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The random cluster model on finite graphs

by

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Contents

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Declarations

I declare that, unless otherwise indicated, the material in this thesis is my own work and has not been submitted for another degree at the University of Warwick or any other university. In particular:

1. Chapter 2 provides an introduction to the key concepts studied in this thesis. The majority of this content is not original, and has been adapted from a variety of sources, most notably [10], [14], [16], [23], and [30]. One key exception is Section 2.1.2, which establishes a sprinkling method for the random cluster model that extends techniques introduced in [1] to study percolation. Section 3.1 is another expository section, heavily based on [6].

2. The contents of Section 3.2, Chapter 4 and Chapter 5 constitute a novel investigation of the random cluster model on the complete graph, which is based upon methods developed for percolation in [3]. A condensed version of this material has been submitted for publication in [34].

3. The contents of Section 2.1.2, Section 3.3, Chapter 6 and Chapter 7 constitute a novel investigation of the random cluster model on the hypercube, which combines our sprinkling method for the random cluster model with an analysis of the Potts model on the hypercube using methods developed for the lattice $\mathbb{Z}^d$ in [30]. A condensed version of this material will be submitted for publication in [32] in collaboration with Roman Kotecký.

Abstract

The abrupt change of the size of the largest connected component is a central quantity of interest in the study of random graphs. For the percolation model, it is well known for a variety of families of finite regular graphs that the largest connected component experiences an asymptotic phase transition marking the emergence of a giant component (that is, one which contains a positive proportion of the total number of vertices) when the edge weight is appropriately rescaled by the vertex degree. For the random cluster model, a similar asymptotic phase transition was established by Bollobás, Grimmett, and Janson on the complete graph in [6]. The problem of establishing a similar phase transition on any family of finite graphs with more complicated geometry had remained open.

In this thesis, we study the emergence of the giant component for the random cluster model on two families of finite regular graphs. Our first result provides an alternative analysis of the random cluster model on the complete graph using a thermodynamic/large deviations approach introduced by Biskup, Chayes, and Smith to study percolation on the complete graph in [3]. In particular, we compute the exponential rate of the large deviations of the size of the largest connected component of the random graph. Our second result establishes an asymptotic phase transition for the random cluster model on the hypercube when the cluster weight is an integer. In particular, we introduce a new concept which we call the sprinkled random cluster measure, which we combine with results obtained from an analysis of the asymptotics of a corresponding Potts model in order to extend the arguments of [1] to the random cluster model.

Chapter 1

Introduction

The initial objective of this thesis was to investigate a model of random permutations known as the *interchange process*, which was introduced by Tóth in [39] as a probabilistic representation of the quantum Heisenberg ferromagnet. In particular, one may show that the expected phase transition of the magnetisation of the Heisenberg ferromagnet on the lattice $\mathbb{Z}^d$ ($d \geq 3$) is closely related to the appearance of an infinite cycle in the corresponding interchange process.

Establishing the appearance of an infinite cycle in the interchange process on $\mathbb{Z}^d$ is hard, so one may instead investigate long cycles in simpler graphs, beginning with the complete graph K_n. This was first done by Schramm in [37], who investigated an *unweighted* version of the interchange process and established that the size of the largest cycle experiences an asymptotic phase transition in the limit $n \to \infty$ by arguing that each cycle is contained within a component of an associated percolation model. In particular, a giant cycle cannot appear in the unweighted interchange process before a giant component appears in the corresponding percolation model. Similar arguments were used by Kotecký, Miłoś, and Ueltschi in [33] to analyse the unweighted interchange process on the hypercube Q_n.

It has been conjectured that for graphs of diverging degree, a similar correspondence exists between the largest cycle of the *weighted* interchange model and the largest component of a weighted version of the percolation model, known as the *random cluster model*. For the complete graph, this conjecture was proven by Björnberg in [4]. For the hypercube, however, even the largest component of the random cluster model had not yet been studied, and so we focused on this open problem.

The random cluster model was introduced by Fortuin and Kasteleyn in [21] as a generalisation of several existing models in statistical physics satisfying certain series and parallel laws. Given an underlying graph G, the random cluster model samples a random edge-set of a given graph G according to a probability measure $\phi_{G,p,q}$ which depends on two parameters; the *edge weight* $p \in [0,1]$ and the *cluster weight* $q > 0$. When $q = 1$, the random cluster measure reduces to the percolation measure $\phi_{G,p}$, which was introduced by Broadbent and Hammersley in [11] in order to study the percolation of liquid through a porous medium. When $q = 2$, we instead recover the Ising model for ferromagnetism from [28]. More generally, we may relate the random-cluster model to the Potts model (introduced in [36]) when $q \in \mathbb{N}_{\geq 2}$ using the Edwards-Sokal coupling given in [17]. By taking an appropriate limit as $q \to 0$, we may recover the measure for the uniform spanning tree, which was related to the theory of electrical networks by Kirchhoff in [31].

When $q \geq 1$, the random cluster model is stochastically ordered by the edge weight p, and we investigate how the random graph defined by the sampled edge-set evolves as p increases. One key quantity of interest in this investigation is the size of the largest connected component of the random graph. This is motivated by the study of percolation on a lattice, wherein one seeks a path of edges in the random graph which spans the lattice, so that liquid may percolate in the underlying physical model. In the limiting case of the infinite lattice, one seeks an infinite path - or equivalently, an infinite component. Typically, one expects to find a critical probability $p_c = p_c(G)$ such that for $p < p_c$, the largest component is finite, while for $p > p_c$, the largest component is infinite. For example, it was famously proven by Kesten in [29] that $p_c(\mathbb{Z}^2) = \frac{1}{2}$. This abrupt change in the size of the largest component is known as a *phase transition*. For more general values of q, this phase transition is related to phase transitions in the Potts model and other spin systems.

When G is finite, there is a positive probability that the random graph is connected, so matters are more complicated. In this instance, one typically fixes a sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs, a sequence $(p_n)_{n\in\mathbb{N}}$ of edge weights, and investigates the largest component of a sequence of random graphs drawn from the sequence $(\phi_{G_n,p_n,q})_{n\in\mathbb{N}}$ of measures. We will be particularly interested in the emergence of a *giant* component containing a fixed proportion of the total number of vertices. One early result in this direction is the groundbreaking paper [19], in which Erdős and Rényi studied the percolation measure $\phi_{K_n,\lambda/n}$ on the complete graph and established the existence of a critical parameter $\lambda_c = 1$ marking the emergence of the giant component asymptotically almost surely. More specifically, they showed that, with probability tending to one as $n \to \infty$, the largest component of the random graph is of order $\log n$ when $\lambda < 1$ and of order n when $\lambda > 1$.

The abrupt change in the size of the largest component (in the limit as $n \to \infty$) established in [19] is known as an *asymptotic phase transition*, and is proven using an *exploration process* whereby one chooses a vertex and sequentially inspects which vertices are connected to it in the random graph. Provided one has not yet explored a large fraction of the vertices, this exploration can be approximated by a Poisson branching process with mean λ in the limit as $n \to \infty$, and the vertex belongs to the giant component if the corresponding branching process survives. Indeed, it can be shown that the density of the giant component converges to the survival probability of a Poisson branching process with mean λ. More detailed results, including the behaviour of the largest component around the critical point $\lambda = 1$, may be found in e.g. [5] and [26].

The exploration process is not crucially dependent on the structure of the complete graph, and has been successfully applied to the study of percolation on a variety of families of finite, regular graphs. One key result is the paper [1], in which Ajtai, Komlós, and Szemerédi studied the percolation measure $\phi_{Q_n,\lambda/n}$ on the hypercube by combining the exploration process with a new technique known as the *sprinkling method*. Consequently, they showed that, with probability tending to one as $n \to \infty$, the largest component of the random graph is of order n when $\lambda < 1$ and of order 2^n when $\lambda > 1$. As for the complete graph, the density of the largest component converges to the survival probability of a Poisson branching process with mean λ, and the behaviour around the critical point $\lambda = 1$ has been investigated in detail in e.g. [7] and [10].

For $q \neq 1$, we encounter an additional complexity - the edges included in the random graph are no longer independent, and it is no longer clear that the exploration process can be used to study the component of a given vertex. Nevertheless, a complete treatment of the random cluster model on the complete graph was given by Bollobás, Grimmett, and Janson in [6], who considered the measure $\phi_{K_n,\lambda/n,q}$ and established the existence of a critical parameter λ_c (depending only on q) marking the emergence of the giant component asymptotically almost surely. More specifically, they showed that, with probability tending to one as $n \to \infty$, the largest component of the random graph is of order $\log n$ when $\lambda < \lambda_c$ and of order n when $\lambda > \lambda_c$. When $q \in \mathbb{N}_{\geq 2}$, this is related to a phase transition in the mean-field Potts model investigated by Wu in [40], and the limiting density of the giant component may be expressed in terms of the mean-field magnetisation.

The asymptotic phase transition established in [6] is proven by randomly colouring the vertices of K_n using two colours (say red and green) in a particular way so that the distribution of edges on the *red vertices* is given by a percolation measure, to which we may apply the exploration process used in [19]. This colouring argument relies on the observation that for a fixed set R of red vertices, the conditional distribution of edges yields a random percolation on the complete graph induced on the set R. This is a particular fact that does not generalise to more structured families of graphs. Consequently, the complete graph is the *only* family of finite graphs for which the random cluster model has been studied in detail.

In this thesis, we extend the study of the random cluster model on finite graphs to a slightly more general setting. We have two main results, which both establish that the largest component of a family of regular graphs undergoes an asymptotic phase transition when we rescale the edge weight p by the vertex degree.

Our first result is a new analysis of the measure $\phi_{K_n,\lambda/n,q}$ on the complete graph, which omits the colouring arguments used in [6] in favour of a thermodynamic / large deviations approach introduced by Biskup, Chayes and Smith in [3] in order to study the percolation measure $\phi_{K_n,\lambda/n}$. In particular, we will compute the *rate function* for large deviations of the size of the largest connected component, thereby recovering the asymptotic phase transition proven in [6]. This rate function is new. In addition, we obtain a limit for the *free energy* of the random cluster model on the complete graph. This was also computed in [6] via the colouring argument, but used to study the large deviations of the *number* of connected components, rather than their size. As a byproduct of our analysis, we also obtain the exponential decay rate for the events that the random graph is connected and acyclic, respectively.

Our second result is an analysis of the largest component of the random graph for the measure $\phi_{Q_n,\lambda/n,q}$ on the hypercube, which uses a new analogue of the sprinkling method for the random cluster model in order to extend the arguments employed in [1]. As a substitute for the exploration process used in [1], we will also investigate the free energy of the corresponding Potts model, using methods developed by Kesten and Schonmann in [30] to study the Potts model on the lattice $\mathbb{Z}^d$. In particular, we will show that the Potts model on the hypercube converges to a mean-field limit when appropriately rescaled. Using this limit, we will establish that the largest connected component of the random cluster model undergoes an asymptotic phase transition for integer q, where the critical parameter $\lambda_c(q)$ is the same parameter established for the complete graph in [6].

Finally, we briefly discuss the structure of the thesis. To begin, we provide a discussion of the necessary mathematical prerequisites in Chapter 2. This includes a formal introduction to the random cluster and Potts models, based upon the treatments given in [16] and [30], and a discussion of some isoperimetric inequalities for the hypercube, taken from [10] and [14]. Crucially, we will also state and prove the new analogue of the sprinkling method for the random cluster model in Section 2.1.2. With these prerequisites in hand, we discuss the main results of the thesis in Chapter 3. Section 3.1 is another expository section which discusses the results established in [6], while Sections 3.2 and 3.3 discuss our new results for the random cluster model on the complete graph and the hypercube, respectively. The remainder of the thesis is dedicated to the proofs of these results. In particular, Chapters 4 and 5 deal with the complete graph, while Chapters 6 and 7 are dedicated to the hypercube.

Chapter 2

Preliminaries

In this chapter, we present the preliminary definitions and results used in this thesis. With the key exception of Section 2.1.2, the content of this chapter is not original, and draws from a variety of sources.

We begin with a formal introduction to the random cluster model in Section 2.1, based upon the formulation of the model given by Duminil-Copin in [16]. In Section 2.1.1, we show that the random cluster model is stochastically ordered in various appropriate ways, again following the methods of [16]. Then, in Section 2.1.2, we establish a version of the sprinkling method (introduced by Ajtai, Komlós, and Szemerédi in [1] for percolation) for the random cluster model. In particular, we introduce a new concept of independent interest which we call the *sprinkled random cluster measure*.

Next, we discuss the Potts model in Section 2.2, using the formulation of the model given by Kesten and Schonmann in [30]. In Section 2.2.1, we discuss the Edwards Sokal coupling, which was introduced in [17] and provides a relationship between the Potts and random cluster models. Then, in Section 2.2.2, we analyse the free energy of the Potts model. The treatment of the Edwards Sokal coupling in Section 2.2.1 is based upon [16], while the treatment of the free energy in Section 2.2.2 is adapted from that of Friedli and Velenik in [23].

2.1 The random cluster model

In this section, we provide a formal introduction to the random cluster model, based upon the formulation of the model given by Duminil-Copin in [16].

Let $G = (V, E)$ be a finite graph, and define $\Omega_G = \{0, 1\}^E$. An element $\omega \in \Omega_G$ is known as a *percolation configuration*, and corresponds to an assignment of a value $\omega_e \in \{0, 1\}$ to each edge $e \in E$. If $\omega_e = 1$, we say the edge e is *open*. Otherwise, e is *closed*. Given $\omega \in \Omega_G$, observe that the set $E(\omega) := \{e \in E : \omega_e = 1\}$ of open edges of ω defines a subgraph $G(\omega) = (V, E(\omega))$ of G. As the map $\omega \to G(\omega)$ is a bijection, any measure on the set Ω_G induces a measure on the set of subgraphs of G. We define the random cluster model by the following measure:

Definition 2.1.1 (Random cluster model)**.** *Let $G = (V, E)$ be a finite graph, $p \in [0, 1]$ and $q > 0$. The random cluster model on G with edge weight p and cluster weight q is defined by the measure*

$$\phi_{G,p,q}[\omega] := \frac{\{\prod_{e\in E} p^{\omega_e}(1-p)^{1-\omega_e}\}q^{k(\omega)}}{Z^{RC}_{G,p,q}}, \tag{2.1}$$

where

$$Z^{RC}_{G,p,q} := \sum_{\omega\in\Omega_G} \{\prod_{e\in E} p^{\omega_e}(1-p)^{1-\omega_e}\}q^{k(\omega)} \tag{2.2}$$

is the normalising partition function, and $k(\omega)$ is the number of connected components in the graph $G(\omega)$.

When $q = 1$, the random cluster measure reduces to a percolation measure, in which case we use the standard notation $\phi_{G,p}$. For percolation, the states of edges are independent, and one may sample a graph from the percolation measure by opening each edge of G independently with probability p. This procedure may be used to extend the percolation measure to infinite graphs in a natural way. Extending the random cluster model to infinite graphs is possible, but more complicated. As this thesis is only concerned with cases where the graph G is finite, we will not discuss the matter further.

2.1.1 Stochastic monotonicity of the random cluster model

Consider the following construction of a configuration ω under the measure $\phi_{G,p}$:

1. To each edge $e \in E$, we associate a uniform random variable U_e on $[0, 1]$.

2. If $U_e \leq p$, we set $\omega_e = 1$. Otherwise, we set $\omega_e = 0$.

Let $p_1 < p_2$ be two edge weights, and construct two configurations $\omega, \tilde{\omega}$ by setting $\omega_e = 1$ if $U_e \leq p_1$ and $\tilde{\omega}_e = 1$ if $U_e \leq p_2$ for each $e \in E$. Then $\omega, \tilde{\omega}$ are distributed according to the measures $\phi_{G,p_1}, \phi_{G,p_2}$ respectively. Moreover, this coupling has the additional property that if $\omega_e = 1$ then $\tilde{\omega}_e = 1$. Informally speaking, the random graph associated to the measure $\phi_{G,p}$ is *growing* as p increases.

The objective of this subsection is to provide a generalisation of the above idea for the random cluster model. To begin, we will need a partial ordering on the set Ω_G with which to compare two configurations $\omega, \tilde{\omega}$ and their corresponding graphs $G(\omega), G(\tilde{\omega})$. We will use the partial ordering on the set Ω_G given by

$$\omega \leq \tilde{\omega} \iff \forall\ e \in E,\ \omega_e \leq \tilde{\omega}_e. \tag{2.3}$$

Equivalently, $\omega \leq \tilde{\omega}$ if and only if $G(\omega)$ is a subgraph of $G(\tilde{\omega})$. We say that a function on Ω_G is *increasing* if it respects this ordering. That is:

Definition 2.1.2 (Increasing functions)**.** *Let $G = (V, E)$ be a finite graph and $\leq$ be the partial ordering of Ω_G defined in (2.3). We say a function $f : \Omega_G \to \mathbb{R}$ is increasing (with respect to the partial ordering $\leq$) if for any pair $\omega, \tilde{\omega} \in \Omega_G$,*

$$\omega \leq \tilde{\omega} \Rightarrow f(\omega) \leq f(\tilde{\omega}). \tag{2.4}$$

We say a set $A \subset \Omega_G$ is increasing if its indicator function $\mathbb{1}_A$ is increasing. Equivalently, A is increasing if for any pair $\omega, \tilde{\omega} \in \Omega_G$,

$$\omega \in A,\ \omega \leq \tilde{\omega} \Rightarrow \tilde{\omega} \in A. \tag{2.5}$$

In particular, we see that an event $A \subset \Omega_G$ is increasing if it still holds whenever we open additional edges in G. Note that $\{\omega_e = 1\}$ is an increasing event, and for any $p_1 < p_2$, our earlier coupling shows that

$$\phi_{G,p_1}[\omega_e = 1] \leq \phi_{G,p_2}[\omega = 1]. \tag{2.6}$$

In fact, (2.6) holds if we replace the event $\{\omega_e = 1\}$ with any increasing event A. This motivates the following definition:

Definition 2.1.3 (Stochastic monotonicity)**.** *Let Ω be a set equipped with a partial ordering $\leq$ and let μ, ν be two probability measures on Ω. We say that μ is stochastically dominated by ν, written $\mu \leq_{st} \nu$, if $\mu[A] \leq \nu[A]$ for any increasing event A.*

Our goal is to establish various forms of stochastic monotonicity for the random cluster model as we vary the edge weight p and/or the cluster weight q. This will be done using the following criterion, taken from [16]:

Lemma 2.1.4 ([16, Lemma 1.5]). *Let μ, ν be two strictly positive measures on Ω_G. Suppose that for any $e \in E$ and any pair of configurations $\psi, \psi' \in \{0,1\}^{E \setminus \{e\}}$ such that $\psi \leq \psi'$, we have*

$$\mu[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] \leq \nu[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi']. \tag{2.7}$$

Then, there exists a measure $\mathbb{P}_{\mu,\nu}$ on pairs $(\omega, \tilde{\omega})$ such that the marginals of $\mathbb{P}_{\mu,\nu}$ on $\omega, \tilde{\omega}$ are equal to μ, ν respectively, and $\mathbb{P}_{\mu,\nu}(\omega \leq \tilde{\omega}) = 1$. In particular, μ is stochastically dominated by ν.

Proof. We follow the proof of [16, Lemma 1.5], constructing the measure $\mathbb{P}_{\mu,\nu}$ using a continuous time Markov chain on pairs $(\omega^t, \tilde{\omega}^t) \subset \Omega_G$ defined in the following way:

1. At time $t = 0$, let ω^0 be the configuration identically equal to 0 (i.e. all edges are closed) and $\tilde{\omega}^0$ be the configuration identically equal to 1 (i.e. all edges are open).

2. To each edge $e \in E$, associate an exponential clock and a sequence $(U_{k,e})_{k\in\mathbb{N}}$ of independent uniform random variables on $[0, 1]$.

3. Suppose the clock associated to the edge e rings for the kth time at time t, at which time the configurations of $\omega_{E\setminus\{e\}}, \tilde{\omega}_{E\setminus\{e\}}$ are given by $\psi_{e,k}, \tilde{\psi}_{e,k}$ respectively. Then, set:

$$\omega_e^t = \begin{cases} 1 & \text{if } U_{e,k} \leq \mu[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi_{e,k}] \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{\omega}_e^t = \begin{cases} 1 & \text{if } U_{e,k} \leq \nu[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \tilde{\psi}_{e,k}] \\ 0 & \text{otherwise} \end{cases}$$

As ω^t is an irreducible continuous time Markov chain, its distribution converges to a stationary measure, which is necessarily μ due to the choice of jump probabilities. Similarly, the distribution of $\tilde{\omega}^t$ converges to the stationary measure ν. Moreover, as $\omega^0 \leq \tilde{\omega}^0$, the choice of jump probabilities ensures that $\omega^t \leq \tilde{\omega}^t$ for every t. Consequently, we see that the measure for the pair $(\omega^t, \tilde{\omega}^t)$ converges to a stationary distribution $\mathbb{P}_{\mu,\nu}$ with marginals μ, ν such that $\mathbb{P}_{\mu,\nu}(\omega \leq \tilde{\omega}) = 1$ in the limit as $t \to \infty$. Finally, we observe that for any increasing event A,

$$\mu[A] = \mathbb{P}_{\mu,\nu}[\omega \in A] = \mathbb{P}_{\mu,\nu}[\omega, \tilde{\omega} \in A] \leq \mathbb{P}_{\mu,\nu}[\tilde{\omega} \in A] = \nu[A]. \tag{2.8}$$

In particular, μ is stochastically dominated by ν. □

The first major stochastic domination result for the random cluster model is the Fortuin-Kasteleyn-Ginibre (FKG) inequality, introduced in [22]:

Theorem 2.1.5 (FKG Inequality)**.** *Let $p \in [0,1]$ and $q \geq 1$. Then, for any pair of increasing events $A, B \subset \Omega_G$, we have*

$$\phi_{G,p,q}[A \cap B] \geq \phi_{G,p,q}[A]\phi_{G,p,q}[B]. \tag{2.9}$$

Proof. We follow the proof of [16, Theorem 1.6]. If $\phi_{G,p,q}[B] = 0$ then (2.9) is trivial, so without loss of generality we may assume that $\phi_{G,p,q}[B] > 0$. As the configuration which is identically equal to 1 belongs to any non-empty increasing event, we may construct the same continuous time Markov chain as in the proof of Lemma 2.1.4 with $\mu = \phi_{G,p,q}[\cdot]$ and $\nu = \phi_{G,p,q}[\cdot \mid B]$. Observing that

$$\nu[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] = \begin{cases} 1 & \text{if } \psi^{(0)} \notin B \\ \mu[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] & \text{if } \psi^{(0)} \in B \end{cases}$$

we see that μ and ν satisfy the condition of (2.7), so we may follow the proof of Lemma 2.1.4 to deduce that μ is stochastically dominated by ν. In particular, for any increasing event A we see that $\phi_{G,p,q}[A] \leq \phi_{G,p,q}[A \mid B]$, which is equivalent to (2.9). □

Note that the condition $q \geq 1$ is important, and the FKG inequality fails without it. A counterexample showing that the FKG inequality fails for $q < 1$ on a graph involving only two vertices with two edges between them is given by Grimmett in [24, Equation 3.9]. We provide an alternative counterexample on a simple graph (that is, one without multiple edges between the same vertices) using only three vertices:

Example (FKG fails for $q < 1$) Let $G = C_3$ be the cyclic graph on three vertices and three edges. If we set $p = \frac{1}{2}$, then the weight of any configuration $\omega \in \{0,1\}^E$ with respect to the random cluster measure $\phi_{G,p,q}$ is given by

$$Z^{\mathrm{RC}}_{G,p,q}\phi_{G,p,q}[\omega] = \tfrac{1}{8}q^{k(\omega)}. \tag{2.10}$$

Label the edges of G as e_1, e_2 and e_3, and consider the events $A = \{\omega_{e_1} = 1\}$ and $B = \{\omega_{e_2} = 1\} \cap \{\omega_{e_3} = 1\}$. Then, one may check that

$$\begin{aligned} Z^{\mathrm{RC}}_{G,p,q} &= \tfrac{1}{8}(q^3 + 3q^2 + 4q), \\ \phi_{G,p,q}[A] &= \frac{q^2 + 3q}{q^3 + 3q^2 + 4q}, \\ \phi_{G,p,q}[B] &= \frac{2q}{q^3 + 3q^2 + 4q}, \\ \phi_{G,p,q}[A \cap B] &= \frac{q}{q^3 + 3q^2 + 4q}. \end{aligned} \tag{2.11}$$

In particular, the FKG inequality $\phi_{G,p,q}[A] \leq \phi_{G,p,q}[A \mid B]$ may be written as

$$\frac{q^2 + 3q}{q^3 + 3q^2 + 4q} = \phi_{G,p,q}[A] \leq \frac{\phi_{G,p,q}[A \cap B]}{\phi_{G,p,q}[B]} = \frac{1}{2}. \tag{2.12}$$

Rearranging (2.12) yields the inequality

$$q(q+2)(q-1) \geq 0 \tag{2.13}$$

which fails for $0 < q < 1$.

In light of Lemma 2.1.4, it will be useful to calculate the probability that an edge e is open when conditioned on the states of the edges in $E \setminus \{e\}$ for the measure $\phi_{G,p,q}$. This is the content of the following proposition:

Proposition 2.1.6 ([16, Equation 1.3])**.** *Let $p \in [0,1]$ and $q > 0$. Then, for any $e = \{x, y\} \in E$ and $\psi \in \{0,1\}^{E \setminus \{e\}}$, we have*

$$\phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] = \begin{cases} p & \text{if } x \leftrightarrow y \text{ in } \psi \\ \frac{p}{p+q(1-p)} & \text{otherwise} \end{cases} \tag{2.14}$$

Proof. Let $\psi^{(0)}$ and $\psi^{(1)}$ denote the configurations on Ω given for each $f \in E$ by

$$\psi^{(i)}_f = \begin{cases} i & \text{if } f = e \\ \psi_f & \text{otherwise} \end{cases} \tag{2.15}$$

Then, observe that

$$\phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] = \frac{\phi_{G,p,q}[\psi^{(1)}]}{\phi_{G,p,q}[\psi^{(0)}] + \phi_{G,p,q}[\psi^{(1)}]}. \tag{2.16}$$

There are two cases to consider:

1. If $x \leftrightarrow y$ in $G(\psi)$, then $q^{k(\psi^{(1)})} = q^{k(\psi^{(0)})}$ and so $\phi^{\mathrm{RC}}_{G,p,q}[\psi^{(1)}] = \frac{p}{1-p}\phi^{\mathrm{RC}}_{G,p,q}[\psi^{(0)}]$.

2. If $x \not\leftrightarrow y$ in $G(\psi)$, then $q^{k(\psi^{(1)})} = q^{k(\psi^{(0)})-1}$ and so $\phi^{\mathrm{RC}}_{G,p,q}[\psi^{(1)}] = \frac{p}{q(1-p)}\phi^{\mathrm{RC}}_{G,p,q}[\psi^{(0)}]$.

In either case, we recover (2.14). □

In the remainder of this subsection, we prove various stochastic orderings for the random cluster model by applying Proposition 2.1.6 in conjunction with Lemma 2.1.4. We begin with the following result, which establishes that for fixed $q \geq 1$, the measure $\phi_{G,p,q}$ is stochastically increasing with respect to p:

Proposition 2.1.7 ([16, Theorem 1.6])**.** *Let $p_1 \leq p_2$ and $q \geq 1$. Then*

$$\phi_{G,p_1,q} \leq_{st} \phi_{G,p_2,q}. \tag{2.17}$$

Proof. We apply Lemma 2.1.4 to the measures $\mu = \phi_{G,p_1,q}$ and $\nu = \phi_{G,p_2,q}$. In order to do this, we need to show that for any edge $e = \{x, y\} \in E$ and any configurations $\psi, \psi' \in \{0,1\}^{E\setminus\{e\}}$ such that $\psi \leq \psi'$, we have

$$\phi_{G,p_1,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] \leq \phi_{G,p_2,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi']. \tag{2.18}$$

There are two cases to consider:

1. If $x \leftrightarrow y$ in $G(\psi)$, then $x \leftrightarrow y$ in $G(\psi')$, as $\psi \leq \psi'$. Thus

$$\phi_{G,p_1,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] = p_1 \leq p_2 = \phi_{G,p_2,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi'].$$

2. If $x \not\leftrightarrow y$ in $G(\psi)$, then using the inequality $\frac{p}{p+q(1-p)} \leq p$ for $q \geq 1$, we have

$$\begin{aligned}\phi_{G,p_1,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] &= \frac{p_1}{p_1 + q(1-p_1)} \\ &\leq \frac{p_2}{p_2 + q(1-p_2)} \leq \phi_{G,p_2,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi'].\end{aligned}$$

In either case, (2.18) holds and (2.17) follows by Lemma 2.1.4. □

Intuitively, Proposition 2.1.7 says that for $q \geq 1$, the random graph associated to the measure $\phi_{G,p,q}$ (and in particular, its largest component) *grows* as p increases. The next proposition establishes a similar stochastic ordering for the measure $\phi_{G,p,q}$ when we instead fix p and vary q:

Proposition 2.1.8. *Let* $p \in [0,1]$ *and* $q_1 \geq q_2 \geq 1$. *Then*

$$\phi_{G,p,q_1} \leq_{st} \phi_{G,p,q_2}. \tag{2.19}$$

Proof. We apply Lemma 2.1.4 to the measures $\mu = \phi_{G,p,q_1}$ and $\nu = \phi_{G,p,q_2}$. In order to do this, we need to show that for any edge $e = \{x,y\} \in E$ and any configurations $\psi, \psi' \in \{0,1\}^{E\setminus\{e\}}$ such that $\psi \leq \psi'$, we have

$$\phi_{G,p,q_1}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] \leq \phi_{G,p,q_2}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi']. \tag{2.20}$$

There are two cases to consider:

1. If $x \leftrightarrow y$ in $G(\psi)$, then $x \leftrightarrow y$ in $G(\psi')$, as $\psi \leq \psi'$. Thus

$$\phi_{G,p,q_1}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] = p = \phi_{G,p,q_2}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi'].$$

2. If $x \not\leftrightarrow y$ in $G(\psi)$, then using the inequality $\frac{p}{p+q(1-p)} \leq p$ for $q \geq 1$, we have

$$\begin{aligned} \phi_{G,p,q_1}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] &= \frac{p}{p+q_1(1-p)} \\ &\leq \frac{p}{p+q_2(1-p)} \leq \phi_{G,p,q_2}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi']. \end{aligned}$$

In either case, (2.20) holds and (2.19) follows by Lemma 2.1.4. □

Proposition 2.1.8 states that the random cluster model is stochastically *decreasing* with respect to q. Intuitively, this is because the weight $q^{k(\omega)}$ provides larger biases towards configurations with more components for larger values of q, and the number of components is negatively correlated with their size.

It will be useful to directly compare the random cluster model to the percolation model, as the states of the edges in the latter model are independent, which greatly simplifies calculations. In order to do this, we will use the following proposition:

Proposition 2.1.9 ([24, Theorem 3.21])**.** *Let* $p \in [0,1]$ *and* $q \geq 1$. *Then*

$$\phi_{G,p/q} \leq_{st} \phi_{G,p,q} \leq_{st} \phi_{G,p}. \tag{2.21}$$

Proof. The right hand side of (2.21) is a consequence of Proposition 2.1.8, so it will suffice to prove the left hand side by applying Lemma 2.1.4 to the measures $\mu = \phi_{G,p/q}$

and $\nu = \phi_{G,p,q}$. In order to do this, we need to show that for any edge $e = \{x, y\} \in E$ and any configurations $\psi, \psi' \in \{0,1\}^{E\setminus\{e\}}$ such that $\psi \leq \psi'$, we have

$$\phi_{G,p/q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] \leq \phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi'], \tag{2.22}$$

which follows as

$$\phi_{G,p/q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] = \frac{p}{q} \leq \frac{p}{p + q(1-p)} \leq \phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi']. \tag{2.23}$$

Thus (2.21) follows by Lemma 2.1.4. □

In percolation, it is a common technique to restrict the measure to a subgraph of the original graph. This is possible as the states of edges inside and outside of the subgraph are independent for the percolation model. In order to apply the same technique to the random cluster model, where the states of edges are no longer independent in general, we will need the following proposition:

Proposition 2.1.10 (Comparison between boundary conditions)**.** *Let $p \in [0,1]$ and $q \geq 1$. Let H be a subgraph of G with edge-set F, and let $\Delta = E \setminus F$. Then, for any pair of configurations $\eta, \eta' \in \{0,1\}^{\Delta}$ satisfying $\eta \leq \eta'$, we have*

$$\phi_{G,p,q}[\cdot \mid \omega_\Delta = \eta] \leq_{st} \phi_{G,p,q}[\cdot \mid \omega_\Delta = \eta']. \tag{2.24}$$

In particular, if we write $\emptyset$ for the configuration on Δ where every edge is closed, then for every configuration $\eta \in \{0,1\}^{\Delta}$ we have

$$\phi_{H,p,q} = \phi_{G,p,q}[\cdot \mid \omega_\Delta = \emptyset] \leq_{st} \phi_{G,p,q}[\cdot \mid \omega_\Delta = \eta]. \tag{2.25}$$

Proof. We write each configuration $\omega \in \Omega_G$ as a pair $(\omega_F, \omega_\Delta)$, and apply Lemma 2.1.4 to the measures $\mu = \phi_{G,p,q}[\cdot \mid \omega_\Delta = \eta]$ and $\nu = \phi_{G,p,q}[\cdot \mid \omega_\Delta = \eta']$. In order to do this, we need to show that for any edge $e = \{x, y\} \in F$ and any configurations $\psi, \psi' \in \{0,1\}^{F\setminus\{e\}}$ such that $\psi \leq \psi'$, we have

$$\phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = (\psi, \eta)] \leq \phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = (\psi', \eta')]. \tag{2.26}$$

There are two cases to consider:

1. If $x \leftrightarrow y$ in $G((\psi, \eta))$, then $x \leftrightarrow y$ in $G((\psi', \eta'))$, as $(\psi, \eta) \leq (\psi', \eta')$. Thus

$$\phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = (\psi, \eta)] = p = \phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = (\psi', \eta')].$$

2. If $x \nleftrightarrow y$ in $G((\psi, \eta))$, then using the inequality $\frac{p}{p+q(1-p)} \leq p$ for $q \geq 1$, we have

$$\phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = (\psi, \eta)] = \frac{p}{p + q(1-p)} \leq \phi_{G,p,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = (\psi', \eta')].$$

In either case, (2.26) holds and (2.24) follows by Lemma 2.1.4. □

2.1.2 The sprinkling method

In the previous subsection, we showed that for $q \geq 1$, the random cluster measure $\phi_{G,p,q}$ is stochastically increasing with respect to p. The objective of this subsection is to quantify the *rate* of this increase. In particular, we will introduce a new concept which we call the *sprinkled random cluster measure*, which may be used to extend arguments developed by Ajtai, Komlós, and Szemerédi in [1] in order to study percolation on the hypercube.

Let $p_1 < p_2$ be edge weights, and let $\delta > 0$ satisfy $(1 - p_1)(1 - \delta) = 1 - p_2$. The sprinkling method constructs a random graph in two steps. In the first step, we open each edge $e \in E$ independently with probability p_1. Then, in the second step, we open any remaining closed edges independently with probability δ. At the end of the first step, the random graph is distributed according to the measure ϕ_{G,p_1}, and at the end of the second step, it is distributed according to the measure ϕ_{G,p_2}.

The sprinkling method allows us to investigate how a random graph changes as we increase p by independently adding edges with some probability δ. We seek an analogue of this technique for the random cluster measure $\phi_{G,p,q}$ with $q \geq 1$. To this end, we define the following measure:

Definition 2.1.11 (Sprinkled random cluster measure)**.** *Let $p, \delta \in [0, 1]$ and $q \geq 1$. Given a pair $(\xi, \omega) \in \Omega_G \times \Omega_G$, define the function*

$$\Pi_\delta[\xi, \omega] = \left\{ \prod_{e \in E : \xi_e = 0} \delta^{\omega_e} (1 - \delta)^{1 - \omega_e} \right\} \mathbb{1}_{\xi \leq \omega} \tag{2.27}$$

and note that

$$\sum_{\omega \in \Omega_G} \Pi_\delta[\xi, \omega] = (\delta + (1 - \delta))^{|E \setminus E(\xi)|} = 1. \tag{2.28}$$

We now introduce the probability measure $\Psi_{G,p,q,\delta}$ on $\Omega_G \times \Omega_G$ defined by

$$\Psi_{G,p,q,\delta}[\xi, \omega] := \phi_{G,p,q}[\xi] \Pi_\delta[\xi, \omega]. \tag{2.29}$$

The ξ-marginal of the measure $\Psi_{G,p,q,\delta}$ is given by the random cluster measure $\phi_{G,p,q}[\xi]$. The sprinkled random cluster measure $\phi_{G,p,q,\delta}$ is defined to be the ω-marginal of the measure $\Psi_{G,p,q,\delta}$, given by

$$\phi_{G,p,q,\delta}[\omega] := \sum_{\xi\in\Omega_G} \Psi_{G,p,q,\delta}[\xi,\omega] = \sum_{\xi\in\Omega_G} \phi_{G,p,q}[\xi]\Pi_\delta[\xi,\omega], \tag{2.30}$$

More concretely, we may construct the sprinkled random cluster measure $\phi_{G,p,q,\delta}$ by first generating a random graph under the measure $\phi_{G,p,q}$, and then opening any closed edges in the random graph independently with probability δ.

Let $p_1 < p_2$ and $q \geq 1$. We claim that for $\delta > 0$ sufficiently small, the measure $\phi_{G,p_2,q}$ stochastically dominates the measure $\phi_{G,p_1,q,\delta}$. First, we compute the probability that an edge $e \in E$ is open when conditioned on the states of the edges in $E \setminus \{e\}$ for the measure $\phi_{G,p,q,\delta}$:

Proposition 2.1.12. *Let $p, \delta \in [0,1]$ and $q \geq 1$. Then, for any $e = \{x,y\} \in E$ and $\psi \in \{0,1\}^{E\setminus\{e\}}$, we have*

$$\phi_{G,p,q,\delta}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] \leq \begin{cases} p + \delta(1-p) & \text{if } x \leftrightarrow y \text{ in } \psi \\ \frac{p}{p+q(1-p)} + \delta\left(1 - \frac{p}{p+q(1-p)}\right) & \text{otherwise} \end{cases} \tag{2.31}$$

To prove Proposition 2.1.12, we essentially condition on the states of the edges in $E \setminus \{e\}$ after the first step, *before* the sprinkling is applied. Indeed, the first inequality of (2.31) says that if $x \leftrightarrow y$ in $G(\psi)$ then we have probability at most p of the edge e being open under the measure $\phi_{G,p,q}$, and if not, then a further probability δ of being opened during the sprinkling.

Proof. Let $e = \{x,y\} \in E$ and $F = E \setminus \{e\}$. For a configuration $\omega \in \{0,1\}^F$, write $\omega^{(0)}, \omega^{(1)}$ for the configurations on $\{0,1\}^E$ agreeing with ω on F and taking values $0,1$ respectively on e. We decompose the conditional probability as

$$\phi_{G,p,q,\delta}[\omega_e = 1 \mid \omega_F = \psi] = \frac{\phi_{G,p,q,\delta}[\psi^{(1)}]}{\phi_{G,p,q,\delta}[\psi^{(0)}] + \phi_{G,p,q,\delta}[\psi^{(1)}]}. \tag{2.32}$$

We may rewrite the numerator by summing over $\xi \leq \psi \in \{0,1\}^F$ to obtain

$$\phi_{G,p,q,\delta}[\psi^{(1)}] = \sum_{\xi\in\{0,1\}^F} \left\{\phi_{G,p,q}[\xi^{(0)}]\Pi_\delta[\xi^{(0)},\psi^{(1)}] + \phi_{G,p,q}[\xi^{(1)}]\Pi_\delta[\xi^{(1)},\psi^{(1)}]\right\}. \tag{2.33}$$

Next, we decompose each of these terms as

$$\phi_{G,p,q}[\xi^{(0)}]\Pi_\delta[\xi^{(0)},\psi^{(1)}] = \phi_{G,p,q}[\omega_e = 0 \mid \omega_F = \xi]\phi_{G,p,q}[\omega_F = \xi]\Pi_\delta[\xi^{(0)},\psi^{(1)}], \quad (2.34)$$

$$\phi_{G,p,q}[\xi^{(1)}]\Pi_\delta[\xi^{(1)},\psi^{(1)}] = \phi_{G,p,q}[\omega_e = 1 \mid \omega_F = \xi]\phi_{G,p,q}[\omega_F = \xi]\Pi_\delta[\xi^{(1)},\psi^{(1)}]. \quad (2.35)$$

Write $p_{\xi,e} = \phi_{G,p,q}[\omega_e = 1 \mid \omega_F = \xi]$. As the δ sprinkling amongst edges is independent, we have the relation

$$\Pi_\delta[\xi^{(0)},\psi^{(1)}] = \delta\Pi_\delta[\xi^{(1)},\psi^{(1)}], \quad (2.36)$$

which allows us to rewrite (2.33) as

$$\phi_{G,p,q,\delta}[\psi^{(1)}] = \sum_{\xi\in\{0,1\}^F}\Big\{p_{\xi,e} + \delta(1-p_{\xi,e})\Big\}\phi_{G,p,q}[\omega_F = \xi]\Pi_\delta[\xi^{(1)},\psi^{(1)}], \quad (2.37)$$

If $x \nleftrightarrow y$ in $G(\psi)$, then $x \nleftrightarrow y$ in $G(\xi)$ for any $\xi \le \psi$. Thus $p_{\xi,e} = \frac{p}{p+q(1-p)}$ and we may extract a factor of $\frac{p}{p+q(1-p)} + \delta(1 - \frac{p}{p+q(1-p)})$ from (2.37). Otherwise, we may uniformly bound $p_{\xi,e}$ by p and extract the uniform bound $p + \delta(1-p) + p$ from (2.37). In either case, we are left with a sum S given by

$$\begin{aligned} S &= \sum_{\xi\in\{0,1\}^F}\phi_{G,p,q}[\omega_F = \xi]\Pi_\delta[\xi^{(1)},\psi^{(1)}] \\ &= \sum_{\xi\in\{0,1\}^F}\Big\{\phi_{G,p,q}[\xi^{(0)}]\Pi_\delta[\xi^{(1)},\psi^{(1)}] + \phi_{G,p,q}[\xi^{(1)}]\Pi_\delta[\xi^{(1)},\psi^{(1)}]\Big\}. \end{aligned} \quad (2.38)$$

Similarly to (2.36), we have the relation

$$\Pi_\delta[\xi^{(0)},\psi^{(0)}] = (1-\delta)\Pi_\delta[\xi^{(1)},\psi^{(1)}], \quad (2.39)$$

which may be combined with (2.36) to obtain the relation

$$\Pi_\delta[\xi^{(1)},\psi^{(1)}] = \Pi_\delta[\xi^{(0)},\psi^{(1)}] + \Pi_\delta[\xi^{(0)},\psi^{(0)}]. \quad (2.40)$$

Plugging (2.40) into the first summand of (2.38) allows us to rewrite S as

$$\begin{aligned} S = &\sum_{\xi\in\{0,1\}^F}\phi_{G,p,q}[\xi^{(0)}]\Big(\Pi_\delta[\xi^{(0)},\psi^{(1)}] + \Pi_\delta[\xi^{(0)},\psi^{(0)}]\Big) + \sum_{\xi\in\{0,1\}^F}\phi_{G,p,q}[\xi^{(1)}]\Pi_\delta[\xi^{(1)},\psi^{(1)}] \\ =&\phi_{G,p,q,\delta}[\psi^{(0)}] + \phi_{G,p,q,\delta}[\psi^{(1)}]. \end{aligned} \quad (2.41)$$

We thus obtain an upper bound on $\phi_{G,p,q,\delta}[\omega_e = 1 \mid \omega_F = \psi]$ corresponding to the bound on the quantity $p_{\xi,e} + \delta(1 - p_{\xi,e})$ extracted earlier. $\square$

We use Proposition 2.1.12 to prove the following result:

Lemma 2.1.13. *Let $p_1 \leq p_2$, $q \geq 1$, and choose $\delta \leq \frac{1}{q}(p_2 - p_1)$. Then*

$$\phi_{G,p_1,q,\delta} \leq_{st} \phi_{G,p_2,q}. \tag{2.42}$$

Proof. We apply Lemma 2.1.4 using the measures $\mu = \phi_{G,p_1,q,\delta}$ and $\nu = \phi_{G,p_2,q}$. In order to do this, we need to show that for any edge $e = \{x, y\} \in E$ and any configurations $\psi, \psi' \in \{0,1\}^{E\setminus\{e\}}$ such that $\psi \leq \psi'$, we have

$$\phi_{G,p_1,q,\delta}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] \leq \phi_{G,p_2,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi']. \tag{2.43}$$

There are two cases to consider:

1. If $x \leftrightarrow y$ in $G(\psi)$, then $x \leftrightarrow y$ in $G(\psi')$, as $\psi \leq \psi'$. Thus

$$\begin{aligned}
\phi_{G,p_1,q,\delta}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] &\leq p_1 + \delta(1 - p_1) \\
&\leq p_1 + \tfrac{1}{q}(p_2 - p_1)(1 - p_1) \\
&\leq p_2 \\
&= \phi_{G,p_2,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi'].
\end{aligned}$$

2. If $x \not\leftrightarrow y$ in $G(\psi)$, then using the inequality $\frac{p}{p+q(1-p)} \leq p$ for $q \geq 1$, we have

$$\begin{aligned}
\phi_{G,p_1,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi] &\leq \frac{p_1}{p_1 + q(1-p_1)} + \delta\left(1 - \frac{p_1}{p_1 + q(1-p_1)}\right) \\
&\leq \frac{p_1 + (p_2 - p_1)(1 - p_1)}{p_1 + q(1-p_1)} \\
&\leq \frac{p_2}{p_2 + q(1-p_2)} \\
&\leq \phi_{G,p_2,q}[\omega_e = 1 \mid \omega_{E\setminus\{e\}} = \psi'].
\end{aligned}$$

In either case, (2.43) holds and (2.42) follows by Lemma 2.1.4. $\square$

In fact, the bound $\delta \leq \frac{1}{q}(p_2 - p_1)$ is tight:

Example (Tightness of δ): Let $p_1 = 0$, $p_2 \in [0, 1]$ and fix $q \geq 1$. Let G be any graph containing a *bridge* e - that is, an edge e whose removal disconnects the graph. By

Proposition 2.1.6, we know that

$$\phi_{G,p_2,q}[\omega_e = 1] = \frac{p_2}{p_2 + q(1-p_2)}. \tag{2.44}$$

Fix $\epsilon > 0$ and suppose $\delta = \frac{p_2}{q} + \epsilon$. Then

$$\phi_{G,p_1,q,\delta}[\omega_e = 1] = \frac{p_2}{q} + \epsilon \tag{2.45}$$

and this exceeds (2.44) provided one chooses $p_2 < q\sqrt{\frac{\epsilon}{q-1}}$.

Lemma 2.1.13 allows us to mimic the arguments used by Ajtai, Komlós, and Szemerédi in [1] to analyse increasing functions and events by generating an initial configuration according to the random cluster model, before randomising remaining closed edges independently with probability δ. In order to achieve an independent sprinkling, we must take an additional factor of $1/q$ in the sprinkling constant δ compared to percolation. This additional factor compensates for the weight of q lost when an edge connects two disjoint connected components.

2.2 The Potts model

In this section, we provide a formal introduction to the Potts model, based upon the formulation of the model given by Kesten and Schonmann in [30]. Throughout, we write $\mathbb{N}_{\geq 2}$ for the set of integers greater than or equal to 2.

Let $G = (V, E)$ be a finite graph, fix $q \in \mathbb{N}_{\geq 2}$ and define $\Sigma_G = \{v_1, \cdots, v_q\}^V$, where $v_1, \cdots, v_q$ are the co-ordinate vectors of $\mathbb{R}^q$. An element $\sigma \in \Sigma_G$ is known as a *spin configuration*, and corresponds to an assignment of a spin $\sigma_x \in \{v_1, \cdots, v_q\}$ to each vertex x of the graph G. By applying a probability measure to Σ_G, we may randomly assign a spin configuration to G. We will be interested in the following measure:

Definition 2.2.1 (Potts Model)**.** *For $x \in V$, let λ_x denote the counting measure on $\{v_1, \cdots, v_q\}$ and define the product measure $\nu_G := \prod_{x \in V} \lambda_x$. The q-state Potts model with inverse temperature $\beta > 0$ assigns a spin configuration to the vertices of G according to the measure*

$$\mu_{G,\beta,q}[\sigma] := \frac{1}{Z^P_{G,\beta,q}} \exp\{-\beta H_G(\sigma)\}\nu_G(\sigma), \tag{2.46}$$

where the Hamiltonian $H_G(\sigma)$ *is given by*

$$H_G(\sigma) := -\sum_{\{x,y\}\in E} \sigma_x \cdot \sigma_y \tag{2.47}$$

and the scalar product is the usual Euclidean product on $\mathbb{R}^q$*. The partition function* $Z^P_{G,\beta,q}$ *ensures that these probabilities are appropriately normalised, and is given by*

$$Z^P_{G,\beta,q} = \int \nu_G(d\sigma)\exp\{-\beta H_G(\sigma)\} = \sum_{\sigma\in\Sigma_G} \exp\{-\beta H_G(\sigma)\}. \tag{2.48}$$

We write $\langle f\rangle_{G,\beta,q}$ *for the expectation of a function* $f : V \to \mathbb{R}$ *with respect to this measure.*

Let $E_\sigma = \{\{x,y\} \in E : \sigma_x \neq \sigma_y\}$. Then we may write the Hamiltonian as

$$H_G(\sigma) = |E_\sigma| - |E|. \tag{2.49}$$

It is common to define the Potts model by assigning each vertex x a numerical spin $\sigma_x \in \{1, \cdots, q\}$, rather than a vector. We adopt a vector representation so that we may leverage certain vector arguments. One may also define a vector representation using the vertices of a $(q-1)$-dimensional tetrahedron. This was done by Duminil-Copin in [16], and is equivalent up to rescaling, but will be less convenient than the co-ordinate vector representation for our purposes.

2.2.1 The Edwards Sokal coupling

In this subsection, we couple the Potts model with an appropriate random cluster model by constructing them on the same probability space. This will be done using the famous Edwards Sokal coupling, introduced in [17].

To begin, we fix $q \in \mathbb{N}_{\geq 2}$ and define $\Xi_G = \{0,1\}^E \times \{v_1, \cdots, v_q\}^V = \Omega_G \times \Sigma_G$. Each configuration $\xi \in \Xi_G$ may be decomposed into a pair (ω, σ), where $\omega \in \Omega_G$ is a percolation configuration and $\sigma \in \Sigma_G$ is a spin configuration. We say that the pair (ω, σ) is *compatible* if $\omega_e = 0$ for every $e \in E_\sigma$.

Theorem 2.2.2 (Edwards Sokal coupling [17])**.** *Define a measure* $\Phi_{G,p,q}$ *on the set* Ξ_G *by*

$$\Phi_{G,p,q}[(\omega,\sigma)] = \frac{1}{Z^{RC}_{G,p,q}}\left\{\prod_{e\in E} p^{\omega_e}(1-p)^{1-\omega_e}\right\}\mathbb{1}_{\{(\omega,\sigma)\ \textit{are compatible}\}}. \tag{2.50}$$

Then $\Phi_{G,p,q}$ *is a probability measure with marginal distributions* $\phi_{G,p,q}$ *on* ω *and* $\mu_{G,\beta(p),q}$ *on* σ*, where* $\beta(p) := -\log(1-p)$*.*

All logarithms in this thesis should be assumed to be natural unless specified otherwise. We may sample a pair (ω, σ) according to the measure $\Phi_{G,p,q}$ in one of the two following ways:

1. Sample ω according to the measure $\phi_{G,p,q}$, and then assign a spin from the set $\{v_1, \cdots, v_q\}$ uniformly and independently to each connected component of $G(\omega)$.
2. Sample σ according to the measure $\mu_{G,\beta(p),q}$, and then open any edge in $E \setminus E_\sigma$ independently with probability p.

These two alternative methods of sampling underpin the following proof of Theorem 2.2.2, which is taken from [16, Proposition 1.2]:

Proof of Theorem 2.2.2. First, we compute the marginal of $\Phi_{G,p,q}$ on ω. Given a percolation configuration ω, a spin configuration σ is compatible with ω if and only if σ is constant on each connected component of $G(\omega)$. Moreover, every compatible spin configuration is equally probable. As there are $k(\omega)$ connected components with q choices of spin for each component, it follows that

$$\Phi_{G,p,q}[\omega] = \frac{1}{Z^{\mathrm{RC}}_{G,p,q}} \left\{ \prod_{e \in E} p^{\omega_e}(1-p)^{1-\omega_e} \right\} q^{k(\omega)}. \tag{2.51}$$

Next, we compute the marginal of $\Phi_{G,p,q}$ on σ. Given a spin configuration σ, recall the definition of the set $E_\sigma = \{\{x, y\} \in E : \sigma_x \neq \sigma_y\}$ and observe that a percolation configuration ω is compatible with σ if and only if $\omega_e = 0$ for every $e \in E_\sigma$. Thus

$$\Phi_{G,p,q}(\sigma) = \frac{(1-p)^{|E_\sigma|}}{Z^{\mathrm{RC}}_{G,p,q}} \sum_{\omega' \in \{0,1\}^{E \setminus E_\sigma}} \left\{ \prod_{e \in E \setminus E_\sigma} p^{\omega'_e}(1-p)^{1-\omega'_e} \right\}. \tag{2.52}$$

As the sum in (2.52) evaluates to 1, we may substitute $e^{-\beta(p)} = 1 - p$ to obtain

$$\begin{aligned} \Phi_{G,p,q}(\sigma) &= \frac{1}{Z^{\mathrm{RC}}_{G,p,q}} \exp\{-\beta(p)|E_\sigma|\} \\ &= \frac{e^{-\beta(p)|E|}}{Z^{\mathrm{RC}}_{G,p,q}} \exp\{-\beta(p) H_G(\sigma)\} \end{aligned} \tag{2.53}$$

Where we recall that $H_G(\sigma) = |E_\sigma| - |E|$ to obtain the final equality.

□

We will use the following two corollaries of Theorem 2.2.2:

Corollary 2.2.3. *Fix $q \in \mathbb{N}_{\geq 2}$ and $p \in [0,1]$. Then*

$$Z^{RC}_{G,p,q} = e^{-\beta(p)|E|} Z^{P}_{G,\beta(p),q}. \tag{2.54}$$

Proof. (2.54) can be seen in the computation of the marginal on σ in the proof of Theorem 2.2.2. □

Corollary 2.2.4. *Fix $q \in \mathbb{N}_{\geq 2}$ and $p \in [0,1]$. Then, for any pair $x, y \in V$:*

$$\langle \sigma_x \cdot \sigma_y \rangle_{G,\beta(p),q} = \frac{1}{q} + \frac{q-1}{q} \phi_{G,p,q}[x \leftrightarrow y]. \tag{2.55}$$

Proof. Observe that

$$\begin{aligned} \langle \sigma_x \cdot \sigma_y \rangle_{G,\beta(p),q} &= \Phi_{G,p,q}[\sigma_x \cdot \sigma_y] \\ &= \Phi_{G,p,q}[\sigma_x \cdot \sigma_y \mathbb{1}_{\{x \leftrightarrow y\}}] + \Phi_{G,p,q}[\sigma_x \cdot \sigma_y \mathbb{1}_{\{x \not\leftrightarrow y\}}]. \end{aligned} \tag{2.56}$$

On the event $\{x \leftrightarrow y\}$, σ_x and σ_y are equal. Otherwise, they are independent, and so their product has expectation $1/q$. Thus

$$\begin{aligned} \langle \sigma_x \cdot \sigma_y \rangle_{G,\beta(p),q} &= \Phi_{G,p,q}[\mathbb{1}_{\{x \leftrightarrow y\}}] + \frac{1}{q} \Phi_{G,p,q}[\mathbb{1}_{\{x \not\leftrightarrow y\}}] \\ &= \phi_{G,p,q}[x \leftrightarrow y] + \frac{1}{q}(1 - \phi_{G,p,q}[x \leftrightarrow y]), \end{aligned} \tag{2.57}$$

which re-arranges to give (2.55). □

The Edwards Sokal coupling has many uses. For example, one may apply Proposition 2.1.7 to Corollary 2.2.4 to deduce that spin correlations are increasing with respect to β. This is difficult to prove directly, as the Potts model lacks a natural ordering.

2.2.2 Free energy, convexity and differentiability

One often analyses the Potts model through a property known as its *free energy.* In this subsection, we define the free energy of the Potts model and introduce some of its basic properties.

Definition 2.2.5 (Free energy)**.** *Let G be a finite graph, $q \in \mathbb{N}_{\geq 2}$, and $\beta > 0$. Then, the free energy of the q-state Potts model with inverse temperature β is given by*

$$\psi_{G,q}(\beta) = \frac{1}{|V|} \log Z^{P}_{G,\beta,q}. \tag{2.58}$$

We view the free energy as a function of β, and use it to extract information about how the properties of spin configurations change as the temperature of the system varies. This will be particularly useful provided the underlying graph is edge-transitive:

Definition 2.2.6 (Edge transitive graphs)**.** *Let $G = (V, E)$ be a finite graph and $\psi : V \to V$ be a bijection. Then:*

1. *An automorphism of G is a bijection $\psi : V \to V$ of the vertices which preserves edges i.e. if $\{x, y\} \in E$ then $\{\psi(x), \psi(y)\} \in E$.*
2. *We say that G is edge-transitive if for every pair of edges $e_1, e_2 \in E$ there exists an automorphism ψ of G for which $\psi(e_1) = e_2$.*

We say a graph is n-regular if every vertex of the graph has precisely n neighbours. Consider the following calculation:

Proposition 2.2.7 (Derivative of free energy)**.** *Let $G = (V, E)$ be a finite, n-regular, edge-transitive graph, $q \in \mathbb{N}_{\geq 2}$, and $\beta > 0$. Then, for any pair $x, y \in V$ such that $\{x, y\} \in E$, we have*

$$\frac{\partial}{\partial \beta} \psi_{G,q}(\beta) = \frac{n}{2} \langle \sigma_x \cdot \sigma_y \rangle_{G,\beta,q}. \tag{2.59}$$

Proof. Observe that

$$\begin{aligned}
\frac{\partial}{\partial \beta} \psi_{G,q}(\beta) &= \frac{\partial}{\partial \beta} \frac{1}{|V|} \log Z^{\mathrm{P}}_{G,\beta,q} \\
&= \frac{1}{|V|} \sum_{\sigma \in \Sigma} \sum_{\{x,y\} \in E} \sigma_x \cdot \sigma_y \frac{e^{-\beta H_G(\sigma)}}{Z^{\mathrm{P}}_{G,\beta,q}} \\
&= \frac{1}{|V|} \sum_{\{x,y\} \in E} \langle \sigma_x \cdot \sigma_y \rangle_{G,\beta,q},
\end{aligned} \tag{2.60}$$

where we have exchanged the order of summation in the final line. For an edge-transitive graph, the spin correlation $\langle \sigma_x \cdot \sigma_y \rangle_{G,\beta,q}$ does not depend on the choice of the pair x, y. If G is n-regular, then $|E| = \frac{n}{2}|V|$ (as each vertex has n edges, and each edge has two vertex endpoints) and so for any fixed $x, y \in V$ such that $\{x, y\} \in E$, we have

$$\frac{\partial}{\partial \beta}\psi_{G,q}(\beta) = \frac{|E|}{|V|}\langle \sigma_x \cdot \sigma_y \rangle_{G,\beta,q} = \frac{n}{2}\langle \sigma_x \cdot \sigma_y \rangle_{G,\beta,q}. \tag{2.61}$$

□

We may use Proposition 2.2.7 to extract information about the spin correlations from the free energy, which can then be translated into information regarding connection probabilities in the random cluster model via Corollary 2.2.4.

Another particularly useful property of the free energy is its convexity:

Lemma 2.2.8 ([23, Lemma 3.5])**.** *The function $\beta \to \psi_{G,q}(\beta)$ is convex. That is, for any $\beta_1, \beta_2 \geq 0$ and $\alpha \in (0, 1)$,*

$$\psi_{G,q}(\alpha\beta_1 + (1-\alpha)\beta_2) \leq \alpha\psi_{G,q}(\beta_1) + (1-\alpha)\psi_{G,q}(\beta_2). \tag{2.62}$$

Proof. By applying Hölder's inequality with exponents α, $1-\alpha$, we see that

$$\begin{aligned} Z^{\mathrm{P}}_{G,\alpha\beta_1+(1-\alpha)\beta_2,q} &= \sum_{\sigma\in\Sigma_G} \exp\left\{ -\alpha\beta_1 H_G(\sigma) - (1-\alpha)\beta_2 H_G(\sigma)\right\} \\ &\leq \left(\sum_{\sigma\in\Sigma_G} \exp\{-\beta_1 H_G(\sigma)\}\right)^{\alpha}\left(\sum_{\sigma\in\Sigma_G} \exp\{-\beta_2 H_G(\sigma)\}\right)^{1-\alpha}. \end{aligned} \tag{2.63}$$

Convexity follows upon taking the logarithm. □

In light of Lemma 2.2.8, it will be useful to recall the following standard result from elementary analysis:

Lemma 2.2.9 (Convergence of convex derivatives [35])**.** *Let $(f_n)_{n\in\mathbb{N}} : \mathbb{R} \to \mathbb{R}$ be a sequence of convex differentiable functions, and suppose* $\lim_{n\to\infty} f_n(x) = f(x)$ *for every $x \in \mathbb{R}$ and some function $f : \mathbb{R} \to \mathbb{R}$. If f is differentiable at x, then*

$$\lim_{n\to\infty} f_n'(x) = f'(x). \tag{2.64}$$

Note that the free energy is clearly differentiable on any finite graph. Moreover, it can be shown that the free energy converges to a differentiable limit on a variety of families of graphs when the inverse temperature is appropriately rescaled. In light of Proposition 2.2.7, Lemma 2.2.9 implies that the spin correlations will converge to the derivative of the limit of the free energy on suitably well behaved graphs.

Chapter 3

The random cluster model on finite graphs

In this chapter, we discuss the random cluster model on two families of finite graphs of diverging degree. In both cases, we see that the largest component undergoes an asymptotic phase transition in the limit as $n \to \infty$ provided the vertex degree is appropriately rescaled.

In the case of the complete graph K_n, the random cluster model was first studied by Bollobás, Grimmett, and Janson in [6] using a colouring argument which does not readily extend to graphs with more complicated geometry. Section 3.1 provides a brief overview of these results, and serves to introduce and motivate some important quantities which recur throughout the rest of the thesis.

In Section 3.2, we detail a new analysis of the random cluster model on the complete graph which extends arguments developed by Biskup, Chayes and Smith in [3] for percolation. In particular, we will obtain the large deviations rate function for the size of the largest component in terms of the exponential rates of the events that the random graph is connected and acyclic, respectively. Crucially, this analysis does not use the colouring argument developed in [6], and thus admits the prospect of generalisation to more complicated families of graphs.

In Section 3.3, we discuss the random cluster model on the hypercube. In particular, we analyse the largest component using the tools developed in Section 2.1.2 in order to extend the arguments employed in [1] to study percolation. As a substitute for the exploration process used in [1], we will also investigate the free energy of a corresponding Potts model, using methods developed by Kesten and Schonmann in [30] to study the Potts model on the lattice $\mathbb{Z}^d$.

3.1 Random cluster model on the complete graph via colouring arguments

In this section, we discuss the analysis of the random cluster model on the complete graph given by Bollobás, Grimmett, and Janson in [6]. This will introduce several quantities which reappear in our analyses of the random cluster model on both the complete graph and the hypercube, given in Sections 3.1 and 3.2 respectively.

We begin with the following definition:

Definition 3.1.1. *Let $(\Omega_n, \mathcal{F}_n, P_n)_{n\in\mathbb{N}}$ be a sequence of probability spaces and $(A_n)_{n\in\mathbb{N}}$ be a sequence of events with $A_n \in \mathcal{F}_n$ for every $n \in \mathbb{N}$. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ happens asymptotically almost surely (written a.a.s.) if*

$$\lim_{n\to\infty} P_n[A_n] = 1. \tag{3.1}$$

More concretely, let $(G_n)_{n\in\mathbb{N}}$ be a sequence of finite graphs, define $\Omega_n = \{0,1\}^{E(G_n)}$ and let $\mathcal{F}_n$ be the set of all subsets of Ω_n. For fixed $\lambda > 0$ and $q > 0$, let $\phi_{n,\lambda,q}$ denote the random cluster probability measure on G_n with edge weight $p = \lambda/n$ and cluster weight q. We will take $P_n = \phi_{n,\lambda,q}$ and investigate the size of the largest component of $G_n(\omega)$ in the limit as $n \to \infty$.

When $G_n = K_n$, this was first studied in [6] by randomly colouring each connected component of $K_n(\omega)$ red independently with probability $1/q$, so that the distribution of edges on the *red vertices* is given by a percolation measure (with the same edge weight). As the subgraph of red vertices is necessarily complete, it may be studied using the same exploration processes used in [19]. Consequently, one may find a critical value λ_c (depending only on q) such that the following statements hold:

1. For $\lambda < \lambda_c(q)$, the largest component of $K_n(\omega)$ is of order $\log n$ a.a.s..

2. For $\lambda > \lambda_c(q)$, the largest component of $K_n(\omega)$ is of order n a.a.s..

The value λ_c is given by

$$\lambda_c(q) = \begin{cases} q & \text{if } q \le 2 \\ 2\left(\frac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2 \end{cases} \tag{3.2}$$

Furthermore, the density of the largest connected component of $K_n(\omega)$ is asymptotically almost surely equal to

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_c(q) \\ \theta_{\max} & \text{if } \lambda \geq \lambda_c(q) \end{cases} \tag{3.3}$$

where $\theta_{\max}$ is the largest solution of the *mean field equation*

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}. \tag{3.4}$$

When $q = 1$, we recover the results of [19]. In particular, we see that $\lambda_c(1) = 1$, and that the mean-field equation reduces to the equation

$$e^{-\lambda\theta} = 1 - \theta \tag{3.5}$$

governing the survival probability of a Poisson branching process with mean λ. Heuristically, (3.4) may be obtained from (3.7) by conditioning on the event that the largest component of $K_n(\omega)$ is coloured red in the aforementioned colouring process. Indeed, suppose that the largest component of $K_n(\omega)$ has density θ. On the event that the largest component is coloured red, the red subgraph has $[\theta + \frac{1}{q}(1-\theta)]n$ vertices on average, and so the edges on the red subgraph are distributed according to a $\phi_{n',\lambda'}$ percolation measure, where

$$n' = [\theta + \tfrac{1}{q}(1-\theta)]n, \ \lambda' = [\theta + \tfrac{1}{q}(1-\theta)]\lambda. \tag{3.6}$$

In particular, the density $\theta' = \theta/[\theta + \frac{1}{q}(1-\theta)]$ of the largest component *in the red subgraph* satisfies the equation

$$e^{-\lambda'\theta'} = 1 - \theta', \tag{3.7}$$

which yields (3.4) after substitution. This argument is formalised in [6, Lemma 4.2].

It remains to analyse the solutions of (3.4). To do this, define the function

$$f(\theta) = \frac{1}{\theta}[\log(1+(q-1)\theta) - \log(1-\theta)]. \tag{3.8}$$

and observe that $\theta \in [0,1]$ is a solution to (3.4) if and only if $f(\theta) = \lambda$. In order to understand this latter equation, we use the following basic properties of the function f:

Lemma 3.1.2 ([6, Lemma 2.4]). *The function f defined in (3.8) is strictly convex on $(0,1)$, with $\lim_{\theta\downarrow 0} f(\theta) = q$ and $\lim_{\theta\uparrow 1} f(\theta) = \infty$. Moreover:*

1. *If $0 < q \leq 2$, then f is strictly increasing.*

2. *If $q > 2$, then there exists $\theta_{min} \in (0,1)$ such that f is strictly decreasing on $(0, \theta_{min})$*

and strictly increasing on $(\theta_{min}, 1)$.

Proof. We follow the proof of [6, Lemma 2.4], beginning by writing f in the form

$$f(\theta) = \int_{-1}^{q-1} (1 + t\theta)^{-1} dt. \tag{3.9}$$

For $t > -1$ and $\theta \in (0, 1)$, the integrand of (3.9) is a strictly convex function of θ, and thus f is convex. Next, we apply a Taylor expansion of f about $\theta = 0$ to obtain

$$f(\theta) = q + \frac{q(2-q)}{2}\theta + O(\theta^2). \tag{3.10}$$

In particular, it follows that $\lim_{\theta \downarrow 0} f'(\theta) = \frac{q(2-q)}{2}$. As this is positive for $q \leq 2$ and negative for $q > 2$, the statements of the lemma follow. □

Using Lemma 3.1.2, the solutions of (3.4) may be summarised as follows:

Lemma 3.1.3 ([6, Lemma 2.5])**.** *In addition to the root* $\theta = 0$, *(3.4) has the following roots:*

1. *Suppose* $0 < q \leq 2$. *Then:*
   (a) *If* $\lambda \leq \lambda_c(q) = q$, *there are no non-zero roots.*
   (b) *If* $\lambda > q$, *there is a unique positive root* $\theta_{max}(\lambda, q)$. *In addition,* $\lim_{\lambda \downarrow q} \theta_{max}(\lambda, q) = 0$.
2. *Suppose* $q > 2$, *and write* $\lambda_{min} = f(\theta_{min})$. *Then:*
   (a) *If* $\lambda < \lambda_{min}$, *there are no non-zero roots.*
   (b) *If* $\lambda = \lambda_{min}$, *there is a unique positive root* θ_{min}.
   (c) *If* $\lambda_{min} < \lambda < q$, *there are two positive roots* $\theta_1(\lambda, q)$ *and* $\theta_{max}(\lambda, q)$.
   (d) *If* $\lambda > q$, *there is a unique positive root* $\theta_{max}(\lambda, q)$.

It remains to show that $\lambda_c > \lambda_{\min}$ when $q > 2$, so that $\theta(\lambda, q) > 0$ for $\lambda > \lambda_c$. As f is convex, it will suffice to check that $f(\frac{q-2}{q-1}) = \lambda_c(q)$ and that the derivative $f'(\frac{q-2}{q-1})$ is strictly positive. Indeed, one may even show that

$$\theta(\lambda_c(q), q) = \begin{cases} 0 & \text{if } q \leq 2 \\ \frac{q-2}{q-1} & \text{if } q > 2 \end{cases} \tag{3.11}$$

When $q \leq 2$, (3.11) follows from the fact that f is strictly increasing and $f(0) = q$. In particular, (3.11) implies that the (asymptotic) phase transition for the size of the largest component is continuous if and only if $q \leq 2$.

For integer values of q, the asymptotic phase transition established in [6] may be interpreted in terms of the Potts model by expressing the probability $\phi_{n,\lambda,q}[x \leftrightarrow y]$ (for any pair $\{x, y\} \in E$) in terms of the size of the largest component $\mathcal{C}_{\max}$. In particular, we may write

$$\phi_{n,\lambda,q}[x \leftrightarrow y] = \phi_{n,\lambda,q}[\{x \leftrightarrow y\} \cap \{x \in \mathcal{C}_{\max}\}] + \phi_{n,\lambda,q}[\{x \leftrightarrow y\} \cap \{x \notin \mathcal{C}_{\max}\}]. \quad (3.12)$$

We know that $|\mathcal{C}_{\max}|/n$ converges to $\theta(\lambda, q)$ a.a.s. as $n \to \infty$. As K_n is edge-transitive, it follows that

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[\{x \leftrightarrow y\} \cap \{x \in \mathcal{C}_{\max}\}] = \theta(\lambda, q)^2. \quad (3.13)$$

On the other hand, it is known (see e.g. [6, Lemma 3.2]) that the second largest component has order at most $n^{3/4}$, leading to the result

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[\{x \leftrightarrow y\} \cap \{x \notin \mathcal{C}_{\max}\}] = 0. \quad (3.14)$$

Combining (3.13) and (3.14) yields

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] = \theta(\lambda, q)^2. \quad (3.15)$$

Observing that $\beta(\frac{\lambda}{n}) = -\log(1 - \frac{\lambda}{n}) = \frac{\lambda}{n} + O(n^{-2})$, the Edwards Sokal coupling may be applied via Corollary 2.2.4 to 3.15 to see that

$$\lim_{n\to\infty} \langle \sigma_x \cdot \sigma_y \rangle_{K_n,\lambda/n,q} = \frac{1}{q} + \frac{q-1}{q}\theta(\lambda, q)^2. \quad (3.16)$$

In fact, (3.16) is a classical result, and has been proven independently by e.g. Kesten and Schonmann in [30]. By reversing the above argument, it is possible to deduce the limit (3.15) of the connection probabilities for the random cluster model from the limit (3.16) of the spin correlations for the corresponding Potts model. This connection is the premise of our results for the hypercube Q_n in Section 3.3, which goes one step further and deduces an asymptotic phase transition for the largest component from the asymptotic phase transition for the nearest neighbour connection probabilities.

3.2 Random cluster model on the complete graph via large deviations

Fix $q > 0$, $\lambda > 0$, and let $\phi_{n,\lambda,q}$ denote the random cluster probability measure with edge weight $p = \lambda/n$ and cluster weight q on the complete graph K_n. In addition, let $\mathcal{V}_{>r}$ be the set of vertices in $K_n(\omega)$ belonging to components of size larger than r. The objective of this section is to calculate a function $I(\theta, \lambda, q)$ such that

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\log\phi_{n,\lambda,q}\big[|\mathcal{V}_{>\epsilon n}| = \lfloor\theta n\rfloor\big] = -I(\theta,\lambda,q). \tag{3.17}$$

In the language of large deviations, (3.17) says that in the limit $\epsilon \downarrow 0$, the sequence of random variables $|\mathcal{V}_{>\epsilon n}|/n$ satisfies the large deviations principle in $[0, 1]$ with rate function $I(\theta, \lambda, q)$. If the function $I(\theta, \lambda, q)$ is minimised at a point θ^* with respect to θ, then (3.17) implies that for $\theta \neq \theta^*$, the probability of the large deviation $|\mathcal{V}_{>\epsilon n}| = \lfloor\theta n\rfloor$ is exponentially small, and the random variable $|\mathcal{V}_{>\epsilon n}|/n$ concentrates around θ^* in the limit as $n \to \infty$.

In order to state our main result, we must introduce some notation. Firstly, recall the *entropy function*, defined for $\theta \in (0, 1)$ by

$$S(\theta) = -\theta\log\theta - (1-\theta)\log(1-\theta). \tag{3.18}$$

In addition, we define two functions on $[0,\infty)$ by

$$\pi_1(x) = 1 - e^{-x},\ \ \Psi(x) = \left(\log x - \frac{1}{2}\left[x - \frac{1}{x}\right]\right)\wedge 0. \tag{3.19}$$

Using (3.18) and (3.19), we may define the function

$$\begin{aligned}\Phi(\theta,\lambda,q) = S(\theta) - \lambda\theta(1-\theta) + \theta\log\pi_1(\lambda\theta)\\ + (1-\theta)\{\Psi(\tfrac{\lambda(1-\theta)}{q}) - (\tfrac{q-1}{2q})\lambda(1-\theta) + \log q\}.\end{aligned} \tag{3.20}$$

Our main result says that the large deviation principle (3.17) holds, with rate function

$$I(\theta,\lambda,q) := \sup_{\theta\in[0,1]}\Phi(\theta,\lambda,q) - \Phi(\theta,\lambda,q). \tag{3.21}$$

Theorem 3.2.1. *Fix $q > 0$ and $\lambda > 0$. Then, for every $\theta \in [0, 1]$,*

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\log\phi_{n,\lambda,q}\big[|\mathcal{V}_{>\epsilon n}| = \lfloor\theta n\rfloor\big] = \Phi(\theta,\lambda,q) - \sup_{\theta\in[0,1]}\Phi(\theta,\lambda,q). \tag{3.22}$$

If $\theta^* \in [0,1]$ maximises $\Phi(\theta,\lambda,q)$, then Theorem 3.2.1 implies that approximately $\theta^* n$ vertices of $K_n(\omega)$ belong to components of order n. Let $\mathcal{N}_{>r}$ be the number of connected components in $K_n(\omega)$ of size larger than r. The following lemma says that, in fact, all of the vertices in the set $|\mathcal{V}_{>\epsilon n}|$ belong to a *single* component of size $\theta^* n$:

Lemma 3.2.2. *Fix $q > 0$ and $\lambda > 0$. Then, for every $\epsilon > 0$ there exists a constant $c = c(\lambda, \epsilon) > 0$ such that for every $\theta > \epsilon > 0$,*

$$\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor,\ \mathcal{N}_{>\epsilon n} = 1] \geq (1 - e^{-cn})\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor]. \tag{3.23}$$

It remains to specify the maximiser θ^*. In Chapter 5, we will see that the value θ^* maximising $\Phi(\theta,\lambda,q)$ is equal to the value $\theta(\lambda,q)$ defined in (3.3). Consequently, Theorem 3.2.1 implies that the largest component of the graph $K_n(\omega)$ is of order n asymptotically almost surely when $\lambda > \lambda_c$. In this way, we recover the asymptotic phase transition for the size of the largest connected component established in [6]. The behaviour for $\lambda = \lambda_c$ is more complicated, and will not be discussed here.

The existence of the rate function $I(\theta,\lambda,q)$ was first established for percolation (the special case $q = 1$) in [3] by conditioning on the set of vertices A contained in the largest component. To this end, define the events K that $K_n(\omega)$ is connected and B_r that $K_n(\omega)$ contains no components of size larger than r. Then, assuming θn is an integer for simplicity, we have the relation

$$\phi_{n,\lambda}[|\mathcal{V}_{>\epsilon n}| = \theta n,\ \mathcal{N}_{>\epsilon n} = 1] = \binom{n}{\theta n}\left(1 - \tfrac{\lambda}{n}\right)^{\theta(1-\theta)n^2} \phi_{\theta n,\lambda\theta}[K]\phi_{(1-\theta)n,\lambda(1-\theta)}[B_{\epsilon n}]. \tag{3.24}$$

Indeed, the first term of (3.24) is precisely the number of choices for the set A, and the second term is the probability that A is disconnected from A^c. Conditionally on this event, the measure $\phi_{n,\lambda}$ restricts to two independent percolation measures $\phi_{\theta n,\lambda\theta}$ and $\phi_{(1-\theta)n,\lambda(1-\theta)}$ on the sets A and A^c respectively. In particular, the term $\phi_{\theta n,\lambda\theta}[K]$ corresponds to the event that A is connected, and the term $\phi_{(1-\theta)n,\lambda(1-\theta)}[B_{\epsilon n}]$ corresponds to the event that A^c does not contain any large components.

For more general values of q, we encounter an additional complexity when we calculate the probability that the sets A and A^c are disconnected:

Proposition 3.2.3. *Fix $q > 0$ and $\lambda > 0$. Let $A \subset [n]$, and suppose that $|A| = k$. Let $E(A, A^c)$ be the set of open edges between A and A^c in $K_n(\omega)$. Then*

$$\phi_{n,\lambda,q}[E(A,A^c) = \emptyset] = \frac{Z^{RC}_{k,\lambda k/n,q} Z^{RC}_{n-k,\lambda(1-k/n),q}}{Z^{RC}_{n,\lambda,q}}(1 - \lambda/n)^{k(n-k)}. \tag{3.25}$$

Proof. Given a set of edges $F \subset E$, write ω_F for the restriction of ω to F. Observe that $\omega \in \{E(A, A^c) = \emptyset\}$ if and only if $\omega_{E(A,A^c)} = 0$, in which case we may decompose ω into the pair $(\omega_{E(A)}, \omega_{E(A^c)})$. Noting further that $k(\omega) = k(\omega_{E(A)}) + k(\omega_{E(A^c)})$, we have

$$\begin{aligned}\phi_{n,\lambda,q}[E(A, A^c) = \emptyset] = &\sum_{\omega_{E(A)}} q^{k(\omega_{E(A)})} \prod_{e\in E(A)} (\tfrac{\lambda}{n})^{(\omega_{E(A)})_e}(1-\tfrac{\lambda}{n})^{1-(\omega_{E(A)})_e} \\ &\times \sum_{\omega_{E(A^c)}} q^{k(\omega_{A^c})} \prod_{e\in E(A^c)} (\tfrac{\lambda}{n})^{(\omega_{E(A^c)})_e}(1-\tfrac{\lambda}{n})^{1-(\omega_{E(A^c)})_e} \\ &\times \frac{(1-\frac{\lambda}{n})^{k(n-k)}}{Z^{\mathrm{RC}}_{n,\lambda,q}}.\end{aligned}$$

We recover (3.25) by observing that the sums on the first two lines yield the partition functions $Z^{\mathrm{RC}}_{k,\lambda k/n,q}$ and $Z^{\mathrm{RC}}_{n-k,\lambda(1-k/n),q}$ respectively. □

In order to accommodate the additional ratio of partition functions introduced in (3.25), we introduce the notation $Z^{\mathrm{RC}}_{n,\lambda,q}[\cdot] := Z^{\mathrm{RC}}_{n,\lambda,q}\phi_{n,\lambda,q}[\cdot]$ for the random cluster measure *before* normalisation. We may then write

$$Z_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \theta n,\ \mathcal{N}_{>\epsilon n} = 1] = \binom{n}{\theta n}(1-\tfrac{\lambda}{n})^{\theta(1-\theta)n^2} Z_{\theta n,\lambda\theta,q}[K] Z_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}]. \tag{3.26}$$

We now estimate each of these factors. To estimate the first, we apply Stirling's Formula

$$\sqrt{2\pi n}n^n e^{-n} \le n! \le \sqrt{e^2 n}n^n e^{-n} \tag{3.27}$$

to each of the factorials in the binomial coefficient $\binom{n}{k}$ to show that

$$\binom{n}{k} = e^{o(n)} e^{nS\left(\frac{k}{n}\right)}. \tag{3.28}$$

The second factor of (3.26) may be estimated as

$$(1-\tfrac{\lambda}{n})^{\theta(1-\theta)n^2} = e^{o(n)}e^{-\lambda\theta(1-\theta)n}. \tag{3.29}$$

To estimate the remaining two terms of (3.26), we have the following three theorems, which generalise [3, Theorems 2.3, 2.4 and 2.5]:

Theorem 3.2.4. *Fix $q > 0$ and $\lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}[K] = \log \pi_1(\lambda). \tag{3.30}$$

Moreover, convergence is uniform for λ *belonging to compact subsets of* $[0,\infty)$.

Observe that the quantity $\log \pi_1(\lambda)$ computed in (3.30) is independent of q. This is no coincidence, as on the event K the weight $q^{k(\omega)}$ is constant and disappears when taking the appropriate limit.

As in [3], one may prove that the exponential rate of the event $B_{\epsilon n}$ coincides with the exponential rate F of the event that $K_n(\omega)$ is a forest (i.e. acyclic) in the limits as $n \to \infty$ and $\epsilon \downarrow 0$. This is not surprising, as in [6] it was shown that almost all vertices outside of the largest component belong to trees. This argument is summarised in the following analogue of [3, Theorem 2.5]:

Theorem 3.2.5. *Fix* $q > 0$ *and* $\lambda > 0$. *Then*

$$\lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[B_r\big] = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[B_r\big] = \lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[F\big] \tag{3.31}$$

and

$$\lim_{\epsilon\downarrow 0} \liminf_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[B_{\epsilon n}\big] = \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[B_{\epsilon n}\big] = \lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[F\big]. \tag{3.32}$$

Moreover, convergence is uniform for λ *belonging to compact subsets of* $(0,\infty)\setminus\{q\}$.

On the event F, we have the correspondence $k(\omega) = n - |E_n(\omega)|$ between the number of components and edges of the graph $K_n(\omega)$. In particular, it is possible to absorb the cluster weight q into the edge weight and extend [3, Theorem 2.4] in the following form:

Theorem 3.2.6. *Fix* $q > 0$ *and* $\lambda > 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[F\big] = \lim_{r\to\infty} \lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[B_r \cap F\big] = \Psi(\tfrac{\lambda}{q}) - (\tfrac{q-1}{2q})\lambda + \log q. \tag{3.33}$$

Moreover, convergence is uniform for λ *belonging to compact subsets of* $(0,\infty)\setminus\{q\}$.

By combining the preceding three theorems, we may compute the rate function for the size of the largest connected component in the following form:

Theorem 3.2.7. *Fix* $q > 0$ *and* $\lambda > 0$. *Then, for every* $\theta \in [0,1]$,

$$\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q}\big[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor\big] = \Phi(\theta,\lambda,q). \tag{3.34}$$

Moreover, convergence is uniform for λ *belonging to compact subsets of* $(0,\infty)\setminus\{q\}$.

In order to turn these theorems into statements about probabilities in the random cluster model, we must reintroduce the partition function. This will be done using the following theorem:

Theorem 3.2.8. *Fix $q > 0$ and $\lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{RC}_{n,\lambda,q} = \sup_{\theta\in[0,1]} \Phi(\theta, \lambda, q). \tag{3.35}$$

By combining Theorems 3.2.7 and 3.2.8, we obtain Theorem 3.2.1. The limit of equation (3.35) is known as the *free energy* of the random cluster model, and is not a new quantity of interest; it was computed in [6, Theorem 2.6]. In Lemma 5.3.2, we show that our computation agrees with theirs.

The structure of the free energy provides some hint as to its derivation. Indeed, for fixed $\epsilon > 0$, one may decompose the partition function $Z^{\mathrm{RC}}_{n,\lambda,q}$ according to the number of vertices in components of size at least ϵn to obtain

$$Z^{\mathrm{RC}}_{n,\lambda,q} = \sum_{k=0}^{n} Z^{\mathrm{RC}}_{n,\lambda,q}\big[|\mathcal{V}_{\epsilon n}| = (\tfrac{k}{n})n\big]. \tag{3.36}$$

We seek to apply Theorem 3.2.7 to each of the terms on the right hand side of (3.36). We may then dominate the sum in (3.36) in terms of its largest summand, and apply the *Laplace Principle* when taking the appropriate limit. However, these steps involve some technicalities, which we will address in Chapter 5. In particular, uniform convergence of the rate function is required in order to pass to a supremum in the limit. This is the reason why it is specified in our theorems, while it was not required for the random graphs in [3].

3.3 Random cluster model on the hypercube

In this section, we formally state the results obtained in this thesis for the random cluster model on the hypercube Q_n. Fix $q \in \mathbb{N}$ and $\lambda > 0$. We denote the random cluster probability measure on Q_n with edge weight $p = \lambda/n$ and cluster weight q by $\phi_{n,\lambda,q}$. Similarly, we denote the q-state Potts model on Q_n with inverse temperature $\beta = \lambda/n$ by $\mu_{n,\lambda,q}$, and the corresponding partition functions as $Z^{\mathrm{RC}}_{n,\lambda,q}$ and $Z^{\mathrm{P}}_{n,\lambda,q}$, respectively.

Let $\mathcal{C}_{\max}$ denote the largest connected component of $Q_n(\omega)$. Our first result says that the size of the largest connected component undergoes an asymptotic phase transition at the value $\lambda_c(q)$ defined in (3.2):

Theorem 3.3.1. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$. Then*

1. *If $\lambda < \lambda_c(q)$, then for any $\epsilon > 0$ we have*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq \epsilon 2^n] = 0. \tag{3.37}$$

2. *If $\lambda > \lambda_c(q)$, then there exists an $\epsilon > 0$ (depending only on λ) such that*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq \epsilon 2^n] = 1. \tag{3.38}$$

When $q = 1$, the asymptotic phase transition in Theorem 3.3.1 has been studied in much greater detail. In particular, for $\lambda < \lambda_c$, it has been shown in e.g. [7] that $\mathcal{C}_{\max}$ is of order n asymptotically almost surely, while for $\lambda > \lambda_c$ it was shown in [1] that $\mathcal{C}_{\max}$ has density $\theta(\lambda, 1)$ asymptotically almost surely. The behaviour for $\lambda = \lambda_c$ is more complicated, and we will not discuss it here.

For $q \in \mathbb{N}_{\geq 2}$, our results are less detailed. For $\lambda < \lambda_c$, we expect $\mathcal{C}_{\max}$ to be of order n, but can only argue by contradiction that $\mathcal{C}_{\max}$ cannot be of order 2^n. For $\lambda > \lambda_c$, we expect $\mathcal{C}_{\max}$ to have density $\theta(\lambda, q)$, but our arguments only provide a lower bound on the density of $\theta(\lambda, q)^4$, which can be improved to $\theta(\lambda, q)^2$ with some care.

When q is not an integer, we may apply Proposition 2.1.8 to compare the measure $\phi_{n,\lambda,q}$ with the measures $\phi_{n,\lambda,\lfloor q \rfloor}$ and $\phi_{n,\lambda,\lceil q \rceil}$, obtaining the following corollary of Theorem 3.3.1:

Corollary 3.3.2. *Fix $q \geq 1$ and $\lambda > 0$. Then*

1. *If $\lambda < \lambda_c(\lfloor q \rfloor)$, then $\forall\ \epsilon > 0$, we have*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq \epsilon 2^n] = 0. \tag{3.39}$$

2. *If $\lambda > \lambda_c(\lceil q \rceil)$, then there exists an $\epsilon > 0$ depending only on λ such that*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq \epsilon 2^n] = 1. \tag{3.40}$$

The previous best bounds for general q assert that asymptotically almost surely, the largest component of $Q_n(\omega)$ is of size

$$\begin{cases} n & \text{if } \lambda < 1 \\ 2^n & \text{if } \lambda > q \end{cases} \tag{3.41}$$

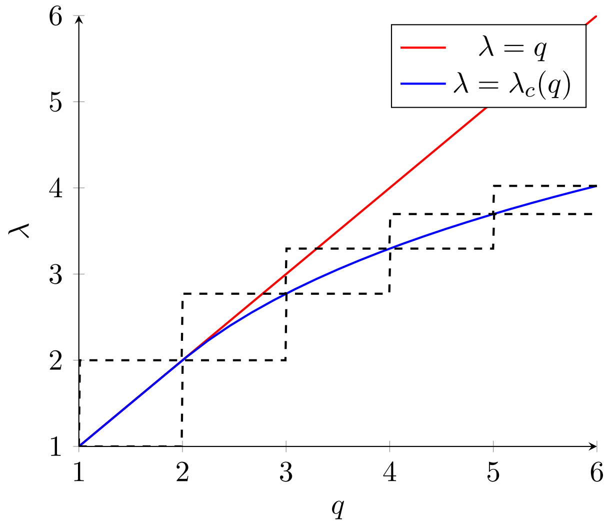

Figure 3.1: A plot of $\lambda_c(q)$, with dashed sections representing $\lambda_c(\lfloor q \rfloor)$ and $\lambda_c(\lceil q \rceil)$. We see that for $q > \lambda_c(4)$, the bounds in Corollary 3.3.2 are a strict improvement on the bounds obtained via comparisons with percolation.

The bounds in (3.41) are obtained by comparing the random cluster measure $\phi_{n,\lambda,q}$ with the percolation measures $\phi_{n,\lambda/q}$ and $\phi_{n,\lambda}$, using the stochastic inequality $\phi_{n,\lambda/q} \leq_{\text{st}} \phi_{n,\lambda,q} \leq_{\text{st}} \phi_{n,\lambda}$ given in Proposition 2.1.7. The bounds in Corollary 3.3.2 are much tighter, establishing the size of the largest component everywhere except for an interval around the critical point of size

$$\begin{aligned}\lambda_c(\lceil q \rceil) - \lambda_c(\lfloor q \rfloor) &= 2\left[\left(\frac{\lfloor q \rfloor}{\lfloor q \rfloor - 1}\right)\log\lfloor q \rfloor - \left(\frac{\lfloor q \rfloor - 1}{\lfloor q \rfloor - 2}\right)\log(\lfloor q \rfloor - 1)\right] \\ &\leq 2\left(\frac{\lfloor q \rfloor - 1}{\lfloor q \rfloor - 2}\right)\left[\log\lfloor q \rfloor - \log(\lfloor q \rfloor - 1)\right] \\ &= O(\lfloor q \rfloor^{-1}). \end{aligned} \tag{3.42}$$

The proof of Theorem 3.3.1 is based upon the methods used by Ajtai, Komlós, and Szemerédi in [1] for percolation. In particular, [1] shows that under the measure $\phi_{n,\lambda}$ with $\lambda > 1$, $Q_n(\omega)$ contains a component of order 2^n asymptotically almost surely, using the following two steps:

1. Given a vertex x, the size of the component C_x containing x may be approximated by the size of a Poisson branching process with mean λ. For $\lambda > 1$, there is a positive probability that the branching process survives. As a result, most of the

vertices in $Q_n(\omega)$ belong to components of order at least n.

2. Increasing λ slightly causes most of the components of order n in $Q_n(\omega)$ to merge, resulting in a single giant component of order 2^n.

The second step of the above argument uses the sprinkling method, described in Section 2.1.2, and may be extended to the random cluster measure using the generalisation of the sprinkling method proved in Lemma 2.1.13. The first step is more complicated, as the branching process approximation used for the percolation measure $\phi_{n,\lambda}$ depends crucially on the fact the states of the edges are independent, which is not true for the random cluster measure $\phi_{n,\lambda,q}$ in general. In order to prove Theorem 3.3.1, we will replace the use of branching processes with an analysis of a corresponding Potts model. In particular, we will show that the free energy of the measure $\mu_{n,\lambda,q}$ converges to a mean field limit:

Theorem 3.3.3. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{2^n} \log Z^P_{n,\lambda,q} = \psi(\lambda) \tag{3.43}$$

where the function $\psi(\lambda)$ is defined by

$$\psi(\lambda) = \max_{v\in\mathbb{R}^q} \log \left[\int \lambda_0(d\sigma) \exp\left\{ -\frac{\|v\|^2}{2\lambda} + v^T \cdot \sigma \right\} \right]. \tag{3.44}$$

We claim that $\psi(\lambda)$ is well defined and increasing. To this end, we introduce the notation

$$\psi(\lambda, v) = \log \left[\int \lambda_0(d\sigma) \exp\left\{ -\frac{\|v\|^2}{2\lambda} + v^{\mathrm{T}} \cdot \sigma \right\} \right]. \tag{3.45}$$

and check that

$$\lim_{\|v\|\to\infty} \psi(\lambda, v) = -\infty. \tag{3.46}$$

Consequently, $\psi(\lambda, v)$ is maximised for some $v_\lambda \in \mathbb{R}^q$, and $\psi(\lambda) = \psi(\lambda, v_\lambda)$ is well defined. Moreover, for any $\epsilon > 0$, we may write

$$\psi(\lambda + \epsilon) - \psi(\lambda) \geq \psi(\lambda + \epsilon, v_\lambda) - \psi(\lambda, v_\lambda). \tag{3.47}$$

As $\psi(\lambda, v)$ is increasing in λ, it follows that $\psi(\lambda)$ is increasing. In fact, we may compute the maximum in (3.44) explicitly:

Theorem 3.3.4. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$. Then*

$$\psi(\lambda) = \frac{g(\theta(\lambda, q))}{2q} + \frac{\lambda}{2q} + \log q \tag{3.48}$$

where $\theta(\lambda, q)$ was defined in (3.3) and the function g is given by

$$g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - [2 + (q-1)\theta]\log[1 + (q-1)\theta]. \tag{3.49}$$

In (3.11), we showed that the function $\theta(\lambda, q)$ is discontinuous at the point $\lambda_c(q)$ when $q > 2$. Otherwise, it is continuous, and the free energy $\psi(\lambda)$ inherits this continuity. In particular, we may apply the Edwards Sokal coupling (in the form of Corollary 2.2.3) to show that the free energy of the random cluster model on Q_n converges to the same limit computed for the random cluster model on the complete graph in [6, Theorem 2.6]:

Corollary 3.3.5. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$ with $\lambda \neq \lambda_c(q)$. Then*

$$\lim_{n\to\infty} \frac{1}{2^n} \log Z^{RC}_{n,\lambda,q} = \frac{g(\theta(\lambda, q))}{2q} - \left(\frac{q-1}{2q}\right)\lambda + \log q. \tag{3.50}$$

An analysis of the limit obtained in Theorem 3.3.3 shows that for any edge $\{x, y\} \in E$, the spin correlations $\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda,q}$ converge to the following limit:

Lemma 3.3.6. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$ with $\lambda \neq \lambda_c(q)$. Then for any pair of vertices $x, y \in Q_n$ with $\{x, y\} \in E$, we have*

$$\lim_{n\to\infty} \langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda,q} = \frac{1}{q} + \frac{q-1}{q}\theta(\lambda, q)^2. \tag{3.51}$$

By applying the Edwards Sokal coupling (in the form given by Corollary 2.2.3) to Lemma 3.3.6, we establish a similar result regarding connection probabilities for the random cluster measure $\phi_{n,\lambda,q}$:

Lemma 3.3.7. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$ with $\lambda \neq \lambda_c(q)$. Then for any pair of vertices $x, y \in Q_n$ with $\{x, y\} \in E$, we have*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] = \theta(\lambda, q)^2. \tag{3.52}$$

Lemma 3.3.7 serves as the starting point for our analysis of the random cluster model of the hypercube. In particular, it will be shown that the asymptotic phase transition in nearest neighbour connection probabilities corresponds to an asymptotic phase transition in the size of the largest connected component using the following arguments:

1. For $\lambda > \lambda_c$, we apply Markov's inequality to the set of vertices in the neighbourhood of a given vertex x to compute a lower bound on the probability that a given vertex x belongs to a component of order n in $Q_n(\omega)$. This will allow us to substitute for the exploration process arguments used in [1, Lemma 1], after which we may apply the sprinkling method developed in Section 2.1.2.

2. For $\lambda < \lambda_c$, we argue that if a component of order 2^n exists, then the probability that two adjacent vertices both belong to it (and hence are connected in $Q_n(\omega)$) is bounded away from 0. As the probability $\phi_{n,\lambda,q}[x \leftrightarrow y]$ vanishes in the limit $n \to \infty$, this presents a contradiction.

Finally, we compute the exponential rate of the event K that $Q_n(\omega)$ is connected under the measure $\phi_{n,\lambda,q}$:

Theorem 3.3.8. *Fix $q > 0$ and $\lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{2^n} \log Z^{RC}_{n,\lambda,q}[K] = \log \pi_1(\lambda). \tag{3.53}$$

Moreover, convergence is uniform for λ belonging to compact subsets of $[0, \infty)$.

Theorem 3.3.8 is analogous to Theorem 3.2.4 from Section 3.1, and will be proven in Section 4.1 using the same methods. Ultimately, we hope the methods of Section 3.1 may be extended to the hypercube more fully, in order to generalise the asymptotic phase transition for the size of the largest component from Theorem 3.3.1 to non-integer values of q.

Chapter 4

The complete graph I: connected and acyclic graphs

In this chapter, we continue the analysis of the random cluster model on the complete graph which began in Section 3.2. Throughout, we will adopt the notation used in Section 3.2, writing $\phi_{n,\lambda,q}$ for the random cluster measure on K_n with edge weight $p = \lambda/n$ and cluster weight q. If $q = 1$, we use the standard notation $\phi_{n,\lambda}$ for the percolation measure.

First, we prove Theorem 3.2.4 in Section 4.1, which gives the exponential rate of the event K that the graph $K_n(\omega)$ is connected under the measure $\phi_{n,\lambda,q}$. As the weight $q^{k(\omega)}$ is constant on the event K, the exponential rate of the event K is actually independent of q, and so Theorem 3.2.4 is an immediate consequence of the rate function given in [3, Theorem 2.3] for the case $q = 1$.

Next, we prove Theorem 3.2.6 in Section 4.2, which gives the exponential rate of the event F that the graph $K_n(\omega)$ is acyclic under the measure $\phi_{n,\lambda,q}$. In particular, we will analyse the measure $Z^{\mathrm{RC}}_{n,\lambda,q}[F \cap B_r]$ of the event that $K_n(\omega)$ is acyclic and contains no components of size greater than r, and then take the limit $r \to \infty$, as was done for the case $q = 1$ in [3, Section 4]. The additional weight $q^{k(\omega)}$ is dealt with using an explicit correspondence between the number of components and number of edges of the acyclic random graph.

Finally, we prove Theorem 3.2.5 in Section 4.3, which shows that the exponential rates of the event that $K_n(\omega)$ is acyclic and the event that $K_n(\omega)$ contains only components of size $o(n)$ coincide, by extending the arguments used when $q = 1$ in [3, Section 5]. In particular, this will allow us to assume the complement of the largest component of $K_n(\omega)$ under the measure $\phi_{n,\lambda,q}$ is acyclic, which makes the weight $q^{k(\omega)}$ much easier to deal with.

4.1 Connected graphs

In this section, we prove Theorem 3.2.4. In particular, we compute the exponential rate of the event K that the graph $K_n(\omega)$ is connected under the measure $\phi_{n,\lambda,q}$. This extends the following result, due to Biskup, Chayes and Smith:

Theorem 4.1.1 ([3, Theorem 2.3])**.** *Fix $\lambda > 0$. Then*

$$\phi_{n,\lambda}[K] = (1 - e^{-\lambda})^n e^{O(\log n)}, \tag{4.1}$$

where $O(\log n)$ is bounded by a constant times $\log n$ uniformly for λ belonging to compact subsets of $[0, \infty)$.

In fact, Theorem 3.2.4 may be proven as a direct corollary of Theorem 4.1.1:

Proof of Theorem 3.2.4. Observe that $K_n(\omega)$ is connected if and only if $k(\omega) = 1$. Thus

$$\begin{aligned} Z^{\text{RC}}_{n,\lambda,q}[K] &= \sum_{\omega \in K} \left\{ \prod_{e \in E} p^{\omega_e}(1-p)^{1-\omega_e} \right\} q^{k(\omega)} \\ &= q \sum_{\omega \in K} \prod_{e \in E} p^{\omega_e}(1-p)^{1-\omega_e} \\ &= q\phi_{n,\lambda}[K]. \end{aligned}$$

In particular,

$$\frac{1}{n} \log Z^{\text{RC}}_{n,\lambda,q}[K] = \frac{1}{n} \log \phi_{n,\lambda}[K] + \frac{1}{n} \log q. \tag{4.2}$$

The first term converges uniformly to $\log \pi_1(\lambda)$ for λ belonging to compact subsets of $[0, \infty)$ by Theorem 4.1.1. The second is independent of λ, and hence converges uniformly to 0. □

We devote the remainder of this section to reproducing the proof of Theorem 4.1.1 from [3] in more generality. In particular, we will show that for any finite connected graph G, one may compute the probability that the random graph is connected under percolation by instead considering a simpler problem on a directed random graph. For the complete graph K_n, this was proven in [3, Lemma 3.2]. Using this generalisation, we will also compute the rate function for connectedness on the hypercube in Section 7.3. We begin by introducing the problem of inhomogeneous percolation:

Definition 4.1.2 (Inhomogeneous percolation)**.** *Let $G = (V, E)$ be a finite, simple graph, $\Omega_G = \{0, 1\}^E$ and $\mathbf{p} \in [0, 1]^E$ be a vector. The inhomogeneous percolation measure with* *weight* $\mathbf{p}$ *is defined by*

$$\phi_{G,\mathbf{p}}[\omega] = \prod_{e \in E} p_e^{\omega_e} (1 - p_e)^{1-\omega_e}, \tag{4.3}$$

Equivalently, the measure declares each edge $e \in E$ *to be open with probability* p_e*, independently of every other edge.*

As in the homogeneous case, the measure $\phi_{G,\mathbf{p}}$ gives a measure on the set of random subgraphs of G via the bijection $\omega \to G(\omega)$.

Next, we consider a directed version of the percolation problem. Let $G = (V, E)$ be a finite, simple graph, and define the directed graph $\vec{G} = (V, \vec{E})$ using the set of directed edges

$$\vec{E} = \{(x, y), (y, x) : \ \{x, y\} \in E\}. \tag{4.4}$$

In other words, $\vec{G}$ is the directed graph which replaces each edge $e \in E$ with directed edges in both directions. Let $\Omega_{\vec{G}} = \{0, 1\}^{\vec{E}}$, and observe that an element $\omega \in \Omega_{\vec{G}}$ corresponds to a directed subgraph $\vec{G}(\omega)$ of $\vec{G}$. We sample a directed subgraph of $\vec{G}$ according to the following measure:

Definition 4.1.3 (Inhomogeneous directed percolation)**.** *Let* $G = (V, E)$ *be a finite, simple graph, and* $\vec{G} = (V, \vec{E})$ *be the corresponding directed graph. Let* $\Omega_{\vec{G}} = \{0, 1\}^{\vec{E}}$ *and* $\vec{\mathbf{p}} \in [0, 1]^{\vec{E}}$ *be a vector. The inhomogeneous directed percolation measure with weight* $\vec{\mathbf{p}}$ *is defined by*

$$\phi_{\vec{G},\vec{\mathbf{p}}}[\omega] = \prod_{e \in \vec{E}} p_e^{\omega_e} (1 - p_e)^{1-\omega_e}, \tag{4.5}$$

Equivalently, the measure declares each directed edge $e \in \vec{E}$ *to be open with probability* p_e*, independently of every other edge.*

In practice, we will only be interested in probability vectors satisfying the additional constraint that for every $\{x, y\} \in E$, we have $p_{(x,y)} = p_{(y,x)}$. Given a vector $\mathbf{p} \in [0, 1]^E$, we write $\phi_{\vec{G},\mathbf{p}}$ for the directed measure where for every $\{x, y\} \in E$, we define $p_{(x,y)} = p_{(y,x)} = p_{\{x,y\}}$. We study the following property of $\vec{G}(\omega)$ under the measure $\phi_{\vec{G},\mathbf{p}}$:

Definition 4.1.4. *A directed graph is grounded at a vertex* $v \in V$ *if for every other vertex* $w \in V$*, there exists a path of directed edges from* w *to* v*.*

The following lemma, which generalises [3, Lemma 3.2], says that for any finite, simple graph G, the probability that $G(\omega)$ is connected under the measure $\phi_{G,\mathbf{p}}$ is equal to the probability that $\vec{G}(\omega)$ is grounded at a given vertex under the measure $\phi_{\vec{G},\mathbf{p}}$:

Lemma 4.1.5 ([3, Lemma 3.2]). *Let $G = (V, E)$ be a finite, simple graph and let $\mathbf{p} \in [0,1]^E$. Let K be the event that $G(\omega)$ is connected, and for $v \in V$, let $\mathcal{G}_v$ be the event that $\vec{G}(\omega)$ is grounded at v. Then*

$$\phi_{G,\mathbf{p}}(K) = \phi_{\vec{G},\mathbf{p}}(\mathcal{G}_v). \tag{4.6}$$

The proof of Lemma 4.1.5 is based heavily upon the proof of [3, Lemma 3.2]:

Proof. We will prove the result by induction on the number of edges $e \in E(G)$ for which $p_e > 0$. If $p_e = 0$ for every $e \in E(G)$, then $\phi_{G,\mathbf{p}}(K) = \phi_{\vec{G},\mathbf{p}}(\mathcal{G}_v) = 0$, and so the base case is clear.

Next, let $F \subset E$ and suppose that $\phi_{G,\mathbf{p}}(K) = \phi_{\vec{G},\mathbf{p}}(\mathcal{G}_v)$ when $p_e = 0$ for every $e \in F$. Fix an edge $e = \{x, y\} \in F$. We claim that (4.6) still holds when $p_e > 0$. To prove this claim, it will suffice to show that the partial derivatives of $\phi_{G,\mathbf{p}}(K)$ and $\phi_{\vec{G},\mathbf{p}}(\mathcal{G}_v)$ with respect to p_e are equal for every $p_e \in [0,1]$.

As K and $\mathcal{G}_v$ are increasing events, we may compute these partial derivatives using *Russo's formula*. In particular, let A be an increasing event which depends only on a finite number of edges. Given an edge e, recall that e is *pivotal* for the event A if A occurs when e is open, and does not occur when e is closed. Let $N(A)$ be the number of edges which are pivotal for A. Then, Russo's formula says that

$$\frac{\partial}{\partial \mathbf{p}_e}\phi_{G,\mathbf{p}}(A) = \phi_{G,\mathbf{p}}(N(A)). \tag{4.7}$$

For the event K, Russo's formula becomes

$$\frac{\partial}{\partial \mathbf{p}_e}\phi_{G,\mathbf{p}}(K) = \phi_{G,\mathbf{p}}(e \text{ is pivotal for } K), \tag{4.8}$$

where e is pivotal for K if and only if we may partition V into two sets, each containing an endpoint of e, such that each set is connected in $G(\omega)$ and every other edge between them is closed.

For the event $\mathcal{G}_v$, Russo's formula instead becomes

$$\frac{\partial}{\partial \mathbf{p}_e}\phi_{\vec{G},\mathbf{p}}(\mathcal{G}_v) = \phi_{\vec{G},\mathbf{p}}((x,y) \text{ is pivotal for } \mathcal{G}_v) + \phi_{\vec{G},\mathbf{p}}((y,x) \text{ is pivotal for } \mathcal{G}_v). \tag{4.9}$$

Let $\mathcal{C}_v$ be the set of vertices which are grounded at v in $G(\omega) \setminus e$. By construction, no directed edge from $\mathcal{C}_v^c$ to $\mathcal{C}_v$ other than (x, y) may be open, and so (x, y) is pivotal for $\mathcal{G}_v$ if and only if:

1. $y \in \mathcal{C}_v$,

2. $x \in \mathcal{C}_v^c$,

3. $\mathcal{C}_v^c$ is grounded at x.

It is not possible for both of the directed edges (x, y) and (y, x) to be pivotal, as this would imply that $x \in \mathcal{C}_v$ and $x \in \mathcal{C}_v^c$. In particular, (4.9) becomes

$$\frac{\partial}{\partial \mathbf{p}_e} \phi_{\vec{G},\mathbf{p}}(\mathcal{G}_v) = \phi_{\vec{G},\mathbf{p}}((x, y) \text{ is pivotal for } \mathcal{G}_v \text{ or } (y, x) \text{ is pivotal for } \mathcal{G}_v). \tag{4.10}$$

Write $A_e = \{(x, y) \text{ is pivotal for } \mathcal{G}_v \text{ or } (y, x) \text{ is pivotal for } \mathcal{G}_v\}$. Observe that A_e corresponds to the event that we may partition $\vec{G}(\omega)$ into two components $\mathcal{C}_v$ and $\mathcal{C}_v^c$, one containing x and one containing y, such that $\mathcal{C}_v$ is grounded at v and $\mathcal{C}_v^c$ is grounded at w, where $w \in \{x, y\}$ is the endvertex of e belonging to $\mathcal{C}_v^c$. Conditioning on the set $\mathcal{C}_v$, we have

$$\phi_{\vec{G},\mathbf{p}}(A_e) = \sum_{\substack{\mathcal{C} \subset V \\ v \in \mathcal{C}, |\{x,y\} \cap \mathcal{C}| = 1}} \phi_{\vec{C},\mathbf{p}}(\mathcal{G}_v) \phi_{\vec{C}^c,\mathbf{p}}(\mathcal{G}_w) \prod_{f \in E(\mathcal{C}, \mathcal{C}^c) \setminus e} (1 - \mathbf{p}_f). \tag{4.11}$$

By induction, $\phi_{\vec{C},\mathbf{p}}(\mathcal{G}_v) = \phi_{C,\mathbf{p}}(K)$ and $\phi_{\vec{C}^c,\mathbf{p}}(\mathcal{G}_w) = \phi_{C^c,\mathbf{p}}(K)$, and so (4.11) becomes

$$\phi_{\vec{G},\mathbf{p}}(A_e) = \sum_{\substack{\mathcal{C} \subset V \\ v \in \mathcal{C}, |\{x,y\} \cap \mathcal{C}| = 1}} \phi_{C,\mathbf{p}}(K) \phi_{C^c,\mathbf{p}}(K) \prod_{f \in E(\mathcal{C}, \mathcal{C}^c) \setminus e} (1 - \mathbf{p}_f). \tag{4.12}$$

Observing that the right hand side of (4.12) is precisely $\phi_{G,\mathbf{p}}(e \text{ is pivotal for } K)$, the partial derivatives are equal and Lemma 4.1.5 follows by induction. □

Using Lemma 4.1.5, we may obtain Theorem 4.1.1 as a consequence of the following lemma, which gives appropriate upper and lower bounds:

Lemma 4.1.6 ([3, Lemmas 3.3 and 3.4])**.** *Fix $\lambda > 0$. Then, we have the upper bound*

$$\phi_{n,\lambda}[K] \leq \left(1 - (1 - \lambda/n)^{n-1}\right)^{n-1} \tag{4.13}$$

and the lower bound

$$\phi_{n,\lambda}[K] \geq \frac{1}{n}\left(1 - (1 - \tfrac{\lambda}{n})^{n-1}\right)^{n-1}. \tag{4.14}$$

In particular, Theorem 4.1.1 holds.

Proof. We follow the proof of [3, Lemmas 3.3 and 3.4], analysing the directed graph $\vec{K}_n(\omega)$ under the measure $\phi_{\vec{K}_n,\mathbf{p}}$ where $p_e = \lambda/n$ for every $e \in E$.

We begin with the upper bound (4.13). Fix a vertex $v \in K_n$ and let $\mathcal{E}_v$ denote the event that every vertex of the graph, except possibly the vertex v, has at least one outgoing edge in $\vec{K}_n(\omega)$. If another vertex with no outgoing edge exists, $\vec{K}_n(\omega)$ cannot be grounded at v, and so $\mathcal{G}_v \subset \mathcal{E}_v$. It follows that

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v) \leq \phi_{\vec{K}_n,\mathbf{p}}(\mathcal{E}_v) = \big(1 - (1-\lambda/n)^{n-1}\big)^{n-1}. \tag{4.15}$$

Applying Lemma 4.1.5 to (4.15) yields (4.13).

For the lower bound, we condition on the event $\mathcal{E}_v$ to obtain

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v) = \phi_{\vec{K}_n,\mathbf{p}}(\mathcal{E}_v)\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{E}_v). \tag{4.16}$$

The factor $\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{E}_v)$ was calculated in the upper bound. Define the event $\mathcal{F}_v$ that every vertex of the graph, except possibly v, has *exactly* one outgoing edge. We claim that the other factor satisfies the lower bound

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{E}_v) \geq \phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{F}_v). \tag{4.17}$$

To see (4.17), let $\omega \in \mathcal{E}_v$. To each vertex $w \neq v$, choose an outgoing edge uniformly at random, colour it red, and let $\mathcal{G}'_v$ be the event that $\mathcal{G}_v$ occurs using only these red edges. Then

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{E}_v) \geq \phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}'_v|\mathcal{E}_v). \tag{4.18}$$

Observing that conditionally on $\mathcal{E}_v$ or $\mathcal{F}_v$, the red edges are distributed identically (indeed, uniformly), we have

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}'_v|\mathcal{E}_v) = \phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}'_v|\mathcal{F}_v). \tag{4.19}$$

Moreover, as there is only one choice of red edge on $\mathcal{F}_v$, the events $\mathcal{G}_v$ and $\mathcal{G}'_v$ coincide, giving

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}'_v|\mathcal{F}_v) = \phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{F}_v). \tag{4.20}$$

It remains to estimate the conditional probability on the right hand side of (4.20). As $\mathcal{G}_v$ does not depend on the outgoing edges from v itself, we may ignore them. To each of the remaining $n-1$ vertices, we have $n-1$ possible outgoing edges, yielding $(n-1)^{n-1}$ equally weighted configurations for $\mathcal{F}_v$. The number of these configurations which result in $K_n(\omega)$ being grounded at v is precisely the number of spanning trees of K_n, which is given by n^{n-2}. Thus

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{F}_v) = \frac{n^{n-2}}{(n-1)^{n-1}} \geq \frac{1}{n}. \tag{4.21}$$

It follows that

$$\phi_{\vec{K}_n,\mathbf{p}}(\mathcal{G}_v) \geq \frac{1}{n}\big(1 - (1 - \tfrac{\lambda}{n})^{n-1}\big)^{n-1}. \tag{4.22}$$

Applying Lemma 4.1.5 to (4.22) yields (4.14). □

4.1.1 Connected subgraphs of the hypercube

In this subsection, we prove Theorem 3.3.8, which gives the exponential rate of the event that the random graph is connected for the random cluster model on the hypercube Q_n. We include this result here because the proof is very similar to that of Theorem 4.1.1, and extends the arguments introduced in Lemma 4.1.6. To begin, we will need the following lower bound on the number of spanning trees of the hypercube:

Proposition 4.1.7. *Let a_n denote the number of spanning trees of the hypercube Q_n. Then, for any $\epsilon > 0$, we have the lower bound*

$$a_n^2 \geq \frac{4n^2}{2^{2n}}[(1-4\epsilon^2)n^2]^{2^n(1-2e^{-\epsilon^2 n/2})}. \tag{4.23}$$

Proof. It is known (e.g. [38]) that the number of spanning trees of Q_n is given by

$$a_n = \frac{2n}{2^n}\prod_{k=1}^{n-1}(2k)^{\binom{n}{k}}. \tag{4.24}$$

By squaring (4.24) and applying symmetry of the binomial coefficients, we obtain

$$a_n^2 = \frac{4n^2}{2^{2n}}\prod_{k=1}^{n-1}(4k(n-k))^{\binom{n}{k}}. \tag{4.25}$$

Given $\epsilon > 0$, we may bound (4.25) below by

$$\begin{aligned} a_n^2 &\geq \frac{4n^2}{2^{2n}}\prod_{k>(\frac{1}{2}-\epsilon)n}^{(\frac{1}{2}+\epsilon)n}(4k(n-k))^{\binom{n}{k}} \\ &\geq \frac{4n^2}{2^{2n}}[(1-4\epsilon^2)n^2]^{s_n}, \end{aligned} \tag{4.26}$$

where

$$s_n = \sum_{k>(\frac{1}{2}-\epsilon)n}^{(\frac{1}{2}+\epsilon)n} \binom{n}{k}. \tag{4.27}$$

Using an exponential Markov inequality (e.g. Lemma 7.1.8), we may bound s_n below by

$$s_n \geq 2^n(1 - 2e^{-\epsilon^2 n/2}), \tag{4.28}$$

which yields the bound (4.23). □

Theorem 3.3.8 is then analogous to the following result:

Lemma 4.1.8. *Fix $\lambda > 0$ and let $\phi_{n,\lambda}$ denote the percolation measure with edge weight $p = \lambda/n$ on Q_n. Then, we have the upper bound*

$$\limsup_{n\to\infty} \frac{1}{2^n} \log \phi_{n,\lambda}[K] \leq \log \pi_1(\lambda). \tag{4.29}$$

and the lower bound

$$\liminf_{n\to\infty} \frac{1}{2^n} \log \phi_{n,\lambda}[K] \geq \log \pi_1(\lambda). \tag{4.30}$$

Proof. We follow the proof of Lemma 4.1.6, analysing the directed graph $\vec{Q}_n(\omega)$ under the measure $\phi_{\vec{Q}_n,\mathbf{p}}$ where $p_e = \lambda/n$ for every $e \in E$ and producing upper and lower bounds on the probability $\phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v)$. Throughout this proof, we denote the percolation measure on Q_n with edge weight λ/n by $\phi_{n,\lambda}$.

We begin with the upper bound (4.29), recalling the definition of the event $\mathcal{E}_v$ from the proof of Lemma 4.1.6 and using the inclusion $\mathcal{G}_v \subset \mathcal{E}_v$ to obtain the bound

$$\phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v) \leq \phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{E}_v) = \left(1 - (1-\lambda/n)^n\right)^{2^n-1}. \tag{4.31}$$

(4.29) follows upon applying Lemma 4.1.5 to (4.31) and taking the appropriate limit. For the lower bound (4.30), we recall the definition of the event $\mathcal{F}_v$ from the proof of Lemma 4.1.6 and use the inequality

$$\phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{E}_v) \geq \phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{F}_v) \tag{4.32}$$

to obtain the lower bound

$$\phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v) \geq \phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{E}_v)\phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{F}_v). \tag{4.33}$$

We now estimate the second factor. As $\mathcal{G}_v$ does not depend on the outgoing edges from v itself, we may ignore them. To each of the remaining $2^n - 1$ vertices, we have n possible outgoing edges, yielding n^{2^n-1} equally weighted configurations for $\mathcal{F}_v$. The number of these configurations which result in $Q_n(\omega)$ being grounded at v is precisely the number a_n of spanning trees of Q_n. By Proposition 4.1.7, it follows that for any $\epsilon > 0$, we have

$$\log\left(\frac{a_n}{n^{2^n-1}}\right) \geq \frac{1}{2^n}\log\frac{2n}{2^n} + (1 - 2e^{-\epsilon^2 n/2})\log\sqrt{1-4\epsilon^2} - 2e^{-\epsilon^2 n/2}\log n. \tag{4.34}$$

Taking the limits as $n \to \infty$ and $\epsilon \downarrow 0$ in that order, we see that

$$\liminf_{n\to\infty}\frac{1}{2^n}\log\phi_{\vec{Q}_n,\mathbf{p}}(\mathcal{G}_v|\mathcal{F}_v) \geq 0 \tag{4.35}$$

which may be combined with (4.31) to yield the lower bound (4.29). □

4.2 Acyclic graphs

In this section, we prove Theorem 3.2.6, which gives the exponential rate of the event F that the graph $K_n(\omega)$ is acyclic under the measure $\phi_{n,\lambda,q}$.

Let $\omega \in F$ be a percolation configuration, and let m_l be the number of connected components of size l in $K_n(\omega)$. Observe that each component of size l has $l-1$ open edges, and there are $\sum_l (l-1)m_l = n - \sum_l m_l$ open edges in total. As $\sum_l m_l = k(\omega)$, it follows that the weight of the configuration ω is given by

$$Z^{\mathrm{RC}}_{n,\lambda,q}[\omega] = q^{\sum_l m_l}(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n+\sum_l m_l}\prod_{l\geq 1}(\tfrac{\lambda}{n})^{(l-1)m_l}. \tag{4.36}$$

We will use (4.36) to obtain a representation of the quantity $Z^{\mathrm{RC}}_{n,\lambda,q}[F\cap B_r]$ by summing over the possible choices of the numbers $m_1, \cdots, m_n$. Given a set $m_1, \cdots, m_n$, there are $\frac{n!}{\prod_l m_l!(l!)^{m_l}}$ ways of partitioning $K_n(\omega)$ into appropriately sized components. As $\omega \in F$, each component is a tree. There are $a_l = l^{l-2}$ possible spanning trees for each component of size l, yielding $\prod_l a_l^{m_l}$ possible spanning forests. Thus

$$Z^{\mathrm{RC}}_{n,\lambda,q}[F\cap B_r] = \sum_{\substack{\sum lm_l = n\\ m_l=0\forall l>r}}\frac{n!}{\prod_l[m_l!(l!)^{m_l}]}q^{\sum m_l}(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n+\sum m_l}\prod_l\left[a_l(\tfrac{\lambda}{n})^{l-1}\right]^{m_l}. \tag{4.37}$$

By factoring out the terms which do not depend on the choice of the numbers $m_1, \cdots, m_n$, we may rewrite (4.37) as

$$Z^{\mathrm{RC}}_{n,\lambda,q}[F \cap B_r] = n!(\tfrac{\lambda}{n})^n(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n}\sum_{k=1}^{n}(\tfrac{\lambda}{n})^{-k}[q(1-\tfrac{\lambda}{n})]^k Q_{n,k,r} \tag{4.38}$$

where

$$Q_{n,k,r} = \sum_{\substack{\sum lm_l = n \\ \sum m_l = k \\ m_l = 0 \forall l > r}} \prod_{l\geq 1}\left(\frac{a_l}{l!}\right)^{m_l}\frac{1}{m_l!}. \tag{4.39}$$

The equation (4.38) is the same re-arrangement employed for percolation in [3], with the random cluster model introducing an extra factor of q^k in the summand. In order to estimate this summand, we use the following proposition, which expresses the asymptotic behaviour of the quantity $Q_{n,k,r}$ in terms of the generating function for the number of spanning trees:

Proposition 4.2.1 ([3, Proposition 4.1])**.** *Consider the generating function*

$$G_r(s) = \sum_{l=1}^{r}\frac{s^l a_l}{l!}. \tag{4.40}$$

Then for all $n, k, r \geq 1$

$$Q_{n,k,r} \leq \frac{1}{k!}\inf_{s>0}\frac{G_r(s)^k}{s^n}. \tag{4.41}$$

Moreover, for each $\eta > 0$ there is an $n_0 < \infty$ and a sequence $(c_r)_{r\geq 1}$ of positive numbers such that for all $n \geq n_0$, $k \geq 1$ and $r \geq 2$ such that $k < (1-\eta)n$ and $rk > n(1+\eta)$, we have

$$Q_{n,k,r} \geq \frac{c_r}{\sqrt{n}}\frac{1}{k!}\inf_{s>0}\frac{G_r(s)^k}{s^n}. \tag{4.42}$$

The upper bound (4.41) of Proposition 4.2.1 may be understood by expanding $G_r(s)^k$ as

$$\left(\sum_{l=1}^{r}\frac{s^l a_l}{l!}\right)^k = \sum_{i=0}^{rk} b_i s^i. \tag{4.43}$$

Each of the coefficients $(b_i)_{i=0}^{rk}$ is positive, and the coefficient b_n of s^n is given by $k!Q_{n,k,r}$. It follows that $G_r(s)^k > k!Q_{n,k,r}s^n$, and the upper bound is obtained by optimising over $s > 0$. The lower bound requires more control over the additional terms, and we will not discuss it here, instead citing the proof from [3, Proposition 4.1].

We now return to (4.38). Using (4.41), we may obtain an upper bound on the summand of

$$(\tfrac{\lambda}{n})^{-k}[q(1-\tfrac{\lambda}{n})]^k Q_{n,k,r} \le (\tfrac{\lambda}{n})^{-k}[q(1-\tfrac{\lambda}{n})]^k \tfrac{1}{k!} \inf_{s>0} \frac{G_r(s)^k}{s^n}. \qquad (4.44)$$

Applying the inequality $1-x \le e^{-x}$ to the factor $(1-\frac{\lambda}{n})^k$ in (4.44) and Stirling's Formula to the factorial yields a further bound of

$$(\tfrac{\lambda}{n})^{-k}[q(1-\tfrac{\lambda}{n})]^k Q_{n,k,r} \le e^{o(n)} \inf_{s>0} \exp\{n\Theta_r(s, k/n, \lambda/q)\} \qquad (4.45)$$

where the error term is bounded uniformly for λ belonging to compact subsets of $(0,\infty)$, and the function $\Theta_r(s,\theta,\alpha)$ is given by

$$\Theta_r(s,\theta,\alpha) = -\theta\log\alpha - \theta\log\theta + \theta + \theta\log G_r(s) - \log s. \qquad (4.46)$$

As a sum of n terms is bounded above by n times its maximal term, (4.38) may be bounded above by

$$\begin{aligned} Z^{\mathrm{RC}}_{n,\lambda,q}[F\cap B_r] &\le n!(\tfrac{\lambda}{n})^n(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n} \times n \sup_{1\le k\le n} (\tfrac{\lambda}{n})^{-k}[q(1-\tfrac{\lambda}{n})]^k Q_{n,k,r} \\ &\le n!(\tfrac{\lambda}{n})^n(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n} e^{o(n)} \sup_{1\le k\le n} \inf_{s>0} \exp\{n\Theta_r(s,k/n,\lambda/q)\}. \qquad (4.47)\end{aligned}$$

Similarly, we may bound a sum of n positive terms below by its maximal summand. In particular, the lower bound of Proposition 4.2.1 (for fixed $\eta > 0$) and the inequality $1-x \ge e^{-x-x^2}$ (valid for sufficiently small x) may be applied to (4.38) to obtain the lower bound

$$Z^{\mathrm{RC}}_{n,\lambda,q}[F\cap B_r] \ge n!(\tfrac{\lambda}{n})^n(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n} e^{o(n)} \sup_{k} \inf_{s>0} \exp\{n\Theta_r(s,k/n,\lambda/q)\}, \qquad (4.48)$$

where the supremum is now taken over k satisfying $\frac{1}{r}(1+\eta)n < k < (1-\eta)n$. If we can show that the supremum of Θ_r is contained in this interval (for sufficiently small η and sufficiently large r) then the bounds (4.47) and (4.48) will coincide. This supremum is evaluated in [3, Lemma 4.2]. We reproduce their lemma here, as it will be important to check that any convergence is uniform:

Lemma 4.2.2 ([3, Lemma 4.2]). *Let $\alpha > 0$ and $r \ge 2$. Then there is a unique $(s_r, \theta_r) \in (0,\infty)\times(1/r, 1)$ for which*

$$\Theta_r(s_r,\theta_r,\alpha) = \sup_{1/r\le\theta\le 1} \inf_{s>0} \Theta_r(s,\theta,\alpha). \qquad (4.49)$$

Moreover, $s_r \in (0,\infty)$ satisfies the limit

$$\lim_{r\to\infty} s_r = \begin{cases} \alpha e^{-\alpha} & \text{if } \alpha \le 1 \\ \frac{1}{e} & \text{if } \alpha > 1 \end{cases} \tag{4.50}$$

and $\theta_r \in (1/r, 1)$ satisfies the limit

$$\lim_{r\to\infty} \theta_r = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } \alpha \le 1 \\ \frac{1}{2\alpha} & \text{if } \alpha > 1 \end{cases} \tag{4.51}$$

Combining these facts yields the limit

$$\lim_{r\to\infty} \Theta_r(s_r, \theta_r, \alpha) = \begin{cases} 1 + \frac{\alpha}{2} - \log\alpha & \text{if } \alpha \le 1 \\ 1 + \frac{1}{2\alpha} & \text{if } \alpha > 1 \end{cases} \tag{4.52}$$

Moreover, convergence is uniform for α belonging to compact subsets of $(0,\infty) \setminus \{1\}$.

As $\alpha = \lambda/q$, the convergence in r of the above three limits will be uniform for λ belonging to compact subsets of $(0,\infty) \setminus \{q\}$, which is precisely the claim of Theorem 3.2.6. We may rewrite the final limit as

$$\lim_{r\to\infty} \Theta_r(s_r, \theta_r, \alpha) = 1 + \tfrac{\alpha}{2} - \log\alpha + \Psi(\alpha). \tag{4.53}$$

Proof. We follow the proof of [3, Lemma 4.2]. By setting the partial derivatives of Θ_r equal to 0, we see that the maximising pair (s_r, θ_r) is a solution of the equations

$$sG_r'(s) = \alpha, \ G_r(s) = \alpha\theta. \tag{4.54}$$

We will obtain the limits (4.50) and (4.51) by analysing the above equations, beginning with the equation $sG_r'(s) = \alpha$. By definition, $sG_r'(s)$ is equal to the sum

$$\sum_{l=1}^{r} a_l \frac{s^l}{(l-1)!} = \sum_{l=1}^{r} \frac{1}{l} \frac{l^l}{l!} s^l. \tag{4.55}$$

When $s > \frac{1}{e}$, we may apply Stirling's Formula to the factorial in (4.55) to see that the sum diverges. When $s \le \frac{1}{e}$, the sum instead converges to the *Lambert* function $W(s)$ satisfying $We^{-W} = s$ (a fact which we cite from [15]). We now consider two cases, depending on the value of α:

1. Suppose first that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_1 < 1$. Define

$$r_1 = \inf\{r : \ (1/e)G_r'(1/e) > \alpha_1\} \tag{4.56}$$

and observe that for every $r \geq r_1$ and $\alpha \leq \alpha_1$, we must have $s_r < 1/e$. In particular, the error

$$\Delta_r(s) := W(s) - sG_r'(s) \tag{4.57}$$

may be uniformly bounded by $\Delta_r := \Delta_r(1/e)$, which converges to 0 as $r \to \infty$. As $s_r G_r'(s_r) = \alpha$, we see that $|W(s_r) - \alpha| \leq \Delta_r$. Moreover, by the Mean Value Theorem, we have

$$|s_r - \alpha e^{-\alpha}| = |W(s_r)e^{-W(s_r)} - \alpha e^{-\alpha}| \leq ce^{-c}|W(s_r) - \alpha| \tag{4.58}$$

for some $c \in (\alpha - \Delta_r, \alpha + \Delta_r)$. As $ce^{-c} \leq 1$, we deduce that $|s_r - \alpha e^{-\alpha}| \leq \Delta_r$, which converges to 0 uniformly.

2. Next, suppose that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_0 \geq 1$. As $s_r G_r'(s_r) < W(s_r)$ for $s_r < \frac{1}{e}$ and $W(\frac{1}{e}) = 1$, it follows that $s_r \geq \frac{1}{e}$. Conversely, we may discard lower order terms in the sum (4.55) to obtain the bound

$$sG_r'(s) \geq r^{-3/2}(es)^r. \tag{4.59}$$

If we write $s = \gamma/e$, then (4.59) exceeds α_1 if

$$\gamma \geq \exp(\tfrac{1}{r}\log\alpha_1 + \tfrac{3}{2r}\log r) := \gamma_r. \tag{4.60}$$

Thus $|s_r - \frac{1}{e}| \leq (\gamma_r - 1)/e$, which converges to 0 uniformly.

We have proven (4.50). Next, we analyse the equation $G_r(s) = \alpha\theta$ in order to prove (4.51). As before, we consider two cases, depending on the value of α:

1. Suppose first that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_1 < 1$. To find a limit for θ_r, we find a limit for $G_r(s_r)$ by integrating the equation

$$sG_r'(s) = W(s) - \Delta_r(s). \tag{4.61}$$

Using the identity $We^{-W} = s$ and its derivative $W'(1 - W)e^{-W} = 1$ with respect to s, we obtain the differential equation

$$G_r'(s) = W'(1-W) - \frac{\Delta_r(s)}{s}, \tag{4.62}$$

which has the solution

$$G_r(s) = W - \tfrac{1}{2}W^2 - \int_0^s \tfrac{\Delta_r(t)}{t} dt. \tag{4.63}$$

Plugging in the equations $G_r(s_r) = \alpha\theta_r$ and $W(s_r) = \alpha + \Delta_r(s_r)$ yields

$$\theta_r = 1 - \tfrac{1}{2}\alpha + \tfrac{1}{\alpha}\left(\Delta_r(s_r) - \alpha\Delta_r(s_r) - \tfrac{1}{2}\Delta_r(s_r)^2 - \int_0^{s_r} \tfrac{\Delta_r(t)}{t} dt\right). \tag{4.64}$$

As α is bounded away from 0, s_r is bounded below $1/e$ for r sufficiently large, and $\Delta_r(t)/t$ converges to 0 as $r \to \infty$, the error term in (4.64) converges to 0 uniformly.

2. Next, suppose that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_0 \geq 1$, and recall that in this case, s_r converges to e^{-1} uniformly. By the triangle inequality, we have

$$|G_r(s_r) - \tfrac{1}{2}| \leq |G_r(s_r) - G_r(e^{-1})| + |G_r(e^{-1}) - \tfrac{1}{2}|. \tag{4.65}$$

The second term converges to 0 independently of α. For the first, we apply the Mean Value Theorem to obtain

$$|G_r(s_r) - G_r(e^{-1})| \leq G_r'(c)|s_r - e^{-1}| \tag{4.66}$$

for some $c \in (e^{-1}, s_r)$. Noting that $G_r'(s)$ is increasing and $G_r'(s_r) = \alpha/s_r$ is bounded, we deduce that the first term must also converge to 0 uniformly. The result follows after substituting $G_r(s_r) = \alpha\theta_r$.

Thus the limit (4.51) holds. Finally, we observe that

$$\Theta_r(s_r, \theta_r, \alpha) = \theta_r - \log s_r. \tag{4.67}$$

The limit (4.52) then follows from the limits (4.50) and (4.51). □

In the above Lemma, we computed the supremum of the function Θ_r over the entire interval $\theta \in [1/r, 1]$. However, the supremums in the bounds (4.47) and (4.48) are taken over discrete subsets of this interval. We claim that the supremums over these sets coincide in the limit as $n \to \infty$. To this end, define $\theta_{r,n} = \frac{1}{n}\lfloor \theta_r n \rfloor$ and let $s_{r,n}$ be the number s satisfying

$$\Theta_r(s_{r,n}, \theta_{r,n}, \alpha) = \inf_{s>0} \Theta_r(s, \theta_{r,n}, \alpha). \tag{4.68}$$

It will be sufficient to prove that the pair $(s_{r,n}, \theta_{r,n})$ converges to the pair (s_r, θ_r) as $n \to \infty$:

Lemma 4.2.3. *Fix $\alpha > 0$ and $r \geq 2$. Then*

$$\lim_{n\to\infty} \Theta_r(s_{r,n}, \theta_{r,n}, \alpha) = \Theta_r(s_r, \theta_r, \alpha). \tag{4.69}$$

Moreover, convergence is uniform for α in compact subsets of $(0, \infty) \setminus \{1\}$.

To prove Lemma 4.2.3, we will need the following proposition:

Proposition 4.2.4. *Let $P(x) = \sum_{r=0}^{n} a_r x^r$ be a polynomial with non-negative coefficients, and consider the function*

$$Q(x) = \frac{P(x)}{xP'(x)}. \tag{4.70}$$

Then for $x > 0$, $Q(x)$ is decreasing.

Proof. Differentiation yields

$$Q'(x) = \frac{1}{x^2 P'(x)^2}\Big(xP'(x)^2 - P(x)P'(x) - xP(x)P''(x)\Big). \tag{4.71}$$

It will be sufficient to show that

$$x^3 P'(x)^2 Q'(x) = \sum_{s=0}^{n}\sum_{t=0}^{n}[sta_s a_t - ta_s a_t - t(t-1)a_s a_t]x^{s+t} \tag{4.72}$$

is negative. Writing $m = s+t$ (and noting that the sum is the same when interchanging the order of summation of s and t) this may be rewritten as

$$\begin{aligned} x^3 P'(x)^2 Q(x) &= \tfrac{1}{2}\sum_{m=0}^{2n}\sum_{r=0}^{n}[2r(m-r) - m - r(r-1) - (m-r)(m-r-1)]a_r a_{m-r} x^m \\ &= -\tfrac{1}{2}\sum_{m=0}^{2n}\sum_{r=0}^{n}(m-2r)^2 a_r a_{m-r} x^m \end{aligned} \tag{4.73}$$

where we have set $a_l = 0$ for any $l < 0$. This is negative, as required. □

Proof of Lemma 4.2.3. Recall that

$$\Theta_r(s,\theta,\alpha) = -\theta\log\alpha - \theta\log\theta + \theta + \theta\log G_r(s) - \log s. \tag{4.74}$$

Using the triangle inequality, we may bound the difference between $\Theta_r(s_{r,n},\theta_{r,n},\alpha)$ and $\Theta_r(s_r,\theta_r,\alpha)$ by the sum of the differences of the terms in (4.74). If we can show that each of these differences converges to 0, we will be done:

1. First, consider the term $|1-\log\alpha||\theta_r - \theta_{r,n}|$, and note that $|\theta_r - \theta_{r,n}| \le 1/n$ by definition. As the function $1-\log\alpha$ is uniformly bounded on compact subsets of $\alpha \in (0,\infty)\setminus\{1\}$, this term converges uniformly to 0 as $n\to\infty$.

2. Next, consider the term $|\theta_r\log\theta_r - \theta_{r,n}\log\theta_{r,n}|$. By the triangle inequality, we have

$$|\theta_r\log\theta_r - \theta_{r,n}\log\theta_{r,n}| \le |\theta_r(\log\theta_r - \log\theta_{r,n})| + |(\theta_r - \theta_{r,n})\log\theta_{r,n}|, \tag{4.75}$$

   where both of the terms in the right hand side converge uniformly to 0 by uniform continuity of the logarithm away from 0.

3. Next, consider the term $|\log s_r - \log s_{r,n}|$. Recall that, at a stationary point, θ is given by

$$\theta = \frac{G_r(s)}{sG_r'(s)}. \tag{4.76}$$

   By Proposition 4.2.4, θ is decreasing when viewed as a function of s. Fix $a<b$ such that, on our chosen compact subset of $\alpha\in(0,\infty)\setminus\{1\}$,

$$\theta(b) < \theta_{r,n} < \theta_r < \theta(a). \tag{4.77}$$

   On $[a,b]$, $\theta(s)$ is continuous and injective. In particular, it has a uniformly continuous inverse $s = h(\theta)$ on the compact set $[\theta(b),\theta(a)]$, and so $s_r - s_{r,n}$ converges uniformly to 0. As the logarithm is uniformly continuous on intervals bounded away from 0, the same holds for $\log s_r - \log s_{r,n}$.

4. Finally, consider the term $|\theta_r\log G_r(s_r) - \theta_{r,n}\log G_r(s_{r,n})|$. By the triangle inequality, we have

$$\begin{aligned}|\theta_r\log G_r(s_r) - \theta_{r,n}\log G_r(s_{r,n})| \le & |\theta_r\log G_r(s_r) - \theta_r\log G_r(s_{r,n})| \\ & + |\theta_r\log G_r(s_{r,n}) - \theta_{r,n}\log G_r(s_{r,n})|. \end{aligned} \tag{4.78}$$

The second term converges to 0 as $n \to \infty$ by continuity of G_r. For the first term, we observe that

$$\log G_r(s_{r,n}) - \log G_r(s_r) = \int_{s_r}^{s_{r,n}} \frac{G_r'(t)}{G_r(t)} dt \leq r \int_{s_r}^{s_{r,n}} t\ dt \leq r s_{r,n}(s_{r,n} - s_r). \quad (4.79)$$

For n sufficiently large, $s_{r,n}$ is uniformly bounded and so this converges to uniformly.

□

We have established that θ_r converges to a limit uniformly for λ belonging to compact subsets of $(0, \infty) \setminus \{q\}$. Moreover, this limit belongs to a compact subset of $(0, 1)$. In particular, one may choose $\eta > 0$ sufficiently small and $r > 0$, $n > 0$ sufficiently large such that $\theta_{r,n} n$ belongs to the interval $[\frac{1}{r}(1+\eta)n, (1-\eta)n]$. Then, we may apply Proposition 4.2.1 to the bounds of equations (4.47) and (4.48) to obtain

$$Z^{\mathrm{RC}}_{n,\lambda,q}[F \cap B_r] = e^{o(n)} \exp\{n(-1 - \tfrac{1}{2}\lambda + \log\lambda + \Theta_r(s_{r,n}, \theta_{r,n}, \lambda/q))\}, \quad (4.80)$$

where the $o(n)$ term is bounded uniformly for λ in compact subsets of $(0, \infty) \setminus \{q\}$.

Proof of Theorem 3.2.6. From (4.80), we have that

$$\begin{aligned} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[F \cap B_r] &= \Theta_r(s_{r,n}, \theta_{r,n}, \lambda/q) - \Theta_r(s_r, \theta_r, \lambda/q) \\ &\quad + \Theta_r(s_r, \theta_r, \lambda/q) - 1 - \tfrac{1}{2}\lambda + \log\lambda \\ &\quad + \tfrac{o(n)}{n}. \end{aligned} \quad (4.81)$$

As the event B_n holds for every possible graph, one may set $r = n$, take the limit as $n \to \infty$, and apply Lemmas 4.2.3 and 4.2.2 to obtain an upper bound of

$$\limsup_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}\big[F\big] \leq \Psi(\tfrac{\lambda}{q}) - (\tfrac{q-1}{2q})\lambda + \log q. \quad (4.82)$$

Conversely, for fixed $r \geq 2$, we may apply the inclusion $F \supset F \cap B_r$ and take the limit as $n \to \infty$ to obtain the lower bound

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}\big[F\big] &\geq \liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}\big[F \cap B_r\big] \\ &= \Theta_r(s_r, \theta_r, \lambda/q) - 1 - \tfrac{1}{2}\lambda + \log\lambda, \end{aligned}$$

which converges to $\Psi(\frac{\lambda}{q}) - (\frac{q-1}{2q})\lambda + \log q$ in the limit as $r \to \infty$ by Lemma 4.2.2. □

4.3 Correspondence between acyclic subgraphs and small components

In this section, we prove Theorem 3.2.5. In particular, we show that for the random cluster measure $\phi_{n,\lambda,q}$, the exponential rates of the event F that $K_n(\omega)$ is acyclic and the event B_r that $K_n(\omega)$ contains no components of size greater than r coincide in an appropriate limit. For the percolation measure $\phi_{n,\lambda}$, this was first done in [3, Theorem 2.5]. We need one preliminary lemma, corresponding to [3, Lemma 5.1]:

Lemma 4.3.1 ([3, Lemma 5.1])**.** *Fix $q > 0$ and $\lambda > 0$. Then*

$$Z^{RC}_{n,\lambda,q}[B_r] \leq Z^{RC}_{n,\lambda,q}[F](1 - \tfrac{\lambda}{n})^{-\frac{1}{2}rn}. \tag{4.83}$$

Proof. We follow the proof of [3, Lemma 5.1], showing that

$$\phi_{n,\lambda,q}[B_r] \leq \phi_{n,\lambda,q}[F](1 - \tfrac{\lambda}{n})^{-\frac{1}{2}rn}. \tag{4.84}$$

Given a set of vertices $S \subset \{1, \cdots, n\}$, let C_S denote the restriction of the graph $K_n(\omega)$ to S, and let T be a tree on S. Conditionally on the event $C_S \supset T$, all vertices of S belong to the same component of $K_n(\omega)$. In particular, any edge in $E(S) \setminus T$ is open independently with probability λ/n, and so

$$\frac{\phi_{n,\lambda,q}[C_S = T]}{\phi_{n,\lambda,q}[C_S \supset T]} = (1 - \tfrac{\lambda}{n})^{\binom{|S|}{2} - |S| + 1} \geq (1 - \tfrac{\lambda}{n})^{\frac{1}{2}|S|^2}. \tag{4.85}$$

Let K_S be the event that C_S is connected. Then

$$\phi_{n,\lambda,q}[K_S] \leq \sum_T \phi_{n,\lambda,q}[C_S \supset T] \leq (1 - \tfrac{\lambda}{n})^{-\frac{1}{2}|S|^2} \phi_{n,\lambda,q}[C_S \text{ is a tree}]. \tag{4.86}$$

Now, let F_r denote the event that each component of $K_n(\omega)$ is either acyclic or has size at most r, and note that $B_r \subset F_r$. Let $\{S_j\}$ be a partition of $\{1, \cdots, n\}$ and let $\phi_{n,\lambda,q}[\{S_j\}]$ denote the probability that $\{S_j\}$ are the connected components of $K_n(\omega)$. Conditioning the event F_r on the partition $\{S_j\}$ of connected components, we have

$$\phi_{n,\lambda,q}[F_r] = \sum_{\{S_j\}} \phi_{n,\lambda,q}[\{S_j\}] \phi_{n,\lambda,q}[F_r \mid \{S_j\}]. \tag{4.87}$$

Moreover, conditionally on the partition $\{S_j\}$ of connected components, the states of edges in different components are independent. In particular, we may write

$$
\begin{aligned}
\phi_{n,\lambda,q}[F_r|\{S_j\}] &= \prod_{j:|S_j|>r} \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}] \\
&= \prod_j \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}] \prod_{j:|S_j|\le r} \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}]^{-1} \\
&\le \prod_j \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}] \prod_{j:|S_j|\le r} (1-\tfrac{\lambda}{n})^{-\frac{1}{2}|S_j|^2}
\end{aligned}
$$

where the inequality in the third line is a consequence of (4.85). As $|S_j| < r$ for every factor in the second product and the sum over $|S_j|$ is at most n, we obtain

$$
\phi_{n,\lambda,q}[F_r \mid \{S_j\}] \le \phi_{n,\lambda,q}[F \mid \{S_j\}](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn}. \tag{4.88}
$$

Finally, we see that

$$
\begin{aligned}
\phi_{n,\lambda,q}[B_r] &\le \phi_{n,\lambda,q}[F_r] \\
&= \sum_{\{S_j\}} \phi_{n,\lambda,q}[\{S_j\}]\phi_{n,\lambda,q}[F_r \mid \{S_j\}] \\
&\le \sum_{\{S_j\}} \phi_{n,\lambda,q}[\{S_j\}]\phi_{n,\lambda,q}[F \mid \{S_j\}](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn} \\
&= \phi_{n,\lambda,q}[F](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn}
\end{aligned}
$$

which establishes (4.84). We obtain the claim (4.83) by multiplying both sides of (4.84) by the partition function. □

Proof of Theorem 3.2.5. The theorem is a consequence of the following inequalities:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[F] &\le \lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_r], \\
\lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_r] &\le \lim_{\epsilon\downarrow 0} \liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}], \\
\lim_{r\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_r] &\le \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}], \\
\lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}] &\le \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[F].
\end{aligned} \tag{4.89}
$$

To prove the first inequality, we apply the inclusion $B_r \supset B_r \cap F$ and Theorem 3.2.6 to see that

$$\liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_r] \geq \liminf_{n\to\infty} \frac{1}{n} Z^{\mathrm{RC}}_{n,\lambda,q}[B_r \cap F] \to \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[F] \tag{4.90}$$

as $r \to \infty$. To prove the second inequality, fix $r \geq 2$, $\epsilon > 0$, and let $N = \lceil r/\epsilon \rceil$. Then, for every $n \geq N$, we have $\epsilon n \geq \epsilon \lceil r/\epsilon \rceil \geq r$. As a result, $B_{\epsilon n} \supset B_r$, and

$$\frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_r] \leq \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}]. \tag{4.91}$$

Taking the limit inferior as $n \to \infty$ on both sides of (4.91) yields

$$\liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_r] \leq \liminf_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}]. \tag{4.92}$$

As r and ϵ were arbitrary, we may take the limits as $r \to \infty$ and $\epsilon \downarrow 0$ to obtain the second inequality. The proof of the third inequality is similar. To prove the fourth inequality, we apply Lemma 4.3.1 to obtain

$$\frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}] \leq \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[F] + \tfrac{1}{2}\lambda\epsilon, \tag{4.93}$$

from which the inequality follows after taking the limit superior as $n \to \infty$ and the limit as $\epsilon \downarrow 0$. □

Chapter 5

The complete graph II: The largest component

In this chapter, we conclude the analysis of the random cluster model on the complete graph which began in Section 3.2. Throughout, we will adopt the notation used in Section 3.2, writing $\phi_{n,\lambda,q}$ for the random cluster measure on K_n with edge weight $p = \lambda/n$ and cluster weight q. If $q = 1$, we use the standard notation $\phi_{n,\lambda}$ for the percolation measure.

First, we prove Lemma 3.2.2 in Section 5.1, which establishes that $K_n(\omega)$ has at most one component of order n under the random cluster measure $\phi_{n,\lambda,q}$ asymptotically almost surely. This will be done by estimating the probability of the event $K_{\epsilon,2}$ that $K_n(\omega)$ is connected or has exactly two connected components each of size at least ϵn, as was done for the case $q = 1$ in [3].

Next, we prove Theorem 3.2.7 in Section 5.2, which gives the rate function for large deviations of the size of the largest component of the graph $K_n(\omega)$ under the random cluster measure $\phi_{n,\lambda,q}$. Following the arguments given for the case $q = 1$ in [3], this will be done by conditioning on the set A of vertices contained in the largest component and combining the exponential rates of the events that A is connected and that A^c does not contain any large components.

Finally, we prove Theorem 3.2.8 in Section 5.3, which states that the limit of the free energy is equal to the maximum of the rate function computed in Theorem 3.2.7. In order to do this, we will decompose the partition function according to the size of the largest component, apply Theorem 3.2.7 to each term, and apply the Laplace Principle in the limit. Crucially, this will depend on a law of large numbers for the percolation measure $\phi_{n,\lambda}$, taken from [26], to handle some estimates of tail probabilities. By computing this maximum, we then recover [6, Theorem 2.6].

5.1 Uniqueness of the large component

In this section, we prove Lemma 3.2.2, which states that the graph $K_n(\omega)$ contains at most one component of order n asymptotically almost surely under the random cluster measure $\phi_{n,\lambda,q}$. In fact, it has already been shown in [6, Lemma 3.2] that the second largest component is of order at most $n^{3/4}$. Rather than citing this result, we will extend [3, Lemma 6.2], which may be applied more directly when computing rate functions in Section 5.2.

First, we prove an analogue of [3, Lemma 6.1], which estimates the probability of the event $K_{\epsilon,2}$ that $K_n(\omega)$ is connected or has exactly two connected components each of size at least ϵn:

Lemma 5.1.1 ([3, Lemma 6.1])**.** *Fix $q > 0$. Then for all $\lambda_0 > 0$ and $\epsilon_0 > 0$ there exists a constant $c_1 = c_1(\lambda_0, \epsilon_0) > 0$ such that for all $\epsilon \geq \epsilon_0$ and $\lambda \leq \lambda_0$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \phi_{n,\lambda,q}[K^c | K_{\epsilon,2}] < -c_1. \tag{5.1}$$

Proof. Observe that

$$\phi_{n,\lambda,q}[K^c | K_{\epsilon,2}] = \frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K] + \phi_{n,\lambda,q}[K]} \leq \frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]}. \tag{5.2}$$

In particular, it will suffice to show that the ratio on the right hand side of (5.2) decays to zero exponentially in n, with a rate that is uniformly bounded in $\epsilon \geq \epsilon_0$ and $\lambda \leq \lambda_0$. Observe that $\omega \in K_{\epsilon,2} \setminus K$ if and only if we may find a set $A \subset K_n$ (where A depends on ω) of vertices of size between ϵn and $n - \epsilon n$ such that A, A^c are connected components of $K_n(\omega)$ and there are no open edges between them. We count the configurations satisfying these conditions. Let $E(A, A^c)$ be the set of open edges between A and its complement in $K_n(\omega)$, and suppose that $|A| = k$. By Proposition 3.2.3, A is disconnected from A^c in $K_n(\omega)$ with probability

$$\phi_{n,\lambda,q}[E(A, A^c) = \emptyset] = \frac{Z^{\mathrm{RC}}_{k,\lambda k/n,q} Z^{\mathrm{RC}}_{n-k,\lambda(1-k/n),q}}{Z^{\mathrm{RC}}_{n,\lambda,q}} (1 - \lambda/n)^{k(n-k)}. \tag{5.3}$$

Conditionally on the event $\{E(A, A^c) = \emptyset\}$, A is connected in $K_n(\omega)$ with probability

$$\phi_{k,\lambda k/n,q}[K] = Z^{\mathrm{RC}}_{k,\lambda k/n,q}[K] / Z^{\mathrm{RC}}_{k,\lambda k/n,q}, \tag{5.4}$$

and A^c is connected in $K_n(\omega)$ with probability

$$\phi_{n-k,\lambda(1-k/n),q}[K] = Z^{\mathrm{RC}}_{n-k,\lambda(1-k/n),q}[K]/Z^{\mathrm{RC}}_{n-k,\lambda(1-k/n),q}. \tag{5.5}$$

Note that there are $\binom{n}{k}$ choices for A, and that we have counted any pair (A, A^c) twice. Thus, we have the equation

$$Z^{\mathrm{RC}}_{n,\lambda,q}[K_{\epsilon,2} \setminus K] = \frac{1}{2} \sum_{\epsilon n \leq k \leq n-\epsilon n} \binom{n}{k} \left(1 - \frac{\lambda}{n}\right)^{k(n-k)} Z^{\mathrm{RC}}_{k,\lambda k/n,q}[K] Z^{\mathrm{RC}}_{n-k,\lambda(1-k/n),q}[K]. \tag{5.6}$$

By Theorem 3.2.4, we know that

$$Z^{\mathrm{RC}}_{n,\lambda,q}[K] = e^{o(n)} \pi_1(\lambda)^n \tag{5.7}$$

where $\pi_1(\lambda) = 1 - e^{-\lambda}$ and the $o(n)$ term is bounded uniformly for λ belonging to compact subsets of $[0, \infty)$. By applying (5.7) to (5.6), we may write

$$\frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]} = e^{o(n)} \sum_{\epsilon n \leq k \leq n-\epsilon n} \binom{n}{k} \frac{\pi_1(\lambda \frac{k}{n})^k \pi_1(\lambda(1-\frac{k}{n}))^{n-k}}{\pi_1(\lambda)^n} \left(1 - \frac{\lambda}{n}\right)^{k(n-k)}. \tag{5.8}$$

Next, we estimate the summand of (5.8). Stirling's Formula may be used to estimate the binomial coefficient (as in (3.28)) as

$$\binom{n}{k} = e^{o(n)} e^{nS\left(\frac{k}{n}\right)}. \tag{5.9}$$

Similarly, we may estimate

$$(1 - \tfrac{\lambda}{n})^{k(n-k)} = e^{o(n)} e^{-\lambda \frac{k}{n}\left(1-\frac{k}{n}\right)n}. \tag{5.10}$$

This allows us to rewrite (5.8) as

$$\frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]} = e^{o(n)} \sum_{\epsilon n \leq k \leq n-\epsilon n} e^{n[\Xi(k/n) - \Xi(0)]}, \tag{5.11}$$

where the function Ξ is defined by

$$\Xi(\theta) = S(\theta) + \theta \log \pi_1(\lambda\theta) + (1-\theta)\log \pi_1(\lambda(1-\theta)) - \lambda\theta(1-\theta). \tag{5.12}$$

We now bound the sum in (5.11) by n times its maximal summand. As Ξ is convex and symmetric around the point 1/2, the summand is maximised at the endpoints, leading to the bound

$$\frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]} \leq e^{o(n)} e^{n[\Xi(\epsilon)-\Xi(0)]}. \tag{5.13}$$

More explicitly, we may take any value $c_1 < \Xi(0) - \Xi(\epsilon)$ provided that n is sufficiently large. □

We may use Lemma 5.1.1 to prove Lemma 3.2.2 as follows:

Proof of Lemma 3.2.2. The proof is identical to that of [3, Lemma 6.2]. In particular, it will suffice to prove that

$$\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor,\ \mathcal{N}_{>\epsilon n} > 1] \leq e^{-cn} \phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor]. \tag{5.14}$$

For a given vertex x, let $\mathcal{C}_x$ denote the component of $K_n(\omega)$ containing x. On the event $\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{\epsilon n} > 1\}$, we may find a pair of vertices $x, y \in [n]$ in $K_n(\omega)$ such that $|\mathcal{C}_x| \geq \epsilon n$, $|\mathcal{C}_y| \geq \epsilon n$ and $x \not\leftrightarrow y$. Define the following two events for the random graph:

$$\begin{aligned} A_1 &= \{|\mathcal{C}_x| \geq \epsilon n\} \cap \{|\mathcal{C}_y| \geq \epsilon n\} \cap \{x \not\leftrightarrow y\}, \\ A_2 &= \{|\mathcal{C}_x| \geq \epsilon n\} \cap \{|\mathcal{C}_y| \geq \epsilon n\}. \end{aligned}$$

As the complete graph is transitive, the probabilities of the events A_1 and A_2 do not depend on the particular choices of x and y. By the union bound, it follows that

$$\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{>\epsilon n} > 1] \leq n^2 \phi_{n,\lambda,q}[\{|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_1]. \tag{5.15}$$

We now condition further on the set $\mathcal{C}_x \cup \mathcal{C}_y$. For a given set $\mathcal{C} \subset [n]$, let D be the event that $\mathcal{C}$ is disconnected from $\mathcal{C}^c$ in $K_n(\omega)$ and that $\mathcal{C}^c$ contains $\lfloor \theta n \rfloor - |\mathcal{C}|$ vertices in components of size at least ϵn. Then

$$\begin{aligned} \phi_{n,\lambda,q}[\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_1] &= \sum_{\mathcal{C} \subset [n]} \phi_{n,\lambda,q}[A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \cap D], \\ &= \sum_{\mathcal{C} \subset [n]} \phi_{n,\lambda,q}[A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \mid D] \phi_{n,\lambda,q}[D]. \end{aligned}$$

Write $m = |\mathcal{C}|$, $\tilde{\lambda} = \lambda\theta$, and $\tilde{\epsilon} = \epsilon/\theta$. On the event D, the measure $\phi_{n,\lambda,q}$ restricts to the measure $\phi_{\theta n,\tilde{\lambda},q}$ on $\mathcal{C}$. Moreover, we have the following correspondences between events:

$$A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} = K^c \cap K_{\tilde{\epsilon},2},$$
$$A_2 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} = K_{\tilde{\epsilon},2}.$$

As $\tilde{\epsilon} \geq \epsilon$ for every $\theta > 0$, we may apply Lemma 5.1.1 to deduce that

$$\phi_{n,\lambda,q}[A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \mid D] \leq e^{-c_1 m} \phi_{n,\lambda,q}[A_2 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \mid D], \tag{5.16}$$

which allows us to rewrite (5.15) as

$$\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{>\epsilon n} > 1] \leq n^2 e^{-c_1 m} \phi_{n,\lambda,q}[\{|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_2]. \tag{5.17}$$

The result follows upon noting the inclusion $\{|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_2 \subset \{|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor\}$ and that $m \geq \epsilon n$. $\square$

5.2 Rate function for the largest component

In this section, we prove Theorem 3.2.7. In particular, we compute the rate function for the size of the largest connected component of $K_n(\omega)$ under the measure $\phi_{n,\lambda,q}$. For the measure $\phi_{n,\lambda}$, this was first done in [3, Theorem 2.1].

Proof of Theorem 3.2.7. By Lemma 3.2.2, it is sufficient to prove that

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \lfloor \theta n \rfloor,\ \mathcal{N}_{>\epsilon n} = 1] = \Phi(\theta, \lambda, q). \tag{5.18}$$

The case $\theta = 1$ reduces to Theorem 3.2.4 and the case $\theta = 0$ reduces to Theorems 3.2.6 and 3.2.5. Let $\theta \in (0,1)$, $\epsilon \in (0,\theta)$ and assume that θn is an integer. Given a configuration ω, observe that $\omega \in \{|\mathcal{V}_{\epsilon n}| = \theta n, N_{\epsilon n} = 1\}$ if and only if we may find a subset $A \subset K_n$ of vertices (where A depends on ω) of size θn such that A is a connected component of $K_n(\omega)$, A^c contains no connected components of $K_n(\omega)$ of size exceeding ϵn, and $E(A, A^c) = \emptyset$. We count the possible configurations which satisfy these conditions. Note that there are $\binom{n}{\theta n}$ possible choices for A and that by Proposition 3.2.3, A is disconnected from A^c in $K_n(\omega)$ with probability

$$\phi_{n,\lambda,q}[E(A, A^c) = \emptyset] = \frac{Z^{\mathrm{RC}}_{\theta n,\lambda\theta,q} Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}}{Z^{\mathrm{RC}}_{n,\lambda,q}} (1 - \lambda/n)^{\theta(1-\theta)n^2}. \tag{5.19}$$

Conditionally on this event, A is connected in $K_n(\omega)$ with probability

$$\phi_{\theta n,\lambda\theta,q}[K] = Z^{\mathrm{RC}}_{\theta n,\lambda\theta,q}[K]/Z^{\mathrm{RC}}_{\theta n,\lambda\theta,q}, \tag{5.20}$$

and A^c does not contain any components of size exceeding ϵn in $K_n(\omega)$ with probability

$$\phi_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}] = Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}]/Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}. \tag{5.21}$$

Thus, we have the equation

$$Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = \theta n,\ N_{>\epsilon n} = 1] = \binom{n}{\theta n}(1-\tfrac{\lambda}{n})^{\theta n(1-\theta)n} Z^{\mathrm{RC}}_{\theta n,\lambda\theta,q}[K] Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}]. \tag{5.22}$$

We now take logarithms, divide by n and take the limit of each term as $n \to \infty$. For the first term, we apply Stirling's Formula to the factorials in the binomial coefficient (as in (3.28)) to obtain the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \binom{n}{\theta n} = S(\theta). \tag{5.23}$$

Similarly, the second term has the limit

$$\lim_{n\to\infty} \frac{1}{n} \log(1-\lambda/n)^{\theta n(1-\theta)n} = (1-\theta)\log[1-\pi_1(\lambda\theta)]. \tag{5.24}$$

For the third term, we apply Theorem 3.2.4, yielding the limit

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{\theta n,\lambda\theta,q}[K] = \theta \log \pi_1(\lambda\theta). \tag{5.25}$$

In order to take the limit of the final term, we first apply Theorem 3.2.5 to see that

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}] = \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}[F]. \tag{5.26}$$

By Theorem 3.2.6, the limit on the right hand side of (5.26) is equal to

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{(1-\theta)n,\lambda(1-\theta),q}[F] =&(1-\theta)\times \\ &\Big\{\Psi(\tfrac{\lambda(1-\theta)}{q}) - (\tfrac{q-1}{2q})\lambda(1-\theta) + \log q\Big\}.\end{aligned} \tag{5.27}$$

Summing these limits yields the result. Convergence is uniform as each of the individual limits converges uniformly. □

5.3 Free energy of the random cluster model

In this section, we prove Theorem 3.2.8, which yields the limit of the free energy of the measure $\phi_{n,\lambda,q}$ as $n \to \infty$ and extends the theorems of the preceding two chapters to the normalised random cluster measure. The proof we provide is new.

To begin, we let $\epsilon > 0$. Then, we may decompose the partition function as

$$Z^{\mathrm{RC}}_{n,\lambda,q} = Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}] + \sum_{k>\epsilon n} Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = k]. \tag{5.28}$$

By Lemma 3.2.2, we may write

$$Z^{\mathrm{RC}}_{n,\lambda,q} = Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}] + (1 - o(1)) \sum_{k>\epsilon n} Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = k,\ \mathcal{N}_{>\epsilon n} = 1]. \tag{5.29}$$

We aim to apply Theorem 3.2.7 to each summand in (5.29). Recall that

$$Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = k,\ N_{>\epsilon n} = 1] = \binom{n}{k} (1 - \tfrac{\lambda}{n})^{k(n-k)} Z^{\mathrm{RC}}_{k,\lambda k/n,q}[K] Z^{\mathrm{RC}}_{n-k,\lambda(1-k/n),q}[B_{\epsilon n}]. \tag{5.30}$$

In order to apply Theorem 3.2.7 to all of the summands in (5.29) simultaneously, we require that the quantity $\lambda(1 - k/n)$ belongs to a compact subset of $(0, \infty) \setminus \{q\}$. It will suffice to prove that the terms for which k/n is close to 1 or $1 - q/\lambda$ have negligible probability, which we do via the following two tail inequalities for sufficiently small ϵ:

$$\begin{aligned} Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] &\leq o(1) Z^{\mathrm{RC}}_{n,\lambda,q}, \\ Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| \leq (1 - q/\lambda + \epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] &\leq o(1) Z^{\mathrm{RC}}_{n,\lambda,q}. \end{aligned} \tag{5.31}$$

Equivalently, we show that

$$\begin{aligned} \phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] &\leq o(1), \\ \phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| \leq (1 - q/\lambda + \epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] &\leq o(1). \end{aligned} \tag{5.32}$$

Both inequalities in (5.32) may be proven via direct comparisons with percolation, using the following law of large numbers for the size of the largest component of $K_n(\omega)$ under the percolation measure $\phi_{n,\lambda}$:

Theorem 5.3.1 ([26, Theorem 4.8]). *Fix $\lambda > 1$ and let $p = \lambda/n$. Then, for every $\nu \in (\frac{1}{2}, 1)$ there exists $\delta = \delta(\lambda, \nu) > 0$ such that*

$$\phi_{n,\lambda}[||\mathcal{C}_1| - \theta(\lambda, 1)n| \geq n^{\nu}] = O(n^{-\delta}) \tag{5.33}$$

where $\theta(\lambda, q)$ was defined in (3.3) and is equal to the largest solution of the equation $e^{-\lambda\theta} = 1 - \theta$ for $q = 1$.

We begin with the first inequality of (5.32). By Proposition 2.1.9, the random cluster measure $\phi_{n,\lambda,q}$ is stochastically dominated by the percolation measure $\phi_{n,\lambda}$ for $q \geq 1$, yielding the upper bound

$$\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] \leq \phi_{n,\lambda}[|\mathcal{V}_{>\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] \tag{5.34}$$

As the percolation measure is stochastically ordered in the edge weight p, we may assume that $\lambda > 1$, in which case the claim follows by Theorem 5.3.1 provided that $\epsilon < 1 - \theta(\lambda, 1)$.

We now turn to the second inequality of (5.32), noting first that it is only relevant if $\lambda > q$. If this is the case, then the random cluster measure $\phi_{n,\lambda,q}$ stochastically dominates the supercritical percolation measure $\phi_{n,\lambda/q}$ by Proposition 2.1.9. As a result, Theorem 5.3.1 may be applied to show that

$$\phi_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| \leq (\theta(\lambda/q, 1) - \epsilon)n,\ \mathcal{N}_{>\epsilon n} = 1] = O(n^{-\delta}). \tag{5.35}$$

Write $\alpha = \lambda/q$, as before. We claim that $\theta(\alpha, 1) > 1 - \alpha^{-1}$. As $\theta(\alpha, 1)$ solves the equation

$$\alpha = -\frac{1}{\theta(\alpha,1)} \log(1 - \theta(\alpha, 1)) \tag{5.36}$$

it will be sufficient to prove that

$$-\frac{1}{\theta(\alpha,1)} \log(1 - \theta(\alpha, 1)) < (1 - \theta(\alpha, 1))^{-1} \tag{5.37}$$

which is a consequence of the inequality $(1 - x)\log(1 - x) + x > 0$ for $x \in (0, 1)$. In particular, the claim follows by Theorem 5.3.1 provided that $\epsilon < \theta(\lambda/q, 1) - 1 + q/\lambda$. Applying both inequalities of (5.31) to (5.29), we see that

$$(1 - o(1))Z^{\mathrm{RC}}_{n,\lambda,q} = Z^{\mathrm{RC}}_{n,\lambda,q}[B_{\epsilon n}] + \sum_{k > \max\{\epsilon, \theta(\lambda/q,1)-\epsilon\}n}^{(1-\epsilon)n} Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = k,\ \mathcal{N}_{>\epsilon n} = 1]. \tag{5.38}$$

We are now in a position to prove Theorem 3.2.8:

Proof of Theorem 3.2.8. By Theorem 3.2.7, we know that

$$\lim_{\epsilon \downarrow 0} \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{RC}}_{n,\lambda,q}[|\mathcal{V}_{>\epsilon n}| = k,\ \mathcal{N}_{>\epsilon n} = 1] = \Phi(\tfrac{k}{n}, \lambda, q) \tag{5.39}$$

where the function Φ was defined as

$$\begin{aligned}\Phi(\theta,\lambda,q) = S(\theta) - \lambda\theta(1-\theta) + \theta\log\pi_1(\lambda\theta) \\ + (1-\theta)\{\Psi(\tfrac{\lambda(1-\theta)}{q}) - (\tfrac{q-1}{2q})\lambda(1-\theta) + \log q\}.\end{aligned} \tag{5.40}$$

In fact, convergence is uniform for $\frac{k}{n} \in [\max\{\epsilon, \theta(\lambda/q,1)-\epsilon\}, 1-\epsilon]$. In particular, the maximal summand s_n of (5.38) converges to the limit

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\log s_n = \sup_{\theta>\theta(\lambda/q,1)}\Phi(\theta,\lambda,q) \tag{5.41}$$

where we have used continuity of the function Φ and density of the rationals to pass to the continuous supremum. Finally, we may bound the sum in (5.38) between its maximal summand and n times its maximal summand to obtain

$$\frac{1}{n}\log s_n \le \frac{1}{n}\log\left((1-o(1))Z^{\mathrm{RC}}_{n,\lambda,q}\right) \le \frac{1}{n}\log s_n + \frac{1}{n}\log n. \tag{5.42}$$

Taking the limits as $n\to\infty$ and $\epsilon\downarrow 0$ in that order yields the result. □

We have proven that the free energy of the random cluster model converges to the supremum of the function $\Phi(\theta,\lambda,q)$. Finally, we evaluate this supremum. In particular, the following lemma shows that our computation of the free energy agrees with the one found in [6, Theorem 2.6]:

Lemma 5.3.2. *Let $q>0$ and $\lambda>0$. Then*

$$\sup_{\theta\in[0,1]}\Phi(\theta,\lambda,q) = \sup_{\theta>\theta(\lambda/q,1)}\Phi(\theta,\lambda,q) = \frac{g(\theta(\lambda,q))}{2q} - \left(\frac{q-1}{2q}\right)\lambda + \log q \tag{5.43}$$

where the function $g:(0,1)\to\mathbb{R}$ is defined by

$$g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - [2+(q-1)\theta]\log[1+(q-1)\theta]. \tag{5.44}$$

Proof of Lemma 5.3.2. We separate the argument into the two cases $\lambda(1-\theta)>q$ and $\lambda(1-\theta)<q$ corresponding to the regions in which the Ψ function is defined. In addition, we define the shorthand notation

$$a = \frac{1-\theta}{q\theta},\ b = \frac{e^{-\lambda\theta}}{1-e^{-\lambda\theta}},\ k=\lambda\theta. \tag{5.45}$$

When $\lambda(1-\theta) > q$, the derivative of Φ with respect to θ is given by

$$\frac{\partial}{\partial\theta}\Phi(\theta,\lambda,q) = (kb) - \log(kb) - 1. \tag{5.46}$$

As $x - \log x - 1 \geq 0$ with equality if and only if $x = 1$, the derivative in (5.46) is equal to zero only if $kb = 1$. This is equivalent to the equation $1 + \lambda\theta = e^{\lambda\theta}$, for which the only solution is $\theta = 0$. When $\lambda(1-\theta) < q$, we obtain the derivative

$$\frac{\partial}{\partial\theta}\Phi(\theta,\lambda,q) = (\log a - ka) - (\log b - kb). \tag{5.47}$$

The function $\log x - kx$ is convex, with a maximum at $x = \frac{1}{k}$. We know that $a \leq \frac{1}{k}$ by assumption, and $b \leq \frac{1}{k}$ is a consequence of the inequality $1 + \lambda\theta \leq e^{\lambda\theta}$. As a result, the derivative in (5.47) is equal to zero only if $a = b$, which may be rearranged to see that the maximising value θ^* satisfies the equation

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}. \tag{5.48}$$

This is the mean field equation, defined in (3.4). Conversely, any solution θ to the mean-field equation satisfies the assumption (and so is a stationary point), as

$$\frac{\lambda(1-\theta)}{q} = ka = kb \leq 1. \tag{5.49}$$

We may now assume θ^* satisfies the mean-field equation. Under this assumption, one may rewrite $\Psi(\theta^*,\lambda,q)$ in the form

$$\Psi(\theta^*,\lambda,q) = \frac{1}{2q}g(\theta^*) - \frac{q-1}{2q}\lambda + \log q. \tag{5.50}$$

It remains to show that this is maximised when we take the solution $\theta(\lambda,q)$ of the mean field equation. We quote the following properties of the function g from [6]:

$$\begin{aligned} g(0) &= g'(0) = 0, \\ g''(\theta) &= -\frac{q(q-1)[q-2-2(q-1)\theta]\theta}{(1-\theta)^2[1+(q-1)\theta]^2}. \end{aligned} \tag{5.51}$$

For $q \leq 2$, $g(\theta)$ is a convex, increasing function and the result is clear. For $q > 2$, $g(\theta)$ is initially decreasing. Moreover, $g''(\theta)$ has a zero at $\theta = \frac{q-2}{2(q-1)}$, and is increasing thereafter. In particular, $g(\theta)$ is convex for $\theta > \frac{q-2}{2(q-1)}$ and has only one zero in this region, which we may compute as $\theta_c = \frac{q-2}{q-1}$. Note that θ_c is the largest solution to the mean-field equation for $\lambda = \lambda_c$.

We claim that $\theta_{\max}$ is increasing as a function of λ. If $\theta_{\max}(\lambda) = 0$ then this is obvious, so we may assume that $\theta_{\max}(\lambda) > 0$. Let $\epsilon > 0$, and define the function

$$h(\theta) := e^{-(\lambda+\epsilon)\theta} - \frac{1-\theta}{1+(q-1)\theta}. \tag{5.52}$$

Noting that $h(\theta_{\max}(\lambda)) < 0$ and $h(1) > 0$, it follows that h has a zero in the interval $(\theta_{\max}(\lambda), 1)$ i.e. $\theta_{\max}(\lambda + \epsilon) > \theta_{\max}(\lambda)$.

We may now conclude. If $\lambda < \lambda_c$, then $\theta_{\max}(\lambda) < \theta_{\max}(\lambda_c)$ and so $g(\theta_{\max}(\lambda)) < 0$. In particular, $\theta^* = 0$ maximises the free energy. Conversely, if $\lambda > \lambda_c$ then it follows that $g(\theta_{\max}(\lambda)) > 0$. As $g(\theta)$ is convex for $\theta > \frac{1}{2}\theta_c$, it follows that $\theta_{\max}$ is the solution maximising the function $g(\theta)$, and so $\theta^* = \theta_{\max}$. □

Chapter 6

The hypercube I: The Potts model

In this chapter, we analyse the free energy of the Potts model on the hypercube. Throughout, we will adopt the notation used in Section 3.3, writing $\mu_{n,\lambda,q}$ for the measure of the q-state Potts model on Q_n with inverse temperature $\beta = \lambda/n$. Similarly, we will write $\phi_{n,\lambda,q}$ for the random cluster measure on Q_n with edge weight $p = \lambda/n$ and cluster weight q.

The main new result of this chapter is a proof of Theorem 3.3.3, which states that the free energy of the measure $\mu_{n,\lambda,q}$ converges to the *mean-field* limit as $n \to \infty$. Intuitively, this mean-field limit replaces all individual interactions with an average, and is typical for a variety of families of graphs of diverging degree where individual fluctuations are averaged out over a large number of neighbours.

The proof of Theorem 3.3.3 consists of an upper bound and a lower bound on the free energy of the measure $\mu_{n,\lambda,q}$ which coincide in the limit $n \to \infty$. These bounds, found in Sections 6.1 and 6.2 respectively, will be proven by adapting arguments used by Kesten and Schonmann in [30] to show that the free energy of an appropriately rescaled Potts model on the lattice $\mathbb{Z}^d$ converges to a mean-field limit as $d \to \infty$.

In Section 6.3, we will investigate the properties of the mean-field limit in Theorem 3.3.3 using the arguments given in [30, Section 3]. In particular, we will show that the limit is differentiable in Lemma 6.3.1. By expressing the spin correlations in terms of the derivative of the free energy, we then prove that the spin correlations undergo the asymptotic phase transition stated in Lemma 3.3.6. Using the Edwards Sokal coupling, we then deduce Lemma 3.3.7, which will be used to analyse the random cluster model in Chapter 7.

6.1 Lower bound for mean-field convergence of free energy

In this section, we establish a lower bound on the free energy of the Potts measure $\mu_{n,\lambda,q}$ on Q_n in the limit $n \to \infty$. More specifically, we prove the following lemma, which is analogous to [30, Section 2, Lemma 1]:

Lemma 6.1.1. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > 0$. Then*

$$\liminf_{n\to\infty} \frac{1}{2^n} \log Z^P_{n,\lambda,q} \geq \psi(\lambda) \tag{6.1}$$

where the function $\psi(\lambda)$ is defined by

$$\psi(\lambda) = \max_{v\in\mathbb{R}^q} \log \left[\int \lambda_0(d\sigma) \exp\left\{ -\frac{\|v\|^2}{2\lambda} + v^T \cdot \sigma \right\}\right]. \tag{6.2}$$

Given a spin configuration $\sigma \in \Sigma_{Q_n}$, write $H_n(\sigma)$ for the Hamiltonian of the measure $\mu_{n,\lambda,q}$. Observe that

$$\begin{aligned} \tfrac{\lambda}{n} H_n(\sigma) &= -\tfrac{\lambda}{2n} \sum_{x\in Q_n} \sum_{y\in Q_n:\{x,y\}\in E} \sigma_x \cdot \sigma_y \\ &= -\tfrac{\lambda}{2} \sum_{x\in Q_n} \sigma_x \cdot \Bigg(\tfrac{1}{n} \sum_{y\in Q_n:\{x,y\}\in E} \sigma_y \Bigg). \end{aligned} \tag{6.3}$$

In other words, we may interpret the Hamiltonian using an external field whose strength at the vertex x is given by the average of the spins at all neighbouring vertices. The main idea in the proof of Lemma 6.1.1 is a *mean-field* approximation, which asserts that the external field may be replaced in the limit as $n \to \infty$ by a field whose strength at every vertex is given by the average of the spins of all vertices in the graph. If the mean-field approximation holds, then the Hamiltonian takes on a Gaussian form, and the partition function may then be computed using a two-sided Laplace transform.

Proof of Lemma 6.1.1. Let S_{Q_n} denote the group of permutations of the vertices of Q_n. For $\sigma \in \Sigma_{Q_n}$, $\tau \in S_{Q_n}$ and $x \in Q_n$, write $(\tau\sigma)_x = \sigma_{\tau(x)}$. Finally, recall the measure $\nu_{Q_n} := \prod_{x\in Q_n} \lambda_x$, where λ_x is the counting measure on the set $\{v_1, \cdots, v_q\}$ of coordinate vectors of $\mathbb{R}^q$. As the measure ν_{Q_n} is invariant under each τ, we have

$$\begin{aligned} Z^{\mathrm{P}}_{n,\lambda,q} &= \int \nu_{Q_n}(d\sigma) \exp\Big\{ -\tfrac{\lambda}{n} H_n(\sigma) \Big\} \\ &= \int \nu_{Q_n}(d\sigma) \frac{1}{(2^n)!} \sum_{\tau \in S_{Q_n}} \exp\Big\{ -\tfrac{\lambda}{n} H_n(\tau\sigma) \Big\}. \end{aligned} \tag{6.4}$$

In order to bound (6.4), we will use Jensen's inequality (see e.g. [23, Appendix B.8.1]), which says that for any probability measure μ, any μ-integrable random variable X and any convex function φ, we have

$$\varphi(\mu[X]) \leq \mu[\varphi(X)]. \tag{6.5}$$

As the exponential function is convex, we may apply (6.5) to (6.4) to obtain the bound

$$Z^{\mathrm{P}}_{n,\lambda,q} \geq \int \nu_{Q_n}(d\sigma) \exp\Big\{ \frac{1}{(2^n)!} \sum_{\tau \in S_{Q_n}} \Big(-\tfrac{\lambda}{n} H_n(\tau\sigma) \Big) \Big\}. \tag{6.6}$$

Next, we analyse the sum of Hamiltonians in (6.6). It will be useful to rewrite the Hamiltonian $H_n(\sigma)$ of the measure $\mu_{n,\lambda,q}$ as

$$H_n(\sigma) = -\frac{1}{2} \sum_{x \in Q_n} \sum_{y \in Q_n : \{x,y\} \in E} \sigma_x \cdot \sigma_y = -\frac{1}{2} \sum_{x \in Q_n} \sum_{u \in \mathcal{V}} \sigma_x \cdot \sigma_{x+u} \tag{6.7}$$

where $\mathcal{V} = \{v_1, \cdots, v_n\}$ is the set of co-ordinate vectors of $\{0,1\}^n$, and we take addition of vectors periodically on each co-ordinate. Using this restated form, we may write

$$\begin{aligned} \sum_{\tau \in S_{Q_n}} \Big(-\tfrac{\lambda}{n} H_n(\tau\sigma) \Big) &= \tfrac{\lambda}{2n} \sum_{\tau \in S_{Q_n}} \sum_{x \in Q_n} \sum_{u \in \mathcal{V}} \sigma_{\tau(x)} \cdot \sigma_{\tau(x+u)} \\ &= \tfrac{\lambda}{2n} \sum_{x \in Q_n} \sum_{u \in \mathcal{V}} \sum_{y,z \in Q_n} c(x,y,z,u) \sigma_y \cdot \sigma_z \end{aligned} \tag{6.8}$$

where $c(x,y,z,u)$ is the number of permutations $\tau \in S_{Q_n}$ such that $\tau(x) = y$ and $\tau(x+u) = z$. This number is given by

$$c(x,y,z,u) = \begin{cases} 0 & \text{if } y = z \\ (2^n - 2)! & \text{otherwise} \end{cases} \tag{6.9}$$

In particular, $c(x,y,z,u)$ is independent of the choices of $x \in Q_n$ and $u \in \mathcal{V}$. As there are 2^n choices for $x \in Q_n$ and n choices for $u \in \mathcal{V}$, we may rewrite (6.8) as

$$\sum_{\tau \in S_{Q_n}} \Big(-\tfrac{\lambda}{n} H_n(\tau\sigma) \Big) = \frac{\lambda}{2} \frac{(2^n)!}{2^n - 1} \sum_{\substack{y,z \in Q_n \\ y \neq z}} \sigma_y \cdot \sigma_z. \tag{6.10}$$

Observing that

$$\begin{aligned} \Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 &= \Big(\sum_{y \in Q_n} \sigma_y \Big) \cdot \Big(\sum_{z \in Q_n} \sigma_z \Big) \\ &= \sum_{\substack{y,z \in Q_n \\ y \neq z}} \sigma_y \cdot \sigma_z + \sum_{\substack{y,z \in Q_n \\ y = z}} \sigma_y \cdot \sigma_z \end{aligned} \tag{6.11}$$

and that

$$\sum_{\substack{y,z \in Q_n \\ y = z}} \sigma_y \cdot \sigma_z = 2^n \tag{6.12}$$

enables us to rewrite (6.10) as

$$\sum_{\tau \in S_{Q_n}} \Big(-\tfrac{\lambda}{n} H_n(\tau\sigma) \Big) = \frac{\lambda}{2} \frac{(2^n)!}{2^n - 1} \Big\{ \Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 - 2^n \Big\}. \tag{6.13}$$

Using (6.13), we may write

$$\frac{1}{(2^n)!} \sum_{\tau \in S_{Q_n}} \Big(-\tfrac{\lambda}{n} H_n(\tau\sigma) \Big) = \frac{\lambda}{2(2^n)} \Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 + R, \tag{6.14}$$

where the remainder term R in (6.14) is given by

$$R = \frac{\lambda}{2(2^n)(2^n - 1)} \Big(\Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 - (2^n)^2 \Big). \tag{6.15}$$

As $0 \leq \Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 \leq (2^n)^2$, it follows that $-\lambda \leq R \leq 0$. In particular, (6.14) implies that

$$Z^{\mathrm{P}}_{n,\lambda,q} \geq e^{-\lambda} \int \nu_{Q_n}(d\sigma) \exp \Big\{ \frac{\lambda}{2(2^n)} \Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 \Big\}. \tag{6.16}$$

It remains to estimate the integral in (6.16). To do this, we will use the following calculation of the two-sided Laplace transform (or, equivalently, the moment generating function) of a Gaussian distribution: let $r \in \mathbb{N}$, let M be a strictly positive definite symmetric $r \times r$ matrix and let ξ be an r-vector. Then

$$\int_{\mathbb{R}^r} \exp\left\{ -\frac{1}{2} v^T M v + v^T \xi \right\} dv = (2\pi)^{r/2} (\det M)^{-1/2} \exp\left\{ \frac{1}{2} \xi^T M^{-1} \xi \right\}. \tag{6.17}$$

We apply (6.17) to the integrand of (6.16), choosing $r = q$, $\xi = \sum_{y \in Q_n} \sigma_y$ and $M = \frac{2^n}{\lambda} I_q$, to obtain

$$\begin{aligned} \exp\left\{ \frac{\lambda}{2(2^n)} \Big\| \sum_{y \in Q_n} \sigma_y \Big\|^2 \right\} &= (2\pi)^{-\frac{q}{2}} \left(\frac{2^n}{\lambda} \right)^{\frac{q}{2}} \int_{\mathbb{R}^q} \exp\left\{ -\frac{2^n}{2\lambda} \|v\|^2 + v^T \sum_{y \in Q_n} \sigma_y \right\} dv \\ &= (2\pi)^{-\frac{q}{2}} \left(\frac{2^n}{\lambda} \right)^{\frac{q}{2}} \int_{\mathbb{R}^q} \prod_{y \in Q_n} \exp\left\{ -\frac{1}{2\lambda} \|v\|^2 + v^T \sigma_y \right\} dv. \end{aligned} \tag{6.18}$$

This yields the bound

$$Z^{\mathrm{P}}_{n,\lambda,q} \geq C(\lambda, q) 2^{\frac{qn}{2}} \int_{\mathbb{R}^q} dv \int \nu_{Q_n}(d\sigma) \prod_{y \in Q_n} \exp\left\{ -\frac{1}{2\lambda} \|v\|^2 + v^T \sigma_y \right\} \tag{6.19}$$

where the constant $C(\lambda, q)$ is equal to $(2\pi\lambda)^{-\frac{q}{2}} e^{-\lambda}$. As $\nu_{Q_n} := \prod_{y \in Q_n} \lambda_y$, the bound (6.19) becomes

$$\begin{aligned} Z^{\mathrm{P}}_{n,\lambda,q} &\geq C(\lambda, q) 2^{\frac{qn}{2}} \int_{\mathbb{R}^q} dv \prod_{y \in Q_n} \int \lambda_y(d\sigma) \exp\left\{ -\frac{1}{2\lambda} \|v\|^2 + v^T \sigma_y \right\} \\ &= C(\lambda, q) 2^{\frac{qn}{2}} \int_{\mathbb{R}^q} dv \left[\int \lambda_0(d\sigma) \exp\left\{ -\frac{1}{2\lambda} \|v\|^2 + v^T \sigma_0 \right\} \right]^{2^n} \end{aligned} \tag{6.20}$$

where the second line follows by symmetry. Taking logarithms yields

$$\begin{aligned} \frac{1}{2^n} \log Z^{\mathrm{P}}_{n,\lambda,q} \geq & \frac{1}{2^n} \log \int_{\mathbb{R}^q} dv \left[\int \lambda_0(d\sigma) \exp\left\{ -\frac{1}{2\lambda} \|v\|^2 + v^T \sigma_0 \right\} \right]^{2^n} \\ & + \frac{1}{2^n} \log C(\lambda, q) + \frac{1}{2^n} \log 2^{\frac{qn}{2}}. \end{aligned} \tag{6.21}$$

The second line of (6.21) vanishes in the limit as $n \to \infty$. To estimate the first line, we apply a version of Laplace's principle, which says that if $\varphi : \mathbb{R}^d \to \mathbb{R}$ is continuous and

$e^{-\varphi}$ is integrable, then

$$\lim_{\theta\to\infty}\frac{1}{\theta}\log\int_{\mathbb{R}^d} e^{-\theta\varphi(v)}dv = -\min_{v\in\mathbb{R}^d}\varphi(v). \tag{6.22}$$

Applying (6.22) to (6.21) yields the bound

$$\liminf_{n\to\infty}\frac{1}{2^n}\log Z^{\mathrm{P}}_{n,\lambda,q} \geq \max_{v\in\mathbb{R}^q}\log\left[\int\lambda_0(d\sigma)\exp\left\{-\frac{\|v\|^2}{2\lambda}+v^{\mathrm{T}}\cdot\sigma\right\}\right] \tag{6.23}$$

which is equal to $\psi(\lambda)$, as required. □

6.2 Upper bound for mean-field convergence of free energy

In this section, we establish an upper bound on the free energy of the Potts measure $\mu_{n,\lambda,q}$ on the hypercube Q_n in the limit $n\to\infty$. More specifically, we prove the following lemma, which is analogous to [30, Section 2, Lemma 4]:

Lemma 6.2.1. *Fix $q\in\mathbb{N}_{\geq 2}$ and $\lambda>0$. Then*

$$\liminf_{n\to\infty}\frac{1}{2^n}\log Z^P_{n,\lambda,q}\leq\psi(\lambda) \tag{6.24}$$

where the function $\psi(\lambda)$ is defined by

$$\psi(\lambda)=\max_{v\in\mathbb{R}^q}\log\left[\int\lambda_0(d\sigma)\exp\left\{-\frac{\|v\|^2}{2\lambda}+v^T\cdot\sigma\right\}\right]. \tag{6.25}$$

In order to prove Lemma 6.2.1, we introduce a matrix representation of the Hamiltonian H_n of the measure $\mu_{n,\lambda,q}$. Let A_{Q_n} denote the adjacency matrix of the hypercube Q_n, and let $\sigma_{x,j}$ denote the jth component of σ_x in $\mathbb{R}^q$. Then, we may write

$$\begin{aligned} H_n(\sigma) &= -\frac{1}{2}\sum_{x,y\in Q_n}(A_{Q_n})_{x,y}\mathbb{1}_{\{\sigma_x=\sigma_y\}} \\ &= -\frac{1}{2}\sum_{j=1}^{q}\sum_{x,y\in Q_n}\sigma_{x,j}(A_{Q_n})_{x,y}\sigma_{y,j}. \end{aligned} \tag{6.26}$$

We will need the following proposition regarding the eigenvalues of the matrix A_{Q_n}:

Proposition 6.2.2. *The eigenvalues of the matrix A_{Q_n} are given by the set*

$$\{-n+2k : 0\leq k\leq n\}, \tag{6.27}$$

where the eigenvalue $-n+2k$ has multiplicity $\binom{n}{k}$.

Proof. Observe that the matrix A_{Q_n} may be written iteratively as

$$A_{Q_n} = \begin{pmatrix} A_{Q_{n-1}} & I_{Q_{n-1}} \\ I_{Q_{n-1}} & A_{Q_{n-1}} \end{pmatrix} \tag{6.28}$$

where $I_{Q_{n-1}}$ is the 2^{n-1} dimensional identity matrix. In particular, the characteristic equation $P_{A_{Q_n}}(\lambda)$ of the matrix A_{Q_n} is equal to

$$P_{A_{Q_n}}(\lambda) = \det \begin{pmatrix} A_{Q_{n-1}} - \lambda I_{Q_{n-1}} & I_{Q_{n-1}} \\ I_{Q_{n-1}} & A_{Q_{n-1}} - \lambda I_{Q_{n-1}} \end{pmatrix}. \tag{6.29}$$

For commuting $r \times r$ matrices M and N, we have the identity

$$\begin{pmatrix} M & N \\ N & M \end{pmatrix} \begin{pmatrix} M & 0 \\ -N & I_r \end{pmatrix} = \begin{pmatrix} M^2 - N^2 & N \\ 0 & M \end{pmatrix} \tag{6.30}$$

which may be applied with $M = A_{Q_{n-1}} - \lambda I_{Q_{n-1}}$ and $N = I_{Q_{n-1}}$ to obtain the iterative formula

$$P_{A_{Q_n}}(\lambda) = P_{A_{Q_{n-1}}}(\lambda - 1) P_{A_{Q_{n-1}}}(\lambda + 1). \tag{6.31}$$

In particular, each eigenvalue of A_{Q_n} is obtained by choosing an eigenvalue of ± 1 of the matrix A_{Q_1} and then shifting it by ± 1 in each of the $n-1$ successive iterations of (6.31), yielding the set of eigenvalues (6.27) with their required multiplicities. □

Note that A_{Q_n} is a real, symmetric matrix, and is therefore diagonalisable. Consequently, we may find an orthogonal matrix O_{Q_n} and a diagonal matrix D_{Q_n} such that

$$A_{Q_n} = O_{Q_n}^T D_{Q_n} O_{Q_n}. \tag{6.32}$$

Define the matrix $D_{Q_n}^+$ consisting of only the positive entries of D_{Q_n} by

$$(D_{Q_n}^+)_{x,y} = \max\{0, (D_{Q_n})_{x,y}\}. \tag{6.33}$$

As $D_{Q_n}^+ - D_{Q_n}$ is positive definite, it follows that for every spin configuration σ, we have the bound

$$-\frac{\lambda}{n} H_n(\sigma) \leq \frac{\lambda}{2n} \sum_{j=1}^{q} \sum_{x,y \in Q_n} \sigma_{x,j} (O_{Q_n}^T D_{Q_n}^+ O_{Q_n})_{x,y} \sigma_{y,j}. \tag{6.34}$$

In the proof of Lemma 6.2.1, we would like to apply the Laplace transform from (6.17) to the matrix $\frac{1}{n}O_{Q_n}^T D_{Q_n}^+ O_{Q_n}$. However, the matrix $\frac{1}{n}O_{Q_n}^T D_{Q_n}^+ O_{Q_n}$ is not necessarily strictly positive definite. In order to overcome this technical difficulty, we introduce the matrix $K_{Q_n,\epsilon}$ defined by

$$K_{Q_n,\epsilon} = \frac{1}{n}O_{Q_n}^T D_{Q_n}^+ O_{Q_n} + \epsilon I_{Q_n}, \tag{6.35}$$

where I_{Q_n} is the $2^n \times 2^n$ identity matrix. Finally, for $z > 1 + \epsilon$, we introduce the matrix

$$R_{Q_n,z,\epsilon} = K_{Q_n,\epsilon}^{-1} - \frac{1}{z}I_{Q_n}. \tag{6.36}$$

The following proposition, which is analogous to [30, Section 2, Lemma 3], establishes some important properties of the matrices $K_{Q_n,\epsilon}$ and $R_{Q_n,z,\epsilon}$:

Proposition 6.2.3. *Let $\epsilon > 0$ and $z > 1 + \epsilon$. Then the matrices $K_{Q_n,\epsilon}$, $K_{Q_n,\epsilon} - \frac{1}{n}A_{Q_n}$, and $R_{Q_n,z,\epsilon}$ are symmetric and strictly positive definite. Furthermore*

$$\lim_{\epsilon \downarrow 0} \lim_{n\to\infty} \frac{1}{2^n} \log\det\Big(K_{Q_n,\epsilon}R_{Q_n,z,\epsilon}\Big) = 0. \tag{6.37}$$

Proof. Symmetry is clear for all three matrices. Next, observe that all eigenvalues of the matrices $K_{Q_n,\epsilon}$ and $K_{Q_n,\epsilon} - \frac{1}{n}A_{Q_n}$ are bounded below by ϵ. As the determinant of a matrix is the product of its eigenvalues, it follows that the matrices $K_{Q_n,\epsilon}$ and $K_{Q_n,\epsilon} - \frac{1}{n}A_{Q_n}$ are strictly positive definite. For the matrix $R_{Q_n,z,\epsilon}$, it is sufficient to check that the matrix

$$K_{Q_n,\epsilon}R_{Q_n,z,\epsilon} = I_{Q_n} - \tfrac{1}{z}K_{Q_n,\epsilon} \tag{6.38}$$

is strictly positive definite. Recall from Proposition 6.2.2 that the eigenvalues of the hypercube adjacency matrix A_{Q_n} are given by

$$\{-n + 2k : 0 \le k \le n\}, \tag{6.39}$$

where the eigenvalue $-n+2k$ has multiplicity $\binom{n}{k}$. Consequently, the matrix $K_{Q_n,\epsilon}R_{Q_n,z,\epsilon}$ has the eigenvalues

$$\left\{\frac{1}{z}\left(z - \epsilon - \max\left\{0, \frac{-n+2k}{n}\right\}\right) : 0 \le k \le n\right\}, \tag{6.40}$$

where the kth eigenvalue has multiplicity $\binom{n}{k}$, as before. As $\max\{0, \frac{-n+2k}{n}\} \le 1$ for $k \le n$ and $z - \epsilon > 1$, these eigenvalues are all strictly positive and so the matrix $I_{Q_n} - \frac{1}{z}K_{Q_n,\epsilon}$ is strictly positive definite. Finally, we prove the limit (6.37). For notational simplicity, define the function

$$f_{z,\epsilon}(x) = \log\Big(z - \epsilon - \max\{0, x\}\Big) - \log z. \tag{6.41}$$

As the determinant of $K_{Q_n,\epsilon} R_{Q_n,z,\epsilon}$ is the product of its eigenvalues, we may write

$$\frac{1}{2^n} \log\det\Big(K_{Q_n,\epsilon} R_{Q_n,z,\epsilon}\Big) = \sum_{k=0}^{n} \frac{1}{2^n}\binom{n}{k} f_{z,\epsilon}\Big(\frac{k - n/2}{n}\Big). \tag{6.42}$$

The equation (6.42) may be interpreted probabilistically as

$$\frac{1}{2^n} \log\det\Big(K_{Q_n,\epsilon} R_{Q_n,z,\epsilon}\Big) = \mathbb{E}\Big[f_{z,\epsilon}\Big(\frac{X_n - n/2}{n}\Big)\Big] \tag{6.43}$$

where X_n is a Binomial random variable with parameters n and $p = 1/2$. By the Strong Law of Large Numbers, $\frac{X_n - n/2}{n}$ converges to 0 almost surely as $n \to \infty$, and thus it also converges in distribution. Moreover, $f_{z,\epsilon}$ is bounded and continuous in the interval $[-1, 1]$. It follows that

$$\lim_{n\to\infty} \mathbb{E}\Big[f_{z,\epsilon}\Big(\frac{X_n - n/2}{n}\Big)\Big] = \mathbb{E}[f_{z,\epsilon}(0)] = \log\Big(\frac{z-\epsilon}{z}\Big). \tag{6.44}$$

The limit (6.37) follows after taking the limit $\epsilon \downarrow 0$. □

Proof of Lemma 6.2.1. Let $\epsilon > 0$ and $z > 1 + \epsilon$. Fix $j \in \{1, \cdots, q\}$ and view $\sigma_{x,j}$ as the component (in position x) of a vector $\sigma_j \in \mathbb{R}^{Q_n}$. As the matrix $K_{Q_n,\epsilon} - \frac{1}{n} A_{Q_n}$ is positive definite, we have

$$\sum_{x,y\in Q_n} \sigma_{x,j}(K_{Q_n,\epsilon} - \tfrac{1}{n} A_{Q_n})_{x,y}\sigma_{y,j} = \sigma_j^{\mathrm{T}}(K_{Q_n,\epsilon} - \tfrac{1}{n} A_{Q_n})\sigma_j \geq 0. \tag{6.45}$$

In particular, it follows that

$$\begin{aligned} Z^{\mathrm{P}}_{n,\lambda,q} &= \int \nu_{Q_n}(d\sigma) \exp\Big\{\frac{\lambda}{2}\sum_{j=1}^{q}\sum_{x,y\in Q_n} \sigma_{x,j}(\tfrac{1}{n} A_{Q_n})_{x,y}\sigma_{y,j}\Big\} \\ &\leq \int \nu_{Q_n}(d\sigma) \exp\Big\{\frac{\lambda}{2}\sum_{j=1}^{q}\sum_{x,y\in Q_n} \sigma_{x,j}(\tfrac{1}{n} A_{Q_n} + (K_{Q_n,\epsilon} - \tfrac{1}{n} A_{Q_n}))_{x,y}\sigma_{y,j}\Big\} \\ &= \int \nu_{Q_n}(d\sigma) \exp\Big\{\frac{\lambda}{2}\sum_{j=1}^{q}\sum_{x,y\in Q_n} \sigma_{x,j}(K_{Q_n,\epsilon})_{x,y}\sigma_{y,j}\Big\}. \end{aligned} \tag{6.46}$$

Next, we apply the Laplace transform from (6.17) to (6.46) q times, choosing $r = 2^n$, $\xi = \sigma_j$ and $M = (\lambda K_{Q_n,\epsilon})^{-1}$ for each fixed $j = \{1, \cdots, q\}$, to obtain

$$\exp\left\{\frac{\lambda}{2}\sum_{j=1}^{q}\sum_{x,y\in Q_n}\sigma_{x,j}(K_{Q_n,\epsilon})_{x,y}\sigma_{y,j}\right\} = (2\pi)^{-\frac{q2^n}{2}}[\det(\lambda K_{Q_n,\epsilon})]^{-\frac{q}{2}}\int_{\mathbb{R}^{2^n}} d\nu_1 \cdots$$
$$\cdots\int_{\mathbb{R}^{2^n}} d\nu_q \exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^{q} v_j^T K_{Q_n,\epsilon}^{-1} v_j + \sum_{j=1}^{q} v_j^T\sigma_j\right\}. \tag{6.47}$$

Substituting (6.47) into (6.46) and writing $K_{Q_n,\epsilon}^{-1} = R_{Q_n,z,\epsilon} + \frac{1}{z}I_{Q_n}$ yields the bound

$$\begin{aligned}
Z_{n,\lambda,q}^{\mathrm{P}} \leq &(2\pi\lambda)^{-\frac{q2^n}{2}}[\det(K_{Q_n,\epsilon})]^{-\frac{q}{2}}\int_{\mathbb{R}^{2^n}} d\nu_1 \\
&\cdots\int_{\mathbb{R}^{2^n}} d\nu_q \exp\left\{-\frac{1}{2\lambda}\sum_{j=1}^{q} v_j^T R_{Q_n,z,\epsilon} v_j\right\} \\
&\cdots\int \nu_{Q_n}(d\sigma)\exp\left\{-\frac{1}{2z\lambda}\sum_{j=1}^{q}\|v_j\|^2 + \sum_{j=1}^{q} v_j^T\sigma_j\right\}.
\end{aligned} \tag{6.48}$$

Next, we estimate the quantity

$$\begin{aligned}
I &= \int \nu_{Q_n}(d\sigma)\exp\left\{-\frac{1}{2z\lambda}\sum_{j=1}^{q}\|v_j\|^2 + \sum_{j=1}^{q} v_j^T\sigma_j\right\} \\
&= \int \nu_{Q_n}(d\sigma)\exp\left\{-\frac{1}{2z\lambda}\sum_{j=1}^{q}\sum_{x\in Q_n}|v_{j,x}|^2 + \sum_{j=1}^{q}\sum_{x\in Q_n} v_{j,x}^T\sigma_{j,x}\right\}.
\end{aligned} \tag{6.49}$$

As $\nu_{Q_n} = \prod_{x\in Q_n}\lambda_x$ is a product measure, we have

$$I = \prod_{x\in Q_n}\int \lambda_x(d\sigma)\exp\left\{-\frac{1}{2z\lambda}\sum_{j=1}^{q}|v_{j,x}|^2 + \sum_{j=1}^{q} v_{j,x}^T\sigma_{j,x}\right\} \tag{6.50}$$

which has the uniform bound

$$\begin{aligned}
I &\leq \left[\max_{w\in\mathbb{R}^q}\int \lambda_0(d\sigma)\exp\left\{-\frac{1}{2z\lambda}\|w\|^2 + w\cdot\sigma\right\}\right]^{2^n} \\
&= \exp\{2^n\psi(z\lambda)\}.
\end{aligned} \tag{6.51}$$

Substituting (6.51) to (6.48) yields the bound

$$Z^{\mathrm{P}}_{n,\lambda,q} \leq (2\pi\lambda)^{-\frac{q2^n}{2}} [\det(K_{Q_n,\epsilon})]^{-\frac{q}{2}} \exp\{2^n \psi(z\lambda)\} \int_{\mathbb{R}^{2^n}} d\nu_1 \cdots \int_{\mathbb{R}^{2^n}} d\nu_q \exp\left\{ -\frac{1}{2\lambda} \sum_{j=1}^{q} v_j^T R_{Q_n,z,\epsilon} v_j \right\}. \tag{6.52}$$

It remains to estimate the integral

$$I_j = \int_{\mathbb{R}^{2^n}} d\nu_j \exp\left\{ -\frac{1}{2\lambda} v_j^T R_{Q_n,z,\epsilon} v_j \right\} \tag{6.53}$$

for each $j \in \{1, \cdots, q\}$. This may be done using the Laplace transform from (6.17), choosing $r = 2^n$, $M = \frac{1}{\lambda} R_{Q_n,z,\epsilon}$ and $\xi = 0$ to obtain

$$I_j = (2\pi\lambda)^{\frac{2^n}{2}} [\det(R_{Q_n,z,\epsilon})]^{-\frac{1}{2}}. \tag{6.54}$$

Applying (6.54) to (6.52), we see that

$$\begin{aligned} Z^{\mathrm{P}}_{n,\lambda,q} &\leq (2\pi\lambda)^{-\frac{q2^n}{2}} [\det(K_{Q_n,\epsilon})]^{-\frac{q}{2}} (2\pi\lambda)^{\frac{q2^n}{2}} [\det(R_{Q_n,z,\epsilon})]^{-\frac{q}{2}} \exp\{2^n \psi(z\lambda)\} \\ &= [\det(K_{Q_n,\epsilon} R_{Q_n,z,\epsilon})]^{-\frac{q}{2}} \exp\{2^n \psi(z\lambda)\}. \end{aligned} \tag{6.55}$$

After taking logarithms and dividing by the volume, we obtain the bound

$$\frac{1}{2^n} \log Z^{\mathrm{P}}_{n,\lambda,q} \leq -\frac{q}{2(2^n)} \log\det(K_{Q_n,\epsilon} R_{Q_n,z,\epsilon}) + \psi(z\lambda). \tag{6.56}$$

Take the limits of (6.56) as $n \to \infty$, $\epsilon \downarrow 0$ and $z \downarrow 1$ in that order. By Proposition 6.2.3, the first term disappears. The second term converges to $\psi(\lambda)$ by right-continuity. □

6.3 Analysis of the mean-field limit

In this section, we analyse the mean-field limit

$$\psi(\lambda) = \max_{v \in \mathbb{R}^q} \log \left[\int \lambda_0(d\sigma) \exp\left\{ -\frac{\|v\|^2}{2\lambda} + v^{\mathrm{T}} \cdot \sigma \right\} \right] \tag{6.57}$$

of the free energy of the measure $\mu_{n,\lambda,q}$ in order to prove Lemma 3.3.7. In particular, we extract the limit of the spin correlation $\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda,q}$ stated in Lemma 3.3.6 from $\psi(\lambda)$, before applying Corollary 2.2.4 to obtain the limit of the connection probability

$\phi_{n,\lambda,q}[x \leftrightarrow y]$.

In Proposition 2.2.7, we showed that the spin correlation $\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda,q}$ is related to the derivative of the free energy. We will argue that the limit of the spin correlations is given by the derivative of the limit $\psi(\lambda)$ of the free energy, which is given in the following lemma:

Lemma 6.3.1 ([30, Section 3, Proposition 2 (3.9)]). *Fix $q \in \mathbb{N}_{\geq 2}$. For $\lambda \neq \lambda_c$, the derivative $\frac{\partial \psi}{\partial \lambda}$ exists and is given by*

$$\frac{\partial \psi}{\partial \lambda} = \frac{1}{2q} + \frac{q-1}{2q}\theta(\lambda, q)^2. \tag{6.58}$$

Proof of Lemma 3.3.6. Let $\psi_{n,q}(\lambda)$ denote the free energy of the measure $\mu_{n,\lambda,q}$, expressed as a function of λ rather than of the inverse temperature $\beta = \lambda/n$. By the chain rule, we have

$$\begin{aligned} \frac{\partial}{\partial \lambda}\psi_{n,q}(\lambda) &= \frac{\partial \beta}{\partial \lambda}\frac{\partial}{\partial \beta}\psi_{n,q}(\lambda) \\ &= \frac{1}{n}\frac{\partial}{\partial \beta}\psi_{n,q}(\lambda). \end{aligned} \tag{6.59}$$

By Proposition 2.2.7, we know that

$$\frac{\partial}{\partial \beta}\psi_{n,q}(\lambda) = \frac{n}{2}\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda,q}, \tag{6.60}$$

leading to the equation

$$\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda,q} = 2\frac{\partial}{\partial \lambda}\psi_{n,q}(\lambda). \tag{6.61}$$

By Lemma 2.2.8, we know that $\psi_{n,q}(\lambda)$ is a convex function of β (and hence λ). Moreover, in Theorem 3.3.3, we showed that $\lim_{n\to\infty} \psi_{n,q}(\lambda) = \psi(\lambda)$. Finally, the function $\psi(\lambda)$ is differentiable for every $\lambda \neq \lambda_c$ by Lemma 6.3.1. As a result, Lemma 2.2.9 may be applied to show that

$$\lim_{n\to\infty} \frac{\partial}{\partial \lambda}\psi_{n,q}(\lambda) = \frac{\partial \psi}{\partial \lambda}(\lambda). \tag{6.62}$$

(3.51) follows by plugging (6.58) into (6.62) and then applying the relation of (6.61). □

Proof of Lemma 3.3.7. By Corollary 2.2.4, we know that

$$\phi_{n,\lambda,q}[x \leftrightarrow y] = \frac{q}{q-1}\left(\langle \sigma_x \cdot \sigma_y \rangle_{n,\beta,q} - \frac{1}{q}\right), \tag{6.63}$$

where the inverse temperature β is given by

$$\beta = -\log\left(1 - \frac{\lambda}{n}\right) = \frac{\lambda}{n} + O(n^{-2}). \tag{6.64}$$

Fix $\epsilon > 0$. For sufficiently large n, we may bound the inverse temperature in (6.64) above by $(\lambda + \epsilon)/n$. As spin correlations are increasing in β, it follows that

$$\phi_{n,\lambda,q}[x \leftrightarrow y] \leq \frac{q}{q-1}\left(\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda+\epsilon,q} - \frac{1}{q}\right). \tag{6.65}$$

Applying Lemma 3.3.6 to (6.65) yields the asymptotic upper bound

$$\limsup_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] \leq \theta(\lambda + \epsilon, q)^2. \tag{6.66}$$

As (6.66) holds for all $\epsilon > 0$ and $\theta(\lambda, q)$ is continuous for all $\lambda \neq \lambda_c$, we deduce that

$$\limsup_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] \leq \theta(\lambda, q)^2. \tag{6.67}$$

Similarly, we may bound the inverse temperature in (6.64) below by $(\lambda - \epsilon)/n$ to obtain

$$\phi_{n,\lambda,q}[x \leftrightarrow y] \geq \frac{q}{q-1}\left(\langle \sigma_x \cdot \sigma_y \rangle_{n,\lambda-\epsilon,q} - \frac{1}{q}\right), \tag{6.68}$$

leading to

$$\liminf_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] \geq \theta(\lambda - \epsilon, q)^2. \tag{6.69}$$

As (6.69) holds for all $\epsilon > 0$, the limit inferior matches the limit superior and (3.52) follows. □

The rest of this section is dedicated to proving Lemma 6.3.1 via a sequence of lemmas which follow the arguments of [30, Section 3]. We begin with the following result:

Lemma 6.3.2 ([30, Lemma 5])**.** *The function $\psi(\lambda)$ may be written in the form*

$$\psi(\lambda) = \max_{x\in\Delta_q}\left\{\sum_{i=1}^{q}(-x_i \log x_i) + \frac{\lambda}{2}\sum_{i=1}^{q} x_i^2\right\}, \tag{6.70}$$

where

$$\Delta_q = \left\{x \in \mathbb{R}^q : x_i \geq 0 \ \forall\ i, \sum_{i=1}^{q} x_i = 1\right\}. \tag{6.71}$$

Proof. Observe that, as a consequence of (6.16) from the proof of Lemma 6.1.1, we have

$$\psi(\lambda) = \lim_{n\to\infty} \frac{1}{2^n} \log \int \nu_{Q_n}(d\sigma) \exp\left\{ \frac{\lambda}{2(2^n)} \Big\| \sum_{y\in Q_n} \sigma_y \Big\|^2 \right\}. \tag{6.72}$$

By definition, ν_{Q_n} is only non-zero for configurations where $\sigma_x \in \{v_1, \cdots, v_q\}$ for each $x \in Q_n$, in which case it attaches the configuration a weight of 1. For $i = 1, \cdots, q$, let $m_i = |\{x \in Q_n : \sigma_x = v_i\}|$ be the number of vertices with spin v_i, and note that $\sum_{i=1}^q m_i = 2^n$. Moreover, the number of configurations σ for which a particular set $m_1, \cdots, m_q$ is obtained is equal to

$$\binom{|Q_n|}{m_1, \cdots, m_q} = \frac{(2^n)!}{m_1! \times \cdots \times m_q!}. \tag{6.73}$$

By applying Stirling's Formula (in the form $n! = n^n e^{-n}(1 + o(1))$) to (6.73), we see that

$$\binom{|Q_n|}{m_1, \cdots, m_q} = \frac{2^{n2^n}}{\prod_{i=1}^q m_i^{m_i}}(1 + o(1)). \tag{6.74}$$

Let $x_i = m_i/|Q_n|$. Plugging (6.74) into (6.72) gives

$$\begin{aligned} \psi(\lambda) &= \lim_{n\to\infty} \frac{1}{2^n} \log \sum_{m_1+\cdots+m_q=2^n} \frac{2^{n2^n}}{\prod_{i=1}^q m_i^{m_i}} \exp\left\{ \frac{\lambda}{2(2^n)} \sum_{i=1}^q m_i^2 \right\} \\ &= \lim_{n\to\infty} \frac{1}{2^n} \log \sum_{m_1+\cdots+m_q=2^n} \exp\left\{ \sum_{i=1}^q (-x_i \log x_i) + \frac{\lambda}{2} \sum_{i=1}^q x_i^2 \right\}^{2^n}. \end{aligned} \tag{6.75}$$

As the sum in (6.75) is bounded between its maximum term and $(2^n)^q$ times its maximum term, we obtain the bounds

$$\begin{aligned} \psi(\lambda) &\geq \max_{m_1+\cdots+m_q=2^n} \left\{ \sum_{i=1}^q (-x_i \log x_i) + \frac{\lambda}{2} \sum_{i=1}^q x_i^2 \right\}, \\ \psi(\lambda) &\leq \max_{m_1+\cdots+m_q=2^n} \left\{ \sum_{i=1}^q (-x_i \log x_i) + \frac{\lambda}{2} \sum_{i=1}^q x_i^2 \right\} + \frac{1}{2^n} \log(2^{qn}). \end{aligned}$$

Moreover, $\mathbb{Q} \cap \Delta_q$ is a dense subset of Δ_q and the function $\sum_{i=1}^q (-x_i \log x_i) + \frac{\lambda}{2} \sum_{i=1}^q x_i^2$ is continuous. As a result, the maximum term in the above bounds converges to the maximum taken over the set Δ_q in the limit as $n \to \infty$, and we recover (6.70). □

Recall the function $\theta(\lambda, q)$ defined by

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_c(q) \\ \theta_{\max} & \text{if } \lambda \geq \lambda_c(q) \end{cases} \tag{6.76}$$

where $\theta_{\max}$ is the largest solution of the mean field equation

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}. \tag{6.77}$$

The next lemma expresses the maximum of (6.70) in terms of the function $\theta(\lambda, q)$:

Lemma 6.3.3 ([30, Lemma 6])**.** *For $\lambda \neq \lambda_c$, the maximum of Lemma 6.3.2 is taken at a vector x of the form*

$$x_1 = \frac{1}{q}(1+(q-1)\theta(\lambda, q)),\ x_2 = \cdots = x_q = \frac{1}{q}(1-\theta(\lambda, q)), \tag{6.78}$$

or a point obtained from such an x by permuting its co-ordinates. In particular, Theorem 3.3.4 holds.

Proof. Without loss of generality, take the variables $x_1 \geq \cdots \geq x_q$ in decreasing order. We begin by showing that the maximum of Lemma 6.3.2 is taken at a vector x of the form

$$x_1 = \frac{1}{q}(1+(q-1)\theta),\ x_2 = \cdots = x_q = \frac{1}{q}(1-\theta). \tag{6.79}$$

By the method of Lagrange multipliers, it is sufficient to compute the maximum of the function

$$F(x, t) = \sum_{i=1}^{q}(-x_i \log x_i) + \frac{\lambda}{2}\sum_{i=1}^{q} x_i^2 + t\Big(\sum_{i=1}^{q} x_i - 1\Big). \tag{6.80}$$

Taking the partial derivatives of (6.80) with respect to x_i gives

$$\frac{\partial F}{\partial x_i} = \lambda x_i - \log x_i - (1-t). \tag{6.81}$$

Define the function $f(x) = \lambda x - \log x$. Setting the partial derivatives in (6.81) equal to 0 yields the equation

$$f(x_i) = 1 - t. \tag{6.82}$$

In particular, $f(x_i) = f(x_1)$ for every $i = 2, \cdots, q$. As f is a strictly convex function, the equation $f(x) = f(x_1)$ has at most one other solution, which we call y. As $f(x)$ is minimised at $x = \lambda^{-1}$, it follows that $y \leq \lambda^{-1} \leq x_1$, with strict inequality if $y \neq x_1$.

We have shown that $x_i \in \{y, x_1\}$ for every $i = 2, \cdots, q$. It remains to prove that if $y < x_1$, then $x_2 = y$. Suppose that we fix $x_3, \cdots, x_q$ and vary only the coordinates x_2, x_1. As $\sum_{i=1}^q x_i = 1$, it follows that $x_1 + x_2 = a$, where $a \in \{y + x_1, 2x_1\}$. The part of the function $F(x, t)$ that varies is then given by

$$F_2(x_1) = -x_1 \log x_1 - (a - x_1)\log(a - x_1) + \frac{\lambda}{2}(x_1^2 + (a - x_1)^2). \tag{6.83}$$

Differentiating (6.83) with respect to x_1, we obtain the derivatives

$$\begin{aligned} F_2'(x_1) &= f(x_1) - f(a - x_1), \\ F_2''(x_1) &= 2\lambda - \frac{1}{x_1} - \frac{1}{a - x_1}. \end{aligned}$$

Suppose that $a = 2x_1$. Then $F_2'(x_1) = 0$, and $F_2''(x_1) = 2(\lambda - 1/x_1) > 0$. In particular, the case $x_2 = x_1$ cannot yield a maximum, and so we must have $x_2 = y$. As we arranged $x_1 \geq \cdots \geq x_q$ in decreasing order, it follows that $x_i = y$ for every $i = 2, \cdots, q$. By applying the constraint $\sum_{i=1}^q x_i = 1$, we obtain (6.79) with $\theta = x_1 - x_2$.

Next, we show that $\theta = x_1 - x_2$ is a solution of the mean-field equation. Recall from (6.82) that

$$\lambda x_1 - \log x_1 = 1 - t = \lambda x_2 - \log x_2, \tag{6.84}$$

which rearranges to show that θ is a solution of the mean-field equation. To show that we take the solution $\theta = \theta(\lambda, q)$, we plug the values $x_1, \cdots, x_q$ back into the equation

$$\psi(\lambda) = \sum_{i=1}^{q}(-x_i \log x_i) + \frac{\lambda}{2}\sum_{i=1}^{q} x_i^2. \tag{6.85}$$

The first term of (6.85) is given by

$$\begin{aligned} \sum_{i=1}^{q} -x_i \log x_i &= -\frac{1 + (q-1)\theta}{q} \log\left(\frac{1 + (q-1)\theta}{q}\right) - \frac{(q-1)(1-\theta)}{q} \log\left(\frac{1-\theta}{q}\right) \\ &= \frac{g(\theta)}{2q} + \log q + \left(\frac{q-1}{2q}\right)\Big[\log(1-\theta) - \log(1 + (q-1)\theta)\Big]\theta \\ &= \frac{g(\theta)}{2q} + \log q - \left(\frac{q-1}{2q}\right)\lambda\theta^2, \end{aligned} \tag{6.86}$$

where $g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - [2 + (q-1)\theta]\log[1 + (q-1)\theta]$ and we have applied (6.77) to obtain the final line. Similarly, the second term of (6.85) is given by

$$\begin{aligned}\frac{\lambda}{2}\sum_{i=1}^{q} x_i^2 &= \frac{\lambda}{2q^2}[(1+(q-1)\theta)^2 + (q-1)(1-\theta)^2] \\ &= \frac{\lambda}{2q}[1+(q-1)\theta^2]. \end{aligned} \tag{6.87}$$

By combining (6.86) and (6.87), (6.85) becomes

$$\psi(\lambda) = \frac{g(\theta)}{2q} + \frac{\lambda}{2q} + \log q. \tag{6.88}$$

This is the form of the free energy given in Theorem 3.3.4. It remains to show that the solution of the mean-field equation which maximises the function g is given by $\theta(\lambda, q)$, which we quote from the proof of Theorem 5.3.2. □

Finally, we compute the derivative described in Lemma 6.3.1:

Proof of Lemma 6.3.1. By combining the first line of (6.86) with the second line of (6.87), we obtain the expression

$$\begin{aligned}\psi(\lambda) =& \log q - (\tfrac{q-1}{q})(1-\theta)\log(1-\theta) \\ & - \tfrac{1}{q}(1+(q-1)\theta)\log(1+(q-1)\theta) + \tfrac{\lambda}{2q}[1+(q-1)\theta^2], \end{aligned} \tag{6.89}$$

where $\theta = \theta(\lambda, q)$ is the largest solution of the mean-field equation, and defined implicitly as a function of λ. By the chain rule, we have

$$\frac{\partial \psi}{\partial \lambda}(\theta(\lambda), \lambda) = \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial \lambda} + \frac{\partial \psi}{\partial \lambda}. \tag{6.90}$$

We compute each of the derivatives $\frac{\partial \psi}{\partial \theta}$ and $\frac{\partial \psi}{\partial \lambda}$. The first derivative is equal to

$$\frac{\partial \psi}{\partial \theta} = (\tfrac{q-1}{q})[\lambda\theta + \log(1-\theta) - \log(1+(q-1)\theta)]. \tag{6.91}$$

As θ satisfies the equation (6.77), the derivative $\frac{\partial \psi}{\partial \theta}$ vanishes. The second derivative is equal to

$$\frac{\partial \psi}{\partial \lambda} = \tfrac{1}{2q}[1+(q-1)\theta^2] \tag{6.92}$$

as required. □

Chapter 7

The hypercube II: The random cluster model

In this chapter, we analyse the random cluster model on the hypercube. Throughout, we will adopt the notation used in Section 3.3, writing $\phi_{n,\lambda,q}$ for the random cluster measure on Q_n with edge weight $p = \lambda/n$ and cluster weight q. If $q = 1$, we use the standard notation $\phi_{n,\lambda}$ for the percolation measure. We call a connected subset of $Q_n(\omega)$ a *cell*, and a connected subset of $Q_n(\omega)$ of order n^2 an *atom*.

We begin this chapter by establishing two isoperimetric inequalities for the hypercube in Section 7.1. The first inequality provides a lower bound on the number of disjoint edges in the boundary of a subset of the hypercube, and is based upon a result by Christofides, Ellis and Keevash in [14]. The second inequality provides a lower bound on the number of disjoint paths of a prescribed length between large subsets of the hypercube, and is taken from [10].

The main result of this chapter is a proof of Theorem 3.3.1, which establishes an asymptotic phase transition for the size of the largest component of the random graph under the measure $\phi_{n,\lambda,q}$ at the point λ_c defined in (3.2). This will be deduced as a consequence of the asymptotic phase transition for nearest neighbour connection probabilities given in Lemma 3.3.7, and proven in two parts. In Section 7.2, we argue by contradiction that in the *sub-critical regime* $\lambda < \lambda_c$, the largest component of $Q_n(\omega)$ is of order $o(2^n)$ asymptotically almost surely. Then, in Section 7.3, we construct a component of order 2^n in the *super-critical regime* asymptotically almost surely. In particular, we will use the generalisation of the sprinkling method to the random cluster model developed in Section 2.1.2 to adapt the arguments of [1], using Lemma 3.3.7 to substitute for their use of the exploration process.

7.1 Isoperimetry of Q_n

We begin this chapter with a section detailing two isoperimetric inequalities for the hypercube Q_n. These inequalities, given in Section 7.1.1 and Section 7.1.2, provide lower bounds on the size of certain boundaries of subsets of the hypercube.

Let $G = (V, E)$ be a finite graph and let $A \subset V$ be a subset of vertices. There are two main notions for the *boundary* of the set A, which are defined as follows:

Definition 7.1.1 (Edge and vertex boundaries)**.** *The edge boundary $\partial_e(A)$ of the set A is defined as*

$$\partial_e(A) = \{\{x, y\} \in E : x \in A, y \in V \setminus A\}, \tag{7.1}$$

and the vertex boundary $b(A)$ of A is defined as

$$b(A) = \{y \in V \setminus A : \{x, y\} \in E \text{ for some } x \in A\}. \tag{7.2}$$

Finally, we define the neighbourhood $\mathcal{N}(A)$ of A as

$$\mathcal{N}(A) = A \cup b(A). \tag{7.3}$$

In other words, the edge boundary $\partial_e(A)$ is the set $E(A, A^c)$ of all edges between A and A^c, and the vertex boundary $b(A)$ is the set of all vertices contained in A^c which can be reached using an edge in $\partial_e(A)$. In particular, we have the inequality $|b(A)| \leq |\partial_e(A)|$.

7.1.1 Disjoint edge isoperimetric inequality

The objective of this subsection is to prove the following isoperimetric inequality, which provides a lower bound on the number of *disjoint* edges - that is, edges which do not share a common vertex - contained in the edge boundary $\partial_e(A)$ of a set $A \subset Q_n$ when both A and A^c are sufficiently large:

Lemma 7.1.2. *Fix $\alpha \in (0, 1)$, and let $A \subset Q_n$ with $|A| \in [\alpha 2^n, (1 - \alpha)2^n]$. Then, provided n is sufficiently large, $\partial_e(A)$ contains at least $\gamma 2^n / \sqrt{n}$ disjoint edges, where*

$$\gamma = \frac{1}{\sqrt{2}} \frac{\alpha}{2} \left(1 - \frac{\alpha}{2}\right). \tag{7.4}$$

The precise value of the constant γ given in (7.4) is not important; we are only interested in the *order* of the number of disjoint edges contained in the edge boundary. Lemma 7.1.2 will be proven as a consequence of the following vertex isoperimetric inequality, given by Christofides, Ellis and Keevash in [14]:

Theorem 7.1.3 ([14, Theorem 3])**.** *Fix* $\alpha \in (0,1)$*, and let* $A \subset Q_n$ *with* $|A| = \alpha 2^n$*. Then,*

$$|b(A)| \geq \sqrt{2}\alpha(1-\alpha)\frac{2^n}{\sqrt{n}}. \tag{7.5}$$

Proof. We follow the proof by induction given in [14]. When $n = 1$, the inequality (7.5) is easily checked.

Next, suppose $A \subset Q_n$ with $|A| = \alpha 2^n$ for some $n \geq 2$. We decompose A into a disjoint union of the following two sets, depending on the value of the final co-ordinate:

$$A_0 = \{v \in A : v_n = 0\},\ A_1 = \{v \in A : v_n = 1\}. \tag{7.6}$$

Note that A_0 and A_1 may be viewed as subsets of Q_{n-1}, and (by inverting coordinates if necessary) we may assume that $|A_0| \geq |A_1|$. Let $\delta = |A_0|/2^{n-1} - \alpha$, and write

$$|A_0| = (\alpha+\delta)2^{n-1},\ |A_1| = (\alpha-\delta)2^{n-1}. \tag{7.7}$$

Next, we decompose $\mathcal{N}(A)$ according to the value of the final coordinate. Suppose that $v \in (\mathcal{N}(A))_0$, and let $w \in A$ be a neighbour of v. If $w_n = 0$, then $v \in \mathcal{N}(A_0)$. Otherwise (if $w_n = 1$) v corresponds to a vertex of A_1 with its final coordinate flipped to 0. In particular, it follows that

$$|(\mathcal{N}(A))_0| = |\mathcal{N}(A_0) \cup A_1|. \tag{7.8}$$

Similarly, we see that $|(\mathcal{N}(A))_1| = |\mathcal{N}(A_1) \cup A_0|$, and so

$$|\mathcal{N}(A)| = |\mathcal{N}(A_0) \cup A_1| + |\mathcal{N}(A_1) \cup A_0|. \tag{7.9}$$

As $A_0 \subset \mathcal{N}(A_0)$, we may further write

$$|\mathcal{N}(A)| \geq \max\left\{2|A_0|, |\mathcal{N}(A_0)| + |\mathcal{N}(A_1)|\right\}. \tag{7.10}$$

Let $f(x) = \sqrt{2}x(1-x)$. By induction, we have

$$|\mathcal{N}(A_0)| = |A_0| + |b(A_0)| \geq (\alpha+\delta)2^{n-1} + f(\alpha+\delta)2^{n-1}/\sqrt{n-1}, \tag{7.11}$$

and

$$|\mathcal{N}(A_1)| = |A_1| + |b(A_1)| \geq (\alpha-\delta)2^{n-1} + f(\alpha-\delta)2^{n-1}/\sqrt{n-1}. \tag{7.12}$$

Using (7.11) and (7.12), we may rewrite (7.10) as

$$|\mathcal{N}(A)| \geq \max\left\{(\alpha+\delta)2^n, \alpha 2^n + [f(\alpha+\delta)+f(\alpha-\delta)]2^{n-1}/\sqrt{n-1}\right\}, \tag{7.13}$$

which yields a lower bound on the vertex boundary of

$$|b(A)| \geq \max\left\{\delta 2^n, [f(\alpha+\delta)+f(\alpha-\delta)]2^{n-1}/\sqrt{n-1}\right\}. \tag{7.14}$$

We claim that if $0 \leq \delta \leq f(\alpha)/\sqrt{n}$, then

$$\frac{f(\alpha+\delta)+f(\alpha-\delta)}{2f(\alpha)} \geq \sqrt{1-1/n}. \tag{7.15}$$

If (7.15) holds, then (7.14) implies that $|b(A)| \geq f(\alpha)2^n/\sqrt{n}$ and thus the theorem is true by induction. To prove (7.15), observe that

$$\begin{aligned}
\frac{f(\alpha+\delta)+f(\alpha-\delta)}{2f(\alpha)} &= \frac{(\alpha+\delta)(1-\alpha-\delta)+(\alpha-\delta)(1-\alpha+\delta)}{2\alpha(1-\alpha)} \\
&= 1-\frac{\delta^2}{\alpha(1-\alpha)} \\
&\geq 1-\frac{1}{\alpha(1-\alpha)}\frac{f(\alpha)^2}{n} \\
&= 1-\frac{2\alpha(1-\alpha)}{n} \\
&\geq 1-\frac{1}{2n} \\
&\geq \sqrt{1-1/n},
\end{aligned} \tag{7.16}$$

Where we have applied the inequalities $\alpha(1-\alpha) \leq \frac{1}{4}$ and $\sqrt{1-x} \leq 1-\frac{x}{2}$ for $\alpha \in (0,1)$ and $x \in (0,1)$. □

Proof of Lemma 7.1.2. Fix $\alpha \in (0,1)$, and let $A \subset Q_n$ with $|A| \in [\alpha 2^n, (1-\alpha)2^n]$. We will prove Lemma 7.1.2 by iteratively constructing a subset S of disjoint edges belonging to the edge boundary $\partial_e(A)$.

To begin, we define the sets $A_0 = A$ and $S_0 = \emptyset$. As $|A_0| \in [\frac{1}{2}\alpha 2^n, (1-\frac{1}{2}\alpha)2^n]$, Theorem 7.1.3 implies that $|b(A_0)| \geq f(\frac{1}{2}\alpha)2^n/\sqrt{n}$, so we may choose an arbitrary vertex $y_1 \in b(A_0)$ and pair it off with an arbitrary neighbour $x_1 \in A_0$. Define $A_1 = A \setminus x_1$ and $S_1 = S_0 \cup \{x_1, y_1\}$.

Now, let $k \in \mathbb{N}$ and suppose that we have constructed the sets A_k and S_k. If

$2k \geq f(\frac{1}{2}\alpha)2^n/\sqrt{n}$, then S_k is a set of disjoint edges of size at least $\gamma 2^n/\sqrt{n}$, and we are done. Otherwise, we must have $|A_k| \in [\frac{1}{2}\alpha 2^n, (1-\frac{1}{2}\alpha)2^n]$, and so Theorem 7.1.3 may be applied to see that $|b(A_k)| \geq f(\frac{1}{2}\alpha)2^n/\sqrt{n}$. In particular, the set $b(A_k)\setminus S_k$ is non empty, so we may choose an arbitrary vertex $y_{k+1} \in b(A_k) \setminus S_k$ and pair it off with an arbitrary neighbour $x_{k+1} \in A_k$. Define $A_{k+1} = A \setminus x_{k+1}$ and $S_{k+1} = S_k \cup \{x_{k+1}, y_{k+1}\}$. As there are finitely many vertices, this procedure eventually terminates at some $K \geq \gamma 2^n/\sqrt{n}$, and we take the corresponding set S_K. □

7.1.2 Disjoint path isoperimetric inequality

The objective of this subsection is to provide a lower bound on the number of *disjoint paths* of a prescribed length between two large subsets of the hypercube. We will use the following isoperimetric inequality, given by Borgs et al. in [10]:

Lemma 7.1.4 ([10, Lemma 2.4])**.** *Fix $\epsilon > 0$, and let $S, T \subset Q_n$ be subsets of vertices of size at least $\epsilon 2^n$ each. Then, for any $\Delta > 0$ satisfying $e^{-\Delta^2/2n} < \epsilon/2$, we may find a collection of $\frac{1}{2}\epsilon 2^n n^{-2\Delta}$ vertex disjoint paths from S to T, each of length at most Δ.*

If $\Delta > 0$ is chosen such that $n = o(\Delta^2)$, then for any $\epsilon > 0$ the condition $e^{-\Delta^2/2n} < \epsilon/2$ will hold provided that n is sufficiently large. We will choose $\Delta = n^{3/4}$ (any choice such that we also have $\Delta = o(n)$ would suffice), and apply Lemma 7.1.4 in the following form:

Corollary 7.1.5. *Fix $\epsilon > 0$, and let $S, T \subset Q_n$ be subsets of vertices of size at least $\epsilon 2^n$ each. Then, for sufficiently large n, we may find a collection of $2^n n^{-2n^{4/5}}$ vertex disjoint paths from S to T, each of length at most $n^{3/4}$.*

Lemma 7.1.4 will be proven using the arguments given in [10]. To begin, we recall the following classical isoperimetric inequality, due to Harper in [25]:

Theorem 7.1.6 (Harper)**.** *Let $A \subset Q_n$ and suppose that $|A| \geq \sum_{i=0}^{u} \binom{n}{i}$. Then*

$$|\mathcal{N}(A)| \geq \sum_{i=0}^{u+1} \binom{n}{i}. \tag{7.17}$$

Given a subset of vertices $A \subset Q_n$ and a positive integer r, we denote the ball of radius r around A by

$$B(A, r) = \{y \in Q_n : \exists\, x \in A \text{ with } d(x, y) \leq r\}, \tag{7.18}$$

where $d(x,y)$ denotes the usual (Hamming) distance on the hypercube. Iterated application of Harper's theorem yields the following isoperimetric inequality on the ball of radius r:

Theorem 7.1.7 ([10, Lemma 2.1])**.** *Let $A \subset Q_n$ and suppose that $|A| \geq \sum_{i=0}^{u} \binom{n}{i}$. Then, for any r, we have*

$$|B(A,r)| \geq \sum_{i=0}^{u+r} \binom{n}{i}. \tag{7.19}$$

In order to estimate the sum of binomial coefficients in (7.19) we will use the following lemma:

Lemma 7.1.8 ([10, Lemma 2.2])**.** *Fix $\Delta > 0$. Then*

$$\sum_{i \leq \frac{n-\Delta}{2}} \binom{n}{i} = \sum_{i \geq \frac{n+\Delta}{2}} \binom{n}{i} \leq 2^n e^{-\Delta^2/2n}. \tag{7.20}$$

Proof. The equality in (7.20) is a result of the symmetry of binomial coefficients. To prove the inequality, we first rewrite it as

$$\sum_{i \geq \frac{n+\Delta}{2}} 2^{-n} \binom{n}{i} \leq e^{-\Delta^2/2n}. \tag{7.21}$$

We recognise the left hand side of (7.21) as the expression $\mathbb{P}(X_n \geq \frac{n+\Delta}{2})$, where X_n is a Binomial$(n, 1/2)$ random variable. Let $t > 0$. By Markov's inequality, we have the bound

$$\begin{aligned} \mathbb{P}(X_n \geq \tfrac{n+\Delta}{2}) &= \mathbb{P}(e^{tX_n} \geq e^{t\frac{n+\Delta}{2}}) \\ &\leq e^{-t\frac{n+\Delta}{2}} \mathbb{E}[e^{tX_n}] \\ &= e^{-t\frac{n+\Delta}{2}} [\tfrac{1}{2}(1+e^t)]^n \\ &= e^{-t\Delta/2} \cosh(t/2)^n \\ &\leq e^{t^2 n/8 - t\Delta/2}, \end{aligned} \tag{7.22}$$

where we apply the inequality $\cosh(t) \leq e^{t^2/2}$ for all $t > 0$ (obtained by comparing Taylor series term for term) to obtain the final line. Setting $t = \Delta/(2n)$ yields the inequality of (7.20). □

Next, we apply Theorem 7.1.7 to show that if we take sufficiently large sets

$S, T \subset Q_n$ and choose $\Delta > 0$ appropriately, then the intersection of $B(S, \Delta)$ and T is large:

Lemma 7.1.9 ([10, Lemma 2.3])**.** *Fix $\epsilon > 0$, and let $S, T \subset Q_n$ be subsets of vertices of size at least $\epsilon 2^n$ each. Then, for any $\Delta > 0$ satisfying $e^{-\Delta^2/2n} < \epsilon/2$, we have*

$$|B(S, \Delta) \cap T| \geq \tfrac{1}{2}|T|. \tag{7.23}$$

Proof. From Lemma 7.1.8, we have that

$$|S| \geq \sum_{i \leq \frac{1}{2}(n-\Delta)} \binom{n}{i}, \tag{7.24}$$

and so by Theorem 7.1.7 it follows that

$$|B(S, \Delta)| \geq \sum_{i \leq \frac{1}{2}(n+\Delta)} \binom{n}{i}. \tag{7.25}$$

Thus

$$|Q_n \setminus B(S, \Delta)| \leq \sum_{i \geq \frac{1}{2}(n+\Delta)} \binom{n}{i}. \tag{7.26}$$

Estimating the sum of binomial coefficients in (7.26) using Lemma 7.1.8, we obtain

$$|Q_n \setminus B(S, \Delta)| < \frac{\epsilon}{2} 2^n \leq \tfrac{1}{2}|T|. \tag{7.27}$$

As $T \setminus B(S, \Delta) \subset Q_n \setminus B(S, \Delta)$, it follows that

$$|B(S, \Delta) \cap T| = |T| - |T \setminus B(S, \Delta)| \geq \tfrac{1}{2}|T|, \tag{7.28}$$

which is precisely (7.23). □

Proof of Lemma 7.1.4. Set $T_1 = B(S, \Delta) \cap T$ and note that $|T_1| \geq \frac{1}{2}\epsilon 2^n$. Let $T_2 \subset T_1$ be a maximal subset such that no $x, y \in T_2$ are within distance 2Δ of each other. Note that every $y \in T_1$ belongs to a ball of radius 2Δ around some $x \in T_1$ (else T_2 would not be maximal) and the size of each ball is bounded by $n^{2\Delta}$. Thus $|T_2| \geq n^{-2\Delta}|T_1|$. Moreover, as $T_2 \subset T_1$, for any $x \in T_2$ we may find a path of length at most Δ from x to S. Given $x, y \in T_2$, observe that the paths obtained must be disjoint; otherwise we obtain a path of length at most 2Δ from x to y. □

7.2 The sub-critical regime $\lambda < \lambda_c$

In this section, we analyse the size of the largest component $\mathcal{C}_{\max}$ of the graph $Q_n(\omega)$ for the random cluster measure $\phi_{n,\lambda,q}$ when $q \in \mathbb{N}_{\geq 2}$ and $\lambda < \lambda_c$, where the critical value λ_c was defined in (3.2). In particular, we prove the following lemma, corresponding to the first half of Theorem 3.3.1:

Lemma 7.2.1. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda < \lambda_c$. Then, for any $\epsilon > 0$, we have*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq \epsilon 2^n] = 0. \tag{7.29}$$

We will prove Lemma 7.2.1 by contradiction. More specifically, we will use the sprinkling methods detailed in Section 2.1.2 to provide a lower bound on the connection probability $\phi_{n,\lambda,q}[x \leftrightarrow y]$ in terms of the probability that a component of order 2^n exists. If Lemma 7.2.1 does not hold, this will imply that the probability $\phi_{n,\lambda,q}[x \leftrightarrow y]$ is bounded below. This is a contradiction, as we showed in Lemma 3.3.7 that for $\lambda < \lambda_c$,

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] = 0. \tag{7.30}$$

Before proving Lemma 7.2.1, we need the following preliminary result:

Lemma 7.2.2. *Fix $q \in \mathbb{N}_{\geq 2}$, $\lambda > 0$, $\alpha > 0$ and $c > 0$. Then, for every $n \in \mathbb{N}$, we have*

$$\phi_{n,\lambda+q\alpha,q}[x \leftrightarrow y] \geq c^2 \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq c2^n]^2 f(\alpha, c, n) \tag{7.31}$$

where the function f satisfies the limit

$$\lim_{n\to\infty} f(\alpha, c, n) = 1. \tag{7.32}$$

Proof. Write $\phi_{n,\lambda,q,\alpha}$ as shorthand for the sprinkled random cluster measure introduced in Definition 2.1.11 with $G = Q_n$, edge weight $p = \lambda/n$ and sprinkling constant $\delta = \alpha/n$. As the event $\{x \leftrightarrow y\}$ is increasing, Lemma 2.1.13 implies that

$$\phi_{n,\lambda+q\alpha,q}[x \leftrightarrow y] \geq \phi_{n,\lambda,q,\alpha}[x \leftrightarrow y]. \tag{7.33}$$

By definition, we may expand the sprinkled random cluster measure as

$$\phi_{n,\lambda,q,\alpha}[x \leftrightarrow y] = \sum_{\omega} \mathbb{1}_{x\leftrightarrow y}(\omega) \sum_{\xi} \phi_{n,\lambda,q}[\xi]\Pi_{\alpha/n}[\xi, \omega]. \tag{7.34}$$

Given a percolation configuration ω, let $\mathcal{C}_x(\omega)$ denote the component of x in the graph $Q_n(\omega)$ and define the event

$$A_{n,c} = \{\omega : |\mathcal{C}_x(\omega)| \geq c2^n, |\mathcal{C}_y(\omega)| \geq c2^n\}. \tag{7.35}$$

We may produce a lower bound on (7.34) by inserting the indicator function $\mathbb{1}_{A_{n,c}}(\xi)$, which is equivalent to adding the constraint that x and y belong to components of size at least $c2^n$ even before the sprinkling is applied. This yields the bound

$$\begin{aligned} \phi_{n,\lambda,q,\alpha}[x \leftrightarrow y] &\geq \sum_{\omega} \mathbb{1}_{x\leftrightarrow y}(\omega) \sum_{\xi} \mathbb{1}_{A_{n,c}}(\xi)\phi_{n,\lambda,q}[\xi]\Pi_{\alpha/n}[\xi,\omega] \\ &= \sum_{\xi} \mathbb{1}_{A_{n,c}}(\xi)\phi_{n,\lambda,q}[\xi] \sum_{\omega} \mathbb{1}_{x\leftrightarrow y}(\omega)\Pi_{\alpha/n}[\xi,\omega]. \end{aligned} \tag{7.36}$$

Observe that the quantity $\sum_\omega \mathbb{1}_{x\leftrightarrow y}(\omega)\Pi_{\alpha/n}[\xi,\omega]$ is equal to the probability that the vertices x and y belong to the same component after sprinkling has been applied to the configuration ξ. We claim that for any $\xi \in A_{n,c}$, this has the bound

$$\sum_{\omega} \mathbb{1}_{x\leftrightarrow y}(\omega)\Pi_{\alpha/n}[\xi,\omega] \geq f(\alpha,c,n) \tag{7.37}$$

where the function $f(\alpha,c,n)$ is given by

$$f(\alpha,c,n) = 1 - \exp\left[-\left(\frac{\alpha}{n}\right)^{n^{3/4}} 2^n n^{-2n^{4/5}}\right] \xrightarrow[n\to\infty]{} 1. \tag{7.38}$$

Indeed, observe that for any $\xi \in A_{n,c}$, the vertices x and y belong to components $\mathcal{C}_x(\xi)$ and $\mathcal{C}_y(\xi)$ of size at least $c2^n$ in $Q_n(\xi)$, which we may assume are disjoint (else they are certainly connected after sprinkling). By Corollary 7.1.5, there are at least $2^n n^{-2n^{4/5}}$ disjoint paths of length at most $n^{3/4}$ between $\mathcal{C}_x(\xi)$ and $\mathcal{C}_y(\xi)$ provided that n is sufficiently large. The probability that none of these paths are opened during the sprinkling represented by the measure $\Pi_{\alpha/n}[\epsilon,\cdot]$ is at most

$$\left[1 - \left(\frac{\alpha}{n}\right)^{n^{3/4}}\right]^{2^n n^{-2n^{4/5}}} \leq \exp\left[-\left(\frac{\alpha}{n}\right)^{n^{3/4}} 2^n n^{-2n^{4/5}}\right]. \tag{7.39}$$

Thus the claim (7.37) holds. Applying this claim to (7.36) yields the bound

$$\phi_{n,\lambda,q,\alpha}[x \leftrightarrow y] \geq \phi_{n,\lambda,q}[|\mathcal{C}_x(\omega)| \geq c2^n, |\mathcal{C}_y(\omega)| \geq c2^n] f(\alpha,c,n). \tag{7.40}$$

We may then apply the FKG inequality (Theorem 2.1.5) to (7.40), obtaining the bound

$$\phi_{n,\lambda,q,\alpha}[x \leftrightarrow y] \geq \phi_{n,\lambda,q}[|\mathcal{C}_x| \geq c2^n]\phi_{n,\lambda,q}[|\mathcal{C}_y| \geq c2^n]f(\alpha, c, n). \tag{7.41}$$

Finally, we note that for any vertex x, we have

$$\begin{aligned}\phi_{n,\lambda,q}[|\mathcal{C}_x| \geq c2^n] &\geq \phi_{n,\lambda,q}[\{x \in \mathcal{C}_{\max}\} \cap \{|\mathcal{C}_{\max}| \geq c2^n\}] \\ &= \phi_{n,\lambda,q}[\{x \in \mathcal{C}_{\max}\} \mid \{|\mathcal{C}_{\max}| \geq c2^n\}]\phi_{n,\lambda,q}[|\mathcal{C}_{\max}| \geq c2^n] \\ &\geq c\phi_{n,\lambda,q}[|\mathcal{C}_{\max}| \geq c2^n] \end{aligned} \tag{7.42}$$

where we have used the fact all vertices are equally likely to belong to the giant component by vertex transitivity of the hypercube and the random cluster measure in order to obtain the final inequality. □

Proof of Lemma 7.2.1. We proceed by contradiction. Suppose that Lemma 7.2.1 does not hold. Then, we may find $\lambda < \lambda_c$, constants $\epsilon_1, \epsilon_2 > 0$ and a subsequence $(n_i)_{i\in\mathbb{N}}$ such that for each i, we have

$$\phi_{n_i,\lambda,q}[|\mathcal{C}_{\max}| \geq \epsilon_1 2^{n_i}] \geq \epsilon_2. \tag{7.43}$$

Choose $\alpha > 0$ such that $\lambda + q\alpha < \lambda_c$. Then, Lemma 7.2.2 implies that

$$\phi_{n_i,\lambda+q\alpha,q}[x \leftrightarrow y] \geq \epsilon_1^2\epsilon_2^2 f(\alpha, \epsilon_1, n_i). \tag{7.44}$$

Taking the limit as $n_i \to \infty$, the left hand side converges to 0 by Lemma 3.3.7 while the right hand side converges to $\epsilon_1^2\epsilon_2^2 > 0$, a contradiction. □

7.3 The super-critical regime $\lambda > \lambda_c$

In this section, we analyse the size of the largest component $\mathcal{C}_{\max}$ of the graph $Q_n(\omega)$ for the random cluster measure $\phi_{n,\lambda,q}$ when $q \in \mathbb{N}_{\geq 2}$ and $\lambda > \lambda_c$, where the critical value λ_c was defined in (3.2). In particular, we prove the following lemma, corresponding to the second half of Theorem 3.3.1:

Lemma 7.3.1. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > \lambda_c$. Then, there exists a constant $\epsilon > 0$ (depending only on λ) such that*

$$\lim_{n\to\infty} \phi_{n,\lambda,q}[|\mathcal{C}_{max}| \geq \epsilon 2^n] = 1. \tag{7.45}$$

We will prove Lemma 7.3.1 by adapting the arguments used by Ajtai, Komlós, and Szemerédi in [1] to study the largest component of $Q_n(\omega)$ for the percolation measure $\phi_{n,\lambda}$. We begin with the following lemma, which provides a lower bound on the probability that e_0 belongs to an atom in $Q_n(\omega)$ for the measure $\phi_{n,\lambda,q}$:

Lemma 7.3.2 (Components of order n^2)**.** *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > \lambda_c(q)$. Let e_0 denote the vertex of Q_n with every coordinate equal to 0. Then, for any $c_1 < \frac{1}{4}\theta(\lambda, q)^4$, we have*

$$\phi_{n,\lambda,q}\Big[|\mathcal{C}_{e_0}| \geq c_1 n^2\Big] \geq c_2 \tag{7.46}$$

for sufficiently large n, where $c_2 := \theta(\lambda, q)^4 - 4c_1$.

Proof. We begin by showing that vertices at a distance 2 apart have positive connection probabilities. Let $x, y \in Q_n$ with $d(x, y) = 2$ and let z be a common neighbour. By the FKG inequality (Theorem 2.1.5), we have

$$\begin{aligned}\phi_{n,\lambda,q}[x \leftrightarrow y] &\geq \phi_{n,\lambda,q}[\{x \leftrightarrow z\} \cap \{y \leftrightarrow z\}] \\ &\geq \phi_{n,\lambda,q}[x \leftrightarrow z]\phi_{n,\lambda,q}[y \leftrightarrow z].\end{aligned} \tag{7.47}$$

Applying Lemma 3.3.7 to (7.47) yields the bound

$$\liminf_{n\to\infty} \phi_{n,\lambda,q}[x \leftrightarrow y] \geq \theta(\lambda, q)^4. \tag{7.48}$$

Fix n_0 such that for every $n \geq n_0$, $\phi_{n,\lambda,q}[x \leftrightarrow y] \geq \theta(\lambda, q)^4 - 2c_1(1 + \frac{2}{n})$. Then, define the following sets:

$$\begin{aligned}X &= \{y \in Q_n : \ d(y, e_0) = 2 \text{ and } y \leftrightarrow e_0 \text{ in } Q_n(\omega)\}, \\ Y &= \{y \in Q_n : \ d(y, e_0) = 2 \text{ and } y \not\leftrightarrow e_0 \text{ in } Q_n(\omega)\}.\end{aligned}$$

Note that there are $\binom{n}{2}$ vertices at distance 2 from e_0 in Q_n, and so $|X| + |Y| = \binom{n}{2}$. By Markov's inequality, we have

$$\phi_{n,\lambda,q}\left[|Y| \geq (1 - 2c_1(1 + \tfrac{2}{n}))\binom{n}{2}\right] \leq \frac{(1 - (\theta(\lambda, q)^4 - 2c_1(1 + \frac{2}{n})))\binom{n}{2}}{(1 - 2c_1(1 + \frac{2}{n}))\binom{n}{2}}. \tag{7.49}$$

Rearranging (7.49), we see that

$$\phi_{n,\lambda,q}\left[|X| \geq 2c_1(1 + \tfrac{2}{n})\binom{n}{2}\right] \geq \theta(\lambda, q)^4 - 4c_1. \tag{7.50}$$

As $X \subset \mathcal{C}_{e_0}$ and $2(1+\frac{2}{n})\binom{n}{2} \geq n^2$ for $n \geq 2$, the claim (7.46) follows. □

In [1, Lemma 1], the authors use a branching process comparison to provide a lower bound on the probability that e_0 belongs to a cell of order n in $Q_n(\omega)$ under the measure $\phi_{n,\lambda}$. The sprinkling method is then used to increase the order of this cell from n to n^2. As Lemma 7.3.2 directly obtains a lower bound on the probability that e_0 belongs to an atom in $Q_n(\omega)$ for the measure $\phi_{n,\lambda,q}$, we will need one less round of sprinkling than [1], which greatly simplifies the proof of Lemma 7.3.1. This simplification comes at a cost, namely the precision of the constant c_1 and hence the density of the giant component. Indeed, the arguments of Lemma 7.3.2 will result in a giant component with density proportional to $\theta(\lambda, q)^4$. It is possible to replace this with a density proportional to $\theta(\lambda, q)^2$ by applying Markov's inequality directly to the nearest neighbours of e_0 and using an extra round of sprinkling. Alternatively, one can show that the free energy of an appropriate *second order* Potts model converges, and hence extend Lemma 3.3.7 to next nearest neighbours. We do not claim the resulting density of $\theta(\lambda, q)^2$ to be optimal, as we expect the density of the giant component to equal $\theta(\lambda, q)$, as for the complete graph.

Our next task is to show that in fact, *most* vertices have *many* neighbours belonging to atoms of order n^2 in $Q_n(\omega)$ under the measure $\phi_{n,\lambda,q}$. In order to make this idea precise, we make the following definition:

Definition 7.3.3 (Rich vertices)**.** *Let $a, b > 0$ and $\omega \in \{0,1\}^E$. We say that a vertex v is (a,b)-rich in $Q_n(\omega)$ if we may find a subset U of the neighbours of v (in Q_n) of size at least an such that*

1. *each $u \in U$ belongs to an atom $B(u)$ of size at least bn^2 in $Q_n(\omega)$, and:*
2. *these atoms are mutually disjoint: if $u_1 \neq u_2$ then $B(u_1) \cap B(u_2) = \emptyset$.*

We label the set of (a,b)-rich vertices as $R(a,b)$.

Our next result is the following analogue of [1, Lemma 3]:

Lemma 7.3.4. *Fix $q \in \mathbb{N}_{\geq 2}$ and $\lambda > \lambda_c(q)$. Then, we may find constants $c_3, c_4, c_5 > 0$ (depending only on λ) such that for n sufficiently large, we have*

$$\phi_{n,\lambda,q}[|R(c_3,c_4)| \geq 2^n - 2^{(1-c_5)n}] \geq 1 - 2^{-c_5 n}. \tag{7.51}$$

Proof. We will show that

$$\phi_{n,\lambda,q}[|R(c_3,c_4)^c| > 2^{(1-c_5)n}] \leq 2^{-c_5 n}. \tag{7.52}$$

Using Markov's inequality and vertex transitivity in the hypercube, we obtain

$$\begin{aligned}\phi_{n,\lambda,q}[|R(c_3,c_4)^c| > 2^{(1-c_5)n}] &\le 2^{-(1-c_5)n}\mathbb{E}_{n,\lambda,q}[|R(c_3,c_4)^c|] \\ &= 2^{-(1-c_5)n}\sum_{v\in Q_n}\phi_{n,\lambda,q}[v \notin R(c_3,c_4)] \\ &= 2^{-(1-c_5)n}2^n\phi_{n,\lambda,q}[e_o \notin R(c_3,c_4)] \\ &= 2^{c_5 n}\phi_{n,\lambda,q}[e_o \notin R(c_3,c_4)]. \end{aligned} \tag{7.53}$$

It thus suffices to show that for appropriate choices of c_3, c_4 and c_5, we have

$$\phi_{n,\lambda,q}[e_o \notin R(c_3,c_4)] \le 2^{-2c_5 n}. \tag{7.54}$$

To prove (7.54), we will express the neighbours of e_0 as the roots of disjoint hypercubes of sufficiently high dimension, apply Lemma 7.3.2 to each, and conclude via an appropriate Chernoff bound.

Consider the first k neighbours $e_1, \cdots, e_k$ of e_0 in Q_n, where $k = \lfloor \delta n \rfloor$ for a constant $\delta > 0$ to be chosen later. We may associate a hypercube $Q^{(i)}$ of dimension $n-k$ to each vertex e_i by fixing the first k coordinates of e_i and varying the remaining $n-k$ coordinates. Moreover, if $i \neq j$ then the hypercubes $Q^{(i)}$ and $Q^{(j)}$ are disjoint. Let $\mathcal{Q}$ denote the union of these k hypercubes, and let $E_{\mathcal{Q}}$ denote the union of the sets of edges contained within these hypercubes. By conditioning on the edges $E \setminus E_{\mathcal{Q}}$ not contained within any of these hypercubes and applying Proposition 2.1.10, we see that for any increasing function f,

$$\begin{aligned}\phi_{n,\lambda,q}[f] &= \sum_{\eta\in\{0,1\}^{E\setminus E_{\mathcal{Q}}}} \phi_{n,\lambda,q}[f \mid \omega_{\{0,1\}^{E\setminus E_{\mathcal{Q}}}} = \eta]\phi_{n,\lambda,q}[\omega_{\{0,1\}^{E\setminus E_{\mathcal{Q}}}} = \eta] \\ &\ge \sum_{\eta\in\{0,1\}^{E\setminus E_{\mathcal{Q}}}} \phi_{n,\lambda,q}[f \mid \omega_{\{0,1\}^{E\setminus E_{\mathcal{Q}}}} = \emptyset]\phi_{n,\lambda,q}[\omega_{\{0,1\}^{E\setminus E_{\mathcal{Q}}}} = \eta] \\ &= \phi_{n,\lambda,q}[f \mid \omega_{\{0,1\}^{E\setminus E_{\mathcal{Q}}}} = \emptyset]. \end{aligned} \tag{7.55}$$

Moreover, conditionally on the event $\{\omega_{\{0,1\}^{E\setminus E_{\mathcal{Q}}}} = \emptyset\}$, the measure $\phi_{n,\lambda,q}$ restricts to an independent $\phi_{n-k,\lambda(1-k/n),q}$ random cluster measure on each hypercube $Q^{(i)}$. Choose $\delta > 0$ such that $(1-\delta)\lambda > \lambda_c$. Then, for n sufficiently large, the measure $\phi_{n-k,\lambda(1-k/n),q}$ is supercritical. In particular, we may apply Lemma 7.3.2 to each hypercube $Q^{(i)}$ to see that the vertex e_i belongs to a cell contained entirely within $Q^{(i)}$ of size at least $c_1(1-\delta)^2 n^2 = c_4 n^2$ in $Q_n(\omega)$ with probability $c_2 > 0$, independently of the other neighbours of e_0. Consequently, the number of neighbours of e_0 belonging to disjoint cells of order at least c_4n^2 is bounded below by a Binomial(k, c_2) random variable, which we call X. To estimate the lower tail of this random variable, we use the *Chernoff* bound (see [13]), which says that if $X_1, \cdots, X_n$ are independent random variables taking values in $\{0, 1\}$ and $X = \sum_{i=1}^n X_i$ then for any $\delta > 0$, we have

$$\mathbb{P}(X \leq (1-\delta)\mathbb{E}[X]) \leq e^{-\frac{\delta^2}{2}\mathbb{E}[X]}. \tag{7.56}$$

In particular, we see that for the variable X defined above:

$$\mathbb{P}(X \leq \tfrac{1}{2}c_2\delta n) \leq e^{-\frac{c_2\delta}{8}n}. \tag{7.57}$$

It follows that for $c_3 = \frac{1}{2}c_2\delta$,

$$\phi_{n,\lambda,q}[e_o \notin R(c_3, c_4)] \leq \mathbb{P}(X \leq \tfrac{1}{2}c_2\delta n) \leq 2^{-2c_5n} \tag{7.58}$$

where the constant c_5 depends only on λ. □

We are now ready to prove Lemma 7.3.1:

Proof of Lemma 7.3.1. Fix $\lambda > \lambda_c$, let $\lambda' = \frac{1}{2}(\lambda_c + \lambda)$, and let $\alpha = \frac{1}{q}(\lambda - \lambda')$. By Lemma 2.1.13, we know that

$$\phi_{n,\lambda,q}[|\mathcal{C}_{\max}| \geq \epsilon 2^n] \geq \phi_{n,\lambda',q,\alpha}[|\mathcal{C}_{\max}| \geq \epsilon 2^n]. \tag{7.59}$$

By definition, we may expand the sprinkled random cluster measure as

$$\phi_{n,\lambda',q,\alpha}[|\mathcal{C}_{\max}| \geq \epsilon 2^n] = \sum_{\omega} \mathbb{1}_{|\mathcal{C}_{\max}| \geq \epsilon 2^n}(\omega) \sum_{\xi} \phi_{n,\lambda',q}[\xi]\Pi_{\alpha/n}[\xi, \omega]. \tag{7.60}$$

Given a percolation configuration ω, define the event

$$R_{n,c_3,c_4,c_5} = \{\omega : |R(c_3, c_4)| \geq 2^n - 2^{(1-c_5)n}\} \tag{7.61}$$

where $R(c_3, c_4)$ is the number of (c_3, c_4)-rich vertices in the graph $Q_n(\omega)$. We may produce a lower bound on (7.60) by inserting the indicator function $\mathbb{1}_{R_{n,c_3,c_4,c_5}}(\xi)$, which is equivalent to adding the constraint that at least $2^n - 2^{(1-c_5)n}$ vertices have c_3n neighbours belonging to atoms of size c_4n^2, even before the sprinkling is applied. This yields the bound

$$\begin{aligned}\phi_{n,\lambda',q,\alpha}[|\mathcal{C}_{\max}| \geq \epsilon 2^n] &\geq \sum_{\omega} \mathbb{1}_{|\mathcal{C}_{\max}|\geq \epsilon 2^n}(\omega) \sum_{\xi} \mathbb{1}_{R_{n,c_3,c_4,c_5}}(\xi)\phi_{n,\lambda',q}[\xi]\Pi_{\alpha/n}[\xi,\omega] \\ &= \sum_{\xi} \mathbb{1}_{R_{n,c_3,c_4,c_5}}(\xi)\phi_{n,\lambda',q}[\xi] \sum_{\omega} \mathbb{1}_{|\mathcal{C}_{\max}|\geq \epsilon 2^n}(\omega)\Pi_{\alpha/n}[\xi,\omega]. \quad (7.62)\end{aligned}$$

Observe that the quantity $\sum_{\omega} \mathbb{1}_{|\mathcal{C}_{\max}|\geq \epsilon 2^n}(\omega)\Pi_{\alpha/n}[\xi,\omega]$ is equal to the probability that the largest component of the random graph is of size at least $\epsilon 2^n$ after sprinkling has been applied to the configuration ξ. We claim that for any $\xi \in R_{n,c_3,c_4,c_5}$, this has the bound

$$\sum_{\omega} \mathbb{1}_{|\mathcal{C}_{\max}|\geq \epsilon 2^n}(\omega)\Pi_{\alpha/n}[\xi,\omega] = 1 - o(1). \quad (7.63)$$

Indeed, observe that for any $\xi \in R_{n,c_3,c_4,c_5}$, at least $\frac{1}{2}c_3 2^n$ vertices have $c_4 n$ neighbours (in Q_n) belonging to atoms in $Q_n(\xi)$. As each vertex has at most n neighbours in total, it follows that at least $\frac{1}{2}c_3 c_4 2^n = c_6 2^n$ vertices belong to atoms in $Q_n(\xi)$. Set $\epsilon = \frac{1}{3}c_6$. To prove the claim (7.63), it is sufficient to show that after sprinkling with probability $\delta = \alpha/n$, there exists a connected component such that even the union of atoms contained within the component has size at least $\epsilon 2^n$ with probability $1 - o(1)$.

If our claim is not true, then there exists some union A of atoms which remains separated from the union B of the remaining atoms after sprinkling. Moreover, as we assumed that at least $3\epsilon 2^n$ vertices belong to atoms and no component contains a union of atoms of size at least $\epsilon 2^n$, we may choose A such that

$$\epsilon 2^n \leq |A| \leq 2\epsilon 2^n. \quad (7.64)$$

Similarly, we have $|B| \geq \epsilon 2^n$. As there are at most $2^n/(c_4 n^2)$ disjoint atoms, there are at most $2^{2^n/(c_4 n^2)}$ choices for A. To prove our claim, it is sufficient to prove that the probability A and B remain separated after sprinkling is at most $\exp(-K_1 2^n/n^2)$, where $K_1 > 0$ is chosen sufficiently large that

$$\lim_{n\to\infty} 2^{2^n/(c_4 n^2)} \exp(-K_1 2^n/n^2) = 0. \quad (7.65)$$

To this end, we define the set

$$D = \mathcal{N}(A) \cap \mathcal{N}(B). \quad (7.66)$$

Let $K_2 > 0$ be a positive constant (to be fixed later). We now consider two further cases, corresponding to "big" and "small" intersections of the neighbourhoods of A and B:

1. $|D| > 2K_2 2^n/n$: On the event $\xi \in R_{n,c_3,c_4,c_5}$, we know that $|R(c_3,c_4)^c| \le 2^{(1-c_5)n}$. In particular, for sufficiently large n, the set $D \cap R(c_3,c_4)$ contains at least $K_2 2^n/n$ vertices neighbouring both A and B, all of which have at least $c_3 n$ neighbours in atoms. As a result, any vertex in the set $D \cap R(c_3,c_4)$ must have at least $\frac{1}{2}c_3 n$ neighbours either A or B. Consequently, A and B are connected through a given vertex in $D \cap R(c_3,c_4)$ during sprinkling with probability at least

$$\tfrac{\alpha}{n}(1-(1-\tfrac{\alpha}{n})^{\frac{1}{2}c_3 n}) \ge c_7/n. \tag{7.67}$$

These randomisations are independent, so A and B are not connected through at least one vertex in the set $D \cap R(c_3,c_4)$ with probability at most

$$(1-c_7/n)^{K_2 2^n/n} \le \exp\{-K_1 2^n/n\} \tag{7.68}$$

provided that K_2 was chosen sufficiently large.

2. $|D| \le 2K_2 2^n/n$: As $\epsilon 2^n \le |A| \le |\mathcal{N}(A)| \le N - |B| \le (1-c_0)2^n$, we may apply Lemma 7.1.2 to deduce that the edge boundary $\partial_e(A)$ contains at least $2c_8 2^n/\sqrt{n}$ disjoint edges. After removing any vertices contained in $D \cup R(c_3,c_4)^c$, we have at least $c_8 2^n/\sqrt{n}$ disjoint edges from $b(A)$ to $b(B)$. The endpoints of each of these edges has $c_3 n$ neighbours in A, B respectively, so A and B are connected through one of these disjoint edges during sprinkling with probability at least

$$\tfrac{\alpha}{n}(1-(1-\tfrac{\alpha}{n})^{c_7 n})^2 \ge c_9/n. \tag{7.69}$$

These randomisations are independent, so we do not connect through one of these edges with probability at most

$$(1-c_9/n)^{c_8 2^n/\sqrt{n}} \le \exp\{-K_1 2^n/n^2\} \tag{7.70}$$

provided n is sufficiently large.

We have shown that the claim (7.63) holds. It follows that

$$\phi_{n,\lambda',q,\alpha}[|\mathcal{C}_{\max}| \ge \epsilon 2^n] \ge (1-o(1))\phi_{n,\lambda,q}[|R(c_3,c_4)| \ge 2^n - 2^{(1-c_5)n}]. \tag{7.71}$$

It remains to observe that we may choose c_3, c_4, c_5 according to Lemma 7.3.4 to ensure that $\phi_{n,\lambda,q}[|R(c_3,c_4)| \ge 2^n - 2^{(1-c_5)n}] \ge 1 - 2^{-2c_5 n}$. □

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