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The infinite-horizon investment–consumption problem for Epstein–Zin stochastic differential utility. I: Foundations

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Abstract

The goal of this article is to provide a detailed introduction to infinite-horizon investment–consumption problems for agents with preferences described by Epstein–Zin (EZ) stochastic differential utility (SDU). In the setting of a Black–Scholes–Merton market, we seek to describe all parameter combinations that lead to a well-founded problem in the sense that the problem is not just mathematically well posed, but the solution is also economically meaningful. The key idea is to consider a novel and slightly different description of EZ SDU under which the aggregator has only one sign. This new formulation clearly highlights the necessity for the coefficients of relative risk aversion and of elasticity of intertemporal complementarity (the reciprocal of the coefficient of intertemporal substitution) to lie on the same side of unity.

Keywords Epstein–Zin stochastic differential utility · Lifetime investment and consumption · Backward stochastic differential equations · Discounted aggregator

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1 Introduction

This paper is the first of a trio of papers by the same authors; see also Herdegen et al. [11, 10]. The collective goal of these papers is to undertake a rigorous study of a Merton-style infinite-horizon investment–consumption problem in the setting of Epstein–Zin (EZ) stochastic differential utility (SDU). In particular, the aim is to study when the problem is mathematically *well posed* and economically *well founded*, and if so, to derive the candidate optimal strategy, the candidate value function and the candidate optimal utility process (this paper), and then to prove *existence* and *uniqueness* (where possible) of the corresponding utility processes and to *verify* that the candidate optimal strategy, value function and utility process are indeed optimal in a large class of admissible investment–consumption strategies; see [11, 10].

Within this general goal, the contributions of this first paper are partly foundational and partly didactic. As already alluded to above, one issue in a stochastic control problem such as an investment–consumption problem is to decide when the problem is mathematically *well posed*: if the value function can become infinite, then it is generally not possible to discuss optimal strategies. In this paper, we also ask when the problem is economically *well founded*. For EZ SDU, the utility process associated to a consumption stream is given as the solution to a backward stochastic differential equation (BSDE), and we ask: *When does this solution have a sound economic interpretation*? We argue that for some parameter combinations, including some which are widely used in the literature, the utility process has the properties of a *utility bubble*. In these settings, the investment–consumption problem is ill founded.

In the economics literature, SDU was introduced by Duffie and Epstein [5] as the continuous-time analogue of recursive utility (see Epstein and Zin [8] and Weil [24]) and further developed by Duffie and Lions [6] as well as Schroder and Skiadas [22]. It can be viewed as an extension of classical additive utility. It is recognised as having the potential to explain several of the inconsistencies between the predictions of the Merton model and agent behaviour (for example the equity premium puzzle, see Mehra and Prescott [16]). However, with several honourable exceptions (including Kraft and Seifried [13], Seiferling and Seifried [23], Xing [25], Matoussi and Xing [15] and Melnyk et al. [17]), SDU has not been widely studied in the mathematical finance literature. Given the deep connections with many areas of modern probability theory (including for example BSDEs), this is in some ways surprising; but given the technical challenges involved, it is also understandable. We consider EZ SDU for infinite-horizon problems and give a clear interpretation of all the parameters, with a focus on the feasible ranges for these parameters. The fact that we concentrate on an infinite horizon brings several issues into focus. Over an infinite horizon, it is not possible to work backwards from the terminal date. Therefore, it is necessary to introduce some form of transversality condition as an alternative. Moreover, integrability (and uniform integrability) become much more significant challenges.

The conventional wisdom (see for example Duffie and Epstein [5] and Melnyk et al. [17]) is that the best technical solution to these challenges is to replace the infinite-horizon problem with a family of finite-horizon problems (but note that this is not the way in which the candidate solution is found). We take a different approach. Key to the definition of EZ SDU is an aggregator, and we introduce a slightly different aggregator to the one which is traditionally used in the literature. The key point is that our aggregator takes only *one sign*. When there exist utility processes associated with both our aggregator and the classical aggregator, then the utility processes agree. But crucially, any utility process associated to the traditional aggregator is also a utility process associated to our modified aggregator, whereas the converse is not true. Moreover, when specialised to the case of additive utility, our aggregator corresponds to the classical formulation of the Merton problem, whereas the traditional aggregator has a non-standard specification in this context.

Our reformulation of the problem brings significant new insights concerning the set of feasible parameters for the problem with EZ SDU preferences. In particular, we conclude that the coefficient R of relative risk aversion (RRA) and the coefficient S of elasticity of intertemporal complementarity (EIC) (the reciprocal of the coefficient of elasticity of intertemporal substitution, see Sect. 4 for details) should lie on the same side of unity, at least for infinite-horizon problems. (In the classical Merton problem for power utility, R and S are necessarily equal and therefore on the same side of unity.) This seems to be a new and significant finding. In particular, our results bring into doubt the conclusions of those parts of the literature which are in the setting of $R > 1 > S$. This includes for example the literature on long-run risk, which builds on the seminal paper by Bansal and Yaron [1]. We argue that the putative solutions which have been found previously in the literature (in the case when R and S are on opposite sides of unity) correspond to a bubble-like behaviour: the value associated with a consumption stream does not come from the utility of consumption in the short and medium term, but rather from a perceived and unrealisable value in the distant future.

While there is strong empirical evidence for the existence of asset price bubbles, supported by a large body of literature (both in finance and mathematical finance), this literature is underpinned by the assumption that bubbles are transitory and exceptions to the norm. (Seminal papers on asset price bubbles include Diba and Grossman [4], Scheinkmann and Xiong [19], Loewenstein and Willard [14] and Cox and Hobson [3]. See also the survey articles by Shiller [21], Scherbina and Schlusche [20] and Protter [18] and the references therein.) Indeed, the effect of a bubble is typically assessed relative to a more classical model in which assets are priced via their fundamentals, and any novel features are highlighted as the impact of the bubble. For this reason, it seems questionable to base long-run investment decisions on the *necessary* and *perpetual* existence of a bubble, at least without explicit recognition that a bubble is present in the setup and driving the conclusions.

The remainder of this paper is structured as follows. In Sects. 2 and 3, we review the classical investment–consumption problem for additive utility and then introduce the corresponding problem for stochastic differential utility (SDU). In Sect. 4, we introduce Epstein–Zin (EZ) SDU and carefully explain how the various parameters should be interpreted, and which parameter combinations lead to a well-founded problem. In Sect. 5, we embed EZ SDU within a Black–Scholes–Merton financial market and derive the candidate value function, utility process and optimal strategy. In Sects. 6 and 7, we compare our formulation with the conventional formulation which has been used heretofore in the literature. We believe that our formulation has significant advantages; first in that it contributes to the understanding of when the problem is ill founded, and second in that it makes it possible to optimise over *all* attainable consumption streams in Herdegen et al. [11], and not just a restricted subclass as in the extant literature.

2 Additive utility

Throughout, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and we assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial. Let $\mathscr{P}$ be the set of progressively measurable processes and $\mathscr{P}_+$, $\mathscr{P}_{++}$ the restrictions of $\mathscr{P}$ to processes that take nonnegative and strictly positive values, respectively. Moreover, denote by $\mathscr{S}$ the set of all semimartingales. We identify processes in $\mathscr{P}$ or $\mathscr{S}$ that agree up to indistinguishability.

Before introducing the notion of stochastic differential utility, we recall the definition of additive expected utility over an infinite horizon. We call $U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ a *utility function* if U is increasing and concave in its second argument, and we call C a *consumption stream* if $C \in \mathscr{P}_+$. The utility associated to a consumption stream is given by $J_U(C) = \mathbb{E}[\int_0^\infty U(t, C_t)\,\mathrm{d}t]$. The *value process* or, as it is called in the SDU literature, the *utility process* $V = V^C \in \mathscr{S}$ associated to the consumption stream C is defined by

$$V_t = V_t^C = \mathbb{E}\bigg[\int_t^\infty U(s, C_s)\,\mathrm{d}s \,\bigg|\, \mathcal{F}_t\bigg]. \tag{2.1}$$

Then $J_U(C) = V_0^C$. The goal is to maximise $J_U(C)$ over an appropriate space of consumption streams. A key example of a utility function is the discounted constant relative risk aversion (CRRA) utility function $U(t, c) = e^{-\delta t}\frac{c^{1-R}}{1-R}$. Under that utility, the utility process associated to C is given by

$$V_t = \mathbb{E}\bigg[\int_t^\infty e^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s \,\bigg|\, \mathcal{F}_t\bigg]. \tag{2.2}$$

It is well known that under CRRA preferences, the parameter R controls the agent's appetite for risk. In particular, since R is a measure of the concavity of the utility function $U(t, c) = e^{-\delta t}\frac{c^{1-R}}{1-R}$ (more precisely, $R = -c\frac{U'(t,c)}{U''(t,c)}$), R captures the agent's aversion to variation of consumption over $\omega \in \Omega$. It is also known, though perhaps less well known, that the parameter R also captures the agent's aversion to variation of consumption over time. (We justify and explain this fact when we study EZ SDU in Sect. 4.)

There is no economic or mathematical justification (beyond mathematical tractability) for restricting attention to preferences in which the same parameter governs preferences over both fluctuations of consumption across sample paths and fluctuations of consumption across time. One of the motivations behind the introduction of SDU is to allow a disentanglement of preferences over these two types of fluctuations of consumption.

3 Stochastic differential utility

Stochastic differential utility (SDU) is a generalisation of time-additive discounted expected utility and is designed to allow a separation of risk preferences from time preferences. The goal in this section is to explain how this statement should be interpreted.

Under discounted expected utility, the value or utility of a consumption stream is given by $J_U(C) = \mathbb{E}[\int_0^\infty U(t, C_t)\,\mathrm{d}t]$, and the value or utility process is given by $V_t = \mathbb{E}[\int_t^\infty U(s, C_s)\,\mathrm{d}s|\mathcal{F}_t]$. Under SDU, the function $U = U(s, C_s)$ is generalised to become an *aggregator* $g = g(s, C_s, V_s)$, and the stochastic differential utility process $V^C = (V_t^C)_{t\geq 0}$ associated to a consumption stream C solves (compare with (2.1))

$$V_t^C = \mathbb{E}\bigg[\int_t^\infty g(s, C_s, V_s^C)\,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg]. \tag{3.1}$$

This creates a feedback effect in which the value at time t may depend in a nonlinear way on the value at future times. This feature (for appropriate choices of the aggregator g) may lead to a separation of the two phenomena mentioned in the previous section: risk aversion and temporal variation aversion.

Note that if g takes positive and negative values, the conditional expectation on the right-hand side of (3.1) need not be well defined. With this in mind, we introduce the following definitions.

Definition 3.1 An *aggregator* is a function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. For $C \in \mathscr{P}_+$, define $\mathbb{I}(g, C) := \{V \in \mathscr{P} :\ \mathbb{E}[\int_0^\infty |g(s, C_s, V_s)|\,\mathrm{d}s] < \infty\}$. Further, let $\mathbb{UI}(g, C)$ be the set of elements of $\mathbb{I}(g, C)$ which are uniformly integrable. Then $V \in \mathbb{I}(g, C)$ is a *utility process* associated to the pair (g, C) if it has càdlàg paths and satisfies (3.1) for all $t \in [0, \infty)$.

Remark 3.2 All utility processes are special semimartingales, and bounded above in absolute value by a uniformly integrable martingale. In particular, if V is a utility process for the pair (g, C), then $V \in \mathbb{UI}(g, C)$. Indeed, let $M = (M_t)_{t\geq 0}$ be the (càdlàg) martingale defined by $M_t = \mathbb{E}[\int_0^\infty g(s, C_s, V_s)\,\mathrm{d}s\,|\,\mathcal{F}_t]$ and $A = (A_t)_{t\geq 0}$ the continuous adapted process given by $A_t = \int_0^t g(s, C_s, V_s)\,\mathrm{d}s$. Then we have $V = M - A \in \mathscr{S}$. Moreover, let $\tilde{M} = (\tilde{M}_t)_{t\geq 0}$ be the uniformly integrable martingale given by $\tilde{M}_t = \mathbb{E}[\int_0^\infty |g(s, C_s, V_s)|\,\mathrm{d}s\,|\,\mathcal{F}_t]$. Then for $t \geq 0$,

$$|V_t| \leq \mathbb{E}\bigg[\int_t^\infty |g(s, C_s, V_s)|\,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg] \leq \mathbb{E}\bigg[\int_0^\infty |g(s, C_s, V_s)|\,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg] = \tilde{M}_t.$$

This immediately also gives $V \in \mathbb{UI}(g, C)$.

Definition 3.3 A consumption stream $C \in \mathscr{P}_+$ is *g-evaluable* for an aggregator g if there exists a utility process $V \in \mathbb{I}(g, C)$ associated to the pair (g, C). The set of g-evaluable C is denoted by $\mathscr{E}(g)$. Furthermore, if the utility process is unique (up to indistinguishability), then C is called *g-uniquely evaluable*. The set of g-uniquely evaluable C is denoted by $\mathscr{E}_u(g)$.

Throughout the paper (with a few exceptions where we explicitly state otherwise), we only consider uniquely evaluable consumption streams. For such C, we may therefore define the *stochastic differential utility* of C and of an aggregator g by $J_g(C) := V_0^C$, where V^C satisfies (3.1).

4 Epstein–Zin stochastic differential utility

The *Epstein–Zin (EZ) aggregator* corresponding to a vector of parameters (b, δ, R, S) is the function $g_{\mathrm{EZ}} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{V}$ given by

$$g_{\mathrm{EZ}}(t, c, v) := be^{-\delta t} \frac{c^{1-S}}{1-S} \big((1-R)v\big)^{\frac{S-R}{1-R}}. \tag{4.1}$$

Here, $\mathbb{V} = (1-R)\overline{\mathbb{R}}_+$ denotes the domain of the EZ utility process and the parameters R and S both lie in $(0,1) \cup (1,\infty)$. Note that some care is required when $\frac{c^{1-S}}{1-S}, ((1-R)v)^{\frac{S-R}{1-R}} \in \{0, \infty\}$.

It is convenient to introduce the parameters $\vartheta := \frac{1-R}{1-S}$ and $\rho = \frac{S-R}{1-R} = \frac{\vartheta - 1}{\vartheta}$ so that (4.1) becomes

$$g_{\mathrm{EZ}}(t, c, v) = be^{-\delta t} \frac{c^{1-S}}{1-S} \big((1-R)v\big)^{\rho}. \tag{4.2}$$

The utility process $V^C = V = (V_t)_{t \geq 0}$ associated to the consumption C and aggregator g_{EZ} solves

$$V_t = \mathbb{E}\bigg[\int_t^\infty be^{-\delta s} \frac{C_s^{1-S}}{1-S} \big((1-R)V_s\big)^\rho \,\mathrm{d}s \,\bigg|\, \mathcal{F}_t \bigg]. \tag{4.3}$$

Standing Assumption 4.1 We assume $b > 0$, $\delta \in \mathbb{R}$ and that $R \neq S$ are both in $(0,\infty) \setminus \{1\}$.

Some comments on this standing assumption are in order.

Remark 4.2 (a) Positivity of the parameter b corresponds to monotone preferences which are increasing in consumption. Beyond that, b has no effect on preferences. To see this, suppose that V is a solution to (4.3) with $b = 1$. For arbitrary $d > 0$, it follows that $d^\vartheta V = (d^\vartheta V_t)_{t \geq 0}$ is a solution to (4.3) with $b = d$. Since preferences remain unchanged by a multiplicative scaling of the utility function, it does not matter which value of b we choose.

(b) The parameter δ is left unrestricted. Based on its interpretation as a discount factor, it is natural to expect δ to be positive. Notwithstanding, when EZ SDU is associated with a financial market model, a deterministic change of consumption units leads to a change in the value of δ and potentially to a change in sign; see Sect. 5.2. Since the choice of accounting units should be arbitrary, there is no economic or mathematical reason to require or expect that $\delta \geq 0$.

(c) The case $S = R$ (equivalently $\vartheta = 1$ or $\rho = 0$) corresponds to CRRA utility. We exclude this case as it has been extensively studied and is well understood. In addition to excluding $R = S$, we also exclude $R = 1$ and $S = 1$. Just as power law utility becomes logarithmic utility when $R = S = 1$, EZ SDU also changes form. The parameter combination when $S = 1$ is considered by Chacko and Viceira [2]. (It is less clear how to extend EZ SDU to the case $R = 1$.)

Remark 4.3 The expression in (4.2) is a reformulation of the classical EZ SDU. Other authors use the *difference* form aggregator g^{Δ}_{EZ} given by

$$g^{\Delta}_{\mathrm{EZ}}(c, v) := b\frac{c^{1-S}}{1-S}\big((1-R)v\big)^{\rho} - \delta\vartheta v. \tag{4.4}$$

When we want to emphasise the distinction between the two formulations, we call (4.2) the *discounted* form of EZ SDU. The naming is explained by the way that δ enters the aggregator (4.2) and (4.4), respectively. As might be expected, there is a very close relationship between solutions of the two different forms, and we discuss this further in Sect. 6. Note immediately, however, that the discounted form is easily recognised as the natural generalisation of CRRA utility as given in (2.2). Indeed, when $R = S$, we recover (2.2) from (4.2) instantly.

Let g_{EZ} be the aggregator in (4.2). We begin by trying to give interpretations of the various parameters and to show that—despite first impressions—R captures the agent's relative risk aversion (RRA), whereas S captures the agent's elasticity of intertemporal complementarity (EIC), which is the reciprocal of the elasticity of intertemporal substitution, or temporal variation aversion. In addition, δ represents the agent's subjective discount rate, and b is a scaling parameter which has no effect on the agent's preferences (as long as it is positive); see Remark 4.2. We have included b to facilitate comparison with other forms of Epstein–Zin SDU used in the literature, but it may be set to 1 without loss of generality (alternatively, it is sometimes set to δ). The Standing Assumption 4.1 restricts the sets of parameters for a rational agent.

4.1 Risk aversion and temporal variation aversion

We proceed to show via a pair of examples that the condition $R > 0$ corresponds to the agent being averse to (rather than seeking) variation of consumption over ω, and the condition $S > 0$ corresponds to the agent being averse to (rather than seeking) variation in consumption over time.

To this end, consider a consumption stream $C = (C_t)_{t\geq 0}$ of the form $C_t = Yc(t)$ where Y is nonnegative and $\mathscr{F}_{0+}$-measurable (for the exposition, we temporarily drop the assumption that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and instead assume $\mathscr{F}_t = \sigma(Y)$ for all $t > 0$), both Y and Y^{1-R} are integrable, and the deterministic function $c = (c(t))_{t\geq 0}$ is such that $e^{-\delta s}c(s)^{1-R}$ is Lebesgue-integrable at infinity. Then, since all uncertainty is resolved at $t = 0$, for $t > 0$, $V^C = V = (V(t))_{t>0}$ can be found by solving the ordinary differential equation

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} = -be^{-\delta t}Y^{1-S}\frac{c(t)^{1-S}}{1-S}\big((1-R)V(t)\big)^{\rho},$$

subject to $\lim_{t\to\infty} V(t) = 0$, where the (limit) boundary condition follows from taking limits in (4.3). Setting $W(t) = (1 - R)V(t)$ and dividing through by $W(t)^\rho$, we find (recall $\vartheta = \frac{1-R}{1-S} = \frac{1}{1-\rho}$)

$$\frac{1}{W(t)^\rho}\frac{\mathrm{d}W(t)}{\mathrm{d}t} = -be^{-\delta t}\vartheta Y^{1-S}c(t)^{1-S}, \qquad \lim_{t\to\infty} W(t) = 0. \tag{4.5}$$

Assuming that $\vartheta > 0$, a solution to (4.5) is

$$W(t) = \left(\int_t^\infty be^{-\delta s}Y^{1-S}c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta = Y^{1-R}\left(\int_t^\infty be^{-\delta s}c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta.$$

Therefore, for $t > 0$, a utility process $V = V^C$ associated to C is

$$V(t) = \frac{Y^{1-R}}{1-R}\left(b\int_t^\infty e^{-\delta s}c(s)^{1-S}\,\mathrm{d}s\right)^\vartheta. \tag{4.6}$$

To consider the time-dependence in more detail, assume (again temporarily and just for the purposes of exposition) that $\delta > 0$ and define a new (probability) measure $\mathbb{Q} = \mathbb{Q}_\delta$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ by

$$\mathbb{Q}_\delta[A] := \int_A \delta e^{-\delta t}\,\mathrm{d}t.$$

The choice of δ accounts for the agent's temporal preferences for consumption in the sense that the higher the value of δ, the greater the weighting on consumption which occurs earlier. Then (4.6) yields

$$V(0+) = \frac{Y^{1-R}}{1-R}\left(\frac{b}{\delta}\right)^\vartheta (\mathbb{E}^{\mathbb{Q}_\delta}[c^{1-S}])^\vartheta.$$

It remains to deal with the random level of consumption. We find

$$\begin{aligned} J_{\mathrm{gEZ}}(C) = V^C(0) &= \frac{\mathbb{E}[Y^{1-R}]}{1-R}\left(\frac{b}{\delta}\right)^\vartheta (\mathbb{E}^{\mathbb{Q}_\delta}[c^{1-S}])^\vartheta \\ &= \vartheta\left(\frac{b}{\delta}\right)^\vartheta \mathbb{E}[Y^{1-R}]\frac{(\mathbb{E}^{\mathbb{Q}_\delta}[c^{1-S}])^\vartheta}{1-S}. \end{aligned}$$

For this particularly simple consumption stream, it is clear that the agent's risk aversion is captured by the parameter R and the aversion to temporal uncertainty is captured by S. Looking at (4.3) or (4.6), one might expect that the risk aversion comes from the value of S, but contrary to naive intuition, this is not the case. This justifies considering S as the parameter governing aversion to variation over time. In the economics literature, S is called the coefficient of elasticity of intertemporal complementarity (EIC), the reciprocal of the coefficient of elasticity of intertemporal substitution. Furthermore, in each case, Jensen's inequality gives that the agent is averse to fluctuations of consumption with respect to both randomness and time.

Note that if $(1 - R)V_t < 0$, the integrand on the right-hand side of (4.3) is ill defined for non-integer ρ. This justifies the choice $\mathbb{V} = (1 - R)\mathbb{R}_+$. Further, the integrand is either positive ($S < 1$) or negative ($S > 1$). It is therefore necessary to impose a link between the coefficient R of RRA and the coefficient S of EIC to ensure agreement in the sign of the left-hand side of (4.3) and the right-hand side.

Theorem 4.4 *For EZ SDU over an infinite horizon with aggregator given by* (4.2), *we must have* $\vartheta = \frac{1-R}{1-S} > 0$ *for there to exist solutions to* (4.3).

The condition $\vartheta > 0$, or equivalently $\rho \in (-\infty, 1)$, means that R and S are either both greater than unity or both smaller than unity.

Remark 4.5 In the finite-horizon problem, the parity issue can be overcome by adding a bequest function so that (4.3) is replaced by

$$V_t = \mathbb{E}\bigg[\int_t^T b e^{-\delta s} \frac{C_s^{1-S}}{1-S} \big((1-R)V\big)^{\rho} \, \mathrm{d}s + e^{-\delta T} \frac{B(X_T)}{1-R} \bigg| \mathcal{F}_t \bigg],$$

where $B : \mathbb{R}_+ \to \mathbb{R}_+$ assigns a value to terminal wealth. But even over a finite horizon, this leads to conceptual issues; for example, when $S < 1 < R$, the utility process can be negative at time t even though the term corresponding to consumption over (t, T) is everywhere positive, because this positive term can be outweighed by the contribution from the bequest. Moreover, if we let the horizon tend to infinity, the problem becomes even more stark—in order to outweigh the increasing contribution from consumption (as T increases), the contribution from the bequest must also grow, and must become more (not less) influential as the horizon increases. In Sect. 6.2, we argue that in the limit $T \nearrow \infty$, we end up with a bubble-like behaviour which cannot be justified economically, and which is not consistent with any notion of transversality. This further justifies the requirement $\vartheta > 0$.

5 Optimal investment and consumption in a Black–Scholes–Merton financial market

5.1 The financial market and attainable consumption streams

The Black–Scholes–Merton financial market consists of a risk-free asset with interest rate $r \in \mathbb{R}$, whose price process $S^0 = (S_t^0)_{t\geq 0}$ satisfies $S_t^0 = \exp(rt)$, together with a risky asset whose price process $S^1 = (S_t^1)_{t\geq 0}$ follows a geometric Brownian motion with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$, and whose initial value is $S_0^1 = s_0^1 > 0$. So $S_t^1 = s_0^1 \exp(\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t)$, where $B = (B_t)_{t\geq 0}$ denotes a Brownian motion.

The agent optimises over the control variables *the proportion of wealth invested in each asset* and the *rate of consumption*. Let Π_t represent the proportion of wealth invested in the risky asset at time t and $\Pi_t^0 = 1 - \Pi_t$ the proportion of wealth held in the riskless asset at time t. Further, let C_t denote the rate of consumption at time t. It then follows that the wealth process $X = (X_t)_{t\geq 0}$ satisfies the SDE

$$\mathrm{d}X_t = X_t \Pi_t \sigma \, \mathrm{d}B_t + \Big(X_t\big(r + \Pi_t(\mu - r)\big) - C_t\Big) \, \mathrm{d}t \tag{5.1}$$

with initial condition $X_0 = x$, where x is the initial wealth. Let $\lambda := \frac{\mu - r}{\sigma}$ be the Sharpe ratio of the risky asset.

Definition 5.1 Given $x > 0$, an *admissible investment–consumption strategy* is a pair $(\Pi, C) = (\Pi_t, C_t)_{t \geq 0}$ of progressively measurable processes, where Π is real-valued and C is nonnegative, such that the SDE (5.1) has a unique strong solution $X^{x,\Pi,C}$ that is $\mathbb{P}$-a.s. nonnegative. We denote the set of admissible investment–consumption strategies for $x > 0$ by $\mathscr{A}(x; r, \mu, \sigma)$.

Note that the criterion (4.3) by which an admissible investment–consumption stream is evaluated only depends upon the consumption stream and not upon the investment in the financial asset. This motivates the following definition.

Definition 5.2 A consumption stream $C \in \mathscr{P}_+$ is called *attainable* for initial wealth $x > 0$ if there exists a progressively measurable process $\Pi = (\Pi_t)_{t \geq 0}$ such that (Π, C) is an admissible investment–consumption strategy. Denote the set of attainable consumption streams for $x > 0$ by $\mathscr{C}(x; r, \mu, \sigma)$.

When it is clear which financial market we are considering, we simplify the notation and write $\mathscr{A}(x) = \mathscr{A}(x; r, \mu, \sigma)$ and $\mathscr{C}(x) = \mathscr{C}(x; r, \mu, \sigma)$.

The goal of an agent with Epstein–Zin stochastic differential utility preferences is to maximise $J_{g_{\mathrm{EZ}}}(C)$ over attainable consumption streams. However, $J_{g_{\mathrm{EZ}}}(C)$ is currently only defined for $C \in \mathcal{E}_u(g_{\mathrm{EZ}})$, and therefore we can currently only optimise over uniquely evaluable consumption streams. Thus, we seek to find

$$V^*_{\mathscr{E}_u(g_{\mathrm{EZ}})}(x) = \sup_{C \in \mathscr{C}(x) \cap \mathscr{E}_u(g_{\mathrm{EZ}})} V_0^C = \sup_{C \in \mathscr{C}(x) \cap \mathscr{E}_u(g_{\mathrm{EZ}})} J_{g_{\mathrm{EZ}}}(C). \tag{5.2}$$

This is very restrictive. For $\vartheta > 1$, we have $\mathscr{E}_u(g_{\mathrm{EZ}}) = \{0\}$ because $V \equiv 0$ is always a solution to (4.3), and so the problem (5.2) is meaningless. Further, even when $\vartheta \in (0, 1)$, there are many attainable consumptions streams which are not evaluable so that we currently cannot assign them a utility. For example, when $S > 1$, the zero consumption stream is not evaluable. Since it might reasonably be argued that zero consumption is clearly suboptimal (and when $S > 1$ should give a utility process with negative infinite utility), we should like to eliminate this choice of consumption stream because it is suboptimal, not because we cannot evaluate it. The same applies to other non-evaluable consumption streams. Ideally, we should like *every* attainable consumption stream to be considered, not just the 'nice' ones for which we can define a unique utility process. For $\vartheta \in (0, 1)$, this problem is considered in Herdegen et al. [11].

5.2 Changes of numéraire

One apparent advantage of the difference form g^{Δ}_{EZ} of the EZ SDU aggregator in (4.4) over the discounted form g_{EZ} in (4.2) is that g^{Δ}_{EZ}, unlike g_{EZ}, has no explicit time-dependence, i.e., $g^{\Delta}_{\mathrm{EZ}} = g^{\Delta}_{\mathrm{EZ}}(c, v)$ whereas $g_{\mathrm{EZ}} = g_{\mathrm{EZ}}(t, c, v)$. However, when we consider EZ SDU in the constant parameter Black–Scholes–Merton model, a simple change of accounting unit leads to a modification of the discount factor δ, but leaves the problem otherwise unchanged. It follows that by an appropriate choice of units, we can switch to a coordinate system in which the aggregator becomes time-independent. The change of accounting units has an effect upon the financial market model, but it remains a Black–Scholes–Merton financial market, albeit with modified interest rate and market drift.

Let C be a consumption stream with utility process V for g_{EZ}. Let $\chi \in \mathbb{R}$ and define the *discounted consumption stream* $\tilde{C}$ by $\tilde{C}_t = e^{-\chi t} C_t$. Then V satisfies

$$V_t = \mathbb{E}\bigg[\int_t^\infty b e^{-\delta s} \frac{C_s^{1-S}}{1-S} \big((1-R)V_s\big)^\rho \, \mathrm{d}s \,\bigg|\, \mathcal{F}_t \bigg]$$
$$= \mathbb{E}\bigg[\int_t^\infty b e^{-(\delta - \chi(1-S))t} \frac{\tilde{C}_s^{1-S}}{1-S} \big((1-R)V_s\big)^\rho \, \mathrm{d}s \,\bigg|\, \mathcal{F}_t \bigg].$$

Thus V is the utility process for $\tilde{C}$ and aggregator $g_{\chi,\mathrm{EZ}}$ defined by

$$g_{\chi,\mathrm{EZ}}(t,c,v) := b e^{-(\delta - \chi(1-S))t} \frac{c^{1-S}}{1-S} \big((1-R)v\big)^\rho.$$

Choosing $\chi = \frac{\delta}{1-S}$, we find that V is the utility process for the time-independent aggregator

$$f_{\mathrm{EZ}} = f_{\mathrm{EZ}}(c,v) = g_{\chi,\mathrm{EZ}}(t,c,v) = b \frac{c^{1-S}}{1-S} \big((1-R)v\big)^\rho.$$

Furthermore, we observe that we have $V \in \mathbb{I}(f_{\mathrm{EZ}}, \tilde{C} = (C_t e^{-\frac{\delta}{1-S}t})_{t\geq 0})$ if and only if $V \in \mathbb{I}(g_{\mathrm{EZ}}, C) = \mathbb{I}(g_{0,\mathrm{EZ}}, C)$ and $\tilde{C} \in \mathscr{E}_u(f_{\mathrm{EZ}})$ if and only if $C \in \mathscr{E}_u(g_{\mathrm{EZ}})$.

If we consider the discounted wealth process $\tilde{X}_t^{\Pi,\tilde{C}} := e^{-\frac{\delta}{1-S}t} X_t^{\Pi,C}$, then by applying Itô's lemma, we find that with $\tilde{r} = r - \frac{\delta}{1-S}$ and $\tilde{\mu} = \mu - \frac{\delta}{1-S}$,

$$d\tilde{X}_t^{\Pi,\tilde{C}} = \quad \tilde{X}_t^{\Pi,\tilde{C}} \Pi_t \sigma \, \mathrm{d}B_t + \Big(\tilde{X}_t^{\Pi,\tilde{C}} \big(\tilde{r} + \Pi_t(\tilde{\mu} - \tilde{r})\big) - \tilde{C}_t \Big) \mathrm{d}t, \qquad \tilde{X}_0^{\Pi,\tilde{C}} = x.$$

This means that our control problem (5.2) admits the equivalent formulation

$$V^*_{\mathscr{E}_u(g_{\mathrm{EZ}})}(x) = \sup_{C \in \mathscr{C}(x;r,\mu,\sigma) \cap \mathscr{E}_u(g_{\mathrm{EZ}})} V_0^{C,g_{\mathrm{EZ}}}$$
$$= \sup_{\tilde{C} \in \mathscr{C}(x;\tilde{r},\tilde{\mu},\sigma) \cap \mathscr{E}_u(f_{\mathrm{EZ}})} V_0^{\tilde{C},f_{\mathrm{EZ}}} = V^*_{\mathscr{E}_u(f_{\mathrm{EZ}})}(x).$$

In particular, by an appropriate change of accounting units, the problem for EZ SDU in discounted form reduces to an equivalent form with no discounting. This simplification result is used extensively in Herdegen et al. [11, 10]. In the following, however, our goal is to compare and contrast the discounted and the difference forms of the aggregator. For this reason, we continue to allow δ to be any real number.

5.3 The candidate optimal strategy

Suppose now $\vartheta > 0$. We seek to heuristically find an admissible (and uniquely evaluable) consumption stream C that maximises the value of V_0^C, where

$$V_t^C = \mathbb{E}\bigg[\int_t^\infty be^{-\delta s}\frac{C_s^{1-S}}{1-S}\big((1-R)V_s^C\big)^\rho \,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg]. \tag{5.3}$$

As in the Merton problem with CRRA utility, it is reasonable to guess that the optimal strategy is to invest a constant proportion of wealth in the risky asset and to consume a constant proportion of wealth. Consider the investment–consumption strategy $\Pi \equiv \pi \in \mathbb{R}$ and $C = \xi X$ for $\xi \in \mathbb{R}_{++}$. Then, solving (5.1), the wealth process $X^{x,\pi,\xi} = X = (X_t)_{t\geq 0}$ is given by

$$X_t = x\exp\bigg(\pi\sigma B_t + \Big(r + \pi(\mu - r) - \xi - \frac{\pi^2\sigma^2}{2}\Big)t\bigg).$$

Using that $\mu - r = \sigma\lambda$, we obtain for $t \geq 0$ that

$$X_t^{1-R} = x^{1-R}\exp\bigg(\pi\sigma(1-R)B_t + (1-R)\Big(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}\Big)t\bigg). \tag{5.4}$$

Now consider a value process of the form $V_t = V(t, X_t) = Ae^{-\beta t}\frac{X_t^{1-R}}{1-R}$ for some constant β to be determined. By substituting this expression into (5.3) and using $1 - S + \rho(1-R) = 1 - R$, we have

$$\begin{aligned} V_t &= \mathbb{E}\bigg[\int_t^\infty be^{-\delta s}\frac{(\xi X_s)^{1-S}}{1-S}(Ae^{-\beta s}X_s^{1-R})^\rho \,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg] \\ &= bA^\rho\frac{\xi^{1-S}}{1-S}\mathbb{E}\bigg[\int_t^\infty e^{-(\delta+\beta\rho)s}X_s^{1-R}\,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg]. \end{aligned} \tag{5.5}$$

Then

$$\mathbb{E}[e^{-(\delta+\beta\rho)s}X_s^{1-R}\,|\,\mathcal{F}_t] = e^{-(\delta+\beta\rho)t}X_t^{1-R}e^{-H_{\delta+\beta\rho}(\pi,\xi)(s-t)} \qquad \text{for } s > t,$$

where for $\nu \in \mathbb{R}$, the function $H_\nu : \mathbb{R}\times\mathbb{R}_{++} \to \mathbb{R}$ is given by

$$H_\nu(\pi, \xi) = \nu + (R-1)\bigg(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}R\bigg). \tag{5.6}$$

Remark 5.3 If we consider the constant proportional investment–consumption strategy (π, ξ), the drift of $(e^{-\nu t}X_t^{1-R})_{t\geq 0}$ is given by $-H_\nu(\pi,\xi)$. Thus $H_\nu(\pi,\xi)$ is a critical quantity not only for the well-definedness of the integral $\mathbb{E}[\int_0^\infty e^{-\nu t}X_t^{1-R}\,\mathrm{d}t]$, but also for the transversality condition $\lim_{t\to\infty}\mathbb{E}[e^{-\nu t}X_t^{1-R}] = 0$, which will feature heavily in Sect. 7.

If $H_{\delta+\beta\rho}(\pi,\xi)>0$ so that the integral in (5.5) is well defined, we obtain

$$V_t=\frac{be^{-(\delta+\beta\rho)t}A^{\rho}\xi^{1-S}}{H_{\delta+\beta\rho}(\pi,\xi)}\frac{X_t^{1-R}}{1-S}.$$

Since V was postulated to be of the form $V_t=Ae^{-\beta t}\frac{X_t^{1-R}}{1-R}$, it must be the case that $\beta=\delta+\beta\rho$ (i.e., $\beta=\delta\vartheta$) and $A=A(\pi,\xi)=(\frac{b\vartheta\xi^{1-S}}{H_\beta(\pi,\xi)})^{\vartheta}>0$. Then $\delta+\beta\rho=\delta\vartheta$, and $H:=H_{\delta\vartheta}$ satisfies

$$H(\pi,\xi)=\delta\vartheta+(R-1)\left(r+\lambda\sigma\pi-\xi-\frac{\pi^2\sigma^2}{2}R\right).$$

It follows that any proportional investment strategy ($\Pi\equiv\pi,\ C=\xi X$) is evaluable, provided that $H(\pi,\xi)$ is positive.

To find the optimal amongst constant proportional strategies (and hence to find a candidate optimal strategy), it remains to maximise $\frac{A(\pi,\xi)}{1-R}=\frac{1}{1-R}(\frac{b\vartheta\xi^{1-S}}{H(\pi,\xi)})^{\vartheta}$ over $(\pi,\xi)\in\mathbb{R}\times\mathbb{R}_{++}$ such that $H(\pi,\xi)>0$. It follows that the *candidate optimal strategy* (expressed as a fraction of wealth) is given by

$$(\hat{\pi},\hat{\xi})=\left(\frac{\lambda}{\sigma R},\eta\right),\tag{5.7}$$

where

$$\eta=\frac{1}{S}\left(\delta+(S-1)r+(S-1)\frac{\lambda^2}{2R}\right).\tag{5.8}$$

This means that the candidate optimal strategy in the original coordinates is given in feedback form by

$$\hat{\Pi}\equiv\hat{\pi}=\frac{\lambda}{\sigma R},\qquad \hat{C}=\hat{\xi}X=\eta X.\tag{5.9}$$

Note that $H(\hat{\pi},\hat{\xi})=H(\frac{\lambda}{\sigma R},\eta)>0$ if and only if $\eta>0$. If η is positive, it is not difficult to check that $(\pi=\frac{\lambda}{\sigma R},\xi=\eta)$ is a maximum of $(1-R)^{-1}A(\pi,\xi)$ over $\{(\pi,\xi):H(\pi,\xi)>0\}$; it then follows that $\max_{\{\xi>0:H(\hat{\pi},\xi)>0\}}V_0=b^{\vartheta}\eta^{-\vartheta S}\frac{x^{1-R}}{1-R}$. Considering this as a function of the initial wealth, for $\eta>0$, a candidate value function is defined by

$$\hat{V}(x)=b^{\vartheta}\eta^{-\vartheta S}\frac{x^{1-R}}{1-R}.\tag{5.10}$$

The results of this section are summarised in the following proposition.

Proposition 5.4 *Define* $D=\{(\pi,\xi)\in\mathbb{R}\times\mathbb{R}_+:H(\pi,\xi)>0\}$. *Consider constant proportional strategies with parameters* $(\pi,\xi)\in D$. *Suppose* $\vartheta>0$ *and the (candidate) wellposedness condition* $\eta>0$ *is satisfied*, $\eta>0$, *where* η *is given in* (5.8).

(i) *For* $(\pi, \xi) \in D$, *one solution* $V = (V_t)_{t \geq 0}$ *to* (5.3) *is given by*

$$V_t = e^{-\delta \vartheta t} \left(\frac{b \vartheta \xi^{1-S}}{H(\pi, \xi)} \right)^{\vartheta} \frac{X_t^{1-R}}{1-R}.$$

(ii) *The global maximum of* $h(\pi, \xi) = \frac{1}{1-R} (\frac{b \vartheta \xi^{1-S}}{H(\pi, \xi)})^{\vartheta}$ *over the set* D *is attained at* $(\pi, \xi) = (\frac{\lambda}{\sigma R}, \eta)$, *and the maximum is* $\frac{b^{\vartheta} \eta^{-\vartheta S}}{1-R}$.

(iii) *The strategy* $(\pi, \xi) = (\frac{\lambda}{\sigma R}, \eta)$ *is such that* $V_0 = b^{\vartheta} \eta^{-\vartheta S} \frac{x^{1-R}}{1-R} = \hat{V}(x)$, *where* x *denotes initial wealth.*

The agent's (candidate) optimal investment in this case is a constant fraction $\hat{\pi} = \frac{\lambda}{\sigma R}$ of their wealth, a proportion which is independent of their EIC. The agent's investment preferences are controlled solely by the risk aversion coefficient R. The agent's (candidate) optimal consumption is a constant proportion η of their wealth.

To understand the interpretation of η, it is insightful to perform a change of numéraire. As in Herdegen et al. [9, Sect. 7], the defining equation (4.3) of a utility process associated to the pair (g_{EZ}, C) may be rewritten in equivalent form as

$$V_t = \mathbb{E}\left[\int_t^{\infty} \frac{b e^{-(\delta + r(S-1))s}}{1-S} \left(\frac{C_s}{S_s^0} \right)^{1-S} \big((1-R) V_s \big)^{\rho} \, \mathrm{d}s \,\middle|\, \mathcal{F}_t \right].$$

With this in mind, it makes sense to call $\phi := \delta + r(S-1)$ the *impatience rate*. Then the optimal proportional consumption rate is given by

$$\eta = \frac{\phi}{S} + \frac{S-1}{S} \frac{\lambda^2}{2R}.$$

This is a linear (convex if $S > 1$) combination of the impatience rate and (half of) the squared Sharpe ratio per unit of risk aversion, with the weights depending on the elasticity of intertemporal complementarity S.

Remark 5.5 The wellposedness condition $\eta > 0$ is equivalent to

$$\delta > (1-S) \left(r + \frac{\lambda^2}{2R} \right)$$

(or $\phi > (1-S) \frac{\lambda^2}{2R}$). This means that when $S > 1$ (or $r < 0$), the problem can be well posed even for negative values of δ (or ϕ).

Remark 5.6 When $\vartheta > 1$, uniqueness of a utility process fails (for example, $V \equiv 0$ always solves (5.3)). In this case, the first issue is to decide *which* utility process to associate to a consumption stream; this in turn has implications for the optimal value function and optimal consumption stream, and ultimately for the wellposedness of the problem. This is a delicate issue which we cover in [10].

6 A comparison of the discounted and difference formulations

The goal of this section is to compare the discounted and difference formulations of the aggregator for EZ SDU. Despite the ubiquity of the latter in the literature, we argue that the discounted form has many advantages. As demonstrated in Sect. 5.2, its main disadvantage, the fact that it has an explicit dependence on time, is easily overcome by a change in accounting unit.

6.1 The difference form of CRRA utility

Additive utilities such as CRRA may be thought of as special cases of SDU in which the aggregator has no dependence on v. In this sense, CRRA utility may be identified with the aggregator

$$g_{\mathrm{CRRA}}(t,c,v) = g_{\mathrm{CRRA}}(t,c) = be^{-\delta t}\frac{c^{1-R}}{1-R}.$$

Note that if $\mathbb{E}[\int_0^\infty e^{-\delta s}|C_s^{1-R}|\,\mathrm{d}s] < \infty$, it follows that

$$V_t^C = \mathbb{E}\bigg[\int_t^\infty be^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg] \tag{6.1}$$

is the unique utility process associated with consumption C for aggregator g_{CRRA}, and then $J_{g_{\mathrm{CRRA}}}(C) = V_0^C$. Further, if $\mathbb{E}[\int_0^\infty e^{-\delta s}|C_s^{1-R}|\,\mathrm{d}s] = \infty$, we can set $J(C) = \infty$ if $R < 1$ and $J(C) = -\infty$ if $R > 1$.

In particular, two subtle but important questions which are crucial to the study of SDU are absent from the additive utility setting: first, what value to assign to non-evaluable strategies, and second, which utility process to assign to consumptions which are not uniquely evaluable.

Now suppose C is such that $\mathbb{E}[\int_0^\infty e^{-\delta s}|C_s^{1-R}|\,\mathrm{d}s] < \infty$. Then the martingale $M = (M_t)_{0\le t\le\infty}$ given by $M_t := \mathbb{E}[\int_0^\infty be^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s\,|\,\mathcal{F}_t]$ is uniformly integrable and satisfies $M_t = \int_0^t be^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s + V_t$, where V is the utility process in (6.1). Using that $M_\infty = \int_0^\infty be^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s$ and rearranging, we find that

$$V_t = \int_t^\infty be^{-\delta s}\frac{C_s^{1-R}}{1-R}\,\mathrm{d}s - \int_t^\infty \mathrm{d}M_t.$$

Then, applying Itô's formula to V^Δ given by $V_t^\Delta := e^{\delta t}V_t$ and integrating yields $V_t^\Delta = \int_t^\infty (b\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta)\,\mathrm{d}s + \int_t^\infty e^{\delta s}\,\mathrm{d}M_s$, provided that the integrals are well defined. Taking expectations and assuming that the process $M^\delta = (M_t^\delta)_{t\ge0}$ given by $M_t^\delta = \int_0^t e^{\delta s}\,\mathrm{d}M_s$ is a uniformly integrable martingale, we get the *difference form* of discounted expected utility as

$$V_t^\Delta = \mathbb{E}\bigg[\int_t^\infty \bigg(b\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta\bigg)\,\mathrm{d}s\,\bigg|\,\mathcal{F}_t\bigg]. \tag{6.2}$$

Modulo the technical issues, under CRRA preferences, it is possible to define the value associated to a consumption stream C as the initial value V_0^Δ of the utility process $V^\Delta = (V_t^\Delta)_{t\ge0}$, where V^Δ solves (6.2), rather than using (6.1). However, doing so brings several immediate disadvantages. It is no longer obvious if solutions to (6.2) are unique or even exist. This may result in a smaller class of evaluable strategies. Indeed, there are simple deterministic counterexamples to the existence of a solution to (6.2); consider Example 6.1 below and set $R = S$ so that $\vartheta = 1$. The counterexamples arise because the integrand $b\frac{C_s^{1-R}}{1-R} - \delta V_s^{\Delta}$ takes both signs and so the integral on the right-hand side of (6.2) may fail to be well defined. (In contrast, $\mathbb{E}[\int_0^\infty b e^{-\delta s} \frac{C_s^{1-R}}{1-R}\,\mathrm{d}s]$ is always well defined, at least in $[-\infty, \infty]$.) Further, whenever $\mathbb{E}[\int_0^\infty e^{-\delta s}\,|C_s^{1-R}|\,\mathrm{d}s]$ is finite, M is a uniformly integrable martingale. But M^δ may fail to be uniformly integrable, and the representation (6.2) may fail.

6.2 The difference form of Epstein–Zin stochastic differential utility

In the previous section, we argued that for additive CRRA preferences, the discounted form was better than the difference form for three reasons: first, existence and uniqueness of the utility process are guaranteed; second, there is a wider class of consumption streams to which it is possible to assign a (finite) value; and third, it is possible to assign a value (possibly infinite) to any consumption stream even when $\int_0^\infty g_{\mathrm{CRRA}}(s, C_s)\,\mathrm{d}s$ is not integrable. The goal in this section is to show that when we move to EZ SDU preferences, the second and third advantages of the discounted form remain; the first advantage does not carry over. Indeed, much of the discussion is as in the additive case.

Suppose that $C \in \mathscr{E}_u(g_{\mathrm{EZ}})$. Following arguments similar to the CRRA case, if V is the value process associated to consumption C and aggregator g_{EZ} and if we define the (upcounted) utility process V^Δ by $V_t^\Delta := e^{\delta\vartheta t} V_t$, then we may reasonably hope that V^Δ is the solution to

$$V_t^\Delta = \mathbb{E}\left[\int_t^\infty \left(b\frac{C_s^{1-S}}{1-S}\big((1-R)V_s^\Delta\big)^\rho - \delta\vartheta V_s^\Delta\right)\mathrm{d}s\,\middle|\,\mathcal{F}_t\right]. \tag{6.3}$$

If so, V^Δ is the utility process associated to the difference form of the EZ aggregator g_{EZ}^Δ.

As discussed in Sect. 6.1, for some consumption streams, (6.3) is not well defined because the integrand may take positive and negative signs. If the utility process is defined via the difference aggregator g_{EZ}^Δ, then it is necessary to restrict the class of consumption streams when compared with those which may be evaluated under g_{EZ}.

Example 6.1 Suppose $\delta > 0$, let $A := \bigcup_{n\geq 0}[2n, 2n+1)$ and consider the deterministic consumption stream $c = (c(t))_{t\geq 0}$ satisfying

$$\frac{c(t)^{1-S}}{1-S} := 2\frac{\delta}{b(1-S)} e^{\delta(\lceil t\rceil - t)}\mathbf{1}_{A^c}(t).$$

Let $V^\Delta = (V^\Delta(t))_{t\geq 0}$ be given by

$$V^\Delta(t) = \frac{1}{1-R}\exp\Big(\delta\vartheta(t-\lfloor t\rfloor)\big(\mathbf{1}_A(t) - \mathbf{1}_{A^c}(t)\big)\Big).$$

Then

$$\mathrm{d}V^{\Delta}(t)=\left(\delta\vartheta V^{\Delta}(t)-b\frac{c(t)^{1-S}}{1-S}\big((1-R)V^{\Delta}(t)\big)^{\rho}\right)\mathrm{d}t.$$

For this consumption stream, both the positive and the negative part of the integral $\int_t^\infty (b\frac{c(t)^{1-S}}{1-S}((1-R)V^{\Delta}(t))^{\rho}-\delta\vartheta V^{\Delta}(s))\,\mathrm{d}s$ (which can then be re-expressed as $\int_t^\infty \delta\vartheta V^{\Delta}(s)(\mathbf{1}_A(s)-\mathbf{1}_{A^c}(s))\,\mathrm{d}s$) are infinite for all $t\geq 0$. Hence it cannot be the case that V^{Δ} solves (6.3). On the other hand, if $V(t)=e^{-\delta\vartheta t}V^{\Delta}(t)$, then

$$\int_0^\infty be^{-\delta t}\frac{c(t)^{1-S}}{1-S}\big((1-R)V(t)\big)^{\rho}\,\mathrm{d}t=\int_0^\infty 2e^{-\delta t}\frac{\delta}{1-S}e^{\delta\vartheta(\lceil t\rceil -t)}\mathbf{1}_{A^c}(t)\,\mathrm{d}t<\infty$$

and $V=(V(t))_{t\geq 0}\in\mathbb{I}(g_{\mathrm{EZ}},c)$. Furthermore, it can be shown that V solves (5.3). Thus $\mathcal{E}(g^{\Delta}_{\mathrm{EZ}})\subsetneq\mathcal{E}(g_{\mathrm{EZ}})$.

7 Alternative formulations of SDU

7.1 A family of finite-horizon problems

Our approach to investment–consumption problems for EZ SDU over an infinite horizon differs from the conventional approach in two important ways. First, we use the discounted aggregator given by (4.2), whereas the standard approach is to use the difference form. Second, we define the value function over an infinite horizon directly (with the natural transversality condition that the value process tends to zero in expectation), whereas the standard approach (formulated by Duffie, Epstein and Skiadas in the Appendix to [5], and developed further by Melnyk et al. [17]) is to look for utility processes which solve a family of finite-horizon problems (where now the form of the transversality condition is not so clear, and may be part of the definition of a utility process). We have already compared the aggregators; so the goal in this section is to explain why we believe that it is better to define utility processes over an infinite horizon directly, and why, as a corollary, parameter combinations corresponding to $\vartheta<0$ cannot make economic sense.

For the sake of exposition, we introduce some additional pieces of notation. Fix an aggregator g and a consumption stream $C\in\mathscr{P}_+$. Then for $T>0$, set $\mathbb{I}_T(g,C):=\{W\in\mathscr{P}:\int_0^T|g(s,C_s,W_s)|\,\mathrm{d}s<\infty\}$, and let $\mathbb{J}_T=\mathbb{J}_T(g,C)$ be a subset of $\mathbb{I}_T(g,C)$ such that elements of $\mathbb{J}_T$ have additional regularity and/or integrability properties. Finally, set $\mathbb{J}:=\bigcap_{T>0}\mathbb{J}_T$. Examples of suitable sets $\mathbb{J}_T$ are given below.

As an alternative to defining utility processes directly over an infinite horizon, [5] and [17] define utility processes as solutions to a family of finite-horizon problems.

Definition 7.1 A $\mathbb{V}$-valued process V is the $(\nu,\mathbb{J})$-*utility process* associated to the consumption stream C and aggregator g if it has càdlàg paths, lies in $\mathbb{J}$, satisfies the transversality condition

$$\lim_{t\to\infty}e^{-\nu t}\mathbb{E}[|V_t|]=0,\tag{7.1}$$

and for all $0 \le t \le T < \infty$,

$$V_t = \mathbb{E}\left[\int_t^T g(s, C_s, V_s)\, \mathrm{d}s + V_T \,\middle|\, \mathcal{F}_t \right]. \tag{7.2}$$

Remark 7.2 It follows as in Remark 3.2 that a $(\nu, \mathbb{J})$-utility process is automatically a semimartingale.

Denote by $\mathscr{E}^{\nu,\mathbb{J}}(g)$ the set of all consumption streams C such that there exists a $(\nu, \mathbb{J})$-utility process associated to C for the aggregator g, and let $\mathscr{E}_u^{\nu,\mathbb{J}}(g)$ be the subset of $\mathscr{E}^{\nu,\mathbb{J}}(g)$ where there exists a unique $(\nu, \mathbb{J})$-utility process. Moreover, let $\mathscr{C}_0(x)$ be some subset of $\mathscr{C}(x)$, the set of attainable consumption streams from initial wealth x. Additional regularity conditions on the consumption streams may be encoded in $\mathscr{C}_0$.

In order to avoid the technical challenges of dealing with the infinite-horizon problem directly, the idea in [5, 17] is to replace the problem of finding $V(x)$ with the problem of finding $V_{\mathscr{C}_0, \mathscr{E}_u^{\nu,\mathbb{J}}(g)}(x) = \sup_{C \in \mathscr{C}_0(x) \cap \mathscr{E}_u^{\nu,\mathbb{J}}(g)} V_0^C$ for an appropriate transversality parameter ν and appropriate sets $\mathscr{C}_0(x)$ and $\mathbb{J}$. But this immediately raises several issues. What exactly are the spaces $\mathscr{C}_0(x)$, $\mathscr{E}^{\nu,\mathbb{J}}(g)$ and $\mathscr{E}_u^{\nu,\mathbb{J}}(g)$? How do we (easily) check whether $C \in \mathscr{C}_0(x)$ and/or $C \in \mathscr{E}_u^{\nu,\mathbb{J}}(g)$?

Regarding the choice of the transversality condition (7.1), two issues arise: First, how do we know that $\mathscr{E}^{\nu,\mathbb{J}}(g)$ is non-empty? Second, how do we know that a utility process V associated with a consumption C makes economic sense? Regarding the first issue, if $\nu < \nu'$, any $(\nu, \mathbb{J})$-utility process is also a $(\nu', \mathbb{J})$-utility process. Hence $\mathscr{E}^{\nu,\mathbb{J}}(g) \subseteq \mathscr{E}^{\nu',\mathbb{J}}(g)$, and if ν is chosen too small, it may easily follow that $\mathscr{E}^{\nu,\mathbb{J}}(g)$ does not contain the candidate optimal consumption. Regarding the second issue, we introduce in Sect. 7.2 below the concept of a *bubble solution* and argue that bubble solutions do not make economic sense.

Duffie et al. [5] impose Lipschitz-style conditions which exclude EZ SDU. Melnyk et al. [17] do study EZ SDU, but their main focus is to understand the impact of market frictions on the investment–consumption problem for SDU preferences. Nonetheless, in the frictionless case which is the subject of this paper, [17] proves some of the most complete results for EZ preferences currently available in the literature. Melnyk et al. [17] only consider $R > 1$, but this is mainly to limit the number of cases rather than because their methods do not extend to the general case. The following definition is from [17, Definition 3.1].

Definition 7.3 Suppose $R > 1$ and $\delta > 0$. For $T > 0$, let

$$\mathbb{S}_T^1 = \left\{ V \in \mathscr{S} : \mathbb{E}\Big[\sup_{0 \le t \le T} |V_t| \Big] < \infty \right\},$$

$$\mathbb{J}_T^1 = \mathbb{S}_T^1 \cap \mathbb{I}_T(g_{\mathrm{EZ}}^{\Delta}, C),$$

$$\mathbb{J}_T^2 = \left\{ V \in \mathbb{J}_T^1 : V_t \le -\frac{C_t^{1-R}}{R-1} \le 0 \text{ for all } 0 \le t \le T \right\}.$$

For $k \in \{1,2\}$, set $\mathbb{J}^k := \bigcap_{T>0} \mathbb{J}^k_T$ and let $\mathscr{C}_0(x)$ be the set of $C \in \mathscr{C}(x)$ for which there exists Π with $\Pi(X^{x,\Pi,C})^{1-R} \in \mathbb{S}^1_T$ for all $T > 0$ and $\frac{1}{1-R}(X^{x,\Pi,C})^{1-R} \in \mathbb{J}^1$. Moreover, if $0 < \vartheta < 1$, set $\mathbb{J}^{\mathrm{MMS}} := \mathbb{J}^1$ and

$$\mathscr{E}^{\mathrm{MMS}} = \mathscr{E}^{\mathrm{MMS}}(g^{\Delta}_{\mathrm{EZ}}) := \mathscr{E}^{\delta\vartheta,\mathbb{J}^{\mathrm{MMS}}}(g^{\Delta}_{\mathrm{EZ}});$$

if $\vartheta > 1$ or $\vartheta \in (-\infty, 0)$, set $\mathbb{J}^{\mathrm{MMS}} := \mathbb{J}^2$ and

$$\mathscr{E}^{\mathrm{MMS}} = \mathscr{E}^{\mathrm{MMS}}(g^{\Delta}_{\mathrm{EZ}}) := \mathscr{E}^{\delta,\mathbb{J}^{\mathrm{MMS}}}(g^{\Delta}_{\mathrm{EZ}}).$$

Note that as we move from $\vartheta \in (0,1)$ to $\vartheta \notin (0,1)$, the transversality parameter for (7.1) changes from $\delta\vartheta$ to δ. Moreover, an additional restriction that $V \le -\frac{C^{1-R}}{R-1}$ is imposed.

Melnyk et al. [17] take $b = \delta$. Then it follows from (5.10) that for $\eta > 0$, the candidate value function is given by $\hat{V}(x) = \eta^{-\vartheta S}\delta^{\vartheta}\frac{x^{1-R}}{1-R}$. The following result is from [17, Corollary 2.3, Theorem 3.4].

Theorem 7.4 *Suppose $R > 1$ and $\delta > 0$. Then $\mathscr{E}^{\mathrm{MMS}} = \mathscr{E}^{\mathrm{MMS}}_u$. Moreover, suppose $\frac{\mu-r}{R\sigma^2} \notin \{0,1\}$ and $\eta > 0$. Then*:

(i) *If $\vartheta \in (0,1)$ (i.e., $1 < R < S$), then we have $V_{\mathscr{C}_0,\mathscr{E}^{\mathrm{MMS}}_u}(x) = \hat{V}(x)$.*

(ii) *If $\vartheta \in (1,\infty)$ (i.e., $1 < S < R$) and $\frac{R-S}{R-1}\delta = \delta\rho < \eta < \delta$, then we have $V_{\mathscr{C}_0,\mathscr{E}^{\mathrm{MMS}}_u}(x) = \hat{V}(x)$.*

(iii) *If $\vartheta \in (-\infty,0)$ (i.e., $S < 1 < R$), then $\delta < \eta < \delta\rho = \delta\frac{R-S}{R-1}$. Then we have $V_{\mathscr{C}_0,\mathscr{E}^{\mathrm{MMS}}_u}(x) = \hat{V}(x)$.*

The results of Melnyk et al. [17] on the frictionless problem are amongst the few rigorous results on the investment–consumption problem over an infinite horizon. Nonetheless, they are incomplete in several respects. For all values of ϑ, there is no existence result; although it is possible (at least under the conditions of the theorem) to verify that the candidate optimal consumption stream is a member of $\mathscr{C}_0(x) \cap \mathscr{E}^{\mathrm{MMS}}_u$, little is said in general about which consumption streams are evaluable by Definition 7.3, and it is unclear if the space of evaluable strategies goes beyond the set of constant proportional strategies. The fact that the wealth process must satisfy transversality and integrability conditions means that many plausible consumption streams are excluded by assumption, rather than because they are sub-optimal.

When $\vartheta \notin (0,1)$, there are additional issues. In that case, Melnyk et al. [17] use the transversality condition (7.1) with $\nu = \delta$. This condition leads to simple mathematics, but does not necessarily make economic sense—we argue in Sect. 7.3 below that the economically correct transversality condition is (7.1) with $\nu = \delta\vartheta$. Moreover, the restriction to consumption streams for which there exists a utility processes with $V \le \frac{1}{1-R}C^{1-R}$ seems both hard to verify in general and hard to interpret. Finally, the analysis in [17] leaves several parameter combinations uncovered, including the case where $\vartheta > 1$ together with $\eta \in (0,\delta\rho] \cup [\delta,\infty)$.

Although the space $\mathscr{E}^{\mathrm{MMS}}$ is difficult to describe, the following result, whose proof is given in the Appendix, says that if C has an associated utility process in the sense of [17], then it automatically has an associated utility process in the sense of a solution to (3.1). The converse is not true.

Proposition 7.5 *Let $\vartheta \in (0, \infty) \setminus \{1\}$ and $\delta > 0$. Suppose $C \in \mathscr{E}^{\mathrm{MMS}}$ and let V^{Δ} be a $(\delta\vartheta, \mathbb{J}^{\mathrm{MMS}})$-utility process associated to the consumption stream C and the aggregator g^{Δ}_{EZ}. Then V given by $V_t = e^{\delta\vartheta t} V^{\Delta}_t$ is a utility process associated to the consumption stream C and the aggregator g_{EZ} in the sense of Definition 3.1. In particular, $\mathscr{E}^{\mathrm{MMS}}(g^{\Delta}_{\mathrm{EZ}}) \subseteq \mathscr{E}(g_{\mathrm{EZ}})$.*

Although Melnyk et al. [17] also define utility processes in the case $\vartheta < 0$, we argue below that there are issues with the economic interpretation of solutions in this case. In the case $\vartheta < 0$, while solutions in the sense of [17] are mathematically correct, they only make sense economically as utility bubbles.

7.2 The transversality condition and utility bubbles in the additive case

Our goal is to show that when coupled with the switch from the infinite-horizon problem to the family of finite-horizon problems approach, a mismatched transversality condition can lead to a peculiar behaviour. We conclude that the modeller is not free to choose the transversality condition, at least in the framework of Definition 7.1, and electing to use the wrong condition can either rule out perfectly reasonable admissible strategies (and possibly rule out all strategies, including the candidate optimal strategy), or can allow utility processes to be defined which have the characteristics of a bubble.

In this section, we consider the simpler case of time-additive CRRA utility, i.e., the case where $R = S$, or equivalently $\vartheta = 1$. We assume **throughout this section** that the wellposedness condition

$$\eta_a := \frac{\delta}{R} - \frac{1-R}{R}\left(r + \frac{\lambda^2}{2R}\right) > 0$$

holds (see for example Herdegen et al. [9, Corollary 6.4] for a discussion of the wellposedness of the Merton problem for additive utility) and also that $R > 1$. The latter condition is only imposed to avoid case distinctions; similar behaviour is observed when $R < 1$.

In the above setting, it is clear that for g_{CRRA}-evaluable consumption streams, the infinite-horizon formulation

$$V_t = \mathbb{E}\left[\int_t^\infty b e^{-\delta s} \frac{C_s^{1-R}}{1-R}\, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \qquad 0 \leq t < \infty,$$

is equivalent to the finite-horizon formulation

$$V_t = \mathbb{E}\left[\int_t^T b e^{-\delta s} \frac{C_s^{1-R}}{1-R}\, \mathrm{d}s + V_T \,\middle|\, \mathcal{F}_t\right], \qquad 0 \leq t \leq T < \infty, \tag{7.3}$$

if and only if the transversality condition $\lim_{T\to\infty}\mathbb{E}[V_T]=0$ is satisfied. Define $V_t^\Delta = e^{\delta t}V_t$. By arguing as in the proof of Proposition 7.5 (specialised to the case $\vartheta = 1$), V satisfies (7.3) if and only if V^Δ satisfies

$$V_t^\Delta = \mathbb{E}\bigg[\int_t^T \bigg(b\frac{C_s^{1-R}}{1-R} - \delta V_s^\Delta\bigg)\,\mathrm{d}s + V_T^\Delta\,\bigg|\,\mathcal{F}_t\bigg], \qquad 0 \le t \le T < \infty, \tag{7.4}$$

where the transversality condition is $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[V_t^\Delta]=0$.

The above observation suggests that the 'correct' transversality condition for the problem with the difference aggregator is $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[V_t^\Delta]=0$. But what happens if the transversality condition is modified to become $\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^\Delta]=0$ for some $\nu \neq \delta$?

For (π,ξ) such that

$$H_\delta(\pi,\xi) = \delta + (R-1)\bigg(r + \lambda\sigma\pi - \xi - \frac{\pi^2\sigma^2}{2}R\bigg) \neq 0,$$

it follows from (5.4) that the constant proportional strategy with $\Pi \equiv \pi$ and $C = \xi X$ satisfies $\mathbb{E}[C_t^{1-R}] = \xi^{1-R}\mathbb{E}[X_t^{1-R}] = \xi^{1-R}x^{1-R}e^{(1-R)(r+\frac{\lambda^2}{2R}-\xi)t}$, and the solution to (7.3) is

$$V_t = V_t^{\pi,\xi} = \frac{K(\pi,\xi)}{1-R}e^{-\delta t}X_t^{1-R},$$

where $K(\pi,\xi) := b\frac{\xi^{1-R}}{H_\delta(\pi,\xi)}$. This implies that a solution to (7.4) is given by

$$V_t^\Delta = V_t^{\Delta,\pi,\xi} = e^{\delta t}V_t^{\pi,\xi} = \frac{K(\pi,\xi)}{1-R}X_t^{1-R}. \tag{7.5}$$

Alternatively, we can see this directly from (7.4): if we look for a solution of the form $V_t^{\Delta,\pi,\xi} = B\frac{\xi^{1-R}X_t^{1-R}}{1-R}$, then $B = B(\pi,\xi)$ solves

$$B = \int_t^T (b - \delta B)e^{-H_0(\pi,\xi)(s-t)}\,\mathrm{d}s + Be^{-H_0(\pi,\xi)(T-t)},$$

which simplifies to $B = (b-\delta B)/H_0(\pi,\xi)$ or, equivalently,

$$BH_\delta(\pi,\xi) = b. \tag{7.6}$$

In the following discussion, we focus on the case $\pi = \hat{\pi} := \frac{\lambda}{\sigma R}$ and abbreviate $V^{\hat{\pi},\xi}$ and $V^{\Delta,\hat{\pi},\xi}$ to V^ξ and $V^{\Delta,\xi}$, respectively. Note that

$$\hat{\pi} = \operatorname*{argmax}_{\pi: H_\delta(\pi,\xi)>0} \frac{1}{(1-R)}\frac{\xi^{1-R}}{H_\delta(\pi,\xi)}$$

and that $H_\delta(\hat{\pi},\xi) = \delta - (1-R)(r + \frac{\lambda^2}{2R} - \xi)$. Furthermore,

$$\eta_a = \frac{\delta}{R} - \frac{1-R}{R}\bigg(r + \frac{\lambda^2}{2R}\bigg) = \operatorname*{argmax}_{\xi: H_\delta(\hat{\pi},\xi)>0} \frac{1}{(1-R)}\frac{\xi^{1-R}}{H_\delta(\hat{\pi},\xi)}$$

so that $(\hat{\pi}, \eta_a)$ is the candidate optimal proportional investment and consumption. This is precisely the reduction of the candidate optimal strategy given in (5.7) to the case $S = R$.

Turning to the transversality condition, note that $\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^{\Delta,\xi}] = 0$ is equivalent to $\lim_{t\to\infty} e^{(\delta-\nu)t}\mathbb{E}[V_t^{\xi}] = 0$, which in turn is equivalent to $H_\nu(\hat{\pi}, \xi) > 0$. We can therefore define the *maximum* value of ξ such that the transversality condition $\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^{\Delta,\xi}] = 0$ is satisfied. This is given by

$$\xi_{\max}^{\nu} := \sup\{\xi > 0 : H_\nu(\hat{\pi}, \xi) > 0\} = \left(r + \frac{\lambda^2}{2} + \frac{\nu}{R-1}\right)^+ < \infty.$$

If the transversality condition

$$\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[V_t^{\Delta}] = 0 \tag{7.7}$$

associated with (7.4) is replaced by the transversality condition

$$\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^{\Delta}] = 0 \tag{7.8}$$

where $\nu < \delta$, then (7.8) is more restrictive than (7.7) since any process which satisfies (7.7) also satisfies (7.8), but the converse is not true. Since $\nu < \delta$, it follows that $H_\delta(\hat{\pi}, \xi) > H_\nu(\hat{\pi}, \xi)$. In this case, if

$$H_\delta(\hat{\pi}, \xi) > 0 \geq H_\nu(\hat{\pi}, \xi)$$

or, equivalently, if ξ is such that $R\eta_a > (R-1)\xi \geq \nu + (R-1)(r + \frac{\lambda^2}{2R})$, then V^Δ defined in (7.5) satisfies (7.7) but does not satisfy (7.8). In particular, if $\eta_a > \xi_{\max}^{\nu}$, then the candidate optimal strategy $(\pi, \xi) = (\hat{\pi}, \eta_a)$ leads to a utility process which does not satisfy (7.8) and hence does not lie in the set of consumption streams over which the optimisation takes place. This is illustrated in Fig. 1(a) for the case $R > 1$ (but can also occur when $R < 1$).

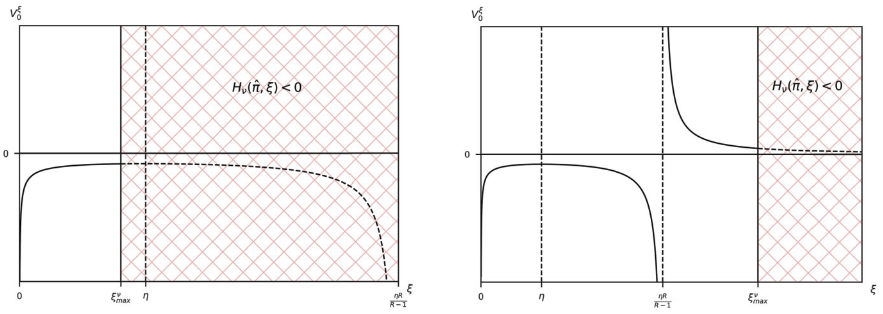
Second, consider solving (7.4) under the transversality condition

$$\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^{\Delta}] = 0$$

for $\nu > \delta$. In this case, $H_\nu(\hat{\pi}, \xi) > H_\delta(\hat{\pi}, \xi)$. Let $\xi \neq \frac{R\eta_a}{R-1}$ be such that

$$H_\nu(\hat{\pi}, \xi) > 0 > H_\delta(\hat{\pi}, \xi),$$

e.g. for concreteness, $\xi = \xi_\varepsilon := \frac{\delta+\varepsilon}{R-1} + (r + \frac{\lambda^2}{2R}) = \frac{\varepsilon + R\eta_a}{R-1} > 0$ for $\varepsilon \in (0, \nu - \delta)$. Then $V^{\Delta,\xi_\varepsilon}$ given by (7.5) solves (7.4) for the investment–consumption strategy $(\hat{\pi}, \xi_\varepsilon)$. As $H_\nu(\hat{\pi}, \xi_\varepsilon) > 0$, the transversality condition $\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^{\Delta,\xi_\varepsilon}] = 0$ is met. Moreover, $V^{\Delta,\xi_\varepsilon} = -\frac{K(\hat{\pi},\xi_\varepsilon)}{R-1}X^{1-R}$, where $K(\hat{\pi}, \xi_\varepsilon) = -\frac{\xi_\varepsilon^{1-R}}{\varepsilon}$. In particular, we have $V_0^{\xi_\varepsilon} = \frac{\xi^{1-R}}{\varepsilon}\frac{x^{1-R}}{R-1} > 0$. By comparison, $V_0^{\eta_a} = b\eta_a{}^{-\vartheta S}\frac{x^{1-R}}{1-R} < 0$. Hence the candidate optimal strategy $(\pi, \xi) = (\hat{\pi}, \eta_a)$ no longer maximises the initial value of the utility



When the transversality parameter is too small ($\nu < \delta$), the candidate optimal strategy need not satisfy the transversality condition (7.1) in Definition 7.1.

When the transversality parameter is too large ($\nu > \delta$), the candidate optimal strategy is not optimal. Furthermore, some consumption streams lead to bubble-like utility processes.

Fig. 1 Plots of the solution to (7.5) associated to the constant proportional investment–consumption strategy $(\hat{\pi}, \xi)$ along with blocked-out region where the transversality condition is not met (i.e., where $H_\nu(\hat{\pi}, \xi) \leq 0$)

process over constant proportional strategies, in contradiction to the well-established theory for this case.

In the case $R > 1$ where we expect to assign a negative utility, we may actually obtain an arbitrarily large *positive* utility (see Fig. 1(b)). This can be seen by letting $\varepsilon \searrow 0$ in the above. What is happening is that—while the integrand in (7.3) is always negative—the discounted expected future utility $\mathbb{E}[V_T^\Delta \,|\, \mathcal{F}_t]$ is diverging to $+\infty$ as $T \nearrow \infty$. The agent is always receiving a negative utility from consumption, but this is offset by an ever increasing positive contribution from expectations of future utility. The endless optimism that things will always be better in the future creates bubble-like behaviour.

Although there are special features in the additive case, the study of CRRA utility does show that some delicacy is needed when defining infinite-horizon utility to be the solution to the finite-horizon utilities paired with a transversality condition. If we wish to define SDU in this manner, we must be very careful that we use an appropriate transversality condition. We return to this issue in the next section.

In preparation for the move beyond the additive case, we record the following definition and proposition summarising the results of this section.

Definition 7.6 A process $V = (V_t)_{t \geq 0}$ is a *CRRA bubble* for a consumption stream C if $V \not\equiv 0$ solves (7.3) for each $0 \leq t \leq T < \infty$, but V and $U = U(t, C)$ are of opposite sign.

Proposition 7.7 (i) *For constant proportional strategies, there do not exist CRRA bubbles which satisfy the transversality condition* $\lim_{t\to\infty} e^{-\delta t}\mathbb{E}[V_t^\Delta] = 0$.

(ii) *If* $\nu < \delta$, *there is a financial market such that the candidate optimal investment–consumption strategy given by* (5.9) *with* $S = R$ *does not satisfy the transversality condition* (7.1).

(iii) *If* $\nu > \delta$, *there is a financial market such that there is an investment–consumption strategy* (*for example* $(\pi, \xi) = (\hat{\pi}, \xi_\varepsilon)$ *from the previous page*) *for which the* *associated utility process satisfies the transversality condition* (7.1), *but is a CRRA bubble. When* $R > 1$, *the candidate optimal investment–consumption strategy given by* (5.9) *with* $S = R$ *does not maximise* V_0^C *over attainable strategies.*

Remark 7.8 It is not difficult to check that Proposition 7.7(i) extends to *all* admissible strategies.

7.3 Transversality, the case $\vartheta < 0$, and the family of finite-horizon problems

For the EZ SDU aggregator in discounted form over an infinite horizon, it is not possible to define a utility process in the case $\vartheta < 0$. However, several authors have attempted to define a utility process for $\vartheta < 0$ using the difference form with the family of finite-horizon problems approach or otherwise. Motivated by the analysis of the additive case, we explain in this section why the mathematical results they find may fail to have a sensible economic interpretation.

The only strategies for which we can hope to find a non-trivial utility process in explicit form are constant proportional investment–consumption strategies. Moreover, the candidate optimal strategy is of this form. In consequence, and for this section only, we make the following assumption so we can explicitly see the issues which arise when $\vartheta < 0$.

Assumption 7.9 (Assumed throughout Sect. 7.3 only) Consumption plans under consideration in this section are generated by constant proportional investment–consumption strategies (π, ξ). If an associated utility process exists for the aggregator in difference form, then it is assumed to be of the form $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ for a positive constant $B = B(\pi, \xi)$. If there is no solution of the form $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ with $B \in (0, \infty)$, then the consumption stream is assumed to be not evaluable.

Remark 7.10 If $\vartheta \in (0, 1)$, Herdegen et al. [11, Corollary 5.9] show that if a utility process exists for a consumption stream C, then it is unique. If $\vartheta \notin [0, 1]$, this need not be the case. In that case, we must decide which utility process to assign to a given consumption stream. Typically, the literature makes additional assumptions to ensure that the time-homogeneous process V^Δ given by $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ is the utility process associated with C, if such a process exists. Without discussing what these assumptions might be, the impact of the temporary standing assumption is to assign the utility process V^Δ given by $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$ to the constant proportional strategy.

Consider now g_{EZ}^Δ and a constant proportional investment–consumption strategy (π, ξ). Suppose $V^\Delta = (V_t^\Delta)_{t\geq 0}$ is a solution to

$$V_t^\Delta = \mathbb{E}\bigg[\int_t^T \bigg(b\frac{\xi^{1-S}X_s^{1-S}}{1-S}\big((1-R)V_s^\Delta\big)^\rho - \delta\vartheta V_s^\Delta\bigg)\,\mathrm{d}s + V_T^\Delta\,\bigg|\,\mathcal{F}_t\bigg] \tag{7.9}$$

for all $0 \leq t \leq T < \infty$. By Assumption 7.9, we look for a solution of the form $V_t^\Delta = B\xi^{1-R}\frac{X_t^{1-R}}{1-R}$, where $B = B(\pi, \xi)$ is a positive constant which we seek to identify. For a constant proportional strategy (π, ξ), we have by Remark 5.3 that $\mathbb{E}[X_s^{1-R}|\mathcal{F}_t] = X_t^{1-R} e^{-H_0(s-t)}$, where $H_0 = H_0(\pi, \xi)$ is as in (5.6) with $\nu = 0$. Then, substituting the candidate form for V^Δ provided by Assumption 7.9 into (7.9) and dividing by $\xi^{1-R} X_t^{1-R}$ yields

$$\frac{B}{1-R} = \int_t^T \left(\frac{b}{1-S} B^\rho - \frac{\delta \vartheta B}{1-R} \right) e^{-H_0(s-t)} \, \mathrm{d}s + \frac{B}{1-R} e^{-H_0(T-t)},$$

and, provided $H_0(\pi, \xi) \neq 0$,

$$B = (b \vartheta B^\rho - \delta \vartheta B) \frac{1 - e^{-H_0(T-t)}}{H_0(\pi, \xi)} + B e^{-H_0(T-t)}. \tag{7.10}$$

It follows that there is a solution V^Δ to (7.9) of the form $V_t^\Delta = B \xi^{1-R} \frac{X_t^{1-R}}{1-R}$ if there is a solution to

$$B H_{\delta\vartheta}(\pi, \xi) = B\big(\delta\vartheta + H_0(\pi, \xi)\big) = b\vartheta B^\rho, \tag{7.11}$$

where $H_{\delta\vartheta}(\pi, \xi)$ is as in (5.6) with $\nu = \delta\vartheta$. (If $H_0(\pi, \xi) = 0$, instead of (7.10), we get $B = (T-t)(b\vartheta B^\rho - \delta\vartheta B) + B$ which means that again B solves (7.11).) Since $b > 0$, there can only be a positive solution to (7.11) if $\vartheta H_{\delta\vartheta}(\pi, \xi) > 0$.

Note that already this is different to the additive case ($\rho = 0$ and $\vartheta = 1$) which was presented in Sect. 7.2. In the additive case (recall (7.6)), we looked for solutions to $B(\delta + H_0(\pi, \xi)) = b$, but did not require that $B > 0$; indeed, we sometimes found (genuine) solutions with $B > 0$ and sometimes bubble solutions with $B < 0$. Solutions in the additive case with $B < 0$ do not satisfy $V \in \mathbb{V}$ and are automatically excluded by Assumption 7.9. We now argue that similar ideas mean that $\vartheta < 0$ does not make sense if bubbles are excluded, where (colloquially) a bubble exists if the value assigned to the combination of a flow and a terminal value arises mainly from the terminal value: for example if $(W_t^Z)_{t \geq 0}$ solves $W_t^Z = \mathbb{E}[\int_t^T Z_s \, \mathrm{d}s + W_T^Z \,|\, \mathcal{F}_t]$ and W_t^Z is *primarily* determined by W_T^Z, and in extremis a positive process W^Z is assigned to a negative flow Z, or vice versa. We give a formal definition of a *bubble solution* in the Epstein–Zin setting in Definition 7.12 below.

Suppose $\vartheta \neq 1$ (equivalently, $\rho \neq 0$ or $R \neq S$) and consider nonnegative solutions to (7.11). If $\vartheta \in (0, 1)$ (equivalently, $\rho < 0$), this equation has a solution if and only if $H_{\delta\vartheta}(\pi, \xi) > 0$, and then the solution is unique and given by $B = (\frac{b\vartheta}{H_{\delta\vartheta}(\pi, \xi)})^\vartheta$. If $\vartheta > 1$, then $B = 0$ is always a solution to (7.11) (and so is $B = \infty$ if $H_{\delta\vartheta}(\pi, \xi) > 0$), and there exists a strictly positive, finite solution if and only if $H_{\delta\vartheta}(\pi, \xi) > 0$, whence again $B = (\frac{b\vartheta}{H_{\delta\vartheta}(\pi, \xi)})^\vartheta$. If $\vartheta < 0$, then $B = 0$ is always a solution to (7.11), $B = \infty$ is a solution if $H_{\delta\vartheta}(\pi, \xi) < 0$, and there exists a further (strictly positive and finite) solution $B = (\frac{b|\vartheta|}{|H_{\delta\vartheta}(\pi, \xi)|})^\vartheta$ if and only if $H_{\delta\vartheta}(\pi, \xi) < 0$. By Assumption 7.9, we exclude 0 and ∞ as solutions.

For a constant proportional strategy $(\hat{\pi} = \frac{\lambda}{\sigma R}, \xi)$, a change of accounting units has the effect of changing the discount parameter. Fix δ and g_{EZ}^Δ, but introduce also $g^\gamma = g_{\mathrm{EZ}}^\gamma$ and V^γ, where

$$g^\gamma := b \frac{c^{1-S}}{1-S} \big((1-R)v\big)^\rho - \gamma \vartheta v \tag{7.12}$$

and $V^\gamma = (V_t^\gamma)_{t\geq 0}$ is a solution to

$$V_t^\gamma = \mathbb{E}\bigg[\int_t^T \bigg(be^{(\gamma-\delta)s}\frac{\xi^{1-S}X_s^{1-S}}{1-S}\big((1-R)V_s^\gamma\big)^\rho - \gamma\vartheta V_s^\gamma\bigg)\,\mathrm{d}s + V_T^\gamma\,\bigg|\,\mathcal{F}_t\bigg] \quad (7.13)$$

for all $0 \leq t \leq T < \infty$. (Then also $(g^\delta, V^\delta) = (g^\Delta_{\mathrm{EZ}}, V^\Delta)$.) As before, we look for a solution of the form $V_t^\gamma = B_\gamma \xi^{1-R}\frac{X_t^{1-R}}{1-R}$, where $B_\gamma = B_\gamma(\pi, \xi) \in (0, \infty)$.

Lemma 7.11 *Let $(X_t^\gamma)_{t\geq 0}$ be given by $X_t^\gamma = X_t e^{-\frac{(\gamma-\delta)}{1-S}t}$ so that X^γ is the wealth process which arises from a change of accounting unit. Then*:

(i) V^Δ *solves* (7.9) *if and only if $V^\gamma = (V_t^\gamma)_{t\geq 0}$ defined by $V_t^\gamma = e^{(\gamma-\delta)\vartheta t}V_t^\Delta$ solves* (7.13).

(ii) V^γ *solves* (7.13) *if and only if it also solves*

$$V_t^\gamma = \mathbb{E}\bigg[\int_t^T \bigg(b\frac{\xi^{1-S}(X_s^\gamma)^{1-S}}{1-S}\big((1-R)V_s^\gamma\big)^\rho - \gamma\vartheta V_s^\gamma\bigg)\,\mathrm{d}s + V_T^\gamma\,\bigg|\,\mathcal{F}_t\bigg].$$

Proof The proof of (i) follows by an argument similar to the one used in the proof of Proposition 7.5. Statement (ii) is a simple renaming of variables. □

In particular, taking $\gamma = 0$, V^0 solves

$$V_t^0 = \mathbb{E}\bigg[\int_t^T b\xi^{1-S}\frac{(X_s^0)^{1-S}}{1-S}\big((1-R)V_s^0\big)^\rho\,\mathrm{d}s + V_T^0\,\bigg|\,\mathcal{F}_t\bigg]. \quad (7.14)$$

Considering solutions to (7.14), it is clear that the aggregator g^0 takes only one sign in the sense that (except possibly on the boundary, where it need not be defined) either $g^0 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{R}_+$ or $g^0 : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{V} \to \mathbb{R}_-$.

Definition 7.12 Let g be an aggregator that takes values in $[0, +\infty]$ or in $[-\infty, 0]$. Then $V \not\equiv 0$ is a *bubble solution* for a consumption stream C if it solves

$$V_t = \mathbb{E}\bigg[\int_t^T g(s, C_s, V_s)\,\mathrm{d}s + V_T\,\bigg|\,\mathcal{F}_t\bigg] \qquad \mathbb{P}\text{-a.s.}$$

for each $0 \leq t \leq T < \infty$, and either $V \leq 0$ and $g \geq 0$, or $V \geq 0$ and $g \leq 0$.

Our contention is that it is not appropriate to value consumption streams using a mechanism which incorporates bubble solutions (unless the purpose is to consider the impact of bubbles). When g is one-signed, we have defined a bubble solution. Our immediate goal is to consider the consequences for the Epstein–Zin aggregator under a variety of accounting units.

Theorem 7.13 *If there are no bubble solutions for the Epstein–Zin aggregator under any choice of accounting units, then $\vartheta > 0$.*

Proof Consider a constant proportional strategy (π, ξ) for which a solution V^0 to (7.14) exists and is non-zero everywhere. It follows from the discussion at the beginning of this section that such a strategy exists for any choice of ϑ. Since there are by hypothesis no bubble solutions for the Epstein–Zin aggregator under any accounting units, V^0 is not a bubble solution and therefore has the same sign as g^0. Since $(1-S)g^0 \geq 0$ and V^0 is non-zero everywhere, it follows that $(1-S)V^0 > 0$. Using again that $V^0 \in \mathbb{V} = (1-R)\overline{\mathbb{R}}_+$ is non-zero everywhere, it follows that $(1-R)V^0 > 0$. We conclude that $\vartheta = \frac{(1-R)V_0}{(1-S)V_0} > 0$. □

Now we want to consider which transversality condition we should associate with (7.9). Suppose the transversality condition is

$$\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^\Delta] = 0. \tag{7.15}$$

It is easy to see from the definition of V^γ in Lemma 7.11 that $e^{-\nu t}\mathbb{E}[V_t^\Delta] \to 0$ if and only if $e^{-(\nu-\delta\vartheta)t}\mathbb{E}[e^{-\gamma\vartheta t}V_t^\gamma] \to 0$, and so the transversality condition (7.15) becomes $\lim_{t\to\infty} e^{-(\nu-\delta\vartheta)t}\mathbb{E}[V_t^0] = 0$.

We make the following hypothesis, which is very intuitive from an economic perspective.

Hypothesis 7.14 The transversality condition associated with the aggregator g depends on the aggregator, but not on the financial market.

For the next lemma, we recall that the candidate optimal consumption stream is given in feedback form in (5.9) by $(\hat{\Pi}, \hat{C}) = (\hat{\pi}, \eta X)$. Furthermore, recall from Proposition 5.4 that the candidate wellposedness condition is $\eta > 0$, where η is defined by (5.8).

Lemma 7.15 *Under Hypothesis 7.14, if the wellposedness condition $\eta > 0$ is satisfied and the utility process associated with the candidate optimal consumption stream satisfies the transversality condition (7.15) for every well-posed problem, then $\nu \geq \delta\vartheta$.*

Proof Suppose $\nu < \delta\vartheta$ and define $\varepsilon := \delta\vartheta - \nu > 0$. Then the candidate optimal strategy $(\hat{\pi}, \eta)$ satisfies the transversality condition $\lim_{t\to\infty} e^{-\nu t}\mathbb{E}[V_t^\Delta] = 0$ if and only if it satisfies $\lim_{t\to\infty} e^{\varepsilon t}\mathbb{E}[e^{-\delta\vartheta t}V_t^\Delta] = 0$, which in turn is equivalent to $H_{\delta\vartheta}(\hat{\pi}, \eta) > \varepsilon$ by Remark 5.3 and Assumption 7.9. Suppose the market parameters are such that $\eta \in (0, \frac{\varepsilon}{\vartheta})$. Then $H_{\delta\vartheta}(\hat{\pi}, \eta) = \vartheta\eta < \varepsilon$, and the candidate optimal utility process fails to satisfy the transversality condition. □

In general, the larger the value of ν, the larger the class of processes which satisfy the transversality condition (7.15). However, for the EZ aggregator, there is a point where increasing ν further makes no difference to the set of evaluable consumption streams.

Lemma 7.16 *Suppose $\vartheta > 0$ and (ξ, π) is a constant proportional investment–consumption strategy. If $V^\Delta = (V_t^\Delta)_{t\geq 0}$ is a $\mathbb{V}$-valued solution to (7.9) for all $0 \leq t \leq T < \infty$, then V^Δ satisfies (7.15) for $\nu = \delta\vartheta$.*

Proof Suppose that V^Δ solves (7.9) for all $0 \le t \le T < \infty$. Then Lemma 7.11 implies that V^0 solves (7.14) for all $0 \le t \le T < \infty$. By Assumption 7.9 and the form for V^γ given in Lemma 7.11,

$$V_t^0 = B\xi^{1-R} e^{-\delta\vartheta t} \frac{X_t^{1-R}}{1-R}.$$

Hence since $\vartheta > 0$, by letting $T \to \infty$ in (7.14), it follows that $\lim_{t\to\infty} \mathbb{E}[V_t^0] = 0$. Equivalently, $\lim_{t\to\infty} e^{-\delta\vartheta t}\mathbb{E}[V_t^\Delta] = 0$. □

Our second hypothesis says that we choose the smallest possible value for ν which allows us to evaluate all the strategies that we want.

Hypothesis 7.17 The transversality parameter should be the smallest parameter ν such that every solution to (7.9) satisfies (7.15).

Combining Lemmas 7.15 and 7.16, we get the following results.

Proposition 7.18 *Under Hypotheses 7.14 and 7.17 and if $\vartheta > 0$, the parameter ν in the transversality condition (7.15) must take the value $\nu = \delta\vartheta$.*

Moreover, we get the following analogue to Proposition 7.7 and converse to Theorem 7.13.

Theorem 7.19 *Suppose Hypotheses 7.14 and 7.17 are satisfied. If $\vartheta > 0$, there are no bubble solutions for constant proportional strategies for the Epstein–Zin aggregator under any choice of accounting units.*

Proof The EZ aggregator g^γ from (7.12) has one sign if and only if $\gamma(1-R) \ge 0$. Suppose first that $\gamma = 0$. We want to show that there are no bubble solutions for g^0 under the transversality condition $\mathbb{E}[V_T^0] \to 0$. But since g^0 has one sign, taking $T \to \infty$ in (7.14) implies that V^0 and g^0 have the same sign. Now consider some other γ for which $\gamma(1-R) > 0$ and hence g^γ has one sign, which is the same sign as that of g^0. Then since they only differ by an exponential pre-factor, V^γ defined in Lemma 7.11 has the same sign as V^0. Hence V^γ and g^γ also have the same sign. □

Remark 7.20 (i) For $\vartheta > 1$, Melnyk et al. [17] take the transversality condition to be (7.15) with $\nu = \delta < \delta\vartheta$. For some parameter values, the candidate optimal strategy from (5.9) may fail to be permitted by [17], in the sense of not being in the set of MMS-evaluable strategies $\mathscr{E}^{\mathrm{MMS}}$ from Definition 7.3, because it fails their transversality condition that uses $\nu = \delta$. However, these parameter combinations are ruled out by the extra parameter restrictions imposed in [17]. In particular, [17] restrict attention to financial models for which $\eta > \delta\rho$. It can be checked that the optimal strategy satisfies $H_\delta(\hat{\pi}, \eta) = (\eta - \delta)\vartheta + \delta$. Hence since the restriction $\eta > \delta\rho$ implies that $(\eta - \delta)\vartheta + \delta > 0$, this is precisely enough to ensure that $e^{-\delta t}\mathbb{E}[X_t^{1-R}] \to 0$ for the candidate optimal strategy by Remark 5.3. For $0 < \eta \le \delta\rho$, the utility process for the candidate optimal strategy would fail the transversality condition. Further, both in the case $\eta > \delta\rho \geq 0$ and in the case $0 < \eta \leq \delta\rho$, many reasonable strategies (consider constant proportional strategies (π, ξ) such that $H_{\delta\vartheta}(\pi, \xi) > 0$ but $H_\delta(\pi, \xi) < 0$, and consequently $e^{-\delta t}\mathbb{E}[X_t^{1-R}] \not\to 0$ for $\vartheta > 1$) are unnecessarily excluded because they fail the transversality condition, and not because they are suboptimal.

For $\vartheta < 0$ (and $R > 1$), Melnyk et al. [17] define candidate solutions V^Δ as solutions to (7.9). It follows that $V = (V_t)_{t\geq 0}$ given by $V_t = e^{-\delta\vartheta t} V_t^\Delta$ solves the family of finite horizon problems given in (7.14). However, relative to the aggregator g^0, the solution V^0 is a bubble solution and therefore questionable from an economic perspective.

7.4 The dual approach

Dual methods have proved spectacularly successful for the Merton problem with additive utility. They work for general utility functions, and in principle make it possible to move beyond the setting of constant-parameter financial markets to non-Markovian settings and incomplete markets. However, it is not immediately clear how to extend dual methods to the SDU setting. One promising idea is based on stochastic variational utility (SVU) as formulated by Dumas et al. [7] and applied in the context of utility maximisation under finite-horizon EZ SDU by Matoussi and Xing [15].

The papers [7] and especially [15] provide great insights and a potential roadmap describing how dual methods might be extended to the investment–consumption problem for SDU. However, there are several obstacles which make it difficult to apply these ideas to the infinite-horizon problem. First, at present, the dual method has little to say about existence of solutions. Second, the equivalence between the SDU and SVU formulations may be challenging to prove in the infinite-horizon setting without imposing substantive technical assumptions. Third, there are major issues of non-uniqueness when $\vartheta > 1$ (cf. Herdegen et al. [10]); these issues do not disappear simply by a change of viewpoint.

8 Summary

The conclusions from this paper are twofold.

First, for EZ SDU over an infinite horizon, certain restrictions on the parameters are necessary to have a well-founded problem. In particular, in addition to $b > 0$, for the problem to make sense, the coefficients of relative risk aversion and of elasticity of intertemporal complementarity both must lie on the same side of unity, i.e., $\vartheta > 0$. The finding that $\vartheta > 0$ is a necessary condition raises fundamental questions over the strand of literature which considers long-run risks and builds on the seminal paper of Bansal and Yaron [1], since these papers assume $S < 1 < R$. Although these papers consider an equilibrium setup, underpinning the analysis is the idea that there is a utility process associated to the equilibrium consumption stream. Our results show that the interpretation of this utility process may be problematic.

Second, for the infinite-horizon problem, it is preferable to consider a discounted aggregator rather than a difference aggregator. The one-sign property of the discounted form of the EZ SDU aggregator means that the integral $\int_0^\infty g(s, C_s, V_s)\,\mathrm{d}s$

and its expectation are always well defined in $[-\infty, \infty]$, whereas this is not always the case for the difference aggregator. Then, in addition to the fact that the discounted aggregator is the natural generalisation of the standard form of the Merton problem for additive utility, there are no issues for the discounted aggregator over bubble solutions. In the companion paper Herdegen et al. [11], we strengthen this result further by showing that at least when $\vartheta \in (0, 1)$, for the discounted aggregator, it is possible to define a (generalised) utility process for *every* consumption stream. This means that we can prove optimality of a candidate optimal strategy within the class of *all* admissible investment–consumption strategies from Definition 5.1, and not just a subclass satisfying certain integrability properties.

Appendix: Proof omitted from the main text

Proof of Proposition 7.5 Let V^Δ be a $(\delta\vartheta, \mathbb{J}^{\mathrm{MMS}})$-utility process associated to the consumption stream C and aggregator g^Δ_{EZ}. Then we have $V^\Delta \in \mathbb{S}^1_T \cap \mathbb{I}_T(g^\Delta_{\mathrm{EZ}}, C)$ as well as $\lim_{t\to\infty} e^{-\delta\vartheta t}\mathbb{E}[V^\Delta_t] = 0$, and V^Δ solves (7.2) with the aggregator g^Δ_{EZ} for all $0 \le t \le T < \infty$.

Define the process $V = (V_t)_{t\ge 0}$ by $V_t := \exp(-\delta t)V^\Delta_t$. Then $V \in \mathbb{S}^1_T$, and the transversality condition (7.1) of V^Δ uses $\nu = \delta$. This gives $\lim_{t\to\infty} \mathbb{E}[V_t] = 0$. We proceed to show that $V \in \mathbb{I}_T(g_{\mathrm{EZ}}, C)$ and V satisfies

$$V_t = \mathbb{E}\left[\int_t^T be^{-\delta u}\frac{C_u^{1-S}}{1-S}\big((1-R)V_u\big)^\rho \,\mathrm{d}u + V_T \,\middle|\, \mathcal{F}_t\right] \tag{A.1}$$

for all $0 \le t \le T < \infty$ and $T > 0$. So fix $T > 0$. Using that $V^\Delta \in \mathbb{S}^1_T \cap \mathbb{I}_T(g^\Delta_{\mathrm{EZ}}, C)$ and $e^{-\delta t}|V_t|^\rho \le e^{|\delta\vartheta|T}|V^\Delta_t|^\rho$ for $t \in [0, T]$, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|be^{-\delta s}\frac{C_s^{1-S}}{1-S}\big((1-R)V_s\big)^\rho\right| \mathrm{d}s\right] \\
&\le e^{|\delta\vartheta|T}\mathbb{E}\left[\int_0^T \left|b\frac{C_s^{1-S}}{1-S}\big((1-R)V^\Delta_s\big)^\rho - \delta\vartheta V^\Delta_s\right| \mathrm{d}s\right] \\
&\quad + e^{|\delta\vartheta|T}T|\delta\vartheta|\mathbb{E}\Big[\sup_{s\in[0,T]}|V^\Delta_s|\Big] \\
&< \infty.
\end{aligned}$$

Thus $V \in \mathbb{I}_T(g_{\mathrm{EZ}}, C)$. Next, define the martingale $M = (M_t)_{t\in[0,T]}$ by

$$M_t = \mathbb{E}\left[\int_0^T \left(b\frac{C_s^{1-S}}{1-S}\big((1-R)V_s\big)^\rho - \delta\vartheta V_s\right) \mathrm{d}s + V_T \,\middle|\, \mathcal{F}_t\right].$$

As V^Δ satisfies (7.2), it satisfies the BSDE

$$V^\Delta_t = V^\Delta_T + \int_t^T \left(b\frac{C_u^{1-S}}{1-S}\big((1-R)V^\Delta_u\big)^\rho - \delta\vartheta V^\Delta_u\right) \mathrm{d}u - \int_t^T \mathrm{d}M_u.$$

Applying the product rule to $V_t = e^{-\delta\vartheta t} V_t^\Delta$, we find that

$$V_t = V_T + \int_t^T b e^{-\delta u} \frac{C_u^{1-S}}{1-S}\big((1-R)V_u\big)^\rho \,\mathrm{d}u + \int_t^T e^{-\delta\vartheta u}\,\mathrm{d}M_u.$$

Since $\mathbb{E}[(1-e^{-\delta\vartheta T})|M_T|] < \infty$, N defined by $N_t = \int_0^t e^{-\delta\vartheta s}\,\mathrm{d}M_s$ is a martingale by Herdegen and Muhle-Karbe [12, Lemma A.1.], and taking conditional expectations gives (A.1).

Next, using that V and the integrand in (A.1) have the same sign, it follows from monotone convergence and $\lim_{T\to\infty}\mathbb{E}[V_T] = 0$ that V satisfies (5.3). Since V_0 is finite, this also gives $V \in \mathbb{I}(g_{\mathrm{EZ}}, C)$. Finally, if $\vartheta > 1$, then $\delta\vartheta > \delta$ and any $(\delta, J^{\mathrm{MMS}})$-utility process is automatically a $(\delta\vartheta, J^{\mathrm{MMS}})$-utility process. Hence $\mathscr{E}^{\mathrm{MMS}}(g_{\mathrm{EZ}}^\Delta) \subseteq \mathscr{E}(g_{\mathrm{EZ}})$. □

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Declarations

Competing Interests The authors declare no competing interests.

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