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VARIETIES OF TROPICAL IDEALS ARE BALANCED

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ABSTRACT. Tropical ideals, introduced in [MR18], define subschemes of tropical toric varieties. We prove that the top-dimensional parts of their varieties are balanced polyhedral complexes of the same dimension as the ideal. This means that every subscheme of a tropical toric variety defined by a tropical ideal has an associated class in the Chow ring of the toric variety. A key tool in the proof is that specialization of variables in a tropical ideal yields another tropical ideal; this plays the role of hyperplane sections in the theory. We also show that elimination theory (projection of varieties) works for tropical ideals as in the classical case. The matroid condition that defines tropical ideals is crucial for these results.

1. INTRODUCTION

This paper is part of a program to develop an intrinsic scheme theory for tropical geometry, begun in [GG16]; see [DR21, L15, GG14, GG18, JM18a, JM18b, MR20, M14, Sil22]. We focus on subschemes of tropical toric varieties. Usual subschemes of affine or projective space are defined by ideals in a polynomial ring. Ideals in the semiring of tropical polynomials, however, are too general with which to build a theory analogous to classical algebraic geometry. For example, the variety of an ideal in this semiring is not necessarily a finite polyhedral complex; see [MR18, Example 5.14].

The remedy proposed in [MR18] is to work with a smaller class of ideals, called *tropical ideals* (see Definition 1.1 below). The class of tropical ideals includes all tropicalizations of classical ideals [GG16], but it is strictly larger. In [MR18], the authors show that the variety of a tropical ideal is a finite \mathbb{R} -rational polyhedral complex, and that tropical ideals satisfy the ascending chain condition and the weak Nullstellensatz. In addition, homogeneous tropical ideals have a Hilbert polynomial, which in particular allows a definition of dimension and degree. This suggests that tropical ideals form a reasonable class with which to work for tropical algebraic geometry.

The main result of this paper is that the variety of a tropical ideal is *balanced* with respect to an intrinsically defined multiplicity on its maximal-dimensional cells. This generalizes the Structure Theorem for tropicalizations of classical varieties. The balancing condition is a combinatorial constraint on a polyhedral complex that can be interpreted as a “zero-tension” condition; see Definition 6.1 for a precise definition. It plays a fundamental role in tropical geometry. Along the way we prove other basic results for tropical ideals, including that they are closed under specializations of the variables, and that the dimension of their varieties agrees with the dimension of the defining ideal.

We now state this more precisely, beginning with the definition of a tropical ideal. We write $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ for the tropical semiring, where \oplus is \min , \odot is regular addition, ∞ is the additive identity, and 0 is the multiplicative identity. For simplicity, we restrict our presentation in the introduction to ideals in the Laurent polynomial semiring.

Definition 1.1. An ideal $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a *tropical ideal* if it satisfies the following “monomial elimination axiom”:

(†) For any $f, g \in I$ and any monomial $\mathbf{x}^{\mathbf{u}}$ for which $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}} \neq \infty$, there exists $h \in I$ such that $[h]_{\mathbf{x}^{\mathbf{u}}} = \infty$ and $[h]_{\mathbf{x}^{\mathbf{v}}} \geq \min([f]_{\mathbf{x}^{\mathbf{v}}}, [g]_{\mathbf{x}^{\mathbf{v}}})$ for all monomials $\mathbf{x}^{\mathbf{v}}$, with the equality holding whenever $[f]_{\mathbf{x}^{\mathbf{v}}} \neq [g]_{\mathbf{x}^{\mathbf{v}}}$.

Here, we use the notation $[f]_{\mathbf{x}^{\mathbf{u}}}$ to denote the coefficient of the monomial $\mathbf{x}^{\mathbf{u}}$ in the polynomial f . As we refer to this condition on f , g , and h several times throughout the paper, we abbreviate it as

$$h \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g).$$

We refer to h as *an* elimination of $\mathbf{x}^{\mathbf{u}}$ from f and g , as it is not uniquely defined by $\mathbf{x}^{\mathbf{u}}$, f , and g .

When X is a d -dimensional irreducible subvariety of the torus $(K^*)^n$ over a valued field K , the tropicalization $\text{trop}(X)$ is the support of a pure d -dimensional balanced polyhedral complex. The following is the main theorem of the paper, which generalizes this fact to varieties of tropical ideals; see §2.3 for the definition.

Theorem 1.2. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal of dimension d . Then the variety $V(I)$ is the support of a polyhedral complex Σ whose maximal-dimensional cells have dimension d . Moreover, the weighted \mathbb{R} -rational polyhedral complex Σ^d consisting of the d -dimensional cells of Σ , with weights given by the multiplicities of Definition 6.3, is balanced.*

The restriction to the d -skeleton of Σ is necessary here, as there is not yet a reasonable definition of an irreducible subscheme of the tropical torus that implies that $V(I)$ is pure. Theorem 1.2 allows us to define a Hilbert-Chow morphism for tropical ideals, taking a subscheme of a tropical toric variety to its class in the Chow ring of the toric variety; see Remark 6.7.

A key tool used is the fact, whose proof is non-trivial, that the class of tropical ideals is closed under specialization of some of the variables.

Theorem 1.3. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal. For any $a \in \mathbb{R}$, the ideal*

$$I|_{x_n=a} := \{f(x_1, \dots, x_{n-1}, a) : f \in I\} \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$$

is a tropical ideal. When $V(I)$ is the support of a pure polyhedral complex, we have

$$V(I|_{x_n=a}) = \pi(V(I) \cap_{st} \{x_n = a\}),$$

where $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ coordinates, and \cap_{st} is the stable intersection.

This theorem plays the role of a hyperplane section in tropical scheme theory, as it allows induction on dimension. In the realizable case, $V(I|_{x_n=a})$ is the tropicalization of the intersection of the variety with a generic translate of a subtorus; see Remark 3.7.

Two important consequences of Theorem 1.3, which were already part of the standard tropical tool-kit in the case that I is the tropicalization of a classical ideal, are the following.

- (1) (Theorem 4.3). The variety of a tropical ideal of dimension d is the support of a polyhedral complex with maximal-dimensional cells of dimension d .

- (2) (Theorem 4.7). If $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a tropical ideal, and $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the projection onto the first $n - 1$ coordinates, then

$$V(I \cap \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]) = \pi(V(I)).$$

We note that this differs slightly from the non-tropical case, where a closure is needed on $\pi(V(I))$. The proof of the projection result uses a Nullstellensatz of Grigoriev and Podolskii [GP18] in a crucial fashion.

The last important ingredient in the proof of Theorem 1.2 is the fact (Theorem 5.10) that the degree of a zero-dimensional ideal is the sum of the multiplicities of the points in its variety. The proof here is more complicated than in the classical case, owing to the lack of primary decomposition (so far) in tropical scheme theory.

The structure of the paper is as follows. In Section 2 we develop more Gröbner theory for tropical ideals. The proofs of these results are fairly similar to the realizable case. Section 3 contains the first deep result, with the proof of the key specialization theorem (Theorem 3.6). The results about dimension and projection (Theorem 4.3 and Theorem 4.7) are proved in Section 4, while Section 5 contains the key facts about degrees of zero-dimensional ideals. Finally, Theorem 1.2 is proved in Section 6 (Theorem 6.6).

Notation. Our conventions for writing tropical polynomials are to write terms using standard notation: $c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$. We also write $\prod_i \mathbf{x}^{\mathbf{u}_i}$ for a product of monomials to avoid switching conventions. For the coefficient of a polynomial we usually use standard arithmetic (so write $\lambda \cdot \mathbf{u}$ for vectors λ and \mathbf{u} rather than $\lambda^{\mathbf{u}}$). The aim of these choices is to make the terms of the tropical polynomial easier to distinguish.

Acknowledgments. DM was partially supported by EPSRC grant EP/R02300X/1. FR was partially supported by the Research Council of Norway grant 239968/F20 and EPSRC grant EP/T031042/1. We thank Alex Fink for useful conversations about Theorem 5.10, and the Institut Mittag-Leffler for hosting those conversations during the program on *Tropical Geometry, Amoebas, and Polytopes*.

2. GRÖBNER THEORY

In this section we prove basic results about initial ideals of tropical ideals and the connection with initial ideals with respect to monomial term orders.

2.1. Variants of tropical ideals. Throughout this paper we will consider tropical ideals in both $\overline{\mathbb{R}}[x_1, \dots, x_n]$ and $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and also homogeneous tropical ideals in $\overline{\mathbb{R}}[x_0, \dots, x_n]$. A slightly more general setting also occurs in §5. In each case the definition of tropical ideal is that it obeys the monomial elimination axiom (†) given in Definition 1.1. In the case of homogeneous tropical ideals in $\overline{\mathbb{R}}[x_0, \dots, x_n]$ it suffices to check the condition when f and g are homogeneous. Equivalently, we require that for any finite selection E of monomials (which can be Laurent in the case $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$), the restriction $I|_E$ is the set of vectors of a *valuated matroid* $\text{Mat}(I|_E)$ on the ground set E . We denote by $\underline{\text{Mat}}(I|_E)$ the underlying matroid of $\text{Mat}(I|_E)$, which is a matroid on the set E . See [MR18, §2] for more on this perspective.

We will also consider tropical ideals where the semiring of coefficients is the *Boolean semiring* $\mathbb{B} := \{0, \infty\} \subseteq \overline{\mathbb{R}}$. Many results also hold for more general additively idempotent

semifields, as in [MR18]; we restrict to \mathbb{B} and $\overline{\mathbb{R}}$ here as the main focus is on the polyhedral structure of varieties.

We now describe the connection between these three versions of tropical ideals. The *homogenization* of a tropical polynomial $f = \bigoplus c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$ is

$$\tilde{f} = \bigoplus c_{\mathbf{u}} x_0^{d-|\mathbf{u}|} \mathbf{x}^{\mathbf{u}} \in \overline{\mathbb{R}}[x_0, x_1, \dots, x_n]$$

where $|\mathbf{u}| := \sum_{i=1}^n u_i$ and $d = \max(|\mathbf{u}| : c_{\mathbf{u}} \neq \infty)$. The homogenization of an ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is the ideal

$$I^h := \langle \tilde{f} : f \in I \rangle \subseteq \overline{\mathbb{R}}[x_0, x_1, \dots, x_n].$$

Conversely, if $f \in \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a homogeneous polynomial, its *dehomogenization* is the polynomial $f|_{x_0=0} := f(0, x_1, \dots, x_n) \in \overline{\mathbb{R}}[x_1, \dots, x_n]$. The dehomogenization of a homogeneous ideal $J \subseteq \overline{\mathbb{R}}[x_0, x_1, \dots, x_n]$ is the ideal

$$J|_{x_0=0} := \{f|_{x_0=0} : f \in J\} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n].$$

If $J \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is an ideal in the Laurent polynomial semiring, the intersection $J \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$ is an ideal in the affine polynomial semiring $\overline{\mathbb{R}}[x_1, \dots, x_n]$. Conversely, any ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ generates an ideal

$$I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Its elements are the Laurent polynomials of the form $f\mathbf{x}^{\mathbf{u}}$ with $f \in I$ and $\mathbf{x}^{\mathbf{u}}$ a Laurent monomial.

If I is an ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ and m is a monomial, then

$$(I : m) := \{f \in \overline{\mathbb{R}}[x_1, \dots, x_n] : fm \in I\}.$$

The *saturation* of I with respect to m is

$$(I : m^\infty) := \{f \in \overline{\mathbb{R}}[x_1, \dots, x_n] : fm^k \in I \text{ for some } k \geq 0\}.$$

The following lemma details the relationships between these ideals.

Lemma 2.1.

- (1) If $J \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a tropical ideal then $J \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal as well. Conversely, if $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal then so is $I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.
- (2) If $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal then $I^h \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a tropical ideal. Conversely, if $J \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a homogeneous tropical ideal, then $J|_{x_0=0} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal.
- (3) If $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal and m is any monomial, then $(I : m)$ and $(I : m^\infty)$ are also tropical ideals. When $m = \prod_{i=1}^n x_i$, then $(I : m^\infty) = I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. In particular, if $J \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $I = J \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$, then $(I : m^\infty) = I$.

Proof. (1) Suppose $J \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a tropical ideal, and let $I = J \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. If $f, g \in I$ with $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}}$ then $f, g \in J$, so there is $h \in J$ with $h \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g)$. This in particular implies that $h \in I$, and so I satisfies the monomial elimination axiom.

Assume now that $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal, and let $J = I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Fix $f, g \in J$ with $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}}$. Take a monomial $\mathbf{x}^{\mathbf{v}}$ such that $f\mathbf{x}^{\mathbf{v}}, g\mathbf{x}^{\mathbf{v}} \in I$. Since $[f\mathbf{x}^{\mathbf{v}}]_{\mathbf{x}^{\mathbf{u}+\mathbf{v}}} = [g\mathbf{x}^{\mathbf{v}}]_{\mathbf{x}^{\mathbf{u}+\mathbf{v}}}$, there is $h \in I$ satisfying $h \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}+\mathbf{v}}}(f\mathbf{x}^{\mathbf{v}}, g\mathbf{x}^{\mathbf{v}})$. It follows that $h\mathbf{x}^{-\mathbf{v}} \in J$ satisfies $h\mathbf{x}^{-\mathbf{v}} \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g)$, showing that J satisfies the monomial elimination axiom.

- (2) Suppose $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal. Since I^h is a homogeneous ideal, it is enough to prove the monomial elimination axiom for homogeneous polynomials $f, g \in I^h$ of the same degree. Suppose $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}}$, and let \mathbf{u}' be the last n coordinates of \mathbf{u} . Then $f' := f|_{x_0=0}, g' := g|_{x_0=0} \in I$, and $[f']_{\mathbf{x}^{\mathbf{u}'}} = [g']_{\mathbf{x}^{\mathbf{u}'}}$. By the monomial elimination axiom for I there is $h' \in I$ with $h' \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}'}}(f', g')$. Then $h := x_0^{\deg(f)-\deg(h')} \tilde{h}'$ satisfies $h \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g)$, showing that I^h is a tropical ideal.

Conversely, suppose $J \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a homogeneous tropical ideal, and let $f, g \in J|_{x_0=0}$ with $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}}$. Without loss of generality we may assume that $d := \deg(\tilde{f}) - \deg(\tilde{g}) \geq 0$. Take $c \in \mathbb{N}$ such that $x_0^c \tilde{f}, x_0^{c+d} \tilde{g} \in J$. Denote $e = c + \deg(\tilde{f}) - |\mathbf{u}|$. By the monomial elimination axiom for J there is $h \in J$ with $h \leftarrow \text{elim}_{x_0^e \mathbf{x}^{\mathbf{u}}}(x_0^c \tilde{f}, x_0^{c+d} \tilde{g})$, and thus $h|_{x_0=0} \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g)$, which shows that $J|_{x_0=0}$ is a tropical ideal.

- (3) Suppose that $f, g \in (I : m)$ with $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}}$. Then $mf, mg \in I$, with $[mf]_{m\mathbf{x}^{\mathbf{u}}} = [mg]_{m\mathbf{x}^{\mathbf{u}}}$, so there exists $h \in I$ with $h \leftarrow \text{elim}_{m\mathbf{x}^{\mathbf{u}}}(mf, mg)$. Thus every term of h is divisible by m , and so $h' := h/m \in (I : m)$ satisfies $h' \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g)$, as desired.

We have $I \subseteq (I : m) \subseteq (I : m^2) \subseteq \dots$. Since these are all tropical ideals, by the ascending chain condition [MR18, Theorem 3.11] this chain stabilizes, so there is $N \geq 0$ for which $(I : m^k) = (I : m^N)$ for all $k \geq N$. We then have $(I : m^\infty) = (I : m^N)$, which shows that $(I : m^\infty)$ is a tropical ideal.

When $m = \prod_{i=1}^n x_i$, if $m^k f \in I$ then $f = 1/m^k(m^k f) \in I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, so $(I : m^\infty) \subseteq I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. Conversely, if $g \in I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$, we have that $g = \mathbf{x}^{\mathbf{u}} f$ for $f \in I$ and $\mathbf{x}^{\mathbf{u}}$ a (possibly Laurent) monomial. Setting $k = -\min\{u_i : u_i < 0\}$, we get $m^k g \in I$.

The last claim follows from the fact that $J = I\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. \square

We next recall the Gröbner theory developed in [MR18, §3]. For $\mathbf{w} \in \mathbb{R}^n$ and $f = \bigoplus_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$, the *initial term* of f with respect to \mathbf{w} is

$$\text{in}_{\mathbf{w}}(f) := \bigoplus_{\mathbf{u} : c_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w} = f(\mathbf{w})} \mathbf{x}^{\mathbf{u}} \in \mathbb{B}[x_1, \dots, x_n].$$

For a tropical ideal I we define the *initial ideal* with respect to \mathbf{w} as

$$\text{in}_{\mathbf{w}}(I) := \langle \text{in}_{\mathbf{w}}(f) \mid f \in I \rangle \subseteq \mathbb{B}[x_1, \dots, x_n].$$

Note that in fact $\text{in}_{\mathbf{w}}(I)$ is equal to the set $\{\text{in}_{\mathbf{w}}(f) \mid f \in I\}$, as this set is already closed under tropical addition, scalar multiplication, and multiplication by any monomial. Analogous definitions apply for polynomials and ideals in the Laurent polynomial ring $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and also when the coefficients are in \mathbb{B} .

We have the following relationships between initial ideals.

Lemma 2.2.

(1) If $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is a tropical ideal and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{in}_{\mathbf{w}}(I \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]) = \text{in}_{\mathbf{w}}(I) \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

(2) If $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a homogeneous tropical ideal and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{in}_{\mathbf{w}}(I|_{x_0=0}) = \text{in}_{(0, \mathbf{w})}(I)|_{x_0=0}.$$

(3) If $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is a tropical ideal and $\mathbf{w} \in \mathbb{R}^n$ then

$$\text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_1, \dots, x_n] = (\text{in}_{\mathbf{w}}(I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]) : (\prod_{i=1}^n x_i)^\infty).$$

Proof. (1) Let $f \in I \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $f = \mathbf{x}^{\mathbf{u}} f'$ for some Laurent monomial $\mathbf{x}^{\mathbf{u}}$ and $f' \in I$, so $\text{in}_{\mathbf{w}}(f) = \mathbf{x}^{\mathbf{u}} \text{in}_{\mathbf{w}}(f') \in \text{in}_{\mathbf{w}}(I) \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Conversely, let $g \in \text{in}_{\mathbf{w}}(I) \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $g = \mathbf{x}^{\mathbf{u}} g'$ for some Laurent monomial $\mathbf{x}^{\mathbf{u}}$ and $g' \in \text{in}_{\mathbf{w}}(I)$. Write $g' = \text{in}_{\mathbf{w}}(f)$ for $f \in I$. Then $\mathbf{x}^{\mathbf{u}} f \in I \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, so $g = \mathbf{x}^{\mathbf{u}} \text{in}_{\mathbf{w}}(f) = \text{in}_{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} f) \in \text{in}_{\mathbf{w}}(I \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$.

(2) If $f \in I$ then $\text{in}_{(0, \mathbf{w})}(f)|_{x_0=0} = \text{in}_{\mathbf{w}}(f|_{x_0=0}) \in \text{in}_{\mathbf{w}}(I|_{x_0=0})$, so $\text{in}_{(0, \mathbf{w})}(I)|_{x_0=0} \subseteq \text{in}_{\mathbf{w}}(I|_{x_0=0})$. Conversely, if $g \in \text{in}_{\mathbf{w}}(I|_{x_0=0})$ then $g = \text{in}_{\mathbf{w}}(f|_{x_0=0})$ for some $f \in I$, so $g = \text{in}_{(0, \mathbf{w})}(f)|_{x_0=0} \in \text{in}_{(0, \mathbf{w})}(I)|_{x_0=0}$.

(3) If $f \in \text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_1, \dots, x_n]$ then $f = \text{in}_{\mathbf{w}}(g)$ for $g \in I$. Choose a monomial $\mathbf{x}^{\mathbf{u}}$ with $\mathbf{x}^{\mathbf{u}} g \in \overline{\mathbb{R}}[x_1, \dots, x_n]$. Then $\mathbf{x}^{\mathbf{u}} f = \text{in}_{\mathbf{w}}(\mathbf{x}^{\mathbf{u}} g) \in \text{in}_{\mathbf{w}}(I \cap \overline{\mathbb{R}}[x_1, \dots, x_n])$, so $f \in (\text{in}_{\mathbf{w}}(I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]) : (\prod_{i=1}^n x_i)^\infty)$. Conversely, if $f \in (\text{in}_{\mathbf{w}}(I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]) : (\prod_{i=1}^n x_i)^\infty)$ then there is $g \in I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$ with $\mathbf{x}^{\mathbf{u}} f = \text{in}_{\mathbf{w}}(g)$. Then $\mathbf{x}^{-\mathbf{u}} g \in I$, and $f = \text{in}_{\mathbf{w}}(\mathbf{x}^{-\mathbf{u}} g) \in \text{in}_{\mathbf{w}}(I)$, as required. \square

If $I \subseteq \mathbb{R}[x_0, \dots, x_n]$ is a homogeneous tropical ideal, its *Hilbert function* is the map $H_I : \mathbb{N} \rightarrow \mathbb{N}$ given by $d \mapsto \text{rank}(\text{Mat}(I_d))$, where I_d is the degree- d part of I . More specifically, $H_I(d)$ is the size of any maximal subset B of monomials of degree d with the property that B does not contain the support of any polynomial in I_d . For $\mathbf{w} \in \mathbb{R}^n$ the initial ideal $\text{in}_{\mathbf{w}}(I) \subseteq \mathbb{B}[x_0, \dots, x_n]$ is a homogeneous tropical ideal, and $H_{\text{in}_{\mathbf{w}}(I)} = H_I$ [MR18, Corollary 3.6].

2.2. Monomial term orders. In commutative algebra over a field, Gröbner theory usually begins with monomial term orders. We now introduce these for the semiring of tropical polynomials.

Definition 2.3. A total order \prec on the monomials in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ is a monomial term order if

$$\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{u}'}$$
 implies $\mathbf{x}^{\mathbf{u}+\mathbf{v}} \prec \mathbf{x}^{\mathbf{u}'+\mathbf{v}}$ for all monomials $\mathbf{x}^{\mathbf{v}}$,

and

$$\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{0}}$$

for all monomials $\mathbf{x}^{\mathbf{u}} \neq \mathbf{x}^{\mathbf{0}}$.

The direction of the inequality in the second condition is to make this compatible with the min convention for initial ideals that we use in this theory. It is the opposite of the usual order, but is *not* the tropical analogue of a local order in usual Gröbner theory.

Example 2.4. Two central examples of monomial term orders are the lexicographic and reverse-lexicographic term orders. The *lexicographic* term order \prec_{lex} on $\overline{\mathbb{R}}[x_1, \dots, x_n]$ has $\mathbf{x}^{\mathbf{u}} \prec_{lex} \mathbf{x}^{\mathbf{v}}$ if the first nonzero entry of $\mathbf{u} - \mathbf{v}$ is positive. The *reverse-lexicographic* term order \prec_{revlex} on $\overline{\mathbb{R}}[x_1, \dots, x_n]$ has $\mathbf{x}^{\mathbf{u}} \prec_{revlex} \mathbf{x}^{\mathbf{v}}$ if $\deg(\mathbf{x}^{\mathbf{u}}) > \deg(\mathbf{x}^{\mathbf{v}})$, or $\deg(\mathbf{x}^{\mathbf{u}}) = \deg(\mathbf{x}^{\mathbf{v}})$ and the last nonzero entry of $\mathbf{u} - \mathbf{v}$ is negative. Note that these are the reverse of the usual orders, to be compatible with the min convention. For example,

$$x_1^2 \prec_{lex} x_1 x_2^2 \prec_{lex} x_2 \prec_{lex} x_3^2 \prec_{lex} 0,$$

and

$$x_3^3 \prec_{revlex} x_2^2 \prec_{revlex} x_1 x_3 \prec_{revlex} 0,$$

where 0 denotes the constant monomial $x_1^0 x_2^0 x_3^0$. \diamond

Definition 2.5. Let \prec be a monomial term order on $\overline{\mathbb{R}}[x_1, \dots, x_n]$. The *initial term* of a tropical polynomial $f = \bigoplus c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$ is $\text{in}_{\prec}(f) = \mathbf{x}^{\mathbf{v}}$, where $\mathbf{x}^{\mathbf{v}} = \min_{\prec} \{\mathbf{x}^{\mathbf{u}} : c_{\mathbf{u}} \neq \infty\}$. The *initial ideal* of an ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is the monomial ideal

$$\text{in}_{\prec}(I) := \langle \text{in}_{\prec}(f) : f \in I \rangle.$$

As with traditional Gröbner bases, one use of monomial initial ideals is that they give distinguished bases for the matroids associated to a tropical ideal.

Lemma 2.6. *Let $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ be a tropical ideal and let \prec be a monomial term order. Then for any finite collection E of monomials in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ the set of monomials in $E \setminus \{\text{in}_{\prec}(f) : f \in I|_E\}$ is a basis of $\underline{\text{Mat}}(I|_E)$. In particular, if I is a homogeneous tropical ideal, $H_I(d) = H_{\text{in}_{\prec}(I)}(d)$ for all $d \geq 0$.*

Proof. If the set $B := E \setminus \{\text{in}_{\prec}(f) : f \in I|_E\}$ were not an independent set of the matroid $\underline{\text{Mat}}(I|_E)$, then there would be $f \in I$ with support in B . We would then have $\text{in}_{\prec}(f) \in B$, contradicting the definition of B . To show that B is a basis we show that for all $\mathbf{x}^{\mathbf{u}} \in E \setminus B$ there is $f \in I$ supported in $B \cup \{\mathbf{x}^{\mathbf{u}}\}$. To see this, fix $\mathbf{x}^{\mathbf{u}} \in E \setminus B$. We have $\mathbf{x}^{\mathbf{u}} = \text{in}_{\prec}(f)$ for $f \in I|_E$. We may assume that f has been chosen so that the smallest $\mathbf{x}^{\mathbf{v}} \in \text{supp}(f) \setminus (B \cup \{\mathbf{x}^{\mathbf{u}}\})$ (if any exists) with respect to \prec is as large as possible; this is possible because E is finite. For such a minimal $\mathbf{x}^{\mathbf{v}} \in \text{supp}(f) \setminus (B \cup \{\mathbf{x}^{\mathbf{u}}\})$, there is a polynomial $f_{\mathbf{v}} \in I|_E$ with $\text{in}_{\prec}(f_{\mathbf{v}}) = \mathbf{x}^{\mathbf{v}}$, where we may assume that the coefficient of $\mathbf{x}^{\mathbf{v}}$ in $f_{\mathbf{v}}$ is 0. Since $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$, we have $\mathbf{x}^{\mathbf{u}} \notin \text{supp}(f_{\mathbf{v}})$. Let α be the coefficient of $\mathbf{x}^{\mathbf{v}}$ in f , and let $f' \in I|_E$ be an elimination $f' \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{v}}}(f, \alpha f_{\mathbf{v}})$. We have $\text{in}_{\prec}(f') = \mathbf{x}^{\mathbf{u}}$, and the smallest monomial in $\text{supp}(f') \setminus (B \cup \{\mathbf{x}^{\mathbf{u}}\})$ is larger than $\mathbf{x}^{\mathbf{v}}$, which contradicts the construction of f . We thus conclude that there is $f \in I$ with $\text{in}_{\prec}(f) = \mathbf{x}^{\mathbf{u}}$, and $\text{supp}(f) \subseteq B \cup \{\mathbf{x}^{\mathbf{u}}\}$, as claimed.

For a homogeneous ideal I and $d \geq 0$, take E to be the collection of monomials of degree d . We then have that the set of monomials of degree d not in $\text{in}_{\prec}(I)$ is a basis for $\underline{\text{Mat}}(I_d)$, as $\text{in}_{\prec}(I_d) = (\text{in}_{\prec}(I))_d$ for homogeneous ideals. This implies the equality of Hilbert functions $H_{\text{in}_{\prec}(I)}(d) = H_I(d)$. \square

The following lemma states that for a *fixed* tropical ideal I every initial ideal with respect to a monomial term order is also an initial ideal with respect to a weight vector $\mathbf{w} \in \mathbb{R}^n$. The proof is very similar to the classical case; see [Stu96, Proposition 1.11].

The *recession cone* of a polyhedron P is the largest cone C for which the Minkowski sum $P + C \subseteq P$. Equivalently, the recession cone of a nonempty polyhedron $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is the cone $\{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{0}\}$.

Lemma 2.7. *Let I be a homogeneous tropical ideal in $\overline{\mathbb{R}}[x_0, \dots, x_n]$, and let \prec be a monomial term order. There is a nonempty polyhedron $C_\prec \subseteq \mathbb{R}^{n+1}$ with an $(n+1)$ -dimensional recession cone, with the property that $\text{in}_{\mathbf{w}}(I) = \text{in}_\prec(I)$ for all \mathbf{w} in the interior of C_\prec .*

Proof. Let $\mathbf{x}^{\mathbf{u}_1}, \dots, \mathbf{x}^{\mathbf{u}_s}$ be the minimal generators of the monomial ideal $\text{in}_\prec(I)$. Write $B_{|\mathbf{u}_i|}$ for the monomials in $\overline{\mathbb{R}}[x_0, \dots, x_n]$ of degree $|\mathbf{u}_i|$ not in $\text{in}_\prec(I)$, which form a basis for $\text{Mat}(I_{|\mathbf{u}_i|})$ by Lemma 2.6. For any $1 \leq i \leq s$, there exists a homogeneous polynomial $g_i \in I$ with $\text{supp}(g_i) \subseteq B_{|\mathbf{u}_i|} \cup \{\mathbf{x}^{\mathbf{u}_i}\}$, corresponding to the fundamental circuit of $\mathbf{x}^{\mathbf{u}_i}$ over $B_{|\mathbf{u}_i|}$. After scaling, we can write $g_i = \mathbf{x}^{\mathbf{u}_i} \oplus \bigoplus_{\mathbf{v} \in B_{|\mathbf{u}_i|}} c_{i\mathbf{v}} \mathbf{x}^{\mathbf{v}}$. Let C_\prec be the closure of the set

$$C_\prec^\circ = \{\mathbf{w} \in \mathbb{R}^{n+1} : \mathbf{w} \cdot \mathbf{u}_i < c_{i\mathbf{v}} + \mathbf{w} \cdot \mathbf{v} \text{ for all } 1 \leq i \leq s \text{ and } \mathbf{x}^{\mathbf{v}} \in \text{supp}(g_i) \setminus \{\mathbf{x}^{\mathbf{u}_i}\}\}.$$

For any $\mathbf{w} \in C_\prec^\circ$ we have $\text{in}_{\mathbf{w}}(g_i) = \mathbf{x}^{\mathbf{u}_i}$ for all i , and so $\text{in}_\prec(I) \subseteq \text{in}_{\mathbf{w}}(I)$. We have $H_{\text{in}_{\mathbf{w}}(I)}(d) = H_I(d) = H_{\text{in}_\prec(I)}(d)$ for all $d \geq 0$ by [MR18, Corollary 3.6] and Lemma 2.6. Thus we cannot have $\text{in}_\prec(I)$ properly contained in $\text{in}_{\mathbf{w}}(I)$, as otherwise we would have a proper containment of the sets of cycles of two matroids with the same rank [Oxl92, Corollary 7.3.4].

It thus remains to show that C_\prec is nonempty and has a full-dimensional recession cone. Form the matrix U with $n+1$ columns whose ℓ rows are the vectors $-\mathbf{v} + \mathbf{u}_i$ for $1 \leq i \leq s$ and $\mathbf{x}^{\mathbf{v}} \in \text{supp}(g_i) \setminus \{\mathbf{x}^{\mathbf{u}_i}\}$. If C_\prec° is empty, then there is no $\mathbf{w} \in \mathbb{R}^{n+1}$ for which $U\mathbf{w} < \mathbf{c}$, where the (i, \mathbf{v}) th entry of \mathbf{c} is $c_{i\mathbf{v}}$, and the inequality is coefficientwise. There is thus also no $\mathbf{w} \in \mathbb{R}_{\leq 0}^{n+1}$ with $U\mathbf{w} \leq \mathbf{c}'$, where $c'_i = \min(c_i, 0) - 1$. Let U' be the $(\ell + n + 1) \times (n + 1)$ matrix with first ℓ rows equal to U , and the last $n + 1$ rows an identity matrix. There is thus no $\mathbf{w} \in \mathbb{R}^{n+1}$ with $U'\mathbf{w} \leq (\mathbf{c}', \mathbf{0})^T$. By the Farkas lemma ([Zie95, Proposition 1.7]) there is thus $\mathbf{b} \in \mathbb{R}_{\geq 0}^{\ell+n+1}$ with $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{b}^T U' = \mathbf{0}$. Since U' has integral entries, we may choose $\mathbf{b} \in \mathbb{N}^{\ell+n+1}$. Write $\mathbf{b}_{i,\mathbf{v}}$ for the component of \mathbf{b} corresponding to the row $-\mathbf{v} + \mathbf{u}_i$ of U' . Then since $\mathbf{b} \geq \mathbf{0}$, we must have $\sum_{i,\mathbf{v}} \mathbf{b}_{i,\mathbf{v}}(-\mathbf{v} + \mathbf{u}_i) \leq \mathbf{0}$. This means that $\prod_{i,\mathbf{v}} (\mathbf{x}^{\mathbf{u}_i})^{\mathbf{b}_{i,\mathbf{v}}}$ divides $\prod_{i,\mathbf{v}} (\mathbf{x}^{\mathbf{v}})^{\mathbf{b}_{i,\mathbf{v}}}$, so $\prod_{i,\mathbf{v}} (\mathbf{x}^{\mathbf{u}_i})^{\mathbf{b}_{i,\mathbf{v}}} \geq \prod_{i,\mathbf{v}} (\mathbf{x}^{\mathbf{v}})^{\mathbf{b}_{i,\mathbf{v}}}$. But $\mathbf{x}^{\mathbf{u}_i} \prec \mathbf{x}^{\mathbf{v}}$ for all $1 \leq i \leq s$ and all $\mathbf{x}^{\mathbf{v}} \in \text{supp}(g_i) \setminus \{\mathbf{x}^{\mathbf{u}_i}\}$, so $\prod_{i,\mathbf{v}} (\mathbf{x}^{\mathbf{u}_i})^{\mathbf{b}_{i,\mathbf{v}}} \prec \prod_{i,\mathbf{v}} (\mathbf{x}^{\mathbf{v}})^{\mathbf{b}_{i,\mathbf{v}}}$. From this contradiction we conclude that C_\prec° is nonempty, and thus C_\prec is nonempty as well.

Finally, note that the argument in the previous paragraph applies verbatim substituting \mathbf{c} by $\mathbf{0}$ to show that the open cone

$$\{\mathbf{w} \in \mathbb{R}^{n+1} : \mathbf{w} \cdot \mathbf{u}_i < \mathbf{w} \cdot \mathbf{v} \text{ for all } 1 \leq i \leq s \text{ and } \mathbf{x}^{\mathbf{v}} \in \text{supp}(g_i) \setminus \{\mathbf{x}^{\mathbf{u}_i}\}\}$$

is nonempty. The recession cone of C_\prec is the closure of this cone, so it is full dimensional. \square

Example 2.8. Let $I \subseteq \overline{\mathbb{R}}[x_0, x_1, x_2]$ be the ideal of the point $[0 : 0 : 0] \in \text{trop}(\mathbb{P}^2)$. This is tropicalization of the ideal $\langle x_1 - x_0, x_2 - x_0 \rangle \subseteq K[x_0, x_1, x_2]$ for any field K . Let \prec be the reverse lexicographic term order with $x_0 \prec x_1 \prec x_2$. Then $\text{in}_\prec(I) = \langle x_0, x_1 \rangle$. The cone C_\prec° from the proof of Lemma 2.7 is $C_\prec^\circ = \{\mathbf{w} \in \mathbb{R}^3 : w_0 < w_2, w_1 < w_2\}$. Note that while $\text{in}_{\mathbf{w}}(I) = \text{in}_\prec(I)$ for all $\mathbf{w} \in C_\prec^\circ$, we do not have $\text{in}_{\mathbf{w}}(f) = \text{in}_\prec(f)$ for all $f \in I$ and $\mathbf{w} \in C_\prec^\circ$. For example, $\text{in}_\prec(x_0 \oplus x_1) = x_0$, while $\text{in}_{(1,0,2)}(x_0 \oplus x_1) = x_1$. \diamond

2.3. Varieties of tropical ideals. The *variety* of a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ is

$$V(I) := \{\mathbf{w} \in \overline{\mathbb{R}}^n : f(\mathbf{w}) = \infty, \text{ or the minimum in } f(\mathbf{w}) \text{ is attained at least twice}\}.$$

The variety of an ideal $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined similarly:

$$V(I) := \{\mathbf{w} \in \mathbb{R}^n : \text{the minimum in } f(\mathbf{w}) \text{ is attained at least twice}\}.$$

For a homogeneous ideal $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$, we can think of its variety as a subset of

$$\text{trop}(\mathbb{P}^n) = (\overline{\mathbb{R}}^{n+1} \setminus (\infty, \dots, \infty)) / \mathbb{R}(1, \dots, 1),$$

namely

$$V(I) := \{[\mathbf{w}] \in \text{trop}(\mathbb{P}^n) : f(\mathbf{w}) = \infty, \text{ or the min in } f(\mathbf{w}) \text{ is attained at least twice}\}.$$

See [MR18, §4] for more on this.

Theorem 5.11 of [MR18] proves that if I is a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $\overline{\mathbb{R}}[x_1, \dots, x_n]$, or a homogeneous tropical ideal in $\overline{\mathbb{R}}[x_0, \dots, x_n]$, the variety $V(I)$ is the support of a finite \mathbb{R} -rational polyhedral complex in either \mathbb{R}^n , $\overline{\mathbb{R}}^n$, or $\text{trop}(\mathbb{P}^n)$ respectively. Here by \mathbb{R} -rational we mean that every polyhedron in it has a rational normal fan (but not necessarily rational vertices). One source of this polyhedral complex structure in the homogeneous case is the *Gröbner complex* of I . This is the finite \mathbb{R} -rational polyhedral complex for which \mathbf{w} and \mathbf{w}' live in the same relatively open polyhedron if and only if $\text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}'}(I)$; see [MR18, §5].

We have the following relationships between the varieties of ideals.

Lemma 2.9.

(1) Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal, and let $J = I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. Then $V(J) \cap \mathbb{R}^n = V(I)$.

(2) Let $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ be a tropical ideal. Then

$$V(I^h) \cap \{[\mathbf{w}] : w_0 = 0\} = \{[0, \mathbf{w}'] : \mathbf{w}' \in V(I)\}.$$

Proof. (1) Since every polynomial in J is also in I , we have the inclusion $V(I) \subseteq V(J) \cap \mathbb{R}^n$. Now suppose $\mathbf{w} \in \mathbb{R}^n$ is not in $V(I)$. Then there is $f \in I$ with $\text{in}_{\mathbf{w}}(f)$ a monomial. Choose \mathbf{x}^u with $\mathbf{x}^u f \in J$. Then $\text{in}_{\mathbf{w}}(\mathbf{x}^u f) = \mathbf{x}^u \text{in}_{\mathbf{w}}(f)$ is also a monomial, so $\mathbf{w} \notin V(I)$.

(2) For any $\mathbf{w} \in \mathbb{R}^{n+1}$, write \mathbf{w}' for the projection of \mathbf{w} onto the last n coordinates. Then for every $f \in I$ and $\mathbf{w} \in \mathbb{R}^{n+1}$ with $w_0 = 0$, the minimum in $\tilde{f}(\mathbf{w})$ is attained twice if and only if the minimum in $f(\mathbf{w}')$ is attained twice, and so $\mathbf{w} \in V(I^h)$ if and only if $\mathbf{w}' \in V(I)$. \square

For realizable tropical ideals the variety of an initial ideal with respect to \mathbf{w} is the *star* of the variety at \mathbf{w} . We now extend this to all tropical ideals.

Let Σ be a polyhedral complex in \mathbb{R}^n , and let σ be a cell of Σ . The *linear span* of σ is the linear subspace

$$\text{span}(\sigma) := \text{span}\{\mathbf{x} - \mathbf{y} : \mathbf{x}, \mathbf{y} \in \sigma\}.$$

The *star* $\text{star}_{\Sigma}(\sigma)$ of Σ at σ is a polyhedral fan whose cones are indexed by the cells τ of Σ containing σ . The cone indexed by such a τ is the convex cone $\bar{\tau} := \text{cone}\{\mathbf{x} - \mathbf{y} : \mathbf{x} \in \tau \text{ and } \mathbf{y} \in \sigma\}$. Equivalently, if $\mathbf{w} \in \text{relint}(\sigma)$, we have

$$\bar{\tau} = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{w} + \epsilon \mathbf{v} \in \tau \text{ for all } 0 < \epsilon \ll 1\}.$$

The fan $\text{star}_{\Sigma}(\sigma)$ has lineality space equal to $\text{span}(\sigma)$.

If $\mathbf{w} \in \mathbb{R}^n$ lies in the support of Σ , we set

$$\text{star}_{\Sigma}(\mathbf{w}) := \text{star}_{\Sigma}(\sigma),$$

where σ is the cell of Σ for which $\mathbf{w} \in \text{relint}(\sigma)$. If \mathbf{w} is not in the support of Σ we set $\text{star}_{\Sigma}(\mathbf{w}) = \emptyset$.

Proposition 2.10. *Fix a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then we have*

$$\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I)$$

for $0 < \epsilon \ll 1$, and thus

$$V(\text{in}_{\mathbf{w}}(I)) = \text{star}_{V(I)}(\mathbf{w}).$$

Proof. For any $f \in \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(f)) = \text{in}_{\mathbf{w}+\epsilon\mathbf{v}}(f)$ for small enough $\epsilon > 0$. Let $I^h \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ denote the homogenization of the ideal $I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$, and consider $\tilde{\mathbf{w}} := (0, \mathbf{w}) \in \mathbb{R}^{n+1}$ and $\tilde{\mathbf{v}} := (0, \mathbf{v}) \in \mathbb{R}^{n+1}$. Since the Gröbner complex of I^h is a finite polyhedral complex, there is $\epsilon > 0$ for which the ideal $\text{in}_{\tilde{\mathbf{w}}+\epsilon'\tilde{\mathbf{v}}}(I^h)$ is constant for all $0 < \epsilon' < \epsilon$. If $\text{in}_{\tilde{\mathbf{w}}+\epsilon'\tilde{\mathbf{v}}}(I^h)$ is different from $\text{in}_{\tilde{\mathbf{v}}}(\text{in}_{\tilde{\mathbf{w}}}(I^h))$ then the two ideals differ in some degree d . Their degree d parts are generated by the corresponding initial forms of the (finitely many) circuits of I^h , and we can take ϵ' small enough so that for any such circuit f we have $\text{in}_{\tilde{\mathbf{v}}}(\text{in}_{\tilde{\mathbf{w}}}(f)) = \text{in}_{\tilde{\mathbf{w}}+\epsilon'\tilde{\mathbf{v}}}(f)$, which is a contradiction. Finally, we have that $I = I^h|_{x_0=0}$, and so by Part 2 of Lemma 2.2 we get that for any $0 < \epsilon' < \epsilon$,

$$\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\tilde{\mathbf{v}}}(\text{in}_{\tilde{\mathbf{w}}}(I^h))|_{x_0=0} = \text{in}_{\tilde{\mathbf{w}}+\epsilon'\tilde{\mathbf{v}}}(I^h)|_{x_0=0} = \text{in}_{\mathbf{w}+\epsilon'\mathbf{v}}(I).$$

The fact that $V(\text{in}_{\mathbf{w}}(I)) = \text{star}_{V(I)}(\mathbf{w})$ then follows directly from the definitions. \square

Later in Proposition 6.4 we show that the equality $V(\text{in}_{\mathbf{w}}(I)) = \text{star}_{V(I)}(\mathbf{w})$ is in fact an equality of *weighted* polyhedral fans.

Any $\mathbf{v} \in \mathbb{Z}^n$ induces a grading of the semiring $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, by setting $\deg(x_i) = v_i$, so the degree of a term $c\mathbf{x}^{\mathbf{u}}$ is $\mathbf{v} \cdot \mathbf{u} \in \mathbb{Z}$. If $L \subseteq \mathbb{R}^n$ is a rational d -dimensional linear subspace, fixing a basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d$ of L with $\mathbf{v}_i \in \mathbb{Z}^n$ for all i gives then rise to a \mathbb{Z}^d -grading on $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where the degree of a term $c\mathbf{x}^{\mathbf{u}}$ is $(\mathbf{v}_1 \cdot \mathbf{u}, \dots, \mathbf{v}_d \cdot \mathbf{u}) \in \mathbb{Z}^d$.

Corollary 2.11. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal, and let $\mathbf{w} \in V(I)$ lie in the relative interior of a cell σ of the Gröbner complex of I . Then $\text{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the grading by \mathbf{v} for any $\mathbf{v} \in \text{span}(\sigma)$. Thus $\text{in}_{\mathbf{w}}(I)$ is homogeneous with respect to a $\mathbb{Z}^{\dim(\sigma)}$ -grading induced by $\text{span}(\sigma)$.*

Proof. By Proposition 2.10, if $\mathbf{v} \in \text{span}(\sigma)$ then for $0 < \epsilon \ll 1$ we have $\text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) = \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))$. This shows that $\text{in}_{\mathbf{w}}(I)$ is generated by polynomials of the form $\text{in}_{\mathbf{v}}(f)$ with $f \in \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and thus homogeneous with respect to the grading by \mathbf{v} . \square

There is a tight connection between the tropicalization of a classical variety over the same field with a nontrivial and trivial valuation. We now extend this to tropical ideals. Let $\varphi : \overline{\mathbb{R}} \rightarrow \mathbb{B}$ be the semiring homomorphism defined by $\varphi(a) = 0$ if $a \neq \infty$, and $\varphi(\infty) = \infty$. This induces a semiring homomorphism

$$\varphi : \overline{\mathbb{R}}[x_1, \dots, x_n] \rightarrow \mathbb{B}[x_1, \dots, x_n].$$

The image $\varphi(I)$ of I is a tropical ideal in $\mathbb{B}[x_1, \dots, x_n]$, called the *trivialization* of I ; in fact, we have $\text{Mat}(\varphi(I)|_E) = \underline{\text{Mat}}(I|_E)$ for any finite collection E of monomials. For a monomial term order \prec we have $\text{in}_{\prec}(I) = \text{in}_{\prec}(\varphi(I))$. The same notions apply to ideals in the Laurent polynomial semiring $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

The set of the recession cones of all polyhedra in a polyhedral complex Σ is not always a fan, as the cones may not intersect correctly; see, for example, [BGS11]. However, when X is a subvariety of $(K^*)^n$ and Σ is a polyhedral complex structure on $\text{trop}(X)$, then this set

is a rational polyhedral fan [MS15, Theorem 3.5.6]. We now show that this generalizes to tropical ideals.

We will make use of the following notation. The *normal complex* $\mathcal{N}(f)$ of a polynomial $f \in \overline{\mathbb{R}}[x_0, \dots, x_n]$ is the \mathbb{R} -rational polyhedral complex in \mathbb{R}^{n+1} whose polyhedra are the closures of the sets $C[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^{n+1} : \text{in}_{\mathbf{w}'}(f) = \text{in}_{\mathbf{w}}(f)\}$ for $\mathbf{w} \in \mathbb{R}^{n+1}$.

Proposition 2.12. *Let $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ be a homogeneous tropical ideal. The Gröbner complex in \mathbb{R}^{n+1} of the trivialization $\varphi(I)$ is the recession fan of the Gröbner complex of I in \mathbb{R}^{n+1} . The maximal cones of this fan correspond to the monomial initial ideals of I of the form $\text{in}_{\prec}(I)$ with \prec a monomial term order. In addition, if Σ is a polyhedral complex with $|\Sigma| = V(I)$, then the support of the recession fan of Σ is $V(\varphi(I))$.*

Proof. For any $d \geq 0$, denote by Mon_d the set of monomials in $\overline{\mathbb{R}}[x_0, \dots, x_n]$ of degree d , and let $p : \binom{\text{Mon}_d}{r_d} \rightarrow \overline{\mathbb{R}}$ be the basis valuation function of the rank r_d valuated matroid $\text{Mat}(I_d)$. Consider the polynomial

$$F_d := \bigoplus_{B \text{ basis of } \underline{\text{Mat}}(I_d)} p(B) \circ \left(\prod_{\mathbf{x}^{\mathbf{u}} \in \text{Mon}_d \setminus B} \mathbf{x}^{\mathbf{u}} \right) \in \overline{\mathbb{R}}[x_0, \dots, x_n].$$

Theorem 5.6 of [MR18] shows that for $D \gg 0$, the Gröbner complex of I is equal to the normal complex $\mathcal{N}(F)$ of the polynomial $F = \bigodot_{d \leq D} F_d$. In a similar way, the Gröbner fan of the trivialization $\varphi(I)$ is the normal complex $\mathcal{N}(G)$ of the polynomial $G = \bigodot_{d \leq D} G_d \in \mathbb{B}[x_0, \dots, x_n]$, where

$$G_d := \bigoplus_{B \text{ basis of } \underline{\text{Mat}}(I_d)} \left(\prod_{\mathbf{x}^{\mathbf{u}} \in \text{Mon}_d \setminus B} \mathbf{x}^{\mathbf{u}} \right) \in \mathbb{B}[x_0, \dots, x_n]$$

and $D \gg 0$. Note that $G = \varphi(F)$. The statement that the Gröbner fan of $\varphi(I)$ is the recession fan of the Gröbner complex of I follows then from the fact that for any polynomial f , the normal complex of $\varphi(f)$ is the recession fan of the normal complex of f . Indeed, the normal complex of f is a polyhedral complex dual to the regular subdivision of the Newton polytope $\text{NP}(f)$ of f induced by the coefficients of f , and its recession fan is the normal fan of $\text{NP}(f)$, which is the normal complex of $\varphi(f)$.

We now show that the maximal cones of the Gröbner fan Σ of $\varphi(I)$ correspond to monomial initial ideals $\text{in}_{\prec}(I)$ with \prec a monomial term order. Lemma 2.7 ensures that any monomial initial ideal $\text{in}_{\prec}(I) = \text{in}_{\prec}(\varphi(I))$ is equal to $\text{in}_{\mathbf{w}}(\varphi(I))$ for \mathbf{w} in the relative interior of a maximal cone of Σ . Conversely, suppose C is a maximal cone of Σ , and take $\mathbf{w} \in C$ with all its coordinates linearly independent over \mathbb{Q} . Since $\varphi(I)$ is homogeneous, we can subtract a large multiple of $(1, \dots, 1)$ and assume that all the entries of \mathbf{w} are negative. The ordering on monomials given by $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$ if $\mathbf{w} \cdot \mathbf{u} \leq \mathbf{w} \cdot \mathbf{v}$ is then a total order, and it satisfies the two conditions for it to be a monomial term order. By definition, we have $\text{in}_{\prec}(I) = \text{in}_{\mathbf{w}}(\varphi(I))$, and thus the cone C corresponds to the monomial initial ideal $\text{in}_{\prec}(I)$.

Finally, to prove the claim about the tropical varieties, we first observe the analogous claim for a single tropical polynomial $f \in \overline{\mathbb{R}}[x_0, \dots, x_n]$. The variety $V(f)$ is the codimension-one skeleton of the normal complex to the subdivision of the Newton polytope $\text{NP}(f)$ of f induced by the coefficients of f . Maximal cells of $V(f)$ are dual to edges of this subdivision. These cells are unbounded, so have a nontrivial recession cone, only if the dual edge is part of an

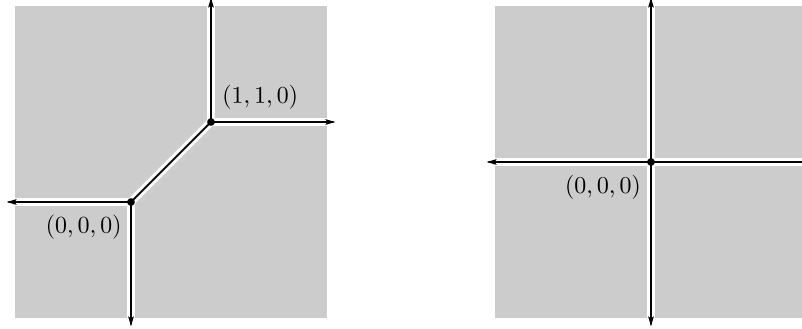


FIGURE 1. The Gröbner complex and its recession fan from Example 2.13.

edge of the Newton polytope $\text{NP}(f)$. In that case the recession cone of the cell is the normal cone to the edge. Since $V(\varphi(f))$ is the codimension-one skeleton of the normal fan of $\text{NP}(f)$, and the maximal cells of $\text{NP}(f)$ are the normal cones to edges; this proves the claim for a single tropical polynomial.

For the general case, by [MR18, Theorem 5.9] there exists a finite collection f_1, \dots, f_s of polynomials in I that form a tropical basis for I and for which $\varphi(f_1), \dots, \varphi(f_s)$ form a tropical basis for $\varphi(I)$, meaning that $V(I) = \bigcap_{i=1}^s V(f_i)$ and $V(\varphi(I)) = \bigcap_{i=1}^s V(\varphi(f_i))$. The result then follows from the fact that the recession cone of the intersection of two polyhedra is the intersection of the two recession cones, and so the recession fan of the intersection of the complexes $V(f_i)$ for $1 \leq i \leq s$ equals the intersection of the fans $V(\varphi(f_i))$, as required. \square

Example 2.13. Let $J = \langle xy + xz + yz + 2z^2 \rangle \subseteq \mathbb{Q}[x, y, z]$, where \mathbb{Q} has the 2-adic valuation, and let $I = \text{trop}(J)$. The Gröbner complex of I is the normal complex $\mathcal{N}(xy \oplus xz \oplus yz \oplus 1 \circ z^2)$, shown on the left of Figure 1 with the lineality space $\mathbb{R}(1, 1, 1)$ quotiented out. The Gröbner complex of the trivialization $\varphi(I)$ is the normal complex $\mathcal{N}(xy \oplus xz \oplus yz \oplus z^2)$, shown on the right of Figure 1. Note that the second complex is the recession fan of the first. \diamond

3. SPECIALIZATION

In this section we prove that the class of tropical ideals is closed under specialization of the variables (Theorem 3.6).

Definition 3.1. If $f \in \overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$ and $\mathbf{a} = (a_1, \dots, a_m) \in \overline{\mathbb{R}}^m$, we write $f|_{\mathbf{y}=\mathbf{a}} := f(x_1, \dots, x_n, a_1, \dots, a_m) \in \overline{\mathbb{R}}[x_1, \dots, x_n]$. For an ideal $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$, we denote by $I|_{\mathbf{y}=\mathbf{a}}$ the set

$$I|_{\mathbf{y}=\mathbf{a}} := \{f|_{\mathbf{y}=\mathbf{a}} : f \in I\} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n],$$

and call it the *specialization* of I at $\mathbf{y} = \mathbf{a}$. Note that $I|_{\mathbf{y}=\mathbf{a}}$ is an ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$.

For any polynomial $f \in \overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$ and any monomial $\mathbf{x}^{\mathbf{u}}$ in the variables x_1, \dots, x_n , we denote by $f_{\mathbf{u}} \in \overline{\mathbb{R}}[y_1, \dots, y_m]$ the coefficient of $\mathbf{x}^{\mathbf{u}}$ in f viewed as a polynomial in x_1, \dots, x_n , so

$$f(x_1, \dots, x_n, y_1, \dots, y_m) = \bigoplus_{\mathbf{u} \in \mathbb{N}^n} f_{\mathbf{u}}(\mathbf{y}) \circ \mathbf{x}^{\mathbf{u}}.$$

Our main result in this section is that if I is a tropical ideal then any specialization of I is also a tropical ideal. The proof boils down to the following lemma. For tropical polynomials f, g we write $f \geq g$ if the inequality holds coefficientwise.

Lemma 3.2. *Let I be a tropical ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$. Suppose that $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}$ are monomials in the variables x_1, \dots, x_n . If $f, g \in I$ satisfy $f_{\mathbf{u}}(\mathbf{0}) \leq g_{\mathbf{u}}(\mathbf{0})$, and $f_{\mathbf{v}}(\mathbf{0}) > g_{\mathbf{v}}(\mathbf{0})$, then there is $h \in I$ with $h_{\mathbf{u}} = \infty$, $h_{\mathbf{v}}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$, $h|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}}$, and $\text{in}_{\mathbf{0}}(h_{\mathbf{v}}) = \text{in}_{\mathbf{0}}(g_{\mathbf{v}})$.*

Proof. If $g_{\mathbf{u}} = \infty$ then we can take $h = g$, so henceforth we assume that $g_{\mathbf{u}} \neq \infty$. The proof is by induction on m . In the case $m = 0$ the polynomials $f_{\mathbf{u}}, f_{\mathbf{v}}, g_{\mathbf{u}}$, and $g_{\mathbf{v}}$ are all constants, and the claim is true for an elimination $h \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}((g_{\mathbf{u}} - f_{\mathbf{u}}) \circ f, g)$. We now assume that $m > 0$ and that the lemma is true for smaller m .

Throughout the proof, we make use of the induction hypothesis and apply the lemma by regarding the variable y_m as an x variable. For a polynomial $h \in I$ we write $h_{(\mathbf{u},j)} \in \overline{\mathbb{R}}[y_1, \dots, y_{m-1}]$ for the coefficient of $\mathbf{x}^{\mathbf{u}} y_m^j$ in h viewed as a polynomial in x_1, \dots, x_n, y_m , so

$$h(x_1, \dots, x_n, y_1, \dots, y_m) = \bigoplus_{\substack{\mathbf{u} \in \mathbb{N}^n \\ j \in \mathbb{N}}} h_{(\mathbf{u},j)}(y_1, \dots, y_{m-1}) \circ \mathbf{x}^{\mathbf{u}} y_m^j.$$

Note that for any $\mathbf{u} \in \mathbb{N}^n$ we have

$$h_{\mathbf{u}}(y_1, \dots, y_m) = \bigoplus_{j \in \mathbb{N}} h_{(\mathbf{u},j)}(y_1, \dots, y_{m-1}) \circ y_m^j.$$

We denote by $\deg_{y_m}(h_{\mathbf{u}})$ the maximum j such that $h_{(\mathbf{u},j)} \neq \infty$.

Now, suppose the lemma is not true for the tropical ideal I and the monomials $\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}$. Choose a counterexample $f, g \in I$ which is minimal in the sense that $(\deg_{y_m}(g_{\mathbf{u}}), \deg_{y_m}(f_{\mathbf{u}})) \in \mathbb{N} \times \mathbb{N}$ is lexicographically minimal among all counterexamples $f, g \in I$. Consider the sets

$$\begin{aligned} \mathcal{F} &:= \{h \in I : h|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}} \text{ and } h_{\mathbf{v}}(\mathbf{0}) > g_{\mathbf{v}}(\mathbf{0})\}, \\ \mathcal{G} &:= \{h \in I : h|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}} \text{ and } h_{\mathbf{v}}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})\}. \end{aligned}$$

Note that $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Moreover, regarding y_m as an x variable, any application of the lemma to monomials $\mathbf{x}^{\mathbf{u}} y_m^j$ and $\mathbf{x}^{\mathbf{v}} y_m^k$ in polynomials $h_1 \in \mathcal{F}$ and $h_2 \in \mathcal{G}$ with $(h_2)_{(\mathbf{v},k)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$ yields a polynomial $h_3 \in \mathcal{G}$. Similarly, any application of the lemma to any two monomials in polynomials $h_1, h_2 \in \mathcal{F}$ gives a polynomial $h_3 \in \mathcal{F}$.

Set

$$d := \deg_{y_m}(f_{\mathbf{u}}) \in \mathbb{N} \quad \text{and} \quad \alpha := f_{(\mathbf{u},d)}(\mathbf{0}) \in \mathbb{R},$$

and similarly

$$e := \deg_{y_m}(g_{\mathbf{u}}) \in \mathbb{N} \quad \text{and} \quad \beta := g_{(\mathbf{u},e)}(\mathbf{0}) \in \mathbb{R}.$$

Claim 3.3. There is no $h \in \mathcal{G}$ with $c := \deg_{y_m}(h_{\mathbf{u}}) \leq d$ and $\gamma := h_{(\mathbf{u},c)}(\mathbf{0}) < \alpha$.

Suppose that such an $h \in \mathcal{G}$ exists. Set $h' = (\alpha - \gamma) \circ y_m^{d-c} \circ h \in \mathcal{F}$, so that both $f_{\mathbf{u}}$ and $h'_{\mathbf{u}}$ are polynomials of degree d in y_m and $h'_{(\mathbf{u},d)}(\mathbf{0}) = \alpha$. Since $f_{\mathbf{u}}(\mathbf{0}) \leq h_{\mathbf{u}}(\mathbf{0}) \leq \gamma < \alpha$, there is $j < d$ with $f_{(\mathbf{u},j)}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})$. We have $h'_{(\mathbf{u},d)}(\mathbf{0}) = \alpha = f_{(\mathbf{u},d)}(\mathbf{0})$ and $h'_{(\mathbf{u},j)}(\mathbf{0}) > h_{(\mathbf{u},j)}(\mathbf{0}) \geq h_{\mathbf{u}}(\mathbf{0}) \geq f_{\mathbf{u}}(\mathbf{0}) = f_{(\mathbf{u},j)}(\mathbf{0})$, so we can apply the lemma to the monomials $\mathbf{x}^{\mathbf{u}} y_m^d$ and $\mathbf{x}^{\mathbf{u}} y_m^j$ in the polynomials h' and f to obtain a polynomial $f' \in \mathcal{F}$ with $f'_{(\mathbf{u},d)} = \infty$ and $f'_{\mathbf{u}}(\mathbf{0}) = f'_{(\mathbf{u},j)}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})$. Since $\deg_{y_m}(f'_{\mathbf{u}}) < d$, our minimality assumption

implies that f', g satisfy the statement of the lemma, so there is $h \in I$ with $h_{\mathbf{u}} = \infty$, $h|_{\mathbf{y}=\mathbf{0}} \geq f'|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}}$ and $\text{in}_{\mathbf{0}}(h_{\mathbf{v}}) = \text{in}_{\mathbf{0}}(g_{\mathbf{v}})$. But then h contradicts the assumption that f and g were a counterexample to the lemma, finishing the proof of the claim.

Claim 3.4. There is no $h \in \mathcal{F}$ with $c := \deg_{y_m}(h_{\mathbf{u}}) \leq e$ and $\gamma := h_{(\mathbf{u},c)}(\mathbf{0}) \leq \beta$.

Suppose that such an $h \in \mathcal{F}$ exists. Set $h' = y_m^{e-c} \circ h \in \mathcal{F}$, so that both $h'_{\mathbf{u}}$ and $g_{\mathbf{u}}$ are polynomials of degree e in y_m . Fix j such that $g_{(\mathbf{v},j)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$. We have $h'_{(\mathbf{u},e)}(\mathbf{0}) = \gamma \leq g_{(\mathbf{u},e)}(\mathbf{0})$ and $h'_{(\mathbf{v},j)}(\mathbf{0}) \geq h'_{\mathbf{v}}(\mathbf{0}) = h_{\mathbf{v}}(\mathbf{0}) > g_{\mathbf{v}}(\mathbf{0}) = g_{(\mathbf{v},j)}(\mathbf{0})$, so we can apply the lemma to the monomials $\mathbf{x}^{\mathbf{u}}y_m^e$ and $\mathbf{x}^{\mathbf{v}}y_m^j$ in the polynomials h' and g to obtain a polynomial $g^j \in \mathcal{G}$ with $g^j_{(\mathbf{u},e)}(\mathbf{0}) = \infty$, $g^j_{(\mathbf{v},j)}(\mathbf{0}) = g_{(\mathbf{v},j)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$, and $\text{in}_{\mathbf{0}}(g^j_{(\mathbf{v},j)}) = \text{in}_{\mathbf{0}}(g_{(\mathbf{v},j)})$. Let $g' \in \mathcal{G}$ be the sum of all g^j over all such choices of j . Then $g'_{(\mathbf{u},e)}(\mathbf{0}) = \infty$, and $g'_{\mathbf{v}}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$. Since $\text{in}_{\mathbf{0}}(g_{\mathbf{v}}) = \bigoplus_j \text{in}_{\mathbf{0}}(g_{(\mathbf{v},j)})$, this implies that $\text{in}_{\mathbf{0}}(g'_{\mathbf{v}}) = \text{in}_{\mathbf{0}}(g_{\mathbf{v}})$. Since $\deg_{y_m}(g'_{\mathbf{u}}) < e$, our minimality assumption implies that f, g' satisfy the statement of the lemma, so there is $h \in I$ such that $h_{\mathbf{u}} = \infty$, $h_{\mathbf{v}}(\mathbf{0}) = g'_{\mathbf{v}}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$, $h|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g'|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}}$, and $\text{in}_{\mathbf{0}}(h_{\mathbf{v}}) = \text{in}_{\mathbf{0}}(g'_{\mathbf{v}}) = \text{in}_{\mathbf{0}}(g_{\mathbf{v}})$. Such an h contradicts the assumption that f and g were a counterexample to the lemma, so this finishes the proof of the claim.

Since $g \in \mathcal{G}$, Claim 3.3 implies that if $\alpha > \beta$ then $d < e$. Also, since $f \in \mathcal{F}$, Claim 3.4 implies that if $\alpha \leq \beta$ then $d > e$. We now show that both of these cases are impossible, which leads to a contradiction to our original assumption that the counterexample exists.

• **Case $\alpha > \beta$ and $d < e$.** We inductively construct an infinite sequence of polynomials $f_1, f_2, \dots \in \mathcal{F}$ satisfying the following conditions:

- (1) For all i we have $(f_i)_{\mathbf{u}}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})$.
- (2) Set $d_i := \deg_{y_m}((f_i)_{\mathbf{u}})$ and $\alpha_i := (f_i)_{(\mathbf{u},d_i)}(\mathbf{0})$. We have $\alpha_i > \beta$ and $d_i < e$ for all i .
- (3) Set $l_i := \max\{j : (f_i)_{(\mathbf{u},j)}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})\}$. We have $l_1 < l_2 < \dots$.

Set $f_1 := f$, and suppose that we have constructed $f_i \in \mathcal{F}$. To construct f_{i+1} , set $f'_i = y_m^{e-d_i} \circ f_i \in \mathcal{F}$ and $g'_i = (\alpha_i - \beta) \circ g \in \mathcal{F}$, so that both $(f'_i)_{\mathbf{u}}$ and $(g'_i)_{\mathbf{u}}$ are polynomials of degree e in y_m and $(f'_i)_{(\mathbf{u},e)}(\mathbf{0}) = \alpha_i = (g'_i)_{(\mathbf{u},e)}(\mathbf{0})$. Since $(f'_i)_{\mathbf{u}}(\mathbf{0}) = (f_i)_{\mathbf{u}}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})$, there is k such that $(f'_i)_{(\mathbf{u},k)}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})$. Take k to be the largest possible such value, which is equal to $(e - d_i) + l_i$. We have $(g'_i)_{(\mathbf{u},e)}(\mathbf{0}) = (f'_i)_{(\mathbf{u},e)}(\mathbf{0})$ and $(g'_i)_{(\mathbf{u},k)}(\mathbf{0}) > g_{(\mathbf{u},k)}(\mathbf{0}) \geq g_{\mathbf{u}}(\mathbf{0}) \geq f_{\mathbf{u}}(\mathbf{0}) = (f'_i)_{(\mathbf{u},k)}(\mathbf{0})$, so $k \neq e$, and we can apply the lemma to the monomials $\mathbf{x}^{\mathbf{u}}y_m^e$ and $\mathbf{x}^{\mathbf{u}}y_m^k$ in the polynomials g'_i and f'_i to obtain a polynomial $f_{i+1} \in \mathcal{F}$ with $(f_{i+1})_{(\mathbf{u},e)} = \infty$ and $(f_{i+1})_{\mathbf{u}}(\mathbf{0}) = (f_{i+1})_{(\mathbf{u},k)}(\mathbf{0}) = f_{\mathbf{u}}(\mathbf{0})$. We have $l_{i+1} := \max\{j : (f_{i+1})_{(\mathbf{u},j)} = f_{\mathbf{u}}(\mathbf{0})\} = k = (e - d_i) + l_i$, so $l_{i+1} > l_i$. Furthermore, note that $d_{i+1} := \deg_{y_m}((f_{i+1})_{\mathbf{u}}) < e$, so by Claim 3.4 we must also have $\alpha_{i+1} > \beta$. This all shows that f_{i+1} satisfies the desired properties. We conclude this case by noting that the sequence $l_1 < l_2 < \dots$ is a strictly increasing infinite sequence of integers bounded above by e , which is a contradiction.

• **Case $\alpha \leq \beta$ and $d > e$.** We inductively construct a sequence of polynomials $g_1, g_2, \dots \in \mathcal{G}$ satisfying the following conditions:

- (1) Set $e_i := \deg_{y_m}((g_i)_{\mathbf{u}})$ and $\beta_i := (g_i)_{(\mathbf{u},e_i)}(\mathbf{0})$. We have $\alpha \leq \beta_i$ and $d > e_i$ for all i .
- (2) Let $l_i := \min\{j : (g_i)_{(\mathbf{v},j)} = g_{\mathbf{v}}(\mathbf{0})\}$. We have $l_1 < l_2 < \dots$.

Set $g_1 := g$, and suppose that we have constructed $g_i \in \mathcal{G}$. To construct g_{i+1} , set $g'_i = y_m^{d-e_i} \circ g_i \in \mathcal{G}$, so that both $f_{\mathbf{u}}$ and $(g'_i)_{\mathbf{u}}$ are polynomials of degree d in y_m . Set $k := (d - e_i) + l_i$, which is the minimum value satisfying $(g'_i)_{(\mathbf{v},k)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$. We have $f_{(\mathbf{u},d)}(\mathbf{0}) = \alpha \leq$

$\beta_i = (g'_i)_{(\mathbf{u},d)}(\mathbf{0})$ and $f_{(\mathbf{v},k)}(\mathbf{0}) \geq f_{\mathbf{v}}(\mathbf{0}) > g_{\mathbf{v}}(\mathbf{0}) = (g'_i)_{(\mathbf{v},k)}(\mathbf{0})$, so we can apply the lemma to the monomials $\mathbf{x}^{\mathbf{u}}y_m^d$ and $\mathbf{x}^{\mathbf{v}}y_m^k$ in the polynomials f and g'_i to obtain a polynomial $g_{i+1} \in \mathcal{G}$ with $(g_{i+1})_{(\mathbf{u},d)}(\mathbf{0}) = \infty$, and $(g_{i+1})_{(\mathbf{v},k)}(\mathbf{0}) = (g'_i)_{(\mathbf{v},k)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$. We have $l_{i+1} := \min\{j : (g_{i+1})_{(\mathbf{v},j)} = g_{\mathbf{v}}(\mathbf{0})\} = k = (d - e_i) + l_i$, so $l_{i+1} > l_i$. This all shows that g_{i+1} satisfies the desired properties. Now, as the sequence of degrees e_1, e_2, \dots is an infinite sequence of integers bounded above by d , it must contain an infinite constant subsequence $e_{i_1} = e_{i_2} = \dots$. This contradicts the following claim.

Claim 3.5. There is no infinite sequence of polynomials $g_1, g_2, \dots \in \mathcal{G}$ satisfying the condition $l_1 < l_2 < \dots$ of (2) with $\deg_{y_m}((g_1)_{\mathbf{u}}) = \deg_{y_m}((g_2)_{\mathbf{u}}) = \dots$.

To prove the claim, suppose such a sequence $g_1, g_2, \dots \in \mathcal{G}$ exists, and assume $c := \deg_{y_m}((g_1)_{\mathbf{u}}) = \deg_{y_m}((g_2)_{\mathbf{u}}) = \dots$ is minimal among all such sequences. Set $\beta_i := (g_i)_{(\mathbf{u},c)}(\mathbf{0})$. For any $j > 0$, let $g'_{2j-1} = \max(0, \beta_{2j} - \beta_{2j-1}) \circ g_{2j-1}$ and $g'_{2j} = \max(0, \beta_{2j-1} - \beta_{2j}) \circ g_{2j}$, so that both $(g'_{2j-1})_{\mathbf{u}}$ and $(g'_{2j})_{\mathbf{u}}$ are polynomials of degree c in y_m satisfying $(g'_{2j-1})_{(\mathbf{u},c)}(\mathbf{0}) = (g'_{2j})_{(\mathbf{u},c)}(\mathbf{0})$. Set $l = l_{2j-1}$ if $\beta_{2j-1} \geq \beta_{2j}$, and $l = l_{2j}$ if $\beta_{2j-1} < \beta_{2j}$. Note that l is the minimum value for which either $(g'_{2j-1})_{(\mathbf{v},l)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$ or $(g'_{2j})_{(\mathbf{v},l)}(\mathbf{0}) = g_{\mathbf{v}}(\mathbf{0})$, and that exactly one of these equalities holds. Since $(g'_{2j-1})_{(\mathbf{u},c)}(\mathbf{0}) = (g'_{2j})_{(\mathbf{u},c)}(\mathbf{0})$ and $(g'_{2j-1})_{(\mathbf{v},l)}(\mathbf{0}) \neq (g'_{2j})_{(\mathbf{v},l)}(\mathbf{0})$, so $c \neq l$ and we can apply the lemma to the monomials $\mathbf{x}^{\mathbf{u}}y_m^c$ and $\mathbf{x}^{\mathbf{v}}y_m^l$ in the polynomials g'_{2j-1} and g'_{2j} (possibly in the reverse order) to obtain a polynomial $g''_j \in I$ with $(g''_j)_{(\mathbf{u},c)}(\mathbf{0}) = \infty$, $(g''_j)_{(\mathbf{v},l)}(\mathbf{0}) = \min((g'_{2j-1})_{(\mathbf{v},l)}(\mathbf{0}), (g'_{2j})_{(\mathbf{v},l)}(\mathbf{0})) = g_{\mathbf{v}}(\mathbf{0})$, and $g''_j|_{\mathbf{y}=\mathbf{0}} \geq g'_{2j-1}|_{\mathbf{y}=\mathbf{0}} \oplus g'_{2j}|_{\mathbf{y}=\mathbf{0}} \geq f|_{\mathbf{y}=\mathbf{0}} \oplus g|_{\mathbf{y}=\mathbf{0}}$. As $\min\{s : (g''_j)_{(\mathbf{v},s)} = g_{\mathbf{v}}(\mathbf{0})\} = l$, which is equal to either l_{2j-1} or l_{2j} , it follows that g''_1, g''_2, \dots is a sequence of polynomials in \mathcal{G} satisfying condition (2).

Now, since the sequence of degrees $\deg_{y_m}((g''_1)_{\mathbf{u}}), \deg_{y_m}((g''_2)_{\mathbf{u}}), \dots$ is an infinite sequence of non-negative integers strictly less than c , it must contain an infinite constant subsequence $\deg_{y_m}((g''_{i_1})_{\mathbf{u}}) = \deg_{y_m}((g''_{i_2})_{\mathbf{u}}) = \dots$. The sequence $g''_{i_1}, g''_{i_2}, \dots$ is then an infinite sequence of polynomials in \mathcal{G} satisfying condition (2) such that $c > \deg_{y_m}((g''_{i_1})_{\mathbf{u}}) = \deg_{y_m}((g''_{i_2})_{\mathbf{u}}) = \dots$, which contradicts the minimality of c , showing that no such sequence exists. \square

We now use Lemma 3.2 to prove the key specialization theorem. We will also need Lemma 3.2 in its full strength in the proof of the projection theorem in Section 4.

Theorem 3.6. Let $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ be a tropical ideal. For any $a \in \overline{\mathbb{R}}$, the ideal $I|_{x_n=a} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_{n-1}]$ is also a tropical ideal.

Proof. If $a = \infty$, the monomial elimination axiom for $I|_{x_n=a}$ follows directly from the monomial elimination axiom for I . Indeed, if $f, g \in I|_{x_n=\infty}$ then there are $F, G \in I$ such that $f = F|_{x_n=\infty}$ and $g = G|_{x_n=\infty}$. Suppose $\mathbf{x}^{\mathbf{u}}$ is a monomial such that $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}} \neq \infty$. As evaluating the variable x_n at ∞ does not change the coefficients of the monomials not divisible by x_n , we also have $[F]_{\mathbf{x}^{\mathbf{u}}} = [G]_{\mathbf{x}^{\mathbf{u}}} \neq \infty$. Since I is a tropical ideal, there exists $H \in I$ with $H \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(F, G)$. This implies that $h := H|_{x_n=\infty} \in I|_{x_n=\infty}$ is an elimination $h \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{u}}}(f, g)$, showing that $I|_{x_n=\infty}$ satisfies the monomial elimination axiom.

Suppose now that $a \neq \infty$. Consider the ideal $I' := \{f(x_1, \dots, x_{n-1}, ax_n) : f \in I\} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$. Since I' is obtained from I by doing an invertible scaling of the variable x_n , the fact that I is a tropical ideal implies that I' is also a tropical ideal. Note that $I|_{x_n=a} = I'|_{x_n=0}$, and thus we may assume that $a = 0$.

To show that $I|_{x_n=0}$ is a tropical ideal, fix two polynomials $f, g \in I|_{x_n=0}$ and a monomial $\mathbf{x}^{\mathbf{u}}$ with $[f]_{\mathbf{x}^{\mathbf{u}}} = [g]_{\mathbf{x}^{\mathbf{u}}} \neq \infty$. Choose $F, G \in I$ such that $f = F|_{x_n=0}$ and $g = G|_{x_n=0}$. For any monomial $\mathbf{x}^{\mathbf{v}}$ for which $F_{\mathbf{v}}(0) \neq G_{\mathbf{v}}(0)$, we can use Lemma 3.2 to construct a polynomial $H^{\mathbf{v}} \in I$ such that $H_{\mathbf{u}} = \infty$, $H_{\mathbf{v}}(0) = \min(F_{\mathbf{v}}(0), G_{\mathbf{v}}(0))$, and $H|_{x_n=0} \geq F|_{x_n=0} \oplus G|_{x_n=0}$. Let $H \in I$ be the sum of all such polynomials $H^{\mathbf{v}}$. Then $h := H|_{x_n=0} \in I|_{x_n=0}$ satisfies $[h]_{\mathbf{x}^{\mathbf{u}}} = \infty$ and $[h]_{\mathbf{x}^{\mathbf{v}}} \geq \min([f]_{\mathbf{x}^{\mathbf{v}}}, [g]_{\mathbf{x}^{\mathbf{v}}})$ for all monomials $\mathbf{x}^{\mathbf{v}}$, with the equality holding whenever $[f]_{\mathbf{x}^{\mathbf{v}}} \neq [g]_{\mathbf{x}^{\mathbf{v}}}$. This shows that $I|_{x_n=0}$ satisfies the monomial elimination axiom, and so it is a tropical ideal. \square

Remark 3.7. If K is a valued field with an uncountable residue field \mathbb{k} , $I \subseteq K[x_1, \dots, x_n]$ is an ideal, and $a \in \mathbb{R}$ is an element of the value group of K , then $\text{trop}(I)|_{x_n=a} = \text{trop}(I|_{x_n=\alpha})$ where α is a sufficiently generic element of K with valuation a . To see this, first fix $\alpha_0 \in K$ of valuation a . For $f \in I$, write f as a polynomial in x_1, \dots, x_{n-1} with coefficients in $K[x_n]$. We claim that for each such coefficient $g = \sum c_i x_n^i$ and α of valuation a we have $\text{val}(g(\alpha)) \geq \text{trop}(g)(a)$, with equality for all but finitely values of $\alpha/\alpha_0 \in \mathbb{k}$. The inequality is immediate from the valuation axioms, so we only need justify the equality condition. Fix j with $\text{val}(c_j \alpha_0^j) = \text{trop}(g)(a)$, and note that $g = c_j \alpha_0^j \sum_i (c_i \alpha_0^{i-j}/c_j)(x_n/\alpha_0)^i$. Set $b_i = c_i \alpha_0^{i-j}/c_j$ and $g' = \sum b_i y^i \in K[y]$. By construction $\text{val}(g(\alpha)) > \text{trop}(g)(a)$ if and only if $\text{val}(g'(\alpha/\alpha_0)) > \text{trop}(g')(0) = 0$. This occurs if and only if α/α_0 is one of the finitely many roots of $\bar{g}' = \sum_i \bar{b}_i y^i \in \mathbb{k}[y]$.

The ideal $\text{trop}(I)|_{x_n=a}$ is generated by its circuits, which are specializations at $x_n = a$ of circuits of $\text{trop}(I)$. These are tropicalizations of polynomials in I , and up to scaling, there are a countable number of them. There are thus, up to scaling, a countable number of polynomials in $\bar{g}' \in \mathbb{k}[x_n]$ where g is a coefficient of a circuit. We have $\text{trop}(I)|_{x_n=a} = \text{trop}(I|_{x_n=\alpha})$ for all $\alpha \in K$ with $\text{val}(\alpha) = a$ and the property that α/α_0 is not a root of any of these polynomials in $\mathbb{k}[x_n]$.

To see that some hypothesis on the field is necessary, consider the trivial valuation on $\mathbb{Z}/2\mathbb{Z}$, and the ideal $I = \langle y^2 + y + x \rangle \subseteq \mathbb{Z}/2\mathbb{Z}[x, y]$. Then $\text{trop}(I) \subseteq \mathbb{B}[x, y]$ contains $y^2 \oplus y \oplus x$ but does not contain any polynomial in $\mathbb{B}[y]$, and is saturated with respect to x . It follows that $\text{trop}(I)|_{y=0}$ contains $x \oplus 0$, but does not contain x . However, the only element of $\mathbb{Z}/2\mathbb{Z}$ with valuation 0 is 1, so $I|_{y=1} = \langle x \rangle$, and thus $\text{trop}(I)|_{y=1}$ is not equal to $\text{trop}(I)|_{y=0}$.

On the other hand, $\text{trop}(I)|_{x_n=\infty} = \text{trop}(I|_{x_n=0})$ holds without any hypothesis on the field.

Corollary 3.8. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal, and let $J = I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. For any $a \in \mathbb{R}$ the ideal*

$$I|_{x_n=a} := \{f|_{x_n=a} : f \in I\} \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$$

equals $J|_{x_n=a} \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. Thus $I|_{x_n=a}$ is a tropical ideal.

Proof. The equality $I|_{x_n=a} = J|_{x_n=a} \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ follows from $I = J \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and from the fact that for any polynomial $f \in J$ and any Laurent monomial $\mathbf{x}^{\mathbf{u}}$ we have $(f\mathbf{x}^{\mathbf{u}})|_{x_n=a} = f|_{x_n=a} \circ \mathbf{x}^{\mathbf{u}}|_{x_n=a}$. Part 1 of Lemma 2.1 and Theorem 3.6 thus imply that $I|_{x_n=a}$ is a tropical ideal. \square

The following example shows that polynomials of degree at most d in a specialization are not necessarily specializations of polynomials of degree at most d in the ideal.

Example 3.9. Let $J = \langle x_1 - 1, x_2 - x_3 \rangle \subseteq \mathbb{C}[x_1, x_2, x_3]$, and let $I = \text{trop}(J)$. Consider the specialization $I' = I|_{x_3=0}$. We have $x_1 \oplus 0 \in I$, and thus in $x_1 \oplus 0 \in I'$. Also, $x_2 \oplus x_3 \in I$, so $x_2 \oplus 0 \in I'$. Since I' is a tropical ideal, the monomial elimination axiom implies that $x_1 \oplus x_2 \in I'$. However there is no polynomial with tropicalization $x_1 \oplus x_2$ in J ; we need to set x_3 to 0 in the polynomial $x_1 x_3 \oplus x_2 = \text{trop}(x_3(x_1 - 1) - (x_2 - x_3)) \in I$. \diamond

Example 3.9 can be homogenized to show that if I is a homogeneous tropical ideal, the operation of specializing $x_n = x_0$, which is an ideal in $\overline{\mathbb{R}}[x_0, \dots, x_{n-1}]$, is not always a tropical ideal. This can be fixed, however, by saturating appropriately.

Proposition 3.10. *Let $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ be a homogeneous tropical ideal, and denote by $I|_{x_0=0} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ its dehomogenization. For any $a \in \overline{\mathbb{R}}$, we have*

$$(I|_{x_n=a x_0} : x_0^\infty) = ((I|_{x_0=0})|_{x_n=a})^h \subseteq \overline{\mathbb{R}}[x_0, \dots, x_{n-1}].$$

In particular, $(I|_{x_n=a x_0} : x_0^\infty)$ is a homogeneous tropical ideal.

Proof. For any homogeneous polynomial $f \in \overline{\mathbb{R}}[x_0, \dots, x_n]$, we have, for some $r \geq 0$, the equality $f|_{x_n=a x_0} = x_0^r \circ (f|_{x_0=0, x_n=a})^h \in \overline{\mathbb{R}}[x_0, \dots, x_{n-1}]$. This implies that $I|_{x_n=a x_0} \subseteq ((I|_{x_0=0})|_{x_n=a})^h \subseteq (I|_{x_n=a x_0} : x_0^\infty)$. As $((I|_{x_0=0})|_{x_n=a})^h$ is saturated with respect to x_0 , the desired equality must hold. \square

We finish this section with some observations about the effect of specialization on initial ideals and varieties. Recall that the trivialization of a tropical ideal I is the image of I under the homomorphism induced by $\varphi: \overline{\mathbb{R}} \rightarrow \mathbb{B}$ defined by $\varphi(a) = 0$ if $a \neq \infty$ and $\varphi(\infty) = \infty$.

Lemma 3.11. *Let I be a tropical ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ or $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.*

- (1) *For all $a \in \overline{\mathbb{R}}$ we have $\varphi(I|_{x_n=a}) = \varphi(I)|_{x_n=\varphi(a)}$.*
- (2) *For all $\mathbf{w} \in \mathbb{R}^{n-1}$ and all $a \in \mathbb{R}$ we have $\text{in}_{\mathbf{w}}(I|_{x_n=a}) = \text{in}_{(\mathbf{w}, a)}(I)|_{x_n=0}$.*

Proof. In both cases, Part 1 is expressing the fact that applying φ to a polynomial commutes with specializing a variable, as φ is a semiring homomorphism.

To prove Part 2 for $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$, note that $\text{in}_{\mathbf{w}}(I|_{x_n=a})$ is equal to the set of polynomials of the form $\text{in}_{\mathbf{w}}(f|_{x_n=a})$ with $f \in I$, while $\text{in}_{(\mathbf{w}, a)}(I)|_{x_n=0}$ is equal to the set of polynomials of the form $\text{in}_{(\mathbf{w}, a)}(f)|_{x_n=0}$ with $f \in I$. It thus suffices to show that

$$(3.1) \quad \text{in}_{\mathbf{w}}(f|_{x_n=a}) = \text{in}_{(\mathbf{w}, a)}(f)|_{x_n=0}$$

for any polynomial $f = \bigoplus c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$. For $\mathbf{u} \in \mathbb{N}^n$, denote by \mathbf{u}' its projection onto the first $n-1$ coordinates. The initial term $\text{in}_{(\mathbf{w}, a)}(f)$ is equal to the sum of those monomials $\mathbf{x}^{\mathbf{u}}$ for which $c_{\mathbf{u}} + (\mathbf{w}, a) \cdot \mathbf{u}$ is smallest, and so $\text{in}_{(\mathbf{w}, a)}(f)|_{x_n=0}$ is the sum of those monomials $\mathbf{x}^{\mathbf{u}'}$ for which there exists u_n with $c_{(\mathbf{u}', u_n)} + \mathbf{w} \cdot \mathbf{u}' + a u_n$ equal to the minimum value $f(\mathbf{w}, a)$. Now, the coefficient of $\mathbf{x}^{\mathbf{u}'}$ in $f|_{x_n=a}$ is $\min(c_{(\mathbf{u}', u_n)} + u_n a) =: b_{\mathbf{u}'}$. The minimum value of $b_{\mathbf{u}'} + \mathbf{w} \cdot \mathbf{u}'$ is thus also $f(\mathbf{w}, a)$, and $\text{in}_{\mathbf{w}}(f|_{x_n=a})$ is the sum of those monomials $\mathbf{x}^{\mathbf{u}'}$ achieving this minimum. The desired equality follows from the fact that \mathbf{u}' achieves the minimum if and only if any choice of u_n such that $\min(c_{(\mathbf{u}', u_n)} + u_n a) = b_{\mathbf{u}'}$ satisfies $c_{(\mathbf{u}', u_n)} + \mathbf{w} \cdot \mathbf{u}' + a u_n = f(\mathbf{w}, a)$.

The case that $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ follows from the above argument using Part 1 of Lemma 2.2, since $I|_{x_n=a} = (I \cap \overline{\mathbb{R}}[x_1, \dots, x_n])|_{x_n=a} \mathbb{R}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. \square

Recall that two polyhedral complexes Σ_1, Σ_2 in \mathbb{R}^n intersect transversely at $\mathbf{w} \in \mathbb{R}^n$ if \mathbf{w} lies in the relative interior of $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, and $\text{span}(\sigma_1) + \text{span}(\sigma_2) = \mathbb{R}^n$. We say that Σ_1 and Σ_2 intersect transversely if they intersect transversely at any $\mathbf{w} \in \Sigma_1 \cap \Sigma_2$.

Proposition 3.12. *Fix a tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $a \in \mathbb{R}$. Then $V(I|_{x_n=a}) \subseteq \mathbb{R}^{n-1}$ is contained in $\pi(V(I) \cap \{x_n = a\})$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the projection onto the first $n-1$ coordinates. Moreover if \mathbf{w} lies in a closed cell σ of a polyhedral complex Σ with $|\Sigma| = V(I)$ with the property that $\text{span}(\sigma) \not\subseteq \{x_n = 0\}$, then $\pi(\mathbf{w}) \in V(I|_{x_n=a})$. Thus if the intersection of $V(I)$ and $\{x_n = a\}$ is transverse at $\mathbf{w} \in \mathbb{R}^n$ then $\pi(\mathbf{w}) \in V(I|_{x_n=a})$.*

Proof. Fix $\mathbf{w}' \in V(I|_{x_n=a}) \subseteq \mathbb{R}^{n-1}$. Then for any $f \in I$, the minimum in $f|_{x_n=a}(\mathbf{w}')$ is achieved at least twice, say at monomials $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$. Write i, j for exponents of x_n at which the minimum in the univariate polynomials $f_{\mathbf{u}}(a)$ and $f_{\mathbf{v}}(a)$ is achieved. Then the minimum in $f(\mathbf{w}', a)$ is achieved at the terms $\mathbf{x}^{\mathbf{u}}x_n^i$ and $\mathbf{x}^{\mathbf{v}}x_n^j$, and so $(\mathbf{w}', a) \in V(I)$.

Now suppose that \mathbf{w} lies in a closed cell σ of a polyhedral complex Σ with $|\Sigma| = V(I)$ with the property that $\text{span}(\sigma) \not\subseteq \{x_n = 0\}$, but assume that $\pi(\mathbf{w}) \notin V(I|_{x_n=a})$. Then there is $f \in I|_{x_n=a}$ with $\text{in}_{\pi(\mathbf{w})}(f)$ equal to a monomial $\mathbf{x}^{\mathbf{u}}$. Fix $F \in I$ with $f = F|_{x_n=a}$. Then $\text{in}_{\mathbf{w}}(F)$ equals $\mathbf{x}^{\mathbf{u}}$ times a polynomial $g \in \mathbb{B}[x_n^{\pm 1}]$ with more than one term, by Equation (3.1). This means that for any $\mathbf{w}' \in V(\text{in}_{\mathbf{w}}(I))$ we must have $w'_n = 0$. By Proposition 2.10, the variety of the initial ideal $\text{in}_{\mathbf{w}}(I)$ is the star of $V(I)$ at \mathbf{w} . But this contradicts the assumption that $\mathbf{w} \in \sigma$ with $\text{span}(\sigma) \not\subseteq \{x_n = 0\}$. The claim about transverse intersection is the special case that $\mathbf{w} \in \text{relint}(\sigma)$. \square

In the case that the variety $V(I)$ is the support of a pure polyhedral complex, we show later in Proposition 6.8 that $V(I|_{x_n=a})$ is the *stable intersection* of $V(I)$ and $\{x_n = a\}$.

4. DIMENSION AND PROJECTIONS

In this section we prove several fundamental results about the dimension (Theorem 4.3) and projections (Theorem 4.7) of varieties of tropical ideals.

For a homogeneous tropical ideal $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$, the Hilbert function $H_I(d) = \text{rk}(\text{Mat}(I_d))$ agrees with a polynomial for $d \gg 0$, called the *Hilbert polynomial* of I [MR18, Proposition 3.8]. The *dimension* of I is defined to be the degree of this polynomial. We can extend this definition to tropical ideals in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ and $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, by setting $\dim(I) = \dim(I^h)$ for $I \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$, and $\dim(I) = \dim(I \cap \overline{\mathbb{R}}[x_1, \dots, x_n])$ for $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Proposition 4.1. *Let I be a proper tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ or $\overline{\mathbb{R}}[x_1, \dots, x_n]$. For any $a \in \mathbb{R}$ we have*

$$\dim(I|_{x_n=a}) \leq \dim(I) - 1.$$

Proof. If I is a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, let $J = I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. By definition, $\dim(J) = \dim(I)$. We also have $J|_{x_n=a} = (I|_{x_n=a}) \cap \overline{\mathbb{R}}[x_1, \dots, x_{n-1}]$, and so $\dim(J|_{x_n=a}) = \dim(I|_{x_n=a})$. Replacing I by J , we see that we can reduce to proving the statement for ideals in $\overline{\mathbb{R}}[x_1, \dots, x_n]$.

Suppose that I is a tropical ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$. The dimension of a tropical ideal is determined by the ranks of the degree- d matroids of its homogenization, and so depends only

on its trivialization. Thus by Part 1 of Lemma 3.11 the dimension of $I|_{x_n=a}$ does not depend on the value of $a \in \mathbb{R}$.

For any $\mathbf{w} \in \mathbb{R}^n$, write \mathbf{w}' for the projection of \mathbf{w} onto the first $n - 1$ coordinates. By Part 2 of Lemma 3.11, we have $\text{in}_{\mathbf{w}'}(I|_{x_n=w_n}) = (\text{in}_{\mathbf{w}}(I))|_{x_n=0}$. In particular, if $\text{in}_{\mathbf{w}}(I)$ is a monomial ideal then $\text{in}_{\mathbf{w}'}(I|_{x_n=w_n})$ is a monomial ideal as well.

We now prove that there is a $\mathbf{w} \in \mathbb{R}^n$ such that $\text{in}_{\mathbf{w}}(I)$ is a monomial ideal with the property that $(\text{in}_{\mathbf{w}'}(I|_{x_n=w_n}))_{\leq d} = \text{in}_{\mathbf{w}'}((I|_{x_n=w_n})_{\leq d})$ for all $d \geq 0$. Consider the homogenization $I^h \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ of I . The Gröbner complex of I^h is a finite polyhedral complex in \mathbb{R}^{n+1} whose maximal cells correspond to monomial initial ideals, and so there exists $w_n \in \mathbb{R}$ such that the set of $(w_0, \dots, w_{n-1}) \in \mathbb{R}^n$ for which $\text{in}_{(w_0, \dots, w_{n-1}, w_n)}(I^h)$ is not a monomial ideal is a polyhedral complex in \mathbb{R}^n of dimension at most $n - 1$. Set $I' = I|_{x_n=w_n}$. Let \prec be a reverse-lexicographic order on $\overline{\mathbb{R}}[x_0, \dots, x_{n-1}]$ with the variables ordered so that $x_i \prec x_0$ for all $i = 1, \dots, n - 1$. By Proposition 2.12, the set of $(w_0, \dots, w_{n-1}) \in \mathbb{R}^n$ such that $\text{in}_{(w_0, \dots, w_{n-1})}((I')^h) = \text{in}_{\prec}((I')^h)$ is an n -dimensional open polyhedron in \mathbb{R}^n , and so our assumption on w_n implies we can pick one such (w_0, \dots, w_{n-1}) with the additional property that $\text{in}_{(w_0, \dots, w_{n-1}, w_n)}(I^h)$ is a monomial ideal. Adding a suitable multiple of $(1, \dots, 1)$, we can assume that $w_0 = 0$. We claim that $\mathbf{w} = (w_1, \dots, w_n)$ chosen in this way satisfies the desired properties. First, by Part 2 of Lemma 2.2, we have $\text{in}_{\mathbf{w}}(I) = (\text{in}_{(0, \mathbf{w})}(I^h))|_{x_0=0}$, which is a monomial ideal. It remains to be checked that $(\text{in}_{\mathbf{w}'}(I'))_{\leq d} = \text{in}_{\mathbf{w}'}(I'_{\leq d})$. The inclusion $(\text{in}_{\mathbf{w}'}(I'))_{\leq d} \supseteq \text{in}_{\mathbf{w}'}(I'_{\leq d})$ holds for any ideal I' and any vector \mathbf{w}' . For the reverse inclusion, take $\mathbf{x}^{\mathbf{u}}$ a generator of the monomial ideal $\text{in}_{\mathbf{w}'}(I')$ satisfying $\deg(\mathbf{x}^{\mathbf{u}}) \leq d$. Again by Part 2 of Lemma 2.2, we have $\text{in}_{\mathbf{w}'}(I') = \text{in}_{(0, \mathbf{w}')}((I')^h)|_{x_0=0}$, and so there is some $k \geq 0$ such that $x_0^k \mathbf{x}^{\mathbf{u}} \in \text{in}_{(0, \mathbf{w}')}((I')^h) = \text{in}_{\prec}((I')^h)$. Let $m = k + |\mathbf{u}|$. By Lemma 2.6, the monomials of degree m not in $\{\text{in}_{\prec}(f) : f \in (I')_m^h\}$ form a basis B for $\underline{\text{Mat}}((I')_m^h)$, and so there is a polynomial $f \in (I')_m^h$ such that $\text{supp}(f) \cap \text{in}_{\prec}((I')^h) = \{x_0^k \mathbf{x}^{\mathbf{u}}\}$, corresponding to the fundamental circuit of $x_0^k \mathbf{x}^{\mathbf{u}}$ over B . We then have $\text{in}_{(0, \mathbf{w}')} (f) = \text{in}_{\prec}(f) = x_0^k \mathbf{x}^{\mathbf{u}}$. Since \prec is reverse-lexicographic with x_0 the last variable, it follows that all monomials in $\text{supp}(f)$ are divisible by x_0^k . As $(I')^h$ is saturated with respect to x_0 , this implies that $f = x_0^k g$ with $g \in (I')^h$. We conclude that $\mathbf{x}^{\mathbf{u}} = \text{in}_{(0, \mathbf{w}')} (g) \in \text{in}_{(0, \mathbf{w}')} ((I')_{\leq d}^h)$, and thus $\mathbf{x}^{\mathbf{u}} \in \text{in}_{\mathbf{w}'} (I'_{\leq d})$, as desired.

Now, for any $d \geq 0$, let B'_d be the set of monomials of degree at most d not in $\text{in}_{\mathbf{w}'}(I')$. Since $(\text{in}_{\mathbf{w}'}(I'))_{\leq d} = \text{in}_{\mathbf{w}'}(I'_{\leq d})$, by [MR18, Lemma 3.3] the set B'_d is a basis of the matroid $\underline{\text{Mat}}(I'_{\leq d})$. Consider the set of monomials $B_d := \{\mathbf{x}^{\mathbf{u}} x_0^k : \mathbf{x}^{\mathbf{u}} \in B'_d \text{ and } 0 \leq k \leq d - |\mathbf{u}|\}$. As $\text{in}_{\mathbf{w}'}(I') = (\text{in}_{\mathbf{w}}(I))|_{x_n=0}$, none of the monomials in B_d can be contained in $\text{in}_{\mathbf{w}}(I)$, and thus B_d is an independent set of the matroid $\underline{\text{Mat}}(I_{\leq d})$. Note that B_{d+1} is the disjoint union $B_{d+1} = B'_{d+1} \sqcup x_n B_d$. We then have $H_{I'}(d+1) = |B'_{d+1}| = |B_{d+1}| - |B_d|$. For $d \gg 0$, the function $H_{I'}(d)$ agrees with a polynomial on d of degree $\dim(I')$ by [MR18, Proposition 3.8]. It follows that for $d \gg 0$, the sequence $(|B_d|)_{d \geq 0}$ agrees with a polynomial of degree $\dim(I') + 1$. Since $H_I(d) \geq |B_d|$, the Hilbert polynomial of I is a polynomial of degree at least $\dim(I') + 1$, and thus $\dim(I) \geq \dim(I') + 1$, as claimed. \square

Remark 4.2. The strict inequality $\dim(I|_{x_n=a}) < \dim(I) - 1$ is possible. An example is given by $I = \text{trop}(\langle x_1 - 1, x_2 - 1 \rangle \cap \langle x_3 - 1 \rangle) \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$, which has dimension two. By Remark 3.7, for any $a \in \mathbb{R}$ we have $I|_{x_3=a} = \text{trop}(\langle x_1 - 1, x_2 - 1 \rangle) \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, x_2^{\pm 1}]$, which is zero-dimensional.

We now prove that the dimension of a tropical ideal agrees with the dimension of its variety.

Theorem 4.3. *Let I be a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of dimension d . Then $V(I) \subseteq \mathbb{R}^n$ is a d -dimensional polyhedral complex.*

We note that this complex need not be pure, so may have maximal cells of dimension less than d .

Proof. We first prove the equality

$$(4.1) \quad \dim(V(I)) = \max(|S| : S \subseteq \{1, \dots, n\} \text{ and } I \cap \overline{\mathbb{R}}[x_j^{\pm 1} : j \in S] = \{\infty\}),$$

where by convention we set this maximum to be -1 if $I \cap \overline{\mathbb{R}} = \overline{\mathbb{R}}$. Denote by e the expression on the right hand side. To show that $\dim(V(I)) \leq e$, take a cell σ of $V(I)$ of maximal dimension. There exists $S \subseteq \{1, \dots, n\}$ of size $\dim(\sigma) = \dim(V(I))$ such that the projection $\pi(\sigma)$ of σ onto the coordinate subspace $\mathbb{R}^S \subseteq \mathbb{R}^n$ is injective. It follows that I does not contain any polynomial in $\overline{\mathbb{R}}[x_j^{\pm 1} : j \in S]$ other than ∞ , as otherwise $\pi(\sigma) \subseteq \mathbb{R}^S$ would have codimension at least 1.

We now prove that $\dim(V(I)) \geq e$ by induction on $\dim(V(I))$. If $\dim(V(I)) = -1$ then $V(I) = \emptyset$, and by the weak Nullstellensatz for tropical ideals [MR18, Corollary 5.17], we also have $e = -1$. For the induction step, suppose $\dim(V(I)) \geq 0$, and fix $S \subseteq \{1, \dots, n\}$ of size $\dim(V(I)) + 1$ and $i \in S$. Choose $a \in \mathbb{R}$ such that the hyperplane $\{x_i = a\}$ intersects $V(I)$ transversely at all points of intersection (or their intersection is empty). This is possible since $V(I)$ is the support of a finite polyhedral complex. By Proposition 3.12 we have $V(I|_{x_i=a}) = \pi(V(I) \cap \{x_i = a\})$, where π is the projection onto the coordinates other than i , and so $V(I|_{x_i=a})$ has dimension at most $\dim(V(I)) - 1$. As $|S \setminus \{i\}| = \dim(V(I))$, the induction hypothesis then guarantees that there is a polynomial $f \neq \infty$ in $I|_{x_i=a} \cap \overline{\mathbb{R}}[x_j^{\pm 1} : j \in S \setminus \{i\}]$. The polynomial f must be the specialization $f = g|_{x_i=a}$ for a polynomial $g \in I \cap \overline{\mathbb{R}}[x_j^{\pm 1} : j \in S]$, which shows that $I \cap \overline{\mathbb{R}}[x_j^{\pm 1} : j \in S] \neq \{\infty\}$. This completes the proof of (4.1).

We now show that (4.1) implies that $\dim(I) = \dim(V(I))$. Take $S \subseteq \{1, \dots, n\}$ of size $\dim(V(I))$ such that $I \cap \overline{\mathbb{R}}[x_j^{\pm 1} : j \in S] = \{\infty\}$. Let $J \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ be the homogenization of $I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. Fix \prec be a term order on $\overline{\mathbb{R}}[x_0, \dots, x_n]$ with the property that $x_i \prec \mathbf{x}^{\mathbf{u}}$ whenever both $i \notin \{0\} \cup S$ and $u_j = 0$ for all $j \notin \{0\} \cup S$. Any polynomial in J involves a variable x_i with $i \notin \{0\} \cup S$, and thus any monomial $\mathbf{x}^{\mathbf{u}} \in \text{in}_{\prec}(J)$ must involve a variable x_i with $i \notin \{0\} \cup S$. This means that $\text{in}_{\prec}(J) \cap \overline{\mathbb{R}}[x_j : j \in \{0\} \cup S] = \{\infty\}$, and so

$$\max(|S'| : S' \subseteq \{0, \dots, n\} \text{ and } \text{in}_{\prec}(J) \cap \overline{\mathbb{R}}[x_j : j \in S'] = \{\infty\}) \geq \dim(V(I)) + 1.$$

In fact, this last inequality must be an equality, as any subset S' such that $\text{in}_{\prec}(J) \cap \overline{\mathbb{R}}[x_j : j \in S'] = \{\infty\}$ must also satisfy $J \cap \overline{\mathbb{R}}[x_j : j \in S'] = \{\infty\}$. The left-hand side is $\dim(\text{in}_{\prec}(J)) - 1$, as our convention for dimension of homogeneous tropical ideals is the degree of the Hilbert polynomial (so one less than the Krull dimension), and this formula for dimension holds for all monomial ideals. It follows that $\dim(I) = \dim(V(I))$, as $\dim(I) = \dim(J) = \dim(\text{in}_{\prec}(J))$ by Lemma 2.6. \square

Remark 4.4. One consequence of Theorem 4.3 is that the definition of dimension we use here, as the degree of the Hilbert polynomial, essentially agrees with a naive notion of Krull dimension, as in [JM18b]. This follows from the proof of Theorem 7.2.1 of [KM16], which shows that for an arbitrary ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$, if $V(I)$ is the support of an \mathbb{R} -rational polyhedral complex then $\dim(\overline{\mathbb{R}}[x_1, \dots, x_n]/\mathcal{B}(I))$ is one more than the maximal dimension of a cell in the complex. The “one more” comes from the fact that $\dim(\overline{\mathbb{R}}) = 1$ in this theory.

However, as shown in [JM18a], varieties of prime ideals are not flexible enough to play the role of irreducible varieties in a tropical scheme theory.

We next consider the effects of changes of coordinates on $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. This is essentially identical to the realizable case (see [MS15, Lemma 2.6.10 and Corollary 3.2.13]).

Let $A \in \text{GL}(n, \mathbb{Z})$ be an $n \times n$ invertible matrix with integer entries, and fix $\lambda \in \mathbb{R}^n$. Let $\psi^*: \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the semiring homomorphism given by $x_i \mapsto \lambda_i \mathbf{x}^{\mathbf{a}_i}$, where \mathbf{a}_i is the i th column of A . Such homomorphisms are precisely the automorphisms of $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Write ψ_A^* for the homomorphism given by $\psi_A^*(x_i) = \mathbf{x}^{\mathbf{a}_i}$. We denote by $\text{trop}(\psi): \mathbb{R}^n \rightarrow \mathbb{R}^n$ the linear map given by $\text{trop}(\psi)(\mathbf{w}) = A^T \mathbf{w} + \lambda$.

Lemma 4.5. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal, and let $I' = \psi^{*-1}(I)$. Then I' is also a tropical ideal, and*

$$\psi_A^*(\text{in}_{\text{trop}(\psi)(\mathbf{w})}(I')) = \text{in}_{\mathbf{w}}(I)$$

for all $\mathbf{w} \in \mathbb{R}^n$. As a consequence, we have

$$V(I') = \text{trop}(\psi)(V(I)).$$

Proof. We first show that I' is a tropical ideal. Write $\psi^* = \psi_\lambda^* \circ \psi_A^*$, where $\psi_\lambda^*(x_i) = \lambda_i x_i$. Let E be a finite collection of monomials in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $E' = \psi_A^*(E)$. Since I is a tropical ideal, the polynomials in I with support in E' are the vectors of a valuated matroid. Applying the invertible map ψ_λ^{*-1} to this collection produces an equivalent valuated matroid on the ground set E' . Since ψ_A^* is an injection, the collection of polynomials in I' with support in E define the same matroid. This shows that I' is a tropical ideal.

Fix $f = \bigoplus c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in I'$, so $\psi^*(f) = \bigoplus c_{\mathbf{u}} \circ (\lambda \cdot \mathbf{u}) \circ \mathbf{x}^{A\mathbf{u}} \in I$. We have

$$\begin{aligned} (4.2) \quad \text{in}_{\mathbf{w}}(\psi^*(f)) &= \bigoplus_{\mathbf{u}: c_{\mathbf{u}} + \lambda \cdot \mathbf{u} + \mathbf{w} \cdot (A\mathbf{u}) \text{ is minimal}} \mathbf{x}^{A\mathbf{u}} \\ &= \bigoplus_{\mathbf{u}: c_{\mathbf{u}} + (\lambda + A^T \mathbf{w}) \cdot \mathbf{u} \text{ is minimal}} \mathbf{x}^{A\mathbf{u}} \\ &= \bigoplus_{\mathbf{u}: c_{\mathbf{u}} + \text{trop}(\mathbf{w}) \cdot \mathbf{u} \text{ is minimal}} \mathbf{x}^{A\mathbf{u}} \\ &= \psi_A^*(\text{in}_{\text{trop}(\psi)(\mathbf{w})}(f)). \end{aligned}$$

This implies that $\psi_A^*(\text{in}_{\text{trop}(\psi)(\mathbf{w})}(I')) \subseteq \text{in}_{\mathbf{w}}(I)$. As A is invertible, we also have $\psi_{A^{-1}}^*(\text{in}_{\mathbf{w}}(I)) \subseteq \text{in}_{\text{trop}(\psi)(\mathbf{w})}(I')$, since $\mathbf{w} = \text{trop}(\psi^{-1})(\text{trop}(\psi)(\mathbf{w}))$, so $\text{in}_{\text{trop}(\psi)(\mathbf{w})}(I') = \text{in}_{\mathbf{w}}(I)$. Thus $V(I') = \text{trop}(\psi)(V(I))$, as required. \square

We conclude this section with Theorem 4.7, which shows that the projection of the variety of a tropical ideal is the variety of the elimination ideal, as is the case for realizable tropical ideals. For this we first require the following result. Recall from [MR18, Theorem 5.9] that any tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ has a finite tropical basis. This is a set $\{f_1, \dots, f_s\} \subset I$ with the property that $V(I) = \bigcap_{i=1}^s V(f_i)$.

Lemma 4.6. *Let I be a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. If $\mathbf{v} \in \mathbb{R}^n$ is not in $V(I)$ then there exists a finite tropical basis $\mathcal{T} \subset I$ such that for all $f \in \mathcal{T}$ we have $\mathbf{v} \notin V(f)$.*

Proof. Fix $f \in I$ such that $\mathbf{v} \notin V(f)$. The set $\mathbb{R}^n \setminus V(f)$ decomposes naturally as a union of finitely many (full-dimensional) open polyhedra, so \mathbf{v} lies in the interior of one such open polyhedron P . Let \mathcal{T} be a finite tropical basis for I . For each $g \in \mathcal{T}$ we can decompose $\mathbb{R}^n \setminus V(g)$ as a union of open polyhedra. After subdividing, we can thus write $\mathbb{R}^n \setminus (V(I) \cup P) = \bigcup P_i$, where for each P_i the closure \overline{P}_i is an n -dimensional pointed polyhedron, there is $g \in \mathcal{T}$ such that $P_i \subseteq \mathbb{R}^n \setminus V(g)$ with $g(\mathbf{w}) = \mathbf{a} \cdot \mathbf{w} + b$ for $\mathbf{w} \in P_i$, and f is linear on P_i .

Fix i , and set $Q = P_i$, with $Q \subseteq \mathbb{R}^n \setminus V(g)$ for $g \in \mathcal{T}$. Since \overline{Q} is a pointed polyhedron not containing \mathbf{v} , there is a hyperplane $\{\mathbf{w} \in \mathbb{R}^n : \mathbf{c} \cdot \mathbf{w} + d = 0\}$ with $\mathbf{c} \cdot \mathbf{w} + d > 0$ for $\mathbf{w} \in \overline{Q}$ and $\mathbf{c} \cdot \mathbf{v} + d < 0$. We may assume that the face of \overline{Q} minimizing \mathbf{c} is a vertex \mathbf{p} .

Consider the polynomial $g' := g \oplus (d\mathbf{x}^{\mathbf{c}})^N f \in I$. We claim that for sufficiently large N , the minimum in $g'(\mathbf{w})$ is achieved at terms in g for $\mathbf{w} \in Q$, and at terms in $(d\mathbf{x}^{\mathbf{c}})^N f$ for $\mathbf{w} = \mathbf{v}$. This implies that $Q \subseteq \mathbb{R}^n \setminus V(g')$, and $\mathbf{v} \notin V(g')$. It follows that the collection of g' for $g \in \mathcal{T}$, together with f , form the desired tropical basis for I .

To see the claim, since f is linear on Q , there are \mathbf{a}' and b with $f(\mathbf{w}) = \mathbf{a}' \cdot \mathbf{w} + b$ for $\mathbf{w} \in Q$. For $N \gg 0$ the face of \overline{Q} minimizing $N\mathbf{c} + \mathbf{a}' - \mathbf{a}$ is still \mathbf{p} , so $(N\mathbf{c} + \mathbf{a}' - \mathbf{a}) \cdot \mathbf{w} \geq (N\mathbf{c} + \mathbf{a}' - \mathbf{a}) \cdot \mathbf{p}$ for $\mathbf{w} \in Q$. Since $\mathbf{c} \cdot \mathbf{p} + d > 0$, for $N \gg 0$ we have

$$\begin{aligned} (N\mathbf{c} + \mathbf{a}' - \mathbf{a}) \cdot \mathbf{w} + Nd + b' - b &\geq (N\mathbf{c} + \mathbf{a}' - \mathbf{a}) \cdot \mathbf{p} + Nd + b' - b \\ &= N(\mathbf{c} \cdot \mathbf{p} + d) + (\mathbf{a}' - \mathbf{a}) \cdot \mathbf{p} + b' - b \\ &> 0 \end{aligned}$$

for all $\mathbf{w} \in Q$. For such N ,

$$g(\mathbf{w}) = \mathbf{a} \cdot \mathbf{w} + b < N(\mathbf{c} \cdot \mathbf{w} + d) + \mathbf{a}' \cdot \mathbf{w} + b' = N(\mathbf{c} \cdot \mathbf{w} + d) + f(\mathbf{w})$$

for all $\mathbf{w} \in P$, which shows the first part of the claim. For the second part, note that any $N > (g(\mathbf{v}) - f(\mathbf{v})) / (\mathbf{c} \cdot \mathbf{v} + d)$ suffices. \square

Theorem 4.7. *Let I be a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]$. Then*

$$V(I \cap \overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_m^{\pm 1}]) = \pi(V(I)),$$

where $\pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the projection onto the second factor.

Proof. Since $V(I) \subseteq V(f)$ for all $f \in I \cap \overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_m^{\pm 1}]$, where we regard f as a polynomial in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]$, we have $\pi(V(I)) \subseteq V(I \cap \overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_m^{\pm 1}])$.

To prove the reverse inclusion, suppose $\mathbf{w} \in \mathbb{R}^m$ satisfies $\mathbf{w} \notin \pi(V(I))$. By tropically scaling the y variables, we may assume that $\mathbf{w} = \mathbf{0}$. As $(\mathbf{0}, \mathbf{0}) \in \mathbb{R}^n \times \mathbb{R}^m$ is not in $V(I)$, by Lemma 4.6 the ideal I has a finite tropical basis $\mathcal{T} \subset I$ with $(\mathbf{0}, \mathbf{0}) \notin V(f)$ for all $f \in \mathcal{T}$. After multiplying each polynomial in \mathcal{T} by a monomial we may assume that \mathcal{T} consists of polynomials in $\overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$.

Now consider the specializations $f|_{\mathbf{y}=\mathbf{0}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$ for $f \in \mathcal{T}$. Since $\mathbf{0} \notin \pi(V(I))$, we claim that the tropical prevariety $\bigcap_{f \in \mathcal{T}} V(f|_{\mathbf{y}=\mathbf{0}}) \subseteq \mathbb{R}^n$ is empty. Indeed, if this prevariety contained a point $\mathbf{w}' \in \mathbb{R}^n$, then $(\mathbf{w}', \mathbf{0}) \in V(f)$ for all $f \in \mathcal{T}$, and so $(\mathbf{w}', \mathbf{0}) \in V(I)$. As $\bigcap_{f \in \mathcal{T}} V(f|_{\mathbf{y}=\mathbf{0}})$ is empty, we can apply the Tropical Primary Nullstellensatz of Grigoriev and Podolskii [GP18, Theorem 10]. This says that there is a tropical polynomial combination

$$g = \bigoplus_{i=1}^s c_i \mathbf{x}^{\mathbf{u}_i} \circ f_i|_{\mathbf{y}=\mathbf{0}} \in \overline{\mathbb{R}}[x_1, \dots, x_n]$$

where $f_i \in \mathcal{T}$ for all i , and for all monomials $\mathbf{x}^{\mathbf{v}} \in \text{supp}(g)$ there is a unique $i = i(\mathbf{v})$ with the coefficient of $\mathbf{x}^{\mathbf{v}}$ in $c_i \mathbf{x}^{\mathbf{u}_i} \circ f_i|_{\mathbf{y}=\mathbf{0}}$ less than the coefficient of $\mathbf{x}^{\mathbf{v}}$ in $c_j \mathbf{x}^{\mathbf{u}_j} \circ f_j|_{\mathbf{y}=\mathbf{0}}$ for $j \neq i(\mathbf{v})$, and for $\mathbf{v} \neq \mathbf{v}'$ we have $i(\mathbf{v}) \neq i(\mathbf{v}')$. Note that these conditions imply that we can take s to be the number of terms of g .

Set $g_i = c_i \mathbf{x}^{\mathbf{u}_i} f_i \in \overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$. We may regard g_i as a polynomial in x_1, \dots, x_n with coefficients in $\overline{\mathbb{R}}[y_1, \dots, y_m]$. Write $g_{i\mathbf{v}} \in \overline{\mathbb{R}}[y_1, \dots, y_m]$ for the coefficient of $\mathbf{x}^{\mathbf{v}}$ in g_i . The monomials $\mathbf{x}^{\mathbf{v}}$ appearing in g_i are a subset of the monomials appearing in g . Fix an order on the monomials $\mathbf{x}^{\mathbf{v}_1}, \dots, \mathbf{x}^{\mathbf{v}_s}$ appearing in g . After reordering, we may assume that the lowest coefficient appearing in $g_i|_{\mathbf{y}=\mathbf{0}}$ for any i appears in g_s , and is the coefficient of $\mathbf{x}^{\mathbf{v}_s}$, and in general the lowest coefficient of $\mathbf{x}^{\mathbf{v}_j}$ in any $g_i|_{\mathbf{y}=\mathbf{0}}$ occurs in $g_j|_{\mathbf{y}=\mathbf{0}}$ for all $1 \leq j \leq s$. Since the coefficient of $\mathbf{x}^{\mathbf{v}_s}$ in $g_s|_{\mathbf{y}=\mathbf{0}}$ is the lowest coefficient of any $g_j|_{\mathbf{y}=\mathbf{0}}$, it is the lowest coefficient appearing in $g_s|_{\mathbf{y}=\mathbf{0}}$. The assumption that $(\mathbf{0}, \mathbf{0}) \notin V(f)$ for all $f \in \mathcal{T}$ then implies that $\mathbf{0} \notin V(g_{s\mathbf{v}_s})$.

We now repeatedly apply Lemma 3.2 to do a form of Gaussian elimination on this system of polynomials. Applying Lemma 3.2 to g_1 and g_i for $i > 1$, with $\mathbf{u} = \mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_i$, we get a new polynomial $g'_i \in I$ with $g'_{i\mathbf{v}_1} = \infty$, $g'_{i\mathbf{v}_i}(\mathbf{0}) = g_{i\mathbf{v}_i}(\mathbf{0})$, and $g'_i|_{\mathbf{y}=\mathbf{0}} \geq g_1|_{\mathbf{y}=\mathbf{0}} \oplus g_i|_{\mathbf{y}=\mathbf{0}}$. Additionally, when $i = s$ we get that $\text{in}_{\mathbf{0}}(g'_{s\mathbf{v}_s}) = \text{in}_{\mathbf{0}}(g_{s\mathbf{v}_s})$, and so $\mathbf{0} \notin V(g'_{s\mathbf{v}_s})$. Note that the smallest coefficient in any $g'_i|_{\mathbf{y}=\mathbf{0}}$ still occurs in g'_s , and the coefficient of \mathbf{v}_i in $g'_i|_{\mathbf{y}=\mathbf{0}}$ is still smaller than that coefficient in $g'_j|_{\mathbf{y}=\mathbf{0}}$ for $j > i$.

We now replace g_i by g'_i for $i > 1$, and iterate, with g_2 playing the role of g_1 . After $s - 1$ iterations we obtain $g_s = \mathbf{x}^{\mathbf{v}_s} h$, where $h \in \overline{\mathbb{R}}[y_1, \dots, y_m]$ satisfies $\mathbf{0} \notin V(h)$. Since $g_s \in I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]$, we have $h \in I$, so $\mathbf{0} \notin V(I \cap \overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_m^{\pm 1}])$, as desired. This proves the reverse inclusion. \square

Example 4.8. Theorem 4.7 is not true verbatim with $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_m^{\pm 1}]$ replaced by $\overline{\mathbb{R}}[x_1, \dots, x_n, y_1, \dots, y_m]$. For example, consider $I = \text{trop}(\langle x_0 x_3 - x_1 x_2 \rangle) \subseteq \overline{\mathbb{R}}[x_0, x_1, x_2, x_3]$. Then $V(I) = \{\mathbf{w} \in \overline{\mathbb{R}}^4 : w_0 + w_3 = w_1 + w_2\}$. We have $I \cap \overline{\mathbb{R}}[x_0, x_1, x_2] = \{\infty\}$, and so $V(I \cap \overline{\mathbb{R}}[x_0, x_1, x_2]) = \overline{\mathbb{R}}^3$, but $\{(\infty, w_1, w_2) : w_1, w_2 \in \mathbb{R}\}$ is not in $\pi(V(I))$. In this example we have $V(I \cap \overline{\mathbb{R}}[x_0, x_1, x_2])$ equal to the closure of $\pi(V(I))$, as in the classical case. \diamond

5. DEGREES OF ZERO-DIMENSIONAL IDEALS

For a zero-dimensional subscheme of \mathbb{P}^n , its degree equals the sum of the multiplicities of the points in its variety. In this section we extend this to tropical ideals. The proof is more complicated than in the classical case, as we do not yet have primary decomposition available as a tool.

Definition 5.1. The *degree* of a d -dimensional homogeneous tropical ideal $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is $d!$ times the leading coefficient of the Hilbert polynomial of I .

When I is zero-dimensional, the Hilbert polynomial of I is a constant, and the degree is equal to that constant. For a zero dimensional tropical ideal I in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ or $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we define the degree to be the degree of the homogenization I^h , or of the homogenization of $I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$ respectively. We make the same definitions with $\overline{\mathbb{R}}$ replaced by \mathbb{B} . This has the following equivalent formulation.

Lemma 5.2. *Let I be a zero-dimensional tropical ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ or $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then the degree of I is the maximum size of a finite collection E of monomials not containing*

the support of any polynomial in I :

$$\deg(I) = \max(\text{rk}(\underline{\text{Mat}}(I_E))).$$

Proof. The degree of $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is by definition the same as the degree of $J = I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. If E is a collection of monomials in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, there is a polynomial in $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with support contained in E if and only if there is some monomial $\mathbf{x}^{\mathbf{u}}$ with $\mathbf{x}^{\mathbf{u}}E \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$ and a polynomial in $J := I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$ with support in $\mathbf{x}^{\mathbf{u}}E$. So it suffices to consider $J \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$.

The degree of J is the degree of J^h , which is the rank of $\underline{\text{Mat}}(J_d^h)$ for $d \gg 0$. If E is a collection of monomials in $\overline{\mathbb{R}}[x_1, \dots, x_n]$, let $E^d = \{x_0^{d-|\mathbf{u}|}\mathbf{x}^{\mathbf{u}} : \mathbf{x}^{\mathbf{u}} \in E\}$ for $d \gg \max_{\mathbf{x}^{\mathbf{u}} \in E} |\mathbf{u}|$. If there is no polynomial in J with support in E , then there is no polynomial in J^h with support in E^d , so $\deg(J^h) \geq |E|$. Conversely, let E' be a basis for $\underline{\text{Mat}}(J_d^h)$ for fixed $d \gg 0$, so there is no polynomial in J^h with support in E' . Set $E = E'|_{x_0=0}$. Then $|E'| = |E|$, and since $J^h|_{x_0=0} = J$, we have that E is a collection of monomials in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ not containing the support of any polynomial in J , thus the maximum such $|E|$ is at least $\deg(J^h)$. \square

The notion of *multiplicity* is central in tropical geometry. We now define it for zero-dimensional ideals. A general definition follows in Definition 6.1. Note that any initial ideal of a zero-dimensional tropical ideal is also zero-dimensional; this follows, for instance, from Theorem 4.3 and Proposition 2.10.

Definition 5.3. Let I be a zero-dimensional tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $\mathbf{w} \in V(I)$. The *multiplicity* $\text{mult}_{V(I)}(\mathbf{w})$ of $V(I)$ at \mathbf{w} is the degree of the zero-dimensional initial ideal $\text{in}_{\mathbf{w}}(I) \subseteq \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

When I is a zero-dimensional tropical ideal in $\overline{\mathbb{R}}[x_1, \dots, x_n]$ or a zero-dimensional homogeneous tropical ideal in $\overline{\mathbb{R}}[x_0, \dots, x_n]$, we define the multiplicity of $V(I)$ at $\mathbf{w} \in \mathbb{R}^n$ or $[\mathbf{w}] \in \text{trop}(\mathbb{P}^n)$ with $\mathbf{w} \in \mathbb{R}^{n+1}$ to be the degree of the saturation $(\text{in}_{\mathbf{w}}(I) : (\prod_{i=0}^n x_i)^\infty)$. We will not need the case where \mathbf{w} has infinite coordinates in this paper.

We first note some basic properties of the multiplicity and degree. Recall from §4 that an automorphism of $\overline{\mathbb{R}}[x_1, \dots, x_n]$ is given by a function ψ^* of the form $\psi^*(x_i) = \lambda_i \mathbf{x}^{\mathbf{a}_i}$, where $\lambda_i \in \mathbb{R}$ for all i and the $n \times n$ matrix A with columns \mathbf{a}_i has determinant ± 1 . This has tropicalization $\text{trop}(\psi)$ given by $\text{trop}(\psi)(\mathbf{w}) = A^T \mathbf{w} + \lambda$.

Proposition 5.4.

- (1) Let $\psi^*: \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ be the semiring homomorphism given by $\psi^*(x_i) = \mathbf{y}^{\mathbf{a}_i}$, where the $n \times n$ matrix A with columns \mathbf{a}_i has rank n . If J is a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then $J_\psi := \psi^*(J)\overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ is a tropical ideal as well. In addition, if J is zero-dimensional then so is J_ψ , and $\deg(J_\psi) = |\det(A)| \deg(J)$. The same holds with $\overline{\mathbb{R}}$ replaced by \mathbb{B} .
- (2) When I is a zero-dimensional ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the degree of I and the multiplicity of a point in $V(I)$ are invariant under automorphisms of $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Explicitly, if $\psi^*: \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is an automorphism and $I' = \psi^{*-1}(I)$, then $\deg(I') = \deg(I)$, and $\text{mult}_{V(I')}(\mathbf{w}) = \text{mult}_{V(I)}(\text{trop}(\psi)(\mathbf{w}))$.
- (3) For a zero-dimensional tropical ideal $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, let $J \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ be the homogenization of $I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. Then for any $\mathbf{w} \in V(I)$, we have $\text{mult}_{V(I)}(\mathbf{w}) = \text{mult}_{V(J)}([0 : \mathbf{w}])$.

Proof. (1) We first show that the ideal $\psi^*(J)\overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ is a tropical ideal. Note that this ideal is homogeneous with respect to the grading of $\overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ by the cokernel G of the matrix A , which is a finite abelian group of size $|\det(A)|$. Explicitly, let L be the sublattice of \mathbb{Z}^n spanned by the columns \mathbf{a}_i of A . Grade $\overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ by $\deg(y_i) = \mathbf{e}_i + L \in G = \mathbb{Z}^n/L$. Since $\psi^*(f)$ has degree zero for every $f \in J$, the ideal J_ψ is homogeneous with respect to this grading. In addition, note that if $\deg(\mathbf{y}^{\mathbf{u}}) = \mathbf{0}$ then $\mathbf{y}^{\mathbf{u}} = \psi^*(\mathbf{x}^{\mathbf{v}})$ for some monomial $\mathbf{x}^{\mathbf{v}}$, so $\mathbf{y}^{\mathbf{u}}\psi^*(f) = \psi^*(\mathbf{x}^{\mathbf{v}}f)$, and thus $(J_\psi)_{\mathbf{0}} = \psi^*(J)$. This also shows that $(J_\psi)_\gamma = \mathbf{y}^{\mathbf{u}}\psi^*(J)$ for any $\gamma \in G$ and $\mathbf{y}^{\mathbf{u}}$ with $\deg(\mathbf{y}^{\mathbf{u}}) = \gamma$.

Now suppose that $f, g \in J_\psi$ with $[f]_{\mathbf{y}^{\mathbf{u}}} = [g]_{\mathbf{y}^{\mathbf{u}}} < \infty$. Since J_ψ is homogeneous with respect to the G -grading, to prove that it is a tropical ideal it is enough to show the elimination axiom holds for homogeneous polynomials f and g of the same degree $\gamma := \deg(\mathbf{y}^{\mathbf{u}})$. By factoring out $\mathbf{y}^{\mathbf{u}}$, we may assume that $\mathbf{u} = \mathbf{0}$ and $\deg(f) = \deg(g) = \mathbf{0}$. This means that $f = \psi^*(f')$ and $g = \psi^*(g')$ for some $f', g' \in J$, with the coefficient of $\mathbf{x}^{\mathbf{0}}$ equal in f' and g' . Since J is a tropical ideal, there is $h' \in J$ satisfying $h' \leftarrow \text{elim}_{\mathbf{x}^{\mathbf{0}}}(f', g')$. Set $h = \psi^*(h')$. Then $h \leftarrow \text{elim}_{\mathbf{y}^{\mathbf{0}}}(f, g)$, since all monomials appearing in f, g , and h have degree $\mathbf{0}$. This shows that J_ψ is a tropical ideal.

Since J_ψ is homogeneous, for any collection of monomials E in $\overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ the matroid $\text{Mat}(J_\psi|_E)$ is the direct sum $\bigoplus_{\gamma \in G} \text{Mat}(J_\psi|_{E_\gamma})$, where E_γ is the collection of monomials in E of degree γ . In addition, if E is a collection of monomials of degree γ , so $E = \mathbf{y}^{\mathbf{v}}\psi^*(E')$ for a collection of monomials E' in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, then $\text{Mat}(J_\psi|_E)$ is isomorphic to $\text{Mat}(J|_{E'})$. Lemma 5.2 then implies that the largest set of monomials E not supporting a polynomial in J_ψ has size $|G| \deg(J) = |\det(A)| \deg(J)$. The proof is verbatim the same in the case that $\overline{\mathbb{R}}$ is replaced by \mathbb{B} .

- (2) Let $\psi^*: \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be an automorphism, given by $\psi^*(x_i) = \lambda_i \mathbf{x}^{\mathbf{a}_i}$ for $1 \leq i \leq n$. Set $I' = \psi^{*-1}(I)$. The fact that $\deg(I') = \deg(I)$ follows from Lemma 5.2. By Lemma 4.5, if $\mathbf{w} \in V(I)$ then $\text{trop}(\psi)(\mathbf{w}) \in V(I')$. In addition, $\text{in}_{\text{trop}(\psi)(\mathbf{w})}(I') = \psi_A^{*-1}(\text{in}_{\mathbf{w}}(I))$. Lemma 5.2 then implies that the degree of $\text{in}_{\text{trop}(\psi)(\mathbf{w})}(I')$ and $\text{in}_{\mathbf{w}}(I)$ are the same, as the automorphism ψ_A^{*-1} does not change the maximal size of a collection of monomials not containing the support of a polynomial in either ideal. As these degrees are the multiplicities by definition, the result follows.
- (3) Set $I' = I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$, so that $J = (I')^h$. By definition, $\text{mult}_{V(J)}([0 : \mathbf{w}])$ is the degree of $J_{in} := (\text{in}_{(0, \mathbf{w})}(J) : (\prod_{i=0}^n x_i)^\infty)$. Since J_{in} is saturated with respect to x_0 , it is the homogenization with respect to x_0 of $(J_{in})|_{x_0=0} \subseteq \overline{\mathbb{R}}[x_1, \dots, x_n]$, so $\deg(J_{in}) = \deg(J_{in}|_{x_0=0})$ by the definition of degree. Note that $J_{in}|_{x_0=0} = (\text{in}_{(0, \mathbf{w})}(J)|_{x_0=0} : (\prod_{i=1}^n x_i)^\infty)$. Thus, by Part 2 of Lemma 2.2, $J_{in}|_{x_0=0} = (\text{in}_{\mathbf{w}}(I') : (\prod_{i=1}^n x_i)^\infty)$. Part 3 of Lemma 2.2 then implies that $(\text{in}_{\mathbf{w}}(I') : (\prod_{i=1}^n x_i)^\infty) = \text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_1, \dots, x_n]$. Thus $\deg(J_{in}) = \deg(\text{in}_{\mathbf{w}}(I)) = \text{mult}_{V(I)}(\mathbf{w})$ as required. \square

We now show that Definition 5.3 agrees with the multiplicity of a root of a univariate polynomial. Many different tropical polynomials give rise to the same function from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$, but given a polynomial f there is a minimum possible choice for the coefficients of a polynomial giving rise to the same function. We call the polynomial with these coefficients the *convexification* of f .

Definition 5.5. For a polynomial $f \in \overline{\mathbb{R}}[x^{\pm 1}]$ or $\overline{\mathbb{R}}[x]$, we can factor the convexification of f as $\alpha \circ \bigodot_{i=1}^s (x \oplus w_i)^{m_i}$, with $\alpha \in \mathbb{R}$, $w_i \in \overline{\mathbb{R}}$, and $m_i \in \mathbb{Z}_{>0}$; see, for example [GM07]. The w_i occurring in the product are the roots of f , and the multiplicity $\text{mult}_{w_i}(f)$ of f at w_i equals m_i . For $w \in \overline{\mathbb{R}}$ with $w \neq w_i$ for all i , we set $\text{mult}_w(f) = 0$.

Example 5.6. The convexification of $f = x^3 \oplus 1 \circ x^2 \oplus x \oplus 1$ is equal to $x^3 \oplus x^2 \oplus x \oplus 1 = (x \oplus 0)^2 \circ (x \oplus 1)$. The multiplicity of f at 0 thus equals 2. \diamond

The following proposition shows that, while tropical ideals in one variable might not be finitely generated, they still behave like principal ideals.

Proposition 5.7. *Let I be a tropical ideal in $\overline{\mathbb{R}}[x^{\pm 1}]$. There exists $h \in I$ such that $V(I) = V(h)$, and the multiplicity of $V(I)$ at any point $w \in \mathbb{R}$ equals $\text{mult}_w(h)$. In addition, for every $f \in I$ the convexification of h divides the convexification of f .*

Proof. Fix $h \in I \cap \overline{\mathbb{R}}[x]$ with $\deg(h)$ minimal. We first prove that for any $f \in I$ and $w \in \mathbb{R}$, the multiplicity of $V(f)$ at w is at least the multiplicity of $V(h)$ at w . Denote by $r_1 < r_2 < \dots < r_j$ the points of $V(h)$, which we call the roots of h . The claim is trivially true for $w < r_1$, and indeed for any w not equal to one of the roots r_i . Suppose now that the claim is true for all $w < r_l$ for some root r_l . After replacing x by $r_l \circ x$ we may assume that $r_l = 0$. If the claim is false for r_l , then there is $g \in I \cap \overline{\mathbb{R}}[x]$ with $g \neq \infty$ and $\text{mult}_0(g) < \text{mult}_0(h)$. Choose such a g of minimal degree. We have $\deg(g) > \deg(h)$, as otherwise we could eliminate $x^{\deg(h)}$ from h and a suitable multiple of g to get a polynomial in I of lower degree than $\deg(h)$. Write $h = \bigoplus a_i x^i$, and $g = \bigoplus b_i x^i$. After scaling we may assume that $h(0) = g(0) = 0$. This means that $a_i, b_i \geq 0$. Let $j = \max\{i : a_i = 0\}$, and $k = \max\{i : b_i = 0\}$. We have $\deg(h) = j + \sum_{i=1}^{l-1} \text{mult}_{r_i}(h)$. The leading coefficient of h is $\text{lc}(h) = -\sum_{i=1}^{l-1} r_i \text{mult}_{r_i}(h)$, where if $l = 1$ we have the empty sum, so the coefficient is 0. An analogous statement holds for g . Our assumption implies that $\text{mult}_{r_i}(g) \geq \text{mult}_{r_i}(h)$ for all $i < l$. This means that $\deg(g) - k \geq \deg(h) - j$, and $\text{lc}(g) \geq \text{lc}(h)$.

Set $h' = (\text{lc}(g) - \text{lc}(h)) \circ x^{\deg(g) - \deg(h)} \circ h$, and let $f = \bigoplus c_i x^i \in I$ be an elimination $f \leftarrow \text{elim}_{x^{\deg(g)}}(g, h')$. We have $\deg(f) < \deg(g)$. If $\text{lc}(g) > \text{lc}(h)$, then $c_i = 0$ if and only if $b_i = 0$, so $\text{mult}_0(f) = \text{mult}_0(g) < \text{mult}_0(h)$, contradicting our choice of g to have minimal degree. If $\text{lc}(g) = \text{lc}(h)$, then we also have $\deg(g) - k = \deg(h) - j$. Note that $c_{k - \text{mult}_0(h)} = 0$, as $0 = a_{j - \text{mult}_0(h)} < b_{k - \text{mult}_0(h)}$, and $c_i > 0$ for $i < k - \text{mult}_0(h)$. Let $l = \max\{i : c_i = 0\}$. By the argument above applied to f we have $\deg(f) - l \geq \deg(h) - j = \deg(g) - k$, so since $\deg(f) < \deg(g)$, we must have $l < k$. But this means that $\text{mult}_0(f) < \text{mult}_0(h)$, contradicting our choice of g to have minimal degree. From these contradictions we conclude that g does not exist, so the claim is true for $w = r_l$, and thus for all w . This shows that the convexification of h divides the convexification of every other polynomial in I .

The multiplicity of $V(I)$ at w is the smallest degree of a polynomial in $\text{in}_w(I) \cap \mathbb{B}[x]$. For a polynomial $f \in I$ with $\text{mult}_w(f) = m$, the convexification of $\text{in}_w(f)$ has the form $x^b(x \oplus 0)^m$, for some $b \leq \deg(f) - m$, so $(x \oplus 0)^m \in \text{in}_w(I)$. Thus the minimal degree polynomial in $\text{in}_w(I) \cap \mathbb{B}[x]$ has degree $\min\{\text{mult}_w(f) : f \in I\}$. The above argument shows that this equals $\text{mult}_w(h)$. \square

The following lemma is a key technical tool in the proof of Theorem 5.10.

Lemma 5.8. *Let I be a zero-dimensional homogeneous tropical ideal in $\overline{\mathbb{R}}[x_0, \dots, x_n]$. Then for all $0 \leq i < j \leq n$ there is a nontrivial polynomial $f_{ij} \in I \cap \overline{\mathbb{R}}[x_i, x_j]$ that factors as a*

product of linear factors. If I is saturated with respect to the product of the variables, then f_{ij} can be chosen to have degree at most $2 \deg(I)$.

Proof. We can assume without loss of generality that $i = 0$ and $j = 1$. Fix $d \gg 0$, so that the Hilbert function of I in degree d equals $\deg(I)$. The set of monomials $E = \{x_0^{d-i}x_1^i : 0 \leq i \leq \deg(I)\}$ cannot be an independent set of $\underline{\text{Mat}}(I_d)$, so there is a polynomial in I with support in E , and thus a polynomial $f \in I \cap \overline{\mathbb{R}}[x_0, x_1]$. If I is saturated with respect to the product of the variables, then we may divide by a power of x_0 to assume that $\deg(f) \leq \deg(I)$. Let f' be the convexification of f . As described in [MR18, Example 4.13], we have $g := (f')^2 = f' \circ f \in I \cap \overline{\mathbb{R}}[x_0, x_1]$. Since this polynomial is equal to its convexification, it factors into linear polynomials, so is the required polynomial. \square

Remark 5.9. If $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ is a homogeneous zero-dimensional tropical ideal saturated with respect to the product of the variables, then $V(I) \subseteq \text{trop}(\mathbb{P}^n)$ is contained in the tropical torus $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$. Indeed, suppose that $[\mathbf{w}] \in V(I)$, where we may assume, after relabelling the coordinates if necessary, that $w_0 = 0$. For all $j > 0$, the polynomial f_{0j} of Lemma 5.8 can be chosen to not be divisible by x_j , which implies that $w_j < \infty$, and thus $\mathbf{w} \in \mathbb{R}^{n+1}$.

The main result of this section is the following theorem, which is the equivalent for tropical ideals of [MS15, Proposition 3.4.13].

Theorem 5.10. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a zero-dimensional tropical ideal. Then*

$$\deg(I) = \sum_{\mathbf{w} \in V(I)} \text{mult}_{V(I)}(\mathbf{w}).$$

Note that by Theorem 4.3, the sum on the right hand side is finite. The overall approach of the proof of Theorem 5.10 is to prove that the sums of the multiplicities of points in certain polyhedral regions are the degrees of related ideals. The proof requires the following technical lemma, which will be used to gradually increase the size of the regions.

Lemma 5.11. *Let $I \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ be a zero-dimensional homogeneous tropical ideal that is saturated with respect to the product of the variables. Fix $\mathbf{w} \in \mathbb{R}^{n+1}$, $\sigma \subseteq \{1, \dots, n\}$ with $\sigma \neq \emptyset$, and $i \in \sigma$. Write $m_\sigma = \prod_{l \in \{0, \dots, n\} \setminus \sigma} x_l$.*

For ϵ sufficiently small we have

$$\deg(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty) = \deg(\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty) + \deg(\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(I) : m_\sigma^\infty)$$

and

$$\deg(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty) = \deg(\text{in}_{\mathbf{w} - \epsilon \mathbf{e}_i}(I) : m_\sigma^\infty).$$

Proof. Let \prec be the reverse lexicographic term order on the monomials in $\overline{\mathbb{R}}[x_0, \dots, x_n]$ with x_i largest (so initial terms of homogeneous polynomials are divisible by the lowest power of x_i) and x_0 second-largest, and let \prec' be the lexicographic term order with x_i smallest (so initial terms of all polynomials are divisible by the largest power of x_i) and x_0 largest. For $0 < \epsilon \ll 1$ we define the monomial ideals

- (1) $J^+ = \text{in}_{\prec}(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty)$,
- (2) $J_\epsilon^+ = \text{in}_{\prec}(\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(I) : m_\sigma^\infty)$,
- (3) $J_{\mathbf{w}} = \text{in}_{\prec}(\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)$,
- (4) $J^- = \text{in}_{\prec'}(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty)$,

$$(5) \quad J_\epsilon^- = \text{in}_{\prec'}(\text{in}_{\mathbf{w}-\epsilon\mathbf{e}_i}(I) : m_\sigma^\infty).$$

Since the Hilbert function, and thus the degree, is preserved on passing to a monomial initial ideal by Lemma 2.6, it suffices to prove that for ϵ sufficiently small we have

$$\deg(J^+) = \deg(J_{\mathbf{w}}) + \deg(J_\epsilon^+) \quad \text{and} \quad \deg(J^-) = \deg(J_\epsilon^-).$$

We assume that ϵ has been chosen sufficiently small so that $\text{in}_{\mathbf{w}+\epsilon\mathbf{e}_i}(I) = \text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(I))$ and $\text{in}_{\mathbf{w}-\epsilon\mathbf{e}_i}(I) = \text{in}_{-\mathbf{e}_i}(\text{in}_{\mathbf{w}}(I))$, and furthermore $\text{in}_{\mathbf{w}\pm\epsilon\mathbf{e}_i}(f) = \text{in}_{\pm\mathbf{e}_i}(\text{in}_{\mathbf{w}}(f))$ for all homogeneous $f \in I$ of degree at most $2\deg(I)$. This is possible by Proposition 2.10.

Step 1: $J^+ \subseteq J_{\mathbf{w}} \cap J_\epsilon^+$, and $J^- \subseteq J_\epsilon^-$. Since $(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty)$ is contained in $(\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty) = ((\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty) : x_i^\infty)$, it follows that $J^+ \subseteq J_{\mathbf{w}}$. Now, we have $\text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty) \subseteq (\text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(I)) : m_\sigma^\infty)$. Since $\text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}+\epsilon\mathbf{e}_i}(I)$ and the term order \prec is a refinement of the partial order given by \mathbf{e}_i , we have $J^+ \subseteq J_\epsilon^+$. The same is true with ϵ replaced by $-\epsilon$, and \prec replaced by \prec' , so $J^- \subseteq J_\epsilon^-$.

Step 2: $(J^+ : x_i^\infty) = J_{\mathbf{w}}$. The containment $\text{in}_{\prec}(J : x_i^\infty) \subseteq (\text{in}_{\prec}(J) : x_i^\infty)$ holds for any ideal J . Taking $J = (\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty)$, we obtain $J_{\mathbf{w}} \subseteq (J^+ : x_i^\infty)$.

Since \prec is the reverse-lexicographic order and $(\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)$ is homogeneous and saturated with respect to x_i , we claim that $J_{\mathbf{w}}$ is also saturated with respect to x_i . To see this, suppose $\mathbf{x}^{\mathbf{v}} = \text{in}_{\prec}(f)$ is a minimal generator of $J_{\mathbf{w}}$ with $f \in (\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)$. If $v_i > 0$ then $u_i > 0$ for all other monomials $\mathbf{x}^{\mathbf{u}}$ in f , by the definition of the reverse-lexicographic order, so $f/x_i \in (\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)$, and thus $\mathbf{x}^{\mathbf{v}}/x_i \in J_{\mathbf{w}}$, contradicting that $\mathbf{x}^{\mathbf{v}}$ was a minimal generator. We thus conclude that no generator of $J_{\mathbf{w}}$ is divisible by x_i , so the claim follows. This means that $J_{\mathbf{w}} \subseteq (J^+ : x_i^\infty) \subseteq (J_{\mathbf{w}} : x_i^\infty) = J_{\mathbf{w}}$, so $(J^+ : x_i^\infty) = J_{\mathbf{w}}$.

Step 3: $J_\epsilon^+ \subseteq (J^+ : x_0^\infty)$, and $J_\epsilon^- \subseteq (J^- : x_0^\infty)$. The proof is the same for both cases, so we give it for $J_\epsilon^+ \subseteq (J^+ : x_0^\infty)$. The only changes needed are to replace each ϵ by $-\epsilon$, and \prec by \prec' .

If $\mathbf{x}^{\mathbf{u}} \in J_\epsilon^+$, then there is $f \in I$ of the form $f = m_\sigma^k(\mathbf{x}^{\mathbf{u}} \oplus f') \oplus f''$, where all terms $c_{\mathbf{v}}\mathbf{x}^{\mathbf{v}}$ in f' have $c_{\mathbf{v}} + \mathbf{w} \cdot \mathbf{v} + \epsilon v_i = \mathbf{w} \cdot \mathbf{u} + \epsilon u_i$ but $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$, and all terms in f'' have greater weight with respect to $\mathbf{w} + \epsilon\mathbf{e}_i$ than $m_\sigma^k\mathbf{x}^{\mathbf{u}}$. This means that $\text{in}_{\mathbf{w}+\epsilon\mathbf{e}_i}(f) = m_\sigma^k(\mathbf{x}^{\mathbf{u}} \oplus \bigoplus_{c_{\mathbf{v}}\mathbf{x}^{\mathbf{v}} \text{ a term of } f'} \mathbf{x}^{\mathbf{v}})$. We can write $f'' = f_1'' \oplus f_2''$, where the terms $\mathbf{x}^{\mathbf{v}'}$ in f_1'' have $c_{\mathbf{v}'} + \mathbf{w} \cdot \mathbf{v}' = \mathbf{w} \cdot \mathbf{u} + k(\sum_{i \notin \sigma} w_i)$, and the equality is an inequality $>$ for the terms in f_2'' . Then $\text{in}_{\mathbf{w}}(f) = \text{in}_{\mathbf{w}+\epsilon\mathbf{e}_i}(f) \oplus \bigoplus_{c_{\mathbf{v}}\mathbf{x}^{\mathbf{v}} \text{ a term of } f_1''} \mathbf{x}^{\mathbf{v}}$. We thus have $v_i > u_i$ for terms $c_{\mathbf{v}}\mathbf{x}^{\mathbf{v}}$ of f_1'' .

We now show that, after multiplying f by a sufficiently high power of x_0 and doing some vector eliminations, we may assume that each term of f_1'' is divisible by m_σ^k . By Lemma 5.8, since I is saturated, for all $j > 0$ there is a polynomial $f_j \in I \cap \overline{\mathbb{R}}[x_0, x_j]$ that factors as a product of linear terms. Thus $\text{in}_{\mathbf{w}}(f_j) = x_0^{a_j}(x_0 \oplus x_j)^{b_j}x_j^{c_j} \in \text{in}_{\mathbf{w}}(I)$ for some $a_j, b_j, c_j \geq 0$. Fix $j > 0$ with $j \notin \sigma$. We may assume that $b_j \neq 0$, as otherwise $x_0^{a_j}x_j^{c_j} \in \text{in}_{\mathbf{w}}(I)$, so $0 \in (\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty)$, and so $J^+ = \langle 0 \rangle$, from which the lemma is immediate. Since $\text{in}_{\mathbf{w}}(I)$ is a tropical ideal, we may use the term $x_0^{a_j+b_j}$ of $\text{in}_{\mathbf{w}}(f_j)$ to eliminate a term $x_0^l\mathbf{x}^{\mathbf{v}}$ of $x_0^l f_1''$ from $\text{in}_{\mathbf{w}}(x_0^l f)$, where $l \gg 0$. This replaces the term $x_0^l\mathbf{x}^{\mathbf{v}}$ with terms with the same exponent on x_i , but higher exponents on x_j . After iterating with all f_j , as j ranges over $\{1, \dots, n\} \setminus \sigma$, we may assume that all terms of $\text{in}_{\mathbf{w}}(x_0^l f)$ are divisible by m_σ^k . Note that these elimination steps did not change $\text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(x_0^l f))$, as they did not change the exponent of x_i in the affected terms, so did not change $\text{in}_{\prec}(\text{in}_{\mathbf{w}}(x_0^l f)) = x_0^l m_\sigma^k \mathbf{x}^{\mathbf{u}}$. We thus conclude that $x_0^l \mathbf{x}^{\mathbf{u}} \in J^+$, so $\mathbf{x}^{\mathbf{u}} \in (J^+ : x_0^\infty)$. This shows that $J_\epsilon^+ \subseteq (J^+ : x_0^\infty)$ as required.

Step 4: J^+ contains a power of every variable except x_0, x_i , and J^- contains a power of every variable except x_0 . From the previous paragraph we have $x_0^{a_j}(x_0 \oplus x_j)^{b_j}x_j^{c_j} \in \text{in}_{\mathbf{w}}(I)$ for all $j > 0$, so $x_j^{b_j+c_j} \in J^+$ for $j \neq i$, and $x_j^{b_j+c_j} \in J^-$ for all $j > 0$.

Step 5: $J_d^+ = (J_{\mathbf{w}})_d \cap (J_{\epsilon}^+)_d$ and $J_d^- = (J_{\epsilon}^-)_d$ for $d \gg 0$. Since all of J^+ , J^- , $J_{\mathbf{w}}$, J_{ϵ}^+ , and J_{ϵ}^- are monomial ideals, they have monomial primary decompositions, viewed as monomial ideals living in a polynomial ring over a field. Since J^+ contains a power of every variable except x_0, x_i and J^- contains a power of every variable except x_0 , and both have constant Hilbert polynomials, the only possible associated primes of J^+ are $P_i = \langle x_j : j \neq i \rangle$, $P_0 = \langle x_j : j \neq 0 \rangle$, and $\mathfrak{m} = \langle x_j : 0 \leq j \leq n \rangle$, and the only possible associated primes of J^- are P_0 and \mathfrak{m} . This means that for $d \gg 0$,

$$(5.1) \quad J_d^+ = (J^+ : x_0^\infty)_d \cap (J^+ : x_i^\infty)_d,$$

and

$$(5.2) \quad J_d^- = (J^- : x_0^\infty)_d.$$

Since $J^- \subseteq J_{\epsilon}^- \subseteq (J^- : x_0^\infty)$, we thus have $J_d^- = (J_{\epsilon}^-)_d$ for $d \gg 0$, and so $\deg(J^-) = \deg(J_{\epsilon}^-)$.

Since $J^+ \subseteq J_{\mathbf{w}} \cap J_{\epsilon}^+$, we have $J_d^+ \subseteq (J_{\mathbf{w}})_d \cap (J_{\epsilon}^+)_d \subseteq (J^+ : x_i^\infty)_d \cap (J^+ : x_0^\infty)_d = J_d^+$ for $d \gg 0$. This means that $\deg(J^+) = \deg(J_{\mathbf{w}} \cap J_{\epsilon}^+)$.

Step 6: *Equality of degrees.* Since $\text{in}_{\mathbf{w}}(f_i) = x_0^{a_i}(x_0 \oplus x_i)^{b_i}x_i^{c_i} \in \text{in}_{\mathbf{w}}(I)$, we have $(x_0 \oplus x_i)^{b_i} \in (\text{in}_{\mathbf{w}}(I) : (m_{\sigma}x_i)^\infty)$, and so $x_0^{b_i} \in J_{\mathbf{w}}$. On the other hand, since $\deg(f_i) \leq 2 \deg(I)$ by Lemma 5.8, $\text{in}_{\mathbf{w}+\epsilon \mathbf{e}_i}(f_i) = \text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(f_i)) = x_0^{a_i+b_i}x_i^{c_i}$, so $x_i^{c_i} \in J_{\epsilon}^+$. This means that the monomials not in J_{ϵ}^+ and not in $J_{\mathbf{w}}$ are disjoint in high degree, so $\deg(J^+) = \deg(J_{\mathbf{w}} \cap J_{\epsilon}^+) = \deg(J_{\mathbf{w}}) + \deg(J_{\epsilon}^+)$. \square

We are now ready to prove Theorem 5.10.

Proof of Theorem 5.10. We first consider the related situation that I is a zero-dimensional homogeneous tropical ideal that is saturated with respect to the product of the variables, and prove the following stronger result: For any $[\mathbf{w}]$ in the tropical torus $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ of $\text{trop}(\mathbb{P}^n)$,

$$(5.3) \quad \deg(\text{in}_{\mathbf{w}}(I) : x_0^\infty) = \sum_{p \in (\mathbf{w} + \text{cone}(\mathbf{e}_l : l > 0)) \cap V(I)} \text{mult}_{V(I)}(p).$$

To see that this implies the theorem, we claim that when \mathbf{w} is in the interior of an unbounded cell $C_{\mathbf{w}}$ of the Gröbner complex of I that contains points \mathbf{w}' with $w'_l \ll w'_0$ for all $l > 0$, then $(\text{in}_{\mathbf{w}}(I) : x_0^\infty) = \text{in}_{\mathbf{w}}(I)$, so the result follows from the facts that $\deg(\text{in}_{\mathbf{w}}(I)) = \deg(I)$ by [MR18, Corollary 3.6], and that for such \mathbf{w}' all of $V(I)$ is contained in $\mathbf{w}' + \text{cone}(\mathbf{e}_l : l > 0)$. To see the claim, note that $\text{in}_{\mathbf{w}}(I)$ is a monomial ideal, and suppose that $\mathbf{x}^{\mathbf{u}}x_0^l \in \text{in}_{\mathbf{w}}(I)$. There is $f = \bigoplus c_{\mathbf{v}}\mathbf{x}^{\mathbf{v}} \in I$ with $\text{in}_{\mathbf{w}}(f) = \mathbf{x}^{\mathbf{u}}x_0^l$ and all other monomials $\mathbf{x}^{\mathbf{v}}$ occurring in f not in $\text{in}_{\mathbf{w}}(I)$; such f corresponds to the fundamental circuit of $\mathbf{x}^{\mathbf{u}}x_0^l$ over the basis $B := \{\mathbf{x}^{\mathbf{v}} : \mathbf{x}^{\mathbf{v}} \notin \text{in}_{\mathbf{w}}(I)_{|\mathbf{u}|+l}\}$ of the matroid $\underline{\text{Mat}}(I_{|\mathbf{u}|+l})$. Since $\text{in}_{\mathbf{w}'}(I) = \text{in}_{\mathbf{w}}(I)$ for all $\mathbf{w}' \in C_{\mathbf{w}}$, we must have $\text{in}_{\mathbf{w}'}(f) = \mathbf{x}^{\mathbf{u}}x_0^l$ for such \mathbf{w}' . By choosing such a \mathbf{w}' with w'_j sufficiently less than w_0 for all $1 \leq j \leq n$, we see that the other terms of f must also be divisible by x_0^l , so since I is $(\prod x_j)$ -saturated, $f/x_0^l \in I$, and so $\mathbf{x}^{\mathbf{u}} \in \text{in}_{\mathbf{w}}(I)$. This shows that $(\text{in}_{\mathbf{w}}(I) : x_0^\infty) = \text{in}_{\mathbf{w}}(I)$ as required.

To prove Equation (5.3), we prove the following stronger formula. For $\sigma \subseteq \{1, \dots, n\}$, set $m_\sigma = \prod_{l \in \{0, \dots, n\} \setminus \sigma} x_l$. Then for any $\sigma \subseteq \{1, \dots, n\}$ and any $\mathbf{w} \in \mathbb{R}^{n+1}$,

$$(5.4) \quad \deg(\text{in}_{\mathbf{w}}(I) : m_\sigma^\infty) = \sum_{p \in (\mathbf{w} + \text{cone}(\mathbf{e}_i : i \in \sigma)) \cap V(I)} \text{mult}_{V(I)}(p).$$

The case that $\sigma = \{1, \dots, n\}$ is Equation (5.3). We prove Equation (5.4) by induction on $|\sigma|$. When $\sigma = \emptyset$, the right-hand side is $\text{mult}_{V(I)}(\mathbf{w})$, and the left-hand side is the definition of this.

Now suppose that $|\sigma| > 0$ and Equation (5.4) holds for all σ' with $|\sigma'| < |\sigma|$. Write \mathcal{C}_σ for the locus of $\mathbf{w} \in \mathbb{R}^{n+1}$ for which (5.4) holds. To prove the claim we need to show that $\mathcal{C}_\sigma = \mathbb{R}^{n+1}$. Suppose $\mathcal{C}_\sigma \neq \mathbb{R}^{n+1}$, and fix $\mathbf{w}_0 \in \mathbb{R}^{n+1} \setminus \mathcal{C}_\sigma$. Fix $i \in \sigma$, and consider the ray $\{\mathbf{w} : \mathbf{w} = \mathbf{w}_0 + \lambda \mathbf{e}_i \text{ with } \lambda \geq 0\}$. We observe that for λ large enough, $\mathbf{w}_0 + \lambda \mathbf{e}_i \in \mathcal{C}_\sigma$. Indeed, by Lemma 5.8 there is a polynomial $f \in I \cap \overline{\mathbb{R}}[x_0, x_i]$ that factors as a product of linear terms, and is saturated with respect to x_0 and x_i . Thus for $\lambda \gg 0$, we have $\text{in}_{\mathbf{w}_0 + \lambda \mathbf{e}_i}(f) = x_0^m$ for some $m > 0$. This means that $(\text{in}_{\mathbf{w}_0 + \lambda \mathbf{e}_i}(I) : m_\sigma^\infty) = \langle 0 \rangle$, so since $\mathbf{w}_0 + \lambda \mathbf{e}_i + \text{cone}(\mathbf{e}_i : i \in \sigma) \cap V(I) = \emptyset$ for sufficiently large λ , (5.4) holds for such a point $\mathbf{w}_0 + \lambda \mathbf{e}_i$.

We thus have that the set of λ such that $\mathbf{w}_0 + \lambda \mathbf{e}_i \notin \mathcal{C}_\sigma$ is bounded above; let $\tilde{\lambda}$ be its supremum, and set $\mathbf{w} = \mathbf{w}_0 + \tilde{\lambda} \mathbf{e}_i$. By Lemma 5.11 the left-hand side of (5.4) does not change when subtracting $\epsilon \mathbf{e}_i$ from \mathbf{w} for sufficiently small $\epsilon > 0$, which implies that $\mathbf{w} \notin \mathcal{C}_\sigma$. Now, since $V(I)$ is finite and $\text{in}_{\mathbf{w}}(I)_{\leq \deg(I)}$ is generated by finitely many polynomials, we can fix $\epsilon > 0$ small enough so that:

- (1) $\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(I) = \text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(I))$, and $\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(f) = \text{in}_{\mathbf{e}_i}(\text{in}_{\mathbf{w}}(f))$ for all $f \in I$ of degree at most $2 \deg(I)$, and
- (2) there are no points $p \in V(I)$ with $p \in \mathbf{w} + \text{cone}(\mathbf{e}_j : j \in \sigma)$ and $w_i < p_i < w_i + \epsilon$.

We now consider $(\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)$ and $(\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(I) : m_\sigma^\infty)$. The degree of $(\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)$ is the left-hand side of (5.4) for the set $\sigma' = \sigma \setminus \{i\}$. By the induction hypothesis we know that this equals $\sum_{p \in (\mathbf{w} + \text{cone}(\mathbf{e}_j : j \in \sigma')) \cap V(I)} \text{mult}_{V(I)}(p)$. Since $\mathbf{w} + \epsilon \mathbf{e}_i \in \mathcal{C}_\sigma$, the degree of $(\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(I) : m_\sigma^\infty)$ equals $\sum_{p \in (\mathbf{w} + \epsilon \mathbf{e}_i + \text{cone}(\mathbf{e}_j : j \in \sigma)) \cap V(I)} \text{mult}_{V(I)}(p)$. By our assumptions on ϵ , we thus have that

$$\deg((\text{in}_{\mathbf{w}}(I) : (m_\sigma x_i)^\infty)) + \deg((\text{in}_{\mathbf{w} + \epsilon \mathbf{e}_i}(I) : m_\sigma^\infty)) = \sum_{p \in (\mathbf{w} + \text{cone}(\mathbf{e}_i : i \in \sigma)) \cap V(I)} \text{mult}_{V(I)}(p),$$

which is the right-hand side of (5.4). Equation (5.4) then follows from Lemma 5.11.

We now consider the case that I is a zero-dimensional ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $I^h \subseteq \overline{\mathbb{R}}[x_0, \dots, x_n]$ be the homogenization of $I \cap \overline{\mathbb{R}}[x_1, \dots, x_n]$. By definition we have $\deg(I^h) = \deg(I)$, and by Remark 5.9 $V(I)$ equals $V(I^h)$ after the identification of $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ with \mathbb{R}^n by choosing the representative with first coordinate 0. In addition, by Part 3 of Proposition 5.4 we have $\text{mult}_{V(I)}(\mathbf{w}) = \text{mult}_{V(I^h)}([0 : \mathbf{w}])$ for all $\mathbf{w} \in V(I)$. The theorem for I then follows from the result for homogeneous ideals proved above. \square

Remark 5.12. The proof of Theorem 5.10 is simpler in the realizable case, as we can use primary decomposition. If $I \subseteq K[x_0, \dots, x_n]$ is zero-dimensional, then $I = J \cap \bigcap_{p \in V(I)} Q_p$, where Q_p is $I(p)$ -primary, and J is $\langle x_0, \dots, x_n \rangle$ -primary. We then have $\deg(I) = \sum_{p \in V(I)} \deg(Q_p)$. The degree of an ideal Q_p primary to the ideal of a point p equals its multiplicity, so the result follows.

6. BALANCING

In this section we prove that the top-dimensional part of the variety of a tropical ideal is *balanced* with respect to intrinsically defined multiplicities.

We first recall the definition of the balancing condition. A *weighted* polyhedral complex is a polyhedral complex Σ with a weight function that assigns a positive integer to each maximal cell of the complex.

Definition 6.1. Let Σ be a pure one-dimensional weighted rational polyhedral fan with s rays. Let \mathbf{u}_i be the first lattice point on the i th ray of Σ , and let m_i be the weight on that ray. We say that Σ is *balanced* if $\sum_{i=1}^s m_i \mathbf{u}_i = \mathbf{0}$.

Let Σ be a pure d -dimensional \mathbb{R} -rational weighted polyhedral complex, and let σ be a $(d - 1)$ -dimensional cell of Σ . The quotient of the star $\text{star}_\Sigma(\sigma)$ by the subspace $\text{span}(\sigma)$ is a pure one-dimensional rational polyhedral fan that inherits a weighting from Σ . We say that Σ is *balanced* if this fan is balanced for *all* $(d - 1)$ -dimensional cells of Σ .

We now define the multiplicities on the variety of a tropical ideal that give $V(I)$ the structure of a weighted polyhedral complex. This needs the following lemma.

Given a \mathbb{Z}^d -grading on the tropical Laurent polynomial semiring $S = \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ or $S = \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we denote by $S_{\mathbf{0}}$ the subsemiring consisting of degree zero elements. An ideal $I \subseteq S_{\mathbf{0}}$ is a tropical ideal if for any finite collection of monomials $E \subseteq S_{\mathbf{0}}$ the restriction $I|_E$ is the collection of vectors of a valuated matroid on the set E ; see [MR18, Definition 4.3]. Note that $S_{\mathbf{0}}$ is isomorphic to a Laurent polynomial semiring in fewer variables. We can use this fact to define the dimension and degree of I . Explicitly, we say that I is zero-dimensional if there is an upper bound on the rank of $\text{Mat}(I|_E)$ for $E \subseteq S_{\mathbf{0}}$ a finite collection of monomials, and in that case we set the degree of I to be the maximum possible such rank.

Lemma 6.2. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal, and fix $\mathbf{w} \in \mathbb{R}^n$ in the relative interior of a maximal cell σ of $V(I)$, where $V(I)$ is given the Gröbner polyhedral complex structure. By Corollary 2.11, $\text{in}_{\mathbf{w}}(I)$ is homogeneous with respect to a $\mathbb{Z}^{\dim(\sigma)}$ -grading of $S := \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $S_{\mathbf{0}}$ be the degree-zero part of S with respect to this grading. Then $\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}}$ is a zero-dimensional tropical ideal.*

Definition 6.3. Under the same setup as in Lemma 6.2, we define the *multiplicity* of $V(I)$ at \mathbf{w} to be the degree of the zero-dimensional tropical ideal $\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}}$.

Note that this definition agrees with the one given for zero-dimensional ideals in Definition 5.3, as the grading is trivial when $\dim(I) = 0$.

Proof of Lemma 6.2. By Proposition 2.10, we have $V(\text{in}_{\mathbf{w}}(I)) = \text{span}(\sigma)$, which is a subspace of \mathbb{R}^n of dimension $d := \dim(\sigma)$. Using Lemma 4.5 we may change coordinates to assume that $\text{span}(\sigma) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$. Explicitly, choose a basis for the lattice $\text{span}(\sigma) \cap \mathbb{Z}^n$, and let $A \in \text{GL}(n, \mathbb{Z})$ be the inverse of an invertible $n \times n$ matrix with first d rows equal to this basis. Set $\psi^*: \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ to be $\psi^*(x_i) = \mathbf{x}^{\mathbf{a}_i}$, where \mathbf{a}_i is the i th column of A . By construction we then have $\text{trop}(\psi)(\text{span}(\sigma)) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$. The ideal $I' = \psi^{*-1}(I)$ satisfies $V(\text{in}_{\text{trop}(\psi)(\mathbf{w})}(I')) = V(\psi^{*-1}(\text{in}_{\mathbf{w}}(I))) = \text{trop}(\psi)(V(\text{in}_{\mathbf{w}}(I))) = \text{trop}(\psi)(\text{span}(\sigma)) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$. The \mathbb{Z}^d -grading induced by $\text{trop}(\psi)(\sigma)$ on S is given by $\deg(x_i) = \mathbf{e}_i$ for $1 \leq i \leq d$ and $\deg(x_i) = \mathbf{0}$ for $i > d$, so the degree $\mathbf{0}$ part of S is $\mathbb{B}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$. By Part 2 of Proposition 5.4, the tropical ideal $\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}}$ is zero-dimensional if and only if the

tropical ideal $\psi^{*-1}(\text{in}_{\mathbf{w}}(I)) \cap \mathbb{B}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$ is zero-dimensional. We can thus assume that $\text{span}(\sigma) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$, and $S_0 = \mathbb{B}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$.

To prove that $\text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$ is zero-dimensional, let $J = \text{in}_{\mathbf{w}}(I) \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Denote by $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^{n-d}$ the projection onto the last $n-d$ coordinates. By Theorem 4.7 we have

$$V(\text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]) = V(J \cap \overline{\mathbb{R}}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]) = \pi(V(J)) = \pi(V(\text{in}_{\mathbf{w}}(I))).$$

As $V(\text{in}_{\mathbf{w}}(I)) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d)$, the last term in the equalities above is equal to $\{\mathbf{0}\}$. Theorem 4.3 then implies that $\text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_{d+1}^{\pm 1}, \dots, x_n^{\pm 1}]$ is zero-dimensional, as claimed. \square

Recall from §2 that if Σ is a polyhedral complex and \mathbf{w} is in the interior of a cell $\sigma \in \Sigma$, the star $\text{star}_{\Sigma}(\mathbf{w})$ is a polyhedral fan with cones $\bar{\tau}$ for any $\tau \in \Sigma$ containing σ . If Σ is a weighted polyhedral complex, the star $\text{star}_{\Sigma}(\mathbf{w})$ inherits weights on its maximal cones, and thus it is a weighted polyhedral fan.

Proposition 6.4. *Suppose I is a tropical ideal in $\overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ or $\mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $\mathbf{w} \in \mathbb{R}^n$. Then*

$$V(\text{in}_{\mathbf{w}}(I)) = \text{star}_{V(I)}(\mathbf{w})$$

as weighted polyhedral complexes, where $V(I)$ and $V(\text{in}_{\mathbf{w}}(I))$ are given their Gröbner complex structures.

Proof. The equality of these fans as polyhedral complexes was proved in Proposition 2.10. To show that they have the same weighting, suppose \mathbf{v} is in the interior of a maximal cone $\bar{\tau}$ of $V(\text{in}_{\mathbf{w}}(I)) = \text{star}_{V(I)}(\mathbf{w})$. The linear subspace $\text{span}(\bar{\tau}) = \text{span}(\tau)$ induces a $\mathbb{Z}^{\dim(\tau)}$ -grading on $S := \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The multiplicity of the cone $\bar{\tau} \in V(\text{in}_{\mathbf{w}}(I))$ is equal to the degree of the zero-dimensional tropical ideal $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) \cap S_0 = \text{in}_{\mathbf{w}+\epsilon\mathbf{v}}(I) \cap S_0$, which is equal to the multiplicity of the cell τ in $V(I)$, as desired. \square

We now show the key special case of Theorem 6.6 when the ideal I is one-dimensional with coefficients in \mathbb{B} .

Lemma 6.5. *Let $I \subseteq \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a one-dimensional tropical ideal. Then $V(I)$ is the support of a one-dimensional rational polyhedral fan that is balanced, with weights on the rays given by the multiplicities of Definition 6.3.*

Proof. Since $I \subseteq \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we have $I = \text{in}_{\mathbf{0}}(I)$ and thus $V(I) = \text{star}_{V(I)}(\mathbf{0})$ by Proposition 2.10. This implies that the variety $V(I)$ is a fan, which is one-dimensional by Theorem 4.3. The same is true for the ideal $I' = I \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, as the circuits are the same for both ideals. For the rest of the proof we replace I by I' , so assume that $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Let $\mathbf{u}_1, \dots, \mathbf{u}_s$ be the first lattice points on the rays of $V(I)$. Set $m_i = \text{mult}_{V(I)}(\mathbf{u}_i)$. Let $\mathbf{u} = \sum m_i \mathbf{u}_i$. We will show that $\mathbf{v} \cdot \mathbf{u} = 0$ for all $\mathbf{v} \in \mathbb{Z}^n$, which implies that $\mathbf{u} = \mathbf{0}$, so $V(I)$ is balanced at the origin. It suffices to show that $\mathbf{v} \cdot \mathbf{u} = 0$ for a basis of \mathbb{Z}^n , so up to reindexing we may assume that $\mathbf{v} = \mathbf{e}_1$.

Let $J_1 = I|_{x_1=1} \subseteq \overline{\mathbb{R}}[x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ and $J_2 = I|_{x_1=-1} \subseteq \overline{\mathbb{R}}[x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. By Proposition 3.12, since the intersections of $V(I)$ with the two hyperplanes $\{x_1 = 1\}$ and $\{x_1 = -1\}$ are transverse, these intersections equal $V(J_1)$ and $V(J_2)$ respectively. By Proposition 4.1, J_1 and J_2 are either the unit ideal $\langle 0 \rangle$ or are zero-dimensional. By Part 1 of Lemma 3.11, $J_1 = \langle 0 \rangle$

if and only if $J_2 = \langle 0 \rangle$. In that case we have $V(J_1) = V(J_2) = \emptyset$, so $V(I) \cap \{x_1 = \pm 1\} = \emptyset$. This means that $\mathbf{e}_1 \cdot \mathbf{u}_i = 0$ for all $1 \leq i \leq s$, and thus $\mathbf{e}_1 \cdot \mathbf{u} = 0$. We may thus restrict to the case that both J_1 and J_2 have dimension zero.

By Theorem 5.10 we know that

$$\deg(J_i) = \sum_{\mathbf{w} \in V(J_i)} \text{mult}_{V(J_i)}(\mathbf{w})$$

for $i = 1, 2$. By Part 2 of Lemma 3.11 we have $\text{in}_{\mathbf{w}}(J_1) = (\text{in}_{(1, \mathbf{w})}(I))|_{x_1=0}$ for all $\mathbf{w} \in \mathbb{R}^{n-1}$. If $(1, \mathbf{w})$ lies on a ray $\mathbb{R}_{\geq 0}\mathbf{u}'$, then $\text{in}_{(1, \mathbf{w})}(I) = \text{in}_{\mathbf{u}'}(I)$, as the initial ideals with respect to \mathbf{u}' and $\lambda\mathbf{u}'$ are the same for any $\lambda > 0$, since all circuits of I have coefficients in \mathbb{B} . Thus

$$\deg(J_1) = \sum_{i: (\mathbf{u}_i)_1 > 0} \deg(\text{in}_{\mathbf{u}_i}(I)|_{x_1=0}) \quad \text{and} \quad \deg(J_2) = \sum_{i: (\mathbf{u}_i)_1 < 0} \deg(\text{in}_{\mathbf{u}_i}(I)|_{x_1=0}).$$

Suppose now that \mathbf{u}' is the first lattice point on a ray of $V(I)$. We next show that

$$(6.1) \quad \deg(\text{in}_{\mathbf{u}'}(I)|_{x_1=0}) = |u'_1| \text{mult}_{V(I)}(\mathbf{u}').$$

Grade $S = \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by $\deg(x_i) = u'_i$. The multiplicity $\text{mult}_{V(I)}(\mathbf{u}')$ is equal to the degree of the zero-dimensional ideal $\text{in}_{\mathbf{u}'}(I) \cap S_0 \subseteq S_0$. We now show that $\deg(\text{in}_{\mathbf{u}'}(I)|_{x_1=0}) = |u'_1| \deg(\text{in}_{\mathbf{u}'}(I) \cap S_0)$. To see this, first note that $S_0 \cong \mathbb{B}[y_1^{\pm 1}, \dots, y_{n-1}^{\pm 1}]$. This isomorphism is given by fixing a basis $\mathbf{b}_1, \dots, \mathbf{b}_{n-1}$ for the kernel of the map $\mathbb{Z}^n \rightarrow \mathbb{Z}$ given by sending \mathbf{w} to $\mathbf{w} \cdot \mathbf{u}'$, and sending y_i to $\mathbf{x}^{\mathbf{b}_i}$. While this isomorphism depends on the choice of basis, different choices differ by an automorphism of $\mathbb{B}[y_1^{\pm 1}, \dots, y_{n-1}^{\pm 1}]$, so by Part 2 of Proposition 5.4, $\text{in}_{\mathbf{u}'}(I) \cap S_0$ can be considered as an ideal in $\mathbb{B}[y_1^{\pm 1}, \dots, y_{n-1}^{\pm 1}]$ for the purpose of computing the degree. The map ψ^* from $\mathbb{B}[y_1^{\pm 1}, \dots, y_{n-1}^{\pm 1}]$ to $\mathbb{B}[x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ then takes y_i to $\mathbf{x}^{\mathbf{b}'_i}$, where \mathbf{b}'_i is the projection of \mathbf{b}_i onto the last $n-1$ coordinates. Write B for the $(n-1) \times (n-1)$ matrix with columns \mathbf{b}'_i . Note that $|\det(B)| = |u'_1|$. One way to see this is to note that since the map $\mathbb{Z}^n / \mathbb{Z}(\mathbf{b}_1, \dots, \mathbf{b}_{n-1}) \rightarrow \mathbb{Z}$ given by $\mathbf{e}_i \mapsto u'_i$ is an isomorphism, the induced map $\mathbb{Z}^{n-1} / \mathbb{Z}(\mathbf{b}'_1, \dots, \mathbf{b}'_{n-1}) \rightarrow \mathbb{Z} / \mathbb{Z}u'_1$ on the last $n-1$ coordinates given by $\mathbf{e}_i \mapsto u'_i$ is an isomorphism as well, so $|\det(B)| = |u'_1|$. Write $\text{in}_{\mathbf{u}'}(I)_\psi$ for the ideal in $\mathbb{B}[x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by $\psi^*(\text{in}_{\mathbf{u}'}(I) \cap S_0)$. By Part 1 of Proposition 5.4, the degree of $\text{in}_{\mathbf{u}'}(I)_\psi$ equals $|\det(B)| \deg(\text{in}_{\mathbf{u}'}(I) \cap S_0)$. Finally, note that $\text{in}_{\mathbf{u}'}(I)_\psi$ is generated by $\{f|_{x_1=0} : f \text{ is a generator of } \text{in}_{\mathbf{u}'}(I) \cap S_0\}$. If f is a homogeneous element of $\text{in}_{\mathbf{u}'}(I)$, then $\mathbf{x}^{\mathbf{v}} f \in \text{in}_{\mathbf{u}'}(I) \cap S_0$, where $\mathbf{x}^{\mathbf{v}}$ is any monomial of degree $-\deg(f)$, so $(\mathbf{x}^{\mathbf{v}} f)|_{x_1=0}$, and thus also $f|_{x_1=0}$, are in $\text{in}_{\mathbf{u}'}(I)_\psi$. Thus $\text{in}_{\mathbf{u}'}(I)_\psi = \text{in}_{\mathbf{u}'}(I)|_{x_1=0}$, so $\deg(\text{in}_{\mathbf{u}'}(I)|_{x_1=0}) = |u'_1| \deg(\text{in}_{\mathbf{u}'}(I) \cap S_0)$ as required. This finishes the proof of (6.1).

Thus $\deg(J_1)$ is a weighted sum of the positive first components of the vectors \mathbf{u}_i :

$$(6.2) \quad \deg(J_1) = \sum_{i: (\mathbf{u}_i)_1 > 0} |(\mathbf{u}_i)_1| \text{mult}_{V(I)}(\mathbf{u}_i),$$

and analogously for J_2 . We conclude that

$$\mathbf{u} \cdot \mathbf{e}_1 = \left(\sum_{i=1}^s m_i \mathbf{u}_i \right) \cdot \mathbf{e}_1 = \sum_{i=1}^s m_i (\mathbf{u}_i)_1 = \deg(J_1) - \deg(J_2) = 0,$$

where the last equality follows from Lemma 3.11, as $\deg(J_1) = \deg(\varphi(J_1)) = \deg(\varphi(J_2)) = \deg(J_2)$. \square

Theorem 6.6. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a d -dimensional tropical ideal, and let Σ be a polyhedral complex with support equal to $V(I)$. Then the collection of closed d -dimensional*

cells of Σ forms an \mathbb{R} -rational polyhedral complex Σ^d that is balanced, with weights given by the multiplicities of Definition 6.3.

Proof. Fix a $(d - 1)$ -dimensional cell σ of Σ^d , and \mathbf{w} in the relative interior of σ . By Proposition 6.4 the variety $V(\text{in}_{\mathbf{w}}(I))$ equals $\text{star}_{V(I)}(\mathbf{w})$ as a weighted rational polyhedral fan. Write $L = \text{span}(\sigma)$ for the $(d - 1)$ -dimensional lineality space of this fan. After a monomial change of coordinates, which by Part 2 of Proposition 5.4 does not change degrees or multiplicities of zero-dimensional ideals, we may assume that L is the span of $\mathbf{e}_1, \dots, \mathbf{e}_{d-1}$. Let $J = \text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_d^{\pm 1}, \dots, x_n^{\pm 1}]$. By Theorem 4.7 we have $V(J) = V(\text{in}_{\mathbf{w}}(I))/L$, so $V(J)$ is a one-dimensional rational polyhedral fan in \mathbb{R}^{n-d+1} . By Theorem 4.3 we have that $\dim(J) = 1$ as well, so $V(J)$ is balanced by Lemma 6.5.

Each ray in $V(J)$ is the quotient of a cone in $\text{star}_{V(I)}(\mathbf{w})$ by the lineality space L . To show that Σ^d is balanced at σ it suffices to show that the multiplicity on a ray in $V(J)$ equals the weight of the corresponding cone in $\text{star}_{V(I)}(\mathbf{w})$. After a monomial change of coordinates we may assume that the ray is $\mathbb{R}_{\geq 0}\mathbf{e}_n$, so the corresponding cone τ is $\text{span}(\mathbf{e}_1, \dots, \mathbf{e}_d) + \mathbb{R}_{\geq 0}\mathbf{e}_n$. The multiplicity of $\mathbb{R}_{\geq 0}\mathbf{e}_n$ in $V(J)$ is the degree of the zero-dimensional ideal obtained by intersecting $\text{in}_{\mathbf{e}_n}(J)$ with $\mathbb{B}[x_d^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. The multiplicity of τ in $\text{star}_{V(I)}(\sigma) = V(\text{in}_{\mathbf{w}}(I))$ is the degree of $\text{in}_{\mathbf{e}_n}(\text{in}_{\mathbf{w}}(I)) \cap \mathbb{B}[x_d^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. The equality then follows from the fact that, since $\text{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the grading by $\deg(x_i) = \mathbf{e}_i \in \mathbb{Z}^{d-1}$ for $1 \leq i \leq d-1$ and $\deg(x_i) = \mathbf{0} \in \mathbb{Z}^{d-1}$ otherwise, we have $\text{in}_{\mathbf{e}_n}(J) = \text{in}_{\mathbf{e}_n}(\text{in}_{\mathbf{w}}(I)) \cap \mathbb{B}[x_d^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. \square

Remark 6.7. In [MR18] tropical ideals were used to define subschemes of arbitrary toric varieties. One consequence of Theorem 6.6 is the existence of a *Hilbert-Chow morphism* that takes a subscheme of an n -dimensional tropical toric variety $\text{trop}(X_{\Sigma})$ given by a locally tropical ideal (see [MR18, §4]) to a class in $A^*(X_{\Sigma})$. More specifically, if I is a locally tropical ideal in $S := \text{Cox}(X_{\Sigma})$, and m is the product of the variables of S , then $I' = (IS_m)_{\mathbf{0}} \subseteq (S_m)_{\mathbf{0}} \cong \overline{\mathbb{R}}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ is a tropical ideal. Set $d = \dim(I')$. By Theorem 6.6, the union Δ of the d -dimensional cones of $V(\varphi(I')) \subseteq \mathbb{R}^n$ forms a finite \mathbb{R} -rational balanced polyhedral complex that is pure of dimension d . Then the techniques of [FS97] and [MS15, §6.7] associate a class $[V(I)] \in A_d(X_{\Sigma})$ to Δ .

For an irreducible variety $X \subseteq (K^*)^n$ and fixed $a \in \mathbb{R}$, the variety $\text{trop}(X \cap V(x_n - a))$ is the *stable intersection* of $\text{trop}(X)$ and the hyperplane $\text{trop}(V(x_n - a)) = \{x_n = a\}$ for most choices of $\alpha \in K$ with $\text{val}(\alpha) = a$. Here “most” means that given a fixed $t \in K$ with $\text{val}(t) = a$, there is a finite set in the residue field \mathbb{k} of K for which if the residue $\overline{\alpha}/t \in \mathbb{k}$ is not in this set then the tropicalization has the given form; see [MS15, Proposition 3.6.15]. Remark 3.7 suggests a connection of this fact to the specialization construction, which we now explain.

Proposition 6.8. *Let $I \subseteq \overline{\mathbb{R}}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a tropical ideal, and fix $a \in \mathbb{R}$. If $V(I)$ is the support of a pure d -dimensional polyhedral complex, we have*

$$V(I|_{x_n=a}) = \pi(V(I) \cap_{st} \{x_n = a\})$$

as weighted polyhedral complexes, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is the projection onto the first $n - 1$ coordinates and \cap_{st} denotes the stable intersection.

Proof. Fix a pure \mathbb{R} -rational polyhedral complex Σ with support $V(I)$. By definition, the stable intersection of Σ with the hyperplane $H := \{x_n = a\}$ is the polyhedral complex $\Sigma \cap_{st} H$

that has a cell $\tau \cap H$ for all cells $\tau \in \Sigma$ with $\dim(\tau + H) = n$, or equivalently with τ not contained in H .

By Proposition 3.12 we know that $\pi(\Sigma \cap_{st} H) \subseteq V(I|_{x_n=a}) \subseteq \Sigma \cap H$. Conversely, if $\mathbf{w} \in \Sigma \cap H$ but $\mathbf{w} \notin \Sigma \cap_{st} H$, then any $\tau \in \Sigma$ containing \mathbf{w} lies in H , so $\text{star}_{V(I)}(\mathbf{w})$ is contained in the hyperplane $\{x_n = 0\}$. By Proposition 2.10 this means that $V(\text{in}_{\mathbf{w}}(I))$ is contained in this hyperplane. Theorem 4.7 then implies that $\text{in}_{\mathbf{w}}(I) \cap \mathbb{B}[x_n^{\pm 1}] \neq \{\infty\}$. This means that $\text{in}_{\mathbf{w}}(I)|_{x_n=0}$, which equals $\text{in}_{\pi(\mathbf{w})}(I|_{x_n=a})$ by Part 2 of Lemma 3.11, equals $\langle 0 \rangle$, and so $\pi(\mathbf{w}) \notin V(I|_{x_n=a})$ as required. This shows that $V(I|_{x_n=a}) = \pi(\Sigma \cap_{st} H)$ as a set.

We now prove that the multiplicities also coincide: for $\mathbf{w} \in \mathbb{R}^n$ with $w_n = a$, we have $\text{mult}_{V(I|_{x_n=a})}(\pi(\mathbf{w})) = \text{mult}_{\Sigma \cap_{st} H}(\mathbf{w})$. We may assume that the polyhedral complex Σ has been chosen so that $\Sigma \cap H$ is a subcomplex of Σ , so every cell $\sigma \in \Sigma \cap_{st} H$ can also be viewed as a cell in Σ . Maximal cells of the stable intersection $\Sigma \cap_{st} H$ have dimension $d - 1$. By definition, if \mathbf{w} is in the relative interior of a maximal cell σ of $\Sigma \cap_{st} H$, the multiplicity of \mathbf{w} is $\sum_{\tau} \text{mult}_{\Sigma}(\tau)[N : N_{\tau} + N_H]$, where N_{τ} and N_H are the sublattices of $N = \mathbb{Z}^n$ generated by the lattice points in $\text{span}(\tau)$ and H respectively, $[N : N_{\tau} + N_H]$ is the lattice index, and the sum is over all maximal cells τ in Σ containing σ with $\tau \cap (\epsilon \mathbf{v} + H) \neq \emptyset$ for fixed generic \mathbf{v} and $0 < \epsilon \ll 1$. Since H is a coordinate hyperplane, we may take $\mathbf{v} = \mathbf{e}_n$. As both sides of the equality in the proposition statement are invariant under changes of coordinates in x_1, \dots, x_{n-1} that fix x_n , we may assume that $\text{span}(\sigma) = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_{d-1})$.

For τ a maximal cell in Σ containing σ , let $\mathbf{v}_{\tau} \in N$ be a representative for a generator of the one-dimensional lattice N_{τ}/N_{σ} . The sublattice $N_{\tau} + N_H$ is generated by $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ and \mathbf{v}_{τ} , so the index $[N : N_{\tau} + N_H]$ is up to sign equal to the last coordinate $(\mathbf{v}_{\tau})_n$. Thus the multiplicity of \mathbf{w} in $\Sigma \cap_{st} H$ equals

$$(6.3) \quad \sum_{\tau} (\mathbf{v}_{\tau})_n \text{mult}_{\Sigma}(\tau),$$

where the sum is over all maximal cells τ containing σ with $\tau \cap (\epsilon \mathbf{e}_n + H) \neq \emptyset$, which are precisely those τ with $(\mathbf{v}_{\tau})_n > 0$. The multiplicities $\text{mult}_{\Sigma}(\tau)$ are the same as the multiplicities of the corresponding cones in $\text{star}_{\Sigma}(\sigma) = V(\text{in}_{\mathbf{w}}(I))$. Grade $S = \mathbb{B}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by $\text{span}(\sigma)$. We have $S_{\mathbf{0}} = \mathbb{B}[x_d^{\pm 1}, \dots, x_n^{\pm 1}]$. The variety of the tropical ideal $\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}}$ is a one-dimensional fan in \mathbb{R}^{n-d+1} which is isomorphic to $\text{star}_{\Sigma}(\sigma)/\text{span}(\sigma)$ by Theorem 4.7.

Equation (6.3) is then the right-hand side of (6.2) applied to the ideal $\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}}$, so the multiplicity of \mathbf{w} in $\Sigma \cap_{st} H$ equals $\deg((\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}})|_{x_n=1})$. This degree is equal to $\deg((\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}})|_{x_n=0})$ by Part 1 of Lemma 3.11, since the degree of a tropical ideal depends only on its trivialization. Now, we have

$$(\text{in}_{\mathbf{w}}(I) \cap S_{\mathbf{0}})|_{x_n=0} = \text{in}_{\mathbf{w}}(I)|_{x_n=0} \cap S_{\mathbf{0}}|_{x_n=0} = \text{in}_{\pi(\mathbf{w})}(I|_{x_n=a}) \cap \mathbb{B}[x_d^{\pm 1}, \dots, x_{n-1}^{\pm 1}],$$

where the first equality follows from the fact that $\deg(x_n) = \mathbf{0}$, and the second from Part 2 of Lemma 3.11. Thus the multiplicity of \mathbf{w} in $\Sigma \cap_{st} H$ equals $\deg(\text{in}_{\pi(\mathbf{w})}(I|_{x_n=a}) \cap \mathbb{B}[x_d^{\pm 1}, \dots, x_{n-1}^{\pm 1}])$, which is by definition the multiplicity of $\pi(\mathbf{w})$ in $V(I|_{x_n=a})$. \square

Stable intersection is only defined for balanced polyhedral complexes, to ensure the resulting multiplicities are independent of choices. The reason we require $V(I)$ to be pure in the previous theorem is because Theorem 6.6 only guarantees that the maximal-dimensional subcomplex of $V(I)$ is balanced, and thus the stable intersection can only be defined when that is the entire complex.

In the case that $I = \text{trop}(J)$ for a *prime* ideal $J \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the Structure Theorem for tropical geometry implies that $V(I)$ is the support of a pure-dimensional polyhedral complex. It would be good to have a more general condition on tropical ideals that still guarantees this. If J is not prime, it still has a primary decomposition, so $V(I)$ decomposes as the support of a union of balanced polyhedral complexes. Proposition 6.8 thus extends to these cases. An analogous construction is currently missing from tropical scheme theory.

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