^{α}-efficient portfolios. We call this situation ES^{α}*-arbitrage*.

Definition 2.2.1. Fix a confidence level $\alpha \in (0,1)$. The market (S^0, S) is said to satisfy ES^{α}*-arbitrage* if there are no ES^{α}-efficient portfolios. It is said to satisfy *strong* ES^{α}*-arbitrage* if there exists a sequence of portfolios $(\pi_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ with

$$\mathbb{E}[X_{\pi_n}] \uparrow \infty \quad \text{and} \quad \mathrm{ES}^{\alpha}(X_{\pi_n}) \downarrow -\infty.$$

Based on Theorem 2.1.12 and Lemma 2.1.6, we can show that the existence of (strong) ES^{α}-arbitrage is fully characterised by the sign of ES^{α}_1.

Theorem 2.2.2. *Fix a confidence level $\alpha \in (0,1)$. We have the following trichotomy:*

(a) *If $\mathrm{ES}^{\alpha}_1 > 0$, then the market (S^0, S) does not admit ES^{α}-arbitrage.*

(b) *If $\mathrm{ES}^{\alpha}_1 = 0$, then the market (S^0, S) admits ES^{α}-arbitrage but does not admit strong ES^{α}-arbitrage.*

(c) *If* $\mathrm{ES}_1^\alpha < 0$*, then the market* (S^0, S) *admits strong* ES^α*-arbitrage.*

Proof. (a) This follows from Theorem 2.1.12.

(b) By Theorem 2.1.12, it suffices to show that (S^0, S) does not admit strong ES^α-arbitrage. Let $\mathbf{0} \in \mathbb{R}^d$ be the riskless portfolio and $\pi' \in \mathbb{R}^d$ be any other portfolio such that $\nu := \mathbb{E}[X_{\pi'}] > 0$. Then $\mathrm{ES}^\alpha(X_{\pi'}) \geq \mathrm{ES}_\nu^\alpha = \nu \mathrm{ES}_1^\alpha = 0$ by the definition of ES_ν^α and Lemma 2.1.6. Thus (S^0, S) does not admit strong ES^α-arbitrage.

(c) Since $\mathrm{ES}_1^\alpha < 0$, by definition there exists a portfolio $\pi \in \Pi_1$ such that $\mathrm{ES}^\alpha(X_\pi) < 0$. Then, $\mathbb{E}[X_{k\pi}] \to \infty$ and $\mathrm{ES}^\alpha(X_{k\pi}) \to -\infty$ as $k \to \infty$. Thus, (S^0, S) admits strong ES^α-arbitrage. $\square$

Remark 2.2.3. Theorem 2.2.2 shows that ES^α-arbitrage corresponds to cases (b) and (c) in Figure 3, whereas strong ES^α-arbitrage corresponds to case (c).

Theorem 2.2.2 provides a full characterisation of (strong) ES^α-arbitrage. However, the criterion is rather indirect as it requires to calculate ES_1^α, which relies on a nontrivial optimisation problem. Inspired by the Dalang-Morton-Willinger theorem [33] and the dual representation of ES, one might want to look at a simpler *dual characterisation* of ES^α-arbitrage in terms of equivalent martingale measures (EMMs) $\mathcal{P}$ for the discounted risky assets S/S^0. To this end, fix $\alpha \in (0,1)$ and pick $\pi_\alpha \in \Pi_1^\alpha$. Combining Theorem 2.2.2 and (2.2) shows that the market (S^0, S) does not admit ES^α-arbitrage if and only if

$$\max_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}^{\mathbb{Q}}[-X_{\pi_\alpha}] > 0.$$

Now if $\mathcal{Q}_\alpha \cap \mathcal{P} \neq \emptyset$, i.e., $\mathcal{Q}_\alpha$ contains an EMM $\tilde{\mathbb{P}}$ for S/S^0, then (1.1) and the fact that $\tilde{\mathbb{P}}$ is an EMM give

$$\mathbb{E}^{\tilde{\mathbb{P}}}[-X_{\pi_\alpha}] = \mathbb{E}^{\tilde{\mathbb{P}}}[-\pi_\alpha \cdot (R - r\mathbf{1})] = -\pi_\alpha \cdot \mathbb{E}^{\tilde{\mathbb{P}}}[R - r\mathbf{1}] = -\pi_\alpha \cdot \mathbf{0} = 0,$$

where the expectation of a vector is understood component wise. As the optimal $\mathbb{Q}^*$ in (2.2) is in general not an equivalent measure, one might expect

$$\max_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}^{\mathbb{Q}}[-X_{\pi_\alpha}] > \mathbb{E}^{\tilde{\mathbb{P}}}[-X_{\pi_\alpha}] = 0,$$

which would imply that (S^0, S) does not admit ES^α-arbitrage.

Conversely, if $\mathcal{Q}_\alpha \cap \mathcal{P} = \emptyset$, one might hope that

$$\mathbb{E}^{\mathbb{Q}}[-X_{\pi_\alpha}] < 0$$

for all $\mathbb{Q} \in \mathcal{Q}_\alpha$, which would imply that the market admits ES^α-arbitrage.

It turns out that the heuristic argument above holds some truth and that the minimal L^∞-norm of Radon-Nikodým derivatives of EMMs is directly linked to the presence or absence of (strong) ES^α-arbitrage. This is the content of the two main results in Chapter 2.

Our first main result provides a dual characterisation of ES^α-arbitrage.

Theorem 2.2.4. *Let $\mathcal{P}$ be the set of equivalent martingale measures for S/S^0. Then for $\alpha \in (0,1)$ the following are equivalent:*

(a) *The market (S^0, S) does not admit* ES^α*-arbitrage.*

(b) *There exists $\mathbb{Q} \in \mathcal{P}$ such that* $\left\| \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right\|_\infty < \frac{1}{\alpha}$.

Our second main result provides a dual characterisation of strong ES^α-arbitrage.

Theorem 2.2.5. *Let $\mathcal{P}$ be the set of equivalent martingale measures for S/S^0. Then for $\alpha \in (0,1)$ the following are equivalent:*

(a) *The market (S^0, S) does not admit strong* ES^α*-arbitrage.*

(b) $\inf_{\mathbb{Q} \in \mathcal{P}} \left\| \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right\|_\infty \leq \frac{1}{\alpha}$.

The proofs of Theorems 2.2.4 and 2.2.5 are rather involved and delegated to the next section.

2.3 Proofs of the Main Results

In this section we provide rigorous proofs for our main results, Theorems 2.2.4 and 2.2.5. To this end, we define the *critical confidence level* $\alpha^* \in (0,1)$ for the market (S^0, S) by

$$\alpha^* := \sup\{\alpha \in (0,1) : \mathcal{Q}_\alpha \cap \mathcal{P} \neq \emptyset\}.$$

Note that $\alpha^* > 0$ by the Dalang-Morton-Willinger theorem [33] and $\alpha^* < 1$ by the fact that $\mathbb{P} \notin \mathcal{P}$ by nondegeneracy. More precisely, if there exists a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ with $\lim_{n\to\infty} \alpha_n = 1$ such that for each n there is $\mathbb{Q}_{\alpha_n} \in \mathcal{Q}_{\alpha_n} \cap \mathcal{P}$ with Radon-Nikodým derivative Z_{α_n}, then Z_{α_n} converges to $1 = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}}$ in L^1 by the fact that

$$\mathbb{E}[|Z_{\alpha_n} - 1|] = 2\mathbb{E}[(Z_{\alpha_n} - 1)^+] \leq 2\left(\frac{1}{\alpha_n} - 1\right),$$

where we have used that $\mathbb{E}[|X|] = 2\mathbb{E}[X^+]$ for $\mathbb{E}[X] = 0$. Hence, by dominated convergence, $\mathbb{P} \in \mathcal{P}$. Also, note for future reference that

$$\frac{1}{\alpha^*} = \inf_{\mathbb{Q}\in\mathcal{P}} \left\| \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right\|_\infty . \tag{2.6}$$

With the help of the critical confidence level α^*, we may combine Theorems 2.2.4 and 2.2.5 as follows:

Theorem 2.3.1. *Let $\alpha^* \in (0,1)$ be the critical confidence level for the market (S^0, S). Then we have the following trichotomy:*

(a) *If $0 < \alpha < \alpha^*$, then the market (S^0, S) does not admit* ES^α*-arbitrage.*

(b) *If $\alpha = \alpha^*$, then the market (S^0, S) admits* ES^α*-arbitrage but not strong* ES^α*-arbitrage.*

(c) *If $1 > \alpha > \alpha^*$, then the market (S^0, S) admits strong* ES^α*-arbitrage.*

Remark 2.3.2. (a) As a consequence of this result, ES^α is not suitable for portfolio selection/risk management. Indeed, one can always construct a market (S^0, S) that satisfies no-arbitrage and for which $\alpha^* < \alpha$.

(b) The critical confidence level α^* depends *only* on the market. An interesting study would be to empirically find values of α^* using real data. We do not do this here, however Figure 6 suggests that within the current regulatory frameworks, the chance to encounter ES-arbitrage is low but possible.

(c) Going back to the motivating example from the introduction, one can describe the set of EMMs for the two-dimensional market by

$$\mathcal{P} = \{(q_1, q_2, q_3) \in (0,1)^3 : q_1 \in (0, 1/3), q_2 = 1 - 3q_1 \text{ and } q_3 = 2q_1\}. \tag{2.7}$$

Using (2.6), it follows that $\alpha^* = 5/6$ (as we computed). We cannot use Theorem 2.3.1 to find the no-(strong)-ES^α-arbitrage price bounds for S^2 since we have no-arbitrage as part of our standing assumption in this chapter. This will be rectified in the next chapter, cf. Remark 3.3.5(b).

The rest of this section will be devoted to proving Theorem 2.3.1.

The Case $\alpha < \alpha^*$

We first show that if $0 < \alpha < \alpha^*$, then the market (S^0, S) does not admit ES^α-arbitrage. This establishes Theorem 2.3.1(a)

Proposition 2.3.3. *Assume that* $0 < \alpha < \alpha^*$. *Then the market* (S^0, S) *does not admit* ES^α*-arbitrage.*

Proof. By Theorem 2.2.2, it suffices to show that $\mathrm{ES}^\alpha_1 > 0$. So let $\pi \in \Pi^\alpha_1$ be arbitrary and set $X := X_\pi$ for convenience. Let $\mathcal{P}$ be set of equivalent martingale measures for S/S^0 and $\mathcal{Q}_\alpha$ be defined as in (2.1). By the definition of ES^α_1 in (2.3) and the dual characterisation of ES in (2.2) it suffices to show that there is $\mathbb{Q} \in \mathcal{Q}_\alpha$ such that $\mathbb{E}^{\mathbb{Q}}[-X] > 0$.

First, note that $\mathbb{E}^{\mathbb{Q}}[-X] = 0$ for all $\mathbb{Q} \in \mathcal{P}$. By the characterisation of α^* in (2.6) and the fact that $\frac{1}{\alpha} > \frac{1}{\alpha^*}$, it follows that there is $\varepsilon > 0$ and $\tilde{\mathbb{Q}} \in \mathcal{P}$ such that $\tilde{Z} := \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} \leq \frac{1}{\alpha} - \varepsilon$ $\mathbb{P}$-a.s. tThe idea is to perturb $\tilde{\mathbb{Q}}$ or more precisely its Radon-Nikodým derivative $\tilde{Z}$ to define a measure $\mathbb{Q} \in \mathcal{Q}_\alpha$ such that $\mathbb{E}^{\mathbb{Q}}[-X] > 0 = \mathbb{E}^{\tilde{\mathbb{Q}}}[-X]$.

Let $A := \{X < 0\}$. Note that $\mathbb{P}[A] \in (0, 1)$ since $\mathbb{P} \approx \tilde{\mathbb{Q}}$, $\mathbb{E}[X] = 1$ and $\mathbb{E}^{\tilde{\mathbb{Q}}}[-X] = 0$. Pick $\delta \in (0, \varepsilon)$ such that $\mathbb{P}[A]/\tilde{\mathbb{Q}}[A^c] < 1/\delta$ and define the random variable Z by

$$Z(\omega) := \begin{cases} \tilde{Z}(\omega) + \delta & \text{if } \omega \in A, \\ \left(1 - \delta \frac{\mathbb{P}[A]}{\tilde{\mathbb{Q}}[A^c]}\right) \tilde{Z}(\omega) & \text{if } \omega \in A^c. \end{cases}$$

Then Z is $\mathbb{P}$-a.s. positive, bounded above by $\frac{1}{\alpha} - \varepsilon + \delta \leq \frac{1}{\alpha}$, and satisfies

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[(\tilde{Z} + \delta)\mathbf{1}_A] + \left(1 - \delta \frac{\mathbb{P}[A]}{\tilde{\mathbb{Q}}[A^c]}\right) \mathbb{E}[\tilde{Z}\mathbf{1}_{A^c}] \\ &= \tilde{\mathbb{Q}}[A] + \delta\mathbb{P}[A] + \tilde{\mathbb{Q}}[A^c] \left(1 - \delta \frac{\mathbb{P}[A]}{\tilde{\mathbb{Q}}[A^c]}\right) = 1. \end{aligned}$$

So we can define the measure $\mathbb{Q} \approx \mathbb{P}$ by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := Z$. Then one can verify that $\mathbb{Q} \in \mathcal{Q}_\alpha$ by the definition of Z. Moreover, using that $\mathbb{E}[X\mathbf{1}_A] = \mathbb{E}[X\mathbf{1}_{\{X<0\}}] < 0$ by the fact that $\mathbb{P}[A] = \mathbb{P}[X < 0] > 0$ and $\mathbb{E}[\tilde{Z}X\mathbf{1}_{A^c}] = \mathbb{E}[\tilde{Z}X\mathbf{1}_{\{X\geq 0\}}] \geq 0$, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}[ZX] &= \mathbb{E}\left[(\tilde{Z}+\delta)X\mathbf{1}_A + \left(1 - \delta\frac{\mathbb{P}[A]}{\tilde{\mathbb{Q}}[A^c]}\right)\tilde{Z}X\mathbf{1}_{A^c}\right] \\
&= \mathbb{E}[\tilde{Z}X] + \delta\mathbb{E}[X\mathbf{1}_A] - \delta\frac{\mathbb{P}[A]}{\tilde{\mathbb{Q}}[A^c]}\mathbb{E}[\tilde{Z}X\mathbf{1}_{A^c}] \\
&< \mathbb{E}[\tilde{Z}X] = \mathbb{E}^{\tilde{\mathbb{Q}}}[X] = 0.
\end{aligned}$$

Thus, $\mathbb{E}^{\mathbb{Q}}[-X] > 0$. □

The Case $\boldsymbol{\alpha \geq \alpha^*}$

We now consider the case that α is greater or equal than the critical confidence level α^*.

We focus on the case that $\alpha > \alpha^*$ first. In order to show that under this condition the market (S^0, S) admits ES^α-arbitrage, we use a separation argument as is customary in dual characterisations of standard no-arbitrage; cf. [45, Theorem 1.6]. To this end, for $\alpha \in (0,1)$, we define

$$C_\alpha := \{\mathbb{E}^{\mathbb{Q}}[R - r\mathbf{1}] : \mathbb{Q} \in \mathcal{Q}_\alpha\} \subset \mathbb{R}^d, \tag{2.8}$$

where $\mathbb{E}^{\mathbb{Q}}[R - r\mathbf{1}]$ is a shorthand for the d-dimensional vector with components $\mathbb{E}^{\mathbb{Q}}[R^i - r]$.

Note that the sets C_α are strongly related to mean-ES^α portfolio optimisation since for any $\pi \in \mathbb{R}^d$,

$$\mathrm{ES}^\alpha(X_\pi) = \max_{c \in C_\alpha}(-\pi \cdot c).$$

We proceed to study some fundamental properties of the set C_α.

Lemma 2.3.4. *Let $\alpha \in (0,1)$. Then the set C_α is non-empty and convex.*

Proof. This follows from the fact that the set $\mathcal{Q}_\alpha$ is nonempty (since $\mathbb{P} \in \mathcal{Q}_\alpha$) and convex. □

Next, note that $\mathbb{E}^{\mathbb{Q}}[R - r\mathbf{1}] = \mathbf{0}$ for any $\mathbb{Q} \in \mathcal{P}$. Conversely if $\mathbb{Q} \approx \mathbb{P}$ and $\mathbb{E}^{\mathbb{Q}}[R - r\mathbf{1}] = \mathbf{0}$, then $\mathbb{Q} \in \mathcal{P}$ by the definition of an equivalent martingale measure. Thus, if $\alpha \in (0, \alpha^*)$, then $\mathbf{0} \in C_\alpha$ because $\mathcal{P} \cap \mathcal{Q}_\alpha \neq \emptyset$. The next result shows that for $\alpha > \alpha^*$, the zero vector is no longer in C_α. This is not completely obvious as $\mathcal{Q}_\alpha$ also contains measures that are not equivalent to $\mathbb{P}$.

Lemma 2.3.5. *Let* $1 > \alpha > \alpha^*$. *Then* $\mathbf{0} \notin C_\alpha$.

Proof. Seeking a contradiction, suppose that $\mathbf{0} \in C_\alpha$. Then there exists a probability measure $\mathbb{Q}_\alpha \in \mathcal{Q}_\alpha$ with Radon-Nikodým derivative $Z_\alpha \in [0, \frac{1}{\alpha}]$ $\mathbb{P}$-a.s. satisfying

$$\mathbb{E}^{\mathbb{Q}_\alpha}[R - r\mathbf{1}] = \mathbb{E}[Z_\alpha(R - r\mathbf{1})] = \mathbf{0}.$$

Choose $\beta \in (0, \alpha^*)$ such that

$$\alpha^* > \beta > \tfrac{\alpha\alpha^*}{2\alpha - \alpha^*} > 0. \tag{2.9}$$

and note that $1 > \alpha > \alpha^* > 0$ implies that $\alpha^*(2\alpha - \alpha^*) > \alpha\alpha^* > 0$. By the definition of α^*, there exists $\mathbb{Q}_\beta \in \mathcal{Q}_\beta \cap \mathcal{P}$ with corresponding Radon-Nikodým derivative $Z_\beta \in (0, \frac{1}{\beta}]$ $\mathbb{P}$-a.s. satisfying

$$\mathbb{E}^{\mathbb{Q}_\beta}[R - r\mathbf{1}] = \mathbb{E}[Z_\beta(R - r\mathbf{1})] = \mathbf{0}.$$

Consider the probability measure given by the mixture $\tilde{\mathbb{Q}} := \frac{1}{2}\mathbb{Q}_\alpha + \frac{1}{2}\mathbb{Q}_\beta$. Then

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[R - r\mathbf{1}] = \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}_\alpha}[R - r\mathbf{1}] + \tfrac{1}{2}\mathbb{E}^{\mathbb{Q}_\beta}[R - r\mathbf{1}] = \frac{1}{2}\mathbf{0} + \frac{1}{2}\mathbf{0} = \mathbf{0}.$$

The corresponding Radon-Nikodým derivative $\tilde{Z} = \frac{1}{2}Z_\alpha + \frac{1}{2}Z_\beta$ satisfies

$$0 < \tilde{Z} \leq \tfrac{1}{2}\tfrac{1}{\alpha} + \tfrac{1}{2}\tfrac{1}{\beta} < \tfrac{1}{\alpha^*} \text{ } \mathbb{P}\text{-a.s.},$$

where the last inequality follows from the choice of β in (2.9). But this means that $\tilde{\mathbb{Q}}$ is an EMM whose Radon-Nikodým derivative satisfies $\tilde{Z} \leq \frac{1}{\tilde{\alpha}}$ $\mathbb{P}$-a.s. for some $\tilde{\alpha} > \alpha^*$, in contradiction to the characterisation of α^* in (2.6). □

We are now in a position to prove that for $\alpha > \alpha^*$ the market (S^0, S) admits ES^α-arbitrage.

Proposition 2.3.6. *Assume that* $1 > \alpha > \alpha^*$. *Then the market* (S^0, S) *admits* ES^α*-arbitrage.*

Proof. By Lemmas 2.3.4 and 2.3.5, the set $C_\alpha \subset \mathbb{R}^d$ is non-empty, convex and does not contain the origin. Therefore by the separating hyperplane theorem

[16, Proposition B.13] applied to the nonempty and convex sets $\{\mathbf{0}\}$ and C_α, there exists $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ such that $\pi \cdot c \geq \pi \cdot \mathbf{0} = 0$ for all $c \in C_\alpha$, i.e.,

$$\mathbb{E}^{\mathbb{Q}}[\pi \cdot (R - r\mathbf{1})] \geq 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}_\alpha.$$

Thus, (1.1) and the dual characterisation of ES in (2.2) give

$$\mathrm{ES}^\alpha(X_\pi) = \mathrm{ES}^\alpha(\pi \cdot (R - r\mathbf{1})) = \max_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}^{\mathbb{Q}}[-\pi \cdot (R - r\mathbf{1})] \leq 0. \tag{2.10}$$

Now set $\nu := \mathbb{E}[X_\pi]$. Then

$$\mathrm{ES}^\alpha_\nu \leq \mathrm{ES}^\alpha(X_\pi) \leq 0.$$

It follows from expectation boundedness that $\nu \geq 0$. Moreover, $\nu \neq 0$. Indeed, seeking a contradiction, suppose that $\nu = 0$. Then $\mathrm{ES}^\alpha_\nu = 0$ by Lemma 2.1.6 and hence $\mathrm{ES}^\alpha(X_\pi) = 0 = \mathrm{ES}^\alpha_0$, which implies that $\pi \in \Pi^\alpha_0$. By Proposition 2.1.5(a), this in turn implies that $\pi = \mathbf{0}$, and we arrive at a contradiction.

Thus, we may conclude that $\nu > 0$. Lemma 2.1.6 then gives that $\mathrm{ES}^\alpha_1 = \frac{1}{\nu}\mathrm{ES}^\alpha_\nu \leq 0$, which together with Theorem 2.2.2(b) and (c) implies that (S^0, S) admits ES^α-arbitrage. □

The following corollary shows that also for $\alpha = \alpha^*$, the market (S^0, S) admits ES^α-arbitrage but not strong ES^α-arbitrage, which establishes Theorem 2.3.1(b)

Corollary 2.3.7. *Assume that $\alpha = \alpha^*$. Then the market (S^0, S) admits ES^α-arbitrage but not strong ES^α-arbitrage.*

Proof. It follows from Propositions 2.3.3 and 2.3.6 that

$$\begin{aligned} \mathrm{ES}^\alpha_1 &> 0 \quad \text{if } \alpha \in (0, \alpha^*), \\ \mathrm{ES}^\alpha_1 &\leq 0 \quad \text{if } \alpha \in (\alpha^*, 1). \end{aligned}$$

Moreover, by Proposition 2.1.7, the function $\alpha \mapsto \mathrm{ES}^\alpha_1$ is continuous. This implies that $\mathrm{ES}^{\alpha^*}_1 = 0$. Now the claim follows from Theorem 2.2.2(b). □

We proceed to strengthen the assertion of Proposition 2.3.6, by showing that for $\alpha > \alpha^*$ the market (S^0, S) admits strong ES^α-arbitrage, which establishes Theorem 2.3.1 and ends this chapter.

Proposition 2.3.8. *Assume that* $1 > \alpha > \alpha^*$. *Then the market* (S^0, S) *admits strong* ES^α*-arbitrage.*

Proof. Let $\pi \in \Pi_1^{\alpha^*}$ and set $X := X_\pi$ for convenience. Then by definition of ES_1^α in (2.3), Corollary 2.3.7 and Remark 2.1.2(a), we obtain

$$\mathrm{ES}_1^\alpha \leq \mathrm{ES}^\alpha(X) \leq \mathrm{ES}^{\alpha^*}(X) = \mathrm{ES}_1^{\alpha^*} = 0. \tag{2.11}$$

Suppose to the contrary that (S^0, S) does not admit strong ES^α-arbitrage. This means that we have equality throughout (2.11). By the dual characterisation of ES in (2.2), this means that there is $\mathbb{Q}^\alpha \in \mathcal{Q}_\alpha$ with corresponding Radon-Nikodým derivative Z^α such that

$$0 = \mathrm{ES}^\alpha(X) = \mathbb{E}^{\mathbb{Q}^\alpha}[-X] = \mathbb{E}[-Z^\alpha X]. \tag{2.12}$$

Now fix $\beta \in (0, \alpha^*)$. Then by Proposition 2.3.3, the market (S^0, S) does not admit ES^β-arbitrage, and so $\mathrm{ES}_1^\beta > 0$ by Theorem 2.2.2. Therefore, there is a probability measure $\mathbb{Q}^\beta \in \mathcal{Q}_\beta$, with corresponding Radon-Nikodým derivative Z^β, such that

$$0 < \mathrm{ES}_1^\beta \leq \mathrm{ES}^\beta(X) = \mathbb{E}^{\mathbb{Q}^\beta}[-X] = \mathbb{E}[-Z^\beta X]. \tag{2.13}$$

Consider the probability measure given by the mixture

$$\hat{\mathbb{Q}} := \frac{\alpha}{\alpha^*}\frac{\alpha^* - \beta}{\alpha - \beta}\mathbb{Q}^\alpha + \frac{\beta}{\alpha^*}\frac{\alpha - \alpha^*}{\alpha - \beta}\mathbb{Q}^\beta,$$

which has Radon-Nikodým derivative

$$\hat{Z} := \frac{\alpha}{\alpha^*}\frac{\alpha^* - \beta}{\alpha - \beta}Z^\alpha + \frac{\beta}{\alpha^*}\frac{\alpha - \alpha^*}{\alpha - \beta}Z^\beta.$$

Then $\hat{Z} \leq \frac{1}{\alpha^*}$ $\mathbb{P}$-a.s., by the fact that $Z^\alpha \leq \frac{1}{\alpha}$ $\mathbb{P}$-a.s. and $Z^\beta \leq \frac{1}{\beta}$ $\mathbb{P}$-a.s.. Whence $\hat{\mathbb{Q}} \in \mathcal{Q}_{\alpha^*}$. It now follows from (2.12) and (2.13) that

$$\begin{aligned} 0 = \mathrm{ES}_1^{\alpha^*} &= \max_{\mathbb{Q} \in \mathcal{Q}_{\alpha^*}} \mathbb{E}^{\mathbb{Q}}[-X] \geq \mathbb{E}^{\hat{\mathbb{Q}}}[-X] = \mathbb{E}[-\hat{Z}X] \\ &= \frac{\alpha}{\alpha^*}\frac{\alpha^* - \beta}{\alpha - \beta}\mathbb{E}[-Z^\alpha X] + \frac{\beta}{\alpha^*}\frac{\alpha - \alpha^*}{\alpha - \beta}\mathbb{E}[-Z^\beta X] > 0, \end{aligned}$$

which is absurd. Thus, $\mathrm{ES}_1^\alpha < 0$, and the claim follows from Theorem 2.2.2. □

3

Mean-ρ Portfolio Selection for Coherent Risk Measures

We have seen in the previous chapter how an ES constraint is flawed in the context of portfolio selection, cf. Remark 2.3.2(a). In this chapter we explore whether there is an alternative *coherent* risk measure that is superior.

Standing Assumption. Throughout the entire chapter we consider the market (S^0, S) described in the introduction. We assume that $L^\infty \subset L \subset L^1$ is a Riesz space with the $\mathbb{P}$-a.s. ordering that contains $\mathcal{X} = \{X_\pi : \pi \in \mathbb{R}^d\}$. We focus on a positively homogeneous risk measure $\rho : L \to (-\infty, \infty]$, which satisfies the following axioms for $X, Y \in L$:

- Monotonicity: If $X \leq Y$ $\mathbb{P}$-a.s., $\rho(X) \geq \rho(Y)$;
- Cash-invariance: If $c \in \mathbb{R}$, then $\rho(X + c) = \rho(X) - c$;
- Positive homogeneity: For $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$.

Unlike Chapter 2, we do not assume that the market satisfies no-arbitrage.

A couple of remarks are in order.

Remark 3.0.1. (a) Along with Value at Risk and Expected Shortfall, the worst-case (WC) risk measure $\mathrm{WC} : L \to (-\infty, \infty]$ given by

$$\mathrm{WC}(X) := \operatorname{ess\,sup}(-X)$$

is one of the most famous examples of a positively homogeneous risk measure. More examples are given in Section 3.3.

(b) The Riesz space L can be seen as an ambient space of $\mathcal{X}$. Key examples include L^p-spaces, for $p \in [1, \infty]$, or more generally Orlicz spaces (cf. Appendix A.2). Of course, the natural choice is to take $L = L^1$, however the *Fatou property* (defined later) on L is weaker than the Fatou property on L^1 and so our results are (slightly) more general by not fixing $L = L^1$. Moreover, not all positively homogeneous risk measures can be naturally extended to L^1.

(c) In some situations, it is useful to allow ρ to take the value ∞. For example, if all the returns R^i are bounded from above but unbounded from below and only in L^1 (so that $L = L^1$), it makes perfect sense to consider for ρ the worst-case risk measure WC. Then $\mathrm{WC}(X_\pi)$ is finite if $\pi^i \geq 0$ for all $i \in \{1, \ldots, d\}$ but it may take the value ∞ if $\pi^i < 0$ for some $i \in \{1, \ldots, d\}$.

3.1 Mean-ρ Portfolio Optimisation

We start our discussion on mean-ρ portfolio selection by introducing a partial preference order on the set of portfolios. Just like Chapter 2, this preference order formalises the idea that return is "desirable" and risk is "undesirable".

Definition 3.1.1. A portfolio $\pi \in \mathbb{R}^d$ is *strictly ρ-preferred* over another portfolio $\pi' \in \mathbb{R}^d$ if $\mathbb{E}[X_\pi] \geq \mathbb{E}[X_{\pi'}]$ and $\rho(X_\pi) \leq \rho(X_{\pi'})$, with at least one strict inequality.

3.1.1 Optimal Portfolios

We approach mean-ρ portfolio selection by first looking at the slightly simplified problem of finding the minimum risk portfolio(s) given a fixed excess return.

Definition 3.1.2. Let $\nu \geq 0$. A portfolio $\pi \in \Pi_\nu$ is called *ρ-optimal* for ν if $\rho(X_\pi) < \infty$ and $\rho(X_\pi) \leq \rho(X_{\pi'})$ for all $\pi' \in \Pi_\nu$. We denote the set of all ρ-optimal portfolios for ν by Π_ν^ρ. Moreover, we set

$$\rho_\nu := \inf\{\rho(X_\pi) : \pi \in \Pi_\nu\} \in [-\infty, \infty], \tag{3.1}$$

and define the *ρ-optimal boundary* by

$$\mathcal{O}_\rho := \{(\rho_\nu, \nu) : \nu \geq 0\} \subset [-\infty, \infty] \times [0, \infty).$$

Remark 3.1.3. If $\rho_\nu = -\infty$, since ρ can only take values in $(-\infty, \infty]$, for every portfolio in Π_ν there is another portfolio in Π_ν with strictly lower risk. Thus, $\Pi_\nu^\rho = \emptyset$. If $\rho_\nu = \infty$, every portfolio in Π_ν has infinite risk, and so $\Pi_\nu^\rho = \emptyset$.

As the riskless portfolio has zero risk, $\rho_0 \leq 0$. Positive homogeneity implies that either $\rho_0 = -\infty$ (in which case $\Pi_0^\rho = \emptyset$) or $\rho_0 = 0$ (in which case $\mathbf{0} \in \Pi_0^\rho$). For $\nu > 0$, positive homogeneity gives $\Pi_\nu^\rho = \nu \Pi_1^\rho$ and $\rho_\nu = \nu \rho_1$. Thus, the ρ-optimal boundary is given by

$$\mathcal{O}_\rho = \{(\rho_0, 0)\} \cup \{(k\rho_1, k) : k > 0\}, \tag{3.2}$$

where $\rho_0 \in \{-\infty, 0\}$ and $\rho_1 \in [-\infty, \infty]$. Note that the ρ-optimal boundary is nonempty even if ρ-optimal portfolios do not exist. Depending on the sign of ρ_1, Figure 4 gives a graphical illustration of the three different shapes $\mathcal{O}_\rho$ can take when $\rho_0 = 0$ and $\rho_1 \in \mathbb{R}$.

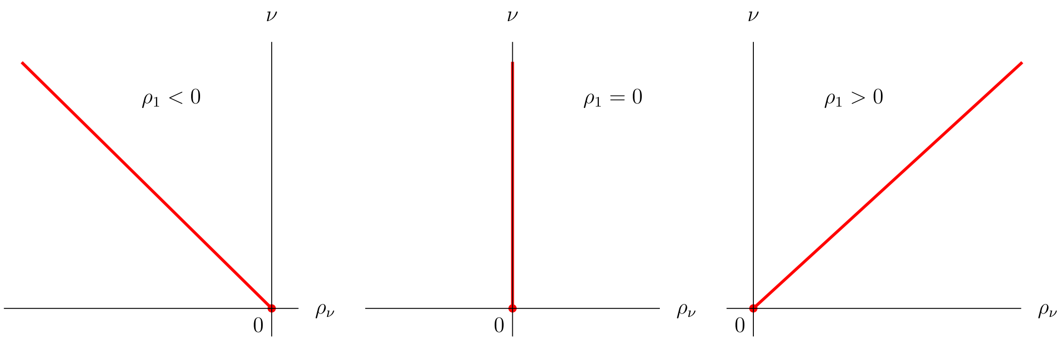


Figure 4: General shapes of the ρ-optimal boundary when $\rho_0 = 0$ and $\rho_1 \in \mathbb{R}$

We now seek to understand under which conditions ρ-optimal portfolios exist and which properties ρ-optimal sets have. First, we consider the case $\nu = 0$, which is also of key importance for the case $\nu > 0$.

Proposition 3.1.4. *$\Pi_0^\rho \neq \emptyset$ if and only if $\rho_0 = 0$. Moreover in this case, either $\Pi_0^\rho = \{\mathbf{0}\}$ or Π_0^ρ fails to be compact.*

Proof. If $\rho_0 = 0$, then $\mathbf{0} \in \Pi_0^\rho$. If $\rho_0 \neq 0$, then $\rho_0 = -\infty$ and $\Pi_0^\rho = \emptyset$. Moreover, if $\rho_0 = 0$ and there is $\pi \neq \mathbf{0}$ with $\rho(X_\pi) = 0$, it follows from positive homogeneity that $\lambda\pi \in \Pi_0^\rho$ for all $\lambda \geq 0$ and hence Π_0^ρ fails to be compact. $\square$

We proceed to find sufficient conditions that guarantee $\rho_0 = 0$ or even $\Pi_0^\rho = \{\mathbf{0}\}$.

Definition 3.1.5. The risk measure ρ is called *expectation bounded* if $\rho(X) \geq \mathbb{E}[-X]$ for all $X \in L$. It is called *strictly expectation bounded* if $\rho(X) > \mathbb{E}[-X]$ for all non-constant $X \in L$.

Remark 3.1.6. (a) Expectation boundedness is implied by, but strictly weaker than, *dilatation monotonicity*, i.e., $\rho(X) \geq \rho(\mathbb{E}[X|\mathcal{G}])$ for all $X \in L$ and all sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$. The latter concept was introduced in [64] and has far reaching implications. For example, every dilatation monotone convex risk measure on an atomless probability space is law-invariant [29], and every dilatation monotone risk measure that satisfies the Fatou property can be extended to L^1 [73].

(b) Strict expectation boundedness – first introduced in [80] – is a natural requirement on a risk measure that is satisfied by Expected Shortfall and a large class of coherent risk measures; see Remark 3.2.1(e) and Proposition 3.2.11. In fact, when the underlying probability space is atomless, L is *rearrangement-invariant* and ρ is law-invariant, coherent and satisfies the Fatou property, then it is automatically strictly expectation bounded unless $\rho(X) = \mathbb{E}[-X]$. This is a simple consequence of the much deeper [12, Proposition 5.12], which is a generalisation of the celebrated Kusuoka representation.

(c) Value at Risk is *not* expectation bounded (apart from degenerate probability spaces). For example, if Z is a standard normal random variable, then $\mathrm{VaR}^{\alpha}(Z) < 0 = \mathbb{E}[-Z]$ for $\alpha > 1/2$. This failure of expectation boundedness for Value at Risk has some undesirable consequences like the non-existence of optimal portfolios; cf. Remark 3.1.29.

(d) By cash-invariance of ρ, it suffices to consider $X \in L$ with $\mathbb{E}[X] = 0$ in the definition of (strict) expectation boundedness.

We proceed to show that under (strict) expectation boundedness of ρ, optimal portfolios for $\nu = 0$ exist (and are unique).

Corollary 3.1.7. *If ρ is expectation bounded, then $\rho_0 = 0$. If ρ is even strictly expectation bounded, then $\Pi_0^{\rho} = \{\mathbf{0}\}$.*

Proof. If ρ is expectation bounded, then for any $\pi \in \Pi_0$, $\rho(X_\pi) \geq \mathbb{E}[-X_\pi] = 0$ and we may conclude that $\rho_0 = 0$. If ρ is strictly expectation bounded, fix $\pi \in \Pi_0 \setminus \{\mathbf{0}\}$. Then X_π is non-constant by nonredundancy of the financial market. Strict expectation boundedness of ρ gives $\rho(X_\pi) > \mathbb{E}[-X_\pi] = 0$. We may conclude that $\Pi_0^{\rho} = \{\mathbf{0}\}$. □

We next consider ρ-optimal sets for $\nu > 0$. To this end, we recall the Fatou property for ρ; for our applications, it sometimes suffices to consider this on a subset $\mathcal{Y} \subset L$.

Definition 3.1.8. The risk measure ρ is said to satisfy the Fatou property on $\mathcal{Y} \subset L$, if $X_n \to X$ $\mathbb{P}$-a.s. for $X_n, X \in \mathcal{Y}$ and $|X_n| \leq Y$ $\mathbb{P}$-a.s. for some $Y \in L$ implies that $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$.

We now come to the main result of this section, which establishes existence of ρ-optimal portfolios under very weak assumptions on ρ, only requiring that ρ satisfies the Fatou property on $\mathcal{X} = \{X_\pi : \pi \in \mathbb{R}^d\}$ and $\Pi_0^\rho = \{\mathbf{0}\}$. In particular, we do *not* require ρ to be convex, which is a key assumption in the extant literature; see e.g. [81, Proposition 4].

Theorem 3.1.9. *Assume $\Pi_0^\rho = \{\mathbf{0}\}$, $\rho_1 \in \mathbb{R}$ and ρ satisfies the Fatou property on $\mathcal{X} = \{X_\pi : \pi \in \mathbb{R}^d\}$. Then for any $\nu \geq 0$, the set Π_ν^ρ of ρ-optimal portfolios for ν is nonempty and compact.*

Proof. The key idea of the proof is to consider the function $f_\rho : \mathbb{R}^d \to [0, \infty]$, defined by

$$f_\rho(\pi) = \begin{cases} \rho(X_\pi) + (|\rho_1| + 1)\mathbb{E}[X_\pi], & \text{if } \pi \in \cup_{k\geq 0}\Pi_k, \\ \infty, & \text{if } \pi \in \cup_{k<0}\Pi_k. \end{cases}$$

Then f_ρ is nonnegative, positively homogeneous and satisfies $f_\rho^{-1}(\{0\}) = \{\mathbf{0}\}$. Moreover, if $\pi_n \to \pi$ in $\mathbb{R}^d$, we have $\mathbb{E}[X_\pi] = \lim_{n\to\infty} \mathbb{E}[X_{\pi_n}]$ as well as $\rho(X_\pi) \leq \liminf_{n\to\infty} \rho(X_{\pi_n})$ because ρ satisfies the Fatou property on $\mathcal{X}$ (and $L \supset \mathcal{X}$ is a Riesz space). This implies that f_ρ is lower semi-continuous.

We proceed to show that f_ρ has compact sublevel sets. As $\rho_1 < \infty$, there is at least one portfolio $\pi^* \in \Pi_1$ with $\rho(X_{\pi^*}) < \infty$. Let $S = \{x \in \mathbb{R}^d : \|x\|_2 = \|\pi^*\|_2\}$. As S is compact and f_ρ lower semi-continuous, $m := \min\{f_\rho(x) : x \in S\}$ is well defined. Note that $m > 0$ since $\|\pi^*\|_2 > 0$ and $f_\pi^{-1}(\{0\}) = \{\mathbf{0}\}$. As f_ρ is positively homogeneous, $f_\rho(\pi) \geq \frac{m}{\|\pi^*\|_2}\|\pi\|_2$ for any portfolio $\pi \in \mathbb{R}^d$. Thus, f_ρ has bounded sublevel sets, which are also closed since f_ρ is lower semi-continuous.

We finish by a standard argument. For $\delta \geq 0$, set $A_\delta := \{\pi \in \mathbb{R}^d : f_\rho(\pi) \leq \delta\} \cap \Pi_1$ and $\delta_1 := \inf\{f_\rho(\pi) : \pi \in \Pi_1\}$. Note that $\delta_1 < \infty$ since $\rho_1 \in \mathbb{R}$. Moreover, each A_δ is compact and nonempty for $\delta > \delta_1$. As the A_δ are nested (i.e., $A_\delta \subset A_{\delta'}$ for $\delta \leq \delta'$), it follows that

$$\Pi_1^\rho = A_{\delta_1} = \bigcap_{\delta > \delta_1} A_\delta$$

is nonempty and compact. Whence, so is $\Pi_\nu^\rho = \nu \Pi_1^\rho$ for any $\nu > 0$. (For $\nu = 0$, the claim is trivial.) □

Remark 3.1.10. (a) The requirement that ρ satisfies the Fatou property on $\mathcal{X}$ is a mild assumption, which is satisfied by VaR, ES and WC. Anticipating ourselves a bit, we note that it is satisfied by any risk measure $\rho : L \to (-\infty, \infty]$ admitting a dual representation $\rho(X) = \sup_{Z \in \mathcal{Q}} \mathbb{E}[-ZX]$ for some nonempty set $\mathcal{Q}$ of Radon-Nikodým derivatives satisfying $ZR^i \in L^1$ for all $Z \in \mathcal{Q}$ and $i \in \{1, \dots, d\}$; cf. Proposition 3.2.3.

(b) By Corollary 3.1.7, the requirement that $\Pi_0^\rho = \{\mathbf{0}\}$ is automatically satisfied if ρ is strictly expectation bounded. By Remark 3.1.6(b), this is not very restrictive.

(c) If ρ is in addition *convex*, i.e., $\rho(\lambda X_1 + (1-\lambda) X_2) \leq \lambda \rho(X_1) + (1-\lambda)\rho(X_2)$ for $X_1, X_2 \in L$ and $\lambda \in [0, 1]$, then we also have convexity of ρ-optimal sets. Indeed, let $\nu \geq 0$, $\pi, \pi' \in \Pi_\nu^\rho$, and $\lambda \in [0, 1]$. Then $\rho(X_{\lambda\pi + (1-\lambda)\pi'}) = \rho(\lambda X_\pi + (1-\lambda) X_{\pi'}) \leq \lambda \rho(X_\pi) + (1-\lambda)\rho(X_{\pi'}) = \rho_\nu$. Therefore, $\lambda \pi + (1-\lambda)\pi' \in \Pi_\nu^\rho$.

(d) If $|\rho_1| = \infty$, then $\Pi_\nu^\rho = \emptyset$ for all $\nu > 0$. If $\rho_1 \in \mathbb{R}$ and $\{\mathbf{0}\} \subsetneq \Pi_0^\rho$, then boundedness of the sublevel sets is lost (since $f_\rho^{-1}(\{0\})$ is unbounded) and Π_ν^ρ can be empty for all $\nu > 0$; see Example A.3.1 for a concrete counterexample.

3.1.2 Efficient Portfolios

We proceed to study the notion of ρ-efficient portfolios.

Definition 3.1.11. A portfolio $\pi \in \mathbb{R}^d$ is called *ρ-efficient* if $\mathbb{E}[X_\pi] \geq 0$ and there is no other portfolio $\pi' \in \mathbb{R}^d$ that is strictly ρ-preferred over π. We denote the *ρ-efficient frontier* by

$$\mathcal{E}_\rho := \{(\rho(X_\pi), \mathbb{E}[X_\pi]) : \pi \text{ is } \rho\text{-efficient}\}.$$

Remark 3.1.12. (a) If $\pi \in \mathbb{R}^d$ is ρ-efficient, it follows that $\rho(X_\pi) < \infty$. Indeed, if $\mathbb{E}[X_\pi] = 0$ and $\rho(X_\pi) = \infty$, then $\mathbf{0}$ is strictly ρ-preferred over π, and if $\mathbb{E}[X_\pi] > 0$ and $\rho(X_\pi) = \infty$, then $\lambda\pi$ is strictly ρ-preferred over π for $\lambda > 1$.

(b) It follows from (a) that every ρ-efficient portfolio is ρ-optimal.

(c) If ρ is expectation bounded, we may drop the assumption that $\mathbb{E}[X_\pi] \geq 0$ for π to be efficient since under expectation boundedness, for any portfolio π with $\mathbb{E}[X_\pi] < 0$, we have $\rho(X_\pi) \geq \mathbb{E}[-X_\pi] > 0$, and so the riskless portfolio $\mathbf{0}$ is strictly ρ-preferred over π.

The mean-ρ portfolio selection problems (1) and (2) are both well-posed and admit solutions when ρ-efficient portfolios exist, i.e., when $\mathcal{E}_\rho \neq \emptyset$. Remark 3.1.12(b) implies that $\mathcal{E}_\rho \subset \mathcal{O}_\rho$. The following result shows that when $\Pi_\nu^\rho \neq \emptyset$ for all $\nu \geq 0$ (which is satisfied under the conditions of Theorem 3.1.9), then the existence of the ρ-efficient frontier depends only on the sign of ρ_1.

Proposition 3.1.13. *Assume* $\Pi_\nu^\rho \neq \emptyset$ *for all* $\nu \geq 0$. *Then the following are equivalent:*

(a) $\rho_1 > 0$.

(b) $\mathcal{E}_\rho \neq \emptyset$.

Moreover, if $\rho_1 > 0$*, the* ρ*-efficient frontier is given by*

$$\mathcal{E}_\rho = \{(k\rho_1, k) : k \geq 0\}.$$

Proof. First assume that $\rho_1 > 0$. We proceed to show that any ρ-optimal portfolio is ρ-efficient. It then follows from Remark 3.1.12(b) and Proposition 3.1.4 that

$$\mathcal{E}_\rho = \mathcal{O}_\rho = \{(k\rho_1, k) : k \geq 0\}.$$

Seeking a contradiction, let $\pi \in \Pi_\nu^\rho$ for some $\nu \geq 0$ and assume that there is $\pi' \in \mathbb{R}^d$ such that $\mathbb{E}[X_{\pi'}] \geq \mathbb{E}[X_\pi] = \nu$ and $\rho(X_{\pi'}) \leq \rho(X_\pi) = \nu\rho_1$, with one inequality being strict. Set $\nu' := \mathbb{E}[X_{\pi'}]$. If $\nu' = \nu$, then $\rho(X_{\pi'}) < \rho(X_\pi)$ and we arrive at a contradiction as $\pi \in \Pi_\nu^\rho$. Otherwise, if $\nu' > \nu$, let $\pi^* \in \Pi_{\nu'}^\rho$. Then $\nu'\rho_1 = \rho(X_{\pi^*}) \leq \rho(X_{\pi'}) \leq \rho(X_\pi) = \nu\rho_1$. Since $\rho_1 > 0$, we arrive at the contradiction that $\nu' > \nu$ and $\nu' \leq \nu$.

Now assume that $\rho_1 \leq 0$. We proceed to show that there does not exist any ρ-efficient portfolio, even though $\Pi_\nu^\rho \neq \emptyset$ for all $\nu \geq 0$. Seeking a contradiction, suppose that $\pi \in \mathbb{R}^d$ is ρ-efficient. Then by Remark 3.1.12(b), $\pi \in \Pi_\nu^\rho$ for some $\nu \geq 0$. Pick $\nu' > \nu$ and let $\pi' \in \Pi_{\nu'}^\rho$. Then $\mathbb{E}[X_{\pi'}] = \nu' > \nu = \mathbb{E}[X_\pi]$ and $\rho(X_{\pi'}) = \nu'\rho_1 \leq \nu\rho_1 = \rho(X_\pi)$ by positive homogeneity of ρ and $\rho_1 \leq 0$. Hence, π' is strictly ρ-preferred over π and we arrive at a contradiction. $\square$

Remark 3.1.14. A close inspection of the proof of Proposition 3.1.13 reveals that the equivalence between (a) and (b) remains true if we only require that $\Pi_\nu^\rho \neq \emptyset$ for all $\nu > 0$. However, if $\Pi_0^\rho = \emptyset$, the ρ-efficient frontier is given by $\mathcal{E}_\rho = \{(k\rho_1, k) : k > 0\}$.

The following figure gives a graphical illustration of Proposition 3.1.13.

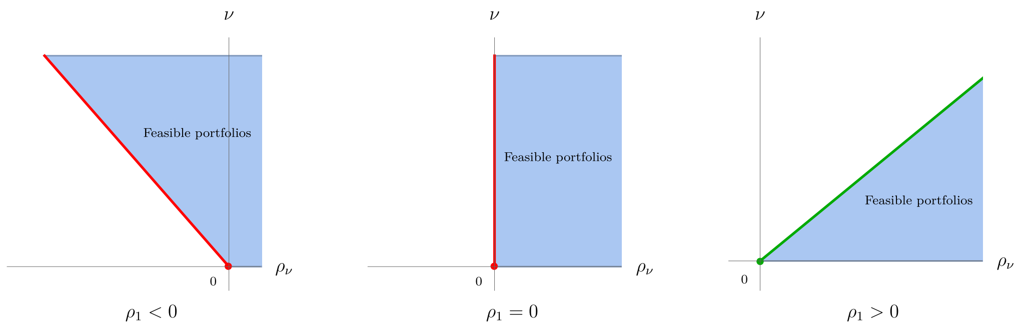


Figure 5: ρ-optimal boundary (red) and ρ-efficient frontier (green) when $\Pi_\nu^\rho \neq \emptyset$ for all $\nu \geq 0$

3.1.3 ρ-Arbitrage

We have seen above that mean-ρ portfolio selection is not always well defined as it can happen that there are no ρ-efficient portfolios. We call this situation ρ-*arbitrage*.

Definition 3.1.15. The market (S^0, S) is said to satisfy ρ-*arbitrage* if there are no ρ-efficient portfolios. It is said to satisfy *strong* ρ-*arbitrage* if there exists a sequence of portfolios $(\pi_n)_{n\in\mathbb{N}} \subset \mathbb{R}^d$ with

$$\mathbb{E}[X_{\pi_n}] \uparrow \infty \quad \text{and} \quad \rho(X_{\pi_n}) \downarrow -\infty.$$

It is clear that strong ρ-arbitrage implies ρ-arbitrage but not vice versa. The following two theorems give primal characterisations. Whereas strong ρ-arbitrage is fully characterised by the sign of ρ_1, defined in (3.1), the case of ρ-arbitrage is more subtle.

Theorem 3.1.16. *The market (S^0, S) admits strong ρ-arbitrage iff $\rho_1 < 0$.*

Proof. First, assume that the market satisfies strong ρ-arbitrage. As the riskless portfolio has zero risk and zero return, by definition of strong ρ-arbitrage, there is a portfolio $\pi \in \mathbb{R}^d$ with $\mathbb{E}[X_\pi] =: \nu > 0$ and $\rho(X_\pi) < 0$. Let $\pi' := \frac{1}{\nu}\pi$. Then $\pi' \in \Pi_1$, and

$$\rho_1 \leq \rho(X_{\pi'}) = \tfrac{1}{\nu}\rho(X_\pi) < 0.$$

Conversely, assume that $\rho_1 < 0$. Then there exists a portfolio $\pi \in \Pi_1$ with $\rho(X_\pi) < 0$. Thus, $\mathbb{E}[X_{k\pi}] \to \infty$ and $\rho(X_{k\pi}) \to -\infty$ as $k \to \infty$. □

Remark 3.1.17. By Theorem 3.1.16, if ρ is a positively homogeneous risk measure that is expectation bounded, the market admits strong ρ-arbitrage if and only if there exists a portfolio $\pi \in \mathbb{R}^d$ (in fractions of wealth) such that $\rho(X_\pi) < 0$. This is equivalent to the existence of a portfolio $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}$ (in numbers of shares) and $\varepsilon > 0$ such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0 \quad \text{and} \quad \rho(\vartheta^0 S_1^0 + \vartheta \cdot S_1 - \varepsilon) \leq 0,$$

which is referred to as a *good-deal (of the second kind)*, see e.g. [55, 28]. Note, however, that this relationship crucially relies on ρ being expectation bounded since otherwise a portfolio with negative risk may have a negative expected excess return. Also note that assuming that ρ is expectation bounded is a real restriction as it is not satisfied by Value at Risk.

Theorem 3.1.18. *We have the following three cases:*

(a) *If $\Pi_1^\rho \neq \emptyset$, then the market (S^0, S) admits ρ-arbitrage if and only if $\rho_1 \leq 0$.*

(b) *If $\Pi_1^\rho = \emptyset$ and $\Pi_0^\rho \neq \emptyset$, then the market (S^0, S) admits ρ-arbitrage if and only if $\rho_1 < 0$.*

(c) *If $\Pi_1^\rho = \emptyset$ and $\Pi_0^\rho = \emptyset$, then the market (S^0, S) admits ρ-arbitrage.*

Proof. (a) This follows from Proposition 3.1.13 and Remark 3.1.14.

(b) If $\rho_1 < 0$, by Theorem 3.1.16 the market admits strong ρ-arbitrage and a fortiori ρ-arbitrage. Conversely, if $\rho_1 \geq 0$, any portfolio $\pi \in \mathbb{R}^d$ with $\mathbb{E}[X_\pi] =: \nu > 0$ has $\rho(X_\pi) > \nu\rho_1 = 0$ because $\Pi_\nu^\rho = \nu\Pi_1^\rho = \emptyset$. Thus, any portfolio in Π_0^ρ is ρ-efficient because $\Pi_0^\rho \neq \emptyset$ (and therefore $\rho_0 = 0$). Thus, the market does not admit ρ-arbitrage.

(c) This follows from Remark 3.1.12(b). □

Remark 3.1.19. A *good-deal of the first kind* is a portfolio $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d} \setminus \{\mathbf{0}\}$ (in numbers of shares) such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0 \quad \text{and} \quad \rho(\vartheta^0 S_1^0 + \vartheta \cdot S_1) \leq 0.$$

In our setting, this corresponds to a portfolio $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ (in fractions of wealth) with $\rho(X_\pi) \leq 0$. Thus, by Theorems 3.1.9 and 3.1.18, when ρ is a positively homogeneous risk measure that satisfies the Fatou property and strict expectation boundedness, the existence of a good-deal of the first kind is equivalent to the market admitting ρ-arbitrage.

The following corollary relates the absence of (strong) ρ-arbitrage to the existence of the mean-ρ portfolio selection problems. The proof is straightforward and hence omitted.

Corollary 3.1.20. *Assume that $\Pi_\nu^\rho \neq \emptyset$ for all $\nu \geq 0$, so that the mean-ρ problem* (1') *is well posed.*

(a) *The mean-ρ portfolio selection problem* (1) *is well posed if any only if the market (S^0, S) does not satisfy strong ρ-arbitrage. In this case, the portfolios that solve* (1) *are in*

$$\begin{cases} \Pi_{\nu^*}^\rho, & \text{if } \rho_1 > 0, \\ \cup_{\nu \geq \nu^*} \Pi_\nu^\rho, & \text{if } \rho_1 = 0. \end{cases}$$

(b) *The mean-ρ portfolio selection problem* (2) *is well posed if and only if the market (S^0, S) does not satisfy ρ-arbitrage. In this case, the portfolios that solve* (2) *are in Π_{ρ^*/ρ_1}^ρ.*

A natural question that arises is how (strong) ρ-arbitrage is related to the ordinary notion of arbitrage. To this end, recall that the market (S^0, S) is said to satisfy

- *arbitrage of the first kind* if there exists a trading strategy $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}$ (parametrised in numbers of shares) such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0, \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \geq 0 \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[\vartheta^0 S_1^0 + \vartheta \cdot S_1 > 0] > 0;$$

- *arbitrage of the second kind* if there exists a trading strategy $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}$ (parametrised in numbers of shares) such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 < 0, \quad \text{and} \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \geq 0 \text{ } \mathbb{P}\text{-a.s.}$$

The following result shows that if ρ is given by the worst-case risk measure WC, (strong) WC-arbitrage is equivalent to arbitrage of the first (second) kind. Thus, ρ-arbitrage can be seen as an extension of the ordinary notion of arbitrage.

Proposition 3.1.21. *The market* (S^0, S) *satisfies (strong)* WC*-arbitrage if and only if the market satisfies arbitrage of the first (second) kind.*

Proof. First note that by Theorem 3.1.9 either $\mathrm{WC}_1 = \infty$ or $\Pi_1^{\mathrm{WC}} \neq \emptyset$.

Now if $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}$ is an arbitrage of the first (second) kind, then $\pi := (\vartheta^1 S_0^1, \ldots, \vartheta^d S_0^d) \neq \mathbf{0}$ satisfies $X_\pi = \pi \cdot (R - r\mathbf{1}) = (\vartheta^0 S_1^0 + \vartheta \cdot S_1) - (1+r)(\vartheta^0 S_0^0 + \vartheta \cdot S_0) \geq 0$ (> 0) $\mathbb{P}$-a.s., which implies that $\mathrm{WC}(X_\pi)$ is nonpositive (negative). Hence, $\mathrm{WC}_1 \leq 0$ $(\mathrm{WC}_1 < 0)$ and $\Pi_1^{\mathrm{WC}} \neq \emptyset$. It follows that the market satisfies (strong) WC-arbitrage by Theorem 3.1.18(a) (Theorem 3.1.16).

Conversely, if the market does not satisfy arbitrage of the first (second) kind, then for all $\pi \in \Pi_1$, $\mathrm{WC}(X_\pi) > 0$ $(\mathrm{WC}(X_\pi) \geq 0)$. Since $\mathrm{WC}_1 = \infty$ or $\Pi_1^{\mathrm{WC}} \neq \emptyset$, it follows that $\mathrm{WC}_1 > 0$ $(\mathrm{WC}_1 \geq 0)$. Hence, the market does not satisfy (strong) WC-arbitrage by Theorem 3.1.18 (Theorem 3.1.16). □

We say that the market is arbitrage-free if it does not admit arbitrage of the first kind. The following result shows that unless ρ is as conservative as the worst-case risk measure, one can *always* construct a financial market that is arbitrage-free but admits strong ρ-arbitrage.

Theorem 3.1.22. *Assume* $\rho : L \to (-\infty, \infty]$ *is not as conservative as the worst-case risk measure. Then there exists a market* (S^0, S) *that is arbitrage-free but admits strong* ρ*-arbitrage.*

Proof. It is enough to construct a random variable $R \in L$ with $\mathbb{E}[R] > 0$, $\mathbb{P}[R < 0] > 0$ and $\rho(R) < 0$. Indeed, we can then define the market (S^0, S) by $S^0 \equiv 1$ and $S := S^1$, where $S_0^1 = 1$ and $S_1^1 = 1 + R$. This is nonredundant, nondegenerate, and arbitrage-free but admits strong ρ-arbitrage by Theorem 3.1.16 since $\rho_1 < 0$.

First, if ρ is not expectation bounded, there exists $X \in L$ such that $\mathbb{E}[-X] - \rho(X) := \varepsilon > 0$. By cash-invariance of ρ, this implies that X cannot be constant so $\operatorname{ess\,sup}(-X + E[X]) > 0$. Set $\delta \in (0, \operatorname{ess\,sup}(-X + E[X]))$ and let $R := X - \mathbb{E}[X] + \delta$. Then $\mathbb{E}[R] = \delta > 0$, $\mathbb{P}[R < 0] > 0$ and $\rho(R) = -\varepsilon - \delta < 0$.

Next, if ρ is expectation bounded but not as conservative as WC, there exists $X \in L$ such that $\rho(X) < \operatorname{ess\,sup}(-X) \leq \infty$. Let $m \in (\rho(X), \operatorname{ess\,sup}(-X))$ and $R := X + m$. Then $\mathbb{P}[R < 0] > 0$, $\rho(R) < 0$ and $\mathbb{E}[R] \geq -\rho(R) > 0$ by expectation boundedness of ρ. □

Remark 3.1.23. The preceding two results together imply that the worst-case risk measure is the only positively homogeneous risk measure suitable for risk management. It is also suitable for portfolio selection if $L = L^\infty$.

3.1.4 ρ-Arbitrage for Elliptical Returns

The primal characterisations of (strong) ρ-arbitrage in Theorems 3.1.16 and 3.1.18 are particularly useful when returns are elliptically distributed with finite second moments and the risk measure is law-invariant. We briefly recall both concepts.

Definition 3.1.24. An $\mathbb{R}^d$-valued random vector $X = (X_1, \dots, X_d)$ has an *elliptical distribution* if there exists a *location vector* $\widetilde{\mu} \in \mathbb{R}^d$, a $d \times d$ non-negative definite *dispersion matrix* $\widetilde{\Sigma} \in \mathbb{R}^{d \times d}$, and a *characteristic generator* $\psi : [0, \infty) \to \mathbb{R}$ such that the characteristic function of X, ϕ_X can be expressed

$$\phi_X(t) = e^{it^\top \widetilde{\mu}} \psi(t^T \widetilde{\Sigma} t) \quad \text{for all } t \in \mathbb{R}^d.$$

In this case we write $X \sim \tilde{E}_d(\widetilde{\mu}, \widetilde{\Sigma}, \psi)$.

Elliptical distributions are generalisations of the multivariate normal distribution, which allow for heavy tail models while possessing many useful properties. Indeed, the fat tails of most of their members make them natural candidates in modelling the distribution of speculative returns. Examples of elliptical distributions include the multivariate normal distribution, the multivariate t-distribution and the multivariate symmetric Laplace distribution. For a thorough description of elliptical distributions refer to [42, 63].

Remark 3.1.25. If X has an elliptical distribution with finite second moments, X is also characterised by its mean vector $\mu \in \mathbb{R}^d$, covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ and characteristic generator ψ. Therefore, we may write $X \sim E_d(\mu, \Sigma, \psi)$; see [70, Remark 3.27] for details.

Definition 3.1.26. A risk measure $\rho : L \to (-\infty, \infty]$ is called *law-invariant* if $\rho(X_1) = \rho(X_2)$ whenever $X_1, X_2 \in L$ have the same law.

The following result shows why elliptical distributions and law-invariant risk measures work particularly nicely together.

Lemma 3.1.27. *Suppose ρ is law-invariant and the return vector R has an elliptical distribution with mean vector $\mu \in \mathbb{R}^d$, covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ and characteristic generator ψ. Assume $\{X \sim E_1(\mu_X, \sigma_X^2, \psi) : \mu_X \in \mathbb{R},\ \sigma_X^2 \geq 0\} \subset L$ and let $Z \sim E_1(0, 1, \psi)$. Then for any $\pi \in \mathbb{R}^d$,*

$$\rho(X_\pi) = -\mathbb{E}[X_\pi] + \rho(Z)\sqrt{\mathrm{Var}(X_\pi)} = -\pi^\top(\mu - r\mathbf{1}) + \rho(Z)\sqrt{\pi^\top\Sigma\pi}, \quad (3.3)$$

where $\infty \times 0 = 0$, so that $\rho(X_\pi) = -\mathbb{E}[X_\pi]$ if $\mathrm{Var}(X_\pi) = 0$. Moreover, $\rho(Z)$ is nonnegative (positive) if ρ is (strictly) expectation bounded.

Proof. Standard properties of elliptical distributions imply that

$$\pi \cdot (R - r\mathbf{1}) \sim E_1\left(\pi \cdot (\mu - r\mathbf{1}), \pi^{\mathrm{T}}\Sigma\pi, \psi\right)$$

for any portfolio $\pi \in \mathbb{R}^d$. This means that $X_\pi \overset{d}{=} \pi^\top(\mu - r\mathbf{1}) + Z\sqrt{\pi^\top\Sigma\pi}$, where $Z \sim E_1(0, 1, \psi)$. As ρ is a law-invariant, $\rho(X_\pi) = -\pi^\top(\mu - r\mathbf{1}) + \rho(Z)\sqrt{\pi^\top\Sigma\pi}$. The final claim follows from the fact that $\mathbb{E}[Z] = 0$ because $Z \sim E_1(0, 1, \psi)$ has a symmetric distribution. □

With the help of Lemma 3.1.27, we can give a very simple characterisation for the absence of (strong) ρ-arbitrage in terms of the maximal *Sharpe ratio*.

Corollary 3.1.28. *Suppose ρ is law-invariant and the return vector R has an elliptical distribution with mean vector $\mu \in \mathbb{R}^d$ satisfying $\mu \neq r\mathbf{1}$, positive definite covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ and characteristic generator ψ. Assume $\{X \sim E_1(\mu_X, \sigma_X^2, \psi) : \mu_X \in \mathbb{R},\ \sigma_X^2 \geq 0\} \subset L$ and let $Z \sim E_1(0, 1, \psi)$. Define the maximal Sharpe ratio as*

$$\mathrm{SR}_{\max} := \max_{\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}} \frac{\mathbb{E}[X_\pi]}{\sqrt{\mathrm{Var}(X_\pi)}} = \sqrt{(\mu - r\mathbf{1})^\top\Sigma^{-1}(\mu - r\mathbf{1})}. \quad (3.4)$$

Then we have the following trichotomy:

(a) *If* $\mathrm{SR}_{\max} < \rho(Z)$*, the market* (S^0, S) *does not admit* ρ*-arbitrage.*

(b) *If* $\mathrm{SR}_{\max} = \rho(Z)$*, the market* (S^0, S) *admits* ρ*-arbitrage but not strong* ρ*-arbitrage.*

(c) *If* $\mathrm{SR}_{\max} > \rho(Z)$*, the market* (S^0, S) *admits strong* ρ*-arbitrage.*

In particular, if $\rho(Z) \leq 0$*, the market* (S^0, S) *admits strong* ρ*-arbitrage, independent of* μ *or* Σ*. Moreover, if* $\rho(Z) < 0$ *and* $d \geq 2$*,* ρ*-optimal portfolios fail to exist for any* $\nu \geq 0$*, independent of* μ *or* Σ*.*

Proof. For $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, set $\mathrm{SR}_\pi := \mathbb{E}[X_\pi]/\sqrt{\mathrm{Var}(X_\pi)}$ and note that this is well defined because $\mu \neq r\mathbf{1}$ and Σ is positive definite. It follows from linearity of the expectation and positive homogeneity of the standard deviation that $\mathrm{SR}_{\max} := \max_{\pi\in\Pi_1} \mathrm{SR}_\pi$. It is not difficult to check that the portfolio

$$\pi^* := \frac{1}{(\mu - r\mathbf{1})^{\mathrm{T}}\Sigma^{-1}(\mu - r\mathbf{1})}\Sigma^{-1}(\mu - r\mathbf{1}) \in \Pi_1$$

has maximal Sharpe ratio given by the right-hand side of (3.4).

If $\rho(Z) \in (0, \infty)$, then by Lemma 3.1.27 for any $\pi \in \Pi_1$,

$$\rho(X_\pi) = -1 + \rho(Z)\sqrt{\mathrm{Var}(X_\pi)} = -1 + \frac{\rho(Z)}{\mathrm{SR}_\pi}.$$

Thus, minimising $\rho(X_\pi)$ over $\pi \in \Pi_1$ is equivalent to maximising SR_π over Π_1. Whence

$$\rho_1 := -1 + \frac{\rho(Z)}{\mathrm{SR}_{\max}} = -1 + \frac{\rho(Z)}{\mathrm{SR}_{\pi^*}} = \rho(X_{\pi^*}).$$

Parts (a), (b) and (c) now follow from Theorems 3.1.16 and Theorem 3.1.18(a).

If $\rho(Z) = \infty$, every portfolio has infinite risk except the riskless portfolio which has zero risk. Whence $\Pi_0^\rho = \{\mathbf{0}\}$, $\Pi_1^\rho = \emptyset$ and $\rho_1 = \infty$. Now part (a) follow from Theorem 3.1.18(b).

If $\rho(Z) = 0$, $\rho(X_\pi) = -\mathbb{E}[X_\pi]$ for every portfolio $\pi \in \mathbb{R}^d$. Thus, $\rho_\nu = -\nu$ for any $\nu \geq 0$ and the market admits strong ρ-arbitrage by Theorem 3.1.16.

Finally, if $\rho(Z) < 0$, Lemma 3.1.27 gives for $\nu \geq 0$,

$$\inf_{\pi\in\Pi_\nu} \rho(X_\pi) = \inf_{\pi\in\Pi_\nu} \{-\nu + \rho(Z)\sqrt{\mathrm{Var}(X_\pi)}\} = -\nu + \rho(Z) \sup_{\pi\in\Pi_\nu} \sqrt{\mathrm{Var}(X_\pi)} < 0.$$

Whence, $\rho_\nu < 0$ and so the market admits strong ρ-arbitrage by Theorem 3.1.16. If $d \geq 2$, it is not difficult to check that $\sup_{\pi\in\Pi_\nu} \sqrt{\mathrm{Var}(X_\pi)} = \infty$, and hence $\rho_\nu = -\infty$, which implies that $\Pi_\nu^\rho = \emptyset$. $\square$

Remark 3.1.29. Corollary 3.1.28 shows that in general it is *not* true that for elliptically distributed returns and a law-invariant risk measure ρ, the ρ-optimal portfolios coincide with the Markowitz optimal portfolios. (This is for instance claimed in [40, Theorem 1].) Indeed, Corollary 3.1.28 shows that in *every* elliptical market, VaR^α-optimal portfolios fail to exist if $\alpha > \mathbb{P}[Z \leq 0] = 1/2 + 1/2\mathbb{P}[Z = 0]$, where $Z \sim E_1(0, 1, \psi)$. (Note that $Z \sim E_1(0, 1, \psi)$ has a symmetric distribution.) In particular, VaR^α-optimal portfolios fail to exist for $\alpha > 1/2$ in every multivariate Gaussian market. The underlying reason is that Value at Risk fails to be expectation bounded.

We illustrate the above result by considering the case that R has multivariate Gaussian returns and the risk measure is either Value at Risk or Expected Shortfall.

Example 3.1.30. Assume the return vector R has a multivariate normal distribution with mean vector $\mu \in \mathbb{R}^d$ satisfying $\mu \neq r\mathbf{1}$ and a positive definite covariance matrix $\Sigma \in \mathbb{R}^{d \times d}$. Let $Z \sim N(0, 1)$. Then for $\alpha \in (0, 1)$, we have

$$\mathrm{VaR}^\alpha(Z) = \Phi^{-1}(1 - \alpha) \quad \text{and} \quad \mathrm{ES}^\alpha(Z) = \frac{\phi(\Phi^{-1}(\alpha))}{\alpha},$$

where ϕ and Φ denote the pdf and cdf of a standard normal distribution, respectively. By Corollary 3.1.28, we can fully characterise (strong) ρ-arbitrage in this market for both risk measures by looking at the maximal Sharpe ratio. Figure 6 gives a graphical illustration.

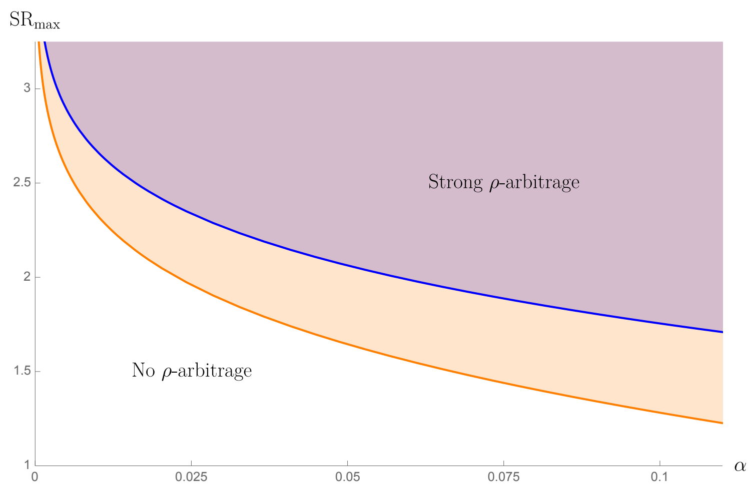


Figure 6: ρ-arbitrage for ES (blue) and VaR (orange), for Gaussian returns

If $\mathrm{SR}_{\max}$ lies above the blue (orange) curve, then this Gaussian market admits strong $\mathrm{ES}^\alpha(\mathrm{VaR}^\alpha)$-arbitrage. If it lies below the blue (orange) curve then the market does not admit $\mathrm{ES}^\alpha(\mathrm{VaR}^\alpha)$-arbitrage. And in the intermediate case, the market admits $\mathrm{ES}^\alpha(\mathrm{VaR}^\alpha)$-arbitrage, but not strong $\mathrm{ES}^\alpha(\mathrm{VaR}^\alpha)$-arbitrage.

Also note that for Value at Risk, if $\alpha > 1/2$, then $\Phi^{-1}(1-\alpha) < 0$. Hence, in this case we always have strong VaR^α-arbitrage and VaR^α-optimal portfolios fail to exist for $d \geq 2$, independent of μ or Σ.

3.2 Dual Characterisation of (Strong) ρ-Arbitrage

Theorems 3.1.16 and 3.1.18 provide a full characterisation of strong ρ-arbitrage and ρ-arbitrage, respectively. However, the criterion is rather indirect as it requires to calculate ρ_1, which relies on a nontrivial optimisation problem. In this section, we consider the case that ρ is in addition convex (and hence coherent), expectation bounded and has a dual representation. We then derive a *dual characterisation* of (strong) ρ-arbitrage.

Let $\mathcal{D} := \{Z \in L^1 : Z \geq 0 \ \mathbb{P}\text{-a.s. and } \mathbb{E}[Z] = 1\}$ be the set of all Radon-Nikodým derivatives of probability measures that are absolutely continuous with respect to $\mathbb{P}$. Throughout this section, we assume that $\rho : L \to (-\infty, \infty]$ is an expectation bounded, coherent risk measure and admits a dual representation

$$\rho(X) = \sup_{Z\in\mathcal{Q}}(\mathbb{E}[-ZX]), \tag{3.5}$$

for some $\mathcal{Q} \subset \mathcal{D}$. Since ρ is expectation bounded, we may assume without loss of generality that $1 \in \mathcal{Q}$. Moreover, taking the supremum over $\mathcal{Q}$ is equivalent to taking the supremum over its convex hull, and therefore, we may assume without loss of generality that $\mathcal{Q}$ is convex.

Remark 3.2.1. (a) Since $-ZX$ may not be integrable, we define $\mathbb{E}[-ZX] := \mathbb{E}[ZX^-] - \mathbb{E}[ZX^+]$, with the conservative convention that if $\mathbb{E}[ZX^-] = \infty$, then $\mathbb{E}[-ZX] = \infty$.

(b) Apart from the (natural) assumption that ρ is expectation bounded, this is the most general class of coherent risk measures on L that admit a dual representation in which the representing set consists of countably additive (and not just finitely additive) set functions. For instance, we do not impose L^1-closedness or uniformly integrability of $\mathcal{Q}$ (which is for instance assumed in

[28]). A wide range of examples of risk measures satisfying a representation of the form (3.5) are given in Section 3.3.

(c) On a general space L, e.g., an Orlicz space whose Young function does not satisfy the Δ_2 condition, a representation of the form (3.5) is generally not possible even for "nicely regular" risk measures. Notably, in a nonatomic probability space, the special representation (3.5) will automatically hold provided the risk measure ρ is law-invariant and satisfies the Fatou property. In this case, the representing set $\mathcal{Q}$ can always be restricted to a subset of $\mathcal{D} \cap L^\infty$ (by [48, Corollary 5.2]), thus simplifying considerably the analysis.

(d) The representation in (3.5) is not unique. However, it is not difficult to check that the *maximal* dual set for which (3.5) is satisfied is given by

$$\mathcal{Q}_\rho := \{Z \in \mathcal{D} : \mathbb{E}[ZX] \geq 0 \text{ and } \mathbb{E}[ZX^-] < \infty \text{ for all } X \in \mathcal{A}_\rho\}, \tag{3.6}$$

where $\mathcal{A}_\rho := \{X \in L : \rho(X) \leq 0\}$ is the *acceptance set* of ρ. Note that in general $\mathcal{Q}_\rho$ is not L^1-closed. It turns out that for the dual characterisation of ρ-arbitrage, it is sometimes useful *not* to consider the maximal dual set; cf. some of the examples in Section 3.3.

(e) If we define ρ by (3.5) for some convex set $\mathcal{Q}$ containing 1, it follows that ρ is $(-\infty, \infty]$-valued, expectation bounded and a coherent risk measure (i.e., it is monotone, cash-invariant, positively homogeneous and convex).

3.2.1 Preliminary Considerations and Conditions

In this section, we introduce and discuss some additional conditions that are needed (and necessary) for the main results in Chapter 3, Theorems 3.2.14 and 3.2.18.

We start by introducing two conditions concerning the (uniform) integrability of the returns under the probability measures "contained" in $\mathcal{Q}$.

Condition I. For all $i \in \{1, \dots, d\}$ and any $Z \in \mathcal{Q}$, $ZR^i \in L^1$.

Condition UI. $\mathcal{Q}$ is uniformly integrable, and for all $i \in \{1, \dots, d\}$, $R^i\mathcal{Q}$ is uniformly integrable, where $R^i\mathcal{Q} := \{R^iZ : Z \in \mathcal{Q}\}$.

Remark 3.2.2. (a) Condition I may depend on the choice of the dual set $\mathcal{Q}$ in the dual representation (3.5) of ρ. In particular, it may not be satisfied for the maximal dual set $\mathcal{Q}_\rho$; cf. Section 3.3.1 for a concrete example. For this reason, one might want to choose a "small" dual set $\mathcal{Q}$ for ρ.

(b) By contrast, Condition UI *essentially* does not depend on the choice of the dual set $\mathcal{Q}$ in the dual representation of ρ. More precisely, this statement is true if ρ is such that all representing dual sets have the same L^1-closure. One important example is when L is an Orlicz space and ρ is real valued; cf. Proposition A.2.5.

While Condition I is quite weak, it has some important consequences.

Proposition 3.2.3. *Suppose that Condition I is satisfied. Then the set*

$$C_{\mathcal{Q}} := \{\mathbb{E}[-Z(R - r\mathbf{1})] : Z \in \mathcal{Q}\} \tag{3.7}$$

is a convex subset of $\mathbb{R}^d$ *and for any portfolio* $\pi \in \mathbb{R}^d$,

$$\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}}} (\pi \cdot c). \tag{3.8}$$

Moreover, ρ *satisfies the Fatou property on* $\mathcal{X} = \{X_\pi : \pi \in \mathbb{R}^d\}$.

Proof. The set $C_{\mathcal{Q}}$ is real valued by Condition I and convex by convexity of $\mathcal{Q}$. This together with linearity of the expectation implies that

$$\rho(X_\pi) = \sup_{Z \in \mathcal{Q}} (\mathbb{E}[-ZX_\pi]) = \sup_{Z \in \mathcal{Q}} (\mathbb{E}[-Z(\pi \cdot (R - r\mathbf{1}))]) = \sup_{c \in C_{\mathcal{Q}}} (\pi \cdot c).$$

Finally, to establish the Fatou property on $\mathcal{X}$, assume that $X_{\pi_n} \to X_\pi$ $\mathbb{P}$-a.s. Nonredundancy of the market implies that $\pi_n \to \pi$. Then for any $Z \in \mathcal{Q}$, Condition I, linearity of the expectation and the definition of ρ in (3.5) gives

$$\begin{aligned}\mathbb{E}[-ZX_\pi] &= \pi \cdot \mathbb{E}[-Z(R - r\mathbf{1})] = \lim_{n \to \infty} \pi_n \cdot \mathbb{E}[-Z(R - r\mathbf{1})] \\ &= \lim_{n \to \infty} \mathbb{E}[-ZX_{\pi_n}] \leq \liminf_{n \to \infty} \rho(X_{\pi_n}).\end{aligned}$$

Taking the supremum over $Z \in \mathcal{Q}$ gives $\rho(X_\pi) \leq \liminf_{n \to \infty} \rho(X_{\pi_n})$. □

Remark 3.2.4. Example A.3.2 shows that without Condition I, the set $C_{\mathcal{Q}}$ may fail to be convex or $\mathbb{R}^d$-valued and (3.8) may break down.

Condition UI is a uniform version of Condition I. (Note that $Z \in L^1$ for all $Z \in \mathcal{Q}$ even though this does not appear explicitly in Condition I.) The following result shows that under Condition UI, the supremum in (3.8) can be replaced by a maximum, if we replace $\mathcal{Q}$ in (3.7) by its L^1-closure.

Proposition 3.2.5. *Suppose that Condition UI is satisfied. Denote by $\bar{\mathcal{Q}}$ the L^1-closure of $\mathcal{Q}$. Then the set*

$$C_{\bar{\mathcal{Q}}} := \{\mathbb{E}[-Z(R - r\mathbf{1})] : Z \in \bar{\mathcal{Q}}\} \tag{3.9}$$

is a convex and compact subset of $\mathbb{R}^d$. Moreover, for any portfolio $\pi \in \mathbb{R}^d$,

$$\rho(X_\pi) = \max_{c \in C_{\bar{\mathcal{Q}}}} (\pi \cdot c). \tag{3.10}$$

Proof. Since Condition UI implies Condition I, (3.8) gives

$$\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}}} (\pi \cdot c) \leq \sup_{c \in C_{\bar{\mathcal{Q}}}} (\pi \cdot c). \tag{3.11}$$

Since $\mathcal{Q}$ is UI and convex, $\bar{\mathcal{Q}}$ is convex and weakly compact by the Dunford-Pettis theorem. To show that the supremum on the right side of (3.11) is attained, let $(Z_n)_{n \in \mathbb{N}}$ be a maximising sequence in $\bar{\mathcal{Q}}$. Since $\bar{\mathcal{Q}}$ is weak sequentially compact by the Eberlein-Šmulian theorem, after passing to a subsequence, we may assume that Z_n converges weakly to some $Z \in \bar{\mathcal{Q}}$. Since the map $\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}(R - r\mathbf{1})]$ is weakly continuous on $\bar{\mathcal{Q}}$ by Proposition A.6.2, Z is a maximiser. The same argument, but now for a maximising sequence in $\mathcal{Q} \subset \bar{\mathcal{Q}}$, shows that we have have equality in (3.11). Finally, using again that the map $\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}(R - r\mathbf{1})]$ is weakly continuous on $\bar{\mathcal{Q}}$ and $\bar{\mathcal{Q}}$ is weakly compact, it follows that $C_{\bar{\mathcal{Q}}}$ is compact. □

Remark 3.2.6. Example A.3.3 shows that without Condition UI (even when Condition I is satisfied), the set $C_{\bar{\mathcal{Q}}}$ may fail to be convex, compact or a subset of $\mathbb{R}^d$ and (3.10) may break down.

Remark 3.2.7. In [28], it is assumed that $\mathcal{Q}$ is uniformly integrable and that $R^i \in L^1(\mathcal{Q})$, where

$$L^1(\mathcal{Q}) := \{X \in L^0 : \lim_{a \to \infty} \sup_{Z \in \mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{|X| > a\}}] = 0\}. \tag{3.12}$$

By Proposition A.6.1, this is equivalent to Condition UI. However, we believe Condition UI better highlights why this is a uniform version of Condition I.

We next aim to introduce a notion of "interior" for $\mathcal{Q}$, which is crucial for the dual characterisation of ρ-arbitrage. This turns out to be rather subtle since neither algebraic nor topological notions of interior work in general;

cf. Remark 3.2.8. Instead, we define our notion of "interior" in an abstract way. More precisely, we look for (nonempty) subsets $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying

Condition POS. $\tilde{Z} > 0$ $\mathbb{P}$-a.s. for all $\tilde{Z} \in \tilde{\mathcal{Q}}$.

Condition MIX. $\lambda Z + (1-\lambda)\tilde{Z} \in \tilde{\mathcal{Q}}$ for all $Z \in \mathcal{Q}$, $\tilde{Z} \in \tilde{\mathcal{Q}}$ and $\lambda \in (0,1)$.

Condition INT. For all $\tilde{Z} \in \tilde{\mathcal{Q}}$, there is an L^∞-dense subset $\mathcal{E}$ of $\mathcal{D} \cap L^\infty$ such that for all $Z \in \mathcal{E}$, there is $\lambda \in (0,1)$ such that $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}$.

A few comments are in order.

Remark 3.2.8. (a) Condition MIX implies in particular that $\tilde{\mathcal{Q}}$ is convex.

(b) Condition INT of $\tilde{\mathcal{Q}}$ is inspired by the definition of the core/algebraic interior. Indeed, recall that for a vector space V, the algebraic interior of a set $M \subset V$ with respect to a vector subspace $X \subset V$ is defined by

$$\begin{aligned}\mathrm{aint}_X M := \{m \in M : &\text{for all } x \in X,\ \text{there is } \lambda > 0 \text{ such that} \\ & m + \delta x \in M \text{ for all } \delta \in [0,\lambda]\};\end{aligned}$$

see [91] for details. When M is convex, one can show that

$$\mathrm{aint}_X M = \{m \in M : \text{for all } x \in X,\ \text{there is } \lambda > 0 \text{ such that } m + \lambda x \in M\},$$

and any strict convex combination of a point in M and $\mathrm{aint}_X M$ belongs to $\mathrm{aint}_X M$. To see the link to our setup, assume that $\mathcal{Q} \subset L^\infty$. Set $M := \mathcal{Q}$, $V := L^\infty$ and $X := \{Z \in L^\infty : \mathbb{E}[Z] = 0\}$. Then $\mathrm{aint}_X M$ satisfies conditions POS, MIX and INT. Moreover, for certain examples (e.g. Expected Shortfall), $\mathrm{aint}_X M \neq \emptyset$. Note, however, that if $\mathcal{Q} \not\subset L^\infty$, it is not possible to define a nonempty set $\tilde{\mathcal{Q}}$ satisfying Conditions POS, MIX and INT via the algebraic interior.

(c) One might wonder if one could define $\tilde{\mathcal{Q}}$ as the topological interior of $\mathcal{Q}$ in a suitable subspace topology of $\mathcal{D} \cap V$, where $L^\infty \subset V \subset L^1$ is a vector subspace. Again if $\mathcal{Q} \subset L^\infty$, for certain examples (e.g. Expected Shortfall), the topological interior of $\mathcal{Q}$ in the subspace topology of $\mathcal{D} \cap L^\infty$ is nonempty and satisfies Conditions POS, MIX and INT. However, if $\mathcal{Q} \not\subset L^\infty$, this approach does not work since the topological interior may fail to satisfy Condition MIX (because $\mathcal{D} \cap V$ is not a vector space).

(d) In light of Propositions 3.2.11 and A.6.6, one could slightly relax Condition INT, by requiring that the sets $\mathcal{E}$ are only $\sigma(L^\infty, L^1)$-dense in $\mathcal{D} \cap L^\infty$.

However, this additional level of generality does not seem to be useful in concrete examples. On the other hand, considering L^∞-dense subsets of $\mathcal{D} \cap L^\infty$ is useful; cf. Section 3.3.3.

We proceed to characterise the maximal subset of $\mathcal{Q}$ satisfying Conditions POS, MIX and INT. This is surprisingly simple and shows that we can expect $\tilde{\mathcal{Q}}_{\max}$ to be nonempty for most risk measures ρ.

Proposition 3.2.9. *Define the set* $\tilde{\mathcal{Q}}_{\max}$ *by*

$$\tilde{\mathcal{Q}}_{\max} := \{\tilde{Z} > 0 \in \mathcal{Q} : \textit{there is an } L^\infty\textit{-dense subset } \mathcal{E} \textit{ of } \mathcal{D} \cap L^\infty \textit{ such that for all } Z \in \mathcal{E}, \textit{there is } \lambda \in (0,1) \textit{ such that } \lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}\}.$$

Then $\tilde{\mathcal{Q}}_{\max}$ *satisfies Conditions POS, MIX and INT. Moreover, if* $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ *satisfies Conditions POS, MIX and INT, then* $\tilde{\mathcal{Q}} \subset \tilde{\mathcal{Q}}_{\max}$.

Proof. $\tilde{\mathcal{Q}}_{\max}$ satisfies Conditions POS and INT by definition. To establish Condition MIX, let $Z \in \mathcal{Q}$, $\tilde{Z} \in \tilde{\mathcal{Q}}_{\max}$ and $\mu \in (0,1)$. Clearly $\mu Z + (1-\mu)\tilde{Z} > 0$ $\mathbb{P}$-a.s. It remains to show that there exists an L^∞-dense subset $\mathcal{E}'$ of $\mathcal{D} \cap L^\infty$ such that for all $Z' \in \mathcal{E}'$, there is $\lambda' > 0$ such that $\lambda' Z' + (1-\lambda')(\mu Z + (1-\mu)\tilde{Z}) \in \mathcal{Q}$. Let $\mathcal{E}$ be the L^∞-dense subset of $\mathcal{D} \cap L^\infty$ for $\tilde{Z}$ in the definition of $\tilde{\mathcal{Q}}_{\max}$. Set $\mathcal{E}' := \mathcal{E}$. Let $Z' \in \mathcal{E}'$. Then there is $\lambda > 0$ such that $\lambda Z' + (1-\lambda)\tilde{Z} \in \tilde{\mathcal{Q}}_{\max} \subset \mathcal{Q}$. Set $\mu' := \frac{1-\mu}{1-\mu\lambda} \in (0,1)$ and $\lambda' := \lambda\mu' \in (0,1)$. Then by convexity of $\mathcal{Q}$,

$$\mu'\big(\lambda Z' + (1-\lambda)\tilde{Z}\big) + (1-\mu')Z = \lambda' Z' + (1-\lambda')(\mu Z + (1-\mu)\tilde{Z}) \in \mathcal{Q}.$$

The additional claim follows immediately from the definition of $\tilde{\mathcal{Q}}_{\max}$. □

Remark 3.2.10. (a) If $\mathcal{Q}' \subset \mathcal{Q}$ are dual sets that can be used to represent ρ, then $\tilde{\mathcal{Q}}'_{\max} \subset \tilde{\mathcal{Q}}_{\max}$.

(b) While Proposition 3.2.9 is insightful from a theoretical perspective, it is very difficult in practise to compute $\tilde{\mathcal{Q}}_{\max}$. For this reason, it is often easier to find a nonempty subset $\tilde{\mathcal{Q}} \in \mathcal{Q}$ satisfying Conditions POS, MIX and INT directly. This is the approach that we take in virtually all of the examples in Section 3.3. Since Condition MIX is easier to satisfy if $\mathcal{Q}$ is smaller, one sometimes might even first have to find a smaller representing dual set $\mathcal{Q}' \subset \mathcal{Q}$ for ρ and then a nonempty subset $\tilde{\mathcal{Q}}' \in \mathcal{Q}'$ satisfying Conditions POS, MIX and INT; see Section 3.3.3 for a concrete example.

We finish this section by explaining the role of Conditions POS and INT for establishing existence of ρ-optimal portfolios.

Proposition 3.2.11. *Suppose Condition I is satisfied. Let $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfy Conditions POS and INT. If $1 \in \tilde{\mathcal{Q}}$, then ρ is strictly expectation bounded and $\Pi_0^\rho = \{\mathbf{0}\}$. If in addition $\rho_1 < \infty$, then for all $\nu \geq 0$, Π_ν^ρ is nonempty, compact and convex.*

Proof. Strict expectation boundedness of ρ follows from Lemma A.6.3 (with $\tilde{Z} = 1$) and Remark 3.1.6(c). Corollary 3.1.7 then gives $\Pi_0^\rho = \{\mathbf{0}\}$. Finally, if $\rho_1 < \infty$, it follows that $\rho_1 \in \mathbb{R}$ since $\rho_1 \geq -1$ by expectation boundedness of ρ. Now the remaining claim follows from Proposition 3.2.3, Theorem 3.1.9 and Remark 3.1.10(c). □

3.2.2 Dual Characterisation of Strong ρ-Arbitrage

In this section, we provide a dual characterisation of strong ρ-arbitrage in terms of absolutely continuous martingale measures (ACMMs) for the discounted risky assets S/S^0. To this end, set

$$\mathcal{M} = \{Z \in \mathcal{D} : \mathbb{E}[Z(R^i - r)] = 0 \text{ for all } i = 1, \dots, d\}, \tag{3.13}$$

and note $Z \in \mathcal{M}$ is the Radon-Nikodým derivative of an ACMM for S/S^0.

A first step towards a dual characterisation is the following equivalent characterisation of strong ρ-arbitrage.

Proposition 3.2.12. *The market (S^0, S) satisfies strong ρ-arbitrage if and only if $\rho(X_\pi) < 0$ for some portfolio $\pi \in \mathbb{R}^d$.*

Proof. If the market admits strong ρ-arbitrage, then $\rho_1 < 0$ by Theorem 3.1.16. Hence, $\rho(X_\pi) < 0$ for some portfolio $\pi \in \mathbb{R}^d$.

Conversely, if $\rho(X_\pi) < 0$ for some portfolio π, $\mathbb{E}[X_\pi] \geq -\rho(X_\pi) > 0$ because $1 \in \mathcal{Q}$. Thus, $\rho_1 < 0$, and the market satisfies strong ρ-arbitrage. □

Our next result shows that if $\mathcal{Q}$ contains an ACMM, the market does not admit strong ρ-arbitrage.

Proposition 3.2.13. *If $\mathcal{Q} \cap \mathcal{M} \neq \emptyset$, then the market (S^0, S) does not admit strong ρ-arbitrage.*

Proof. Let $Z \in \mathcal{Q} \cap \mathcal{M}$. Then for any portfolio $\pi \in \mathbb{R}^d$,

$$\rho(X_\pi) \geq \mathbb{E}[-ZX_\pi] = 0.$$

Thus, by Proposition 3.2.12 the market does not admit strong ρ-arbitrage. □

The converse of Proposition 3.2.13 is in general false. Example A.3.4 shows that even under Condition I, $\mathcal{Q} \cap \mathcal{M} = \emptyset$ is not enough to imply strong ρ-arbitrage. However, under Condition UI, it is essentially true.

Theorem 3.2.14. *Assume $\mathcal{Q}$ satisfies UI. Denote by $\bar{\mathcal{Q}}$ the L^1-closure of $\mathcal{Q}$. The following are equivalent:*

(a) *The market (S^0, S) does not admit strong ρ-arbitrage.*

(b) $\bar{\mathcal{Q}} \cap \mathcal{M} \neq \emptyset$.

Proof. First, assume $\bar{\mathcal{Q}} \cap \mathcal{M} \neq \emptyset$. Let $Z \in \bar{\mathcal{Q}} \cap \mathcal{M}$. Then Proposition 3.2.5 gives $\rho(X_\pi) \geq \mathbb{E}[-ZX_\pi] = 0$ for any $\pi \in \mathbb{R}^d$. Therefore, the market does not admit strong ρ-arbitrage by Proposition 3.2.12.

Conversely, assume $\bar{\mathcal{Q}} \cap \mathcal{M} = \emptyset$. By Proposition 3.2.5, $\{\mathbf{0}\}$ and $C_{\bar{\mathcal{Q}}}$ are two nonempty disjoint convex and compact subsets of $\mathbb{R}^d$. By the strict separation theorem (cf. [16, Proposition B.14]), there exists $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ with $\pi \cdot c < b < 0$ for all $c \in C_{\bar{\mathcal{Q}}}$. Thus, Proposition 3.2.5 gives

$$\rho(X_\pi) = \max_{c \in C_{\bar{\mathcal{Q}}}} (\pi \cdot c) < 0,$$

and so the market admits strong ρ-arbitrage by Proposition 3.2.12. □

Remark 3.2.15. (a) By virtue of Proposition 3.2.12 and Remark 3.1.17, Theorem 3.2.14 is identical to the equivalent characterisation of no-good-deals in [28, Theorem 3.1]. However, our proof is simpler since we are working with a finite number of assets. We have included it for the convenience of the reader.

(b) Example A.3.5 shows that when $\mathcal{Q}$ is uniformly integrable but $R\mathcal{Q}$ is not, then Theorem 3.2.14 is false. Example A.3.6 shows that Theorem 3.2.14 is also false if $R\mathcal{Q}$ is uniformly integrable but $\mathcal{Q}$ is not. Thus, we need both parts of Condition UI simultaneously.

Characterising the absence of strong ρ-arbitrage is important. However, to see the whole picture, it is important to also have a dual characterisation of ρ-arbitrage.

3.2.3 Dual Characterisation of ρ-Arbitrage

In this section, we provide a dual characterisation of ρ-arbitrage in terms of equivalent martingale measures (EMMs) for the discounted risky assets S/S^0. To this end, set

$$\mathcal{P} = \{Z \in \mathcal{M} : Z > 0 \ \mathbb{P}\text{-a.s.}\}.$$

and note $Z \in \mathcal{P}$ is the Radon-Nikodým derivative of an EMM for S/S^0.

As we did for strong ρ-arbitrage, we start by providing an equivalent characterisation of ρ-arbitrage. However, for ρ-arbitrage, we need to assume that $\mathbf{0}$ is the unique ρ-optimal portfolio.

Proposition 3.2.16. *Assume $\Pi_0^\rho = \{\mathbf{0}\}$. Then the market (S^0, S) satisfies ρ-arbitrage if and only if $\rho(X_\pi) \leq 0$ for some portfolio $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.*

Proof. First assume the market satisfies ρ-arbitrage. As the riskless portfolio $\mathbf{0}$ has zero risk, by definition of ρ-arbitrage there must be another portfolio $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ with $\rho(X_\pi) \leq 0$.

Conversely, if $\rho(X_\pi) \leq 0$ for some portfolio $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, then $\mathbb{E}[X_\pi] \geq 0$ by expectation boundedness of ρ, which in turn gives $\mathbb{E}[X_\pi] > 0$ because $\Pi_0^\rho = \{\mathbf{0}\}$. It follows that either $\rho_1 < 0$ (in which case Π_1^ρ may or may not be empty) or $\rho_1 = 0$ (in which case $\Pi_1^\rho \neq \emptyset$). In either case the market admits ρ-arbitrage by Theorem 3.1.18. □

We proceed to give a preliminary dual characterisation of ρ-arbitrage. Note that this characterisation does not rely on the set $\tilde{\mathcal{Q}}_{\max}$ to be nonempty.

Proposition 3.2.17. *Assume $\Pi_0^\rho = \{\mathbf{0}\}$ and $\mathcal{Q}$ satisfies Condition I. If $\mathcal{Q} \cap \mathcal{M} = \emptyset$, then the market (S^0, S) admits ρ-arbitrage.*

Proof. Condition I implies that the set $C_{\mathcal{Q}}$ in (3.7) is convex. If $\mathcal{Q} \cap \mathcal{M} = \emptyset$ then $\mathbf{0} \notin C_{\mathcal{Q}}$. By the supporting hyperplane theorem (cf. [16, Proposition B.12]), there exists $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ with $\pi \cdot c \leq 0$ for all $c \in C_{\mathcal{Q}}$. By (3.8),

$$\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}}} (\pi \cdot c) \leq 0,$$

and the claim follows from Proposition 3.2.16. □

We are now in a position to state and prove the main result of this chapter, the dual characterisation of ρ-arbitrage.

Theorem 3.2.18. *Suppose $\mathcal{Q}$ satisfies Condition I, $\Pi_0^\rho = \{\mathbf{0}\}$, and $\tilde{\mathcal{Q}}_{\max} \neq \emptyset$. Then the following are equivalent:*

(a) *The market (S^0, S) does not admit ρ-arbitrage.*

(b) *$\tilde{\mathcal{Q}} \cap \mathcal{P} \neq \emptyset$ for some $\emptyset \neq \tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying POS, MIX and INT.*

(c) *$\tilde{\mathcal{Q}} \cap \mathcal{P} \neq \emptyset$ for all $\emptyset \neq \tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT.*

Proof. $(b) \implies (a)$. Let $\emptyset \neq \tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT and $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. By Proposition 3.2.16, we have to show that $\rho(X_\pi) > 0$. Let $\tilde{Z} \in \tilde{\mathcal{Q}} \cap \mathcal{P}$. Then $\mathbb{E}[-\tilde{Z} X_\pi] = 0$. Since $X_\pi \neq 0$ by nonredundancy of the market, this implies that X_π is a non-constant random variable. Now the claim follows from Lemma A.6.3.

$(a) \implies (c)$. We argue by contraposition. So assume that there exists $\emptyset \neq \tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT such that $\tilde{\mathcal{Q}} \cap \mathcal{P} = \emptyset$. This implies that $\tilde{\mathcal{Q}} \cap \mathcal{M} = \emptyset$ by Condition POS. Refining the argument of Proposition 3.2.17, it suffices to show that $\mathbf{0}$ is not in the interior of $C_{\mathcal{Q}}$. Seeking a contradiction, assume that $\mathbf{0} \in C_{\mathcal{Q}}^{\mathrm{o}}$. Then there is $\varepsilon > 0$ such that $B(\mathbf{0}, \varepsilon) \subset \mathcal{Q}$, where $B(\mathbf{0}, \varepsilon)$ denotes the open ball of of radius $\varepsilon > 0$ around $\mathbf{0}$ with respect to some norm $\|\cdot\|$. Set

$$C_{\tilde{\mathcal{Q}}} := \{\mathbb{E}[-Z(R - r\mathbf{1})] : Z \in \tilde{\mathcal{Q}}\} \subset C_{\mathcal{Q}} \subset \mathbb{R}^d.$$

Then $C_{\tilde{\mathcal{Q}}}$ is convex by Remark 3.2.8(a) and does not contain the origin because $\tilde{\mathcal{Q}} \cap \mathcal{M} = \emptyset$. Hence, $B(\mathbf{0}, \varepsilon) \not\subset C_{\tilde{\mathcal{Q}}}$. As $\tilde{\mathcal{Q}} \neq \emptyset$, there is $\mathbf{x} \in C_{\tilde{\mathcal{Q}}}$. Set $\mathbf{y} := -\varepsilon/(2\|\mathbf{x}\|)\mathbf{x} \in B(\mathbf{0}, \varepsilon)$. Then $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} = \mathbf{0}$ for $\lambda := \varepsilon/(2\|\mathbf{x}\| + \varepsilon)$. Letting $Z_{\mathbf{x}} \in \tilde{\mathcal{Q}}$ and $Z_{\mathbf{y}} \in \mathcal{Q}$ denote Radon-Nikodým derivatives corresponding to $\mathbf{x}$ and $\mathbf{y}$, respectively, it follows from definition of $\mathcal{M}$ in (3.13) and Condition MIX that $\lambda Z_{\mathbf{x}} + (1 - \lambda) Z_{\mathbf{y}} \in \tilde{\mathcal{Q}} \cap \mathcal{M}$, in contradiction to $\tilde{\mathcal{Q}} \cap \mathcal{M} = \emptyset$.

$(c) \implies (b)$. This is trivial. □

Remark 3.2.19. (a) While $\tilde{\mathcal{Q}}_{\max} \neq \emptyset$ is the minimal theoretical condition for Theorem 3.2.18 to hold (see Example A.3.7 for a counterexample if is not satisfied), it is difficult to check in practise since we rarely can compute $\tilde{\mathcal{Q}}_{\max}$; cf. Remark 3.2.10(b). Instead, it is easier (and of course sufficient) to check that $\tilde{\mathcal{Q}} \neq \emptyset$ for some $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT. In all our examples, the latter is done by showing that $1 \in \tilde{\mathcal{Q}}$, which by Proposition 3.2.11 also implies that $\Pi_0^\rho = \{\mathbf{0}\}$.

(b) If we choose for ρ the worst-case risk measure, we recover a *refined version* of the fundamental theorem of asset pricing in a one-period model; see Theorem 3.3.2 below for details. In this case, the proof is particularly simple. To the best of our knowledge, the argument (for the nontrivial direction) is new, even simpler than any of the existing proofs (cf. e.g. [45, Theorem 1.7]) and yields a much sharper result.

3.3 Examples

In this section, we apply the preceding sections to various examples of risk measures. Recall that we have already investigated the case of elliptically distributed returns in Section 3.1.4. Here, we do not make any assumptions on the returns, other than our standing assumptions that returns are contained in some Riesz space and that the market (S^0, S) is nonredundant and nondegenerate.

3.3.1 Worst-Case Risk Measure

We start our discussion by looking at the worst-case risk measure WC : $L^1 \to (-\infty, \infty]$ given by $\mathrm{WC}(X) := \operatorname{ess\,sup}(-X)$. It is a coherent risk measure and admits a dual representation with maximal dual set $\mathcal{Q}_\rho = \mathcal{D}$. However, if the returns do not lie in L^∞, Condition I is not satisfied. Therefore, we look for a smaller dual set, and it turns out that a good choice is $\mathcal{Q} := \mathcal{D} \cap L^\infty$; see Proposition A.6.6. Using this $\mathcal{Q}$, Condition I is always satisfied. By contrast, Condition UI is never satisfied unless Ω is finite. It is not difficult to check that $\tilde{\mathcal{Q}} = \{Z \in \mathcal{D} \cap L^\infty : Z > 0 \ \mathbb{P}\text{-a.s.}\}$ satisfies conditions POS, MIX and INT. However, it turns out that we get a stronger dual characterisation of WC-arbitrage if we consider the set

$$\hat{\mathcal{Q}} := \{Z \in \mathcal{D} \cap L^\infty : Z \geq \varepsilon \ \mathbb{P}\text{-a.s. for some } \varepsilon > 0\},$$

which also satisfies Conditions POS, MIX and INT. Since $1 \in \hat{\mathcal{Q}}$, it follows from Proposition 3.2.11 that $\Pi_0^{\mathrm{WC}} = \{\mathbf{0}\}$. Theorems 3.2.18 and 3.2.14 now give the following result.

Corollary 3.3.1. *The market (S^0, S) does not admit* WC*-arbitrage if and only if there is $Z \in \mathcal{P} \cap L^\infty$ with $Z \geq \varepsilon$ $\mathbb{P}$-a.s. for some $\varepsilon > 0$. Moreover, if Ω*

is finite, (S^0, S) *does not admit strong* WC-*arbitrage if and only if* $\mathcal{M} \neq \emptyset$.

Combining Proposition 3.1.21 with Corollary 3.3.1 gives a *refined* version of the one-period fundamental theorem of asset pricing for L^1-markets (with trivial initial information). The refinement is that we show the existence of an EMM with a *positive lower bound*.

Theorem 3.3.2. *Suppose that the market* (S^0, S) *has finite first moments.*

(a) *The market does not admit arbitrage of the first kind if and only if there exists* $Z \in \mathcal{P} \cap L^\infty$ *with* $Z \geq \varepsilon$ $\mathbb{P}$*-a.s. for some* $\varepsilon > 0$.

(b) *If* Ω *is finite, the market does not admit arbitrage of the second kind if and only if* $\mathcal{M} \neq \emptyset$.

Remark 3.3.3. To the best of our knowledge, a simple proof for the existence of an EMM with positive lower bound for arbitrage-free L^1-markets (with trivial initial information) has not been given before. In fact, the only extant result that we are aware of that gives this lower bound for L^1-markets is [82, Corollary 2], which uses very heavy machinery from functional analysis. (Under stronger integrability conditions on the market, the result has also been established by [74, Remark 7.5].) By contrast our proof is elementary and short, and might even be given in a classroom setting.

3.3.2 Value at Risk and Expected Shortfall

We have already introduced VaR and ES in Definition 2.1.1. Since VaR has no dual representation, we cannot apply the results from Section 3.2. However, using the inequality $\mathrm{VaR}^\alpha(X) \leq \mathrm{ES}^\alpha(X)$ for $\alpha \in (0, 1)$ and $X \in L^1$, it follows that if there is (strong) ES^α-arbitrage, then there is (strong) VaR^α-arbitrage.

As mentioned in Remark 2.1.2, unlike VaR, ES is coherent and admits for $\alpha \in (0, 1)$ the following dual representation:

$$\mathrm{ES}^\alpha(X) = \sup_{Z \in \mathcal{Q}^\alpha} \mathbb{E}[-ZX], \quad \text{where} \quad \mathcal{Q}^\alpha := \{Z \in \mathcal{D} : \|Z\|_\infty \leq \tfrac{1}{\alpha}\}. \tag{3.14}$$

This can be extended to include $\alpha \in \{0, 1\}$, where $\mathcal{Q}^0 := \mathcal{D} \cap L^\infty$ and $\mathcal{Q}^1 := \{1\}$ only "contains" the real-world measure $\mathbb{P}$. Note that $\mathrm{ES}^1(X) = \mathbb{E}[-X]$; ES^0 corresponds to the worst-case risk measure considered in Section 3.3.1.

For $\alpha \in (0,1)$, Conditions I and UI are satisfied for ES^α and $\mathcal{Q}^\alpha$ is closed in L^1. Moreover, Proposition A.6.7 shows that

$$\tilde{\mathcal{Q}}^\alpha := \{Z \in \mathcal{D} : Z > 0 \ \mathbb{P}\text{-a.s. and } \|Z\|_\infty < \tfrac{1}{\alpha}\} \tag{3.15}$$

satisfies Conditions POS, MIX and INT. Note that $1 \in \tilde{\mathcal{Q}}^\alpha$. Using Proposition 3.2.11 together with Theorems 3.2.14 and 3.2.18, we arrive at the following complete description of mean-ES portfolio selection:

Theorem 3.3.4. *Fix $\alpha \in (0,1)$. Then $\Pi_\nu^{\mathrm{ES}^\alpha}$ is nonempty, compact and convex for $\nu \geq 0$. Moreover:*

(a) *The market (S^0, S) does not admit strong ES^α-arbitrage if and only if there exists $Z \in \mathcal{M}$ such that $\|Z\|_\infty \leq \frac{1}{\alpha}$.*

(b) *The market (S^0, S) does not admit ES^α-arbitrage if and only if there exists $Z \in \mathcal{P}$ such that $\|Z\|_\infty < \frac{1}{\alpha}$.*

Remark 3.3.5. (a) It straightforward to check that

$$\hat{\mathcal{Q}}^\alpha := \{Z \in \mathcal{D} : \text{there exists } \varepsilon > 0 \text{ such that } Z \geq \varepsilon \ \mathbb{P}\text{-a.s. and } \|Z\|_\infty < \tfrac{1}{\alpha}\}$$

is nonempty and satisfies POS, MIX and INT. Thus, Theorem 3.3.4(b) can be strengthened: The market (S^0, S) does not admit ES^α-arbitrage if and only if there exists $Z \in \mathcal{P}$ with $Z \geq \varepsilon$ $\mathbb{P}$-a.s. for some $\varepsilon > 0$ and $\|Z\|_\infty < \frac{1}{\alpha}$.

(a) Revisiting the toy example from the introduction; using this result together with (2.7) and the fact

$$\mathcal{M} = \{(q_1, q_2, q_3) \in [0,1]^3 : q_1 \in [0, 1/3], q_2 = 1 - 3q_1 \text{ and } q_3 = 2q_1\}$$

yields that the no-strong-ES^α-arbitrage pricing interval for S^2 is given by

$$\{\mathbb{E}[Z S_1^2] : Z \in \mathcal{M} \text{ and } \|Z\|_\infty \leq \tfrac{1}{\alpha}\} = [\max(0, \tfrac{1}{3} - \tfrac{1}{9\alpha}), \min(\tfrac{1}{3}, \tfrac{1}{6\alpha})].$$

And the no-ES^α-arbitrage pricing interval for S^2 is given by

$$\{\mathbb{E}[Z S_1^2] : Z \in \mathcal{P} \text{ and } \|Z\|_\infty < \tfrac{1}{\alpha}\} = (\max(0, \tfrac{1}{3} - \tfrac{1}{9\alpha}), \min(\tfrac{1}{3}, \tfrac{1}{6\alpha})).$$

3.3.3 Spectral Risk Measures

Spectral risk measures are mixtures of Expected Shortfall risk measures that were introduced by [1]. Here, we follow the definition of [27], who has studied their finer properties in great detail. For a probability measure μ on $([0,1], \mathcal{B}_{[0,1]})$, the *spectral risk measure* $\rho^\mu : L^1 \to (-\infty, \infty]$ with respect to μ is given by

$$\rho^\mu(X) := \int_{[0,1]} \mathrm{ES}^\alpha(X)\, \mu(\mathrm{d}\alpha).$$

Remark 3.3.6. (a) If μ does not have an atom at 0, we can define the non-increasing function $\phi^\mu : [0,1] \to \mathbb{R}_+$ by $\phi^\mu(u) := \int_{[u,1]} \frac{1}{\alpha}\, \mu(\mathrm{d}\alpha)$ and write $\rho^\mu(X) := \int_0^1 \phi^\mu(u) \mathrm{VaR}^u(X)\, \mathrm{d}u$. This is the original definition of [1]. Some explicit examples for the choice of μ (or more precisely ϕ^μ) are given in [38].

(b) It was shown in [62, Theorem 7] for the domain L^∞ that on a standard probability space where $\mathbb{P}$ is non-atomic, spectral risk measures coincide with law-invariant, comonotone, coherent risk measures that satisfy the Fatou property. It was then shown in [56] that the Fatou property is automatically satisfied by law-invariant coherent risk measures. The result has then been generalised to L^1 by [72, Theorem 2.45].

Spectral risk measures admit a dual representation. It follows from [27, Theorem 4.4] that the maximal dual set $\mathcal{Q}_{\rho^\mu}$ is L^1-closed and given by

$$\mathcal{Q}_{\rho^\mu} = \left\{ \int_{[0,1]} \zeta_\alpha\, \mu(\mathrm{d}\alpha) : \zeta_\alpha(\omega) \text{ is jointly measurable; } \zeta_\alpha \in \mathcal{Q}^\alpha \text{ for } \alpha \in [0,1] \right\},$$

where $\mathcal{Q}^\alpha$ is as in (3.14). Here, we are in a situation, where it is useful to consider a smaller dual set $\mathcal{Q}' \subset \mathcal{Q}_{\rho^\mu}$ so we can explicitly construct a nonempty subset $\tilde{\mathcal{Q}}' \subset \mathcal{Q}'$ satisfying Conditions POS, MIX and INT. It turns out that a good choice is

$$\mathcal{Q}_\mu = \left\{ \int_{[0,1]} \zeta_\alpha\, \mu(\mathrm{d}\alpha) : \zeta_\alpha(\omega) \text{ is jointly measurable and there is } 1 > \varepsilon > 0 \right.$$
$$\left. \text{such that } \zeta_\alpha \in \mathcal{Q}^\alpha \text{ for } \alpha \le 1 - \varepsilon \text{ and } \zeta_\alpha \equiv 1 \text{ for } \alpha > 1 - \varepsilon \right\},$$

which is an L^1-dense subset of $\mathcal{Q}_{\rho^\mu}$. It is shown in Proposition A.6.8(a) that $\mathcal{Q}_\mu$ also represents ρ^μ. If μ does not have an atom at 0 and $\int_{(0,1]} \frac{1}{\alpha}\, \mu(\mathrm{d}\alpha) < \infty$, it follows that $\mathcal{Q}_\mu$ (and $\mathcal{Q}_{\rho^\mu}$) is bounded in L^∞. Hence, ρ^μ is real-valued and

Condition I and UI are satisfied.

If μ does not have an atom at 1, it follows from Proposition A.6.8(b) that

$$\tilde{\mathcal{Q}}_\mu = \Big\{ \int_{[0,1)} \zeta_\alpha \, \mu(\mathrm{d}\alpha) : \zeta_\alpha(\omega) \text{ is jointly measurable; there is } 0 < \varepsilon < 1 \text{ and}$$
$$0 < \delta < \tfrac{\varepsilon}{1-\varepsilon} \text{ such that } \zeta_\alpha \in \tilde{\mathcal{Q}}^{\alpha(1+\delta)} \text{ for } \alpha \leq 1 - \varepsilon \text{ and } \zeta_\alpha \equiv 1 \text{ for } \alpha > 1 - \varepsilon \Big\},$$

where $\tilde{\mathcal{Q}}^{\alpha(1+\delta)}$ is as in (3.15), satisfies Conditions POS, MIX and INT. Note that $1 \in \tilde{\mathcal{Q}}_\mu$. Using Proposition 3.2.11 together with Theorems 3.2.14 and 3.2.18 we arrive at the following result:

Corollary 3.3.7. *Let μ be a probability measure on $([0,1], \mathcal{B}_{[0,1]})$ such that $\mu(\{0\}) = 0$ and $\int_{(0,1]} \frac{1}{\alpha} \, \mu(\mathrm{d}\alpha) < \infty$. Then $\Pi_\nu^{\rho^\mu}$ is nonempty, compact and convex for $\nu \geq 0$. Moreover:*

(a) *The market (S^0, S) does not admit strong ρ^μ-arbitrage if and only if there exists $Z \in \mathcal{M}$ such that $Z = \int_{[0,1]} \zeta_\alpha \, \mu(\mathrm{d}\alpha)$, where $\zeta_\alpha(\omega)$ is jointly measurable and satisfies $\zeta_\alpha \in \mathcal{D}$ and $\|\zeta_\alpha\|_\infty \leq \frac{1}{\alpha}$.*

(b) *If μ does not have an atom at 1, the market (S^0, S) does not admit ρ^μ-arbitrage if and only if there exists $Z \in \mathcal{P}$, $0 < \varepsilon < 1$ and $0 < \delta < \frac{\varepsilon}{1-\varepsilon}$ such that $Z = \mu((1-\varepsilon, 1)) + \int_{[0,1-\varepsilon]} \zeta_\alpha \, \mu(\mathrm{d}\alpha)$, where $\zeta_\alpha(\omega)$ is jointly measurable and satisfies $\zeta_\alpha \in \mathcal{D}$ and $\|\zeta_\alpha\|_\infty \leq \frac{1}{\alpha(1+\delta)}$ for $\alpha \in [0, 1-\varepsilon]$.*

3.3.4 Coherent Risk Measures on Orlicz Spaces

We proceed to discuss how our main results can be applied to the case where the returns lie in some Orlicz space L^Φ and ρ is real-valued on L^Φ. Risk measures on Orlicz spaces/Orlicz hearts are well studied; see e.g. [25, 49]. Not only do these spaces allow for the inclusion of *unbounded* random variables, there is also an elegant duality theory. For a brief overview of some key definition and results, see Appendix A.2.

We consider the following setup: Let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function and $\rho : L^\Phi \to \mathbb{R}$ a coherent risk measure that is expectation bounded.

We first consider $L^\Phi = L^\infty$, i.e., when Φ jumps to infinity, which is different from all other Orlicz spaces in that the corresponding Orlicz heart is the null space. In this case, ρ admits a dual representation if it satisfies the Fatou property (cf. Theorem A.2.3(c)) and we have the following result.

Corollary 3.3.8. *Let $\rho : L^\infty \to \mathbb{R}$ be an expectation bounded coherent risk measure on L^∞ that satisfies the Fatou property. Let $\mathcal{Q} \subset \mathcal{Q}_\rho$ be a convex subset with $1 \in \mathcal{Q}$ and $\bar{\mathcal{Q}} = \mathcal{Q}_\rho$. Suppose that $R^i \in L^\infty$. If ρ is strictly expectation bounded, then Π^ρ_ν is nonempty, compact and convex for all $\nu \geq 0$. Moreover:*

(a) *If ρ is continuous from below (that is $\rho(X_n) \searrow \rho(X)$ whenever $X_n \nearrow X$ $\mathbb{P}$-a.s.), the market (S^0, S) does not admit strong ρ-arbitrage if and only if $\mathcal{Q}_\rho \cap \mathcal{M} \neq \emptyset$.*

(b) *If there exists $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT with $1 \in \tilde{\mathcal{Q}}$, then the market (S^0, S) does not admit ρ-arbitrage if and only if $\tilde{\mathcal{Q}} \cap \mathcal{P} \neq \emptyset$.*

Proof. The first assertion follows from Theorem 3.1.9 and Corollary 3.1.7. Next, since $R^i \in L^\infty$, Condition UI is satisfied if and only if the dual set $\mathcal{Q}$ is uniformly integrable, which by [45, Corollary 4.35] is equivalent to ρ being continuous from below. Since $\bar{\mathcal{Q}} = \mathcal{Q}_\rho$, part (a) follows from Proposition A.2.5 and Theorem 3.2.14. Finally, Condition I is trivially satisfied and so part (b) follows from Theorem 3.2.18. □

We now consider the case of Orlicz spaces for a finite Young function. See Theorem A.2.3 for conditions under which ρ admits a dual representation.

Corollary 3.3.9. *Let $\Phi : [0, \infty) \to [0, \infty)$ be a finite Young function with conjugate Ψ and $\rho : L^\Phi \to \mathbb{R}$ an expectation bounded coherent risk measure that admits a dual representation. Let $\mathcal{Q} \subset \mathcal{Q}_\rho$ be a convex subset with $1 \in \mathcal{Q}$ and whose closure in L^Ψ is $\mathcal{Q}_\rho$. Suppose that $R^i \in L^\Phi$. If ρ is strictly expectation bounded, then Π^ρ_ν is nonempty, compact and convex for all $\nu \geq 0$. Moreover:*

(a) *If $R^i \in H^\Phi$, the market does not admit strong ρ-arbitrage if and only if $\bar{\mathcal{Q}}_\rho \cap \mathcal{M} \neq \emptyset$, where $\bar{\mathcal{Q}}_\rho$ denotes the closure of $\mathcal{Q}_\rho$ in L^1.*

(b) *If there exists $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT with $1 \in \tilde{\mathcal{Q}}$, then the market (S^0, S) does not admit ρ-arbitrage if and only if $\tilde{\mathcal{Q}} \cap \mathcal{P} \neq \emptyset$.*

Proof. The first assertion follows from Theorem 3.1.9 and Corollary 3.1.7. Next, since $R^i \in H^\Phi$, Condition UI is satisfied by Proposition A.6.5(b), and (a) follows from Proposition A.2.5 and Theorem 3.2.14. Finally, Condition I follows from $R^i \in L^\Phi$ and the generalised Hölder inequality (A.5). Now part (b) follows from Theorem 3.2.18. □

Remark 3.3.10. If Φ does not satisfy the Δ_2-condition and $R^i \in L^\Phi \setminus H^\Phi$ for some $i \in \{1, \dots, d\}$, then it is in general not possible to provide a dual characterisation of strong ρ-arbitrage since condition UI is not satisfied. The reason for this is that Proposition A.6.5 does not extend to L^Φ. However, we can often provide a dual characterisation of ρ-arbitrage since finding $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying Conditions POS, MIX and INT with $1 \in \tilde{\mathcal{Q}}$ is possible in many cases; cf. Corollary 3.3.11.

g-Entropic Risk Measures

We proceed to apply the above results to the class of g-entropic risk measures. The class of g-entropic risk measures was introduced by [3, Definition 5.1]. It is best understood when presented in the context of Orlicz spaces. Let $\Phi : [0, \infty) \to \mathbb{R}$ be a finite superlinear Young function and Ψ its conjugate. Let $g : [0, \infty) \to [0, \infty)$ be a convex function that dominates Ψ. For $\beta > g(1)$, define the risk measure $\rho^{g,\beta} : L^\Phi \to \mathbb{R}$ by

$$\rho^{g,\beta}(X) = \sup_{Z \in \mathcal{Q}^{g,\beta}} \mathbb{E}[-ZX], \quad \text{where} \quad \mathcal{Q}^{g,\beta} := \{Z \in \mathcal{D} : \mathbb{E}[g(Z)] \leq \beta\},$$

and call it the *g-entropic risk measure with divergence level β*. (Note that our definition slightly differs from the definition in [3]; there the domain is L^∞ and it is assumed that g is convex, $(-\infty, \infty]$-valued and satisfies $g(1) = 0$.) By convexity and nonnegativity of g and the fact that g dominates Ψ, it follows that $\mathcal{Q}^{g,\beta}$ is convex, L^Ψ-bounded and L^1-closed. (More precisely, $\|Z\|_\Psi \leq \max(1, \beta)$ for all $Z \in \mathcal{Q}^{g,\beta}$ and L^1-closedness follow from Fatou's lemma.) By Proposition A.2.5, we may deduce that $\mathcal{Q}^{g,\beta} = \mathcal{Q}_{\rho^{g,\beta}}$. Moreover, Proposition A.6.9 shows that

$$\tilde{\mathcal{Q}}^{g,\beta} := \{Z \in \mathcal{D} : Z > 0 \ \mathbb{P}\text{-a.s. and } \mathbb{E}[g(Z)] < \beta\}$$

satisfies Conditions POS, MIX and INT. Note that $1 \in \tilde{\mathcal{Q}}^{g,\beta} \subset \mathcal{Q}^{g,\beta}$. Applying Corollary 3.3.9, we get the following result:

Corollary 3.3.11. *Let $\Phi : [0, \infty) \to \mathbb{R}$ be a superlinear finite Young function with conjugate Ψ, $g : [0, \infty) \to [0, \infty)$ a convex function that dominates Ψ and*

$\beta > g(1)$. *Suppose that* $R^i \in L^\Phi$. *Then* $\Pi_\nu^{\rho^{g,\beta}}$ *is nonempty, compact and convex for all* $\nu \geq 0$. *Moreover:*

(a) *If* $R^i \in H^\Phi$, *the market* (S^0, S) *does not admit strong* $\rho^{g,\beta}$*-arbitrage if and only if there is* $Z \in \mathcal{M}$ *with* $\mathbb{E}[g(Z)] \leq \beta$.

(b) *The market* (S^0, S) *does not admit* $\rho^{g,\beta}$*-arbitrage if and only if there is* $Z \in \mathcal{P}$ *with* $\mathbb{E}[g(Z)] < \beta$.

We finish this section, by providing two specific examples of g-entropic risk measures.

Transformed Norm Risk Measure. Let $p \in (1, \infty)$ and $\alpha \in (0, 1)$. (The case $p = 1$ corresponds to Expected Shortfall, see Section 3.3.2.) Define the *transformed L^p-norm risk measure* with sensitivity parameter α as

$$\rho(X) := \min_{s \in \mathbb{R}} \{ \tfrac{1}{\alpha} \|(s - X)^+\|_p - s\}, \quad X \in L^p.$$

It is shown in [25, Section 5.3] that this is a real-valued coherent risk measure on L^p and admits the dual representation with

$$\mathcal{Q}_\rho = \{Z \in \mathcal{D} : \|Z\|_q \leq \tfrac{1}{\alpha}\},$$

where $q := p/(p-1)$. Hence $\rho = \rho^{g,\beta}$, where $\Phi(x) = x^p/p$, $\Psi(y) = y^q/q$, $g = \Psi$ and $\beta := (\frac{1}{\alpha})^q/q$.

Entropic Value at Risk. The entropic value at risk (EVaR) was introduced in [3] and further studied in [4]. Consider the Young function $\Phi(x) = \exp(x) - 1$ and fix $\alpha \in (0, 1)$. Then the *entropic value at risk at level* α is a risk measure on L^Φ given by

$$\mathrm{EVaR}^\alpha(X) := \inf_{z > 0} \left\{ \frac{1}{z} \log \left(\mathbb{E} \left[\frac{\exp(-zX)}{\alpha} \right] \right) \right\}.$$

(Note that the parametrisation in [4] is different: α is replaced by $1 - \alpha$ and X by $-X$.) By [4, Section 4.4] it admits a dual representation with dual set

$$\mathcal{Q} := \{Z \in \mathcal{D} : \mathbb{E}[Z \log(Z)] \leq -\log(\alpha)\}.$$

Hence, $\mathrm{EVaR}^\alpha = \rho^{g,\beta}$, where $\Psi(y) = (y \log(y) - y + 1)\mathbf{1}_{\{y \geq 1\}}$, $g(y) = y \log(y) - y + 1$ and $\beta := -\log(\alpha)$.

4

Mean-ρ Portfolio Selection for Convex Risk Measures

We have seen in Chapter 2 that it is possible for ES constraints to be ineffective. Chapter 3 shows that the root of this issue stems specifically from positive homogeneity. More precisely, Theorem 3.1.22 implies that regulators cannot exclude (strong) ρ-arbitrage *a priori* when imposing a *positively homogeneous* risk measure – unless ρ is as conservative as the worst-case risk measure. Since a worst-case approach to risk is infeasible in practise, this indicates that one should move *beyond* the class of positively homogeneous risk measures for effective risk constraints in the context of portfolio selection.

Positive homogeneity has been questioned on economic grounds from the very beginning. It triggered the introduction of *convex* risk measures by [44] and [47]. For empirical evidence that decision makers become *more* risk averse in the face of large losses, see [18, 19]. It is easy to check that convexity together with normalisation implies that the risk measure is *star-shaped*, i.e., $\rho(\lambda X) \geq \lambda\rho(X)$ for $\lambda \geq 1$, and this rather than convexity turns out to be minimal property required in the context of portfolio selection. Star-shaped risk measures have recently been studied by [22] in a setting where there is no underlying probability measure.

The goal of this chapter is to study mean-ρ portfolio selection when ρ is a *risk functional*, i.e., star-shaped, monotone and normalised. For some of our results, in particular, for our dual characterisations, we assume in addition that ρ is cash-invariant or convex or satisfies the Fatou property.

Standing Assumption. Throughout the entire chapter we consider the market (S^0, S) described in the introduction. We assume that $L^\infty \subset L \subset L^1$ is a Riesz space with the $\mathbb{P}$-a.s. ordering that contains $\mathcal{X} = \{X_\pi : \pi \in \mathbb{R}^d\}$. We focus on a *risk functional* $\rho : L \to (-\infty, \infty]$, which satisfies the following axioms for $X, Y \in L$:

- Monotonicity: If $X \leq Y$ $\mathbb{P}$-a.s., $\rho(X) \geq \rho(Y)$;
- Normalisation: $\rho(0) = 0$;
- Star-shaped: For $\lambda \geq 1$, $\rho(\lambda X) \geq \lambda \rho(X)$.

Monotonicity together with normalisation imply that 0 lies in the *acceptance set* $\mathcal{A}_\rho := \{X \in L : \rho(X) \leq 0\}$ of ρ and $\mathcal{A}_\rho + L_+ \subset \mathcal{A}_\rho$. Star-shapedness (technically speaking, star-shapedness *about the origin*; see Appendix A.4 for details) captures the idea that a position's risk should increase *at least* proportionally when scaled by a factor greater than one and is economically sounder and strictly weaker than positive homogeneity, where the inequality is replaced by an equality (and λ valued in $(0, \infty)$). It implies that $\mathcal{A}_\rho$ is star-shaped (about the origin).

Our definition of a risk functional is very general, but for some of our results, in particular for our dual characterisations, we also need some of the following axioms:

- *Cash-invariance:* For all $X \in L$ and $c \in \mathbb{R}$, $\rho(X + c) = \rho(X) - c$;
- *Convexity:* For any $X, Y \in L$ and $\lambda \in [0, 1]$,

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y);$$

- *Fatou property on $\mathcal{Y} \subset L$:* If $X_n \to X$ $\mathbb{P}$-a.s. for $X_n, X \in \mathcal{Y}$ and $|X_n| \leq Y$ $\mathbb{P}$-a.s. for some $Y \in L$ then $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$.

All three axioms are widely used in the literature. Cash-invariance allows us to write $\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_\rho\}$ and interpret the value as the minimal amount of capital required to make the position X acceptable. Such risk functionals are fully characterised by their acceptance set. Convexity represents the idea that diversification should not increase risk and implies $\mathcal{A}_\rho$

is convex. Note that under normalisation, convexity implies star-shapedness (about the origin) but the converse is false. Finally, the Fatou property ensures that risk is never underestimated by approximations; for our applications, it sometimes suffices to consider this on a subset $\mathcal{Y} \subset L$.

It will be made clear whenever an additional axiom is assumed. In line with the extant literature, we refer to cash-invariant risk functionals as *risk measures* and positively homogeneous convex risk measures as *coherent risk measures*.

While the key point of this chapter is *not* to insist on positive homogeneity of ρ, it turns out that its smallest positively homogeneous majorant $\rho^\infty : L \to (-\infty, \infty]$ plays a key role. This is also a risk functional. The notation comes from the fact that ρ^∞ is the recession function (see Appendix A.4) of ρ. It is explicitly given by

$$\rho^\infty(X) := \lim_{t\to\infty} \frac{\rho(tX)}{t}. \tag{4.1}$$

For future reference, note that $\mathcal{A}_{\rho^\infty} = (\mathcal{A}_\rho)^\infty$ (where the latter is the recession cone of $\mathcal{A}_\rho$; see Appendix A.4), and if ρ satisfies convexity, cash-invariance or the Fatou property on some $\mathcal{Y} \subset L$, then so does ρ^∞.

4.1 Sensitivity to Large Losses

We start our discussion on mean-ρ portfolio optimisation (concurrently, mean-ρ^∞ portfolio optimisation) by introducing a partial preference order on the set of portfolios.

Definition 4.1.1. A portfolio $\pi \in \mathbb{R}^d$ is *strictly ρ-preferred* over another portfolio $\pi' \in \mathbb{R}^d$ if $\mathbb{E}[X_\pi] \geq \mathbb{E}[X_{\pi'}]$ and $\rho(X_\pi) \leq \rho(X_{\pi'})$, with at least one strict inequality.

As in the previous chapters, we approach mean-ρ portfolio selection by first looking at the slightly simplified problem (1') of finding the minimum risk portfolio(s) given a fixed nonnegative excess return.

Definition 4.1.2. Let $\nu \geq 0$. A portfolio $\pi \in \Pi_\nu$ is called *ρ-optimal* for ν if $\rho(X_\pi) < \infty$ and $\rho(X_\pi) \leq \rho(X_{\pi'})$ for all $\pi' \in \Pi_\nu$. We denote the set of all ρ-optimal portfolios for ν by Π^ρ_ν. Moreover, we set

$$\rho_\nu := \inf\{\rho(X_\pi) : \pi \in \Pi_\nu\},$$

and define the *ρ-optimal boundary* by

$$\mathcal{O}_\rho := \{(\rho_\nu, \nu) : \nu \geq 0\}.$$

4.1.1 Weak Sensitivity to Large Losses

We seek to understand under which conditions ρ-optimal portfolios exist (i.e., address (Q1)) and what properties ρ-optimal sets have. To that end, we introduce the following axiom.

Definition 4.1.3. The risk functional ρ is said to satisfy *weak sensitivity to large losses* on $\mathcal{Y} \subset L$ if for each $X \in \mathcal{Y}$ with $\mathbb{P}[X < 0] > 0$ and $\mathbb{E}[X] = 0$, there exists $\lambda > 0$ such that $\lambda X \notin \mathcal{A}_\rho$.

Remark 4.1.4. (a) To the best of our knowledge, the axiom of weak sensitivity to large losses has not been considered in the literature before; cf. also Remark 4.1.21 below.

(b) Note that ρ satisfies weak sensitivity to large losses on $\mathcal{Y} \subset L$ if and only if ρ^∞ does. When $\mathcal{Y} = L$, this is equivalent to $\mathcal{A}_{\rho^\infty} \setminus \{0\} \subset \{X \in L : \mathbb{E}[X] > 0\}$, from which it follows that $\mathcal{A}_{\rho^\infty}$ is *pointed*, i.e., $\mathcal{A}_{\rho^\infty} \cap (-\mathcal{A}_{\rho^\infty}) = \{0\}$. Pointedness of the recession cone of the acceptance set plays an important role in [8, Section 4]; cf. also Remark 4.2.10(c).

(c) It is often the case (see the examples in Section 4.3) that ρ is weakly sensitive to large losses on the entire space L. This is a more general concept than *strict expectation boundedness* (i.e., $\rho(X) > \mathbb{E}[-X]$ for all non-constant $X \in L$), which was important in Chapter 3. The two properties are equivalent when ρ is a positively homogeneous risk measure.

The financial interpretation of weak sensitivity to large losses on $\mathcal{X}$ is clear: For any portfolio $\pi \in \Pi_0 \setminus \{\mathbf{0}\}$, there is eventually a point where the scaled portfolio $\lambda\pi$ is considered unacceptable. By Theorem 3.1.9, if ρ is a positively homogeneous risk measure, then weak sensitivity to large losses together with the Fatou property implies that Π^ρ_ν is nonempty and compact for all ν with $\rho_\nu < \infty$. The same result also holds for risk functionals and this answers (Q1).

Theorem 4.1.5. *Assume ρ is a risk functional that satisfies the Fatou property on $\mathcal{X}$ and weak sensitivity to large losses on $\mathcal{X}$. Then for any $\nu \geq 0$ with $\rho_\nu < \infty$, Π^ρ_ν is nonempty and compact.*

Proof. Define the function $f_\rho : \mathbb{R}^d \to [0, \infty]$ by

$$f_\rho(\pi) = \max\{\rho(X_\pi), 0\} + |\mathbb{E}[X_\pi]|.$$

Then f_ρ is lower semi-continuous by the Fatou property of ρ on $\mathcal{X}$ (and the fact that $L \supset \mathcal{X}$ is a Riesz space) and linearity of the expectation. Moreover, it is star-shaped, i.e., $f_\rho(\lambda\pi) \geq \lambda f_\rho(\pi)$ for all $\lambda \geq 1$ and $\pi \in \mathbb{R}^d$, by the star-shapedness of ρ and linearity of the expectation.

For $\delta \geq 0$, set $A_\delta := \{\pi \in \mathbb{R}^d : f_\rho(\pi) \leq \delta\}$. Then each A_δ is closed by lower semi-continuity of f_ρ. We proceed to show that each A_δ is also bounded and hence compact.

For $\delta = 0$, using $f_\rho(\pi) \geq |\mathbb{E}[X_\pi]| > 0$ for any $\pi \in \mathbb{R}^d \setminus \Pi_0$, it follows that $A_0 \subset \Pi_0$. Also note that for each $\pi \in A_0$, $X_\pi \in \mathcal{A}_\rho$. If A_0 were unbounded, then Proposition A.6.10(a) would imply the existence of a portfolio $\pi \in \Pi_0 \setminus \{\mathbf{0}\}$ with $\rho(\lambda X_\pi) \leq 0$ for all $\lambda > 0$. But this would contradict ρ being weakly sensitive to large losses on $\mathcal{X}$. Therefore, A_0 must be bounded.

For $\delta > 0$, we argue as follows: Since A_0 is bounded, there exists $d > 0$ such that $f_\rho(\pi) > 0$ for any portfolio π belonging to the set $D := \{x \in \mathbb{R}^d : \|x\|_2 = d\}$. Compactness of D and lower semi-continuity of f_ρ give $m := \min\{f_\rho(x) : x \in D\} \in (0, \infty]$. Star-shapedness of f_ρ in turn implies that $f_\rho(\pi) \geq m\|\pi\|_2/d$ for all $\pi \in \mathbb{R}^d$ with $\|\pi\|_2 \geq d$, which in turn implies that each A_δ is bounded.

We finish by a standard argument. Fix $\nu \geq 0$ and assume $\rho_\nu < \infty$. By definition, there exists a sequence of portfolios $(\pi_n)_{n\geq 1} \subset \Pi_\nu$ such that $\rho(X_{\pi_n}) \searrow \rho_\nu$ and $\rho_\nu + 1 \geq \rho(X_{\pi_n})$ for all n. Setting $\delta^* := \max\{\rho_\nu + 1, 0\} + \nu$, it follows that $(\pi_n)_{n\geq 1} \subset A_{\delta^*}$. Compactness of A_{δ^*}, closedness of Π_ν and the Fatou property of ρ imply the existence of a portfolio $\pi \in \Pi_\nu$ with $\rho(X_\pi) \leq \rho_\nu$, i.e., Π^ρ_ν is nonempty. Furthermore, Π^ρ_ν is bounded since it is a subset of A_{δ^*}, and closed since ρ satisfies the Fatou property. □

Remark 4.1.6. (a) If ρ is convex, then so is Π^ρ_ν. If ρ is even *strictly convex on* Π_ν (i.e., $\rho(\lambda X_\pi + (1-\lambda)X_{\pi'}) < \lambda\rho(X_\pi) + (1-\lambda)\rho(X_{\pi'})$ for all $\lambda \in (0,1)$ and $\pi, \pi' \in \Pi_\nu$ with $\rho(X_\pi), \rho(X_{\pi'}) < \infty$), then Π^ρ_ν is a singleton.

(b) One might wonder what happens when ρ is not weakly sensitive to large losses on $\mathcal{X}$. Then A_0 may be unbounded, in which case we lose the boundedness of the sublevel sets of f_ρ. Then for $\nu > 0$, even if $\rho_\nu < \infty$, Π^ρ_ν can be empty; see Example A.3.1.

(c) The proof of Theorem 4.1.5 does not rely on the monotonicity of ρ and so it can be applied to the class of *deviation risk measures* from [79].

4.1.2 Optimal Boundary

We next seek to understand the map $\nu \mapsto \rho_\nu$ from $\mathbb{R}_+$ to $[-\infty, \infty]$ whose graph corresponds exactly to the ρ-optimal boundary (but with the axes reversed). To this end, it turns out useful to relate the map $\nu \mapsto \rho_\nu$ to the map $\nu \mapsto \rho_\nu^\infty$. We start by stating some basic properties.

Proposition 4.1.7. *For a risk functional ρ, the map $\nu \mapsto \rho_\nu$ from $\mathbb{R}_+$ to $[-\infty, \infty]$ is star-shaped about the origin, i.e., for all $\nu \in \mathbb{R}_+$ and $\lambda \geq 1$, $\rho_{\lambda\nu} \geq \lambda\rho_\nu$. Moreover, $\rho_0 \leq 0$ and the map $\nu \mapsto \rho_\nu^\infty$ from $\mathbb{R}_+$ to $[-\infty, \infty]$ is a positively homogeneous majorant.*

As a consequence of this result, $\mathcal{O}_\rho$ lies to the left of $\mathcal{O}_{\rho^\infty}$ in the mean-risk plane. Moreover, the function $\nu \mapsto \rho_\nu$ is increasing on the interval $\{\nu \in [\nu^+, \infty) : \rho_\nu < \infty\}$ where

$$\nu^+ := \inf\{\nu \geq 0 : \rho_\nu > 0\} \in [0, \infty].$$

However, we lack knowledge concerning its behaviour on $(0, \nu^+)$. The next result shows that weak sensitivity to large losses together with the Fatou property yields a stronger connection between $\mathcal{O}_\rho$ and $\mathcal{O}_{\rho^\infty}$ and gives us information concerning

$$\nu_{\min} := \inf\{\nu \geq 0 : \rho_{\nu'} > \rho_\nu \text{ for all } \nu' > \nu\} \in [0, \nu^+] \quad \text{and}$$
$$\rho_{\min} := \inf\{\rho_\nu : \nu \geq 0\} \in [-\infty, 0].$$

Proposition 4.1.8. *Assume ρ is a risk functional that satisfies the Fatou property on $\mathcal{X}$ and weak sensitivity to large losses on $\mathcal{X}$. Then $\nu \mapsto \rho_\nu$ is $(-\infty, \infty]$-valued, lower semi-continuous, its positively homogeneous majorant is given by $\nu \mapsto \rho_\nu^\infty$ and*

$$\rho_1^\infty > 0 \iff \nu^+ < \infty.$$

Moreover, we have the following three cases:

(a) *If $\rho_1^\infty > 0$, then $\nu_{\min} < \infty$ and $\rho_{\min} = \rho_{\nu_{\min}} \in (-\infty, 0]$.*

(b) *If* $\rho_1^\infty = 0$*, then* $\nu_{\min} \in [0,\infty]$ *and* $\rho_{\min} \in [-\infty, 0]$.

(c) *If* $\rho_1^\infty < 0$*, then* $\nu_{\min} = \infty$ *and* $\rho_{\min} = -\infty$.

Proof. First note that by Theorem 4.1.5, the map $\nu \mapsto \rho_\nu$ is $(-\infty,\infty]$-valued.

Next we establish lower semi-continuity. Fix $y \in \mathbb{R}$ and let $B_y := \{\nu \in \mathbb{R}_+ : \rho_\nu \leq y\}$. We must show that this set is closed. So let $(\nu_n)_{n\geq 1} \subset B_y$ and assume $\nu_n \to \nu$. By Theorem 4.1.5, for each n there exists a portfolio π_n such that $\rho(X_{\pi_n}) = \rho_{\nu_n} \leq y$ and $\mathbb{E}[X_{\pi_n}] = \nu_n$. We proceed to show that the sequence $(\pi_n)_{n\geq 1}$ belongs to a compact set. To this end, let $c \in \mathbb{R}$ be such that $|\nu_n| \leq c$ for all n. Setting $\delta := \max\{y, 0\} + c$ it follows that each π_n lies in $A_\delta := \{\pi \in \mathbb{R}^d : \max\{\rho(X_\pi), 0\} + |\mathbb{E}[X_\pi]| \leq \delta\}$, which is compact by the proof of Theorem 4.1.5. Passing to a subsequence, we may assume that $(\pi_n)_{n\geq 1}$ converges to some $\pi \in \mathbb{R}^d$, and by dominated convergence and the Fatou property, it follows that $\mathbb{E}[X_\pi] = \nu$ and $\rho(X_\pi) \leq y$. Whence, $\rho_\nu \leq y$ and so $\nu \in B_y$.

We now show that $\nu \mapsto \rho_\nu^\infty$ is the smallest positively homoegeneous majorant of $\nu \mapsto \rho_\nu$. Since ρ^∞ is weakly sensitive to large losses, $\rho_0^\infty = 0 \geq \rho_0$. Thus, it suffices to show that

$$\rho_1^\infty = \lim_{t\to\infty} \rho_t / t. \tag{4.2}$$

The key idea is to consider the risk functionals $\rho^t : L \to (-\infty,\infty]$ defined by $\rho^t(X) = \rho(tX)/t$ for $t \geq 1$. They satisfy the Fatou property on $\mathcal{X}$, weak sensitivity to large losses on $\mathcal{X}$ and $\rho_t/t = \rho_1^t$. By star-shapedness of ρ and definition of ρ^∞ in (4.1), we have $\rho^{t+1}(X) \geq \rho^t(X)$ and $\lim_{t\to\infty} \rho^t(X) = \rho^\infty(X)$ for all $X \in L$. This implies $(\rho_1^t)_{t\geq 1}$ is a nondecreasing sequence and

$$\rho_1^\infty \geq m := \lim_{t\to\infty} \rho_1^t.$$

If $m = \infty$, the reverse inequality is clear, so assume $m < \infty$. Then as $\rho^t \geq \rho$ and $m \geq \rho_1^t$ for each $t \geq 1$, it follows that

$$\begin{aligned}\Pi_1^{\rho^t} &\subset \{\pi \in \Pi_1 : \rho(X_\pi) \leq m\} \\ &\subset \{\pi \in \Pi_1 : \max\{\rho(X_\pi), 0\} + |\mathbb{E}[X_\pi]| \leq \max\{m, 0\} + 1\} := K.\end{aligned}$$

Since K is compact by the proof of Theorem 4.1.5, (4.2) follows by applying Proposition A.6.11 to the sequence of functions $f_t : K \to (-\infty,\infty]$ given by

$f_t(\pi) := \rho^t(X_\pi)$.

The statements in (b) and (c) as well as the equivalence between $\rho_1^\infty > 0$ and $\nu^+ < \infty$ follow directly from the fact $\nu \mapsto \rho_\nu^\infty$ is the smallest positively homogeneous majorant of $\nu \mapsto \rho_\nu$.

Finally, we establish (a). If $\rho_1^\infty > 0$, then $\nu^+ < \infty$ by the above and hence $\rho_\nu > 0$ for all $\nu > \nu^+$. By lower semi-continuity of $\nu \mapsto \rho_\nu$ and compactness of $[0, \nu^+]$, there exists a global minimum $m \leq \rho_0 \leq 0$ that is attained at $\nu^* := \sup\{\nu \in [0, \nu^+] : \rho_\nu = m\}$. By construction, $\rho_\nu > m$ for all $\nu > \nu^*$. Whence, by definition $\nu_{\min} = \nu^* < \infty$ and $\rho_{\min} = \rho_{\nu_{\min}} \in (-\infty, 0]$. □

Example A.3.8 shows that even under the setting of Proposition 4.1.8, the shape of $\mathcal{O}_\rho$ may be very irregular. From an economic standpoint, we would like more regularity – in particular, convexity (to account for diversification) and continuity (there should be some continuous progression between risk and return). The next result shows that these properties hold if ρ is convex.

Proposition 4.1.9. *Suppose ρ is a convex risk functional that satisfies the Fatou property on $\mathcal{X}$ and weak sensitivity to large losses on $\mathcal{X}$. Then the map $\nu \mapsto \rho_\nu$ from $\mathbb{R}_+$ to $(-\infty, \infty]$ is convex, continuous on the closed set $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$ and*

$$\rho_1^\infty > 0 \iff \nu^+ < \infty \iff \nu_{\min} < \infty.$$

Moreover, we have the following three cases:

(a) *If $\rho_1^\infty > 0$, the map $\nu \mapsto \rho_\nu$ is nonincreasing on $[0, \nu_{\min}]$, increasing on the closed interval $\{\nu \in [\nu_{\min}, \infty) : \rho_\nu < \infty\}$ and bounded below by $\rho_{\min} = \rho_{\nu_{\min}} \in (-\infty, 0]$.*

(b) *If $\rho_1^\infty = 0$, the map $\nu \mapsto \rho_\nu$ is nonincreasing on $\mathbb{R}_+$ and $\rho_{\min} \in [-\infty, 0]$.*

(c) *If $\rho_1^\infty < 0$, the map $\nu \mapsto \rho_\nu$ is decreasing on $\mathbb{R}_+$ and $\rho_{\min} = -\infty$.*

Proof. First, we establish convexity of $\nu \mapsto \rho_\nu$. Let $\nu, \nu' \in \mathbb{R}_+$, $\lambda \in [0, 1]$ and $A := \Pi_\nu \times \Pi_{\nu'}$. Using convexity of ρ and the fact $(\pi, \pi') \in A$ implies $\lambda\pi + (1-\lambda)\pi' \in \Pi_{\lambda\nu + (1-\lambda)\nu'}$, we obtain

$$\begin{aligned}\rho_{\lambda\nu+(1-\lambda)\nu'} &\leq \inf_{(\pi,\pi')\in A} \{\rho(X_{\lambda\pi+(1-\lambda)\pi'})\} \\ &\leq \inf_{(\pi,\pi')\in A} \{\lambda\rho(X_\pi) + (1-\lambda)\rho(X_{\pi'})\} \leq \lambda\rho_\nu + (1-\lambda)\rho_{\nu'}.\end{aligned}$$

Next, since $\nu \mapsto \rho_\nu$ is convex, it is continuous in the interior of its effective domain $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$, which is an interval. This together with lower semi-continuity shown in Proposition 4.1.8, implies that $\nu \mapsto \rho_\nu$ is finite and continuous on the closure of $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$, which a fortiori implies $\{\nu \in \mathbb{R}_+ : \rho_\nu < \infty\}$ is closed. The other claims follow directly from Propositions 4.1.7 and 4.1.8 together with standard properties of convex functions. □

The above result shows that for a convex risk functional ρ satisfying the Fatou property and weak sensitivity to large losses, $\mathcal{O}_\rho$ is continuous (except where it jumps to ∞) and convex. It has a strong connection with $\mathcal{O}_{\rho^\infty}$ by Proposition 4.1.8, and by Theorem 4.1.5 every point on the ρ-optimal boundary (with finite risk) corresponds to a ρ-optimal portfolio.

4.1.3 Efficient Portfolios

We proceed to study the notion of ρ-efficient portfolios.

Definition 4.1.10. A portfolio $\pi \in \mathbb{R}^d$ is called *ρ-efficient* if $\mathbb{E}[X_\pi] \geq 0$ and there is no other portfolio $\pi' \in \mathbb{R}^d$ that is strictly ρ-preferred over π. We denote the *ρ-efficient frontier* by

$$\mathcal{E}_\rho := \{(\rho(X_\pi), \mathbb{E}[X_\pi]) : \pi \text{ is } \rho\text{-efficient}\} \subset (-\infty, \infty) \times [0, \infty).$$

Remark 4.1.11. It is not difficult to show that if $\pi \in \mathbb{R}^d$ is ρ-efficient then $\rho(X_\pi) \in \mathbb{R}$. Whence, every ρ-efficient portfolio is ρ-optimal.

We begin by looking at the existence of the ρ^∞-efficient frontier. Since ρ^∞ is a positively homogeneous risk functional, when it satisfies the Fatou property and weak sensitivity to large losses on $\mathcal{X}$, it follows by a similar argument as in Proposition 3.1.13 (using also Remark 4.1.4(c)) and Remark 4.1.11 that

$$\mathcal{E}_{\rho^\infty} = \begin{cases} \emptyset, & \text{if } \rho^\infty_1 \leq 0, \\ \{(\nu\rho^\infty_1, \nu) : \nu \geq 0\}, & \text{if } 0 < \rho^\infty_1 < \infty, \\ \{(0,0)\}, & \text{if } \rho^\infty_1 = \infty. \end{cases} \tag{4.3}$$

The case for ρ (which is only star-shaped) is slightly more involved but the existence of the ρ-efficient frontier still crucially depends on the sign of ρ^∞_1.

Proposition 4.1.12. *Assume the risk functional ρ satisfies the Fatou property on $\mathcal{X}$ and weak sensitivity to large losses on $\mathcal{X}$. Then $\rho_1^\infty > 0$ implies $\mathcal{E}_\rho \neq \emptyset$. When ρ is also convex,*

$$\mathcal{E}_\rho = \begin{cases} \emptyset, & \text{if } \rho_1^\infty \leq 0, \\ \{(\rho_\nu, \nu) : \nu \geq \nu_{\min} \text{ and } \rho_\nu < \infty\} \neq \emptyset, & \text{if } \rho_1^\infty > 0. \end{cases}$$

Proof. First, assume $\rho_1^\infty > 0$. Then by Proposition 4.1.8(a), $\nu_{\min} < \infty$ and $\rho_{\min} = \rho_{\nu_{\min}}$. By Theorem 4.1.5, there exists $\pi_{\nu_{\min}} \in \Pi^\rho_{\nu_{\min}}$. It follows by definition that $\pi_{\nu_{\min}}$ is ρ-efficient and so $\mathcal{E}_\rho \neq \emptyset$. If ρ is in addition convex,

$$\mathcal{E}_\rho = \{(\rho_\nu, \nu) : \nu \geq \nu_{\min} \text{ and } \rho_\nu < \infty\}.$$

Indeed, for $\nu < \nu_{\min}$ and $\pi \in \Pi_\nu$, $\pi_{\nu_{\min}}$ is strictly ρ-preferred over π. If $\nu > \nu_{\min}$ and $\rho_\nu = \infty$, then by Remark 4.1.11, $(\rho_\nu, \nu) \notin \mathcal{E}_\rho$. Finally, if $\nu \geq \nu_{\min}$ and $\rho_\nu < \infty$, then there exists $\pi_\nu \in \Pi^\rho_\nu$ by Theorem 4.1.5, and this portfolio must be ρ-efficient as otherwise it would contradict Proposition 4.1.9(a). Whence $(\rho_\nu, \nu) \in \mathcal{E}_\rho$.

Next, assume that ρ is convex and $\rho_1^\infty \leq 0$. When $\rho_1^\infty \leq 0$, by Proposition 4.1.9(b) and (c), any portfolio in Π^ρ_ν is strictly ρ-preferred over the portfolios in $\Pi^\rho_{\nu'}$ for $\nu > \nu' \geq 0$. Therefore, there cannot be any ρ-efficient portfolios, i.e., $\mathcal{E}_\rho = \emptyset$. □

Combining the above result with (4.3) indicates a further relationship between ρ and ρ^∞: Under the assumptions of Proposition 4.1.12, $\mathcal{E}_\rho \neq \emptyset$ if (and only if, when ρ is also convex) $\mathcal{E}_{\rho^\infty} \neq \emptyset$. Figure 7 gives a graphical illustration.

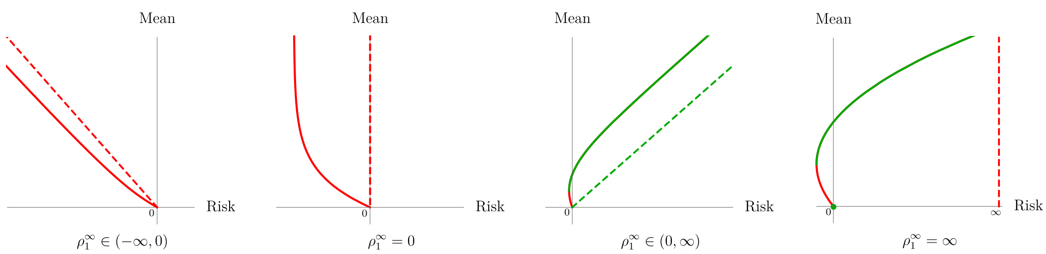


Figure 7: The ρ-optimal boundary (solid) and ρ-efficient frontier (green solid) with the ρ^∞-optimal boundary (dashed) and ρ^∞-efficient frontier (green dashed) when ρ satisfies convexity, the Fatou property on $\mathcal{X}$, weak sensitivity to large losses on $\mathcal{X}$, $\rho_0 = 0$ and $\rho_\nu < \infty$ for all $\nu \geq 0$. The green dot in the right lower panel indicates the efficient frontier for ρ^∞.

4.1.4 (Strong) ρ-Arbitrage

We have seen above that mean-ρ portfolio selection is not always well-defined as it can happen that there are no ρ-efficient portfolios (even if ρ-optimal portfolios exist). To reiterate, this is highly undesirable since it means that for every portfolio there exists another one that dominates it. An even worse scenario would be the existence of a sequence of portfolios whose expectation increases to ∞ whilst simultaneously the risk decreases to $-\infty$. Just as before, we refer to these situations as *ρ-arbitrage* and *strong ρ-arbitrage*, respectively.

Definition 4.1.13. The market (S^0, S) is said to satisfy *ρ-arbitrage* if there are no ρ-efficient portfolios. It is said to satisfy *strong ρ-arbitrage* if there exists a sequence of portfolios $(\pi_n)_{n\geq 1} \subset \mathbb{R}^d$ such that

$$\mathbb{E}[X_{\pi_n}] \uparrow \infty \quad \text{and} \quad \rho(X_{\pi_n}) \downarrow -\infty.$$

It is clear that strong ρ-arbitrage implies ρ-arbitrage but not vice versa. The following two results give primal characterisations in terms of the sign of ρ_1^∞. In particular, note that ρ-arbitrage is fully characterised by the sign of ρ_1^∞ when ρ is convex and satisfies the Fatou property and weak sensitivity to large losses on $\mathcal{X}$.

Proposition 4.1.14. *We have that* (a) $\iff$ (b) $\implies$ (c) *for the statements:*

(a) $\rho_1^\infty < 0$.

(b) *The market (S^0, S) admits strong ρ^∞-arbitrage.*

(c) *The market (S^0, S) admits strong ρ-arbitrage.*

Proof. "(a) $\iff$ (b)". This is Theorem 3.1.16.

"(b) $\implies$ (c)". This follows from the definition of strong ρ-arbitrage and the fact ρ^∞ dominates ρ. □

Remark 4.1.15. It follows from Remark 4.3.18 that the implication "(c) $\implies$ (b)" does not hold even if ρ is convex, satisfies the Fatou property and weak sensitivity to large losses on $\mathcal{X}$.

Proposition 4.1.16. *Assume ρ is convex and satisfies the Fatou property on $\mathcal{X}$ and weak sensitivity to large losses on $\mathcal{X}$. Then the following are equivalent:*

(a) $\rho_1^\infty > 0$.

(b) *The market* (S^0, S) *does not admit* ρ^∞*-arbitrage.*

(c) *The market* (S^0, S) *does not admit* ρ*-arbitrage.*

If ρ *is not convex, then* (a) $\iff$ (b) $\implies$ (c).

Proof. This follows directly from (4.3), Proposition 4.1.12 and the definition of ρ-arbitrage. □

Remark 4.1.17. (a) The direction "(c) $\implies$ (b)" is wrong without convexity; see Example A.3.9 for a counterexample.

(b) By Proposition 4.1.16 and Remark 4.1.4(b), when ρ is convex, weakly sensitive to large losses and satisfies the Fatou property, then the market admits ρ-arbitrage if and only if there exists a portfolio $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ (in fractions of wealth) with $X_\pi \in \mathcal{A}_{\rho^\infty}$. This is equivalent to the existence of a portfolio $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d} \setminus \{\mathbf{0}\}$ (in numbers of shares) such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0 \quad \text{and} \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \in \mathcal{A}_\rho^\infty.$$

This is referred to as a *(strong) scalable acceptable deal* by [8].

Proposition 4.1.16 goes a long way in providing an answer to (Q2) from the introduction. Indeed, when ρ satisfies weak sensitivity to large losses and the Fatou property on $\mathcal{X}$, the market does not admit ρ-arbitrage if (and only if, when ρ is also convex) $\rho_1^\infty > 0$. However, this criterion is rather indirect. We now focus on giving more explicit criteria.

Remark 4.1.18. As in Chapter 3, the primal characterisations of (strong) ρ-arbitrage are particularly useful when returns are *elliptically distributed* with finite second moments and the risk functional satisfies cash-invariance and *law-invariance.* (Note that ρ^∞ inherits these.) We do not give details here.

4.1.5 Strong Sensitivity to Large Losses

One clear case where $\rho_1^\infty \leq 0$ is when the market admits *arbitrage* (of the first kind), i.e., there exists a trading strategy $(\vartheta^0, \vartheta) \in \mathbb{R}^{1+d}$ (parametrised in numbers of shares) such that

$$\vartheta^0 S_0^0 + \vartheta \cdot S_0 \leq 0, \quad \vartheta^0 S_1^0 + \vartheta \cdot S_1 \geq 0 \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[\vartheta^0 S_1^0 + \vartheta \cdot S_1 > 0] > 0.$$

Proposition 4.1.19. *If the market* (S^0, S) *admits arbitrage and* ρ *is a risk functional, then* $\rho_1^\infty \leq 0$ *and the market admits* ρ*-arbitrage.*

This result shows that it is necessary that the market is arbitrage-free in order for $\rho_1^\infty > 0$. However, it is not sufficient: For example, if ρ is any positively homogeneous risk measure that is not the worst-case risk measure, it follows from Theorems 3.1.16 and 3.1.22 that there exists an arbitrage-free market (S^0, S) such that $\rho_1 = \rho_1^\infty < 0$. To obtain a sufficient condition, we introduce the following axiom, which is a stronger version of weak sensitivity to large losses.

Definition 4.1.20. The risk functional ρ is said to satisfy *strong sensitivity to large losses* on $\mathcal{Y} \subset L$ if for each $X \in \mathcal{Y}$ with $\mathbb{P}[X < 0] > 0$, there exists $\lambda > 0$ such that $\lambda X \notin \mathcal{A}_\rho$.

Remark 4.1.21. (a) By star-shapedness, ρ satisfies strong sensitivity to large losses if and only if for each $X \in \mathcal{Y}$ with $\mathbb{P}[X < 0] > 0$, $\lim_{\lambda \to \infty} \rho(\lambda X) = \infty$. This (up to a different sign convention) is the formulation in which this axiom was considered by [24]. Note that we have added the qualifier "strong" to better distinguish it from our axiom of weak sensitivity to large losses. To the best of our knowledge, the property (unnamed) was first considered by [30, Remark 2.7], where the authors write that this "could be quite a desirable feature for potential applications".

(b) It follows directly from (4.1) that ρ satisfies strong sensitivity to large losses on $\mathcal{Y} \subset L$ if and only if ρ^∞ does. When $\mathcal{Y} = L$, this is equivalent to $\mathcal{A}_{\rho^\infty} = L_+$. If we further assume that ρ is cash-invariant, this implies that ρ^∞ is the worst-case (WC) risk measure

$$\mathrm{WC}(X) := \operatorname{ess\,sup}(-X), \quad X \in L.$$

(c) Strong sensitivity to large losses implies weak sensitivity to large losses but the converse is not true; e.g., consider ES^α for $\alpha \in (0, 1)$, which is weakly, but not strongly, sensitive to large losses on L^1.

The financial interpretation of strong sensitivity to large losses is that scaling magnifies gains but it also *amplifies losses*, and at some point, very large losses outweigh very large gains. It ensures that no matter how "good" a portfolio π is, if there is a possibility that it makes a loss, then for λ large enough, the scaled portfolio $\lambda\pi$ is unacceptable as it may leave you with an extreme amount of debt. The following result shows that this property is *exactly* what is required on top of absence of arbitrage to ensure that $\rho_1^\infty > 0$.

Lemma 4.1.22. *Assume the market (S^0, S) is arbitrage-free and ρ is a risk functional that satisfies the Fatou property on $\mathcal{X}$. If ρ is strongly sensitive to large losses on $\mathcal{X}$, then $\rho_1^\infty > 0$. The converse is also true if ρ is weakly sensitive to large losses on L.*

Proof. Suppose ρ and hence ρ^∞ is strongly sensitive to large losses on $\mathcal{X}$. If $\rho_1^\infty = \infty$, we are done, so assume $\rho_1^\infty < \infty$. By Theorem 4.1.5, there exists $\pi^* \in \Pi_1^{\rho^\infty}$ with $\rho_1^\infty = \rho^\infty(X_{\pi^*})$. By the no-arbitrage assumption, $\mathbb{P}[X_{\pi^*} < 0] > 0$, and so by strong sensitivity to large losses, there exists $\lambda > 0$ such that $\rho^\infty(\lambda X_{\pi^*}) > 0$. As ρ^∞ is positively homogeneous, this means $\lambda\rho_1^\infty > 0$, i.e., $\rho_1^\infty > 0$.

Conversely, if ρ and hence ρ^∞ is not strongly sensitive to large losses on $\mathcal{X}$, then there exists $\pi \in \mathbb{R}^d$ with $\mathbb{P}[X_\pi < 0] > 0$ and $\rho^\infty(\lambda X_\pi) \leq 0$ for all $\lambda > 0$. If $\mathbb{E}[X_\pi] \leq 0$, then by monotonicity, $\rho^\infty(\lambda Y) \leq 0$ for all $\lambda > 0$ where $Y := X - \mathbb{E}[X] \geq X$ $\mathbb{P}$-a.s. and $\mathbb{E}[Y] = 0$. But this contradicts the fact ρ and hence ρ^∞ is weakly sensitive to large losses on L. Whence, $\mathbb{E}[X_\pi] > 0$ and so $\rho_1^\infty \leq \rho^\infty(X_\pi/\mathbb{E}[X_\pi]) \leq 0$. □

With this, we have the following more direct answer to (Q2) for the primal characterisation of ρ-arbitrage.

Theorem 4.1.23. *Assume ρ is a convex risk functional that satisfies the Fatou property on $\mathcal{X}$ and is weakly sensitive to large losses on L. Then the following are equivalent:*

(a) *The market (S^0, S) is arbitrage-free and ρ satisfies strong sensitivity to large losses on $\mathcal{X}$.*

(b) *The market (S^0, S) does not admit ρ^∞-arbitrage.*

(c) *The market (S^0, S) does not admit ρ-arbitrage.*

If ρ is not convex, then (a) $\iff$ (b) $\implies$ (c).

Proof. By Lemma 4.1.22 and Proposition 4.1.19, (a) is equivalent to $\rho_1^\infty > 0$. This in turn is equivalent to (b) by Proposition 4.1.16. Furthermore, by Proposition 4.1.16, (b) implies (c), and the converse is also true if ρ is in addition convex. □

Remark 4.1.24. Strong sensitivity to large losses is also important if we want a risk regulation which protects liability holders. Indeed, if this axiom is satisfied, one cannot exploit the acceptability of a certain position by rescaling it without consequences. Protection of liability holders was an argument used in [59] to highlight that a regulation based on coherent risk measures, such as ES, may be ineffective.

4.1.6 Suitability for Risk Management and Portfolio Selection

We now focus on (Q3) from the introduction. To that end, we introduce the following concept.

Definition 4.1.25. A risk functional $\rho : L \to (-\infty, \infty]$ is called *suitable for risk management* if every nonredundant nondegenerate market (S^0, S) with returns in L that satisfies no-arbitrage does not admit strong ρ-arbitrage.

The absence of strong ρ-arbitrage is the main priority for a risk manager. They want to avoid situations where there is a sequence of portfolios whose reward increases to ∞ and risk decreases to $-\infty$. The following result shows that strong sensitivity to large losses is a sufficient (and also necessary under cash-invariance) condition to ensure that a risk functional is suitable for risk management.

Lemma 4.1.26. *Let $\rho : L \to (-\infty, \infty]$ be a risk functional. If ρ satisfies strong sensitivity to large losses, then ρ is suitable for risk management. The converse is also true if ρ is cash-invariant.*

Proof. Assume ρ is strongly sensitive to large losses on L but suppose that ρ is not suitable for risk management, i.e., there exists a nonredundant nondegenerate market (S^0, S) with returns in L that satisfies no-arbitrage and admits strong ρ-arbitrage. Then by definition, there exists a sequence of portfolios $(\pi_n)_{n \geq 1} \subset \mathbb{R}^d$ such that $\rho(X_{\pi_n}) \downarrow -\infty$ and $\mathbb{E}[X_{\pi_n}] \uparrow \infty$. As $\mathbb{E}[X_{\pi_n}] \uparrow \infty$, this means $\|\pi_n\| \to \infty$ so by Proposition A.6.10(b) there exists $Y \in \mathcal{A}_{\rho^\infty}$ such that $\mathbb{P}[Y < 0] > 0$. By Remark 4.1.21(b), this contradicts the fact ρ satisfies strong sensitivity to large losses on L.

When ρ is cash-invariant and not strongly sensitive to large losses, then by Remark 4.1.21(b), ρ^∞ is not the worst-case risk measure. Then Theorem 3.1.22

implies the existence of a nonredundant nondegenerate market with returns in L that admits strong ρ^∞-arbitrage. As ρ is dominated by ρ^∞, the market must also admit strong ρ-arbitrage. $\square$

Remark 4.1.27. In the absence of cash-invariance, if ρ is suitable for risk management, then it is not necessarily strongly sensitive to large losses, e.g., $\rho \equiv 0$.

While suitability for risk management is an important concept, it says nothing about the mean-ρ problems (1) and (2). Thus, we introduce a (seemingly) stronger notion of suitability.

Definition 4.1.28. A risk functional $\rho : L \to (-\infty, \infty]$ is called *suitable for portfolio selection* if for every nonredundant nondegenerate market (S^0, S) with returns in L that satisfies no-arbitrage, and every $\nu^* \geq 0$ and $\rho^* \geq 0$, the mean-ρ problems (1) and (2) each have at least one solution with finite risk.

Remark 4.1.29. (a) In order for ρ to be suitable for portfolio selection, it must be finite on $\{X \in L : \mathbb{E}[X] > 0\}$. Indeed, otherwise there is $Y \in L$ with $\mathbb{E}[Y] > 0$ and $\rho(Y) = \infty$. By normalisation and monotonicity, it must be that $\mathbb{P}[Y < 0] > 0$. Consider the market (S^0, S) with $S^0 \equiv 1$ and $S := S^1$, where $S^1_0 = 1$ and $S^1_1 = 1 + Y$. This is nonredundant, nondegenerate and arbitrage-free. However, the mean-ρ problem (1) has no solution with finite risk for any $\nu^* > 0$.

(b) Note that if ρ is a risk measure, then by cash-invariance, ρ is real-valued on $\{X \in L : \mathbb{E}[X] > 0\}$ if and only if it is real-valued everywhere.

A risk functional ρ that is suitable for portfolio selection is desirable from an investor's perspective as efficient portfolios exist, and from a regulatory point of view since it is clearly suitable for risk management. The following result in conjunction with the previous lemma give a complete primal answer to (Q3).

Lemma 4.1.30. *Let $\rho : L \to (-\infty, \infty]$ be a convex risk functional. If ρ is suitable for portfolio selection, then ρ is strongly sensitive to large losses on L and real-valued on $\{X \in L : \mathbb{E}[X] > 0\}$. The converse is also true if ρ satisfies the Fatou property.*

Proof. If ρ is not real-valued on $\{X \in L : \mathbb{E}[X] > 0\}$, then ρ is not suitable for portfolio selection by Remark 4.1.29. If ρ, and hence ρ^∞, does not satisfy strong sensitivity to large losses on L, then there exists $X \in \mathcal{A}_{\rho^\infty}$ with $\mathbb{P}[X < 0] > 0$. Without loss of generality, we may assume X is not a constant $\mathbb{P}$-a.s., otherwise we can simply replace X by $X + \mathbb{1}_A$ where $A \in \mathcal{F}$ is such that $\mathbb{P}[A] \in (0,1)$. Since X is not a constant, $\mathbb{P}[B] \in (0,1)$ where $B := \{X \geq \mathbb{E}[X]\}$. Now let

$$Y := \begin{cases} X + \frac{(1-\mathbb{E}[X])\mathbb{1}_B}{\mathbb{P}[B]}, & \text{if } \mathbb{E}[X] \leq 0, \\ X, & \text{if } \mathbb{E}[X] > 0. \end{cases}$$

Then $\mathbb{P}[Y < 0] > 0$, $\mathbb{E}[Y] > 0$, and by monotonicity, $Y \in \mathcal{A}_{\rho^\infty}$. Consider the market (S^0, S) with $S^0 \equiv 1$ and $S := S^1$, where $S^1_0 = 1$ and $S^1_1 = 1 + Y$. This is nonredundant, nondegenerate and arbitrage-free. However, as $\rho^\infty_1 = \rho^\infty(Y)/\mathbb{E}[Y] \leq 0$, the market admits ρ-arbitrage by Proposition 4.1.16. Thus, ρ is not suitable for portfolio selection.

Conversely, assume that ρ is convex, strongly sensitive to large losses, real-valued on $\{X \in L : \mathbb{E}[X] > 0\}$, and satisfies the Fatou property. Let (S^0, S) be a nonredundant nondegenerate market with returns in L that satisfies no-arbitrage. Then $\rho_\nu < \infty$ for all $\nu > 0$, and hence $\mathcal{E}_\rho = \{(\rho_\nu, \nu) : \nu \geq \nu_{\min}\} \neq \emptyset$ by Proposition 4.1.12 and Lemma 4.1.22. Moreover, by Proposition 4.1.9, the map $\nu \mapsto \rho_\nu$ is increasing for $\nu \geq \nu_{\min}$. It follows that for any $\nu^* \geq 0$ and $\rho^* \geq 0$, the mean-ρ problems both admit solutions with finite risk. $\square$

A particularly striking consequence of the previous two results is that for a wide class of practically important risk measures, suitability for risk management is equivalent to suitability for portfolio selection.

Theorem 4.1.31. *Let $\rho : L \to \mathbb{R}$ be a convex risk measure that satisfies the Fatou property. The following are equivalent:*

(a) *ρ is suitable for risk management.*

(b) *ρ is suitable for portfolio selection.*

(c) *ρ satisfies strong sensitivity to large losses on L.*

Remark 4.1.32. Convex real-valued risk measures on Orlicz hearts automatically satisfy the Fatou property as a consequence of [25, Theorem 4.3]. Thus, in Theorem 4.1.31 we can drop the Fatou property when L is an Orlicz heart. Similarly in Lemma 4.1.30, if we start with a convex risk measure on an Orlicz heart, then we can drop the Fatou property.

Convex risk measures typically admit a dual representation. Therefore we turn to the dual characterisation of (strong) ρ-arbitrage in the next section and provide a dual description of when they are suitable for portfolio selection.

4.2 Dual Results

In this section, we consider the case that ρ is a convex risk measure on L that admits a dual representation. There are many relevant examples that fall into this category, cf. Section 4.3.

Let $\mathcal{D} := \{Z \in L^1 : Z \geq 0 \ \mathbb{P}\text{-a.s. and } \mathbb{E}[Z] = 1\}$ be the set of all Radon-Nikodým derivatives of probability measures that are absolutely continuous with respect to $\mathbb{P}$. Throughout this section, we assume that $\rho : L \to (-\infty, \infty]$ admits a dual representation

$$\rho(X) = \sup_{Z \in \mathcal{D}} \{\mathbb{E}[-ZX] - \alpha(Z)\} = \sup_{Z \in \mathcal{Q}^\alpha} \{\mathbb{E}[-ZX] - \alpha(Z)\}, \tag{4.4}$$

for some *penalty function* $\alpha : \mathcal{D} \to [0, \infty]$ with effective domain $\mathcal{Q}^\alpha := \operatorname{dom} \alpha = \{Z \in \mathcal{D} : \alpha(Z) < \infty\} \neq \emptyset$. The penalty function determines how seriously we treat probabilistic models in $\mathcal{D}$. Since ρ is normalised, $\inf \alpha = 0$. Moreover, replacing α if necessary by its quasi-convex hull, we may assume without loss of generality that $\mathcal{Q}^\alpha$ is convex; see Remark 4.2.1 for details.

Remark 4.2.1. (a) Since $-ZX$ may not be integrable, we define $\mathbb{E}[-ZX] := \mathbb{E}[ZX^-] - \mathbb{E}[ZX^+]$, with the conservative convention that $\mathbb{E}[-ZX] := \infty$ if $\mathbb{E}[ZX^-] = \infty$. Moreover, if $\alpha(Z) = \infty$, we set $\mathbb{E}[-ZX] - \alpha(Z) := -\infty$ so that the second equality in (4.4) is preserved. This portrays the idea that only the measures "contained" in $\mathcal{Q}^\alpha$ are seen as plausible.

(b) The class of risk measures satisfying (4.4) is very large. In particular, we do not impose lower semicontinuity on α, or L^1-closedness or uniform integrability on $\mathcal{Q}^\alpha$.

(c) The representation in (4.4) is not unique. However, it is not difficult to check that the *minimal* penalty function for which (4.4) is satisfied is given by

$$\alpha^\rho(Z) := \sup\{\mathbb{E}[-ZX] - \rho(X) : X \in L \text{ and } \rho(X) < \infty\}.$$

Note that α^ρ is automatically convex. Moreover, its effective domain $\mathcal{Q}^\rho := \{Z \in \mathcal{D} : \alpha^\rho(Z) < \infty\}$ is also convex and the *maximal* dual set. Notwithstanding, it turns out that it is sometimes useful not to consider α^ρ or $\mathcal{Q}^\rho$; cf. Remark 4.2.1(c).

(d) If $\alpha : \mathcal{D} \to [0,\infty]$ represents ρ and $\alpha' : \mathcal{D} \to [0,\infty]$ satisfies $\alpha \geq \alpha' \geq \alpha^\rho$, then α' represents ρ and $\mathcal{Q}^\alpha \subset \mathcal{Q}^{\alpha'} \subset \mathcal{Q}^\rho$. This follows directly by comparing the right hand side (4.4) for α, α' and α^ρ.

(e) If $\mathcal{Q}^\alpha$ is not convex, we may replace α by its quasi-convex hull $\operatorname{qco}\alpha$, i.e., the largest function dominated by α that is quasi-convex; see Appendix A.5 for details. Since α^ρ is convex and dominated by α, we have $\alpha^\rho \leq \operatorname{qco}\alpha \leq \alpha$. Hence, by part (d), ρ is represented by $\operatorname{qco}\alpha$ and $\mathcal{Q}^\alpha \subset \mathcal{Q}^{\operatorname{qco}\alpha} \subset \mathcal{Q}^\rho$. It follows from the definition of quasi-convexity that $\mathcal{Q}^{\operatorname{qco}\alpha}$ is convex. Moreover, note that $\mathcal{Q}^{\operatorname{qco}\alpha}$ is the convex hull of $\mathcal{Q}^\alpha$.

(f) If we define ρ through (4.4) for some function $\alpha : \mathcal{D} \to [0,\infty]$ satisfying $\inf \alpha = 0$ and for which there exists $Z \in \mathcal{Q}^\alpha$ such that $ZX \in L^1$ for all $X \in L^1$ (often there is no penalty associated with the real-world measure $\mathbb{P}$, i.e., $\alpha(1) = 0$), then ρ is a $(-\infty,\infty]$-valued convex risk measure.

If $\rho : L \to (-\infty,\infty]$ admits a dual representation as in (4.4), then its recession function $\rho^\infty : L \to (-\infty,\infty]$ is a coherent risk measure that admits the dual representation

$$\rho^\infty(X) = \sup_{Z \in \mathcal{Q}^\alpha} (\mathbb{E}[-ZX]). \tag{4.5}$$

4.2.1 Preliminary Considerations and Conditions

We start by recalling the key conditions that were introduced in Section 3.2 and extend some of their consequences to the current setup.

Condition I. For all $i \in \{1,\dots,d\}$ and any $Z \in \mathcal{Q}^\alpha$, $ZR^i \in L^1$.

Condition UI. $\mathcal{Q}^\alpha$ is uniformly integrable, and for all $i \in \{1,\dots,d\}$, $R^i\mathcal{Q}^\alpha$ is uniformly integrable, where $R^i\mathcal{Q}^\alpha := \{R^iZ : Z \in \mathcal{Q}^\alpha\}$.

Condition I is weak, but has some important consequences. Arguing as in Proposition 3.2.3, we obtain the following result.

Proposition 4.2.2. *Suppose that Condition I is satisfied. Then for any portfolio $\pi \in \mathbb{R}^d$,*

$$\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}^\alpha}} (\pi \cdot c - f_\alpha(c)), \tag{4.6}$$

where $C_{\mathcal{Q}^\alpha} := \{\mathbb{E}[-Z(R-r\mathbf{1})] : Z \in \mathcal{Q}^\alpha\} \subset \mathbb{R}^d$ *is convex and* $f_\alpha : \mathbb{R}^d \to [0,\infty]$, *defined by*

$$f_\alpha(c) = \inf\{\alpha(Z) : Z \in \mathcal{Q}^\alpha \text{ and } \mathbb{E}[-Z(R-r\mathbf{1})] = c\}$$

satisfies $\operatorname{dom} f_\alpha = C_{\mathcal{Q}^\alpha}$. *Moreover,* ρ *satisfies the Fatou property on* $\mathcal{X}$.

Condition UI is a uniform version of Condition I. For $X \in \mathcal{X}$, it allows us to replace α in (4.4) by its L^1-lower semi-continuous convex hull $\overline{\mathrm{co}}\,\alpha$, and the infimum by a minimum.

Proposition 4.2.3. *Suppose that Condition UI is satisfied. Denote by* $\overline{\mathrm{co}}\,\alpha : \mathcal{D} \to [0,\infty]$ *the* L^1*-lower semi-continuous convex hull of* α. *Then for* $X \in \mathcal{X}$,

$$\rho(X) = \max_{Z \in \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}} \{\mathbb{E}[-ZX] - \overline{\mathrm{co}}\,\alpha(Z)\}. \tag{4.7}$$

Proof. Let $X \in \mathcal{X}$, i.e., there is $\pi \in \mathbb{R}^d$ such that $X = X_\pi = \pi \cdot (R - r\mathbf{1})$. By Condition UI, this implies that $\mathcal{Q}^\alpha$ and $X\mathcal{Q}^\alpha$ are UI.

First, $\mathrm{co}\,\alpha$ (whose effective domain is $\mathcal{Q}^\alpha$) represents ρ by Remark 4.2.1(d) since $\alpha^\rho \leq \mathrm{co}\,\alpha \leq \alpha$ by Remark 4.2.1(c) and the definition of the convex hull. This together with $\overline{\mathrm{co}}\,\alpha \leq \mathrm{co}\,\alpha$ and Remark 4.2.5(a) implies that $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha} \subset \bar{\mathcal{Q}}^\alpha$ and

$$\rho(X) = \sup_{Z \in \mathcal{Q}^\alpha} \{\mathbb{E}[-ZX] - \mathrm{co}\,\alpha(Z)\} \leq \sup_{Z \in \bar{\mathcal{Q}}^\alpha} \{\mathbb{E}[-ZX] - \overline{\mathrm{co}}\,\alpha(Z)\}. \tag{4.8}$$

If we can show that the supremum on the right side of (4.8) is attained and the inequality is an equality, then (4.7) follows.

To see that the supremum on the right side of (4.8) is attained, let $(Z_n)_{n\in\mathbb{N}}$ be a maximising sequence in $\bar{\mathcal{Q}}^\alpha$. As $\mathcal{Q}^\alpha$ is uniformly integrable and convex, $\bar{\mathcal{Q}}^\alpha$ is convex and $\sigma(L^1, L^\infty)$-sequentially compact by the Dunford-Pettis and the Eberlein-Šmulian theorems. After passing to a subsequence, we may assume that Z_n converges weakly to some $Z^* \in \bar{\mathcal{Q}}^\alpha$. Then because the map $\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}X]$ is weakly continuous on $\bar{\mathcal{Q}}^\alpha$ (by Proposition A.6.2) and $\overline{\mathrm{co}}\,\alpha$ is also $\sigma(L^1, L^\infty)$-lower semi-continuous by [91, Theorem 2.2.1], Z^* is a maximiser.

Finally, we show that the inequality in (4.8) is an equality. We may assume without loss of generality that the right hand side of (4.8) is larger than $-\infty$. Hence, $\overline{\mathrm{co}}\,\alpha(Z^*)$ is finite. Let $\varepsilon > 0$. Since $\overline{\mathrm{co}}\,\alpha$ is the L^1-lower semi-continuous hull of $\mathrm{co}\,\alpha$ and $\mathrm{co}\,\alpha(Z) = \infty$ for $Z \notin \mathcal{Q}^\alpha$ and $\overline{\mathrm{co}}\,\alpha(Z^*) < \infty$, by (A.6), there exists a sequence $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{Q}^\alpha$ that converges in L^1 to the maximiser Z^* and for which $\lim_{n\to\infty} \operatorname{co}\alpha(Z_n) \leq \overline{\operatorname{co}}\,\alpha(Z^*) + \varepsilon$. Using again that the map $\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}X]$ is weakly and hence strongly continuous yields

$$\rho(X) \geq \lim_{n\to\infty} \{\mathbb{E}[-Z_n X] - \operatorname{co}\alpha(Z_n)\} \geq \mathbb{E}[-Z^* X] - \overline{\operatorname{co}}\,\alpha(Z^*) - \varepsilon.$$

Now the claim follows by letting $\varepsilon \to 0$. □

As a consequence of Proposition 4.2.3, we obtain the following result which is is crucial in establishing the dual characterisation of strong ρ-arbitrage.

Proposition 4.2.4. *Suppose that Condition UI is satisfied. Then for any portfolio $\pi \in \mathbb{R}^d$,*

$$\rho(X_\pi) = \max_{c \in C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}}} (\pi \cdot c - f_{\overline{\operatorname{co}}\,\alpha}(c)), \tag{4.9}$$

where $C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}} := \{\mathbb{E}[-Z(R - r\mathbf{1})] : Z \in \mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}\} \subset \mathbb{R}^d$ is convex and bounded, and $f_{\overline{\operatorname{co}}\,\alpha} : \mathbb{R}^d \to [0, \infty]$, defined by

$$f_{\overline{\operatorname{co}}\,\alpha}(c) = \inf\{\overline{\operatorname{co}}\,\alpha(Z) : Z \in \mathcal{Q}^{\overline{\operatorname{co}}\,\alpha} \text{ and } \mathbb{E}[-Z(R - r\mathbf{1})] = c\} \tag{4.10}$$

is the lower semi-continuous convex hull of f_α defined in Proposition 4.2.2 and satisfies $\operatorname{dom} f_{\overline{\operatorname{co}}\,\alpha} = C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}}$. *Moreover, the infimum in* (4.10) *is a minimum if $c \in C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}}$.*

Proof. It is clear by the definition of $C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}}$ that $\operatorname{dom} f_{\overline{\operatorname{co}}\,\alpha} = C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}}$. By Remark 4.2.5(a), $\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha} \subset \bar{\mathcal{Q}}^\alpha$ and as $\overline{\operatorname{co}}\,\alpha(Z) = \infty$ for $Z \in \bar{\mathcal{Q}}^\alpha \setminus \mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}$ it follows that

$$f_{\overline{\operatorname{co}}\,\alpha}(c) = \inf\{\overline{\operatorname{co}}\,\alpha(Z) : Z \in \bar{\mathcal{Q}}^\alpha \text{ and } \mathbb{E}[-Z(R - r\mathbf{1})] = c\}.$$

Since $\bar{\mathcal{Q}}^\alpha$ is $\sigma(L^1, L^\infty)$-sequentially compact by Eberlein-Šmulian theorem and Dunford-Pettis, and $\overline{\operatorname{co}}\,\alpha$ is $\sigma(L^1, L^\infty)$-lower semi-continuous by [91, Theorem 2.2.1], it follows that the infimum is attained and (finite) if $c \in C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}}$. Moreover, since $\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha} \subset \bar{\mathcal{Q}}^\alpha$ is convex, it follows that

$$C_{\mathcal{Q}^{\overline{\operatorname{co}}\,\alpha}} \subset C_{\bar{\mathcal{Q}}^\alpha} = \{\mathbb{E}[-Z(R - r\mathbf{1})] : Z \in \bar{\mathcal{Q}}^\alpha\}$$

is convex and bounded since $C_{\bar{\mathcal{Q}}^\alpha} = \operatorname{cl}(C_{\mathcal{Q}^\alpha})$ is a (convex) compact subset of $\mathbb{R}^d$ by Proposition 3.2.5.

Next, we show that $f_{\overline{\operatorname{co}}\,\alpha}$ is convex and lower semi-continuous. Convexity follows easily from convexity of $\overline{\operatorname{co}}\,\alpha$. To argue lower semi-continuity let

$(c_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$ that converges to $c \in \mathbb{R}^d$. Without loss of generality, we may assume that each c_n and c lies in $C_{\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}}$. Let $(Z_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}$ be a corresponding sequence of minimisers. Since $\bar{\mathcal{Q}}^\alpha$ is $\sigma(L^1, L^\infty)$-sequentially compact by the Dunford–Pettis and the Eberlein-Šmulian theorems, after passing to a subsequence, we may assume that $(Z_n)_{n\in\mathbb{N}}$ converges weakly to some $Z \in \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}$. As the map $\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}X]$ is $\sigma(L^1, L^\infty)$-continuous on $\bar{\mathcal{Q}}^\alpha$ by Proposition A.6.2, it follows that $\mathbb{E}[-Z(R - r\mathbf{1})] = c$. By $\sigma(L^1, L^\infty)$-lower semicontinuity of $\overline{\mathrm{co}}\,\alpha$ this implies that $f_{\overline{\mathrm{co}}\,\alpha}(c) \leq \overline{\mathrm{co}}\,\alpha(Z) \leq \liminf_{n\to\infty} \overline{\mathrm{co}}\,\alpha(Z_n) = \liminf_{n\to\infty} f_{\overline{\mathrm{co}}\,\alpha}(c_n)$.

We proceed to show that $f_{\overline{\mathrm{co}}\,\alpha}$ is the the lower semi-continuous convex hull of f_α. To this end, for $g : \mathbb{R}^d \to [0, \infty]$, define the map $\alpha^g : \mathcal{D} \to [0, \infty]$ by

$$\alpha^g(Z) = \begin{cases} g(\mathbb{E}[-Z(R - r\mathbf{1})]), & \text{if } Z \in \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha} \\ \infty, & \text{otherwise,} \end{cases}$$

If g is convex and lower-semicontinuous, then α^g is convex and $\sigma(L^1, L^\infty)$-lower semi-continuous because the map $\tilde{Z} \mapsto \mathbb{E}[-\tilde{Z}(R - r\mathbf{1})]$ is linear and $\sigma(L^1, L^\infty)$-continuous on $\bar{\mathcal{Q}}^\alpha \supset \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}$ by Proposition A.6.2.

Seeking a contradiction, suppose now that there exists a convex lower semi-continuous function $g : \mathbb{R}^d \to [0, \infty]$ such that $g \leq f_\alpha$ and $f_{\overline{\mathrm{co}}\,\alpha}(c^*) < g(c^*)$ for some $c^* \in C_{\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}}$. Then

$$\alpha^g(Z) \leq \alpha^{f_\alpha}(Z) \leq \alpha(Z), \quad Z \in \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha},$$

and hence $\alpha^g \leq \overline{\mathrm{co}}\,\alpha$. Let $Z^* \in \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}$ be such that $\mathbb{E}[-Z^*(R - r\mathbf{1})] = c^*$ and $\overline{\mathrm{co}}\,\alpha(Z^*) = f_{\overline{\mathrm{co}}\,\alpha}(c^*)$. Then

$$\overline{\mathrm{co}}\,\alpha(Z^*) = f_{\overline{\mathrm{co}}\,\alpha}(c^*) < g(c^*) = \alpha^g(Z^*)$$

and we arrive at a contradiction.

Finally, (4.9) follows from Proposition 4.2.3. □

Remark 4.2.5. (a) By the fact that $\overline{\mathrm{co}}\,\alpha \leq \alpha$ and (A.7) it follows that $\mathcal{Q}^\alpha \subset \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha} \subset \bar{\mathcal{Q}}^\alpha$, where $\bar{\mathcal{Q}}^\alpha$ is the L^1-closure of $\mathcal{Q}^\alpha$. Moreover, if α is bounded from above on its effective domain, then $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha} = \bar{\mathcal{Q}}^\alpha$.

(b) Since the L^1-lower semi-continuous convex hull of α is equal to its $\sigma(L^1, L^\infty)$-lower semi-continuous convex hull by [91, Theorem 2.2.1], $\overline{\mathrm{co}}\,\alpha$ coincides with α^{**}, the biconjungate of α under the pairing $\langle\cdot,\cdot\rangle : L^1 \times L^\infty \to \mathbb{R}$, $\langle Z, X\rangle \mapsto \mathbb{E}[-ZX]$, by the Fenchel-Moreau theorem (and the fact that α is nonnegative); see Appendix A.5 for details.

The final object that we need to recall is the "interior" of $\mathcal{Q}^\alpha$, which will be crucial in the dual characterisation of ρ-arbitrage. This is done in an abstract way. More precisely, we look for (nonempty) subsets $\tilde{\mathcal{Q}}^\alpha \subset \mathcal{Q}^\alpha$ satisfying:

Condition POS. $\tilde{Z} > 0$ $\mathbb{P}$-a.s. for all $\tilde{Z} \in \tilde{\mathcal{Q}}^\alpha$.

Condition MIX. $\lambda Z + (1-\lambda)\tilde{Z} \in \tilde{\mathcal{Q}}^\alpha$ for all $Z \in \mathcal{Q}^\alpha$, $\tilde{Z} \in \tilde{\mathcal{Q}}^\alpha$ and $\lambda \in (0,1)$.

Condition INT. For all $\tilde{Z} \in \tilde{\mathcal{Q}}^\alpha$, there is an L^∞-dense subset $\mathcal{E}$ of $\mathcal{D} \cap L^\infty$ such that for all $Z \in \mathcal{E}$, there is $\lambda \in (0,1)$ such that $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}^\alpha$.

By Proposition 3.2.9, the maximal subset $\tilde{\mathcal{Q}}^\alpha_{\max}$ of $\mathcal{Q}^\alpha$ satisfying Conditions POS, MIX and INT can be described explicitly by

$$\begin{aligned}\tilde{\mathcal{Q}}_{\max} := \{\tilde{Z} > 0 \in \mathcal{Q} : &\text{there is an } L^\infty\text{-dense subset } \mathcal{E} \text{ of } \mathcal{D} \cap L^\infty \text{ such that for} \\ &\text{all } Z \in \mathcal{E}, \text{there is } \lambda \in (0,1) \text{ such that } \lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}\}.\end{aligned}$$

4.2.2 (Strong) ρ-Arbitrage

We are now in a position to state and prove the dual characterisation of strong ρ-arbitrage in terms of absolutely continuous martingale measures (ACMMs) for the discounted risky assets,

$$\mathcal{M} := \{Z \in \mathcal{D} : \mathbb{E}[Z(R^i - r)] = 0 \text{ for all } i = 1, \dots, d\};$$

and the dual characterisation of ρ-arbitrage in terms of equivalent martingale measures (EMMs) for the discounted risky assets,

$$\mathcal{P} := \{Z \in \mathcal{M} : Z > 0 \ \mathbb{P}\text{-a.s.}\}.$$

We first consider the dual characterisation of strong ρ-arbitrage.

Theorem 4.2.6. *Assume Condition UI is satisfied and $1 \in \mathcal{Q}^\alpha$. Denote by $\overline{\mathrm{co}}\,\alpha : \mathcal{D} \to [0,\infty]$ the L^1-lower semi-continuous convex hull of α. Then the following are equivalent:*

(a) *The market (S^0, S) does not admit strong ρ-arbitrage.*

(b) $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha} \cap \mathcal{M} \neq \emptyset$.

Proof. First we show that the market admits strong ρ-arbitrage if and only if $\inf_{\pi\in\mathbb{R}^d} \rho(X_\pi) = -\infty$. For the nontrivial direction, let $(\pi_n)_{n\geq 1} \subset \mathbb{R}^d$ be a sequence of portfolios such that $\rho(X_{\pi_n}) \searrow -\infty$. By the dual representation of ρ, this implies that $\mathbb{E}[-X_{\pi_n}] - \alpha(1) \searrow -\infty$, and since $\alpha(1) < \infty$, this gives $\mathbb{E}[X_{\pi_n}] \nearrow \infty$.

Now let $f_{\overline{\mathrm{co}}\,\alpha}$ be as in Proposition 4.2.4. Since $\mathrm{dom}\, f_{\overline{\mathrm{co}}\,\alpha} = C_{\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}}$, the convex conjugate of $f_{\overline{\mathrm{co}}\,\alpha}$ is given by

$$f^*_{\overline{\mathrm{co}}\,\alpha}(\pi) = \sup_{c\in\mathbb{R}^d}(\pi\cdot c - f_{\overline{\mathrm{co}}\,\alpha}(c)) = \sup_{c\in C_{\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}}}(\pi\cdot c - f_{\overline{\mathrm{co}}\,\alpha}(c)), \quad \pi\in\mathbb{R}^d.$$

By (4.9), this implies

$$f^*_{\overline{\mathrm{co}}\,\alpha}(\pi) = \rho(X_\pi), \quad \pi\in\mathbb{R}^d. \tag{4.11}$$

Since $f_{\overline{\mathrm{co}}\,\alpha}$ is a nonnegative lower semi-continuous convex function, the Fenchel-Moreau theorem (cf. Appendix A.5) and (4.11) give

$$-f_{\overline{\mathrm{co}}\,\alpha}(\mathbf{0}) = -f^{**}_{\overline{\mathrm{co}}\,\alpha}(\mathbf{0}) = -\sup_{\pi\in\mathbb{R}^d}(-f_{\overline{\mathrm{co}}\,\alpha^*}(\pi)) = \inf_{\pi\in\mathbb{R}^d}\rho(X_\pi).$$

Now the result follows from using that $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}\cap\mathcal{M} = \emptyset$ if and only if $f_{\overline{\mathrm{co}}\,\alpha}(\mathbf{0}) = \infty$, and the market admits strong ρ-arbitrage iff $\inf_{\pi\in\mathbb{R}^d}\rho(X_\pi) = -\infty$. $\square$

Remark 4.2.7. (a) Note that in order to apply Theorem 4.2.6, we do not necessarily need to find $\overline{\mathrm{co}}\,\alpha$ but rather only its effective domain $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha}$.

(b) Theorem 4.2.6 is the first of its kind for convex risk measures. When ρ is coherent, then $\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha} = \bar{\mathcal{Q}}^\alpha$ by Remark 4.2.5(a), and we arrive at Theorem 3.2.14.

The interpretation of Theorem 4.2.6 from a pricing perspective is as follows.

Corollary 4.2.8. *Suppose $\mathcal{Q}^\alpha$ is uniformly integrable and $1\in\mathcal{Q}^\alpha$. Let*

$$\tilde{L} := \{X\in L^1 : \lim_{a\to\infty}\sup_{Z\in\mathcal{Q}^\alpha}\mathbb{E}[Z|X|\mathbb{1}_{|X|>a}] = 0\}$$

and assume (S^0, S) is a $(1+d)$-dimensional market with returns in $\tilde{L}$ that does not admit strong ρ-arbitrage. Then, the set of no-strong-ρ-arbitrage prices for any *financial contract outside the original market* $S_1^{d+1} \in \tilde{L}\backslash\text{span}(\{1, S_1^1, \ldots, S_1^d\})$ *is nonempty and given by the interval*

$$I(S_1^{d+1}) = \{\mathbb{E}[ZS_1^{d+1}/(1+r)] : Z \in \mathcal{Q}^{\overline{\text{co}}\,\alpha} \cap \mathcal{M}\} \tag{4.12}$$

where $\mathcal{M}$ is the set of ACMMs for the original market.

Proof. By Theorem 4.2.6, S_0^{d+1} is a no-strong-ρ-arbitrage free price for S_1^{d+1} if and only if there exists an ACMM $Z \in \mathcal{Q}^{\overline{\text{co}}\,\alpha}$ for the extended market, i.e.,

$$S_0^i = \mathbb{E}[ZS_1^i/(1+r)], \quad \text{for } i = 1, \ldots, d+1.$$

In particular, Z is necessarily contained in $\mathcal{Q}^{\overline{\text{co}}\,\alpha} \cap \mathcal{M}$, and we obtain the inclusion $\subset$ in (4.12). Conversely, if $S_0^{d+1} = \mathbb{E}[\hat{Z}S_1^{d+1}/(1+r)]$ for some $\hat{Z} \in \mathcal{Q}^{\overline{\text{co}}\,\alpha} \cap \mathcal{M}$, then this $\hat{Z}$ is also an ACMM for the extended market model, and so the two sets in (4.12) are equal. □

We next consider the dual characterisation of ρ-arbitrage.

Theorem 4.2.9. *Suppose that Condition I is satisfied, ρ satisfies weak sensitivity to large losses on L and $\tilde{\mathcal{Q}}^\alpha_{\max} \neq \emptyset$. Then the following are equivalent:*

(a) *The market (S^0, S) does not admit ρ-arbitrage.*

(b) *$\tilde{\mathcal{Q}}^\alpha \cap \mathcal{P} \neq \emptyset$ for some $\emptyset \neq \tilde{\mathcal{Q}}^\alpha \subset \mathcal{Q}^\alpha$ satisfying POS, MIX and INT.*

(c) *$\tilde{\mathcal{Q}}^\alpha \cap \mathcal{P} \neq \emptyset$ for all $\emptyset \neq \tilde{\mathcal{Q}}^\alpha \subset \mathcal{Q}^\alpha$ satisfying POS, MIX and INT.*

Proof. The result follows from Proposition 4.1.16 and Theorem 3.2.18, noting that by Remark 4.1.4(c), ρ satisfying weak sensitivity to large losses on L implies that ρ^∞ is strictly expectation bounded. □

Remark 4.2.10. (a) Usually (at least in all the examples we consider) there is an "interior" of $\mathcal{Q}$ which contains 1 (the real world measure). This implies $\tilde{\mathcal{Q}}^\alpha_{\max} \neq \emptyset$. Furthermore, by Proposition 3.2.11 it follows that ρ^∞ is strictly expectation bounded and so by Remark 4.1.4(c), ρ is weakly sensitive to large losses on the entire space L. In such cases, we only need to check when Condition I holds in order to apply Theorem 4.2.9.

(b) Theorem 4.2.9 is the first of its kind for convex risk measures. When ρ is coherent, we arrive at Theorem 3.2.18, which is strongly related to [8, Theorem 4.14] by Remark 4.1.4(b) and Remark 4.1.17(b).

The interpretation of Theorem 4.2.9 from a pricing perspective is as follows.

Corollary 4.2.11. *Suppose ρ is weakly sensitive to large losses and admits a dual representation* (4.4) *where $\emptyset \neq \tilde{\mathcal{Q}}^\alpha \subset \mathcal{Q}^\alpha$ satisfies Conditions POS, MIX and INT. Let*

$$\tilde{L} := \{X \in L^1 : ZX \in L^1 \text{ for all } Z \in \mathcal{Q}^\alpha\}$$

and assume (S^0, S) is a $(1+d)$-dimensional market with returns in $\tilde{L}$ that does not admit ρ-arbitrage. Then, the set of no-ρ-arbitrage prices for any *financial contract outside the original market $S_1^{d+1} \in \tilde{L} \setminus \mathrm{span}(\{1, S_1^1, \ldots, S_1^d\})$ is nonempty and given by the interval*

$$I(S_1^{d+1}) = \{\mathbb{E}[ZS_1^{d+1}/(1+r)] : Z \in \tilde{\mathcal{Q}}^\alpha \cap \mathcal{P}\} \tag{4.13}$$

where $\mathcal{P}$ is the set of EMMs for the original market.

Proof. By Theorem 4.2.9, S_0^{d+1} is a no-ρ-arbitrage free price for S_1^{d+1} if and only if there exists an EMM $Z \in \tilde{\mathcal{Q}}^\alpha$ for the extended market, i.e.,

$$S_0^i = \mathbb{E}[ZS_1^i/(1+r)], \quad \text{for } i = 1, \ldots, d+1.$$

In particular, Z is necessarily contained in $\tilde{\mathcal{Q}}^\alpha \cap \mathcal{P}$, and we obtain the inclusion $\subset$ in (4.13). Conversely, if $S_0^{d+1} = \mathbb{E}[\hat{Z}S_1^{d+1}/(1+r)]$ for some $\hat{Z} \in \tilde{\mathcal{Q}}^\alpha \cap \mathcal{P}$, then this $\hat{Z}$ is also an EMM for the extended market model, and so the two sets in (4.12) are equal. □

Theorem 4.2.9 (Theorem 4.2.6) provides a dual characterisation of (strong) ρ-arbitrage for convex risk measures that admit a dual representation. The criterion for the absence of ρ-arbitrage and ρ^∞-arbitrage is the same, but when it comes to the absence of strong ρ-arbitrage and strong ρ^∞-arbitrage, they may differ. In view of Propositions 4.1.14 and 4.1.16, this is as expected.

4.2.3 Risk Measures Suitable for Portfolio Selection

Having provided a dual characterisation of (strong) ρ-arbitrage, we now focus on giving a dual representation of risk measures that are suitable for portfolio selection. We restrict our attention to Orlicz hearts. To that end, let $\Phi : [0, \infty) \to [0, \infty)$ be a finite Young function and H^Φ its corresponding Orlicz heart. Let Ψ be the convex conjugate of Φ and denote its corresponding

Orlicz space by L^Ψ. Recall that the norm dual of $(H^\Phi, \|\cdot\|_\Phi)$ is $(L^\Psi, \|\cdot\|_\Phi^*)$, where $\|\cdot\|_\Phi$ denotes the Luxemburg norm and $\|\cdot\|_\Phi^*$ denotes the Orlicz norm. For a summary of key definitions and result on Orlicz spaces and Orlicz hearts, we refer the reader to Appendix A.2.

By Section 4.1, it is clear that strong sensitivity to large losses is the key axiom for a risk functional to possess. Thus, we first give a dual version of this property.

Proposition 4.2.12. *Assume $\rho : H^\Phi \to (-\infty, \infty]$ is a convex risk measure that admits a dual representation*

$$\rho(X) = \sup_{Z \in \mathcal{Q}^\alpha} \{\mathbb{E}[-ZX] - \alpha(Z)\},$$

for some quasi-convex penalty function $\alpha : \mathcal{D} \to [0, \infty]$ with effective domain $\mathcal{Q}^\alpha \subset \mathcal{D} \cap L^\Psi$. Then ρ is strongly sensitive to large losses on H^Φ if and only if $\mathcal{Q}^\alpha$ is $\sigma(L^\Psi, H^\Phi)$-dense in $\mathcal{D} \cap L^\Psi$.

Proof. By (4.5), the recession function of ρ is

$$\rho^\infty(X) = \sup_{Z \in \mathcal{Q}^\alpha} (\mathbb{E}[-ZX]), \quad X \in H^\Phi.$$

By Remark 4.1.21(b), ρ is strongly sensitive to large losses on H^Φ if and only if $\rho^\infty \equiv \mathrm{WC}$. The result then follows by combining Proposition A.6.12 with the fact $\mathcal{Q}^\alpha$ is a convex subset of $\mathcal{D} \cap L^\Psi$. □

This result together with Lemma 4.1.26 allows us to immediately check whether or not a convex risk measure on H^Φ that admits a dual representation (with a quasi-convex penalty function where $\mathcal{Q}^\alpha \subset \mathcal{D} \cap L^\Psi$) is suitable for risk management. When it comes to being suitable for portfolio selection, we can say even more.

Theorem 4.2.13. *Let $\rho : H^\Phi \to (-\infty, \infty]$ be a convex risk measure. The following are equivalent:*

(a) *ρ is suitable for portfolio selection.*

(b) *ρ admits a dual representation with a quasi-convex penalty function $\alpha : \mathcal{D} \to [0, \infty]$ where $\mathcal{Q}^\alpha$ is a $\sigma(L^\Psi, H^\Phi)$-dense subset of $\mathcal{D} \cap L^\Psi$ and there exists $a \in \mathbb{R}$ and $b > 0$ such that $\alpha(Z) \geq a + b\|Z\|_\Psi$ for all $Z \in \mathcal{Q}^\alpha$.*

(c) *ρ admits a dual representation, and for every quasi-convex penalty function $\alpha : \mathcal{D} \to [0, \infty]$ associated to ρ, $\mathcal{Q}^\alpha$ is a $\sigma(L^\Psi, H^\Phi)$-dense subset of $\mathcal{D} \cap L^\Psi$ and there exists $a \in \mathbb{R}$ and $b > 0$ such that $\alpha(Z) \geq a + b\|Z\|_\Psi$ for all $Z \in \mathcal{Q}^\alpha$.*

Proof. "(b) $\implies$ (a)". By [25, Theorem 4.2] and Proposition 4.2.12, ρ is a convex real-valued risk measure that satisfies the Fatou property on H^Φ and strong sensitivity to large losses on H^Φ. Note that [25, Theorem 4.2] asserts that ρ is Lipschitz-continuous with respect to $\|\cdot\|_\Phi$ which is stronger than the Fatou property on H^Φ. It follows that ρ is suitable for portfolio selection by Theorem 4.1.31.

"(a) $\implies$ (c)". Assume ρ is suitable for portfolio selection. Then by Remark 4.1.29 ρ must be real-valued on H^Φ. Then the statement in (c) is a consequence of: first applying [25, Theorem 4.3]; then using the fact that the growth condition in [25, Definition 4.1] only requires that $\alpha(Z) \geq a + b\|Z\|_\Psi$ for all $Z \in \mathcal{D} \cap L^\Psi$, however, it follows that since $(L^\Psi, \|\cdot\|_\Phi^*)$ is the norm dual of $(H^\Phi, \|\cdot\|_\Phi)$ (and $\|\cdot\|_\Phi^*$ and $\|\cdot\|_\Psi$ are equivalent) that if ρ is real-valued, any penalty function associated with it must be ∞ outside of $\mathcal{D} \cap L^\Psi$; and finally using Proposition 4.2.12 together with Lemma 4.1.30.

"(c) $\implies$ (b)". This is trivial. □

This result is powerful as it characterises *all* convex risk measures that live on Orlicz hearts and are suitable for portfolio selection. Of particular interest is when $H^\Phi = L^1$ and this will be further explored in the next section, as well as the application of our theory to other examples of convex risk measures.

4.3 Examples

In this section, we apply our theory to various examples of risk functionals. Our main focus is on convex risk measures that are not coherent since the latter have been discussed in Section 3.3. We do not make any assumptions on the returns, other than our standing assumptions that they are contained in a Riesz space and that the market is nonredundant and nondegenerate.

4.3.1 Risk Functionals Based on Loss Functions

The examples in this section are based around the theme of *loss functions*, namely: the expected weighted loss, which is closely related to expected utility theory; shortfall risk first introduced in [44, Section 3]; and the optimised certainty equivalent, which comes from [13].

Definition 4.3.1. A function $l : \mathbb{R} \to \mathbb{R}$ is called a *loss function* if it is nondecreasing, convex, $l(0) = 0$ and $l(x) \geq x$ for all $x \in \mathbb{R}$.

A loss function l reflects how risk averse an agent is, and so it is natural to assume that it is nondecreasing and $l(0) = 0$. The assumption $l(x) \geq x$ means that compared to the risk neutral evaluation, there is more weight on losses and less on gains. Finally, convexity of l encodes that diversified positions are less risky than concentrated ones. Some of these properties can be relaxed in what follows and we will make it clear whenever this is possible.

Expected Weighted Loss. The expected weighted loss of $X \in H^{\Phi_l}$ with respect to a loss function l is

$$\mathrm{EW}^l(X) := \mathbb{E}[l(-X)],$$

where H^{Φ_l} is the Orlicz heart corresponding to the Young function $\Phi_l := l|_{\mathbb{R}_+}$. By the properties of l and the definition of H^{Φ_l}, EW^l is a real-valued convex risk functional (but never cash-invariant unless $l(x) = x$). It is also not difficult to check that it satisfies the Fatou property on H^{Φ_l}. Therefore, by Lemmas 4.1.26 and 4.1.30 and Proposition A.6.13 we have the following result:

Corollary 4.3.2. *The following are equivalent:*

(a) EW^l *is suitable for risk management.*

(b) EW^l *is suitable for portfolio selection.*

(c) $\lim_{x\to\infty} l(x)/x = \infty$ *or* $\lim_{x\to-\infty} l(x)/x = 0$.

Remark 4.3.3. One can check that the result (including Proposition A.6.13) extends to functions $l : \mathbb{R} \to \mathbb{R}$ that are nondecreasing, convex, and satisfy $l(0) = 0$ as well as $\lim_{x\to\infty} l(x) = \infty$ (which is weaker than $l(x) \geq x$ for all $x \in \mathbb{R}$).

Shortfall Risk Measures. Shortfall risk measures were first introduced as a case study on L^∞ in [44, Section 3]. Here, we work on Orlicz hearts. To that end, let l be a loss function and define the acceptance set

$$\mathcal{A}_l := \{X \in H^{\Phi_l} : \mathrm{EW}^l(X) \leq 0\},$$

where H^{Φ_l} is the Orlicz heart corresponding to the Young function $\Phi_l := l|_{\mathbb{R}_+}$. Then the shortfall risk measure associated with l is given by $\mathrm{SR}^l : H^{\Phi_l} \to (-\infty, \infty]$ where

$$\mathrm{SR}^l(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_l\} = \inf\{m \in \mathbb{R} : \mathrm{EW}^l(X+m) \leq 0\}.$$

This is a convex risk measure. It can be interpreted as the cash-invariant analogue of EW^l in the sense that it is cash-invariant and $\mathrm{SR}^l(X) \leq 0$ if and only if $\mathrm{EW}^l(X) \leq 0$. When $l|_{\mathbb{R}_-} \equiv 0$, then $\mathcal{A}_l = H^{\Phi_l}_+$ and so $\mathrm{SR}^l \equiv \mathrm{WC}$. This is suitable for risk management by Lemma 4.1.26, but not suitable for portfolio selection by Lemma 4.1.30 and Remark 4.1.29 since it is not real-valued on $H^{\Phi_l} \supsetneq L^\infty$. When $l|_{\mathbb{R}_-} \not\equiv 0$, it is easy to check that SR^l is real-valued. Therefore, by Theorem 4.1.31, Remark 4.1.32 and Proposition A.6.13, we have the following result:

Corollary 4.3.4. *Let l be a loss function and assume $l|_{\mathbb{R}_-} \not\equiv 0$. Then the following are equivalent:*

(a) SR^l *is suitable for risk management.*

(b) SR^l *is suitable for portfolio selection.*

(c) $\lim_{x\to\infty} l(x)/x = \infty$ *or* $\lim_{x\to-\infty} l(x)/x = 0$.

Shortfall risk measures admit a dual representation, which we now recall.

Proposition 4.3.5. *Let l be a loss function and l^* its convex conjugate. Then*

$$\mathrm{SR}^l(X) = \sup_{Z\in\mathcal{D}}\{\mathbb{E}[-ZX] - \alpha^l(Z)\} \tag{4.14}$$

where $\alpha^l(Z) = \inf_{\lambda>0} \frac{1}{\lambda}\mathbb{E}[l^*(\lambda Z)]$.

Proof. If $l|_{\mathbb{R}_-} \equiv 0$, $\alpha^l(Z) = 0$ for $Z \in D \cap L^\infty$, and the result follows from the dual representation of the worst case risk measure. Otherwise, if $l|_{\mathbb{R}_-} \not\equiv 0$, SR^l is a real-valued convex risk measure on H^{Φ_l} and the result follows from [25, Theorem 4.3] and the proof of [44, Theorem 10]. □

When $a_l := \lim_{x\to-\infty} l(x)/x > 0$ and $b_l := \lim_{x\to\infty} l(x)/x < \infty$, then $\operatorname{dom} l^* = [a_l, b_l]$ and it follows that

$$\mathcal{Q}^{\alpha^l} = \{Z \in \mathcal{D} : \text{there exists } k > 0 \text{ such that } kZ \in [a_l, b_l]\ \mathbb{P}\text{-a.s.}\}.$$

(Note that by the convexity of l we have $a_l \leq b_l$, where the inequality is strict unless $a_l = b_l = 1$, in which case l is the identify function and $\mathrm{SR}^l(X) = \mathbb{E}[-X]$.) The dual characterisation of (strong) SR^l-arbitrage now follows from Theorem 4.2.6, Propositions A.6.15 and A.6.16, noting that Conditions I and UI are satisfied since $\|Z\|_\infty \leq b_l/a_l$ for any $Z \in \mathcal{Q}^{\alpha^l}$ by the fact that $kZ \in [a_l, b_l]$ for some $k > 0$ and $\mathbb{E}[Z] = 1$.

Corollary 4.3.6. *Let l be a loss function where $0 < a_l := \lim_{x\to-\infty} l(x)/x < b_l := \lim_{x\to\infty} l(x)/x < \infty$.*

(a) *The market (S^0, S) does not admit SR^l-arbitrage if and only if there exists $Z \in \mathcal{P}$ such that $a_l + \varepsilon < kZ < b_l - \varepsilon$ $\mathbb{P}$-a.s. for some $k, \varepsilon > 0$.*

(b) *The market (S^0, S) does not admit strong SR^l-arbitrage if and only if there exists $Z \in \mathcal{M}$ such that $a_l \leq kZ \leq b_l$ $\mathbb{P}$-a.s. for some $k > 0$.*

Remark 4.3.7. (a) All of the above results for shortfall risk measures hold for functions $l : \mathbb{R} \to \mathbb{R}$ that are nondecreasing, convex and satisfy $l(0) = 0$ and $l(x) > 0$ for all $x > 0$.

(b) Using numerical examples, it was shown in [51, Section 5] that shortfall risk measures corresponding to functions of the form $l(x) = cx^\alpha \mathbb{1}_{\{x>0\}}$ where $\alpha > 1$ and $c > 0$ “adequately account for event risk”. In light of part (a), Corollary 4.3.4 is a generalisation of this result.

OCE Risk Measures. Optimised Certainty Equivalents (OCEs) were first introduced by Ben-Tal and Teboulle [13] and later linked to risk measures on L^∞ by the same authors in [14]. (Note, however, that the name optimised *certainty equivalent* is somewhat misleading as $\mathbb{E}[l(\eta - X)]$ is *not* a certainty equivalent (since this would require to apply l^{-1} from the outside).) We follow [25, Section 5.4] and define the OCE risk measure associated with a loss function l as the map $\mathrm{OCE}^l : H^{\Phi_l} \to \mathbb{R}$,

$$\mathrm{OCE}^l(X) := \inf_{\eta\in\mathbb{R}} \{\mathbb{E}[l(\eta - X)] - \eta\}, \tag{4.15}$$

where H^{Φ_l} is the Orlicz heart corresponding to the Young function $\Phi_l := l|_{\mathbb{R}_+}$. By [25, Section 5.1] (with $V \equiv \mathrm{EW}^l$), OCE^l is the largest real-valued convex risk measure on H^{Φ_l} that is dominated by EW^l. (More generally, cash-invariant hulls of convex functionals have been studied by [43, 61].) By [25, Theorem 4.3], it also satisfies the Fatou property on H^{Φ_l}.

Remark 4.3.8. Normalisation of OCE^l is equivalent to $l(x) \geq x$ for all $x \in \mathbb{R}$. If $l(x) > x$ for all x with $|x|$ sufficiently large, then $\lim_{|x|\to\infty}(l(x) - x) = \infty$ by convexity of x, and the infimum in (4.15) is attained; cf. [25, Lemma 5.2]. However, if $l(x) = x$ for either $x \geq 0$ or $x \leq 0$, then the infimum is not necessarily attained, and it is easy to check that $\mathrm{OCE}^l(X) = \mathbb{E}[-X]$ for all $X \in H^{\Phi_l}$.

Like shortfall risk measures, OCE risk measures admit a dual representation.

Proposition 4.3.9. *Let l be a loss function and denote its convex conjugate by l^*. Then*

$$\mathrm{OCE}^l(X) = \sup_{Z\in\mathcal{D}}\{\mathbb{E}[-ZX] - \alpha^l(Z)\}, \tag{4.16}$$

where $\alpha^l(Z) = \mathbb{E}[l^*(Z)]$.

Proof. This follows from [25, Equation (5.23)] and Remark 4.3.8 in the case that $l(x) > x$ for all x with $|x|$ sufficiently large and from $\alpha^l(1) = 0$ and $\alpha^l(Z) = \infty$ for all $Z \in \mathcal{D} \setminus \{1\}$ in the case that l is equal to the identity either on $\mathbb{R}_+$ or $\mathbb{R}_-$. □

Remark 4.3.10. Shortfall risk measures and OCE risk measures are intimately linked. Indeed, combining (4.14) and (4.16) gives

$$\begin{aligned}
\mathrm{SR}^l(X) &= \sup_{Z\in\mathcal{D}}\{\mathbb{E}[-ZX] - \inf_{\lambda>0}\tfrac{1}{\lambda}\mathbb{E}[l^*(\lambda Z)]\} = \sup_{Z\in\mathcal{D}}\sup_{\lambda>0}\{\mathbb{E}[-ZX] - \tfrac{1}{\lambda}\mathbb{E}[l^*(\lambda Z)]\} \\
&= \sup_{\lambda>0}\sup_{Z\in\mathcal{D}}\{\mathbb{E}[-ZX] - \tfrac{1}{\lambda}\mathbb{E}[l^*(\lambda Z)]\} = \sup_{\lambda>0}\sup_{Z\in\mathcal{D}}\{\mathbb{E}[-ZX] - \mathbb{E}[(l_\lambda)^*(Z)]\} \\
&= \sup_{\lambda>0}\mathrm{OCE}^{l_\lambda}(X),
\end{aligned}$$

where $l_\lambda = l/\lambda$ for $\lambda > 0$. This shows that shortfall risk measures can be understood as the supremum of certain OCE risk measures.

Using the dual representation (4.16), we obtain the following corollary for OCE risk measures.

Corollary 4.3.11. *Let l be a loss function. The following are equivalent:*

(a) OCE^l *is suitable for risk management.*

(b) OCE^l *is suitable for portfolio selection.*

(c) $\lim_{x\to\infty} l(x)/x = \infty$ *and* $\lim_{x\to-\infty} l(x)/x = 0$.

Proof. The penalty function α^l associated with OCE^l is convex since l^* is convex. Moreover, by Proposition A.6.14 and the fact $(l^*)_+ \geq \Psi_l$, α^l satisfies the growth condition

$$\alpha^l(Z) = \mathbb{E}[l^*(Z)] = \mathbb{E}[(l^*)_+(Z)] \geq \mathbb{E}[\Psi_l(Z)] \geq \|Z\|_{\Psi_l} - 1.$$

Now let $a_l := \lim_{x\to-\infty} l(x)/x$ and $b_l := \lim_{x\to\infty} l(x)/x$. Then the $\sigma(L^{\Psi_l}, H^{\Phi_l})$-closure of $\mathcal{Q}^{\alpha^l}$ satisfies

$$\operatorname{cl}\mathcal{Q}^{\alpha^l} \begin{cases} = \mathcal{D} \cap L^{\Psi_l}, & \text{if } a_l = 0 \text{ and } b_l = \infty, \\ \subsetneq \mathcal{D} \cap L^{\Psi_l}, & \text{if } a_l > 0 \text{ or } b_l < \infty. \end{cases} \tag{4.17}$$

Indeed, since $(a_l, b_l) \subset \operatorname{dom} l^* \subset [a_l, b_l]$, it follows that $\{Z \in \mathcal{D} \cap L^{\Psi_l} : Z \in (a_l, b_l)\ \mathbb{P}\text{-a.s.}\} \subset \mathcal{Q}^{\alpha^l} \subset \{Z \in \mathcal{D} \cap L^{\Psi_l} : Z \in [a_l, b_l]\ \mathbb{P}\text{-a.s.}\}$. Thus, $a_l > 0$ or $b_l < \infty$, it follows that $\operatorname{cl}\mathcal{Q}^{\alpha^l} \subsetneq \mathcal{D} \cap L^{\Psi_l}$, and if $a_l = 0$ and $b_l = \infty$, then $\mathcal{D} \cap L^\infty \subset \operatorname{cl}\mathcal{Q}^{\alpha^l} \subset \mathcal{D} \cap L^{\Psi_l}$. Hence, (4.17) follows from the fact that $\mathcal{D} \cap L^\infty$ is $\sigma(L^{\Psi_l}, H^{\Phi_l})$-dense in $\mathcal{D} \cap L^{\Psi_l}$. The result then follows from Theorems 4.2.13 and 4.1.31. □

When $a_l := \lim_{x\to-\infty} l(x)/x > 0$ or $b_l := \lim_{x\to\infty} l(x)/x < \infty$, we can derive a dual characterisation of (strong) OCE^l-arbitrage. By Remark 4.3.8, it suffices to consider the case $a_l < 1 < b_l$.

Corollary 4.3.12. *Let l be a loss function and assume $a_l := \lim_{x\to-\infty} l(x)/x > 0$ or $b_l := \lim_{x\to\infty} l(x)/x < \infty$, and $a_l < 1 < b_l$. Then,*

(a) *The market (S^0, S) does not admit OCE^l-arbitrage if and only if there exists $Z \in \mathcal{P}$ such that $\mathbb{E}[l^*(Z)] < \infty$ and $a_l + \varepsilon < Z < b_l - \varepsilon$ $\mathbb{P}$-a.s. for some $\varepsilon > 0$.*

(b) *If in addition $b_l < \infty$, the market (S^0, S) does not admit strong OCE^l-arbitrage if and only if there exists $Z \in \mathcal{M}$ such that $a_l \leq Z \leq b_l$ $\mathbb{P}$-a.s.*

Proof. (a) This follows from Proposition A.6.17 and Theorem 4.2.9, noting that Condition I follows from the generalised Hölder inequality (A.5).

(b) First, note that $b_l < \infty$ implies $\alpha^l(Z) \leq l^*(b_l) < \infty$ for all $Z \in \mathcal{Q}^{\alpha^l}$. Hence, Remark 4.2.5(a) yields

$$\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha^l} = \bar{\mathcal{Q}}^{\alpha^l} = \{Z \in \mathcal{D} : a_l \leq Z \leq b_l\ \mathbb{P}\text{-a.s.}\}.$$

This shows UI is satisfied. Now the result follows from Theorem 4.2.6. □

4.3.2 g-Adjusted Expected Shortfall

In order to introduce the next class of examples, we first recall the definition of the two most prominent examples of risk measures, Value at Risk (VaR) and Expected Shortfall (ES). For $X \in L^1$ and a confidence level $\alpha \in (0,1)$,

$$\begin{aligned} \mathrm{VaR}^\alpha(X) &:= \inf\{m \in \mathbb{R} : \mathbb{P}[m + X < 0] \leq \alpha\} \quad \text{and} \\ \mathrm{ES}^\alpha(X) &:= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^u(X)\,\mathrm{d}u. \end{aligned}$$

VaR is simple and intuitive, but it completely ignores the behaviour of the loss tail beyond the reference quantile. ES is an improvement, but it still fails to distinguish across different tail behaviours with the same mean. In order to enhance how tail risk is captured, Burzoni, Munari and Wang [21] recently developed a new class of risk measures, which builds on ES. To introduce this class, let $\mathcal{G}$ be the set of all nonincreasing functions $g : (0,1] \to [0,\infty]$ with $g(1) = 0$ and $\{1\} \subsetneq \operatorname{dom} g$.

Definition 4.3.13. Let $g \in \mathcal{G}$. The map $\mathrm{ES}^g : L^1 \to (-\infty, \infty]$, defined by

$$\mathrm{ES}^g(X) := \sup_{\alpha \in (0,1]} \{\mathrm{ES}^\alpha(X) - g(\alpha)\},$$

is called the *g-adjusted Expected Shortfall* (g-adjusted ES).

Remark 4.3.14. Our definition of g-adjusted Expected Shortfall is based on [21, Proposition 2.2], which considers nondecreasing functions that are not identically ∞. However, in line with the way we defined ES, the functions g must be nonincreasing for us. We assume $g(1) = 0$ to achieve normalisation. But this is without loss of generality since otherwise, we simply replace $g(\cdot)$ by $g(\cdot) - g(1)$, leaving identical preference orders (see Definition 4.1.1). Moreover, the case $\operatorname{dom} g = \{1\}$ corresponds to the expected loss risk measure $X \mapsto \mathbb{E}[-X]$ and is not interesting.

This is a family of convex risk measures. The function g may be interpreted as a *target risk profile*. Indeed, a position is acceptable if and only if $\mathrm{ES}^\alpha(X) \leq g(\alpha)$ for all $\alpha \in (0,1]$. In this way, we achieve greater control of the loss tail.

We proceed to state the dual representation of g-adjusted ES. To this end, for $\beta \in [0,1)$ set $\mathcal{G}_\beta := \{g \in \mathcal{G} : \inf \operatorname{dom} g = \beta\}$.

Proposition 4.3.15. *Let $g \in \mathcal{G}$. Then $\mathrm{ES}^g : L^1 \to (-\infty, \infty]$ satisfies the dual representation*

$$\mathrm{ES}^g(X) = \sup_{Z\in\mathcal{D}}\{\mathbb{E}[-ZX] - \alpha^g(Z)\} = \sup_{Z\in\mathcal{Q}^{\alpha^g}}\{\mathbb{E}[-ZX] - g(\|Z\|_\infty^{-1})\}$$

where the penalty function $\alpha^g : \mathcal{D} \to [0,\infty]$ is given by $\alpha^g(Z) = g(\|Z\|_\infty^{-1})$ if $Z \in \mathcal{D} \cap L^\infty$ and $\alpha^g(Z) = \infty$ otherwise. Moreover, $\mathcal{Q}^{\alpha^g} = \{Z \in \mathcal{D} \cap L^\infty : g(\|Z\|_\infty^{-1}) < \infty\}$ is convex and satisfies

$$\mathcal{Q}^{\alpha^g} = \begin{cases} \mathcal{D} \cap L^\infty, & \text{if } g \in \mathcal{G}_0, \\ \{Z \in \mathcal{D} : \|Z\|_\infty \le \frac{1}{\beta}\}, & \text{if } g \in \mathcal{G}_\beta,\ \beta \in (0,1) \text{ and } g(\beta) < \infty, \\ \{Z \in \mathcal{D} : \|Z\|_\infty < \frac{1}{\beta}\}, & \text{if } g \in \mathcal{G}_\beta,\ \beta \in (0,1) \text{ and } g(\beta) = \infty. \end{cases} \tag{4.18}$$

Proof. The dual representation has been shown in [21, Proposition 3.7]. (4.18) follows directly from the definition of α^g, which also gives convexity of $\mathcal{Q}^{\alpha^g}$. □

Combining this dual representation with Proposition 4.2.12 and Theorem 4.2.13 allows us to immediately classify those g-adjusted ES risk measures that are suitable for risk management/portfolio selection.

Corollary 4.3.16. *Let $g \in \mathcal{G}$. Then,*

(a) ES^g *is suitable for risk management if and only if $g \in \mathcal{G}_0$.*

(b) ES^g *is suitable for portfolio selection if and only if $g \in \mathcal{G}_0$ and there exists $a \in \mathbb{R}$ and $b > 0$ such that $g(x) \ge a + b/x$ for all $x \in (0,1]$.*

We can further provide a dual characterisation of (strong) ES^g-arbitrage when $g \in \mathcal{G}_\beta$ and $\beta \in (0,1)$. In this case, since $\mathcal{Q}^{\alpha^g}$ is L^∞-bounded, Conditions I and UI are both satisfied if the returns lie in L^1. Moreover, it is not difficult to check that

$$\tilde{\mathcal{Q}}^{\alpha^g} = \{Z > 0 \in \mathcal{D} : \|Z\|_\infty < 1/\beta\}$$

is a subset of $\mathcal{Q}^{\alpha^g}$ that satisfies Conditions POS, MIX and INT and contains

1; see Proposition A.6.7 for details. Finally, Proposition A.6.19 shows that

$$\mathcal{Q}^{\overline{\text{co}}\,\alpha^g} = \begin{cases} \{Z \in \mathcal{D} : \|Z\|_\infty \leq \frac{1}{\beta}\}, & \text{if } g \in \mathcal{G}_\beta^\infty, \\ \{Z \in \mathcal{D} : \|Z\|_\infty < \frac{1}{\beta}\}, & \text{if } g \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty, \end{cases}$$

where $\mathcal{G}_\beta^\infty := \{g \in \mathcal{G}_\beta : g \text{ is bounded on its effective domain}\}$. Thus, Theorems 4.2.6 and 4.2.9 yield the following result.

Corollary 4.3.17. *Let $g \in \mathcal{G}_\beta$ where $\beta \in (0,1)$ and assume the market (S^0, S) has returns in L^1.*

(a) *(S^0, S) does not admit ES^g-arbitrage if and only if there exists $Z \in \mathcal{P}$ such that $\|Z\|_\infty < \frac{1}{\beta}$.*

(b) *When $g \in \mathcal{G}_\beta^\infty$ ($g \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty$), (S^0, S) does not admit strong ES^g-arbitrage if and only if there exists $Z \in \mathcal{M}$ with $\|Z\|_\infty \leq (<) \frac{1}{\beta}$.*

Remark 4.3.18. This result shows that the implication "(c) $\Longrightarrow$ (b)" in Proposition 4.1.14 does not hold. Indeed, if $g \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty$ for $\beta \in (0,1)$ and there exists no $Z \in \mathcal{M}$ with $\|Z\|_\infty < \frac{1}{\beta}$ but a $Z \in \mathcal{M}$ with $\|Z\|_\infty = \frac{1}{\beta}$, then the market admits strong ρ-arbitrage for $\rho = \mathrm{ES}^g$. However, since the L^1-closure of $\mathcal{Q}^{\alpha^g}$ is $\{Z \in \mathcal{D} : \|Z\|_\infty \leq \frac{1}{\beta}\}$, it follows from (4.5) and Theorem 3.2.14 that the market does not admit strong ρ^∞-arbitrage.

4.3.3 Loss Sensitive Expected Shortfall

Since the minimal requirement for mean-risk portfolio selection is that the returns lie in L^1, we are particularly interested in studying risk measures defined on L^1 that are suitable for portfolio selection. By Theorem 4.2.13, a convex risk measure $\rho : L^1 \to (-\infty, \infty]$ is suitable for portfolio selection if and only if

$$\rho(X) = \sup_{Z \in \mathcal{D}} \{\mathbb{E}[-ZX] - \alpha(Z)\} = \sup_{Z \in \mathcal{Q}^\alpha} \{\mathbb{E}[-ZX] - \alpha(Z)\},$$

for some quasi-convex penalty function $\alpha : \mathcal{D} \to [0, \infty]$ where the set $\mathcal{Q}^\alpha$ is $\sigma(L^\infty, L^1)$-dense in $\mathcal{D} \cap L^\infty$ and there exists $a \in \mathbb{R}$ and $b > 0$ such that $\alpha(Z) \geq a + b\|Z\|_\infty$ for all $Z \in \mathcal{D}$. This class is large since the restrictions on α are not very limiting. Nevertheless, to the best of our knowledge, it has never considered before in the literature. We would like to find risk measures in this class that are "close" to ES.

The natural way to go about this is to assume (just like in the dual representation of ES) that the penalty function depends only on $\|Z\|_\infty$. And from an economic perspective, it seems sensible to assume that measures that are "further away" from $\mathbb{P}$ are punished more severely, i.e., α depends on $\|Z\|_\infty$ in a nondecreasing way. This sub-family coincides exactly with the class of g-adjusted Expected Shortfall risk measures that are suitable for portfolio selection.

Proposition 4.3.19. *Let ρ be a convex risk measure on L^1. The following are equivalent:*

(a) *ρ is suitable for portfolio selection and admits a dual representation that depends only on $\|Z\|_\infty$ in a nondecreasing way.*

(b) *$\rho \equiv \mathrm{ES}^g$ for some $g \in \mathcal{G}_0$ where there exists $a \in \mathbb{R}$ and $b > 0$ such that $g(x) \geq a + b/x$.*

Of particular interest is the case when the penalty function is linear in $\|Z\|_\infty$. By the fact that $g(1) = 0$, this implies that $g(x) = -b + b/x = b(1/x - 1)$ for some $b > 0$.

Definition 4.3.20. Let $b > 0$ be a sensitivity parameter. The *Loss Sensitive Expected Shortfall* (LSES) of $X \in L^1$ at level b is defined by

$$\mathrm{LSES}^b(X) := \sup_{\alpha \in (0,1]} \left\{ \mathrm{ES}^\alpha(X) - b\left(\frac{1}{\alpha} - 1\right)\right\} \tag{4.19}$$

$$= \sup_{Z \in D \cap L^\infty} \left\{ \mathbb{E}[-ZX] - b(\|Z\|_\infty - 1)\right\}, \tag{4.20}$$

where (4.20) follows from Proposition 4.3.15.

Remark 4.3.21. To the best of our knowledge, the risk measure LSES^b has first been considered in [26, Example 8.3], where it was introduced as an example (without name) in the class of so-called *Delta spectral risk measures*.

The smaller the parameter b, the more conservative LSES^b is. The following result shows that the supremum in (4.19) is attained at some $\alpha^* \in (0, 1]$ and LSES^b is a convex combination between $\mathrm{ES}^{\alpha^*}(X)$ and $\mathrm{VaR}^{\alpha^*}(X)$, where the confidence level α^* is chosen endogenously depending on b and X.

Proposition 4.3.22. *Let $X \in L^1$ and $b > 0$. Then*

$$\mathrm{LSES}^b(X) = \max_{\alpha\in(0,1]} \{\mathrm{ES}^\alpha(X) - b(\tfrac{1}{\alpha} - 1)\},$$

where the maximum is attained for α^* given by

$$\alpha^* := \sup\left\{\alpha \in (0,1] : \int_0^\alpha \big(\mathrm{VaR}^u(X) - \mathrm{VaR}^\alpha(X)\big)\,\mathrm{d}u \le b\right\}. \tag{4.21}$$

Moreover, if X has a continuous distribution, then α^* is the unique maximum and

$$\mathrm{LSES}^b(X) = \alpha^*\mathrm{ES}^{\alpha^*}(X) + (1-\alpha^*)\mathrm{VaR}^{\alpha^*}(X). \tag{4.22}$$

Proof. First, note the supremum in the definition of LSES^b is attained because the function $\alpha \mapsto \mathrm{ES}^\alpha(X)$ is continuous on $(0,1]$ (by the definition of Expected Shortfall) and $\lim_{\alpha\to 0}(\mathrm{ES}^\alpha(X) - b(\frac{1}{\alpha}-1)) = -\infty$ (by the fact that $\lim_{\alpha\to0}\alpha\mathrm{ES}^\alpha(X) = 0$).

Next, we show that the maximum is attained for α^* defined by (4.21). To that end, consider the function $I_X : (0,1] \to \mathbb{R}_+$ given by

$$I_X(\alpha) := \alpha\mathrm{ES}^\alpha(X) + \alpha\mathrm{VaR}^\alpha(X) = \int_0^\alpha (\mathrm{VaR}^u(X) - \mathrm{VaR}^\alpha(X))\,\mathrm{d}u.$$

Since $u \mapsto \mathrm{VaR}^u(X)$ is nonincreasing, it follows I_X is nondecreasing. By the definition of Expected Shortfall, and the fundamental theorem of calculus for absolutely continuous functions, it follows that

$$G_X(\alpha) := \mathrm{ES}^\alpha(X) - b(\tfrac{1}{\alpha}-1) = \mathbb{E}[-X] + \int_\alpha^1 \frac{1}{u^2}(I_X(u) - b)\,\mathrm{d}u.$$

By the definitions of α^* and the fact that I_X is nondecreasing, it follows that G_X attains its maximum at α^*.

Finally, if X has a continuous distribution, then $\alpha \mapsto \mathrm{VaR}^\alpha(X)$ and $\alpha \mapsto I_X(\alpha)$ are continuous and strictly monotone on $(0,1]$. Hence, the maximal α^* is unique. If $\alpha^* = 1$, then (4.22) is automatically satisfied. Otherwise, if $\alpha^* \in (0,1)$, then $I_X(\alpha^*) = b$ and

$$\begin{aligned}\mathrm{LSES}^b(X) &= \mathrm{ES}^{\alpha^*}(X) - (\alpha^*\mathrm{ES}^{\alpha^*}(X) - \alpha^*\mathrm{VaR}^{\alpha^*}(X))\left(\frac{1}{\alpha^*} - 1\right)\\ &= \alpha^*\mathrm{ES}^{\alpha^*}(X) + (1-\alpha^*)\mathrm{VaR}^{\alpha^*}(X).\end{aligned}$$

□

Remark 4.3.23. (a) It is not difficult to check that $\alpha \in (0,1]$ is a maximiser of (4.19) if and only if $\alpha \in [\alpha_-^*, \alpha_+^*]$ where

$$\alpha_-^* := \inf \left\{ \alpha \in (0,1] : \int_0^\alpha \big(\mathrm{VaR}^u(X) - \mathrm{VaR}^\alpha(X)\big)\, \mathrm{d}u \geq b \right\},$$
$$\alpha_+^* := \sup \left\{ \alpha \in (0,1] : \int_0^\alpha \big(\mathrm{VaR}^u(X) - \mathrm{VaR}^\alpha(X)\big)\, \mathrm{d}u \leq b \right\},$$

with the convention that $\inf \emptyset := 1$.

(b) Since $Z \mapsto \frac{1}{\beta}(\|Z\|_\infty - 1)$ is convex and $\sigma(L^\infty, L^1)$-lower semi-continuous, it follows from [25, Theorem 4.3] that the supremum in (4.20) is also a maximum.

The following example computes $\mathrm{LSES}^b(X)$ for a normal distribution and illustrates the dependence of α^* on b.

Example 4.3.24. Let $X \sim N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. Denote by ϕ and Φ the pdf and cdf of a standard normal distribution. Then for any $b > 0$, by (4.21), it is not difficult to check that the corresponding $\alpha^* \in (0,1)$ satisfies

$$\phi(\Phi^{-1}(\alpha^*)) - \alpha^* \Phi^{-1}(1 - \alpha^*) = \frac{b}{\sigma}.$$

Figure 8 gives a graphical illustration of the dependence of α^* on $\frac{b}{\sigma}$. In particular it shows that for fixed b, α^* is decreasing in σ as expected.

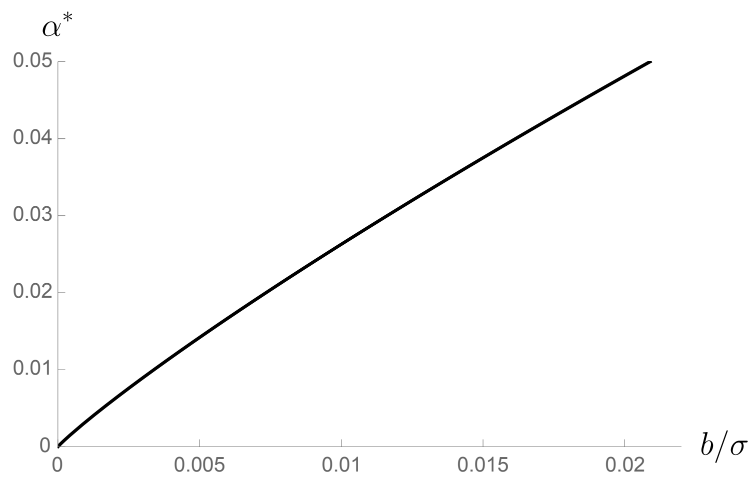


Figure 8: Dependence of α^* on b/σ.

Closing Remarks

The goal of this thesis has been to answer the three questions posed in the introduction. We have seen that essentially (Q1) and (Q2) have positive answers if and only if ρ satisfies weak and strong sensitivity to large losses on the set of excess returns, respectively.

En passant, we have also discovered the key relationship between mean-ρ portfolio selection and mean-ρ^∞ portfolio selection, where ρ^∞ is the *smallest* positively homogeneous risk functional that dominates ρ. This relationship is in particular crucial for the dual characterisation of ρ-arbitrage (and hence "no-ρ-arbitrage" pricing, cf. Remark 4.2.10(b)) when ρ is a convex risk measure that admits a dual representation. Indeed, under mild assumptions on the dual set $\mathcal{Q}$, the market does not admit ρ-arbitrage if and only if $\mathcal{P} \cap \tilde{\mathcal{Q}} \neq \emptyset$ for some nonempty $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ satisfying POS, MIX and INT. We have also demonstrated that $\tilde{\mathcal{Q}}$ can be computed explicitly for a large variety of risk measures.

But most importantly, the relationship between mean-ρ portfolio selection and mean-ρ^∞ portfolio selection has allowed us to fully answer (Q3), which is arguably the most important question, both in a primal and a dual fashion, culminating in Theorem 4.2.13. As a key example of a risk measure suitable for portfolio selection on L^1, we have introduced the new risk measure *Loss Sensitive Expected Shortfall* which is "close" to ES but strongly sensitive to large losses.

Future Directions

The results and methodology in this thesis open the way for many advances in risk management. For example the interplay between ρ (star-shaped) and ρ^∞ (positively homogeneous) is interesting, and has the potential to be utilised in other applications. Furthermore, the axioms of weak and strong sensitivity to large losses lead to a *new* class of risk measures that are suitable for risk management/portfolio selection. It would be interesting to apply these risk measures to other problems in financial mathematics. In particular, it would be very worthwhile to study the properties of Loss Sensitive Expected Shortfall in more detail.

As for mean-ρ portfolio selection, there is a large literature when working under a *fixed* probability measure. However, in practise the exact distribution of the future outcomes is difficult to get. Incorporating *model uncertainty* is a natural next step. Robust mean-*variance* portfolio selection has already been considered in the literature, cf. [67, 20]. But to the best of our knowledge, there is so far no work on *robust mean-risk portfolio selection* for a coherent or convex risk measure.

A

Appendix

A.1 Expected Shortfall Deviation

In this appendix we recall the definition and key properties of the so-called *Expected Shortfall Deviation*, using general results on deviation risk measures from [81, Sections 3 and 4]. We then apply these properties to show that the set Π_ν^α from Definition 2.1.4 is nonempty, compact and convex.

Definition A.1.1. Let $\alpha \in (0, 1)$ be a confidence level and X an integrable random variable. The *Expected Shortfall Deviation* (ESD) of X at level α is given by

$$\mathrm{ESD}^\alpha(X) := \mathrm{ES}^\alpha(X - \mathbb{E}[X]) = \mathrm{ES}^\alpha(X) + \mathbb{E}[X], \tag{A.1}$$

where the second equality follows from cash-invariance of ES.

It is shown in [79, Example 4] that ESD is an example of a so-called *deviation risk measure*, which generalises the notion of the standard deviation. (Note that ESD is referred to as $\mathrm{CVaR}_\alpha^\Delta$ in [79].) It therefore satisfies the following axioms:

(D1) $\mathrm{ESD}^\alpha(X + c) = \mathrm{ESD}^\alpha(X)$ for all $X \in L^1$ and $c \in \mathbb{R}$,

(D2) $\mathrm{ESD}^\alpha(\lambda X) = \lambda \mathrm{ESD}^\alpha(X)$ for all $X \in L^1$ and $\lambda \geq 0$,

(D3) $\mathrm{ESD}^\alpha(X_1 + X_2) \leq \mathrm{ESD}^\alpha(X_1) + \mathrm{ESD}^\alpha(X_2)$ for all X_1 and X_2 in L^1

(D4) $\mathrm{ESD}^\alpha(X) \geq 0$, for all $X \in L^1$, where the inequality is strict if and only if X is not $\mathbb{P}$-a.s. constant.

Note that (D4) together with (A.1) implies in particular that

$$\mathrm{ES}^{\alpha}(X) \geq \mathbb{E}[-X] \tag{A.2}$$

where the inequality is strict if and only if X is not $\mathbb{P}$-a.s. constant.

For $\alpha \in (0,1)$, we define the function $f_{\mathrm{ESD}^{\alpha}} : \mathbb{R}^d \to \mathbb{R}_+$ by

$$f_{\mathrm{ESD}^{\alpha}}(\pi) := \mathrm{ESD}^{\alpha}(\pi \cdot R). \tag{A.3}$$

The following result is a special case of [81, Proposition 4].

Proposition A.1.2. *Fix $\alpha \in (0,1)$. Then the function $f_{\mathrm{ESD}^{\alpha}}$ is convex. Moreover, it has the following properties:*

(a) $f_{\mathrm{ESD}^{\alpha}}(\pi) = 0$ *if and only if* $\pi = 0$.

(b) $f_{\mathrm{ESD}^{\alpha}}(\lambda\pi) = \lambda f_{\mathrm{ESD}^{\alpha}}(\pi)$ *for* $\pi \in \mathbb{R}^d$ *and* $\lambda \geq 0$.

(c) $f_{\mathrm{ESD}^{\alpha}}(\pi + \pi') \leq f_{\mathrm{ESD}^{\alpha}}(\pi) + f_{\mathrm{ESD}^{\alpha}}(\pi')$ *for all* $\pi, \pi' \in \mathbb{R}^d$

(d) *For each* $\delta \geq 0$, *the set* $\{\pi \in \mathbb{R}^d : f_{\mathrm{ESD}^{\alpha}}(\pi) \leq \delta\}$ *is compact.*

We proceed to show the sets Π_{ν}^{α} of optimal portfolios from Definition 2.1.4 are minimum level sets of $f_{\mathrm{ESD}^{\alpha}}$ restricted to Π_{ν}.

Lemma A.1.3. *Let $\alpha \in (0,1)$, $\nu \in \mathbb{R}$ and $\pi \in \Pi_{\nu}$. Then $\pi \in \Pi_{\nu}^{\alpha}$ if and only if $\pi \in \operatorname*{argmin}_{\pi \in \Pi_{\nu}} f_{\mathrm{ESD}^{\alpha}}(\pi)$.*

Proof. The definitions of ESD^{α} in (A.1), (1.1), the fact that $\pi \in \Pi_{\nu}$, property (D1) of ESD^{α} and the definition of $f_{\mathrm{ESD}^{\alpha}}$ in (A.3) give

$$\begin{aligned} \mathrm{ES}^{\alpha}(X_{\pi}) &= \mathrm{ESD}^{\alpha}(X_{\pi}) - \mathbb{E}[X_{\pi}] = \mathrm{ESD}^{\alpha}(\pi \cdot (R - r\mathbf{1})) - \nu \\ &= \mathrm{ESD}^{\alpha}(\pi \cdot R) - \nu = f_{\mathrm{ESD}^{\alpha}}(\pi) - \nu \end{aligned} \tag{A.4}$$

Since $\pi \in \Pi_{\nu}^{\alpha}$ if and only if $\pi \in \operatorname*{argmin}_{\pi \in \Pi_{\nu}} \mathrm{ES}^{\alpha}(X_{\pi})$, the claim follows. □

The following Corollary follows from combining Proposition A.1.2 and Lemma A.1.3.

Corollary A.1.4. *Fix $\nu \in \mathbb{R}$. Then Π_{ν}^{α} is non-empty, compact and convex. Moreover, $\Pi_0^{\alpha} = \{\mathbf{0}\}$ and $\Pi_{\nu}^{\alpha} = \nu \Pi_1^{\alpha}$.*

A.2 Coherent Risk Measures on Orlicz Spaces

The goal of this appendix is to recall some key definitions and results on Orlicz spaces and summarise the main results on the dual representation of a real-valued coherent risk measure defined on an Orlicz space.

Key Definitions and Results on Orlicz Spaces

We begin by recalling some key definitions and results relating to Orlicz spaces and Orlicz hearts; see [90, Chapter 10] and [39, Chapter 2] for details.

- A function $\Phi : [0, \infty) \to [0, \infty]$ is called a *Young function* if it is convex and satisfies $\lim_{x\to\infty} \Phi(x) = \infty$ and $\lim_{x\to 0} \Phi(x) = \Phi(0) = 0$. A Young function Φ is called *superlinear* if $\Phi(x)/x \to \infty$ as $x \to \infty$. (Note that a Young function is continuous except possibly at a single point, where it jumps to ∞. Thus a finite Young function is continuous.)

- Given a Young function Φ, the *Orlicz space* corresponding to Φ is given by
$$L^\Phi := \{X \in L^0 : \mathbb{E}[\Phi(c|X|)] < \infty \text{ for some } c > 0\},$$
and the *Orlicz heart* is the linear subspace
$$H^\Phi := \{X \in L^\Phi : \mathbb{E}[\Phi(c|X|)] < \infty \text{ for all } c > 0\}.$$

- L^Φ and H^Φ are Banach spaces under the *Luxemburg norm* given by
$$\|X\|_\Phi := \inf\left\{\lambda > 0 : \mathbb{E}\left[\Phi\left(\left|\tfrac{X}{\lambda}\right|\right)\right] \leq 1\right\}.$$

- For any Young function Φ, its convex conjugate $\Psi : [0, \infty) \to [0, \infty]$ defined by
$$\Psi(y) := \sup_{x\geq 0}\{xy - \Phi(x)\}$$
is also a Young function and its conjugate is Φ.

- If $X \in L^\Phi$ and $Y \in L^\Psi$, we have the generalised Hölder inequality:
$$\mathbb{E}[|XY|] \leq 2\|X\|_\Phi\|Y\|_\Psi. \tag{A.5}$$

- Using the conjugate Ψ and (A.5), we may define the *Orlicz norm* on L^Φ by
$$\|X\|_\Psi^* := \sup\{\mathbb{E}[XY] : Y \in L^\Psi,\ \|Y\|_\Psi \le 1\}.$$
This norm is equivalent to the Luxemburg norm on L^Φ.

- When Φ jumps to infinity, then $L^\Phi = L^\infty$ (and $\|\cdot\|_\Phi$ is equivalent to $\|\cdot\|_\infty$) and $H^\Phi = \{0\}$.

- When Φ is finite, the norm dual of the Orlicz heart $(H^\Phi, \|\cdot\|_\Phi)$ (with the Luxemburg norm) is the Orlicz space $(L^\Psi, \|\cdot\|_\Phi^*)$ (with the Orlicz norm).

- Φ is said to satisfy the Δ_2-condition if there exists a finite constant $K > 0$ such that $\Phi(2x) \le K\Phi(x)$ for all $x \in [0,\infty)$. Φ satisfies the Δ_2 condition if and only if $L^\Phi = H^\Phi$.

Dual Representation of Coherent Risk Measures on Orlicz Spaces

After these preparations, we consider the following setup: Let $\Phi : [0,\infty) \to [0,\infty]$ be a Young function and $\rho : L^\Phi \to \mathbb{R}$ a coherent risk measure. To give a review of when ρ admits a dual representation, we first consider two versions of the Fatou property.

Definition A.2.1. Let $\Phi : [0,\infty) \to [0,\infty]$ be a Young function and $\rho : L^\Phi \to (-\infty,\infty]$ a map. Then ρ is said to satisfy the

- *Fatou property* on L^Φ, if $X_n \to X$ $\mathbb{P}$-a.s. for $X_n, X \in L^\Phi$ and $|X_n| \le Y$ $\mathbb{P}$-a.s. for some $Y \in L^\Phi$ implies that $\rho(X) \le \liminf_{n\to\infty} \rho(X_n)$.

- *strong Fatou property* on L^Φ, if $X_n \to X$ $\mathbb{P}$-a.s. for $X_n, X \in L^\Phi$ and $\|X_n\|_\Phi \le K < \infty$ implies that $\rho(X) \le \liminf_{n\to\infty} \rho(X_n)$.

The strong Fatou property implies the Fatou property but the converse is not true. Note, however, that the two are equivalent if $L^\Phi = L^\infty$.

Remark A.2.2. The notion of strong Fatou property has been introduced by [49] who noted in [50] that for a general normed vector space L, the Fatou property for risk measures (which was originally only formulated on L^∞) could either be understood in terms of *order bounded* sequences (giving the Fatou property) or *norm bounded* sequences (giving the strong Fatou property).

We proceed to summarise the existing dual representation results for (finite) coherent risk measures on Orlicz spaces from the literature. (For Orlicz hearts, the representation result for (finite) coherent risk measures is given in [25, Corollary 4.2].)

Theorem A.2.3. *Let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function with conjugate Ψ and $\rho : L^\Phi \to \mathbb{R}$ a coherent risk measure. Then ρ admits a dual representation under the following conditions:*

(a) *Φ satisfies the Δ_2-condition.*

(b) *Ψ satisfies the Δ_2-condition and ρ satisfies the Fatou property.*

(c) *Φ is a superlinear Young function and ρ satisfies the strong Fatou property.*

Proof. (a) In this case $L^\Phi = H^\Phi$ and the result follows from [25, Corollary 4.2].

(b) This follows from [49, Theorem 2.5] or [36, Proposition 2.5] and Fenchel-Moreau duality. (Note that for "(4) $\Rightarrow$ (1)" in [49, Theorem 3.7], the assumption of an atomless probability space is not needed.)

(c) This follows from [35, Theorem 3.2] in the case that $L^\Phi = L^\infty$ and from [50, Theorem 2.4] in the general case. □

Remark A.2.4. (a) If a coherent risk measure $\rho : L^\Phi \to \mathbb{R}$ admits a dual representation, it is straightforward to check that it satisfies the Fatou property. The converse is false if both Φ and Ψ fail to satisfy the Δ_2-condition; see [49, Theorem 4.2] for a generic counterexample. (Note, however, that ρ in [49, Theorem 4.2] is $(-\infty, \infty]$-valued.)

(b) A coherent risk measure that admits a dual characterisation does not need to satisfy the strong Fatou property; in fact if Φ is a superlinear Young function and $\rho : L^\Phi \to \mathbb{R}$ admits a dual characterisation such that $\mathcal{Q}_\rho \not\subset H^\Psi$, then ρ fails to satisfy the strong Fatou property by [50, Theorem 2.4].

Finally, we show that all coherent risk measures on Orlicz spaces that satisfy a dual representation (independent of whether one of the conditions of Theorem A.2.3 is satisfied) have a nice maximal dual set.

Proposition A.2.5. *Let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function with conjugate Ψ and $\rho : L^\Phi \to \mathbb{R}$ a coherent risk measure. If ρ admits a dual representation, then the maximal dual set $\mathcal{Q}_\rho$ is L^Ψ-closed and L^Ψ-bounded if Φ*

is finite. If Φ satisfies the Δ_2-condition, $\mathcal{Q}_\rho$ is also L^1-closed. Moreover, if $\mathcal{Q} \subset \mathcal{Q}_\rho$ has L^Ψ-closure $\mathcal{Q}_\rho$, then $\mathcal{Q}$ represents ρ, and if $\mathcal{Q} \subset \mathcal{Q}_\rho$ represents ρ, then $\mathcal{Q}_\rho \subset \bar{\mathcal{Q}}$, where $\bar{\mathcal{Q}}$ denotes the L^1-closure.

Proof. If Φ jumps to ∞, i.e., $L^\Phi = L^\infty$, then the result follows from [35, Theorem 3.2]. So assume for the rest of the proof that Φ is finite. Denote by ρ_H the restriction of ρ to H^Φ. Then $\mathcal{A}_{\rho_H} \subset \mathcal{A}_\rho$ and hence $\mathcal{Q}_{\rho_H} \supset \mathcal{Q}_\rho$. It follows from [25, Corollary 4.2] and Proposition A.6.5(a) that $\mathcal{Q}_{\rho_H}$ is L^Ψ-bounded and L^1-closed. Hence, $\mathcal{Q}_\rho$ is L^Ψ-bounded. This together with the definition of $\mathcal{Q}_\rho$ and the generalised Hölder inequality (A.5) implies that $\mathcal{Q}_\rho$ is L^Ψ-closed. If Φ satisfies the Δ_2-condition then $\mathcal{A}_{\rho_H} = \mathcal{A}_\rho$ and so $\mathcal{Q}_\rho = \mathcal{Q}_{\rho_H}$ is L^1-closed. Moreover, if $\mathcal{Q} \subset \mathcal{Q}_\rho$ has L^Ψ-closure $\mathcal{Q}_\rho$, then $\mathcal{Q}$ represents ρ by the generalised Hölder inequality (A.5) and if $\mathcal{Q} \subset \mathcal{Q}_\rho$ represents ρ, then $\bar{\mathcal{Q}}_\rho = \bar{\mathcal{Q}}$ because otherwise by the Hahn-Banach separation theorem (for the pairing (L^∞, L^1)), there exists $X \in L^\infty$ such that $\sup_{Z \in \bar{\mathcal{Q}}_\rho} \mathbb{E}[-ZX] \neq \sup_{Z \in \bar{\mathcal{Q}}} \mathbb{E}[-ZX]$. □

A.3 Counterexamples

In this appendix, we give counterexamples to complement our theory.

Example A.3.1. In this example we show that if all assumptions of Theorem 3.1.9 hold, but $\{\mathbf{0}\} \subsetneq \Pi_0^\rho$, the result fails.

Let $\Omega = [-5, 5] \times [1, 7] \subset \mathbb{R}^2$ with the Borel σ-algebra and the uniform probability measure $\mathbb{P}$. Let $r = 0$ and assume there are two risky assets with returns $R^i(\omega) := \omega_i$ for $\omega = (\omega_1, \omega_2) \in \Omega$ and $i \in \{1, 2\}$. Let C be the closed ball of radius 2 centred at $(2, 4)$, and for each $(x, y) \in C$, let $C_{(x,y)}$ be the closed ball of radius 1 centred at (x, y), and $Z_{(x,y)}$ the Radon-Nikodým derivative of the uniform probability measure on $C_{(x,y)}$ with respect to $\mathbb{P}$. Define the risk measure ρ via its dual set

$$\mathcal{Q} = \{Z_{(x,y)} : (x, y) \in C\},$$

and note that $\mathbb{E}[Z_{(x,y)} R^1] = x$ and $\mathbb{E}[Z_{(x,y)} R^2] = y$. For this financial market (that is nonredundant and nondegenerate), $\mathbb{E}[R^1] = 0$, $\mathbb{E}[R^2] = 4$, and $\Pi_1 = \{(\pi^1, \pi^2) : \pi^1 \in \mathbb{R},\ \pi^2 = 1/4\}$. Thus, for every $\pi \in \Pi_1$ and $(x, y) \in C$,

$$\mathbb{E}[-Z_{(x,y)} X_\pi] = -\pi \cdot (x, y) = -(\pi^1 x + \tfrac{1}{4} y).$$

It follows that for any $\pi \in \Pi_1$,

$$\rho(X_\pi) = \sup_{(x,y)\in C} -(\pi^1 x + \tfrac{1}{4}y) = \tfrac{1}{2}\sqrt{16(\pi^1)^2+1} - 2\pi^1 - 1 =: g(\pi^1).$$

Therefore, $\rho_1 = \inf\{\rho(X_\pi) : \pi \in \Pi_1\} = \inf\{g(\pi^1) : \pi^1 \in \mathbb{R}\} = -1$ is not attained, since g is strictly decreasing. Thus, Π_1^ρ is empty, even though ρ satisfies the Fatou property on $\{X_\pi : \pi \in \mathbb{R}^d\}$ and $\rho_1 \in \mathbb{R}$. The reason Theorem 3.1.9 fails is because $\Pi_0^\rho = \{(\pi^1,\pi^2) : \pi^1 \geq 0, \pi^2 = 0\} \supsetneq \{\mathbf{0}\}$.

Example A.3.2. In this example we show that when Condition I is not satisfied, the set $C_{\mathcal{Q}}$ from (3.7) may fail to be a convex subset of $\mathbb{R}^d$ and (3.8) may break down.

Take $\Omega = [0,1]$, with the Borel σ-algebra and the Lebesgue measure $\mathbb{P}$. Suppose $r = 0$ and there are two risky assets with returns

$$R^1(\omega) := \begin{cases} \frac{3}{\sqrt{\omega}}, & \text{if } \omega < \frac{1}{16}, \\ -\frac{8}{15}, & \text{if } \omega \geq \frac{1}{16}, \end{cases} \quad \text{and} \quad R^2(\omega) := \begin{cases} -\frac{1}{\sqrt{\omega}}, & \text{if } \omega < \frac{1}{16}, \\ \frac{24}{15}, & \text{if } \omega \geq \frac{1}{16}. \end{cases}$$

Let $\mathcal{Q} := \{\lambda Z + (1-\lambda) : \lambda \in [0,1]\}$, where

$$Z(\omega) := \begin{cases} \frac{2}{\sqrt{\omega}}, & \text{if } \omega < \frac{1}{16}, \\ 0, & \text{if } \omega \geq \frac{1}{16}. \end{cases}$$

Note that $\mathbb{E}[-R^1] = \mathbb{E}[-R^2] = -1$, $\mathbb{E}[-ZR^1] = -\infty$ and $\mathbb{E}[-ZR^2] = \infty$. Thus,

$$C_{\mathcal{Q}} = \{(-1,1), (-\infty,\infty)\},$$

which is neither convex nor a subset of $\mathbb{R}^2$.

Moreover, the portfolio $\pi = (\frac{1}{4}, \frac{3}{4})$ satisfies $\rho(X_\pi) = \mathbb{E}[-ZX_\pi] = 0$ but

$$\sup_{c\in C_{\mathcal{Q}}} (\pi \cdot c) = \max\{-1, -\infty\tfrac{1}{4} + \infty\tfrac{3}{4}\} \neq 0 = \rho(X_\pi),$$

and so (3.8) does not hold.

Example A.3.3. In this example we show that when only Condition I is satisfied but Condition UI is not, the set $C_{\bar{\mathcal{Q}}}$ from (3.9) may fail to be a subset of $\mathbb{R}^d$, whence (3.10) breaks down.

Take $\Omega = [0,1]$, with the Borel σ-algebra and the Lebesgue measure $\mathbb{P}$.

Suppose $r \neq 0$ (so the market is nondegenerate) and there is one risky asset with return

$$R(\omega) := \begin{cases} \frac{1}{\sqrt{\omega}}, & \text{if } \omega \in (0, \frac{1}{2}), \\ -\frac{1}{\sqrt{\omega - 1/2}}, & \text{if } \omega \in (\frac{1}{2}, 1). \end{cases}$$

Note that $R(\omega) = -R(\omega + \frac{1}{2})$ for $\omega \in (0, \frac{1}{2})$. Let $\mathcal{Q} = \{Z \in \mathcal{D} \cap L^\infty : Z(\omega) = Z(\omega + \frac{1}{2}) \text{ for all } \omega \in (0, 1/2)\}$. Then Condition I is satisfied and $\mathbb{E}[-Z(R - r)] = r$ for all $Z \in \mathcal{Q}$, whence $C_{\mathcal{Q}} = \{r\}$. Moreover, $\bar{\mathcal{Q}} = \{Z \in \mathcal{D} : Z(\omega) = Z(\omega + \frac{1}{2}) \text{ for all } \omega \in (0, 1/2)\}$ and $\frac{|R|}{\sqrt{8}} \in \bar{\mathcal{Q}}$. Since $\mathbb{E}[R(R)^-] = +\infty$, it follows that $C_{\bar{\mathcal{Q}}} = \{r, \infty\}$, which is neither convex, compact, nor a subset of $\mathbb{R}$. Finally, for $\pi = 1$, $\sup_{c \in C_{\mathcal{Q}}}(\pi \cdot c) = r \neq \infty = \sup_{c \in C_{\bar{\mathcal{Q}}}}(\pi \cdot c)$.

Example A.3.4. In this example we show that the converse of Proposition 3.2.17 fails.

Take $\Omega = [0, 1]$, with the Borel σ-algebra and the Lebesgue measure $\mathbb{P}$. Let $r = 0$ and assume there is one risky asset whose return R^1 is uniformly distributed on $[0, 1]$. Let ρ be the worst-case risk measure, cf. Section 3.3.1. Then Condition I is satisfied, $\mathcal{Q} \cap \mathcal{M} = \emptyset$ (because $\mathcal{M} = \emptyset$), but $\rho(X_\pi) \geq 0$ for any portfolio π. Therefore, by Theorem 3.1.16, this market does not admit strong ρ-arbitrage, even though $\mathcal{Q} \cap \mathcal{M} = \emptyset$.

Example A.3.5. In this example we show that when $\mathcal{Q}$ is uniformly integrable but $R\mathcal{Q}$ is not, Theorem 3.2.14 may fail.

Take $\Omega = [0, 1]$, with the Borel σ-algebra and the Lebesgue measure $\mathbb{P}$. Let the risk-free rate be given by $r = 1 + 12c$, where $c := \int_{1/4}^{1/3} \log(1/x) \, \mathrm{d}x$. Suppose there is one risky asset whose return is given by

$$R(\omega) = \begin{cases} \ln\left(\frac{1}{\omega}\right), & \text{if } \omega < \frac{1}{3}, \\ 0, & \text{if } \omega \in [\frac{1}{3}, \frac{2}{3}], \\ -1, & \text{if } \omega > \frac{2}{3}. \end{cases}$$

Next, for $n \geq 4$, set

$$Z_n(\omega) = \begin{cases} \frac{n}{\ln(1/\omega)}, & \text{if } \omega < \frac{1}{n}, \\ 0, & \text{if } \omega \in [\frac{1}{n}, \frac{1}{4}], \\ k_n, & \text{if } \omega \in (\frac{1}{4}, \frac{1}{3}], \\ 0, & \text{if } \omega > \frac{1}{3}, \end{cases}$$

where k_n is chosen so that $\mathbb{E}[Z_n] = 1$. Note that $k_n \uparrow 12$, and that Z_n converges in L^1 to $Z = 12\mathbb{1}_{(1/4,1/3]}$. Therefore, $(\cup_{n\geq 4}\{Z_n\}) \cup \{Z\}$ is uniformly integrable, and whence, if we let $\mathcal{Q}$ be the L^1-closed convex hull of $(Z_n)_{n\geq 4}$, Z and 1, it will also be uniformly integrable. Moreover,

$$\mathbb{E}[Z_n R] = 1 + k_n c \uparrow 1 + 12c \quad \text{but} \quad \mathbb{E}[ZR] = 12c.$$

It follows that the set $C_{\mathcal{Q}}$ is given by

$$C_{\mathcal{Q}} = \{\mathbb{E}[-Y(R-r)] : Y \in \mathcal{Q}\} = (0, d],$$

where $d := \mathbb{E}[-(R-r)] > 1$. Thus Condition I is satisfied, $\mathcal{Q}$ is uniformly integrable and $\bar{\mathcal{Q}} \cap \mathcal{M} = \mathcal{Q} \cap \mathcal{M} = \emptyset$, but the market does not admit strong ρ-arbitrage:

$$\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}}} (\pi \cdot c) \geq 0, \quad \text{for any portfolio } \pi \in \mathbb{R}.$$

Example A.3.6. In this example we show that when $R\mathcal{Q}$ is uniformly integrable but $\mathcal{Q}$ is not, Theorem 3.2.14 may fail.

Take $\Omega = [0, 1]$, with the Borel σ-algebra and the Lebesgue measure $\mathbb{P}$. Let $r = 0$ and suppose there is one risky asset whose return is given by

$$R(\omega) = \begin{cases} 1, & \text{if } \omega \leq \frac{1}{4}, \\ 0, & \text{if } \omega \in (\frac{1}{4}, \frac{3}{4}), \\ -\frac{1}{2}, & \text{if } \omega \geq \frac{3}{4}. \end{cases}$$

For $n \geq 2$, define the intervals $A_n := (\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n})$ and set

$$Z_n(\omega) = \begin{cases} 2^n - \frac{1}{n}, & \text{if } \omega \in A_n, \\ 0, & \text{if } \omega \in (\frac{1}{4}, \frac{3}{4}) \setminus A_n, \\ k_n, & \text{if } \omega \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \end{cases}$$

where k_n is chosen so that $\mathbb{E}[Z_n] = 1$. Note that $k_n \downarrow 0$. Let $\mathcal{Q}$ be the closed convex hull of $(Z_n)_{n\geq 2}$ and 1. Then $\mathcal{Q}$ is not uniformly integrable but $R\mathcal{Q}$ is. Moreover, $\mathbb{E}[R] = \frac{1}{8}$, $\mathbb{E}[Z_n R] = \frac{1}{2}k_n \downarrow 0$ and $\frac{1}{2}k_2 < \frac{1}{8}$. It follows that

$$C_{\mathcal{Q}} = \{\mathbb{E}[-Z(R-r)] : Z \in \mathcal{Q}\} = [-\tfrac{1}{8}, 0).$$

Thus $\bar{\mathcal{Q}} \cap \mathcal{M} = \mathcal{Q} \cap \mathcal{M} = \emptyset$, but the market does not admit strong ρ-arbitrage:

$$\rho(X_\pi) = \sup_{c \in C_{\mathcal{Q}}} (\pi \cdot c) \geq 0, \quad \text{for any portfolio } \pi \in \mathbb{R}.$$

Example A.3.7. In this example we show that when $\tilde{\mathcal{Q}}_{\max} = \emptyset$, Theorem 3.2.18 fails.

Take $\Omega = [0,1]$, with the Borel σ-algebra and the Lebesgue measure $\mathbb{P}$. Consider the financial market described in Example A.3.6. Let $\mathcal{Q}$ be the convex hull of the two densities 1 and $Y(\omega) = 2\mathbb{1}_{(1/2,1]}(\omega)$,

$$\mathcal{Q} = \{\mu Y + (1-\mu) : \mu \in [0,1]\}.$$

Then $\mathbb{E}[YR] = -\frac{1}{4}$ and $\mathbb{E}[R] = \frac{1}{8}$, so $C_{\mathcal{Q}} = [-\frac{1}{8}, \frac{1}{4}]$ and there is no ρ-arbitrage.

However, $\tilde{\mathcal{Q}} \cap \mathcal{P} = \emptyset$ because $\tilde{\mathcal{Q}}_{\max} = \emptyset$. Indeed, any $Z \in \mathcal{Q}$ is of the form

$$Z(\omega) = \begin{cases} 1-\mu, & \text{if } \omega \leq \frac{1}{2}, \\ 1+\mu, & \text{if } \omega > \frac{1}{2}, \end{cases}$$

for some $\mu \in [0,1]$. Therefore if $Z \in \mathcal{Q}$ and $\lambda > 0$, then $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}$ for some $\tilde{Z} \in \mathcal{D} \cap L^\infty$ implies that $\tilde{Z} \in \mathcal{Q}$ and since $\mathcal{Q}$ is not dense in $\mathcal{D} \cap L^\infty$, the result follows.

Example A.3.8. This example shows that without convexity the shape of the ρ-optimal boundary can be very irregular even though ρ satisfies the Fatou property and weak sensitivity to large losses on $\mathcal{X}$.

Consider a two-dimensional market, where $r = 0$ and the risky asset has return $R \in L^1$ with $\mathbb{P}[R < 0] > 0$ and $\mathbb{E}[R] = 1$. Let $\mathcal{X} := \{\nu R : \nu \in \mathbb{R}\}$, $l := \operatorname{ess\,sup}(-R)$ and define $\eta : \mathcal{X} \to (-\infty, \infty]$ by

$$\eta(\nu R) = \begin{cases} f(\nu), & \text{if } \nu \geq 0, \\ -\nu, & \text{if } \nu < 0, \end{cases}$$

where $f : \mathbb{R}_+ \to (-\infty, \infty]$ is a lower semi-continuous function with $f(0) = 0$ and for all $\nu \in \mathbb{R}_+$ and $\lambda \geq 1$, $f(\nu) \leq l\nu$ and $f(\lambda\nu) \geq \lambda f(\nu)$. It is not difficult to check that η can be extended to a risk measure $\rho : L^1 \to (-\infty, \infty]$ such that $\rho|_{\mathcal{X}} = \eta$. Moreover, ρ satisfies the Fatou property and weak sensitivity to large losses on $\mathcal{X}$. The ρ-optimal boundary is given by

$$\mathcal{O}_\rho = \{(f(\nu), \nu) : \nu \in \mathbb{R}_+\}.$$

This can be very irregular. For example when $l = 1$ and

$$\begin{aligned} f(\nu) = &-\nu \mathbb{1}_{[0,10]}(\nu) + (\tfrac{\nu}{2} - 15)\mathbb{1}_{(10,15]}(\nu) - 6\mathbb{1}_{(15,20]}(\nu) - \tfrac{\nu}{4}\mathbb{1}_{(20,40]}(\nu) \\ &+ (\tfrac{1}{10}(\nu - 39)^2 - 8)\mathbb{1}_{(40,47]}(\nu) + (10 - (\nu - 53)^2)\mathbb{1}_{(50,53]}(\nu) \\ &+ (\nu - 40)\mathbb{1}_{(53,\infty)}(\nu), \end{aligned}$$

the ρ-optimal boundary takes the form in Figure 9.

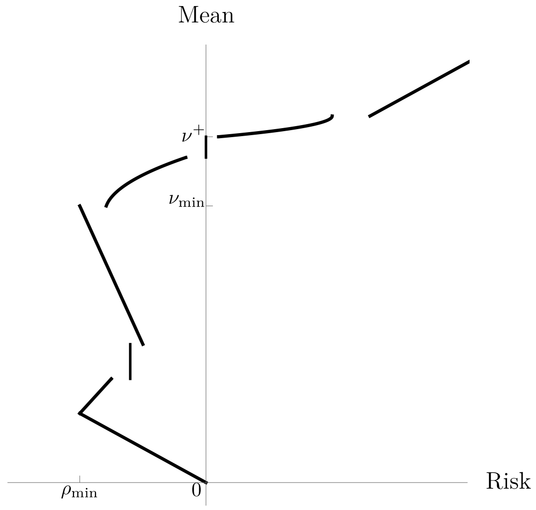


Figure 9: The ρ-optimal boundary corresponding to Example A.3.8

Example A.3.9. This example shows that in the absence of convexity, ρ-arbitrage and ρ^∞-arbitrage may not be equivalent.

Consider the two-dimensional market and risk measure $\rho : L^1 \to (-\infty, \infty]$ defined in Example A.3.8 with $f : \mathbb{R}_+ \to (-\infty, 0]$ given by $f(\nu) = -\nu \mathbb{1}_{[0,1]}$. Then ρ is a risk measure that satisfies the Fatou property and weak sensitivity to large losses on $\mathcal{X}$, but it is not convex. Moreover, $\mathcal{E}_\rho = \{(-1, 1)\} \neq \emptyset$ and the market does not satisfy ρ-arbitrage. However, ρ^∞ satisfies

$$\rho^\infty(\nu R) = \begin{cases} 0, & \text{if } \nu \geq 0, \\ -\nu, & \text{if } \nu < 0, \end{cases}$$

and it follows that $\mathcal{E}_{\rho^\infty} = \emptyset$, i.e., the market admits ρ^∞-arbitrage.

A.4 Star-Shapedness, Recession Cones and Recession Functions

The goal of this appendix is to state some results that relate star-shaped sets and functions (about the origin) with their corresponding recession cone and function, respectively. For a recent survey on star-shaped sets, see [53]. The definition of star-shaped functions varies in the literature. We follow [83] and view them as natural generalisations of convex functions. Finally, for an overview on recession cones and recession functions see [75, Chapter 8].

Let V be a vector space, S a nonempty subset of V and $f : V \to [-\infty, \infty]$ a function that is not identically infinity so that its epigraph, $\text{epi}\, f := \{(x, t) \in X \times \mathbb{R} : f(x) \le t\}$ is nonempty. Note that any function can be reconstructed from its epigraph.

- Given two points $x, y \in S$, we say *x sees y via S* if $\lambda x + (1 - \lambda)y \in S$ for all $\lambda \in [0, 1]$. The set S is called *star-shaped* if there exists $x \in S$ which sees every point $y \in S$ via S. We say the set S is *star-shaped about $s \in S$* if s sees every point $y \in S$ via S. Clearly, if S is convex, it is star-shaped about every one of its elements.

- The *recession cone* of the set S is defined by

  $$S^\infty := \{y \in V : \text{for all } x \in S \text{ and } \lambda \ge 0,\ x + \lambda y \in S\}.$$

  It contains all $y \in V$ such that S *recedes* in that direction. When S is star-shaped about the origin, then its recession cone is the largest cone contained in S, i.e., $S^\infty = \cap_{\lambda \in (0,\infty)} \lambda S$.

- The function f is called *star-shaped* if its epigraph is star-shaped. Of particular importance is when f is *star-shaped about the origin*, that is, when its epigraph is star-shaped about the origin. This is equivalent to the condition that $f(\lambda x) \ge \lambda f(x)$ for all $x \in V$ and $\lambda \ge 1$.

- The *recession function* of f is the function $f^\infty : V \to [-\infty, \infty]$ whose epigraph is the recession cone of the epigraph of f, i.e., $\text{epi}\,(f^\infty) = (\text{epi}\, f)^\infty$. When f is star-shaped about the origin, f^∞ is the positively homogeneous majorant of f and explicitly given by

$$f^\infty(x) = \lim_{t\to\infty} \frac{f(tx)}{t}.$$

Note that because f is star-shaped about the origin, for any $x \in V$, $k \geq 1$ and $s > 0$,

$$f((ks)x)/(ks) \geq kf(sx)/(ks) = f(sx)/s.$$

Whence, $f(tx)/t$ is nondecreasing in t and so f^∞ is well-defined as a $[-\infty, \infty]$-valued map on V.

A.5 Key Definitions and Results on Convex Analysis

In this appendix, we recall some key definitions and results regarding convex functions and convex conjugates.

Let X be a topological vector space and $f : X \to [-\infty, \infty]$ a function.

- The *epigraph* of f is given by

$$\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\}.$$

  Note that f can be recovered from its epigraph, $f(x) = \inf\{t \in \mathbb{R} : (x, t) \in \operatorname{epi} f\}$. Also, a function $g : X \to [-\infty, \infty]$ is dominated by f if and only if $\operatorname{epi} f \subset \operatorname{epi} g$.

- The *effective domain* of f is given by

$$\operatorname{dom} f := \{x \in X : f(x) < \infty\}.$$

  We say f is *proper* if $\operatorname{dom} f \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$.

- We say f is *convex* if $\operatorname{epi} f$ is a convex subset of $X \times \mathbb{R}$. Note that if f is convex, $\operatorname{dom} f$ is a convex subset of X.

- We say f is *quasi-convex* if $\{x \in X : f(x) \leq t\}$ is a convex subset of X for all $t \in \mathbb{R}$. Every convex function is quasi-convex, but the converse is not true. However, if f is quasi-convex, $\operatorname{dom} f$ is a convex subset of X.

- We say f is *lower semi-continuous* if $\operatorname{epi} f$ is a closed subset of $X \times \mathbb{R}$.

- The *convex hull* of f, $\operatorname{co} f : X \to [-\infty, \infty]$, is the largest convex function majorised by f,

$$\operatorname{co} f(x) := \sup\{g(x)|\, g : X \to [-\infty, \infty] \text{ is convex and } g \leq f\}.$$

By [76, Equation (3.5)],

$$\operatorname{epi} \operatorname{co} f = \{(x, t) \in X \times \mathbb{R} : (x, s) \in \operatorname{co} \operatorname{epi} f \text{ for all } s > t\}$$

where $\operatorname{co} \operatorname{epi} f := \bigcap\{C \subset X \times \mathbb{R} : \operatorname{epi} f \subset C \text{ and } C \text{ is convex}\}$. Moreover, one can check that $\operatorname{dom} \operatorname{co} f = \operatorname{co} \operatorname{dom} f$, where $\operatorname{co} \operatorname{dom} f = \bigcap\{C \subset X : \operatorname{dom} f \subset C \text{ and } C \text{ is convex}\}$.

- The *quasi-convex hull* of f, $\operatorname{qco} f : X \to [-\infty, \infty]$, is the largest quasi-convex function majorised by f,

$$\operatorname{qco} f(x) := \sup\{g(x)|\, g : X \to [-\infty, \infty] \text{ is quasi-convex and } g \leq f\}.$$

Since every convex function is quasi-convex, it follows $\operatorname{co} f \leq \operatorname{qco} f \leq f$. Moreover, one can verify that $\operatorname{dom} \operatorname{co} f = \operatorname{dom} \operatorname{qco} f = \operatorname{co} \operatorname{dom} f$.

- The *lower semi-continuous hull* of f, $\operatorname{lsc} f : X \to [-\infty, \infty]$ is the largest lower semi-continuous function majorised by f,

$$\operatorname{lsc} f(x) := \sup\{h(x)|\, h \text{ is lower semi-continuous and dominated by } f\}.$$

By [76, Equation (3.6)], $\operatorname{epi} \operatorname{lsc} f = \operatorname{cl} \operatorname{epi} f$, or equivalently we have [76, Equation (3.7)],

$$\operatorname{lsc} f(x) = \inf\{\liminf_{i \in I} f(x_i) : \lim_{i \in I} x_i = x\}. \tag{A.6}$$

In particular, this implies that $\operatorname{dom} \operatorname{lsc} f \subset \operatorname{cl} \operatorname{dom} f$.

- The *lower semi-continuous convex hull* of f, $\overline{\operatorname{co}} f : X \to [-\infty, \infty]$ is given by $\overline{\operatorname{co}} f := \operatorname{lsc} \operatorname{co} f$ (which may not be the same as $\operatorname{co} \operatorname{lsc} f$). Since the closure of a convex set is again convex and $\operatorname{epi} \overline{\operatorname{co}} f = \operatorname{cl} \operatorname{co} \operatorname{epi} f$, it follows that $\overline{\operatorname{co}} f$ is the largest lower semi-continuous convex function majorised by f. Moreover,

$$\operatorname{dom}\overline{\operatorname{co}}\, f \subset \operatorname{cl}\operatorname{co}\operatorname{dom} f. \tag{A.7}$$

- If Y is a nonempty subset of X and $f : Y \to [-\infty, \infty]$ a function, we can extend f to X by considering the function $\bar{f} : X \to [-\infty, \infty]$ defined by

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in Y, \\ \infty, & \text{if } x \in X \setminus Y. \end{cases}$$

  This extension is *natural* in that $\operatorname{epi}\bar{f} \subset Y \times \mathbb{R}$, $\operatorname{dom}\bar{f} \subset Y$, $\operatorname{dom}\operatorname{co}\bar{f}$, $\operatorname{dom}\operatorname{qco}\bar{f} \subset Y$ if Y is convex, $\operatorname{dom}\operatorname{lsc}\bar{f} \subset Y$ if Y is closed and also $\operatorname{dom}\overline{\operatorname{co}}\,\bar{f} \subset Y$ if Y is convex and closed. For this reason, if Y is convex, we may define the functions $\operatorname{co} f, \operatorname{qco} f : Y \to [-\infty, \infty]$ by $\operatorname{co} f(x) := \operatorname{co}\bar{f}(x)$, $\operatorname{qco} f(x) := \operatorname{qco}\bar{f}(x)$ and call this the convex hull and quasi-convex hull of f, respectively. Similarly, if Y is closed (and convex), we may define the functions $\operatorname{lsc} f : Y \to [-\infty, \infty]$ (and $\overline{\operatorname{co}}\, f : Y \to [-\infty, \infty]$) by $\operatorname{lsc} f(x) := \operatorname{lsc}\bar{f}(x)$ (and $\overline{\operatorname{co}}\, f(x) := \overline{\operatorname{co}}\,\bar{f}(x)$) and call this the lower-semi-continuous (convex) hull of f.

In order to discuss convex conjugates, we assume that $\langle X, X'\rangle$ is a dual pair under the duality $\langle\cdot,\cdot\rangle : X \times X' \to \mathbb{R}$, i.e., X and X' are vector spaces together with a bilinear functional $(x, x') \mapsto \langle x, x'\rangle$ such that

- If $\langle x, x'\rangle = 0$ for each $x' \in X'$, then $x = 0$;
- If $\langle x, x'\rangle = 0$ for each $x \in X$, then $x' = 0$.

We endow X with the weak topology, $\sigma(X, X')$,

$$x_\alpha \xrightarrow{w} x \text{ in } X \text{ if and only if } \langle x_\alpha, x'\rangle \to \langle x, x'\rangle \text{ in } \mathbb{R} \text{ for each } x' \in X',$$

and X' with the weak* topology, $\sigma(X', X)$,

$$x'_\alpha \xrightarrow{w^*} x' \text{ in } X' \text{ if and only if } \langle x, x'_\alpha\rangle \to \langle x, x'\rangle \text{ in } \mathbb{R} \text{ for each } x \in X.$$

They are locally convex and Hausdorff; the topological dual of $(X, \sigma(X, X'))$ is X'; and the topological dual of $(X', \sigma(X', X))$ is X; see [6, Section 5.14] for details.

- The *convex conjugate* of f, $f^* : X' \to [-\infty, \infty]$ is defined as

$$f^*(x') := \sup\{\langle x, x' \rangle - f(x) : x \in X\}$$

and the *biconjugate* of f, $f^{**} : X \to [-\infty, \infty]$ is defined as

$$f^{**}(x) := \sup\{\langle x, x' \rangle - f^*(x') : x' \in X'\}.$$

- It follows from [76, Theorem 5] that $\operatorname{epi} f^{**}$ is the intersection of all the "non-vertical" closed half spaces in $X \times \mathbb{R}$ that contain $\operatorname{epi} f$, i.e.,

$$f^{**}(x) = \sup\{a(x) \,|\, a : X \to \mathbb{R} \text{ is affine, continuous and } a \leq f\}, \quad \text{(A.8)}$$

where a function $a : X \to \mathbb{R}$ is *affine and continuous* if it is of the form $a(x) = \langle x, x' \rangle + c$ for some $x' \in X'$ and $c \in \mathbb{R}$.

- If $\overline{\text{co}}\, f(x) > -\infty$ for all $x \in X$, then $f^{**} = \overline{\text{co}}\, f$ by [76, Theorems 4 and 5]. In particular if f is convex, lower semi-continuous and proper, then $f = f^{**}$, which is the famous Fenchel-Moreau theorem.

A.6 Additional Results

Proposition A.6.1. *For $\mathcal{Q} \subset \mathcal{D}$, set*

$$L^1(\mathcal{Q}) := \{X \in L^0 : \lim_{a \to \infty} \sup_{Z \in \mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] = 0\}.$$

If $\mathcal{Q}$ is UI and $X \in L^1$, the following are equivalent:

(a) $X \in L^1(\mathcal{Q})$

(b) *$X\mathcal{Q}$ is UI.*

Proof. First assume that $X\mathcal{Q}$ is uniformly integrable. Then, for any $a > 0$ and $Z \in \mathcal{Q}$,

$$\begin{aligned}\mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] &= \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}\mathbb{1}_{\{Z\leq 1\}}] + \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}\mathbb{1}_{\{Z>1\}}] \\ &\leq \mathbb{E}[|X|\mathbb{1}_{\{|X|>a\}}] + \mathbb{E}[Z|X|\mathbb{1}_{\{Z|X|>a\}}].\end{aligned}$$

Taking the supremum over $\mathcal{Q}$ on both sides, letting $a \to \infty$ and using that $X \in L^1$ and $X\mathcal{Q}$ is UI yields

$$\lim_{a\to\infty} \sup_{Z\in\mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] \leq \lim_{a\to\infty} \mathbb{E}[|X|\mathbb{1}_{\{|X|>a\}}] + \lim_{a\to\infty} \sup_{Z\in\mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{Z|X|>a\}}] = 0.$$

Conversely, assume that $X \in L^1(\mathcal{Q})$. For any $a, b > 0$ and $Z \in \mathcal{Q}$,

$$\begin{aligned}\mathbb{E}[Z|X|\mathbb{1}_{\{Z|X|>a^2\}}] &\leq \mathbb{E}[Z|X|\mathbb{1}_{\{Z>a\}}] + \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] \leq \mathbb{E}[Z|X|\mathbb{1}_{\{Z>a;\,|X|\leq b\}}] \\ &\quad + \mathbb{E}[Z|X|\mathbb{1}_{\{Z>a\}}\mathbb{1}_{\{|X|>b\}}] + \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] \\ &\leq b\mathbb{E}[Z\mathbb{1}_{\{|Z|>a\}}] + \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>b\}}] + \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}].\end{aligned}$$

Taking the supremum over $\mathcal{Q}$ on both sides, letting $a \to \infty$ and using that $\mathcal{Q}$ is UI and $X \in L^1(\mathcal{Q})$ yields

$$\lim_{a\to\infty} \sup_{Z\in\mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{Z|X|>a\}}] \leq \sup_{Z\in\mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>b\}}].$$

Now, the result follows when letting $b \to \infty$ and using again $X \in L^1(\mathcal{Q})$. □

Proposition A.6.2. *Suppose Condition UI is satisfied.*

(a) *The set $\bar{\mathcal{Q}}$ and $R^i\bar{\mathcal{Q}}$ for $i \in \{1, \dots, d\}$ are uniformly integrable.*

(b) *The $\mathbb{R}^d$-valued map $F : \bar{\mathcal{Q}} \to \mathbb{R}^d$ given by $F(Z) = \mathbb{E}[-Z(R - r\mathbf{1})]$ is weakly continuous.*

Proof. (a) Fix $i \in \{1, \dots, d\}$. The Dunford-Pettis theorem implies that $\bar{\mathcal{Q}}$ and $\overline{R^i\mathcal{Q}}$ are UI. It suffices to show that $R^i\bar{\mathcal{Q}} \subset \overline{R^i\mathcal{Q}}$. So let $Z \in \bar{\mathcal{Q}}$. Then there exists a sequence $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{Q}$ such that Z_n converges to Z in L^1 and hence in probability. It follows that R^iZ_n converges to R^iZ in probability and hence also in L^1 as $(R^iZ_n)_{n\in\mathbb{N}} \subset R^i\mathcal{Q}$ is UI. It follows that $R^iZ \in \overline{R^i\mathcal{Q}}$.

(b) Since $F(\lambda Z^1 + (1-\lambda)Z^2) = \lambda F(Z^1) + (1-\lambda)F(Z^2)$ for $Z^1, Z^2 \in \bar{\mathcal{Q}}$ and $\lambda \in [0, 1]$, preimages under F of convex sets are convex. Since $\bar{\mathcal{Q}}$ is convex as $\mathcal{Q}$ is convex, it therefore suffices to show that F is strongly continuous. So let $(Z_n)_{n\in\mathbb{N}} \subset \bar{\mathcal{Q}}$ be a sequence that converges to Z in L^1 and hence in probability. Then $-Z_n(R^i - r)$ converges to $-Z(R^i - r)$ in probability and hence also in L^1 by part (a) for each $i \in \{1, \dots, d\}$. □

Lemma A.6.3. *Assume $\tilde{\mathcal{Q}} \subset \mathcal{D}$ satisfies Conditions POS and INT. Let $\tilde{Z} \in \tilde{\mathcal{Q}}$ and $X \in L^1$ be a non-constant random variable. If $\mathbb{E}[-\tilde{Z}X] = 0$, then there exists $Z \in \mathcal{Q}$ such that $\mathbb{E}[-ZX] > 0$.*

Proof. Note that $\tilde{Z} > 0$ $\mathbb{P}$-a.s. by Condition POS. Define $\tilde{\mathbb{Q}} \approx \mathbb{P}$ by $\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} := \tilde{Z}$ and $A := \{X < 0\}$. Since X is non-constant and $\mathbb{E}^{\tilde{\mathbb{Q}}}[X] = 0$, it follows that $\tilde{\mathbb{Q}}[A] \in (0, 1)$. Seeking a contradiction, suppose that $\mathbb{E}[-ZX] \leq 0$ for all $Z \in \mathcal{Q}$. Let $\mathcal{E}$ be an L^∞-dense subset of $\mathcal{D} \cap L^\infty$ corresponding to $\tilde{Z}$ in Condition INT. Let $Z' \in \mathcal{E}$. Then there exists $\lambda > 0$ such that $\lambda Z' + (1 - \lambda)\tilde{Z} \in \mathcal{Q}$. Thus, Since Z' was chosen arbitrarily, we may deduce that

$$\sup_{Z \in \mathcal{E}} (\mathbb{E}[-ZX]) \leq 0,$$

which together with Proposition A.6.6 below implies that $X \geq 0$ $\mathbb{P}$-a.s. Since $\tilde{\mathbb{Q}} \approx \mathbb{P}$, it follows that $\tilde{\mathbb{Q}}[A] = 0$ and we arrive at a contradiction. □

Proposition A.6.4. *Let $\Phi : [0, \infty) \to [0, \infty]$ be a Young function. Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence in L^Φ that converges in probability to some random variable Y. Then*

$$\|Y\|_\Phi \leq \liminf_{n \to \infty} \|Y_n\|_\Phi.$$

Proof. Set $K := \liminf_{n \to \infty} \|Y_n\|_\Phi$. We may assume without loss of generality that $K < \infty$. After passing to a subsequence, we may assume without loss of generality that $(Y_n)_{n \in \mathbb{N}}$ converges to Y $\mathbb{P}$-a.s. If Φ jumps to infinity, then $\|\cdot\|_\Phi$ is equivalent to $\|\cdot\|_\infty$ and the result follows. So assume that Φ is finite and hence continuous. For any $\varepsilon > 0$, we can pass to a further subsequence and assume without loss of generality that $\|Y_n\|_\Phi \leq K + \varepsilon$ for all n. Then by the definition of the Luxemburg norm, $\mathbb{E}[\Phi(|Y_n/(K + \varepsilon)|)] \leq 1$ for all n. Fatou's lemma gives

$$\mathbb{E}\left[\Phi\left(\left|\tfrac{Y}{K+\varepsilon}\right|\right)\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[\Phi\left(\left|\tfrac{Y_n}{K+\varepsilon}\right|\right)\right] \leq 1.$$

This implies $\|Y\|_\Phi \leq K + \varepsilon$. By letting $\varepsilon \to 0$, we conclude $\|Y\|_\Phi \leq K$. □

Proposition A.6.5. *Let $\Phi : [0, \infty) \to [0, \infty)$ be a finite Young function with conjugate Ψ. Let $\rho : H^\Phi \to \mathbb{R}$ be a coherent risk measure. Denote by $\mathcal{Q}_\rho$ the maximal dual set. Then*

(a) *$\mathcal{Q}_\rho$ is L^1-closed and L^Ψ-bounded.*

(b) *If $R \in H^\Phi$, then $R\mathcal{Q}_\rho$ is uniformly integrable.*

Proof. (a) It follows from [25, Corollary 4.2] that $\mathcal{Q}_\rho \cap L^\Psi$ is L^Ψ bounded and represents ρ. It suffices to show that $\mathcal{Q}_\rho \cap L^\Psi$ is L^1-closed. Indeed, this implies that $\mathcal{Q}_\rho \subset L^\Psi$ because otherwise, by the Hahn-Banach separation theorem (for the pairing (L^1, L^∞)), there exists $X \in L^\infty$ such that $\sup_{Z\in\mathcal{Q}_\rho\cap L^\Psi} \mathbb{E}[-ZX] < \sup_{Z\in\mathcal{Q}_\rho} \mathbb{E}[-ZX]$, in contradiction to the fact that both $\mathcal{Q}_\rho \cap L^\Psi$ and $\mathcal{Q}_\rho$ represent ρ on L^∞.

Set $K := \sup_{Z\in\mathcal{Q}_\rho} \|Z\|_\Psi < \infty$. Let $(Z_n)_{n\geq 1}$ be a sequence in $\mathcal{Q}_\rho \cap L^\Psi$ that converges to $Z \in L^1$. Then $Z \in \mathcal{D}$ and $\|Z\|_\Psi \leq K$ by Proposition A.6.4. Let $X \in \mathcal{A}_\rho \subset H^\Phi$. We have to show that $\mathbb{E}[ZX] \geq 0$. Since $\mathbb{E}[Z_nX] \geq 0$ by the fact that $Z_n \in \mathcal{Q}_\rho \cap L^\Psi$, it suffices to show that $\mathbb{E}[ZX] = \lim_{n\to\infty} \mathbb{E}[Z_nX]$. For any $n \in \mathbb{N}$ and $a_n > 0$, the generalised Hölder inequality (A.5) yields

$$\begin{aligned}
|\mathbb{E}[Z_nX] - \mathbb{E}[ZX]| &\leq \mathbb{E}[|X||Z_n - Z|] \\
&= \mathbb{E}[|X||Z_n - Z|\mathbb{1}_{\{|X|>a_n\}}] + \mathbb{E}[|X||Z_n - Z|\mathbb{1}_{\{|X|\leq a_n\}}] \\
&\leq \mathbb{E}[|X|Z_n\mathbb{1}_{\{|X|>a_n\}}] + \mathbb{E}[|X|Z\mathbb{1}_{\{|X|>a_n\}}] + a_n\|Z_n - Z\|_1 \\
&\leq (2K + 2K)\|X\mathbb{1}_{\{|X|>a_n\}}\|_\Phi + a_n\|Z_n - Z\|_1. \qquad \text{(A.9)}
\end{aligned}$$

Now if we choose $a_n := \min(n, \frac{1}{\sqrt{\|Z_n - Z\|_1}})$ and let $n \to \infty$, the right hand side of (A.9) converges to 0 by order continuity of of H^Φ (see e.g. [39, Theorem 2.1.14]).

(b) First, consider the case that $R = 1$. If Φ is not superlinear, then Ψ jumps to infinity, and hence $\mathcal{Q}_\rho$ is L^∞-bounded by part (a) and therefore UI. If Φ is superlinear (and finite), then Ψ is superlinear and finite. Set $K := \sup_{Y\in\mathcal{Q}} \|Y\|_\Psi < \infty$ and define the superlinear function $\tilde{\Psi}$ by $\tilde{\Psi}(y) := \Psi(y/K)$. By the definition of the Luxemburg norm,

$$\mathbb{E}[\tilde{\Psi}(Y)] = \mathbb{E}[\Psi(Y/K)] \leq 1, \quad \text{for all } Y \in \mathcal{Q}_\rho.$$

This implies $\sup_{Y\in\mathcal{Q}_\rho} \mathbb{E}[\tilde{\Psi}(Y)] \leq 1 < \infty$. Since $\tilde{\Psi}$ is superlinear, the de la Vallée-Poussin theorem implies that $\mathcal{Q}_\rho$ is UI.

Next, assume that $R \in H^\Phi$. By Proposition A.6.1, it is enough to show that $R \in L^1(\mathcal{Q}_\rho)$ where

$$L^1(\mathcal{Q}_\rho) := \{X \in L^0 : \lim_{a\to\infty} \sup_{Z\in\mathcal{Q}_\rho} \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] = 0\}.$$

Since $R \in H^\Phi$, the generalised Hölder inequality and order continuity of H^Φ

give

$$\lim_{a\to\infty} \sup_{Z\in\mathcal{Q}} \mathbb{E}[Z|X|\mathbb{1}_{\{|X|>a\}}] \leq \lim_{a\to\infty} 2\sup_{Z\in\mathcal{Q}} \|Z\|_\Psi \|X\mathbb{1}_{\{|X|>a\}}\|_\Phi = 0. \qquad \square$$

Proposition A.6.6. *Let $\mathcal{E}$ be an $\sigma(L^\infty, L^1)$-dense subset of $\mathcal{D}\cap L^\infty$. Then for all $X \in L^1$.*

$$\sup_{Z\in\mathcal{E}} \mathbb{E}[-ZX] = \mathrm{WC}(X).$$

Proof. Define the coherent risk measure $\rho : L^1 \to (-\infty, \infty]$ by

$$\rho(X) := \sup_{Z\in\mathcal{E}}(\mathbb{E}[-ZX]).$$

To show that $\rho = \mathrm{WC}$, let $X \in L^1$ and set $c := \operatorname{ess\,sup}(-X) = \mathrm{WC}(X)$.

First, assume that $c < \infty$. Then monotonicity of the expectation gives $\rho(X) \leq \mathrm{WC}(X)$. For the reverse inequality, let $\varepsilon > 0$ and set

$$Z := \mathbb{1}_{\{-X\geq c-\varepsilon\}}/\mathbb{P}[-X \geq c-\varepsilon] \in \mathcal{D}\cap L^\infty.$$

Then $\mathbb{E}[-ZX] \geq c - \varepsilon$. Since $\mathcal{E}$ is $\sigma(L^\infty, L^1)$-dense in $\mathcal{D}\cap L^\infty$, there exists a net $(Z_i)_{i\in I}$ in $\mathcal{E}$ which converges to Z in $\sigma(L^\infty, L^1)$. Thus,

$$\rho(X) \geq \lim_{i\in I} \mathbb{E}[-Z_i X] = \mathbb{E}[-ZX] \geq c - \varepsilon = \mathrm{WC}(X) - \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\rho(X) \geq \mathrm{WC}(X)$.

Finally, assume that $c = \infty$. Let $N > 0$ be given. Set $X_N := \max(X, -N)$. Then $X_N \geq X$ and $\mathrm{WC}(X_N) = N$. By monotonicity of ρ and the first part,

$$\rho(X) \geq \rho(X_N) = \mathrm{WC}(X_N) = N.$$

Letting $N \to \infty$ yields $\rho(X) = \infty = \mathrm{WC}(X)$. $\square$

Proposition A.6.7. *Fix $\alpha \in (0,1)$. Then*

$$\tilde{\mathcal{Q}}^\alpha := \{Z \in \mathcal{D} : Z > 0 \ \mathbb{P}\text{-a.s. and } \|Z\|_\infty < \tfrac{1}{\alpha}\}$$

is a nonempty subset of $\mathcal{Q}^\alpha$ satisfying Conditions POS, MIX and INT.

Proof. It is clear that $1 \in \tilde{\mathcal{Q}}^\alpha \subset \mathcal{Q}^\alpha$, and by definition $\tilde{\mathcal{Q}}^\alpha$ satisfies POS. If $Z \in \mathcal{Q}^\alpha$, $\tilde{Z} \in \tilde{\mathcal{Q}}^\alpha$ and $\lambda \in (0,1)$, then $\lambda Z + (1-\lambda)\tilde{Z} > 0$ $\mathbb{P}$-a.s., and by the triangle inequality

$$\|\lambda Z + (1-\lambda)\tilde{Z}\|_\infty \le \lambda \|Z\|_\infty + (1-\lambda)\|\tilde{Z}\|_\infty < \tfrac{1}{\alpha},$$

so $\tilde{\mathcal{Q}}^\alpha$ satisfies Condition MIX. To show Condition INT, let $\tilde{Z} \in \tilde{\mathcal{Q}}^\alpha$. Set $\mathcal{E} := \mathcal{D} \cap L^\infty$ and let $Z \in \mathcal{E}$. Since $\|Z\|_\infty < \infty$ and $\|\tilde{Z}\|_\infty < \frac{1}{\alpha}$ there is $\lambda \in (0,1)$ such that $\lambda\|\tilde{Z}\|_\infty + (1-\lambda)\|Z\|_\infty \le \frac{1}{\alpha}$. By the triangle inequality it follows that $\lambda\tilde{Z} + (1-\lambda)Z \in \mathcal{Q}^\alpha$. □

Proposition A.6.8. *Assume μ is a probability measure on $([0,1], \mathcal{B}_{[0,1]})$ and ρ^μ the corresponding spectral risk measure.*

(a) *ρ^μ is represented by*

$$\mathcal{Q}_\mu = \Big\{ \int_{[0,1]} \zeta_\alpha \, \mu(\mathrm{d}\alpha) : \zeta_\alpha(\omega) \text{ is jointly measurable; there is } 1 > \varepsilon > 0 \\ \text{such that } \zeta_\alpha \in \mathcal{Q}^\alpha \text{ for } \alpha \le 1-\varepsilon \text{ and } \zeta_\alpha \equiv 1 \text{ otherwise} \Big\}.$$

(b) *If μ does not have an atom at 1, the set*

$$\tilde{\mathcal{Q}}_\mu = \Big\{ \int_{[0,1)} \tilde{\zeta}_\alpha \, \mu(\mathrm{d}\alpha) : \tilde{\zeta}_\alpha(\omega) \text{ is jointly measurable; there is } \varepsilon \in (0,1), \\ \delta \in (0, \tfrac{\varepsilon}{1-\varepsilon}) \text{ such that} \tilde{\zeta}_\alpha \in \tilde{\mathcal{Q}}^{\alpha(1+\delta)} \text{ for } \alpha \le 1-\varepsilon \text{ and else } \tilde{\zeta}_\alpha \equiv 1 \Big\},$$

is nonempty and satisfies Conditions POS, MIX and INT.

Proof. (a) It follows from [27] that ρ^μ is represented by

$$\mathcal{Q}_{\rho^\mu} = \Big\{ \int_{[0,1]} \zeta_\alpha \, \mu(\mathrm{d}\alpha) : \zeta_\alpha(\omega) \text{ is jointly measurable and } \zeta_\alpha \in \mathcal{Q}^\alpha \text{ for all } \alpha \Big\}.$$

Let $Z = \int_{[0,1]} \zeta_\alpha \, \mu(\mathrm{d}\alpha) \in \mathcal{Q}_{\rho^\mu}$. Set $Z_n := \int_{[0,1-1/n]} \zeta_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-1/n, 1]) \in \mathcal{Q}_\mu$. Then

$$\lim_{n\to\infty} \|Z_n - Z\|_\infty \le \lim_{n\to\infty} \frac{n}{n-1} \mu((1-1/n, 1)) = 0.$$

This implies that $\mathbb{E}[-ZX] = \lim_{n\to\infty} \mathbb{E}[-Z_n X]$ for all $X \in L^1$.

(b) Since $1 \in \tilde{\mathcal{Q}}^\beta$ for all $\beta \in [0,1)$ and $\tilde{\mathcal{Q}}^\beta$ only contains positive random variables, it follows that $1 \in \tilde{\mathcal{Q}}_\mu$ and Condition POS is satisfied.

To show Condition MIX, let $Z \in \mathcal{Q}_\mu$, $\tilde{Z} \in \tilde{\mathcal{Q}}_\mu$ and $\lambda \in (0,1)$. Then there is $\varepsilon \in (0,1)$ and $\delta \in (0, \frac{\varepsilon}{1-\varepsilon})$ such that $Z = \int_{[0,1-\varepsilon]} \zeta_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-\varepsilon,1))$ and $\tilde{Z} = \int_{[0,1-\varepsilon]} \tilde{\zeta}_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-\varepsilon,1))$, where $\zeta_\alpha \in \mathcal{Q}^\alpha$ and $\tilde{\zeta}_\alpha \in \tilde{\mathcal{Q}}^{\alpha(1+\delta)}$ for $\alpha \in [0, 1-\varepsilon]$. Set $\delta' := \frac{\delta(1-\lambda)}{1+\delta\lambda} \in (0,\delta)$. A simple calculation shows that $\lambda\zeta_\alpha + (1-\lambda)\tilde{\zeta}_\alpha \in \tilde{\mathcal{Q}}^{\alpha(1+\delta')}$ for all $\alpha \in [0,1-\varepsilon]$. Thus,

$$\lambda Z + (1-\lambda)\tilde{Z} = \int_{[0,1-\varepsilon]} \lambda\zeta_\alpha + (1-\lambda)\tilde{\zeta}_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-\varepsilon,1)) \in \tilde{\mathcal{Q}}_\mu.$$

Finally, to show Condition INT, let $\tilde{Z} \in \tilde{\mathcal{Q}}_\mu$ and set

$$\mathcal{E} := \Big\{ \int_{[0,1)} \zeta_\alpha \, \mu(\mathrm{d}\alpha) : \zeta_\alpha(\omega) \text{ is jointly measurable and there is } 1 > \gamma, \varepsilon > 0 \\ \text{such that } \zeta_\alpha \in \mathcal{Q}^\gamma \text{ for } \alpha \leq 1-\varepsilon \text{ and } \zeta_\alpha \equiv 1 \text{ for } \alpha > 1-\varepsilon \Big\}.$$

It is straightforward to check that $\mathcal{E}$ is a dense subset of $\mathcal{D} \cap L^\infty$. Let $Z \in \mathcal{E}$. Then there exists $\varepsilon, \gamma \in (0,1)$ and $\delta \in (0, \frac{\varepsilon}{1-\varepsilon})$ such that $\tilde{Z} = \int_{[0,1-\varepsilon]} \tilde{\zeta}_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-\varepsilon,1))$ and $Z = \int_{[0,1-\varepsilon]} \zeta_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-\varepsilon,1))$, where $\tilde{\zeta}_\alpha \in \tilde{\mathcal{Q}}^{\alpha(1+\delta)}$ and $\zeta_\alpha \in \mathcal{Q}^\gamma$ for $\alpha \in [0,1-\varepsilon]$. Set $\lambda' := \frac{\delta\gamma}{(2+\delta)(1+\delta-\gamma)} \in (0,1)$. A simple calculation shows that $\lambda'\zeta_\alpha + (1-\lambda')\tilde{\zeta}_\alpha \in \tilde{\mathcal{Q}}^{\alpha(1+\delta/2)}$ for all $\alpha \in [0,1-\varepsilon]$. Thus,

$$\lambda' Z + (1-\lambda')\tilde{Z} = \int_{[0,1-\varepsilon]} \lambda'\zeta_\alpha + (1-\lambda')\tilde{\zeta}_\alpha \, \mu(\mathrm{d}\alpha) + \mu((1-\varepsilon,1)) \in \tilde{\mathcal{Q}}_\mu \subset \mathcal{Q}_\mu. \quad \square$$

Proposition A.6.9. *Let $g : [0,\infty) \to \mathbb{R}$ be a convex function and $\beta > g(1)$. Let $\mathcal{Q}^{g,\beta} := \{Z \in \mathcal{D} : \mathbb{E}[g(Z)] \leq \beta\}$. Then*

$$\tilde{\mathcal{Q}}^{g,\beta} := \{Z \in \mathcal{D} : Z > 0 \ \mathbb{P}\text{-a.s. and } \mathbb{E}[g(Z)] < \beta\}$$

is nonempty and satisfies Conditions POS, MIX and INT.

Proof. It is clear that $\tilde{\mathcal{Q}}^{g,\beta}$ satisfies Condition POS and $1 \in \tilde{\mathcal{Q}}^{g,\beta}$ since $\beta > g(1)$. To show condition MIX, let $Z \in \mathcal{Q}^{g,\beta}$, $\tilde{Z} \in \tilde{\mathcal{Q}}^{g,\beta}$ and $\lambda \in (0,1)$. By the convexity of g,

$$\mathbb{E}[g(\lambda\tilde{Z} + (1-\lambda)Z)] \leq \mathbb{E}[\lambda g(\tilde{Z}) + (1-\lambda)g(Z)] = \lambda\mathbb{E}[g(\tilde{Z})] + (1-\lambda)\mathbb{E}[g(Z)] < \beta.$$

To show Condition INT, let $\tilde{Z} \in \tilde{\mathcal{Q}}^{g,\beta}$. Set $\mathcal{E} := \mathcal{D} \cap L^\infty$ and let $Z \in \mathcal{E}$. Since $\mathbb{E}[g(Z)] < \infty$ and $\mathbb{E}[g(\tilde{Z})] < \beta$ there is $\lambda \in (0,1)$ such that $\lambda\mathbb{E}[g(\tilde{Z})] + (1-$

$\lambda)\mathbb{E}[g(Z)] \leq \beta$. Now convexity of g implies that $\lambda\tilde{Z} + (1-\lambda)Z \in \mathcal{Q}^{g,\beta}$. □

Proposition A.6.10. *Let ρ be a risk functional and assume there exists an unbounded sequence of portfolios $(\pi_n)_{n\geq 1} \subset \mathbb{R}^d$ with $X_{\pi_n} \in \mathcal{A}_\rho$ for all $n \in \mathbb{N}$.*

(a) *If ρ satisfies the Fatou property on $\mathcal{X}$, then there exists a portfolio $\pi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ with $\rho(\lambda X_\pi) \leq 0$ for all $\lambda > 0$. Moreover, if $\mathbb{E}[X_{\pi_n}] = 0$ for all n, we may further assume $\mathbb{E}[X_\pi] = 0$.*

(b) *If the market (S^0, S) satisfies no-arbitrage, then there exists $Y \in \mathcal{A}_{\rho^\infty}$ such that $\mathbb{P}[Y < 0] > 0$.*

Proof. (a) By passing to a subsequence and relabelling the assets, we may assume without loss of generality that $|\pi_n^1| \geq |\pi_n^i|$ for all $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$. As $\|\pi_n\| \to \infty$ we must have that $|\pi_n^1| \to \infty$, and by shifting the sequence we may assume $|\pi_n^1| > 0$ for all $n \in \mathbb{N}$. Then for all $i \in \{1, \dots, d\}$ we have $\pi_n^i/\pi_n^1 \in [-1, 1]$ and by compactness we can pass to a further subsequence and assume that $\pi_n^i/|\pi_n^1| \to \pi^i \in [-1, 1]$, where $\pi^1 \in \{-1, 1\}$. It follows that

$$X_{\pi_n}/|\pi_n^1| \to X_\pi \ \mathbb{P}\text{-a.s.}, \tag{A.10}$$

where $\pi \neq \mathbf{0}$ since $\pi^1 \in \{-1, 1\}$. Since $|\pi_n^1| \to \infty$, for any $\lambda > 0$, there exists N such that $\lambda/|\pi_n^1| \in (0, 1)$ for all $n \geq N$. Now star-shapedness of ρ gives

$$\rho(\lambda X_{\pi_n}/|\pi_n^1|) \leq \lambda\rho(X_{\pi_n})/|\pi_n^1| \leq 0, \quad n \geq N.$$

By the Fatou property ($L \supset \mathcal{X}$ being a Riesz space),

$$\rho(\lambda X_\pi) \leq \liminf_{n\to\infty} \rho(\lambda X_{\pi_n}/|\pi_n^1|) \leq 0.$$

Hence $\rho(\lambda X_\pi) \leq 0$ for all $\lambda > 0$.

If in addition $\mathbb{E}[X_{\pi_n}] = 0$ for all n, then linearity of the expectation and the dominated convergence theorem gives $\mathbb{E}[X_\pi] = 0$. Indeed, since $\pi_n^i/\pi_n^1 \in [-1, 1]$ we have

$$|X_{\pi_n}/|\pi_n^1|| = |X^1 + \tfrac{\pi_n^2}{|\pi_n^1|}X^2 + \dots \tfrac{\pi_n^d}{|\pi_n^1|}X^d| \leq |X^1| + |X^2| + \dots |X^d|$$

where $X^i := R^i - r \in L^1$ for $i \in \{1, \dots, d\}$. This, together with (A.10) and the dominated convergence theorem gives $\mathbb{E}[X_\pi] = \lim_{n\to\infty} \mathbb{E}[X_{\pi_n}/|\pi_n^1|] = 0$.

(b) By passing to a subsequence, relabelling the assets and multiplying their corresponding excess return by -1 if necessary, we may assume without loss of generality that $\pi_n^1 \geq \pi_n^i \geq 0$ for all $n \in \mathbb{N}$ and $i \in \{1, \dots, d\}$. As $\|\pi_n\| \to \infty$ this means $\pi_n^1 \to \infty$, and by shifting the sequence we may assume $\pi_n^1 > 0$ for all $n \in \mathbb{N}$. Then for all $i \in \{1, \dots, d\}$ we have $\pi_n^i / \pi_n^1 \in [0, 1]$, and by compactness we can pass to a further subsequence and assume that $\pi_n^i / \pi_n^1 \to \pi^i \in [0, 1]$, where $\pi^1 = 1$. By relabelling the assets we may assume there exists $k \in \{1, \dots, d\}$ such that for $i > k$, $\pi^i = 0$ and for $i \leq k$, $\pi^i > 0$. And by passing to another subsequence, we may assume for $i > k$, $\pi_n^i / \pi_n^1 \searrow 0$. Now for all $n \in \mathbb{N}$,

$$\rho(\pi_n^1 X^1 + \cdots + \pi_n^d X^d) = \rho(\pi_n^1 (X^1 + \tfrac{\pi_n^2}{\pi_n^1} X^2 + \cdots + \tfrac{\pi_n^d}{\pi_n^1} X^d)) \leq 0.$$

If we can construct a random variable $Y \in L$ such that $\mathbb{P}[Y < 0] > 0$ and for which there exists $M \in \mathbb{N}$ such that

$$Y \geq X^1 + \tfrac{\pi_n^2}{\pi_n^1} X^2 + \cdots + \tfrac{\pi_n^d}{\pi_n^1} X^d \ \mathbb{P}\text{-a.s.}, \quad n \geq M,$$

then we would be done. Indeed, by the monotonicity of ρ, $\rho(\pi_n^1 Y) \leq 0$ for all n, and this would imply $Y \in \mathcal{A}_{\rho^\infty}$. To that end, for $\mathbb{N} \ni N > \max\{\frac{1}{\pi^1}, \dots, \frac{1}{\pi^k}\}$ and $i \in \{1, \dots, k\}$ define

$$V_N^i := \begin{cases} (\pi^i - \frac{1}{N}) X^i, & \text{if } X^i \leq 0, \\ (\pi^i + \frac{1}{N}) X^i, & \text{if } X^i > 0, \end{cases}$$

noting that there exists $M_N \in \mathbb{N}$ such that $V_N^i \geq \frac{\pi_n^i}{\pi_n^1} X^i$ $\mathbb{P}$-a.s. for all $n \geq M_N$. And for $N \in \mathbb{N}$ and $i \in \{k+1, \dots, d\}$ define

$$W_N^i := \begin{cases} 0, & \text{if } X^i \leq 0, \\ \frac{\pi_N^i}{\pi_N^1} X^i, & \text{if } X^i > 0, \end{cases}$$

noting that $W_N^i \geq \frac{\pi_n^i}{\pi_n^1} X^i$ $\mathbb{P}$-a.s. for all $n \geq N$. Now for $N > \max\{\frac{1}{\pi^1}, \dots, \frac{1}{\pi^k}\}$, let

$$Y_N := \sum_{i=1}^k V_N^i + \sum_{i=k+1}^d W_N^i.$$

As $N \to \infty$, $Y_N \to X_\pi$ $\mathbb{P}$-a.s. where $\pi = (1, \pi^2, \dots, \pi^d) \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. Since the market is nonredundant and satisfies no-arbitrage, $\mathbb{P}[X_\pi < 0] > 0$. Therefore, there exists N^* such that $\mathbb{P}[Y_{N^*} < 0] > 0$. Furthermore,

$$Y_{N^*} \geq X^1 + \tfrac{\pi_n^2}{\pi_n^1} X^2 + \cdots + \tfrac{\pi_n^d}{\pi_n^1} X^d \ \mathbb{P}\text{-a.s.}$$

for all $n \geq \max\{M_{N^*}, N^*\}$. Setting $Y = Y_{N^*}$ completes the proof. □

Proposition A.6.11. *Suppose X is a topological space and $K \subset X$ is compact. Then for any nondecreasing sequence of lower semi-continuous functions $f_t : K \to [-\infty, \infty]$ with $f(x) := \lim_{t\to\infty} f_t(x)$ for all $x \in K$, we have*

$$\min_{x\in K} f(x) = \lim_{t\to\infty} \min_{x\in K} f_t(x).$$

Furthermore, if $(x_t)_{t\geq 1}$ is a sequence where $\min_{x\in K} f_t(x) = f_t(x_t)$, then any limit point is a minimiser for f.

Proof. First note that f is lower semi-continuous because it is the supremum of lower semi-continuous functions. By the compactness of K and lower semi-continuity, f and f_t attain their minimum values. Now since f_t is a nondecreasing sequence, it is easy to see that

$$\min_{x\in K} f(x) \geq \lim_{t\to\infty} \min_{x\in K} f_t(x) =: m.$$

For the reverse inequality, consider the sets $A_t := \{x \in K : f_t(x) \leq m\}$. These are nonempty (because $\emptyset \neq \operatorname{argmin} f_t \subset A_t$), closed (by the lower semi-continuity of f_t) and compact (since K is compact and A_t is closed). Moreover, they are nested in the sense that $A_t \supset A_{t+1}$. It follows by Cantor's intersection theorem that

$$A := \bigcap_{t=1}^{\infty} A_t \neq \emptyset,$$

i.e., there exists $x^* \in K$ such that $f_t(x^*) \leq m$ for all t. Taking the limit as $t \to \infty$ yields

$$\min_{x\in K} f(x) \leq f(x^*) \leq m = \lim_{t\to\infty} \min_{x\in K} f_t(x).$$

To prove the final claim, note that $\operatorname{argmin} f = A$ because $f(x) \leq m$ if and only if $f_t(x) \leq m$ for all t. Whence, any limit point of a sequence of minimisers $(x_t)_{t\in\mathbb{N}}$ – that is where $x_t \in \operatorname{argmin} f_t$ for all $t \geq 1$ – is contained in A, and hence, is a minimiser for f. □

Proposition A.6.12. *Let $\mathcal{E}$ be a convex subset of $\mathcal{D} \cap L^\Psi$. Then for all $X \in H^\Phi$,*

$$\sup_{Z \in \mathcal{E}} \mathbb{E}[-ZX] = \mathrm{WC}(X),$$

if and only if $\mathcal{E}$ is a $\sigma(L^\Psi, H^\Phi)$-dense subset of $\mathcal{D} \cap L^\Psi$.

Proof. Note that $\mathcal{D} \cap L^\Psi$ is $\sigma(L^\Psi, H^\Phi)$-closed. Suppose $\mathcal{E}$ is a $\sigma(L^\Psi, H^\Phi)$-dense subset of $\mathcal{D} \cap L^\Psi$. Define the risk measure $\rho : H^\Phi \to (-\infty, \infty]$ by $\rho(X) := \sup_{Z \in \mathcal{E}}(\mathbb{E}[-ZX])$. To show that $\rho \equiv \mathrm{WC}|_{H^\Phi}$, let $X \in H^\Phi$ and set $c := \operatorname{ess\,sup}(-X) = \mathrm{WC}(X)$.

First, assume that $c < \infty$. Then monotonicity of the expectation gives $\rho(X) \leq \mathrm{WC}(X)$. For the reverse inequality, let $\varepsilon > 0$ and set

$$Z := \mathbb{1}_{\{-X \geq c - \varepsilon\}} / \mathbb{P}[-X \geq c - \varepsilon] \in \mathcal{D} \cap L^\infty \subset \mathcal{D} \cap L^\Psi.$$

Then $\mathbb{E}[-ZX] \geq c - \varepsilon$. Since $\mathcal{E}$ is $\sigma(L^\Psi, H^\Phi)$-dense in $\mathcal{D} \cap L^\Psi$, there exists a net $(Z_i)_{i \in I}$ in $\mathcal{E}$ which converges to Z in $\sigma(L^\Psi, H^\Phi)$. Thus,

$$\rho(X) \geq \lim_{i \in I} \mathbb{E}[-Z_i X] = \mathbb{E}[-ZX] \geq c - \varepsilon = \mathrm{WC}(X) - \varepsilon.$$

Letting $\varepsilon \to 0$ yields $\rho(X) \geq \mathrm{WC}(X)$.

Now assume that $c = \infty$. Let $N > 0$ be given and set $X_N := \max(X, -N)$. Then $X_N \geq X$ and $\mathrm{WC}(X_N) = N$. By monotonicity of ρ and the first part,

$$\rho(X) \geq \rho(X_N) = \mathrm{WC}(X_N) = N.$$

Letting $N \to \infty$ yields $\rho(X) = \infty = \mathrm{WC}(X)$.

The converse is an application of the Hahn-Banach separation theorem (for the pairing (L^Ψ, H^Φ)). Indeed, if the $\sigma(L^\Psi, H^\Phi)$-closure of $\mathcal{E}$, $\mathcal{E}^*$ (which is convex), is not equal to $\mathcal{D} \cap L^\Psi$, then there exists $X \in H^\Phi$ such that

$$\sup_{Z \in \mathcal{E}^*} \mathbb{E}[-ZX] < \sup_{Z \in \mathcal{D} \cap L^\Psi} \mathbb{E}[-ZX],$$

i.e., $\sup_{Z \in \mathcal{E}} \mathbb{E}[-ZX] < \mathrm{WC}(X)$. □

Proposition A.6.13. (a) *The expected weighted loss risk measure* EW^l *is strongly sensitive to large losses on H^{Φ_l} if and only if* $\lim_{x \to \infty} l(x)/x = \infty$ *or* $\lim_{x \to -\infty} l(x)/x = 0$.

(b) *When* EW^l *is not strongly sensitive to large losses on* H^{Φ_l}*, then there exists* $X \in H^{\Phi_l}$ *with* $\mathbb{E}[X] > 0$, $\mathbb{P}[X < 0] > 0$ *and* $\mathrm{EW}^l(\lambda X) \to -\infty$ *as* $\lambda \to \infty$.

Proof. Assume first that $\lim_{x\to\infty} l(x)/x = \infty$ and suppose that $X \in H^{\Phi_l}$ and $\mathbb{P}[X < 0] > 0$. Then for any $\lambda > 0$,

$$\mathrm{EW}^l(\lambda X) = \mathbb{E}[l(-\lambda X)] \geq \mathbb{E}[-\lambda X \mathbb{1}_{\{X\geq 0\}}] + \mathbb{E}[l(-\lambda X)\mathbb{1}_{\{X<0\}}] \geq \lambda k_1 + pl(\lambda k_2)$$

where $k_1 := \mathbb{E}[-X\mathbb{1}_{\{X\geq 0\}}] \leq 0$, $k_2 := \min\{1, -\operatorname{ess\,inf}(X)/2\} > 0$ and $p := \mathbb{P}[X \leq -k_2] > 0$. Now as $\lambda \to \infty$, $(\lambda k_1 + pl(\lambda k_2))/\lambda \to \infty$ since $l(x)/x \to \infty$ as $x \to \infty$. Therefore, there exists $\tilde{\lambda} \geq 1$ such that $\mathrm{EW}^l(\tilde{\lambda}X) > 0$ and so EW^l is strongly sensitive to large losses on H^{Φ_l}.

Now assume that $\lim_{x\to-\infty} l(x)/x = 0$ and suppose $X \in H^{\Phi_l}$ and $\mathbb{P}[X < 0] > 0$. If $X \leq 0$, then since $l(x) \geq x$ for all $x \in \mathbb{R}$, $\mathrm{EW}^l(X) \geq \mathbb{E}[-X] > 0$. So assume $\operatorname{ess\,sup}(X) > 0$. Then for any $\lambda > 0$,

$$\mathrm{EW}^l(\lambda X) = \mathbb{E}[l(-\lambda X)] \geq \mathbb{E}[-\lambda X\mathbb{1}_{\{X<0\}}] + \mathbb{E}[l(-\lambda X)\mathbb{1}_{\{X\geq 0\}}] \geq \lambda c_1 + ql(\lambda c_2)$$

where $c_1 := \mathbb{E}[-X\mathbb{1}_{\{X<0\}}] > 0$, $c_2 := \max\{-1, -\operatorname{ess\,sup}(X)/2\} < 0$ and $q := \mathbb{P}[X \geq -c_2] > 0$. As $\lambda \to \infty$, $(\lambda c_1 + ql(\lambda c_2))/\lambda \to c_1 > 0$ since $\lim_{x\to-\infty} l(x)/x = 0$. Therefore, there exists $\tilde{\lambda} \geq 1$ such that $\mathrm{EW}^l(\tilde{\lambda}X) > 0$ and so EW^l is strongly sensitive to large losses on H^{Φ_l}.

On the other hand, if $\beta := \lim_{x\to\infty} l(x)/x \neq \infty$ and $\alpha := \lim_{x\to-\infty} l(x)/x \neq 0$, then by the properties of l, it must be that $\beta \in [1, \infty)$ and $\alpha \in (0, 1]$. Define the loss function $\tilde{l} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{l}(x) = \begin{cases} \beta x, & \text{if } x \geq 0, \\ \alpha x, & \text{if } x < 0. \end{cases}$$

Then $\tilde{l} \geq l$, so to complete the proof, it suffices to find $X \in H^{\Phi_l} = H^{\Phi_{\tilde{l}}} = L^1$ such that $\mathbb{P}[X < 0] > 0$, $\mathbb{E}[X] > 0$ and $\mathrm{EW}^{\tilde{l}}(\lambda X) \to -\infty$ as $\lambda \to \infty$. To that end, let $A \in \mathcal{F}$ be a nontrivial event, $p := \mathbb{P}[A] \in (0, 1)$ and consider the random variable $X = a\mathbb{1}_A - b\mathbb{1}_{A^c}$ where $a, b > 0$ satisfy $-p\alpha a + (1-p)\beta b < 0$ and $pa - b(1-p) > 0$. Then $X \in L^1$, $\mathbb{E}[X] > 0$, $\mathbb{P}[X < 0] > 0$ and as $\lambda \to \infty$,

$$\mathrm{EW}^{\tilde{l}}(\lambda X) = \mathbb{E}[\tilde{l}(-\lambda X)] = \lambda[-p\alpha a + (1-p)\beta b] \to -\infty. \qquad \square$$

Proposition A.6.14. *Suppose Φ is a Young function and $X \geq 0$. Then,*

$$\|X\|_{\Phi} \leq 1 + \mathbb{E}[\Phi(X)].$$

Proof. Set $k := 1 + \mathbb{E}[\Phi(X)] \geq 1$. We may assume without loss of generality that $k < \infty$. Since $\Phi(X/k) \leq \Phi(X)/k$ by nonnegativity of X, $\Phi(0) = 0$ and convexity of Φ, we obtain

$$\mathbb{E}[\Phi(X/k)] \leq \mathbb{E}[\Phi(X)]/k \leq 1.$$

Thus, $\|X\|_{\Phi} \leq k$ by the definition of the Luxemburg norm. □

Proposition A.6.15. *Let l be a loss function and assume that*

$$0 < a_l = \lim_{x \to -\infty} l(x)/x < b_l = \lim_{x \to \infty} l(x)/x < \infty.$$

Consider the penalty function $\alpha^l(Z) = \inf_{\lambda > 0} \frac{1}{\lambda}\mathbb{E}[l^*(\lambda Z)]$. Then

$$\tilde{\mathcal{Q}}^{\alpha^l} = \{Z \in \mathcal{D} : \text{there is } k > 0 \text{ and } \varepsilon > 0 \text{ such that } a_l + \varepsilon < kZ < b_l - \varepsilon \ \mathbb{P}\text{-a.s.}\}$$

is a nonempty subset of $\mathcal{Q}^{\alpha^l}$ satisfying Conditions POS, MIX and INT.

Proof. It is clear that $1 \in \tilde{\mathcal{Q}}^{\alpha^l} \subset \mathcal{Q}^{\alpha^l}$, and by definition $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies Condition POS.

To show Condition MIX, let $Z \in \mathcal{Q}^{\alpha^l}$, $\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$ and $\lambda \in (0,1)$. Then there exists $k, \tilde{k}, \tilde{\varepsilon} > 0$ such that $a_l \leq kZ \leq b_l$ $\mathbb{P}$-a.s. and $a_l + \tilde{\varepsilon} \leq \tilde{k}\tilde{Z} \leq b_l - \tilde{\varepsilon}$ $\mathbb{P}$-a.s. It follows that

$$a_l + \varepsilon^* < k^*(\lambda Z + (1-\lambda)\tilde{Z}) < b_l - \varepsilon^*$$

where $k^* := k\tilde{k}/(\lambda\tilde{k} + (1-\lambda)k) > 0$ and $\varepsilon^* := (1-\lambda)k^*\tilde{\varepsilon} > 0$. Therefore, $\lambda Z + (1-\lambda)\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$ and $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies Condition MIX.

Finally we show Condition INT is satisfied. Let $\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$, set $\mathcal{E} := \mathcal{D} \cap L^\infty$ and $Z \in \mathcal{E}$. Then there exists $k > 0$ such that $\operatorname{ess\,inf} k\tilde{Z} > a_l$ and $\operatorname{ess\,sup} k\tilde{Z} < b_l$. Let

$$\lambda_1 := \begin{cases} \frac{b_l - k\|\tilde{Z}\|_\infty}{k(\|Z\|_\infty - \|\tilde{Z}\|_\infty)}, & \text{if } \|Z\|_\infty > \|\tilde{Z}\|_\infty, \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

and $\lambda_2 := 1 - a_l / \operatorname{ess\,inf} k\tilde{Z}$. Then setting $\lambda := \min\{\lambda_1, \lambda_2\}$ yields $\operatorname{ess\,inf} k(\lambda Z + (1-\lambda)\tilde{Z}) \geq a_l$ and $\operatorname{ess\,sup} k(\lambda Z + (1-\lambda)\tilde{Z}) \leq b_l$. Therefore, $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}^{\alpha^l}$

and $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies INT. $\square$

Proposition A.6.16. *Let l be a loss function and assume that*

$$0 < a_l = \lim_{x\to-\infty} l(x)/x < b_l = \lim_{x\to\infty} l(x)/x < \infty.$$

Consider the penalty function $\alpha^l(Z) = \inf_{\lambda>0} \frac{1}{\lambda}\mathbb{E}[l^*(\lambda Z)]$. Then $\mathcal{Q}^{\overline{\text{co}}\,\alpha^l} = \mathcal{Q}^{\alpha^l}$.

Proof. We first show α^l is bounded on its effective domain. So let $Z \in \mathcal{Q}^{\alpha^l}$ and note that there exists $k > 0$ such that $kZ \in [a_l, b_l]$ $\mathbb{P}$-a.s. Now for any $\lambda \in (0, a_l)$, $\mathbb{E}[l^*(\lambda Z)] = \infty$ since $\mathbb{P}[Z \leq 1] > 0$ and $\operatorname{dom} l^* = [a_l, b_l]$. Similarly, since $\mathbb{P}[Z \geq 1] > 0$, for any $\lambda \in (b_l, \infty)$, $\mathbb{E}[l^*(\lambda Z)] = \infty$. Thus, $k \in [a_l, b_l]$ and since l^* is nondecreasing on $[a_l, b_l]$

$$\alpha^l(Z) \leq \frac{1}{k}\mathbb{E}[l^*(kZ)] \leq \frac{1}{a_l} l^*(b_l) < \infty.$$

It follows by Remark 4.2.5(a) that $\mathcal{Q}^{\overline{\text{co}}\,\alpha^l} = \bar{\mathcal{Q}}^{\alpha^l}$, and so to complete the proof we must show that $\mathcal{Q}^{\alpha^l}$ is L^1-closed. To that end, let $(Z_n)_{n\geq 1} \subset \mathcal{Q}^{\alpha^l}$ and assume the sequence converges in L^1 to Z^*. We must show $Z^* \in \mathcal{Q}^{\alpha^l}$. Clearly $Z^* \in \mathcal{D}$, and by what we have shown above, there exists $k_n \in [a_l, b_l]$ such that $k_n Z_n \in [a_l, b_l]$ $\mathbb{P}$-a.s. By restricting to a subsequence we may assume without loss of generality that k_n converges to $k^* \in [a_l, b_l]$ and $k_n Z_n$ converges to $k^* Z^*$ $\mathbb{P}$-a.s. Thus, $k^* Z^* \in [a_l, b_l]$ $\mathbb{P}$-a.s., and $Z^* \in \mathcal{Q}^{\alpha^l}$ as desired. $\square$

Proposition A.6.17. *Let l be a loss function where $a_l := \lim_{x\to-\infty} l(x)/x > 0$ or $b_l := \lim_{x\to\infty} l(x)/x < \infty$, and $a_l < 1 < b_l$. Consider the penalty function $\alpha^l(Z) = \mathbb{E}[l^*(Z)]$. Then*

$$\tilde{\mathcal{Q}}^{\alpha^l} = \{Z \in \mathcal{Q}^{\alpha^l} : a_l + \varepsilon < Z < b_l - \varepsilon \ \mathbb{P}\text{-a.s. for some } \varepsilon > 0\},$$

is a nonempty subset of $\mathcal{Q}^{\alpha^l}$ satisfying Conditions POS, MIX and INT.

Proof. It is clear that $1 \in \tilde{\mathcal{Q}}^{\alpha^l} \subset \mathcal{Q}^{\alpha^l}$, and by definition $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies POS.

To show Condition MIX, let $Z \in \mathcal{Q}^{\alpha^l}$, $\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$ and $\lambda \in (0,1)$. Since l^* is convex, $\mathbb{E}[l^*(\lambda Z + (1-\lambda)\tilde{Z})] < \infty$ so $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}^{\alpha^l}$. Furthermore, since $a_l \leq Z \leq b_l$ $\mathbb{P}$-a.s. and $a_l + \varepsilon < \tilde{Z} < b_l - \varepsilon$ $\mathbb{P}$-a.s. for some $\varepsilon > 0$, it follows that

$$a_l + (1-\lambda)\varepsilon < \lambda Z + (1-\lambda)\tilde{Z} < b_l - (1-\lambda)\varepsilon \quad \mathbb{P}\text{-a.s.}$$

Therefore, $\lambda Z + (1-\lambda)\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$ and $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies Condition MIX.

Finally we show Condition INT is satisfied. Assume first that $b_l = \infty$ and let $\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$. Set $\mathcal{E} := \mathcal{D} \cap L^\infty$ and $Z \in \mathcal{E}$. Then there exists $\varepsilon > 0$ such that $\tilde{Z} > a_l + \varepsilon$ $\mathbb{P}$-a.s. By choosing $\lambda \in (0, \varepsilon/(a_l+\varepsilon)]$, it follows that

$$\lambda Z + (1-\lambda)\tilde{Z} \geq (1-\lambda)(a_l + \varepsilon) \geq a_l \quad \mathbb{P}\text{-a.s.}$$

Now since l^* is convex, $l^*(y) \geq \Psi_l(y) \geq 0$ for all $y \geq 0$ and l^* is real-valued on $[a_l, \infty)$, it follows that there exists $c \in [a_l, \infty)$ such that l^* is nonincreasing on $[a_l, c)$ and nondecreasing on (c, ∞). Whence, setting $A := \{\lambda Z + (1-\lambda)\tilde{Z} \leq c\}$, $B := \{Z \leq \tilde{Z}\}$ and $Y := \lambda Z + (1-\lambda)\tilde{Z}$ have

$$\begin{aligned}\mathbb{E}[l^*(Y)] &= \mathbb{E}[l^*(Y)\mathbb{1}_A] + \mathbb{E}[l^*(Y)\mathbb{1}_{A^c}] \\ &\leq l^*(a_l) + \mathbb{E}[l^*(Y)\mathbb{1}_{A^c \cap B}] + \mathbb{E}[l^*(Y)\mathbb{1}_{A^c \cap B^c}] \\ &\leq l^*(a_l) + \mathbb{E}[l^*(\tilde{Z})] + l^*(\|Z\|_\infty)\mathbb{P}[Z > c] < \infty.\end{aligned}$$

Therefore, $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}^{\alpha^l}$ and $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies Condition INT when $b_l = \infty$. Now assume $1 < b_l < \infty$ and let $\tilde{Z} \in \tilde{\mathcal{Q}}^{\alpha^l}$. Set $\mathcal{E} := \mathcal{D} \cap L^\infty$ and $Z \in \mathcal{E}$. Then there exists $\varepsilon > 0$ such that $a_l + \varepsilon < \tilde{Z} < b_l - \varepsilon$. By choosing $\lambda \in (0, \varepsilon/(a_l+\varepsilon)]$ if $\|Z\|_\infty \leq b_l$ and choosing $\lambda \in (0, \min\{\varepsilon/(a_l+\varepsilon), \varepsilon/(\varepsilon + \|Z\|_\infty - b_l)\}]$ otherwise, it follows that

$$a_l \leq \lambda Z + (1-\lambda)\tilde{Z} \leq b_l \quad \mathbb{P}\text{-a.s.}$$

Therefore, $\lambda Z + (1-\lambda)\tilde{Z} \in \mathcal{Q}^{\alpha^l}$ and $\tilde{\mathcal{Q}}^{\alpha^l}$ satisfies Condition INT. □

Proposition A.6.18. *Let $\beta \in (0,1)$ and $g \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty$. Define $\hat{g} : (0,1] \to [0,\infty]$ by $\hat{g}(x) = \overline{\mathrm{co}}\, \tilde{g}(1/x)$ where $\tilde{g} : [1,\infty) \to [0,\infty]$ is given by $\tilde{g}(x) = g(1/x)$. Then $\hat{g} \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty$ and $\hat{g}(\beta) = \infty$.*

Proof. Since $g \in \mathcal{G}_\beta$, it follows that $\tilde{g}$ is nondecreasing, real-valued on $[1, 1/\beta)$, ∞ on $(1/\beta, \infty)$ and $\tilde{g}(1) = 0$. By the definition of the lower semi-continuous convex hull, it is not difficult to check that $\overline{\mathrm{co}}\, \tilde{g}$ has the same properties and so $\hat{g} \in \mathcal{G}_\beta$. It remains to show that $\overline{\mathrm{co}}\, \tilde{g}(1/\beta) = \infty$.

Seeking a contradiction, suppose that $\overline{\mathrm{co}}\, \tilde{g}(1/\beta) =: k < \infty$. As $\overline{\mathrm{co}}\, \tilde{g}$ is a proper lower semi-continuous convex function, the Fenchel-Moreau theorem gives $\overline{\mathrm{co}}\, \tilde{g} = \tilde{g}^{**}$ where $\tilde{g}^{**}$ is the biconjugate of $\tilde{g}$. Since $\tilde{g}$ is nondecreasing and $\lim_{x \uparrow 1/\beta} \tilde{g}(x) = \infty$, there exists $c \in [1, 1/\beta)$ such that $\tilde{g}(x) > k+1$ for all $x \in [c, 1/\beta)$. Thus, the affine (and continuous) function $a : [1,\infty) \to \mathbb{R}$ with

$a(c) = 0$ and $a(1/\beta) = k + 1$ satisfies $a \le \tilde{g}$ and $a(1/\beta) > k = \tilde{g}^{**}(1/\beta)$. This is in contradiction to the fact that by (A.8), $\tilde{g}^{**}$ dominates any affine (and continuous) function dominated by $\tilde{g}$. □

Proposition A.6.19. *Let $\beta \in (0,1)$ and $g \in \mathcal{G}_\beta$. Let $\overline{\mathrm{co}}\,\alpha^g$ be the L^1-lower semi-continuous convex hull of α^g. Then its effective domain is given by*

$$\mathcal{Q}^{\overline{\mathrm{co}}\,\alpha^g} = \begin{cases} \{Z \in \mathcal{D} : \|Z\|_\infty \le \frac{1}{\beta}\}, & \text{if } g \in \mathcal{G}_\beta^\infty, \\ \{Z \in \mathcal{D} : \|Z\|_\infty < \frac{1}{\beta}\}, & \text{if } g \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty. \end{cases}$$

Proof. If $g \in \mathcal{G}_\beta^\infty$, then the result follows from Remark 4.2.5(a). So assume $g \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty$. Define the function $\hat{g} : (0,1] \to [0,\infty]$ by $\hat{g}(x) = \overline{\mathrm{co}}\,\tilde{g}(1/x)$ where $\tilde{g} : [1,\infty) \to [0,\infty]$ is given by $\tilde{g}(x) = g(1/x)$. By Proposition A.6.18, $\hat{g} \in \mathcal{G}_\beta \setminus \mathcal{G}_\beta^\infty$ and $\hat{g}(\beta) = \infty$. Moreover, $\hat{g}(x) \le \tilde{g}(1/x) = g(x)$ for $x \in (0,1]$. Moreover, by the fact that $\overline{\mathrm{co}}\,\tilde{g}$ is convex and lower semi-continuous, nondecreasing, real-valued on $[1, 1/\beta)$ and ∞ on $[1/\beta, \infty]$, it follows $\alpha^{\hat{g}} : \mathcal{D} \to [0,\infty]$, given by

$$\alpha^{\hat{g}}(Z) = \begin{cases} \hat{g}(\|Z\|_\infty^{-1}) = \overline{\mathrm{co}}\,\tilde{g}(\|Z\|_\infty), & \text{if } Z \in \mathcal{Q}^{\hat{g}} = \{Z \in \mathcal{D} : \|Z\|_\infty < 1/\beta\}, \\ \infty, & \text{otherwise}, \end{cases}$$

is convex and L^1-lower semi-continuous. Thus, $\alpha^g \ge \overline{\mathrm{co}}\,\alpha^g \ge \overline{\mathrm{co}}\,\alpha^{\hat{g}} = \alpha^{\hat{g}}$, which implies $\mathcal{Q}^{\alpha^g} \subset \mathcal{Q}^{\overline{\mathrm{co}}\,\alpha^g} \subset \mathcal{Q}^{\alpha^{\hat{g}}}$. Since $\mathcal{Q}^{\alpha^g} = \mathcal{Q}^{\alpha^{\hat{g}}} = \{Z \in \mathcal{D} : \|Z\|_\infty < 1/\beta\}$, the result follows. □

Bibliography

[1] C. Acerbi, *Spectral measures of risk: A coherent representation of subjective risk aversion*, J. Bank. Finance **26** (2002), no. 7, 1505–1518.

[2] A. Adam, M. Houkari, and J. P. Laurent, *Spectral risk measures and portfolio selection*, J. Bank. Finance **32** (2008), no. 9, 1870–1882.

[3] A. Ahmadi-Javid, *Entropic value-at-risk: A new coherent risk measure*, J. Optim. Theory Appl. **155** (2012), no. 3, 1105–1123.

[4] A. Ahmadi-Javid and A. Pichler, *An analytical study of norms and banach spaces induced by the entropic value-at-risk*, Math. Financial Econ. **11** (2017), no. 4, 527–550.

[5] G. J. Alexander and A. M. Baptista, *Economic implications of using a mean-var model for portfolio selection: A comparison with mean-variance analysis*, J. Econ. Dyn. Control **26** (2002), no. 7-8, 1159–1193.

[6] C. Aliprantis and K. Border, *Infinite dimensional analysis: A hitchhiker's guide*, Springer, 2006.

[7] T. Arai, *Good deal bounds induced by shortfall risk*, SIAM J. Financial Math. **2** (2011), no. 1, 1–21.

[8] M. Arduca and C. Munari, *Fundamental theorem of asset pricing with acceptable risk in markets with frictions*, arXiv preprint arXiv:2012.08351, 2020.

[9] J. Armstrong and D. Brigo, *Risk managing tail-risk seekers: Var and expected shortfall vs s-shaped utility*, J. Bank. Finance **101** (2019), 122–135.

[10] J. Armstrong and D. Brigo, *Coherent risk measures alone are ineffective in constraining portfolio losses*, J. Bank. Finance (2021), forthcoming.

[11] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, *Coherent measures of risk*, Math. Finance **9** (1999), no. 3, 203–228.

[12] F. Bellini, P. Koch-Medina, C. Munari, and G. Svindland, *Law-invariant functionals on general spaces of random variables*, SIAM J. Financial Math. **12** (2021), no. 1, 318–341.

[13] A. Ben-Tal and M. Teboulle, *Expected utility, penalty functions, and duality in stochastic nonlinear programming*, Manage. Sci. **32** (1986), no. 11, 1445–1466.

[14] A. Ben-Tal and M. Teboulle, *An old-new concept of convex risk measures: The optimized certainty equivalent*, Math. Finance **17** (2007), no. 3, 449–476.

[15] A. E. Bernardo and O. Ledoit, *Gain, loss, and asset pricing*, J. Polit. Econ. **108** (2000), no. 1, 144–172.

[16] D. P. Bertsekas, *Nonlinear programming*, Athena Scientific, Belmont, MA, 1995.

[17] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Polit. Econ. **81** (1973), 637–659.

[18] A. Bosch-Domènech and J. Silvestre, *Reflections on gains and losses: A* $2 \times 2 \times 7$ *experiment*, J. Risk Uncertainty **33** (2006), no. 3, 217–235.

[19] A. Bosch-Domènech and J. Silvestre, *Averting risk in the face of large losses: Bernoulli vs. Tversky and Kahneman*, Econ. Lett. **107** (2010), no. 2, 180–182.

[20] P. Boyle, L. Garlappi, R. Uppal, and T. Wang, *Keynes meets Markowitz: The trade-off between familiarity and diversification*, Manage. Sci. **58** (2012), no. 2, 253–272.

[21] M. Burzoni, C. Munari, and R. Wang, *Adjusted expected shortfall*, J. Bank. Finance (2021), forthcoming.

[22] E. Castagnoli, G. Cattelan, F. Maccheroni, C. Tebaldi, and R. Wang, *Star-shaped risk measures*, arXiv preprint arXiv:2103.15790, 2021.

[23] A. Černỳ and S. Hodges, *The theory of good-deal pricing in financial markets*, Mathematical Finance—Bachelier Congress 2000, Springer, 2002, pp. 175–202.

[24] P. Cheridito, U. Horst, M. Kupper, and T. A. Pirvu, *Equilibrium pricing in incomplete markets under translation invariant preferences*, Math. Oper. Res. **41** (2016), no. 1, 174–195.

[25] P. Cheridito and T. Li, *Risk measures on Orlicz hearts*, Math. Finance **19** (2009), no. 2, 189–214.

[26] P. Cheridito and T. Li, *Dual characterization of properties of risk measures on orlicz hearts*, Math. Financial Econ. **2** (2008), no. 1, 29–55.

[27] A. S. Cherny, *Weighted V@R and its properties*, Finance Stoch. **10** (2006), no. 3, 367–393.

[28] A. S. Cherny, *Pricing with coherent risk*, Theory Probab. its Appl. **52** (2008), no. 3, 389–415.

[29] A. S. Cherny and P. G. Grigoriev, *Dilatation monotone risk measures are law invariant*, Finance Stoch. **11** (2007), no. 2, 291–298.

[30] A. S. Cherny and M. Kupper, *Divergence utilities*, Available at SSRN 1023525, 2007.

[31] S. Ciliberti, I. Kondor, and M. Mézard, *On the feasibility of portfolio optimization under expected shortfall*, Quant. Finance **7** (2007), no. 4, 389–396.

[32] J. H. Cochrane and J. Saa-Requejo, *Beyond arbitrage: Good-deal asset price bounds in incomplete markets*, J. Polit. Econ. **108** (2000), no. 1, 79–119.

[33] R. C. Dalang, A. Morton, and W. Willinger, *Equivalent martingale measures and no-arbitrage in stochastic securities market models*, Stochastics **29** (1990), no. 2, 185–201.

[34] E. De Giorgi, *A note on portfolio selections under various risk measures*, Working paper/Institute for Empirical Research in Economics **122** (2002).

[35] F. Delbaen, *Coherent risk measures on general probability spaces*, Advances in finance and stochastics, Springer, 2002, pp. 1–37.

[36] F. Delbaen and K. Owari, *Convex functions on dual Orlicz spaces*, Positivity **23** (2019), no. 5, 1051–1064.

[37] F. Delbaen and W. Schachermayer, *The mathematics of arbitrage*, Springer Science & Business Media, 2006.

[38] K. Dowd, J. Cotter, and G. Sorwar, *Spectral risk measures: properties and limitations*, J. Financial Serv. Res. **34** (2008), no. 1, 61–75.

[39] G. A. Edgar and L. Sucheston, *Stopping Times and Directed Processes*, Encyclopedia of Mathematics and its Applications, vol. 47, Cambridge University Press, 1992.

[40] P. Embrechts, A. McNeil, and D. Straumann, *Correlation and dependence in risk management: properties and pitfalls*, Risk management: value at risk and beyond **1** (2002), 176–223.

[41] D. H. Erkens, M. Hung, and P. Matos, *Corporate governance in the 2007–2008 financial crisis: Evidence from financial institutions worldwide*, J. Corp. Finance **18** (2012), no. 2, 389–411.

[42] K. T. Fang, S. Kotz, and K. W. Ng, *Symmetric Multivariate and Related Distributions*, Chapman & Hall, London, 1987.

[43] D. Filipović and M. Kupper, *Monotone and cash-invariant convex functions and hulls*, Insur. Math. Econ. **41** (2007), no. 1, 1–16.

[44] H. Föllmer and A. Schied, *Convex measures of risk and trading constraints*, Finance Stoch. **6** (2002), no. 4, 429–447.

[45] H. Föllmer and A. Schied, *Stochastic finance*, fourth ed., de Gruyter Studies in Mathematics, vol. 27, Walter de Gruyter & co., Berlin, 2016.

[46] H. Föllmer and S. Weber, *The axiomatic approach to risk measures for capital determination*, Annu. Rev. Financial Econ. **7** (2015), 301–337.

[47] M. Frittelli and E. R. Gianin, *Putting order in risk measures*, J. Bank. Finance **26** (2002), no. 7, 1473–1486.

[48] N. Gao, D. Leung, C. Munari, and F. Xanthos, *Fatou property, representations, and extensions of law-invariant risk measures on general orlicz spaces*, Finance Stoch. **22** (2018), no. 2, 395–415.

[49] N. Gao, D. H. Leung, and F. Xanthos, *Closedness of convex sets in Orlicz spaces with applications to dual representation of risk measures*, Studia Math. **249** (2019), no. 3, 329–347.

[50] N. Gao and F. Xanthos, *On the C-property and w^*-representations of risk measures*, Math. Finance **28** (2018), no. 2, 748–754.

[51] K. Giesecke, T. Schmidt, and S. Weber, *Measuring the risk of large losses*, J. invest. manag. **6** (2008), no. 4, 1–15.

[52] X. Han, B. Wang, R. Wang, and Q. Wu, *Risk concentration and the mean-expected shortfall criterion*, arXiv preprint arXiv:2108.05066, 2021.

[53] G. Hansen, I. Herburt, H. Martini, and M. Moszyńska, *Starshaped sets*, Aequationes Math. **94** (2020), 1001–1092.

[54] M. Herdegen and N. Khan, *Mean-ρ portfolio selection and ρ-arbitrage for coherent risk measures*, Math. Finance **32** (2022), no. 1, 226–272.

[55] S. Jaschke and U. Küchler, *Coherent risk measures and good-deal bounds*, Finance Stoch. **5** (2001), no. 2, 181–200.

[56] E. Jouini, W. Schachermayer, and N. Touzi, *Law invariant risk measures have the Fatou property*, Advances in mathematical economics, vol. 9, Springer, Tokyo, 2006, pp. 49–71.

[57] P. Klibanoff, M. Marinacci, and S. Mukerji, *A smooth model of decision making under ambiguity*, Econometrica **73** (2005), no. 6, 1849–1892.

[58] S. Klöppel and M. Schweizer, *Dynamic utility-based good deal bounds*, Stat. Dec. **25** (2007), no. 4, 285–309.

[59] P. Koch-Medina and C. Munari, *Unexpected shortfalls of expected shortfall: Extreme default profiles and regulatory arbitrage*, J. Bank. Finance **62** (2016), 141–151.

[60] I. Kondor, S. Pafka, and G. Nagy, *Noise sensitivity of portfolio selection under various risk measures*, J. Bank. Finance **31** (2007), no. 5, 1545–1573.

[61] P. A. Krokhmal and R. Murphey, *Modeling and implementation of risk-averse preferences in stochastic programs using risk measures*, Robust optimization-directed design, Springer, 2006, pp. 95–116.

[62] S. Kusuoka, *On law invariant coherent risk measures*, Advances in mathematical economics, Springer, 2001, pp. 83–95.

[63] Z. M. Landsman and E. A. Valdez, *Tail conditional expectations for elliptical distributions*, N. Am. Actuar. J. **7** (2013), 55–71.

[64] J. Leitner, *Balayage monotonous risk measures*, Int. J. Theor. Appl. Finance **7** (2004), no. 7, 887–900.

[65] J. Lintner, *Security prices, risk, and maximal gains from diversification*, J. Finance **20** (1965), no. 4, 587–615.

[66] J. Lintner, *The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets*, Rev. Econ. Stat. **47** (1965), 13–37.

[67] F. Maccheroni, M. Marinacci, and D. Ruffino, *Alpha as ambiguity: Robust mean-variance portfolio analysis*, Econometrica **81** (2013), no. 3, 1075–1113.

[68] H. Markowitz, *Portfolio selection*, J. Finance **7** (1952), no. 1, 77–91.

[69] E. Mastrogiacomo and E. R. Gianin, *Portfolio optimization with quasiconvex risk measures*, Math. Oper. Res. **40** (2015), no. 4, 1042–1059.

[70] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management*, Princeton University Press, Princeton, New Jersey, 2005.

[71] J. Mossin, *Equilibrium in a capital asset market*, Econometrica (1966), 768–783.

[72] G. C. Pflug and W. Römisch, *Modeling, measuring and managing risk*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[73] M. Rahsepar and F. Xanthos, *On the extension property of dilatation monotone risk measures*, Stat. Risk Model. **1** (2020), no. ahead-of-print.

[74] M. Rásonyi and L. Stettner, *On utility maximization in discrete-time financial market models*, Ann. Appl. Probab. **15** (2005), no. 2, 1367–1395.

[75] R. T. Rockafellar, *Convex analysis*, no. 28, Princeton university press, 1970.

[76] R. T. Rockafellar, *Conjugate duality and optimization*, no. 16, Society for Industrial and Applied Mathematics, 1974.

[77] R. T. Rockafellar and S. Uryasev, *Optimization of conditional value-at-risk*, J. Risk Finance **2** (2000), 21–42.

[78] R. T. Rockafellar and S. Uryasev, *Conditional value-at-risk for general loss distributions*, J. Bank. Finance **26** (2002), no. 7, 1443–1471.

[79] R. T. Rockafellar, S. Uryasev, and M. Zabarankin, *Deviation measures in risk analysis and optimization*, University of Florida, Department of Industrial & Systems Engineering Working Paper (2002).

[80] R. T. Rockafellar, S. Uryasev, and M. Zabarankin, *Generalized deviations in risk analysis*, Finance Stoch. **10** (2006), no. 1, 51–74.

[81] R. T. Rockafellar, S. Uryasev, and M. Zabarankin, *Master funds in portfolio analysis with general deviation measures*, J. Bank. Finance **30** (2006), no. 2, 743–778.

[82] D. Rokhlin and W. Schachermayer, *A note on lower bounds of martingale measure densities*, Illinois J. Math. **50** (2006), no. 1-4, 815–824.

[83] A. M. Rubinov and A. P. Shveidel, *Radiant and star-shaped functions*, Pacific J. Optim. **3** (2007), 193–212.

[84] A. Ruszczyński and A. Shapiro, *Optimization of convex risk functions*, Math. Oper. Res. **31** (2006), no. 3, 433–452.

[85] W. F. Sharpe, *Capital asset prices: A theory of market equilibrium under conditions of risk*, J. Finance **19** (1964), no. 3, 425–442.

[86] J. L. Treynor, *Market value, time, and risk*, Unpublished manuscript, 1961.

[87] R. Wang, *Regulatory arbitrage of risk measures*, Quant. Finance **16** (2016), no. 3, 337–347.

[88] M. Willesson, *What is and what is not regulatory arbitrage? A review and syntheses*, Financial Markets, SME Financing and Emerging Economies, Springer, 2017, pp. 71–94.

[89] M. T. Williams, *Uncontrolled risk: the lessons of lehman brothers and how systemic risk can still bring down the world financial system*, McGraw-Hill, 2010.

[90] M. Wilson, *Weighted Littlewood-Paley Theory and Exponential-Square Integrability*, Springer Science & Business Media, 2008.

[91] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.