A BOURGAIN–BREZIS–MIRONESCU REPRESENTATION FOR FUNCTIONS WITH BOUNDED DEFORMATION

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Abstract. We establish a non-local integral difference quotient representation for symmetric gradient semi-norms in $BD(\Omega)$ and $LD(\Omega)$, which does not require the manipulation of distributional derivatives. Our representation extends the formulas for the symmetric gradient established by Mengesha for vector-fields in $W^{1,p}(\Omega;\mathbb{R}^d)$, which are inspired by the gradient semi-norm formulas introduced by Bourgain, Brezis and Mironescu in $W^{1,p}(\Omega)$ and by Dávila in $BV(\Omega)$.

1. Introduction

Let Ω be a connected open subset of $\mathbb{R}^d$ with uniformly Lipschitz boundary and let $M^{d\times d}_{\mathrm{sym}}$ be the space of symmetric $(d\times d)$ real valued matrices. The distributional symmetric gradient of an integrable vector-field $u:\Omega\to\mathbb{R}^d$ is defined as the $M^{d\times d}_{\mathrm{sym}}$-valued distribution

$$\begin{aligned} Eu &:= \frac{1}{2}(Du + Du^T) \\ &= \frac{1}{2}(\partial_i u^j + \partial_j u^i), \qquad i,j = 1,\dots,d, \end{aligned} \tag{1.1}$$

where ∂_i denotes the distributional partial derivative in the e_i canonical direction of $\mathbb{R}^d$. Analogously to the classical Sobolev spaces, one may define spaces of functions with L^p symmetric gradients as follows: if $p\in[1,\infty)$, then

$$LD^p(\Omega) := \left\{ u\in L^p(\Omega;\mathbb{R}^d) : Eu \in L^p(\Omega; M^{d\times d}_{\mathrm{sym}}) \right\}$$

is the space of p-integrable vector-fields u such that Eu can be represented by a p-integrable $M^{d\times d}_{\mathrm{sym}}$-valued field on Ω.[1] The introduction of more general spaces, where one considers functions with symmetric gradients represented by a symmetric matrix-valued Radon measure, has been a crucial landmark in the understanding of plasticity and fracture models in linear elasticity (we refer the interested reader to [**3, 8, 13, 19, 20**] and references therein for a

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[1]Or simply $LD(\Omega)$ when $p=1$.

deeper discussion on this topic). Precisely for such purposes, Christiansen, Matthies and Strang [8] and Suquet [19] independently introduced the space

$$BD(\Omega) := \left\{ u \in L^1(\Omega;\mathbb{R}^d) : Eu \in \mathcal{M}_b(\Omega; M^{d\times d}_{\text{sym}}) \right\},$$

of *functions with bounded deformation* over Ω, which consists of all integrable vector-fields $u : \Omega \to \mathbb{R}^d$ such that their distributional symmetric gradient Eu can be represented by a $M^{d\times d}_{\text{sym}}$-valued *bounded Radon measure*.

Functions in $BD(\Omega)$ possess *similar* functional and fine properties to the ones exhibited by functions in $BV(\Omega;\mathbb{R}^d)$ (see, e.g., [1, 4, 11, 12]). Here,

$$BV(\Omega;\mathbb{R}^d) = \left\{ u \in L^1(\Omega;\mathbb{R}^d) : Du \in \mathcal{M}_b(\Omega; M^{d\times d}) \right\},$$

is the space of *vector-fields with bounded variation* over Ω. However, the kernel of E is strictly larger than the kernel of the gradient operator D. This property substantially separates these two operators from a functional viewpoint. Indeed, a vector-field $u : \Omega \to \mathbb{R}^d$ satisfies $Eu = 0$ in the sense of distributions on Ω if and only if u is a rigid motion, i.e., $u = Rx + c$, where $R \in M^{d\times d}_{\text{skew}}$ is a $(d \times d)$ skew-symmetric matrix and $c \in \mathbb{R}^d$. For this reason, one cannot expect, in general, to control Du in terms of Eu alone. In order to control the L^p norm of Du in terms of the one of Eu, one has to translate by all possible rigid motions (modulo constant displacements). This reasoning applies *only* when we restrict ourselves to the range $p \in (1,\infty)$, as it is reflected in the following version of Korn's inequality

$$\inf_{R\in M^{n\times n}_{\text{skew}}} \|Du - R\|_{L^p(\Omega)} \le K(\Omega,p)\|Eu\|_{L^p(\Omega)}, \qquad p \in (1,\infty).^{2}$$

As a consequence, for $p > 1$, the definition of $LD^p(\Omega)$ is superfluous from a functional point of view, as it is straightforward to verify that $LD^p(\Omega)$ coincides with the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^d)$. On the other hand, as $p \to 1^+$, the optimal constant $K(\Omega,p)$ in Korn's inequality blows-up to infinity and this points at the fact that the symmetric gradient and the gradient *are truly different* operators from a functional perspective. This is formalized through Ornstein's non-inequality [15], which conveys that neither $LD(\Omega)$ embeds into $W^{1,1}(\Omega;\mathbb{R}^d)$, nor $BD(\Omega)$ embeds into the space $BV(\Omega;\mathbb{R}^d)$.

The goal of this paper is to prove a limiting non-local integral formula for a total variation semi-norm of the symmetric gradient of an integrable vector-field, which avoids the direct manipulation of the distributions in (1.1). The results presented here are inspired by similar formulas for gradients first established by Bourgain, Brezis and Mironescu for functions in $W^{1,p}(\Omega)$, by Dávila for $BV(\Omega)$, and for the symmetric gradient operator by Mengesha

[2]This (rigidity) version is a consequence of Korn's second inequality

$$\|u\|_{W^{1,p}(\Omega)} \le C\left(\|u\|_{L^p(\Omega)} + \|Eu\|_{L^p(\Omega)}\right),$$

whose first proof is arguably contained as a particular case of the coercive estimates established by Smith [18].

for functions $W^{1,p}(\Omega;\mathbb{R}^d)$. Therefore, to contextualize and motivate our findings, we shall first recall the theory for gradients.

1.1. **Background theory for the gradient operator.** In [5] (see also [7]), Bourgain, Brezis and Mironescu established the following (BBM) limiting difference quotient representation for Sobolev functions: if $u \in W^{1,p}(\Omega)$ for some $p \in [1,\infty)$, then

$$\lim_{\varepsilon\to 0^+} \iint_{\Omega\times\Omega} \frac{|u(y)-u(x)|^p}{|y-x|^p}\,\rho_\varepsilon(y-x)\,dy\,dx = K_{p,d}\int_\Omega |\nabla u|^p, \tag{1.2}$$

where $K_{p,d}$ is a positive constant depending on p, d and $\{\rho_\varepsilon\}_{\varepsilon>0} \subset L^1(\mathbb{R}^d)$ is a family of non-negative radial probability mollifiers

$$\|\rho_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1, \qquad \rho_\varepsilon(x) = \hat{\rho}_\varepsilon(|x|), \tag{1.3}$$

which approximate the Dirac mass at zero in the sense that

$$\lim_{\varepsilon\to 0^+} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^d\setminus B_\delta)} = 0 \qquad \text{for all } \delta > 0. \tag{1.4}$$

The authors also show that the converse holds in the range $p \in (1,\infty)$. More precisely, that if $u \in L^p(\Omega)$ and

$$\liminf_{\varepsilon\to 0^+} \iint_{\Omega\times\Omega} \frac{|u(y)-u(x)|^p}{|y-x|^p}\,\rho_\varepsilon(y-x)\,dy\,dx < \infty, \tag{1.5}$$

then automatically $u \in W^{1,p}(\Omega)$. The analysis of the limiting case $p = 1$ is more delicate since one must take into account the appearance of mass concentrations in the gradient. In this regard, Dávila [9] established a related representation for $BV(\Omega)$. He proved that $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and (1.5) holds with $p = 1$; in that case the limit exists and is given by

$$\lim_{\varepsilon\to 0^+} \iint_{\Omega\times\Omega} \frac{|u(y)-u(x)|}{|y-x|}\,\rho_\varepsilon(y-x)\,dy\,dx = K_{1,d}|Du|(\Omega), \tag{1.6}$$

where $|Du|$ is the total variation measure associated to $Du \in \mathcal{M}_b(\Omega;\mathbb{R}^d)$.

1.2. **Background theory for the symmetric gradient.** Now that we have recalled the representations for Sobolev and bounded variation functions, we shall center on the theory concerning the symmetric gradient operator. In order to do this, we shall first introduce an auxiliary family of p-(quasi)norms in the $M^{d\times d}_{\mathrm{sym}}$ as follows: for $p \in (0,\infty)$, we set

$$\mathbf{Q}_p(A) := \left(\fint_{\mathbb{S}^{d-1}} |\langle A\omega,\omega\rangle|^p\,dS(\omega)\right)^{\frac{1}{p}}, \qquad A \in M^{d\times d}_{\mathrm{sym}},$$

where S stands for the surface measure on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$. It is easy to check that if $p \geq 1$, then $\mathbf{Q}_p(\cdot)$ defines a norm on $M^{d\times d}_{\mathrm{sym}}$ (more

details will be given in Sect. 2.3). Furthermore, for $p \in [1,\infty)$, a vector-field $u \in L^p(\Omega;\mathbb{R}^d)$ and a Borel set $U \subset \mathbb{R}^d$, we introduce the following short-hand notation:

$$\mathscr{F}_{p,\varepsilon}(u,U) := \iint_{U\times U} \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\,dx,$$

where the right-hand side is well-defined as the integral of a non-negative function and may take the value ∞.

In [**14, Theorem 2.2**] Mengesha proved (under slightly more restrictive assumptions) the following analogue of (1.2) for the symmetric gradient and functions in $W^{1,p}(\Omega;\mathbb{R}^d)$:

Theorem 1.1. *Let $p \in (1,\infty)$ and let $u \in L^p(\Omega;\mathbb{R}^d)$. Then, u belongs to $W^{1,p}(\Omega;\mathbb{R}^d)$ if and only if*

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_{p,\varepsilon}(u,\Omega) < \infty. \tag{1.7}$$

Moreover, the (extended) limit always exists and equals

$$\lim_{\varepsilon\to 0^+} \iint_{\Omega\times\Omega} \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\,dx = \int_\Omega \mathbf{Q}_p(Eu(x))^p\,dx,$$

with the convention that the right-hand side integral equals ∞ whenever $u \notin W^{1,p}(\Omega;\mathbb{R}^d)$.[3]

Following Dávila's ideas, and as a direct consequence of Mengesha's representation, one has the following strong compactness and convergence result (which is somehow implicit in [**14**]):

Corollary 1.2. *Let $p \in (1,\infty)$ and let $u \in L^p(\Omega;\mathbb{R}^d)$. The following are equivalent:*

(a) $u \in W^{1,p}(\Omega;\mathbb{R}^d)$,
(b) the family of functions

$$\mu_{p,\varepsilon}(x) := \left(\int_\Omega \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\right)^{\frac{1}{p}}, \quad \varepsilon \in (0,1),$$

is uniformly bounded in $L^p(\Omega)$,
(c) $\mu_{p,\varepsilon} \longrightarrow \mathbf{Q}_p(Eu)$ *in $L^p(\Omega)$ as $\varepsilon \to 0^+$.*

Remark 1.3 (An alternative proof)**.** For the convenience of the reader and since our proofs depart in crucial points from the ones given by Bourgain, Brezis and Mironescu, Davila and Mengesha, we have decided to include here the proof of Theorem 1.1 and of Corollary 1.2. Notice also that we do not require Ω to be bounded in any of these or the forthcoming results.

[3]Notice that our $\mathbf{Q}_p$-norms differ by a multiplicative constant $|\mathbb{S}^{d-1}|$ with respect to Mengesha's original norms. This, however, seems to stem from a minor normalization typo.

In addition to Theorem 1.1, Mengesha proved the following criterion for functions of bounded deformation in terms of the symmetric difference quotient energy: a map u belongs to $BD(\Omega)$ if and only if $u \in L^1(\Omega;\mathbb{R}^d)$ and (1.7) is finite for $p = 1$. More precisely, he showed that there exist positive constants γ_1, γ_2 satisfying

$$\begin{aligned} \gamma_1 \|u\|_{BD(\Omega)} &\leq \liminf_{\varepsilon\to 0^+} \mathscr{F}_{1,\varepsilon}(u,\Omega) \\ &\leq \limsup_{\varepsilon\to 0^+} \mathscr{F}_{1,\varepsilon}(u,\Omega) \leq \gamma_2 \|u\|_{BD(\Omega)}, \end{aligned} \tag{1.8}$$

with the convention that these norms may take the value ∞ and where $\|u\|_{BD(\Omega)} := \|u\|_{L^1(\Omega)} + |Eu|(\Omega)$ is the standard norm in $BD(\Omega)$ (see below). As it is already suggested by (1.6) and (1.8), the analysis and characterization of (1.7) (with $p = 1$), requires one to relax the the statement to functions with bounded deformation, rather than to elements of $LD(\Omega)$ or $W^{1,1}(\Omega)$. In other words, the sufficiency of the first statement of Theorem 1.1 fails for $p = 1$ because one must take into account the appearance of mass concentrations on the symmetric gradient.

1.3. **Main results.** In order to state our results, we need to recall the following basic geometric measure theory facts: if $u \in BD(\Omega)$, then Eu is a bounded $M^{d\times d}_{\mathrm{sym}}$-valued Radon measure and hence, by Riesz' representation theorem and the Radon–Nikodým differentiation theorem, we may write Eu in polar form

$$Eu = e_u\,|Eu| \qquad \text{as measures on } \Omega,$$

where $|Eu| \in \mathcal{M}^+(\Omega)$ is the total variation measure of Eu (induced by the classical Frobenius inner product of matrices) and

$$e_u(x) := \frac{d\,Eu}{d\,|Eu|}(x) = \lim_{r\to 0^+} \frac{Eu(B_r(x))}{|Eu|(B_r(x))}, \qquad x \in \Omega,$$

is a norm-1 density function in $L^\infty(\Omega, |Eu|; M^{d\times d}_{\mathrm{sym}})$. We may then define the $\mathbf{Q}_1$-total variation measure of Eu as

$$[Eu](U) := \int_U \mathbf{Q}_1(e_u(x))\, d|Eu|(x), \qquad U \subset \Omega \text{ Borel}.$$

Having set this notation, we are finally ready to state our main result:

Theorem 1.4. *Let $u \in L^1(\Omega;\mathbb{R}^d)$. Then, the (extended) limit*

$$\lim_{\varepsilon\to 0^+} \mathscr{F}_{1,\varepsilon}(u,\Omega) \in [0,\infty]$$

always exists and equals

$$\lim_{\varepsilon\to 0^+} \iint_{\Omega\times\Omega} \frac{|\langle u(y)-u(x), y-x\rangle|}{|y-x|^2}\, \rho_\varepsilon(y-x)\, dy\, dx = [Eu](\Omega),$$

with the convention that the right-hand side equals ∞ whenever $u \notin BD(\Omega)$.

In particular, we obtain the following strengthening of (1.8):

Corollary 1.5. *There exists a dimensional constant $C_d > 0$ such that*

$$C_d|Eu|(\Omega) \leq \lim_{\varepsilon\to 0^+} \mathscr{F}_{1,\varepsilon}(u,\Omega) \leq |Eu|(\Omega),$$

under the convention that the semi-norms may attain the value ∞.

Remark 1.6. One can draw a parallelism between the BV-theory and the BD-theory in the following way: Theorem 1.4 extends Theorem 1.1 to $BD(\Omega)$, just as Dávila's representation (1.6) extends (1.2) to $BV(\Omega)$.

Remark 1.7. Notice that if $u \in LD(\Omega)$, we still get

$$\lim_{\varepsilon\to 0^+} \iint_{\Omega\times\Omega} \frac{|\langle u(y)-u(x), y-x\rangle|}{|y-x|^2}\,\rho_\varepsilon(y-x)\,dy\,dx = \int_\Omega \mathbf{Q}_1(Eu(x))\,dx.$$

See also Corollary 1.12 and remark below for a representation formula with $p \in (0,1)$ and $u \in BD(\Omega)$.

In general, Corollary 1.2 does not have an L^1-convergence analog. However, we can still deduce the following compactness and strict convergence (in the sense of measures) results:

Corollary 1.8. *Let $u \in L^1(\Omega;\mathbb{R}^d)$. The following are equivalent:*

(a) $u \in BD(\Omega)$,

(b) the family of functions

$$\mu_{1,\varepsilon}(x) := \int_\Omega \frac{|\langle u(y)-u(x), y-x\rangle|}{|y-x|^2}\,\rho_\varepsilon(y-x)\,dy, \quad \varepsilon \in (0,1),$$

is uniformly bounded in $L^1(\Omega)$,

(c) $\mu_\varepsilon\,\mathscr{L}^d \overset{*}{\rightharpoonup} [Eu]$ *as measures in $\mathcal{M}(\Omega)$ and*

$$\mu_{1,\varepsilon}(\Omega) \longrightarrow [Eu](\Omega) \qquad \text{as } \varepsilon \to 0^+.$$

We close the exposition of our results with some consequences of the representation for BD-spaces that are inspired by the work of Ponce and Spector [17] on BV-spaces. Let us recall (see [1, 11]) that a function $u \in BD(\Omega)$ is approximately differentiable almost everywhere, that is, there exists a measurable matrix-field $x \mapsto \operatorname{ap}\nabla u(x) \in M^{d\times d}$ satisfying

$$\lim_{r\to 0^+} \fint_{B_r(x)} \frac{|u(y)-u(x)-\operatorname{ap}\nabla u(x)[y-x]|}{r} = 0 \quad \text{for } \mathscr{L}^d\text{-a.e. } x \in \Omega.$$

In this case, the matrix $\operatorname{ap}\nabla u(x)$ is called the *approximate differential* of u at x. On the other hand, it is also well-known (see [1]) that if $u \in BD(\Omega)$, then Eu can be decomposed into an absolutely continuous and a singular part as

$$Eu = \mathcal{E}u\,\mathscr{L}^d + E^s u, \quad \mathcal{E}u(x) := \frac{1}{2}(\operatorname{ap}\nabla u(x) + \operatorname{ap}\nabla u(x)^T),$$

where $|E^s u| \perp \mathscr{L}^d$.

Following verbatim the ideas contained in [17, **Sect. 2**], we give a criterion for the absolute continuity of symmetric gradient measures in the terms of its *approximate first-order Taylor expansion*:

Corollary 1.9. *Let $u \in BD(\Omega;\mathbb{R}^d)$ and let $\operatorname{ap}\nabla u(x) \in M^{d\times d}$ denote the approximate differential of u at a point x, which exists $\mathscr{L}^d$-almost everywhere in Ω. Then, the extended limit*

$$\lim_{\varepsilon\to 0}\iint_{\Omega\times\Omega}\frac{|\langle u(y)-u(x)-\operatorname{ap}\nabla u(x)[y-x],y-x\rangle|}{|y-x|^2}\,\rho_\varepsilon(y-x)\,dy\,dx$$

exists and equals $[E^s u](\Omega)$.

In particular, $u \in LD^1(\Omega)$ if and only if

$$\lim_{\varepsilon\to 0}\iint_{\Omega\times\Omega}\frac{|\langle u(y)-u(x)-F(x)[y-x],y-x\rangle|}{|y-x|^2}\,\rho_\varepsilon(y-x)\,dy\,dx = 0$$

for some Borel measurable function $F\colon \Omega \to M^{d\times d}$.

Remark 1.10. The assertions of the previous corollary remain unchanged if instead we consider the integrand

$$\frac{|\langle u(y)-u(x)-\mathcal{E}u(x)[y-x],y-x\rangle|}{|y-x|^2}\,\rho_\varepsilon(y-x).$$

Finally, following [16] we prove the following representation for general nonlinear integrands that have linear growth limits at infinity:

Theorem 1.11. *Let $f\colon [0,\infty)\to[0,\infty)$ be a continuous function satisfying*

$$f^\infty := \lim_{t\to\infty}\frac{f(t)}{t} \in [0,\infty). \tag{1.9}$$

If $u \in BD(\Omega)$, then the limit

$$\lim_{j\to\infty}\iint_{\Omega\times\Omega} f\left(\frac{|\langle u(x)-u(y),x-y\rangle|}{|x-y|^2}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy$$

exists and equals

$$\int_\Omega \mathbf{Q}_f(\mathcal{E}u)\,dx + f^\infty\int_\Omega \mathbf{Q}_1(E^s u)$$

where

$$\mathbf{Q}_f(A) := \fint_{\mathbb{S}^{d-1}} f(|\langle A\omega,\omega\rangle|)\,dS(\omega), \qquad A \in M^{d\times d}_{\mathrm{sym}}.$$

A direct consequence of this result with $f(t) = t^p$ and $p \in (0,1)$ is the following representation of non-convex p-difference quotients:

Corollary 1.12. *Let $p \in (0,1)$ and let $u \in BD(\Omega)$. Then*

$$\lim_{j\to\infty}\iint_{\Omega\times\Omega}\frac{|\langle u(x)-u(y),x-y\rangle|^p}{|x-y|^{2p}}\rho_{\varepsilon_j}(x-y)\,dx\,dy = \int_\Omega \mathbf{Q}_p(\mathcal{E}u(x))^p\,dx.$$

2. Preliminaries

Here and in all that follows we assume that $p \geq 1$ unless otherwise is stated. In this section we briefly recall the properties of the $\mathbf{Q}$-norms defined in the introduction and we also review some well-known results about $BD(\Omega)$ and $W^{1,p}(\Omega;\mathbb{R}^d)$ spaces, where Ω is a Lipschitz (possibly unbounded) open set of $\mathbb{R}^d$.

In all that follows, we write $|\cdot|$ to denote the classical Frobenius inner product norm on $M^{d\times d}$ (and $M^{d\times d}_{\mathrm{sym}}$), that is,

$$|A|^2 := \mathrm{trace}(A^T A) = \sum_{i,j=1}^{d} A_{ij}^2, \qquad A = (A_{ij}).$$

2.1. **Strict convergence.** We say that a sequence $(u_k) \in BD(\Omega)$ converges *strictly* to u in $BD(\Omega)$ provided that

$$u_k \to u \text{ in } L^1(\Omega;\mathbb{R}^d), \quad Eu_k \overset{*}{\rightharpoonup} Eu \text{ in } \mathcal{M}(\Omega; M^{d\times d}_{\mathrm{sym}}),$$

and

$$|Eu_k|(\Omega) \to |Eu|(\Omega).$$

To denote this, we write

$$u_k \xrightarrow{s} u \quad \text{in } BD(\Omega)$$

2.2. **Extension operators.** When $p > 1$, a direct consequence of Korn's inequality is the embedding

$$LD^p(\Omega) \hookrightarrow W^{1,p}(\Omega;\mathbb{R}^d).$$

This, in particular, allows one to make use of a plethora of extension operators $T\colon W^{1,p}(\Omega;\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ whenever $p > 1$. If $p = 1$, it is well known that neither $LD^1(\Omega)$ nor $BD(\Omega)$ embed into $W^{1,1}(\Omega;\mathbb{R}^d)$, not even locally. However, $BD(\Omega)$ does possess trace operators [4] and in particular it possesses an extension operator $T : BD(\Omega) \to BD(\mathbb{R}^d)$ that does not charge the boundary, i.e., such that

$$|E(Tu)|(\partial\Omega) = 0.$$

2.3. **The Q-norms on LD^p and BD.** As it has already been advanced in the previous section, we will work with certain Rayleigh-type norms on $M^{d\times d}_{\mathrm{sym}}$. For the convenience of the reader, let us recall its definition:

Definition 2.1. Let $p \in [1,\infty)$. We define a norm on $M^{d\times d}_{\mathrm{sym}}$ by letting

$$\mathbf{Q}_p(A) := \left(\fint_{\mathbb{S}^{d-1}} |\langle A\omega, \omega\rangle|^p \, dS(\omega) \right)^{\frac{1}{p}}$$

$$= \kappa_{p,d} \|\langle A\cdot, \cdot\rangle\|_{L^p(\mathbb{S}^{d-1})},$$

where $\kappa_{p,d} := |\mathbb{S}^{d-1}|^{-1/p}$ and $|\mathbb{S}^{d-1}|$ is the measure of the $(d-1)$-dimensional sphere in $\mathbb{R}^d$.

That $\mathbf{Q}_p$ defines a norm for every $p \in [1, \infty)$ is an immediate consequence of the triangle inequality in L^p and the spectral theorem for matrices in $M^{d\times d}_{\mathrm{sym}}$. Indeed, it is straightforward to verify that $\mathbf{Q}_p$ is invariant under the conjugation with orthogonal matrices and hence

$$\mathbf{Q}_p(A)^p = \fint_{\mathbb{S}^{d-1}} (\lambda_1\omega_1^2 + \dots \lambda_d\omega_d^2)^p \, dS(\omega), \tag{2.1}$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of A. In particular, $\mathbf{Q}_p(A) = 0$ if and only if all the eigenvalues are zero, and by homogeneity and Cauchy–Schwarz' inequality it follows that

$$C_{d,p}|A| \le \mathbf{Q}_p(A) \le |A|,$$

for some constant $C_{d,p}$. In a natural manner, this defines an equivalent norm for functions $F \in L^p(\Omega; M^{d\times d}_{\mathrm{sym}})$ by setting

$$[F]_p(\Omega) \coloneqq \left(\int_\Omega \mathbf{Q}_p(F)^p \, dx \right)^{\frac{1}{p}}.$$

For $(M^{d\times d}_{\mathrm{sym}})$-valued Radon measures $\mu \in \mathcal{M}_b(\Omega; M^{d\times d}_{\mathrm{sym}})$, we may consider the $\mathbf{Q}_1$-variation measure $[\mu] \in \mathcal{M}^+(\Omega)$, which on Borel sets $U \subset \Omega$ is defined as the non-negative Radon measure taking the values

$$[\mu](U) \coloneqq \int_U \mathbf{Q}_1(\mu) = \int_U \mathbf{Q}_1\left(\frac{\mu}{|\mu|}(x)\right) d|\mu|(x),$$

where $\mu/|\mu|$ is the Radon–Nikodým derivative of μ with respect to $|\mu|$. Notice that $[\cdot]$ and $|\cdot|$ are equivalent norms in $\mathcal{M}_b(\Omega; M^{d\times d}_{\mathrm{sym}})$. Hence, both

$$|u|_{LD^p(\Omega)} \coloneqq \|u\|_{L^p(\Omega)} + [Eu]_p(\Omega),$$
$$|u|_{BD(\Omega)} \coloneqq \|u\|_{L^1(\Omega)} + [Eu](\Omega),$$

define equivalent norms on $LD^p(\Omega; \mathbb{R}^d)$ and $BD(\Omega)$ respectively.

Remark 2.2 (Strict convexity and strict convergence)**.** Every $\mathbf{Q}_p$ is a convex 1-homogenenous function (each of these being norms). However, it is worthwhile to mention that $\mathbf{Q}_1$ *is not* a strictly convex norm. Indeed, it can be seen from (2.1) that $\mathbf{Q}_1(A) = \alpha_d \operatorname{tr}(A)$ for all positive definite matrices $A \in M^{d\times d}_{\mathrm{sym}}$, which implies that this norm behaves linearly on this connected open set of matrices. In particular, $\mathbf{Q}_1$ *is not* the norm associated to an inner product on $M^{d\times d}_{\mathrm{sym}}$, and the $[\cdot]$-strict convergence of measures

$$\mu_\varepsilon \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(\Omega; M^{d\times d}_{\mathrm{sym}}), \qquad [\mu_\varepsilon](\Omega) \to [\mu](\Omega),$$

does not necessarily imply that $\mu_\varepsilon \longrightarrow \mu$ in the classical $|\cdot|$-strict sense of measures; this last assertion follows from [**10, Theorem 1.3**].

3. Proof of the main result

3.1. **Proof of the upper bound.** The proof of the upper bound inequality is somewhat standard as it follows closely the ideas from [5] and [9], with the exception that we are considering slightly different integrands here.

The first step will be to show a suitable ε-independent upper bound for $u \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ (see Lemma 3.1 below). Once the scale-independent bound contained in Lemma 3.1 has been established, the sought upper bound for $u \in LD^p(\Omega)$ and $u \in BD(\Omega)$ will follow from the existence of suitable extension operators for these spaces.

For the next lemma, we write

$$\mu_{p,\varepsilon}(x) := \left(\int_{\mathbb{R}^d} \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\right)^{\frac{1}{p}}. \tag{3.1}$$

Lemma 3.1. *Let $u \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ and let $U \subset \mathbb{R}^d$ be a Borel set. For a positive radius $R > 0$, we shall write*

$$U_R := U + B_R,$$

to denote the set whose complement is at distance R from U. Then, it holds

$$\int_U \mu_{p,\varepsilon}^p \le ([Eu]_p(U_R))^p + \frac{2}{R^p}\|u\|^p_{L^p(U)} \int_{\mathbb{R}^d\setminus B_R(0)} \rho_\varepsilon(x)\,dx. \tag{3.2}$$

Proof of the lemma. For a fixed $R > 0$, we split

$$\int_U \mu_{p,\varepsilon}^p = \int_U \int_{\mathbb{R}^d} \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\,dx = \ I_1 + I_2,$$

where

$$I_1 := \int_U \int_{B_R(x)} \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\,dx$$

$$I_2 := \int_U \int_{\mathbb{R}^d\setminus B_R(x)} \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|y-x|^{2p}}\,\rho_\varepsilon(y-x)\,dy\,dx.$$

Clearly, the second term in the right hand side of (3.2) is an upper bound for I_2. We shall hence focus on showing that $([Eu]_p(U_R))^p$ is an upper bound for I_1. To this end, let us recall the path integral identity

$$u(y) - u(x) = \int_0^1 \nabla u(ty + (1-t)x)\cdot(y-x)\,dt, \qquad y, x \in \mathbb{R}^d.$$

Together with Jensen's inequality, this yields that

$$\int_U \int_{B_R(x)} \int_0^1 \left|\left\langle \nabla u(ty+(1-t)x)\cdot \frac{y-x}{|y-x|}, \frac{y-x}{|y-x|}\right\rangle\right|^p dt\,\rho_\varepsilon(y-x)\,dy\,dx$$

is an upper bound for I_1.

Fixing x, we apply the change of variables $h := y - x$ and apply Tonelli's Theorem to permute the integrals and obtain

$$\begin{aligned} I_1 &\leq \int_{B_R}\int_0^1\int_U \left|\left\langle \nabla u(x+th)\cdot\frac{h}{|h|},\frac{h}{|h|}\right\rangle\right|^p dx\,dt\,\rho_\varepsilon(h)\,dh \\ &= \int_{B_R}\int_0^1\int_{U+th} \left|\left\langle \nabla u(z)\cdot\frac{h}{|h|},\frac{h}{|h|}\right\rangle\right|^p dz\,dt\,\rho_\varepsilon(h)\,dh \\ &\leq \int_{B_R}\int_{U_R} \left|\left\langle \nabla u(z)\cdot\frac{h}{|h|},\frac{h}{|h|}\right\rangle\right|^p dz\,\rho_\varepsilon(h)\,dh. \end{aligned}$$

Observe that if $A \in M^{d\times d}$, then $\langle A\omega,\omega\rangle = \frac{1}{2}\langle (A^T + A)\omega,\omega\rangle$ for all $\omega \in \mathbb{R}^d$. We shall use this to express the integrand on the right-hand side of the estimate in terms of $Eu(x)$ rather than $Du(x)$. Therefore, from the change of variables $\omega = \frac{h}{|h|}$, the coarea formula on balls and the radial symmetry of the mollifier we deduce that

$$\begin{aligned} I_1 &\leq |\partial B_r| \int_0^R \widehat{\rho_\varepsilon}(r)\, r^{d-1} dr \times \left(\int_{U_R}\int_{\mathbb{S}^{d-1}} |\langle Eu(z)\omega,\omega\rangle|^p\, d\mathcal{H}^{d-1}(\omega)\, dz\right) \\ &\leq \|\rho\|_{L^1}\int_{U_R} \mathbf{Q}_p(Eu)^p\, dz \leq ([Eu]_p(U_R))^p\,. \end{aligned}$$

This completes the proof of the lemma. □

Proof of the upper bound. Let $U \subset \Omega$ be an open set. Let $u \in LD^p(\Omega)$ or $u \in BD(\Omega)$. We aim to show that

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{p,\varepsilon}(u,\Omega) \leq [Eu]_p(\Omega)^p \quad \text{when } p > 1$$

or

$$\limsup_{\varepsilon\to 0} \mathscr{F}_{1,\varepsilon}(u,\Omega) \leq [Eu](\Omega),$$

respectively. Let us recall from the preliminaries that for $p > 1$ there exists an extension operator $T : W^{1,p}(\Omega;\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$, and, for $p = 1$, there also exists an extension operator, which for the sake of simplicity we shall also denote by $T\colon BD(\Omega) \to BD(\mathbb{R}^d)$, that does not charge the boundary, that is, $|E(Tu)|(\partial\Omega) = 0$. On either case, a standard mollification argument yields an approximating sequence $(u_k) \subset C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ satisfying

$$u_k \longrightarrow Tu \text{ in } W^{1,p}(\mathbb{R}^d) \qquad \text{when } p > 1,$$

or

$$u_k \xrightarrow{s} Tu \quad \text{in } BD(\mathbb{R}^d) \qquad \text{when } p = 1.$$

In the latter case, there exists a full L^1-measure set $I \subset (0,\infty)$ for which it holds $|Eu|(\partial U_R) = 0$ for all $R \in I$. In particular, from the strict convergence above we get

$$u_k \xrightarrow{s} Tu \quad \text{in } BD(U_R) \qquad \text{for all } R \in I.$$

Claim. Let $R \in I$ and let $\varepsilon > 0$. Then

$$\lim_{k\to\infty} \mathscr{F}_{p,\varepsilon}(u_k, U_R) = \mathscr{F}_{p,\varepsilon}(Tu, U_R) \tag{3.3}$$

and

$$\lim_{k\to\infty} [Eu_k]_p(U_R) = M(u, U_R) := \begin{cases} [E(Tu)]_p(U_R) & \text{if } u \in LD^p(U), \\ [E(Tu)](U_R) & \text{if } u \in BD(U). \end{cases} \tag{3.4}$$

The first limit follows directly from Tonelli's theorem and the strong convergence $u_k \to u$ in $L^p(U)$. Let us address the convergence for the second limit. For $p = 1$, the argument follows directly from the strict convergence $u \xrightarrow{s} Tu$ in $BD(U_R)$, Reshetnyak's continuity theorem ([**2, Thm. 2.39**]) and the fact that $\mathbf{Q}_1$ is 1-homogeneous:

$$\begin{aligned} \lim_{k\to\infty} [Eu_k]_1(U_R) &= \lim_{k\to\infty} \int_{U_R} \mathbf{Q}_1\left(\frac{Eu_k}{|Eu_k|}(x)\right) d|Eu_k|\mathscr{L}^n(x) \\ &= \int_{U_R} \mathbf{Q}_1\left(\frac{E(Tu)}{|E(Tu)|}(x)\right) d|E(Tu)|(x) = [E(Tu)](U_R). \end{aligned}$$

For $p > 1$, we recall that $\mathbf{Q}_p$ convex, so that $[\cdot]_p$ is lower semicontinuous with respect to weak convergence in L^p. This implies the lower bound

$$[E(Tu)]_p(U) \le \liminf_{k\to\infty} [Eu_k]_p(U).$$

The upper bound follows directly from the strong convergence $Eu_k \to E(Tu)$ in $L^p(\Omega)$ and the triangle inequality for $\mathbf{Q}_p$, namely

$$\mathbf{Q}_p(Eu_k) \le \mathbf{Q}_p(E(Tu)) + \|Eu_k - E(Tu)\|_{L^p(U)} \to \mathbf{Q}_p(E(Tu)) \quad \text{in } L^p(U).$$

This proves the claim.

Conclusion. Let $R \in I$. Using the estimates from Step 1 on u_k we get (recall that $Tu|_U = u$),

$$\begin{aligned} \mathscr{F}_{p,\varepsilon}(u, U) &= \lim_{k\to\infty} \mathscr{F}_{\varepsilon,p}(u_k, U) \\ &\le \lim_{k\to\infty} [Eu_k]_p(U_R)^p + \frac{2}{R^p}\|u_k\|^p_{L^p(U)}\|\rho_\varepsilon\|_{L^1(\mathbb{R}^d\setminus B_R)} \\ &= M(Tu, U_R) + \frac{2}{R^p}\|u\|^p_{L^p(U)}\|\rho_\varepsilon\|_{L^1(\mathbb{R}^d\setminus B_R)}. \end{aligned}$$

Letting $\varepsilon \to 0^+$ on both sides of the inequality and recalling (1.4) yields the estimate

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_{p,\varepsilon}(u, U) \le M(Tu, U_R).$$

Now we use that $\mathbf{Q}_p(A) \le |A|$, to deduce

$$\begin{aligned} \limsup_{\varepsilon\to 0^+} \mathscr{F}_{p,\varepsilon}(u, U) &\le M(u, U) + \limsup_{R\in I, R\to 0^+} M(E(Tu), U_R \setminus U) \\ &\le M(u, U) + M(E(Tu), \partial U). \end{aligned} \tag{3.5}$$

Since $M(E(Tu), \partial\Omega) = 0$, choosing $U = \Omega$ in the estimate above yields the sought upper bound inequality. □

As a immediate corollary we establish pre-compactness (either in L^p or $\mathcal{M}$) for the family $\{\mu_{p,\varepsilon}\}_{\varepsilon>0}$. Moreover, we note that each of its limit points lies below $|Eu|$.

Corollary 3.2. *Let $1 \le p < \infty$ and assume that*

$$u \in \begin{cases} BD(\Omega) & \text{if } p = 1, \\ LD^p(\Omega) & \text{if } 1 \le p < \infty. \end{cases}$$

Then, the family

$$\mathscr{U} := \{\mu^p_{p,\varepsilon}\,\mathscr{L}^d\}_\varepsilon, \qquad \varepsilon \in (0,1)$$

is sequentially pre-compact in $\mathcal{M}^+(\Omega)$ with respect to the weak convergence of measures. Moreover, for every limit*

$$\mu^p_{p,\varepsilon_i} \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}^+(\Omega), \qquad \varepsilon_i \to 0^+,$$

there exists a Borel function $g : \Omega \to [0,1]$ satisfying

$$\mu = g\,\mathbf{Q}_p(Eu)^p.$$

Proof. We give the argument for $p = 1$ as the one for $p > 1$ is analogous. The equi-boundedness follows directly from the upper bound and the fact that the measures are positive. Now, let μ be a limit point as above and let $U \subset \Omega$ be an Borel set. Then, in light of (3.5) we get

$$\mu(U) \le \lim_{\varepsilon_i \to 0^+} \int_\Omega \mu^p_{p,\varepsilon}(U) = \limsup_{\varepsilon \to 0^+} \mathscr{F}_{p,\varepsilon}(u,U) \le \int_{\bar{U}\cap\Omega} \mathbf{Q}_1(Eu)$$

for all Borel sets $U \subset \Omega$. A standard measure theoretic argument implies that $\mu \ll \mathbf{Q}_p(Eu)$. Therefore, by the Radon–Nikodým theorem there exists $g \in L^1(\Omega, \mathbf{Q}_1(Eu); \mathbb{R}^+)$ such that

$$\mu = g\,\mathbf{Q}_1(Eu), \qquad g \le 1.$$

This finishes the proof. □

3.2. **Proof of the lower bound.** We show that if $p \in (1,\infty)$ and $u \in L^p(\Omega;\mathbb{R}^d)$, then (under the conventions discussed in the introduction)

$$[Eu]_p(\Omega)^p \le \liminf_{\varepsilon \to 0^+} \mathscr{F}_{p,\varepsilon}(u,\Omega) \in [0,\infty],$$

and

$$[Eu](\Omega) \le \liminf_{\varepsilon \to 0^+} \mathscr{F}_{1,\varepsilon}(u,\Omega) \in [0,\infty] \quad \text{for all } u \in L^1(\Omega;\mathbb{R}^d).$$

For this step, we give a proof by means of a simple mollification argument. This, in turn, differs from the proof by duality originally given in [5] for $W^{1,p}$-gradients. Our proof follows from the observation that the energy is convex with respect to translations, and hence mollification. In particular, our argument also presents an alternative proof to the one contained in [9] for gradients, which dispenses with the need of performing certain technical measure theoretic density arguments.

3.2.1. *The regular case.* We begin with a lemma, which establishes a local version of the lower bound for regular functions.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $u \in C^2(\Omega;\mathbb{R}^d)$. Then, for every compactly contained connected open set $A \Subset \Omega$ it holds*

$$[Eu]_p(A)^p \leq \liminf_{\varepsilon\to 0^+} \mathscr{F}_{p,\varepsilon}(u,A).$$

Proof. Fix A as in the statement and let $x \in A$. Observe that, since u is of class $C^2(\overline{A})$ and $\bar{A}$ is connected and compact, we may appeal to its Taylor's expansion there. We can thus write, for every $y \in A$

$$u(y) = u(x) + \nabla u(x)\cdot(y-x) + r(|x-y|^2),$$

where $|r(s)| \leq Cs$ for every $s \in [0,(\operatorname{diam} A)^2]$, for some constant $C > 0$ depending only on $\|u\|_{C^2(\bar{A})}$ and A. From this we get

$$\begin{aligned}\frac{\langle u(y)-u(x), y-x\rangle}{|y-x|^2} &= \frac{\langle \nabla u(x)\cdot(y-x), y-x\rangle}{|y-x|^2} + \frac{\langle r(|x-y|^2), y-x\rangle}{|y-x|^2}\\ &= \frac{\langle Eu(x)\cdot(y-x), y-x\rangle}{|y-x|^2} + \frac{\langle r(|x-y|^2), y-x\rangle}{|y-x|^2}.\end{aligned}$$

Rearranging and taking the absolute values we obtain

$$\begin{aligned}\frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|}{|x-y|^2} &\leq \frac{|\langle u(y)-u(x), y-x\rangle|}{|x-y|^2} + \frac{|\langle r(|y-x|^2), y-x\rangle|}{|y-x|^2}\\ &\leq \frac{|\langle u(y)-u(x), y-x\rangle|}{|x-y|^2} + C|y-x|.\end{aligned}$$

Recall that, for every $p \geq 1$, for every $a, b \geq 0$ and for every $\zeta > 0$, by Young's inequality we have the estimate

$$(a+b)^p \leq (1+\zeta)a^p + c_{p,\zeta} b^p,$$

for a suitable (large) constant $c_{p,\zeta} > 0$. Therefore, we have

$$\begin{aligned}&\frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}}\\ &\quad\leq \left(\frac{|\langle u(y)-u(x), y-x\rangle|}{|x-y|^2} + C|x-y|\right)^p\\ &\quad\leq (1+\zeta)\frac{|\langle u(y)-u(x), y-x\rangle|^p}{|x-y|^{2p}} + c_{p,\zeta}C|x-y|^p.\end{aligned}$$

Now we multiply times $\rho_\varepsilon(y-x)$ and we integrate w.r.t. x, y on A. We obtain

$$\begin{aligned}&\int_A\int_A \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}}\rho_\varepsilon(y-x)\,dy\,dx\\ &\quad\leq (1+\zeta)\int_A\int_A \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|x-y|^{2p}}\rho_\varepsilon(y-x)\,dy\,dx\\ &\quad+ c_{p,\zeta}C\int_A\int_A |x-y|^p\rho_\varepsilon(y-x)\,dy\,dx.\end{aligned} \tag{3.6}$$

We observe that the left-hand side of (3.6), for $\delta > 0$ sufficiently small, can be estimated from below by

$$\begin{aligned}
&\int_A \int_A \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx \\
&\geq \int_A \int_{A\cap B_\delta(x)} \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx \\
&\geq \int_{A_{-\delta}} \int_{B_\delta(x)} \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx
\end{aligned} \tag{3.7}$$

where $A_{-\delta} := \{x \in A : B_\delta(x) \subset A\}$. Combining (3.6) and (3.7) we have

$$\begin{aligned}
&\int_{A_{-\delta}} \int_{B_\delta(x)} \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx \\
&\qquad \leq (1+\zeta) \int_A \int_A \frac{|\langle u(y)-u(x), y-x\rangle|^p}{|x-y|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx \\
&\qquad + c_{p,\zeta} C \int_A \int_A |x-y|^p \rho_\varepsilon(y-x)\,dy\,dx.
\end{aligned} \tag{3.8}$$

We now send $\varepsilon \to 0^+$ in (3.8) and we study the terms separately.

Claim 1. We have

$$\lim_{\varepsilon\to 0^+} \int_{A_{-\delta}} \int_{B_\delta(x)} \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx = [Eu]_p(A_{-\delta})^p.$$

Indeed, by the change of variables $h := y - x$ (in the y-variable) we can re-write

$$\begin{aligned}
&\int_{A_{-\delta}} \int_{B_\delta(x)} \frac{|\langle Eu(x)\cdot(y-x), y-x\rangle|^p}{|y-x|^{2p}} \rho_\varepsilon(y-x)\,dy\,dx \\
&\qquad = \int_{A_{-\delta}} \int_{B_\delta(0)} \frac{|\langle Eu(x)\cdot h, h\rangle|^p}{|h|^{2p}} \rho_\varepsilon(h)\,dh\,dx.
\end{aligned}$$

Therefore, by the coarea formula on balls and the radial symmetry of the mollifier, we further obtain

$$\begin{aligned}
&\int_{A_{-\delta}} \int_{B_\delta} \frac{|\langle Eu(x)\cdot h, h\rangle|^p}{|h|^{2p}} \rho_\varepsilon(h)\,dh\,dx \\
&\qquad = \int_{A_{-\delta}} \fint_{\mathbb{S}^{d-1}} |\langle Eu(x)\cdot\omega, \omega\rangle|^p\,dS(\omega)\,dx \int_0^\delta |\partial B_1| \widehat{\rho_\varepsilon}(r) r^{d-1}\,dr \\
&\qquad = \|\rho_\varepsilon\|_{L^1(B_\delta)} \int_{A_{-\delta}} \mathbf{Q}_p(Eu(x))^p\,dx \overset{(1.3)}{\longrightarrow} [Eu]_p(A_{-\delta})^p
\end{aligned}$$

as $\varepsilon \to 0^+$ and this concludes the proof of the Claim.

Claim 2. We have

$$\lim_{\varepsilon\to 0^+} \int_A \int_A |x-y|^p \rho_\varepsilon(y-x)\,dy\,dx = 0.$$

Changing variables as in the previous claim, we get

$$\int_A \int_A |x-y|^p \rho_\varepsilon(y-x)\,dy\,dx \le \mathscr{L}^d(A) \int_{B_R} |h|^p \rho_\varepsilon(h)\,dh,$$

where $R := \operatorname{diam} A$. Now, for any fixed $\sigma > 0$, we can estimate the integral on the right-hand side by

$$\int_{\mathbb{R}^d} |h|^p \rho_\varepsilon(h)\,dh \le \int_{B_\sigma} |h|^p \rho_\varepsilon(h)\,dh + R^p \int_{B_R \setminus B_\sigma} \rho_\varepsilon(h)\,dh.$$

The second term on the right-hand side vanishes as $\varepsilon \to 0^+$ due to (1.4). Recalling the normalization condition (1.3), the first term on the right-hand side can be roughly estimated by

$$\limsup_{\varepsilon \to 0^+} \int_{B_\sigma} |h|^p \rho_\varepsilon(h)\,dh \le \sigma^p$$

and, since $\sigma > 0$ is an arbitrary positive real number, we conclude the assertion of Claim 2.

Combining Claim 1 and Claim 2, we have from (3.8)

$$[Eu]_p(A_{-\delta})^p \le (1+\zeta) \liminf_{\varepsilon \to 0^+} \mathscr{F}_{\varepsilon,p}(u, A) \tag{3.9}$$

and, since $\zeta > 0$ is arbitrary,

$$[Eu]_p(A_{-\delta})^p \le \liminf_{\varepsilon \to 0^+} \mathscr{F}_{\varepsilon,p}(u, A).$$

We now observe that, since A is an open set, $A_{-\delta} \uparrow A$ as $\delta \to 0^+$. Since positive measures are continuous along monotone sequences [**2, Remark 1.3**], we can pass to the limit in (3.9), obtaining

$$[Eu]_p(A)^p \le \liminf_{\varepsilon \to 0^+} \mathscr{F}_{\varepsilon,p}(u, A),$$

which is the sought estimate. □

3.2.2. *The general case.* We are now ready to discuss the lower bound in the general case of a function $u \in L^p(\Omega; \mathbb{R}^d)$. We will make use of the following observation (which seems to be originally due to E. Stein as mentioned in [**6, 17**]). We will denote by $(\psi_\eta)_\eta \subset C_c^\infty(\mathbb{R}^d)$ a family of non-negative smoothing kernels, with $\operatorname{supp} \psi_\eta \subset B_\eta(0)$ and $\int_{\mathbb{R}^d} \psi_\eta = 1$ for every $\eta > 0$. For every $u \in L^1(\Omega; \mathbb{R}^d)$ we define $u_\eta(x) := (u * \psi_\eta)(x)$, which is well-defined and smooth in the set $\Omega_\eta := \{x \in \Omega : d(x, \partial\Omega) > \eta\}$.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $u \in L^p(\Omega; \mathbb{R}^d)$. Let A be an open set with $A \Subset \Omega$. Then for every $p \in [1, \infty)$, for every $\varepsilon > 0$ and for every $0 < \eta < \frac{1}{2} \operatorname{dist}(A, \partial\Omega)$ it holds*

$$\mathscr{F}_{p,\varepsilon}(u_\eta, A) \le \mathscr{F}_{p,\varepsilon}(u, A_\eta),$$

where $A_\eta := A + B_\eta(0)$ is the open neighborhood of A of radius η.

Proof. The claim is a rather easy consequence of the convexity of $\mathscr{F}_{p,\varepsilon}$ which in turn follows by Jensen's inequality. Indeed, we have

$$\begin{aligned}
&\iint_{A\times A} |\langle u_\eta(x)-u_\eta(y), x-y\rangle|^p \frac{\rho_\varepsilon(x-y)}{|x-y|^{2p}}\,dxdy\\
&= \iint_{A\times A} \left|\left\langle \int_{\mathbb{R}^d} [u(x-z)-u(y-z)]\psi_\eta(z)\,dz, x-y\right\rangle\right|^p \frac{\rho_\varepsilon(x-y)}{|x-y|^{2p}}\,dxdy\\
&= \iint_{A\times A} \left|\int_{\mathbb{R}^d} \langle u(x-z)-u(y-z), x-y\rangle\,\psi_\eta(z)\,dz\right|^p \frac{\rho_\varepsilon(x-y)}{|x-y|^{2p}}\,dxdy\\
&\le \iint_{A\times A}\int_{\mathbb{R}^d} |\langle u(x-z)-u(y-z), x-y\rangle|^p\,\psi_\eta(z)\,dz\,\frac{\rho_\varepsilon(x-y)}{|x-y|^{2p}}\,dxdy
\end{aligned}$$

where the last inequality indeed follows by Jensen's inequality with respect to the probability measure $\psi_\eta\mathscr{L}^d$. An application of Tonelli's theorem and a change of variables yield

$$\begin{aligned}
&\int_{\mathbb{R}^d}\iint_{A\times A} |\langle u(x-z)-u(y-z), x-y\rangle|^p\,\frac{\rho_\varepsilon(x-y)}{|x-y|^{2p}}\,dxdy\,\psi_\eta(z)\,dz\\
&\quad\le \int_{\mathbb{R}^d}\iint_{A_\eta\times A_\eta} |\langle u(w)-u(s), w-s\rangle|^p\,\frac{\rho_\varepsilon(w-s)}{|w-s|^{2p}}\,dwds\,\psi_\eta(z)\,dz
\end{aligned}$$

and, recalling that $\|\psi_\eta\|_{L^1}=1$, this concludes the proof. □

We are now ready to present the proof of the lower bound.

Proof of the lower bound. Let $u\in L^p(\Omega;\mathbb{R}^d)$. It is not restrictive to assume that

$$L := \liminf_{\varepsilon\to 0^+} \mathscr{F}_{p,\varepsilon}(u,\Omega)$$

is finite (otherwise there is nothing to prove). Let $A \Subset \Omega$ be an open set whose closure is compact and contained in Ω. Consider convolution kernels $(\psi_\eta)_\eta$ as above. For sufficiently small $\eta>0$, the associated smooth approximations u_η are well defined and smooth in $A\subset\Omega_\eta\subset\Omega$. We can therefore use Lemma 3.3 and obtain

$$[Eu_\eta]_p(A)^p \le \liminf_{\varepsilon\to 0^+} \mathscr{F}_{p,\varepsilon}(u_\eta, A). \tag{3.10}$$

In turn, by Lemma 3.4, we get

$$\mathscr{F}_{p,\varepsilon}(u_\eta, A) \le \mathscr{F}_{p,\varepsilon}(u, A_\eta) \le \mathscr{F}_{p,\varepsilon}(u,\Omega), \tag{3.11}$$

where we have used the fact that $A_\eta = A + B_\eta \subset \Omega$ for sufficiently small $\eta>0$. Passing to the limit in $\varepsilon\to 0^+$ in (3.11) and combining it with (3.10), we obtain

$$C_{d,p}|Eu_\eta|(A)^p \le [Eu_\eta]_p(A)^p \le L.$$

In particular, the family $\{u_\eta\}_\eta$ has L^p-bounded symmetric gradients on A and hence by the lower semicontinuity of the classical total variation we conclude that $Eu\in\mathcal{M}_b(A;M^{d\times d}_{\mathrm{sym}})$. Now, we may use the lower semicontinuity

of the $\mathbf{Q}_p$-variation (cf. the argument given in p. 10) to deduce that

$$\begin{aligned} [Eu]_p(A)^p &\leq L \text{ if } p > 1, \\ [Eu](A) &\leq L \text{ if } p = 1. \end{aligned} \tag{3.12}$$

Since Ω connected and open, it can be written as the monotone limit $A_j \uparrow \Omega$ of connected open sets, whose closure is compact and contained in Ω. Exploiting again the continuity of positive measures along monotone sequences [**2, Remark 1.3**], we may then pass to the limit as $A_j \uparrow \Omega$ in (3.12) and obtain the desired lower bound

$$\begin{aligned} [Eu]_p(\Omega)^p &\leq L \text{ if } p > 1, \\ [Eu](\Omega) &\leq L \text{ if } p = 1. \end{aligned} \tag{3.13}$$

This implies that $u \in BD(\Omega)$ if $p = 1$ or $u \in LD^p(\Omega)$ if $p > 1$ and concludes the proof. □

3.3. **Proof of Theorems 1.1 and 1.4.** Let $p \in [1, \infty)$. The existence and characterization of the extended limits

$$\lim_{\varepsilon \to 0^+} \mathscr{F}_{p,\varepsilon}(u, \Omega) \in [0, \infty],$$

for arbitrary functions $u \in L^p(\Omega; \mathbb{R}^d)$, follows directly from the lower and upper bounds.

3.4. **Proof of Corollaries 1.2 and 1.8.** The proof of these two convergence results follows directly from Corollary 3.2 and Theorems 1.1, 1.4. Indeed, if $\mu = g\mathbf{Q}_p(Eu)$ is the measure from Corollary 3.2, then the characterizations imply that (in both cases) $g \equiv 1$. Since Corollary 3.2 is valid for arbitrary subsequences of $(\mu_{p,\varepsilon})$, this shows that

$$\mu_{p,\varepsilon} \rightharpoonup \mathbf{Q}_p(Eu) \quad \text{in } L^p(\Omega) \text{ for } p \in (1, \infty),$$

and

$$\mu_{1,\varepsilon} \mathscr{L}^d \overset{*}{\rightharpoonup} [Eu] \quad \text{in } \mathcal{M}(\Omega).$$

Moreover, from Theorems 1.1 and 1.4 it follows that

$$\|\mu_{p,\varepsilon}\|_{L^p(\Omega)} \to \|\mathbf{Q}_p\|_{L^p(\Omega)} \quad \text{for } p > 1,$$

and

$$|\mu_{1,\varepsilon}\mathscr{L}^d|(\Omega) \to [Eu](\Omega),$$

for the extended values of the norm. Hence, the equivalences in Corollary 1.2 follow directly from the convergence of the L^p-norms, while Corollary 1.8 follows verbatim from the aforementioned convergences when $p = 1$.

4. A functional for the singular part

We now show that subtracting the first-order term of the approximate Taylor polynomial of u to our symmetric difference quotient leads to a limiting functional representation of the singular part $E^s u$. By appealing to similar ideas as the ones introduced in the previous section via the extension operator $T\colon BD(\Omega) \to BD(\mathbb{R}^d)$, it suffices to prove the following proposition:

Proposition 4.1. *Let $u \in BD(\mathbb{R}^d;\mathbb{R}^d)$ and let $\nabla u(x) \in M^{d\times d}$ denote the approximate differential of u at a point x, which exists $\mathscr{L}^d$-almost everywhere in $\mathbb{R}^d$. Then, the extended limit*

$$\lim_{\varepsilon\to 0^+} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\langle u(x)-u(y)-\mathcal{E}u(x)[y-x], y-x\rangle|}{|x-y|^2}\,\rho_\varepsilon(x-y)\,dx\,dy$$

exists and equals $[E^s u](\mathbb{R}^d)$.

Proof. We follow closely the ideas contained in [**17, Sect. 2**]. From the Radon–Nikodým–Lebesgue decomposition of Eu, we find that (with the same notation of the previous section)

$$E(u_\eta) = (\mathcal{E}u)_\eta + (E^s u)_\eta. \tag{4.1}$$

Define the energies

$$\mathcal{R}^\varepsilon u(x) := \int_{\mathbb{R}^d} \frac{|\langle u(y)-u(x)-\mathcal{E}u(x)[y-x], y-x\rangle|}{|x-y|^2}\,\rho_\varepsilon(y-x)\,dy.$$

Appealing to similar Jensen inequalities as the ones in the previous section, we find

$$\int_{\mathbb{R}^d} \frac{|\langle u_\eta(x+h)-u_\eta(x)-(\mathcal{E}u)_\eta(x)[h], h\rangle|}{|h|^2}\,\rho_\varepsilon(h)\,dh \le (\mathcal{R}^\varepsilon u)_\eta(x)$$

and hence Young's inequality gives

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\langle u_\eta(y)-u_\eta(x)-(\mathcal{E}u)_\eta(x)[y-x], y-x\rangle|}{|x-y|^2}\,\rho_\varepsilon(y-x)\,dy \le \|\mathcal{R}^\varepsilon u\|_{L^1}. \tag{4.2}$$

In light of (4.1), the triangle inequality and the identity

$$\begin{aligned}\mathbf{Q}_1(A) &= \int_0^\infty \fint_{\partial B_1} |\langle A\omega,\omega\rangle|\,dS(\omega)\,\hat{\rho}_\varepsilon(r)|\partial B_1| r^{d-1}\,dr \\ &= \int_{\mathbb{R}^d} \frac{|\langle Ah,h\rangle|}{|h|^2}\,\rho_\varepsilon(h)\,dh, \qquad \varepsilon>0,\end{aligned}$$

we deduce that $\mathcal{R}^\varepsilon u_\eta(x)$ is a bound for the energy $(\mathrm{J})_\varepsilon(x)$, defined by

$$\left|\int_{\mathbb{R}^d} \frac{|\langle u_\eta(x+h)-u_\eta(x)-(\mathcal{E}u)_\eta(x)[h], h\rangle|}{|h|^2}\,\rho_\varepsilon(h)\,dh - \mathbf{Q}_1((E^s u)_\eta(x))\right|.$$

Applying once more the triangle inequality and integrating over x, we deduce from the previous bound, (4.2) and Young's convolution inequality that

$$\begin{aligned}[(E^s u)_\eta](\mathbb{R}^d) &= \int_{\mathbb{R}^d} \mathbf{Q}_1((E^s u)_\eta(x))\,dx \\ &\le \int_{\mathbb{R}^d} (\mathrm{J})_\varepsilon(x)\,dx + \int_{\mathbb{R}^d} (\mathcal{R}^\varepsilon u)_\eta(x)\,dx \\ &\le \|\mathcal{R}^\varepsilon u_\eta\|_{L^1} + \|\mathcal{R}^\varepsilon u\|_{L^1}.\end{aligned}$$

Now, since $u_\eta \in (C^\infty \cap W^{1,1})(\Omega)$, then the first term on the right-hand side vanishes as $\varepsilon \to 0^+$. On the other hand, by Reshetnyak's continuity theorem and the convergence $(E^s u)_\eta \overset{*}{\rightharpoonup} E^s u$, we find that

$$[E^s u](\mathbb{R}^d) \le \liminf_{\varepsilon \to 0^+} \|\mathcal{R}^\varepsilon u\|_{L^1}.$$

This proves the lower bound.

For the upper bound, we continue following [**17**] and we claim that for every non-negative bounded continuous function $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ it holds

$$\int_{\mathbb{R}^d} \mathcal{R}^\varepsilon u(x)\varphi(x)\,dx \le \int_{\mathbb{R}^d} \varphi d[E^s u] + (\mathrm{II})_\varepsilon + (\mathrm{III})_\varepsilon \tag{4.3}$$

where

$$(\mathrm{II})_\varepsilon := \int_0^1 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\varphi(x+th) - \varphi(x)|\,\rho_\varepsilon(h)\,dh \right) d|E^s u|(x)\,dt$$

and

$$(\mathrm{III})_\varepsilon := \|\varphi\|_\infty \int_0^1 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\mathcal{E}u(x+th) - \mathcal{E}u(x)|\,dx \right) \rho_\varepsilon(h)\,dh\,dt.$$

The conclusion will then follow observing that $(\mathrm{II})_\varepsilon \to 0^+$ (because φ is bounded and continuous) and $(\mathrm{III})_\varepsilon \to 0$ as $\varepsilon \to 0$ (because $\mathcal{E}u$ is an L^1-function). To establish (4.3), we rely once again on an approximation argument: by the Fundamental Theorem of Calculus and (4.1) we infer

$$\begin{aligned}|\langle u_\eta(x+h) - u_\eta(x) - (\mathcal{E}u)_\eta(x)\cdot h, h\rangle| \\ \le \int_0^1 |\langle (E^s u)_\eta(x+th)\cdot h, h\rangle|\,dt \\ + \int_0^1 |(\mathcal{E}u)_\eta(x+th) - (\mathcal{E}u)_\eta(x)||h|^2\,dt.\end{aligned} \tag{4.4}$$

Now, exactly as in [17], we observe that, adding and subtracting the term $\varphi(x+th)$ and changing variables $z := x+th$, one has

$$\begin{aligned}\int_{\mathbb{R}^d} &|\langle (E^s u)_\eta(x+th)\cdot h, h\rangle|\varphi(x)\,dx \\ &\le \int_{\mathbb{R}^d} |\langle (E^s u)_\eta(z)h, h\rangle|\varphi(z)\,dz \\ &\quad + |h|^2 \int_{\mathbb{R}^d} |(E^s u)_\eta(z)||\varphi(z)-\varphi(z-th)|\,dz.\end{aligned}$$

In conclusion, by Fubini Theorem

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 \frac{|\langle (E^s u)_\eta(x+th)\cdot h, h\rangle|}{|h|^2} dt\,\rho_\varepsilon(h)\,dh\,\varphi(x)dx \\ &\qquad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\langle (E^s u)_\eta(z)h, h\rangle|}{|h|^2}\rho_\varepsilon(h)\,dh\,\varphi(z)\,dz + (\mathrm{II})_{\varepsilon,\eta} \\ &\qquad = \int_{\mathbb{R}^d} \varphi(z)\mathbf{Q}_1((E^s u)_\eta(z))\,dz + (\mathrm{II})_{\varepsilon,\eta}\end{aligned} \tag{4.5}$$

where we have denoted by

$$(\mathrm{II})_{\varepsilon,\eta} := \int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi(z)-\varphi(z-th)|\,|(E^s u)_\eta(z)|\,dz\,\rho_\varepsilon(h)\,dh\,dt.$$

Observe that due to Jensen's inequality we have the point-wise inequalities $|(E^s u)_\eta| \le |E^s u| * \psi_\eta$ and $[(E^s u)_\eta] \le [E^s u] * \psi_\eta$. In particular, $(\mathrm{II})_{\varepsilon,\eta} \le (\mathrm{II})_\varepsilon$ which, combined with (4.5) and (4.4), yields (4.3) and the proof is complete. □

Using Proposition 4.1, we can finally prove Theorem 1.11:

Proof of Theorem 1.11. The proof is indeed a simple adaptation of the original argument in [16]. First, we observe that it is enough to prove the theorem in the case when $f^\infty = 0$ (the general case follows from applying this special case to the function $\tilde{f}(t) := f(t) - tf^\infty + C$, where C is a sufficiently large constant). In light of the previous sections (and of the usual considerations about extension operators), it is clear that we may also assume without loss of generality that $\Omega = \mathbb{R}^d$. We thus have to prove that for every $u \in BD(\mathbb{R}^d)$ it holds

$$\lim_{j\to\infty} \iint_{\mathbb{R}^d\times\mathbb{R}^d} f\left(\frac{|\langle u(x)-u(y), x-y\rangle|}{|x-y|^2}\right)\rho_{\varepsilon_j}(x-y)\,dx\,dy = \int_{\mathbb{R}^d} \mathbf{Q}_f(\mathcal{E}u)\,dx.$$

Let us introduce the absolutely continuous measures

$$\nu_j(x) := \int_{\mathbb{R}^d} f\left(\frac{|\langle u(x+h)-u(x), h\rangle|}{|h|^2}\right)\rho_{\varepsilon_j}(h)\,dh, \qquad x\in\mathbb{R}^d,$$

which are easily seen to be locally equi-bounded. Let ν be any weak limit of ν_j in the sense of measures. The goal is to characterize the singular

part ν^s (as the zero measure) and the absolutely continuous part ν^{ac} (as $\mathbf{Q}_f(\mathcal{E}u)\mathscr{L}^d$).

In view of (1.9) and our assumption that $f^\infty = 0$, for each $\delta > 0$ we can find a constant $C_\delta > 0$ such that

$$f(s) \leq \delta s + C_\delta \quad \forall s \geq 0.$$

From this and Corollary 1.8 it follows that

$$\nu \leq \delta[Eu] + C_\delta \mathscr{L}^d$$

as measures. In particular, we conclude that

$$\nu^s \leq \delta[E^s u]$$

as measures and hence, letting $\delta \downarrow 0$, we have $\nu^s \equiv 0$.

It remains to characterize ν^{ac}. In order to do this, we need to exploit the sublinear growth of the integrand to get the following estimate (see [**16, Formula (83), pag. 248**]): for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(s) - f(t)| \leq C_\delta |s - t| + \delta(1 + s + t) \qquad \forall s, t \geq 0.$$

Now we set

$$A_j(U) := \int_U \int_{\mathbb{R}^d} \left| f\left(\frac{|\langle u(x+h) - u(x), h\rangle|}{|h|^2}\right) - f\left(\frac{|\langle \mathcal{E}u(x)[h], h\rangle|}{|h|^2}\right)\right| \rho_{\varepsilon_j}(h)\, dh\, dx$$

so that we can bound

$$A_j(U) \leq C_\delta A_j^1(U) + \delta A_j^2(U)$$

where

$$A_j^1(U) := \int_U \int_{\mathbb{R}^d} \frac{|\langle u(x+h) - u(x) - \mathcal{E}u(x)[h], h\rangle|}{|h|^2} \rho_{\varepsilon_j}(h)\, dh\, dx$$

and

$$A_j^2(U) := \int_U \int_{\mathbb{R}^d} \left(1 + \frac{|\langle u(x+h) - u(x), h\rangle|}{|h|^2} + |\mathcal{E}u(x)|\right) \rho_{\varepsilon_j}(h)\, dh\, dx,$$

for all Borel sets $U \subset \mathbb{R}^d$. By Proposition 4.1 we know that $A_j^1(U)$ has a limsup as $j \to \infty$, which is bounded by $[E^s u](\mathbb{R}^d)$, while $A_j^2(U)$ is easily seen to be equi-bounded in j (using again Corollary 1.8). Letting $\delta \downarrow 0$, we deduce that

$$\nu^{ac} = \lim_{j\to\infty} \left[\int_{\mathbb{R}^d} f\left(\frac{|\langle \mathcal{E}u(x)[h], h\rangle|}{|h|^2}\right) \rho_{\varepsilon_j}(h)\, dh\right] \mathscr{L}^d.$$

In particular, for every open and bounded set $U \subset \mathbb{R}^d$ it holds

$$\begin{aligned}\nu^{ac}(U) &= \lim_{j\to\infty} \int_U \int_{\mathbb{R}^d} f\left(\frac{|\langle \mathcal{E}u(x)[h], h\rangle|}{|h|^2}\right) \rho_{\varepsilon_j}(h)\, dh\, dx \\ &= \lim_{j\to\infty} \int_U \int_0^\infty \mathbf{Q}_f(\mathcal{E}u(x))\, \hat{\rho}_{\varepsilon_j}(s) n\omega_n s^{n-1}\, ds\, dx = \int_U \mathbf{Q}_f(\mathcal{E}u)\, dx.\end{aligned}$$

This precisely means that $\nu^{ac} = \mathbf{Q}_f(\mathcal{E}u)\mathscr{L}^d$ in the sense of measures, as desired. □

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