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NUMBER RIGIDITY OF THE STOCHASTIC AIRY OPERATOR

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ABSTRACT. We prove that the spectrum of the stochastic Airy operator is rigid in the sense of Ghosh and Peres [22] for Dirichlet and Robin boundary conditions. This proves the rigidity of the Airy- β point process and the soft-edge limit of rank-1 perturbations of Gaussian β -Ensembles for any $\beta > 0$, and solves an open problem mentioned in [9]. Our proof uses a combination of the semigroup theory of the stochastic Airy operator and the techniques for studying insertion and deletion tolerance of point processes developed in [24].

1. INTRODUCTION

In this paper, we are interested in the spectrum of a random Schrödinger operator called the stochastic Airy operator. The stochastic Airy operator acts on functions $f : [0, \infty) \rightarrow \mathbb{R}$ on the positive half line satisfying a general boundary condition of the form $af'(0) + bf(0) = 0$ for some $a, b \in \mathbb{R}$. The operator in question, which we denote by H_β , is defined as

$$(1.1) \quad H_\beta f(x) := -f''(x) + \left(x + \frac{2}{\sqrt{\beta}}W'(x)\right)f(x),$$

where $\beta > 0$ is a fixed parameter and W is a standard Brownian motion. Given that Brownian motion sample paths are not differentiable, a rigorous definition of H_β is nontrivial. Nevertheless, it can be shown that once a suitable boundary condition is fixed, H_β is a bonafide semibounded self-adjoint operator with compact resolvent when defined through its quadratic form [3, 17, 26, 30]. We refer to Section 2.1 below for more details.

Much of the interest of studying the spectrum of H_β comes from the theories of random matrices and interacting particle systems. Following the seminal work of Dumitriu and Edelman [15], Edelman and Sutton [16] introduced the stochastic Airy operator as a means of characterizing the soft-edge scaling limits of interacting particle systems known as the β -Hermite and β -Laguerre ensembles (see, e.g., [16, Sections 2.1.1 and 2.1.2]). This program was later realized by Ramírez, Rider and Virág [30], and then extended to more general ensembles by Bloemendal, Krishnapur, Rider, and Virág [3, 25] (respectively, rank-1 spiked β -ensembles and general Dyson β -ensembles).

In this paper, we prove that H_β 's eigenvalue point process is number rigid for any choice of $\beta > 0$, both for Dirichlet and Robin boundary conditions. Roughly speaking, a point process Π on \mathbb{R}^d is number rigid if there exists a class of subsets $B \subset \mathbb{R}^d$ such that the number of Π 's points inside B is uniquely determined by the configuration of points outside B . We prove that this rigidity property holds for H_β 's spectrum for every Borel set $B \subset \mathbb{R}$ that is bounded above, that is, for which there exists some $c \in \mathbb{R}$ such that $B \subset (-\infty, c]$. We refer to Theorem 2.7 below for a precise statement.

The result proved in this paper contributes to the program that is concerned with understanding the spatial conditioning of point processes with dependence. That is, if we condition a point process on having some fixed configuration outside of some set B , then what is the conditional distribution of points inside B ? In the specific context of random matrix theory, such problems have been motivated by imperfect observations in nuclear physics experiments and the connection between energy levels of heavy nuclei and random matrix eigenvalues; see [13, Section 1.1] and references therein for more details. From this point of view, number rigidity can be seen as a first step towards exploring questions of spatial conditioning for edge β -Ensembles, and naturally leads to further investigations in this direction, some of which we discuss in Section 1.2.

1.1. Previous Results. The concept of number rigidity was formally introduced by Ghosh and Peres in [22] (see also [2, 24]) in the context of the Ginibre process and Gaussian analytic function zeroes, and has since then led to a number of interesting developments. Using a variety of methods, number rigidity has been established for a number of point processes, including for example perturbed lattices [24, 27], determinantal/Pfaffian point processes [8, 9], hyperuniform point processes [20], Sine_β point processes [14, 21, 10], and more.

Looking more specifically at the stochastic Airy operator, until now the number rigidity of H_β 's spectrum was only known in the case $\beta = 2$ with Dirichlet boundary condition (i.e., $f(0) = 0$). This result, which is due to Bufetov [8], relies crucially on the fact that in this particular case, H_β 's eigenvalues form a determinantal point process. Thus, the method used in [8] is not amenable to extension to $\beta \neq 2$. The number rigidity of H_β 's spectrum with $\beta = 1, 4$ and Dirichlet boundary condition was raised as an open problem in [9] in the context of Pfaffian point processes; to the best of our knowledge, this question remained open until now.

In particular, the methods previously used to prove number rigidity for general (i.e., $\beta \neq 2$) Sine- β processes in [14, 10] do not appear to be directly applicable to the edge. On the one hand, the method used in [10] relies on moment estimates of the circular β -Ensembles, from which the Sine- β process can be obtained as a scaling limit. On the other hand, the proof in [14] relies on establishing a much stronger property encapsulated by the canonical DLR equation ([14, Theorem 2.1]). The proof of this DLR property in [14] also relies on using the circular β -Ensemble approximation, which is not available at the edge. Rigidity properties of general β -Ensembles is addressed in [5] and further refined in [1, 6]; however, these estimates of the rigidity do not pass to the limit under the edge scaling and thus do not appear to immediately yield number rigidity for the Airy- β process.

Motivated in part by the desire to prove the number rigidity of H_β for $\beta \neq 2$, in [18] a new strategy to prove number rigidity in the spectrum of general random Schrödinger operators was developed. The strategy in question, which is based on a sufficient condition for rigidity due to Ghosh and Peres [22], consists of proving that the variance of the trace of the semigroup e^{-tH} of a Schrödinger operator H vanishes as $t \rightarrow 0$ using the Feynman-Kac formula. We refer to Section 3.1 for more details. While the main result of [18] establishes the number rigidity of the spectrum of a wide class of one-dimensional continuous random Schrödinger operators, it unfortunately does not apply to H_β , as in this case the variance of the trace of H_β 's semigroup is merely bounded as $t \rightarrow 0$.

In this context, the main improvement in this paper consists of replacing the sufficient condition for number rigidity by Ghosh and Peres with a more general result inspired by the work of Holroyd and Soo [24]. The result in question, which we state as Lemma 3.2 in the context of the stochastic Airy operator, allows to replace the vanishing of the trace of the semigroup by a weaker covariance condition.

Lastly, we note that the rigidity properties of point processes is an important tool in the derivation of a variety of fundamental results in random matrix theory and interacting particle systems, such as local laws and universality. The most commonly-used notion of rigidity considers the maximal deviation of the point process from a set of deterministic values that, in some sense, captures the typical behavior of each point. Many of the recent works in this direction have focused on the rigidity of general β -Ensembles (e.g, [1, 5, 6, 12]). The results of this paper studies the rigidity of β -Ensembles from a different point of view—the number rigidity of the edge-limit—and thus complements these previous investigations (in particular, number rigidity is not a corollary of this other notion of rigidity).

1.2. Open Problems. The results in this paper raise a number of interesting questions for future research. Most notably, given that number rigidity is now established for both the bulk and soft-edge limits of β -Ensembles and their rank-1 perturbations (by combining [8, 9, 14, 21, 10] with the present paper), it is natural to investigate whether or not number rigidity occurs in the other point processes that describe the scaling limits of random matrix ensembles:

Problem 1.1. For a given integer $r \geq 2$ and a rank- r spike matrix $S = \sum_{i=1}^r w_i u_i u_i^\dagger$, consider the $r \times r$ multivariate stochastic Airy operator

$$H_\beta^{(r)} := -\frac{d^2}{dx^2} + \sqrt{2}B'_x + rx$$

with initial condition $f'(0) = Sf(0)$, as defined in [4] for $\beta = 1, 2, 4$ (i.e., the soft-edge scaling limit of rank- r perturbations of Gaussian β -Ensembles). Is the eigenvalue point process of $H_\beta^{(r)}$ number rigid?

Problem 1.2. Consider the hard-edge scaling limit of the β -Laguerre ensembles, as characterized by the stochastic Bessel operator or its spiked version (i.e., [16, 28, 29, 32]). Is this point process number rigid?

While it seems plausible that these problems could be approached with semigroup methods, substantial modifications would need to be implemented in order to extend our arguments to this setting. We thus leave these two problems open for future investigations.

Finally, it would be interesting to see if more detailed information about the spatial conditioning of the general Airy- β process could be obtained. In particular, a natural starting point would be the following:

Problem 1.3. Fix a boundary condition for H_β , and let $B \subset \mathbb{R}$ be a Borel set that is bounded above. Suppose that we condition on the configuration of H_β 's eigenvalues outside B . By number rigidity, there exists a deterministic $N \in \mathbb{N}$ (which only depends on the configuration of eigenvalues outside B) that is equal to the number of H_β 's eigenvalues inside B . Let $\zeta \in B^N$ be the random vector whose components are H_β 's eigenvalues inside B (conditional on the configuration outside B), taken in a uniformly random order. What is the support of ζ 's probability distribution on B^N ? In particular, is H_β 's eigenvalue point process *tolerant* in the sense of Ghosh and Peres [23]; that is, are ζ 's probability distribution and the Lebesgue measure on B^N mutually absolutely continuous?

Number rigid point processes for which tolerance has been proved include the Ginibre point process [23, Theorem 1.2] and the Sine- β process for every $\beta > 0$ [14, Corollary 1.2] (see also [7] for the Sine-2 process). To the best of our knowledge, no such result is known for the Airy- β process for any choice of $\beta > 0$.

1.3. Organization. The remainder of this paper is organized as follows: In Section 2, we provide a precise definition of the stochastic Airy operator and state our main result, namely, Theorem 2.7. In Section 3, we provide an outline of the main steps of our proof of Theorem 2.7. Then, in Section 4, we provide the proofs of a variety of technical results stated in Section 3.

2. SETUP AND MAIN RESULT

2.1. Stochastic Airy Operator. The main goal of this section is to introduce the stochastic Airy operator and state the main result of this paper. We begin by introducing a few notations. We use $V : [0, \infty) \rightarrow \mathbb{R}$ to denote the linear function $V(x) = x$. Given $f, g \in L^2([0, \infty))$, we use the standard notations

$$\langle f, g \rangle := \int_0^\infty f(x)g(x) \, dx \quad \text{and} \quad \|f\|_2 := \left(\int_0^\infty f(x)^2 \, dx \right)^{1/2}.$$

We define the norm

$$\|f\|_*^2 := \|f'\|_2^2 + \|V^{1/2}f\|_2^2 + \|f\|_2^2.$$

We use $C_0^\infty([0, \infty))$ to denote the set of smooth and compactly supported functions from $[0, \infty)$ to \mathbb{R} .

Let W be a standard Brownian motion. Given a continuous and compactly supported map $\varphi : [0, \infty) \rightarrow \mathbb{R}$, we denote the stochastic integral

$$\xi_\beta(\varphi) := \frac{2}{\sqrt{\beta}} \int_0^\infty \varphi(x) \, dW(x)$$

for every $\beta > 0$. In particular, $\varphi \mapsto \xi_\beta(\varphi)$ is Gaussian process with covariance given by the inner product $\frac{4}{\beta} \langle \cdot, \cdot \rangle$. The basis of the definition of H_β 's spectrum is the following quadratic form:

Definition 2.1. Let $\beta > 0$ and $w \in (-\infty, \infty]$. Define the set

$$\text{FC}_{\beta,w} := \begin{cases} C_0^\infty([0, \infty)) & \text{if } w < \infty \\ \{\varphi \in C_0^\infty([0, \infty)) : \varphi(0) = 0\} & \text{if } w = \infty. \end{cases}$$

Then, for every $\varphi \in \text{FC}_{\beta,w}$, we define the quadratic form

$$(2.1) \quad \mathcal{E}_{\beta,w}(\varphi) := \begin{cases} -w\varphi(0)^2 + \|\varphi'\|_2^2 + \|V^{1/2}\varphi\|_2^2 + \xi_\beta(\varphi^2) & \text{if } w < \infty \\ \|\varphi'\|_2^2 + \|V^{1/2}\varphi\|_2^2 + \xi_\beta(\varphi^2) & \text{if } w = \infty. \end{cases}$$

The spectrum of the stochastic Airy operator was first constructed using the form (2.1) in [3, 30]. The following statement is a special case of [17, Propositions 3.2 and 3.4] (see also [26]):

Theorem 2.2 ([3, 17, 26, 30]). *Let $\beta > 0$ and $w \in (-\infty, \infty]$. The quadratic form $\mathcal{E}_{\beta,w}$ can be continuously extended with respect to $\|\cdot\|_*$ to a domain $D(\mathcal{E}_{\beta,w})$ on which it is closed and semibounded. In particular, there exists a unique self-adjoint operator $H_{\beta,w}$ on $L^2([0, \infty))$ whose quadratic form is given by (2.1). Almost surely, $H_{\beta,w}$ has compact resolvent.*

By examining (2.1), we note that $w < \infty$ corresponds to the case where H_β acts on functions with Robin boundary condition $f'(0) = wf(0)$, and $w = \infty$ corresponds to the Dirichlet boundary condition $f(0) = 0$. As an immediate consequence of the above result, we conclude the following regarding $H_{\beta,w}$'s spectrum:

Corollary 2.3. *Let $\beta > 0$ and $w \in (-\infty, \infty]$. Almost surely, the operator $H_{\beta,w}$ has a purely discrete spectrum of eigenvalues*

$$(2.2) \quad -\infty < \Lambda_{\beta,w}(1) \leq \Lambda_{\beta,w}(2) \leq \Lambda_{\beta,w}(3) \leq \dots$$

that is bounded below and without accumulation point.

Remark 2.4. The eigenvalues can be defined through the min-max formula using the quadratic form (2.1). Following [3, 30], $\Lambda_{\beta,w}(1)$ can be described as

$$(2.3) \quad \Lambda_{\beta,w}(1) := \inf \{ \langle f, H_{\beta,w}f \rangle : f \in D(\mathcal{E}_{\beta,w}), \|f\|_2 = 1 \}.$$

Suppose $f_1^* \in \mathcal{E}_{\beta,w}$ is such that $\Lambda_{\beta,w}(1)$ is equal to $\langle f_1^*, H_{\beta,w}f_1^* \rangle$. Then, $\Lambda_{\beta,w}(2)$ is the infimum of the quadratic form over $f \in D(\mathcal{E}_{\beta,w})$ with $\|f\|_2 = 1$ that are orthogonal to f_1^* w.r.t. $\langle \cdot, \cdot \rangle$, and so on for higher eigenvalues.

2.2. Main Result. We first define the eigenvalue point process of $H_{\beta,w}$ and then state our main result.

Definition 2.5 (Point Process). Let $A \subset \mathbb{R}$ be a Borel set and \mathcal{B}_A be the associated Borel σ -algebra. We use $\mathcal{N}_A^\#$ to denote the set of integer-valued measures μ on (A, \mathcal{B}_A) such that $\mu(K) < \infty$ for every bounded set $K \subset A$. We equip $\mathcal{N}_A^\#$ with the topology generated by the maps

$$\mu \mapsto \int f(x) d\mu(x), \quad f \text{ continuous and compactly supported,}$$

as well as the associated Borel σ -algebra. For every $\mu \in \mathcal{N}_A^\#$, there exists a sequence $(x(k))_{k \in \mathbb{N}} \subset A$ without accumulation point such that $\mu = \sum_{k=1}^{\infty} \delta_{x(k)}$, where δ_x denotes the Dirac mass at a point $x \in \mathbb{R}$; that is, for any Borel set $B \subset \mathbb{R}$, one has $\delta_x(B) = \mathbf{1}_{\{x \in B\}}$. A point process on A is a random element that takes values in $\mathcal{N}_A^\#$.

Definition 2.6 (Eigenvalue point process and its restrictions). For every $\beta > 0$ and $w \in (-\infty, \infty]$, we define $H_{\beta,w}$'s eigenvalue point process as

$$\Pi_{\beta,w} := \sum_{k=1}^{\infty} \delta_{\Lambda_{\beta,w}(k)} \in \mathcal{N}_{\mathbb{R}}^\#.$$

For every Borel set $A \subset \mathbb{R}$, we denote the restriction

$$\Pi_{\beta,w}|_{A^c} := \sum_{k=1}^{\infty} \mathbf{1}_{\{\Lambda_{\beta,w}(k) \in A^c\}} \delta_{\Lambda_{\beta,w}(k)} \in \mathcal{N}_{A^c}^\#.$$

That is, the same random measure as $\Pi_{\beta,w}$, but excluding the masses in A .

We say that a Borel set $B \subset \mathbb{R}$ is *bounded above* if there exists some $c \in \mathbb{R}$ such that $B \subset (-\infty, c]$. By Corollary 2.3, $\Pi_{\beta,w}$ is a random counting measure on \mathbb{R} such that, almost surely, $\Pi_{\beta,w}(B) < \infty$ for every Borel set B that is bounded above. With this in hand, the main result of this paper is the following:

Theorem 2.7. *For every $\beta > 0$, $w \in (-\infty, \infty]$, and any bounded above Borel set $B \subset \mathbb{R}$, there exists a deterministic measurable function $F_B : \mathcal{N}_{B^c}^\# \rightarrow \mathbb{R}$ such that*

$$(2.4) \quad \Pi_{\beta,w}(B) = F_B(\Pi_{\beta,w}|_{B^c})$$

almost surely, where $\Pi_{\beta,w}(B)$ is the number of points of $\Pi_{\beta,w}$ inside the Borel set B and $\Pi_{\beta,w}|_{B^c} \in \mathcal{N}_{B^c}^\#$ is the restriction of the point process Π to the complement of B .

Remark 2.8. Theorem 2.7 shows that the number of $H_{\beta,w}$'s eigenvalues inside B is uniquely determined by the configuration of $H_{\beta,w}$'s eigenvalues outside of B . As per the terminology introduced by Ghosh and Peres in [22], any point process which satisfies this property for any bounded Borel set is number rigid. Since any bounded Borel set on \mathbb{R} is bounded above, Theorem 2.7 proves that $H_{\beta,w}$ is number rigid in the sense of Ghosh and Peres [22].

3. PROOF OUTLINE

In this section, we provide an outline of the proof of Theorem 2.7. In order to keep the argument readable, we defer the proof of several technical lemmas to Section 4. We also take this opportunity to showcase how the proof method used in this paper improves on that of [18].

Remark 3.1. Unless otherwise mentioned, for the remainder of this section we assume that $\beta > 0$ and $w \in (-\infty, \infty]$ are fixed and arbitrary.

3.1. Step 1 - The Semigroup Approach: A New Sufficient Condition. Our method to prove Theorem 2.7 is based on the semigroup of the stochastic Airy operator. More specifically, the key objects of study are the traces

$$\mathrm{Tr}[e^{-tH_{\beta,w}}] = \sum_{k=1}^{\infty} e^{-t\Lambda_{\beta,w}(k)} = \int_{\mathbb{R}} e^{-tx} d\Pi_{\beta,w}(x)$$

for some positive $t > 0$. Our proof of rigidity is based on the following sufficient condition:

Lemma 3.2. *Suppose that*

$$\limsup_{t \rightarrow 0} \mathbf{Var}[\mathrm{Tr}[e^{-tH_{\beta,w}}]] < \infty,$$

and that for every fixed $u > 0$,

$$(3.1) \quad \lim_{t \rightarrow 0} \mathbf{Cov}[\mathrm{Tr}[e^{-tH_{\beta,w}}], \mathrm{Tr}[e^{-uH_{\beta,w}}]] = 0.$$

Then, $H_{\beta,w}$ satisfies (2.4) for any bounded above Borel set B .

Lemma 3.2 is proved in Section 4.1. This result is inspired by the work of Holroyd and Soo [24] on *deletion-tolerance* (see [24, Page 2] for a definition of deletion-tolerance), as well as the connection between deletion-tolerance and number rigidity that was pointed out in [27, Proposition 1.2]. More specifically, a sufficient condition similar to Lemma 3.2 was used in [24, Lemma 7.2] to prove that the two-dimensional perturbed lattice is not deletion-tolerant. We note that in [24, Lemma 7.2], the trace $\text{Tr}[e^{-tH_{\beta,w}}]$ is replaced by a different linear functional of the perturbed lattice, denoted $\Lambda(h_k)$ therein; see Remark 3.5 for more details on the distinction between traces of semigroups and general linear statistics in questions of rigidity. We refer to Section 4.1 for the details of how the argument used in [24, Lemma 7.2] is adapted to our setting.

Semigroup theory is an attractive strategy to prove number rigidity for general Schrödinger operators (including $H_{\beta,w}$) because of the Feynman-Kac formula. In short, the Feynman-Kac formula states that the semigroups of Schrödinger operators admit an explicit formulation in terms of elementary stochastic processes; see, e.g., [33]. The Feynman-Kac formula for $H_{\beta,w}$ specifically was proved in [23] in the case $w = \infty$ and [19] in the case $w < \infty$. Following [17, Theorem 2.24], we first introduce the necessary notations, and then proceed to state the Feynman-Kac formula for the trace of $e^{-tH_{\beta,w}}$.

Let B denote a standard Brownian motion on \mathbb{R} , and Let X denote a reflected Brownian motion on $(0, \infty)$ (i.e., the absolute value of a Brownian motion). Let $Z = B$ or $Z = X$; for every $x, y, t > 0$ we denote the conditioned process

$$Z^x := (Z | Z(0) = x) \quad \text{and} \quad Z_t^{x,y} := (Z | Z(0) = x \text{ and } Z(t) = y).$$

We use $L_t^a(Z)$ ($a, t > 0$) to denote the continuous version of the local time of some process Z (or its conditioned versions) on the time interval $[0, t]$, that is, the stochastic process such that for every Borel measurable $f : (0, \infty) \rightarrow \mathbb{R}$, one has

$$\int_0^t f(Z(s)) \, ds = \int_0^\infty L_t^a(Z) f(a) \, da = \langle L_t(Z), f \rangle.$$

(See, e.g., [31, Chapter VI, Corollary 1.6 and Theorem 1.7].)

We use $\mathfrak{L}_t^0(Z)$ ($t > 0$) to denote the boundary local time of a process Z (or its conditioned versions) on the time interval $[0, t]$, that is,

$$\mathfrak{L}_t^0(Z) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{Z(s) \in [0, \varepsilon]\}} \, ds.$$

Theorem 3.3 ([17, 19, 23]). *For every $t > 0$,*

$$(3.2) \quad \text{Tr}[e^{-tH_{\beta,w/2}}] = \int_0^\infty \frac{1 + e^{-2x^2/t}}{\sqrt{2\pi t}} \mathbf{E} \left[e^{-(L_t(X_t^{x,x}), \frac{1}{2}V) - \frac{1}{2}\xi_\beta(L_t(X_t^{x,x})) - w\mathfrak{L}_t^0(X_t^{x,x})} \Big| \xi_\beta \right] dx,$$

where X is assumed to be independent of ξ_β in the above conditional expectation. We use the convention $e^{-\infty \cdot \mathfrak{L}_t^0(X)} = \mathbf{1}_{\{\mathfrak{L}_t^0(X)=0\}}$ in the case where $w = \infty$, and we recall that $V(x) = x$.

In Sections 3.2 and 3.3, we outline how Theorem 3.3 can be used to show that the conditions in Lemma 3.2 hold, and thus that $H_{\beta,w}$'s spectrum is number rigid. However, before moving on to this outline, we briefly comment on how Lemma 3.2 improves on our previous work in [18]. In [18], the sufficient condition for number rigidity that we used was the following special case of the general strategy devised by Ghosh and Peres in [22, Theorem 6.1]:

Proposition 3.4 ([22]). *Let H be a random Schrödinger operator with (almost surely) compact resolvent. If*

$$(3.3) \quad \lim_{t \rightarrow 0} \mathbf{Var} [\text{Tr}[e^{-tH}]] = 0,$$

then H 's spectrum is number rigid.

Remark 3.5. The strategy in [22, Theorem 6.1] is more general than what is stated in Proposition 3.4: Let Π be a point process. To prove that Π is number rigid, it suffices to show that for any bounded Borel set $B \subset \mathbb{R}$, there is a sequence of functions $(f_n)_{n \in \mathbb{N}}$ that converge uniformly to 1 on B such that the variance of the *linear statistics* $\mathbf{Var} \left[\int f_n(x) \, d\Pi(x) \right]$ vanishes as $n \rightarrow \infty$. In this paper and [18], we use the functions $f(x) = e^{-tx}$ ($t > 0$) because the Feynman-Kac formula makes it possible to compute explicitly the variance and covariance of the trace of e^{-tH} .

Unfortunately, the best that we could achieve in [18] using (3.2) was the following result:

Proposition 3.6 ([18]). *It holds that*

$$(3.4) \quad \limsup_{t \rightarrow 0} \mathbf{Var} [\text{Tr}[e^{-tH_{\beta,w}}]] < \infty.$$

Using the notation/terminology of [18], the above proposition follows from [18, Theorem 4.1] in **Case 2** (i.e., operators on the positive half-line) with white noise (whereby $\mathfrak{d} = 3/2$ by [18, (2.16)]), and $\mathfrak{a} = 1$, $\kappa = 1/2$, and $\nu = 0$, the latter of which corresponds to the case $\frac{1}{2}V = \frac{\mathfrak{a}}{2}$. Moreover, by exploiting the special integrable structure available in the case $\beta = 2$, it can be shown that the finiteness of the limsup in (3.4) cannot be improved to a vanishing limit in general:

Proposition 3.7 ([18, Proposition 2.27]). *It holds that*

$$\lim_{t \rightarrow 0} \mathbf{Var}[\mathrm{Tr}[e^{-tH_{2,\infty}}]] = (4\pi)^{-1}.$$

Proposition 3.7 shows that (3.3) cannot hold for $H_{\beta,w}$ when $\beta = 2$ and $w = \infty$. Thus, the general number rigidity of $H_{\beta,w}$ cannot be proved using Proposition 3.4 since the condition (3.3) cannot be met for every choice of β and w . In particular, one of the key innovations in this paper is to replace the vanishing variance condition (3.3) with the weaker conditions (3.1) and (3.4). We now explain how we use the Feynman-Kac formula to achieve these weaker conditions.

3.2. Step 2 - Covariance Formula. Thanks to (3.4) and Lemma 3.2, the proof of Theorem 2.7 is reduced to proving (3.1). For this purpose, we derive a formula of the covariance in (3.1). Before proceeding to that derivation, we introduce a few notations.

For every $t, u, x, y > 0$, we let $X_t^{x,x}$ and $\bar{X}_u^{y,y}$ be independent, where $\bar{X}_u^{y,y}$ has the same law as $X_u^{y,y}$. We define the shorthands

$$\begin{aligned} \mathcal{P}_{t,u}(x,y) &:= \left(\frac{1 + e^{-2x^2/t}}{\sqrt{2\pi t}} \right) \left(\frac{1 + e^{-2y^2/u}}{\sqrt{2\pi u}} \right), \\ \mathcal{A}_{t,u}(x,y) &:= -\langle L_t(X_t^{x,x}) + L_u(\bar{X}_u^{y,y}), \frac{1}{2}V \rangle \\ \mathcal{B}_{t,u}(x,y) &:= -w\mathfrak{L}_t^0(X_t^{x,x}) - w\mathfrak{L}_u^0(\bar{X}_u^{y,y}), \\ \mathcal{C}_{t,u}(x,y) &:= \frac{1}{2\beta} (\|L_t(X_t^{x,x})\|_2^2 + \|L_u(\bar{X}_u^{y,y})\|_2^2) \\ \mathcal{D}_{t,u}(x,y) &:= \frac{\langle L_t(X_t^{x,x}), L_u(\bar{X}_u^{y,y}) \rangle}{\beta}. \end{aligned}$$

In the following lemma, we state the formula of the covariance in (3.1) in terms of the above notations.

Lemma 3.8. *For every $t, u > 0$,*

$$\mathbf{Cov}[\mathrm{Tr}[e^{-tH_{\beta,w/2}}], \mathrm{Tr}[e^{-uH_{\beta,w/2}}]] = \int_{(0,\infty)^2} \mathcal{P}_{t,u}(x,y) \mathbf{E} \left[e^{\mathcal{A}_{t,u}(x,y) + \mathcal{B}_{t,u}(x,y) + \mathcal{C}_{t,u}(x,y)} (e^{\mathcal{D}_{t,u}(x,y)} - 1) \right] dx dy.$$

Lemma 3.8 is proved in Section 4.2. Applying Hölder's inequality on the right hand side of the above identity, we thus obtain that

$$\mathbf{Cov}[\mathrm{Tr}[e^{-tH_{\beta,w/2}}], \mathrm{Tr}[e^{-uH_{\beta,w/2}}]]$$

is bounded above by

$$(3.5) \quad \sup_{x,y>0} \left\{ \mathcal{P}_{t,u}(x,y) \mathbf{E} \left[e^{4\mathcal{B}_{t,u}(x,y)} \right]^{1/4} \mathbf{E} \left[e^{4\mathcal{C}_{t,u}(x,y)} \right]^{1/4} \right\} \cdot \int_{(0,\infty)^2} \mathbf{E} \left[e^{4\mathcal{A}_{t,u}(x,y)} \right]^{1/4} \mathbf{E} \left[\left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right)^4 \right]^{1/4} dx dy.$$

The following step, which is the last step of the proof, focuses on showing that the expression in (3.5) vanishes as $t \rightarrow 0$.

3.3. Step 3 - Technical Estimates and Conclusion. In order to provide a quantitative control on the terms appearing in (3.5), we need a number of technical estimates. First, it is easy to see by definition of \mathcal{P} that for every fixed $u > 0$,

$$(3.6) \quad \limsup_{t \rightarrow 0} t^{1/2} \sup_{x,y>0} \mathcal{P}_{t,u}(x,y) < \infty.$$

For the remaining terms in (3.5), we have the following:

Lemma 3.9. *For every $u > 0$, there exists constant $c > 0$ such that*

$$\limsup_{t \rightarrow 0} \sup_{x,y>0} \mathbf{E} \left[e^{4\mathcal{B}_{t,u}(x,y)} \right] \leq c < \infty$$

and

$$\limsup_{t \rightarrow 0} \sup_{x,y>0} \mathbf{E} \left[e^{4\mathcal{C}_{t,u}(x,y)} \right] \leq c < \infty.$$

Lemma 3.10. *For every $u > 0$, there exists a constant $c > 0$ such that for every $t \in (0, 1]$ and $x, y > 0$, one has*

$$\mathbf{E} \left[\left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right)^4 \right]^{1/4} \leq c e^{-(x-y)^2/c} \mathbf{E} \left[\left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right)^8 \right]^{1/8}.$$

Lemma 3.11. *For every $u, c > 0$,*

$$\limsup_{t \rightarrow 0} \int_{(0, \infty)^2} \mathbf{E} \left[e^{4A_{t,u}(x,y)} \right]^{1/4} e^{-(x-y)^2/c} dx dy < \infty.$$

Lemma 3.12. *For every $u > 0$,*

$$\limsup_{t \rightarrow 0} t^{-3/4} \sup_{x, y > 0} \mathbf{E} \left[\left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right)^8 \right]^{1/8} < \infty.$$

At this point, we insert (3.6) and Lemma 3.9 in the first line of (3.5), and then apply Lemmas 3.10–3.12 to the integral on the second line on (3.5). This yields that for every $u > 0$,

$$\mathbf{Cov} [\mathrm{Tr}[e^{-tH_{\beta,w}}], \mathrm{Tr}[e^{-uH_{\beta,w}}]] = O_u(t^{-1/2} \cdot t^{3/4}) = O_u(t^{1/4}),$$

as $t \rightarrow 0$, where the constant in O_u depends on u . This then proves Theorem 2.7.

4. TECHNICAL RESULTS

We now conclude the proof of Theorem 2.7 by providing proofs for the technical estimates stated in the previous section. Many of these results follow from the analysis performed in [18]; we include detailed references when such is the case.

4.1. Proof of Lemma 3.2. This proof draws inspiration from [24, Proposition 7.1 and Lemma 7.2], which the authors had used to study the insertion and deletion tolerance of point processes. We nevertheless provide the argument in full for the convenience of the readers. The argument in question is based on the following generalization of the L^2 weak law of large numbers:

Lemma 4.1. *Let X_1, X_2, \dots be a sequence of random variables with $\mathbf{E}[X_i] = 0$. Suppose there is $C \in (0, \infty)$ so that $\sup_{i \in \mathbb{N}} \mathbf{E}[X_i^2] \leq C$. Moreover, assume that for each fixed i , we have $\mathbf{E}[X_i X_j] \rightarrow 0$ as $j \rightarrow \infty$. Then there exists a subsequence $(r_n)_{n \in \mathbb{N}}$ so that $(X_{r_1} + \dots + X_{r_n})/n$ converges to zero in probability as $n \rightarrow \infty$.*

We provide a proof of Lemma 4.1 in Section 4.1.2. Before doing so, however, we explain how Lemma 4.1 is used to complete the proof of Lemma 3.2.

4.1.1. Proof of Lemma 3.2. To prove Lemma 3.2, we first fix a bounded above Borel set $B \subset \mathbb{R}$. Note that for any $t > 0$, we can write

$$(4.1) \quad \mathrm{Tr} [e^{-tH_{\beta,w}}] = \int_{\mathbb{R}} e^{-tx} d\Pi_{\beta,w}(x) = \int_B e^{-tx} d\Pi_{\beta,w}(x) + \int_{B^c} e^{-tx} d\Pi_{\beta,w}(x).$$

Let t_n be a sequence that converges to zero. By adding and subtracting $\frac{1}{n} \sum_{i=1}^n \mathrm{Tr} [e^{-t_i H_{\beta,w}}]$ and $\frac{1}{n} \sum_{i=1}^n \mathbf{E} [\mathrm{Tr}[e^{-t_i H_{\beta,w}}]]$, and then split the traces $-\mathrm{Tr} [e^{-t_i H_{\beta,w}}]$ using (4.1), we can write

$$\Pi_{\beta,w}(B) = \int_B 1 d\Pi_{\beta,w}(x)$$

as the sum of the following three terms:

$$(4.2) \quad \int_B 1 d\Pi_{\beta,w}(x) - \frac{1}{n} \sum_{i=1}^n \int_B e^{-t_i x} d\Pi_{\beta,w}(x)$$

$$(4.3) \quad \frac{1}{n} \sum_{i=1}^n (\mathrm{Tr} [e^{-t_i H_{\beta,w}}] - \mathbf{E} [\mathrm{Tr}[e^{-t_i H_{\beta,w}}]])$$

$$(4.4) \quad \frac{1}{n} \sum_{i=1}^n \left(\mathbf{E} [\mathrm{Tr}[e^{-t_i H_{\beta,w}}]] - \int_{B^c} e^{-t_i x} d\Pi_{\beta,w}(x) \right).$$

Our goal is to make (4.2) and (4.3) vanish along a subsequence. Note that for (4.3), using the uniformly bounded variance assumption and the vanishing covariance assumption in Lemma 3.2, the sequence of random variables $\mathrm{Tr} [e^{-t_i H_{\beta,w}}] - \mathbf{E} [\mathrm{Tr}[e^{-t_i H_{\beta,w}}]]$ satisfies precisely the assumptions of Lemma 4.1. Therefore, we can always choose a sparse enough sequence of t_i 's such that

$$\frac{1}{n} \sum_{i=1}^n (\mathrm{Tr} [e^{-t_i H_{\beta,w}}] - \mathbf{E} [\mathrm{Tr}[e^{-t_i H_{\beta,w}}]]) \rightarrow 0$$

as $n \rightarrow \infty$ in probability. It follows that there exists a sequence of positive integers $(n_k)_{k \geq 1}$ so that

$$\frac{1}{n_k} \sum_{i=1}^{n_k} (\mathrm{Tr}[e^{-t_i H_{\beta,w}}] - \mathbf{E}[\mathrm{Tr}[e^{-t_i H_{\beta,w}}]]) \rightarrow 0$$

as $k \rightarrow \infty$ almost surely. For (4.2), consider the difference

$$(4.5) \quad \int_B 1 \, d\Pi_{\beta,w}(x) - \int_B e^{-t_i x} \, d\Pi_{\beta,w}(x)$$

for a fixed i . Since B is bounded above, there exists some $C \in (0, \infty)$ such that $C = \sup(B)$. Moreover, by Corollary 2.3, we have that $\Lambda_{\beta,w}(1) > -\infty$ and $\Pi_{\beta,w}(B) < \infty$ almost surely. Thus on this event, (4.5) is bounded above by

$$(4.6) \quad \left| \int_B 1 \, d\Pi_{\beta,w}(x) - \int_B e^{-t_i x} \, d\Pi_{\beta,w}(x) \right| \leq \int_B |1 - e^{-t_i x}| \, d\Pi_{\beta,w}(x) \\ \leq \Pi_{\beta,w}(B) \sup_{x \in B \cap \{\Lambda_{\beta,w}(k): k \geq 1\}} |1 - e^{-t_i x}| \leq \Pi_{\beta,w}(B) \max\{|1 - e^{-t_i \Lambda_{\beta,w}(1)}|, |1 - e^{-t_i C}|\},$$

where the last inequality comes from the fact that the maximum of $x \mapsto |1 - e^{-t_i x}|$ on any interval is achieved on the boundary of that interval. The right-hand side of (4.6) goes to zero almost surely as $i \rightarrow \infty$. Since $\int_B e^{-t_i x} \, d\Pi_{\beta,w}(x)$ converges to $\int_B 1 \, d\Pi_{\beta,w}(x)$ almost surely as $i \rightarrow \infty$, so does the average $\frac{1}{n_k} \sum_{i=1}^{n_k} \int_B e^{-t_i x} \, d\Pi_{\beta,w}(x)$ as $k \rightarrow \infty$. We have now shown that (4.2) and (4.3) both vanish along this subsequence n_k , and therefore we have

$$(4.7) \quad \Pi_{\beta,w}(B) = \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} \left(\mathbf{E}[\mathrm{Tr}[e^{-t_i H_{\beta,w}}]] - \int_{B^c} e^{-t_i x} \, d\Pi_{\beta,w}(x) \right) \quad \text{almost surely.}$$

At this point, if we define the deterministic measurable map $F_B : \mathcal{N}_{B^c}^{\#} \rightarrow \mathbb{R}$ as

$$F_B(\mu) := \begin{cases} \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} \left(\mathbf{E}[\mathrm{Tr}[e^{-t_i H_{\beta,w}}]] - \int_{B^c} e^{-t_i x} \, d\mu(x) \right) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

for all $\mu \in \mathcal{N}_{B^c}^{\#}$, then we get the almost-sure equality (2.4) by (4.7).

4.1.2. *Proof of Lemma 4.1.* Set $r_1 = 1$. Inductively, since $\mathbf{E}[X_{r_i} X_j] \rightarrow 0$ as $j \rightarrow \infty$ for each r_1, \dots, r_k , there exists $r_{k+1} > r_k$ so that $\mathbf{E}[X_{r_i} X_{r_{k+1}}] \leq 1/(k+1)$ for all $i = 1, \dots, k$. This way, we obtain a subsequence that satisfies $\mathbf{E}[X_{r_i} X_{r_j}] \leq 1/i \leq 1/(i-j)$ for every $i > j$. Then, for any fixed $N \in \mathbb{N}$,

$$\mathbf{E} \left[\left(\frac{X_{r_1} + \dots + X_{r_n}}{n} \right)^2 \right] = \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbf{E}[X_{r_i} X_{r_j}] \\ \leq \frac{1}{n^2} \sum_{i=1}^n C + \frac{1}{n^2} \sum_{\substack{1 \leq i, j \leq n \\ 1 \leq |i-j| \leq N}} \frac{1}{|i-j|} + \frac{1}{n^2} \sum_{\substack{1 \leq i, j \leq n \\ |i-j| > N}} \frac{1}{|i-j|} \\ \leq \frac{C}{n} + \frac{2N}{n} + \frac{1}{N}.$$

The right-hand side of the above inequality converges to $1/N$ as $n \rightarrow \infty$, which can be made arbitrarily small by taking $N \rightarrow \infty$. Therefore, $(X_{r_1} + \dots + X_{r_n})/n$ converges to zero in L^2 , hence in probability.

4.2. **Proof of Lemma 3.8.** This follows from a similar calculation as [18, Lemma 4.5] using the Feynman-Kac formula of $e^{-tH_{\beta,w}}$ as shown in Theorem 3.3. We first write

$$\mathrm{Cov}[\mathrm{Tr}[e^{-tH_{\beta,w}/2}], \mathrm{Tr}[e^{-uH_{\beta,w}/2}]] = \mathbf{E}[\mathrm{Tr}[e^{-tH_{\beta,w}/2}] \cdot \mathrm{Tr}[e^{-uH_{\beta,w}/2}]] - \mathbf{E}[\mathrm{Tr}[e^{-tH_{\beta,w}/2}]] \cdot \mathbf{E}[\mathrm{Tr}[e^{-uH_{\beta,w}/2}]].$$

Then, by (3.2) and Tonelli's theorem, we have

$$\mathbf{E}[\mathrm{Tr}[e^{-tH_{\beta,w}/2}]] = \int_0^\infty \frac{1 + e^{-2x^2/t}}{\sqrt{2\pi t}} \mathbf{E} \left[\mathbf{E} \left[e^{-\langle L_t(X_t^{x,x}), \frac{1}{2}V \rangle - \frac{1}{2}\xi_\beta(L_t(X_t^{x,x})) - w\mathfrak{L}_t^0(X_t^{x,x})} \middle| \xi_\beta \right] \right] dx \\ = \int_0^\infty \frac{1 + e^{-2x^2/t}}{\sqrt{2\pi t}} \mathbf{E} \left[\mathbf{E} \left[e^{-\langle L_t(X_t^{x,x}), \frac{1}{2}V \rangle - \frac{1}{2}\xi_\beta(L_t(X_t^{x,x})) - w\mathfrak{L}_t^0(X_t^{x,x})} \middle| X_t^{x,x} \right] \right] dx \\ = \int_0^\infty \frac{1 + e^{-2x^2/t}}{\sqrt{2\pi t}} \mathbf{E} \left[e^{-\langle L_t(X_t^{x,x}), \frac{1}{2}V \rangle + \frac{1}{2\beta} \|L_t(X_t^{x,x})\|_2^2 - w\mathfrak{L}_t^0(X_t^{x,x})} \right] dx.$$

Similarly,

$$\mathbf{E}[\mathrm{Tr}[e^{-uH_{\beta,w}/2}]] = \int_0^\infty \frac{1 + e^{-2y^2/t}}{\sqrt{2\pi t}} \mathbf{E} \left[e^{-(L_u(\bar{X}_u^{y,y}), \frac{1}{2}V) + \frac{1}{2\beta} \|L_u(\bar{X}_u^{y,y})\|_2^2 - w\Omega_u^0(\bar{X}_u^{y,y})} \right] dy$$

and

$$\mathbf{E}[\mathrm{Tr}[e^{-tH_{\beta,w}/2}] \cdot \mathrm{Tr}[e^{-uH_{\beta,w}/2}]] = \int_{(0,\infty)^2} \mathcal{P}_{t,u}(x,y) \mathbf{E} \left[e^{\mathcal{A}_{t,u}(x,y) + \mathcal{B}_{t,u}(x,y) + \mathcal{C}_{t,u}(x,y) + \mathcal{D}_{t,u}(x,y)} \right] dx dy.$$

Subtracting $\mathbf{E}[\mathrm{Tr}[e^{-tH_{\beta,w}/2}]] \cdot \mathbf{E}[\mathrm{Tr}[e^{-uH_{\beta,w}/2}]]$ from the above expression we complete the proof of Lemma 3.8.

Notation 4.2. Throughout the next few subsections we will use C, C', c, c' to denote absolute constants that are independent of t . Their exact values may change from line to line.

4.3. Proof of Lemma 3.9. The proof of this Lemma shares similar ideas as in [18, Lemma 4.6]. To avoid repetitions, we will omit details in few occasions and refer to relevant places in the proof of [18, Lemma 4.6]. We start with the bound for the $\mathcal{C}_{t,u}(x,y)$ part. By independence we have

$$\mathbf{E} \left[e^{4\mathcal{C}_{t,u}(x,y)} \right] = \mathbf{E} \left[e^{\frac{2}{\beta} \|L_t(X_t^{x,x})\|_2^2} \right] \cdot \mathbf{E} \left[e^{\frac{2}{\beta} \|L_u(\bar{X}_u^{y,y})\|_2^2} \right].$$

We will show that there exists constants $C', c' > 0$ (that only depend on β) such that for all $t > 0$,

$$\sup_{x>0} \mathbf{E} \left[e^{\frac{2}{\beta} \|L_t(X_t^{x,x})\|_2^2} \right] \leq C e^{c'(t^{3/2} + t^3)}.$$

This is a slightly stronger statement comparing to the one in [18, Lemma 4.6] which only establishes the bound for $t > 0$ sufficiently small. However the original proof still works in this new set up. We first condition on the value $X_t^{x,x}(t/2)$ and use Doob's h -transform to write

$$\mathbf{E} \left[e^{\frac{2}{\beta} \|L_t(X_t^{x,x})\|_2^2} \right] = \int_0^\infty \mathbf{E} \left[e^{\frac{2}{\beta} \|L_t(X_t^{x,x})\|_2^2} \mid X_t^{x,x}(t/2) = y \right] \frac{\Pi_X(t/2; x, y) \Pi_X(t/2; y, x)}{\Pi_X(t; x, x)} dy,$$

where $\Pi_X(t; x, y) = \frac{e^{-(x-y)^2/2} + e^{-(x+y)^2/2}}{\sqrt{2\pi t}}$ is the transition kernel for the reflected Brownian motion. Now by [18, (4.22)] we have

$$\mathbf{E} \left[e^{\frac{2}{\beta} \|L_t(X_t^{x,x})\|_2^2} \mid X_t^{x,x}(t/2) = y \right] \leq \mathbf{E} \left[e^{\frac{8}{\beta} \|L_{t/2}(X_{t/2}^{x,y})\|_2^2} \right].$$

On the other hand we have the elementary bounds

$$\Pi_X(t; x, x) \geq \frac{1}{\sqrt{2\pi t}}, \quad \Pi_X(t/2; y, x) \leq \frac{2}{\sqrt{\pi t}},$$

for all $t, x, y > 0$. Hence

$$\sup_{t>0} \sup_{x,y>0} \frac{\Pi_X(t/2; y, x)}{\Pi_X(t; x, x)} \leq 2\sqrt{2}.$$

Combine the bounds together we see that

$$\begin{aligned} \mathbf{E} \left[e^{\frac{2}{\beta} \|L_t(X_t^{x,x})\|_2^2} \right] &\leq 2\sqrt{2} \int_0^\infty \mathbf{E} \left[e^{\frac{8}{\beta} \|L_{t/2}(X_{t/2}^{x,y})\|_2^2} \right] \Pi_X(t/2; x, y) dy \\ (4.8) \quad &= 2\sqrt{2} \mathbf{E} \left[e^{\frac{8}{\beta} \|L_{t/2}(X^x)\|_2^2} \right]. \end{aligned}$$

Hence it suffices to show that for all $t > 0$ we have

$$\sup_{x>0} \mathbf{E} \left[e^{\frac{8}{\beta} \|L_{t/2}(X^x)\|_2^2} \right] \leq C e^{c'(t^{3/2} + t^3)},$$

where $C', c' > 0$ only depend on β . For this we couple X with a standard Brownian motion B so that $X^x(t) = |B^x(t)|$. Then for every $a > 0$ we have $L_t^a(X^x) = L_t^a(B^x) + L_t^{-a}(B^x)$. Hence

$$(4.9) \quad \|L_t(X^x)\|_2^2 = \int_0^\infty L_t^a(X^x)^2 da \leq 2 \int_{\mathbb{R}} L_t^a(B^x)^2 da \stackrel{\text{distr.}}{=} 2 \int_{\mathbb{R}} L_t^a(B^0)^2 da \stackrel{\text{distr.}}{=} 2t^{3/2} \|L_1(B^0)\|_2^2,$$

where the second-to-last equality in distribution follows from the change of variables $a \mapsto a - x$, and the last equality in distribution follows from the Brownian scaling identity (e.g., [11, Propositions 2.3.3 and 2.3.5]). By [11, Theorem 4.2.1] we have the following sub-Gaussian tail bound for $\|L_1(B^0)\|_2^2$:

$$\mathbf{P}[\|L_1(B^0)\|_2^2 > s] \leq C e^{-cs^2},$$

for absolute constants $c, C > 0$. Recall that a real random variable Z is sub-Gaussian if and only if there exist constants $B, b > 0$ such that

$$(4.10) \quad \mathbf{E}[e^{t(Z - \mathbf{E}Z)}] \leq B e^{bt^2} \quad \text{for all } t \in \mathbb{R}$$

(e.g. [34, Proposition 2.5.2-(v)]). Thus, applying this property and (4.9) we conclude that

$$\begin{aligned} \sup_{x>0} \mathbf{E} \left[e^{\frac{8}{\beta} \|L_{t/2}(X^x)\|_2^2} \right] &\leq \mathbf{E} \left[e^{\frac{4\sqrt{2}t^{3/2}}{\beta} \|L_1(B^0)\|_2^2} \right] \\ &= \mathbf{E} \left[e^{\frac{4\sqrt{2}t^{3/2}}{\beta} \mathbf{E} \|L_1(B^0)\|_2^2 + \frac{4\sqrt{2}t^{3/2}}{\beta} (\|L_1(B^0)\|_2^2 - \mathbf{E} \|L_1(B^0)\|_2^2)} \right] \leq C' e^{c'(t^{3/2} + t^3)} \end{aligned}$$

for constants $c', C' > 0$ that only depend on β . Hence

$$\limsup_{t \rightarrow 0} \sup_{x, y > 0} \mathbf{E} \left[e^{4C_{t,u}(x,y)} \right] \leq c < \infty.$$

The proof for the $\mathcal{B}_{t,u}(x, y)$ part is similar, see also [17, Lemma 5.6] for a similar statement.

4.4. Proof of Lemma 3.10. By the definition of $\mathcal{D}_{t,u}$ (see Section 3.1), we have $\mathcal{D}_{t,u} \neq 0$ if and only if $\langle L_t(X_t^{x,x}), L_u(\bar{X}_u^{y,y}) \rangle \neq 0$ where $\langle \cdot, \cdot \rangle$ is the standard L^2 inner product. Hence by Cauchy-Schwarz inequality we have

$$\begin{aligned} (4.11) \quad \mathbf{E} \left[\left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right)^4 \right]^{1/4} &= \mathbf{E} \left[\left(\mathbf{1}_{\{\langle L_t(X_t^{x,x}), L_u(\bar{X}_u^{y,y}) \rangle \neq 0\}} \left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right) \right)^4 \right]^{1/4} \\ &\leq \mathbf{E} \left[\left(e^{\mathcal{D}_{t,u}(x,y)} - 1 \right)^8 \right]^{1/8} \cdot \mathbf{P} \left[\langle L_t(X_t^{x,x}), L_u(\bar{X}_u^{y,y}) \rangle \neq 0 \right]^{1/8}. \end{aligned}$$

To estimate $\mathbf{P} \left[\langle L_t(X_t^{x,x}), L_u(\bar{X}_u^{y,y}) \rangle \neq 0 \right]$ we first couple the processes X and \bar{X} with two independent standard Brownian motions B and \bar{B} so that

$$(4.12) \quad X^x(s) = |x + B^0(s)|, \quad \bar{X}^y(s) = |y + \bar{B}^0(s)|.$$

Then by [17, (5.9)] we have for any nonnegative path functional F ,

$$(4.13) \quad \mathbf{E}[F(X_t^{x,x})] \leq 2\mathbf{E}[F(|x + B_t^{0,0}|)].$$

In particular

$$\mathbf{P} \left[\langle L_t(X_t^{x,x}), L_u(\bar{X}_u^{y,y}) \rangle \neq 0 \right] \leq 2\mathbf{P} \left[\langle L_t(|x + B_t^{0,0}|), L_u(|y + \bar{B}_u^{0,0}|) \rangle \neq 0 \right].$$

Now it is easy to check the following inclusion of events:

$$\begin{aligned} (4.14) \quad &\{ \langle L_t(|x + B_t^{0,0}|), L_u(|y + \bar{B}_u^{0,0}|) \rangle \neq 0 \} \\ &\subset \{ \text{the range of } (|x + B_t^{0,0}(s)|)_{s \in [0,t]} \text{ and } (|y + \bar{B}_u^{0,0}(s)|)_{s \in [0,u]} \text{ intersect} \} \\ &\subset \left\{ \max_{s \in [0,t]} |B_t^{0,0}(s)| \geq \frac{|x-y|}{2} \right\} \cup \left\{ \max_{s \in [0,u]} |\bar{B}_u^{0,0}(s)| \geq \frac{|x-y|}{2} \right\}. \end{aligned}$$

By Brownian scaling and the fact that maximum of Brownian bridge have sub-Gaussian tails we have

$$\mathbf{P} \left[\max_{s \in [0,t]} |B_t^{0,0}(s)| \geq \frac{|x-y|}{2} \right] = \mathbf{P} \left[\max_{s \in [0,1]} |B_1^{0,0}(s)| \geq \frac{|x-y|}{2t^{1/2}} \right] \leq C e^{-c(x-y)^2/t}$$

for some $C, c > 0$ independent of t . Therefore,

$$\begin{aligned} \mathbf{P} \left[\langle L_t(|x + B_t^{0,0}|), L_u(|y + \bar{B}_u^{0,0}|) \rangle \neq 0 \right] &\leq C \left(e^{-c(x-y)^2/t} + e^{-c(x-y)^2/u} \right) \\ &\leq C' e^{-c'(x-y)^2} \end{aligned}$$

for any $t \in (0, 1]$, where $c' = c'(u) = c \cdot \min\{1, 1/u\}$. Combining this with (4.11) we complete the proof of Lemma 3.10.

4.5. Proof of Lemma 3.11. By the definition of $\mathcal{A}_{t,u}(x, y)$ and basic properties of local time we have

$$\mathcal{A}_{t,u}(x, y) = -\langle L_t(X_t^{x,x}) + L_u(\bar{X}_u^{y,y}), \frac{1}{2}V \rangle = -\frac{1}{2} \left(\int_0^t X_t^{x,x}(s) ds + \int_0^u \bar{X}_u^{y,y}(s) ds \right).$$

Under the same coupling as in (4.12) we have by (4.13)

$$\mathbf{E} \left[e^{-2 \int_0^t X_t^{x,x}(s) ds} \right] \leq 2\mathbf{E} \left[e^{-2 \int_0^t |x + B_t^{0,0}(s)| ds} \right] \leq 2e^{-2xt} \mathbf{E} \left[e^{2 \int_0^t |B_t^{0,0}(s)| ds} \right].$$

Now by a change of variable $s \mapsto st$ and Brownian scaling we have

$$\mathbf{E} \left[e^{2 \int_0^t |B_t^{0,0}(s)| ds} \right] = \mathbf{E} \left[e^{2t^{3/2} \int_0^1 |B_1^{0,0}(s)| ds} \right] \leq \mathbf{E} \left[e^{2t^{3/2} \max_{s \in [0,1]} |B_1^{0,0}(s)|} \right].$$

Since the maximum of the Bessel bridge $|B_1^{0,0}(s)|$ has a sub-Gaussian tail, we have by (4.10) that

$$\mathbf{E} \left[e^{2t^{3/2} \max_{s \in [0,1]} |B_1^{0,0}(s)|} \right] \leq C e^{c(t^{3/2} + t^3)}$$

for some constants $c, C > 0$ independent of t . Thus by independence we have

$$\begin{aligned} \mathbf{E} \left[e^{4A_{t,u}(x,y)} \right]^{1/4} &= \left(\mathbf{E} \left[e^{-2 \int_0^t X_t^{x,x}(s) ds} \right] \cdot \mathbf{E} \left[e^{-2 \int_0^u \bar{X}_u^{y,y}(s) ds} \right] \right)^{1/4} \\ &\leq C e^{-\frac{1}{2}xt - \frac{1}{2}yu + c(t^{3/2} + t^3 + u^{3/2} + u^3)} \end{aligned}$$

Therefore, we can find constants $C, c > 0$ such that for any $t \in (0, 1]$ and $u > 0$ fixed we have

$$\int_{(0,\infty)^2} \mathbf{E} \left[e^{4A_{t,u}(x,y)} \right]^{1/4} e^{-(x-y)^2/c} dx dy \leq C \int_{(0,\infty)^2} e^{-yu/2 - (x-y)^2/c} dx dy < \infty.$$

This completes the proof of Lemma 3.11.

4.6. Proof of Lemma 3.12. This is again a similar calculation as [18, Lemma 4.8]. Using Cauchy-Schwarz and the elementary inequality $|e^z - 1| \leq |z|e^{|z|}$ we have

$$(4.15) \quad \left| e^{\mathcal{D}_{t,u}(x,y)} - 1 \right| \leq \frac{1}{\beta} \|L_t(X_t^{x,x})\|_2 \|L_u(\bar{X}_u^{y,y})\|_2 \cdot e^{\frac{1}{\beta} \|L_t(X_t^{x,x})\|_2 \|L_u(\bar{X}_u^{y,y})\|_2}.$$

By Cauchy-Schwarz inequality we have

$$\mathbf{E} \left[\|L_t(X_t^{x,x})\|_2^8 e^{\frac{8}{\beta} \|L_t(X_t^{x,x})\|_2} \right] \leq (\mathbf{E} \|L_t(X_t^{x,x})\|_2^{16})^{1/2} \left(\mathbf{E} e^{\frac{16}{\beta} \|L_t(X_t^{x,x})\|_2} \right)^{1/2}.$$

Now a similar argument as in Lemma 3.9 shows that

$$\mathbf{E} \|L_t(X_t^{x,x})\|_2^{16} \leq C \mathbf{E} \|L_{t/2}(B^x)\|_2^{16} = C \left(\frac{t}{2} \right)^{12} \mathbf{E} \|L_1(B^0)\|_2^{16} \leq C' t^{12},$$

B^0 is a standard Brownian motion and we couple X and B as in (4.12). Similarly

$$\mathbf{E} e^{\frac{16}{\beta} \|L_t(X_t^{x,x})\|_2} \leq C \mathbf{E} e^{\frac{64}{\beta} \|L_{t/2}(X^x)\|_2} \leq C \mathbf{E} e^{c \|L_{t/2}(B^x)\|_2} = C' \mathbf{E} e^{c' t^{3/4} \|L_1(B^0)\|_2}.$$

By [11, Theorem 4.2.1] we have

$$\mathbf{P}[\|L_1(B^0)\|_2 > s] \leq C e^{-cs^4}.$$

Hence using once again the sub-Gaussian bound (4.10) we get

$$\mathbf{E} e^{\frac{16}{\beta} \|L_t(X_t^{x,x})\|_2} \leq C \mathbf{E} e^{\frac{8}{\beta} t^{3/4} \|L_1(B^0)\|_2} \leq C' e^{c'(t^{3/2} + t^3)}.$$

Combine the estimates together we see that

$$(4.16) \quad \mathbf{E} \left[\|L_t(X_t^{x,x})\|_2^8 e^{\frac{8}{\beta} \|L_t(X_t^{x,x})\|_2} \right] \leq C t^6 e^{c(t^{3/2} + t^3)}.$$

Finally using (4.15), (4.16) and independence we conclude that

$$\left(\mathbf{E} \left| e^{\mathcal{D}_{t,u}(x,y)} - 1 \right|^8 \right)^{1/8} \leq C_{u,\beta} \cdot t^{3/4},$$

uniformly for all $t \in (0, 1]$ where the constant $C_{u,\beta}$ depends only on $u, \beta > 0$ but not t . This completes the proof of Lemma 3.12.

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