REGULAR ORBITS OF FINITE PRIMITIVE SOLVABLE GROUPS, THE FINAL CLASSIFICATION

DEREK HOLT AND YONG YANG

Abstract. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V, and G is not metacyclic. Then G always has a regular orbit on V except for a few "small" cases. We completely classify these cases in this paper.

1. Introduction

Let G be a finite group and V a finite, faithful and completely reducible G-module. One of the most important and natural questions about the orbit structure of G on V is to establish the existence of an orbit of a certain size. For a long time, there has been a deep interest in the size of the largest possible orbits in linear group actions. For $v \in V$, the orbit $v^G := \{v^g : g \in G\}$ is called *regular* if $\mathbf{C}_G(v) = 1$ holds or, equivalently, the size of v^G is $|G|$. The existence of regular orbits has been studied extensively in the literature with many applications to some important questions of character theory and conjugacy classes of finite groups.

Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V of dimension d over a finite field of order q and characteristic p. (So $G \leq \mathrm{GL}(V) = \mathrm{GL}(d, q)$.) Then, as we shall see in Theorem 2.1 below, G has a uniquely determined characteristic subgroup E which is a direct product of extraspecial p_i-groups E_i for various primes p_i. Now each $|E_i/Z(E_i)|$ is an even power of p_i, so $|E/Z(E)|$ is a perfect square, and it is convenient to define $e := \sqrt{|E/\mathbf{Z}(E)|}$.

It is proved in [9, Theorem 3.1] and [10, Theorem 3.1] that, if $e = 5, 6, 7$ or $e \geq 10$ and $e \neq 16$, then G always has at least one regular orbit on V. The information on the existence of a regular orbit has been used by several authors to study a variety of problems in the field (for example [2, 3, 6, 8, 11, 12]).

If $e = 1$, then E is trivial and G is a subgroup of the group $\Gamma(q^d)$ of order $d(q^d - 1)$, which will be defined below, but is equal to the normalizer in $\mathrm{GL}(d, q)$ of a Singer cycle of order $q^d - 1$, which acts regularly on $V \setminus \{0\}$. Note that since $|\Gamma(q^d)| > |V| - 1$ for $d \geq 2$, it is clear that it has no regular orbit. So for $e = 1$ one cannot expect that G necessarily possesses a regular orbit. In this case, the group G is metacyclic and thus there are infinitely many metacyclic primitive linear groups that do not have regular orbits.

There are also other examples for $e > 1$, when G does not possess a regular orbit. In [14], some more detailed calculations are carried out in the outstanding cases $e = 2, 3, 4, 8, 9, 16$. The main result of [14] implies that there are only finite number of examples in these cases.

Note that we know only a few examples of maximal irreducible primitive solvable subgroups of $\mathrm{GL}(V)$ that are not metacyclic and do not possess a regular orbit. In [14], Yang et al. provide a much smaller list of possible groups without regular orbit in [14, Table 3.4].

2000 *Mathematics Subject Classification.* 20C20, 20C15.

In this paper, with the help of the computer algebra system MAGMA [1], we are able to obtain a complete classification for these remaining cases. It appears that this classification can not only simplify many proofs of the past results, but also have future applications.

By combining the results of our computer calculations with the existing results that we have just summarized, we prove the following result.

Theorem 1.1. *Let G be a solvable group, acting faithfully, irreducibly, and quasi-primitively on a finite vector space V. Assume also that G is not metacyclic. Then either G has a regular orbit on V, or (G, V) is listed in Table 4.1.*

2. Notation and Preliminary Results

If V is a finite vector space of dimension d over $\mathrm{GF}(q)$, where q is a prime power, we denote by $\Gamma(q^d) = \Gamma(V)$ the semilinear group of V, i.e.,

$$\Gamma(q^d) = \{x \mapsto ax^\sigma \mid x \in \mathrm{GF}(q^d), a \in \mathrm{GF}(q^d)^\times, \sigma \in \mathrm{Gal}(\mathrm{GF}(q^d)/\mathrm{GF}(q))\}.$$

As mentioned earlier, this group has a normal cyclic subgroup N of order $q^d - 1$ (a so-called *Singer cycle*) consisting of those elements with $\sigma = 1$, which acts regularly on $V \setminus \{0\}$, and $\Gamma(q^d)/N$ is cyclic of order d.

We recall that G is said to act *quasi-primitively* on V, if all nontrivial normal subgroups of G act homogeneously on V. In particular, if G acts primitively then it acts quasi-primitively. We now describe the structure of a finite solvable group G that acts faithfully, irreducibly and quasi-primitively on an d-dimensional finite vector space V over a finite field $\mathbb{F}$ of characteristic p. The following result is from [14, Theorem 2.1].

Theorem 2.1. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a d-dimensional finite vector space V over a finite field $\mathbb{F}$ of characteristic p. Then every normal abelian subgroup of G is cyclic and G has normal subgroups $Z \leq U \leq F \leq A \leq G$ and a characteristic subgroup $E \leq F$ such that,*

(1) *$F = EU$ is a central product where $Z = E \cap U = \mathbf{Z}(E)$ and $\mathbf{C}_G(F) \leq F$;*
(2) *$F/U \cong E/Z$ is a direct sum of completely reducible G/F-modules;*
(3) *There is decomposition $E = E_1 \times E_2 \times \cdots \times E_k$, where E_i is an extraspecial r_i-group for $i = 1, \ldots, s$ for some distinct primes r_i, and $|E_i| = r_i^{2n_i+1}$ for some $n_i \geq 1$. Denoting $e_i = r_i^{n_i}$, we have $e = e_1 \cdots e_s$ divides d and $\gcd(p, e) = 1$;*
(4) *$A = \mathbf{C}_G(U)$ and $G/A \lesssim \mathrm{Aut}(U)$, A/F acts faithfully on E/Z;*
(5) *$A/\mathbf{C}_A(E_i/Z_i) \lesssim \mathrm{Sp}(2n_i, r_i)$;*
(6) *U is cyclic and acts fixed point freely on W where W is an irreducible submodule of V_U;*
(7) *$|V| = |W|^{eb}$ for some integer b;*
(8) *G/A is cyclic and $|G : A| \mid \dim(W)$. We have $G = A$ when $e = d$;*
(9) *Let $g \in G \backslash A$, assume that $o(g) = t$ where t is a prime and let $|W| = p^a$. Then $t \mid a$ and we can view the action of g on U as follows: $U \leq \mathbb{F}_{p^a}^*$ and $g \in \mathrm{Gal}(\mathbb{F}_{p^a} : \mathbb{F}_p)$.*

[14, Table 3.4] consists of a list of the parameters of all examples of non-metacyclic quasi-primitive solvable linear groups with $d = eab$ that might not have a regular orbit, and we reproduce it here (as two tables) for convenience.

Table 2.2. Parameters of examples with $d = ea$ that might not have a regular orbit.

No.	e	p	d	a
1	16	3	16	1
2	16	5	16	1
3	9	2	18	2
4	9	7	9	1
5	9	13	9	1
6	9	2	36	4
7	9	19	9	1
8	9	5	18	2
9	8	3	8	1
10	8	5	8	1
11	8	7	8	1
12	8	3	16	2
13	8	11	8	1
14	8	13	8	1
15	8	17	8	1
16	8	19	8	1
17	8	5	16	2
18	8	3	24	3
19	4	3	4	1
20	4	5	4	1
21	4	7	4	1
22	4	3	8	2
23	4	11	4	1
24	4	13	4	1
25	4	17	4	1
26	4	19	4	1
27	4	23	4	1
28	4	5	8	2
29	4	3	12	3
30	4	29	4	1
31	4	31	4	1
32	4	37	4	1
33	4	41	4	1
34	4	43	4	1
35	4	47	4	1
36	4	7	8	2
37	4	53	4	1
38	4	59	4	1
39	4	61	4	1
40	4	67	4	1
41	4	71	4	1
42	4	73	4	1
43	4	3	16	4
44	4	11	8	2
45	4	5	12	3
46	4	13	8	2
47	4	3	20	5
48	3	2	6	2
49	3	7	3	1
50	3	13	3	1
51	3	2	12	4
52	3	19	3	1
53	3	5	6	2
54	3	7	6	2
55	3	2	18	6
56	3	11	6	2
57	3	13	6	2
58	3	2	24	8
59	3	17	6	2
60	3	7	9	3
61	3	19	6	2
62	2	3	2	1
63	2	5	2	1
64	2	7	2	1
65	2	3	4	2
66	2	11	2	1
67	2	13	2	1
68	2	17	2	1
69	2	19	2	1
70	2	23	2	1
71	2	5	4	2
72	2	3	6	3
73	2	29	2	1
74	2	7	4	2
75	2	3	8	4
76	2	11	4	2
77	2	5	6	3
78	2	13	4	2
79	2	3	10	5
80	2	17	4	2
81	2	7	6	3
82	2	19	4	2
83	2	23	4	2
84	2	5	8	4
85	2	3	12	6
86	2	29	4	2
87	2	31	4	2
88	2	11	6	3
89	2	37	4	2
90	2	41	4	2
91	2	43	4	2
92	2	3	14	7
93	2	13	6	3
94	2	47	4	2
95	2	7	8	4
96	2	53	4	2
97	2	5	10	5
98	2	59	4	2
99	2	61	4	2
100	2	67	4	2
101	2	17	6	3
102	2	71	4	2
103	2	73	4	2

Table 2.3. Parameters of examples with $b = \frac{d}{ea} > 1$ that might not have a regular orbit

No.	e	p	d	a	b
104	2	3	4	1	2
105	2	5	4	1	2
106	2	7	4	1	2
107	2	11	4	1	2
108	2	13	4	1	2
109	2	17	4	1	2
110	2	3	8	2	2
111	2	3	6	1	3
112	2	5	6	1	3
113	2	3	8	1	4
114	3	7	6	1	2
115	3	2	12	2	2
116	3	2	18	2	3
117	4	3	8	1	2
118	4	5	8	1	2
119	4	7	8	1	2
120	4	11	8	1	2
121	4	3	12	1	3
122	4	3	16	1	4
123	4	3	16	2	2
124	8	3	16	1	2
125	8	5	16	1	2
126	8	3	24	1	3
127	9	2	36	2	2

Note that, although our vector space V was originally defined as being of dimension d over the field $\mathbb{F}_q$ with q a power of p, it is convenient to regard V as being a space of dimension da over $\mathbb{F}_p$, where $q = p^a$. This allows us to consider G as a subgroup of the larger group $\Gamma L(d, p^a)$ rather than $\mathrm{GL}(d, q)$. Some of the examples listed in Table 4.1 with $a > 1$ are not absolutely irreducible, which means that they could also be considered as subgroups of $\mathrm{GL}(d/a, p^a)$ (or $\mathrm{GL}(4, 9)$ in Line 75).

3. Computations

In this section, we describe how we constructed candidates for groups G with parameters equal to one of the entries in Tables 2.2 and 2.3 on a computer, and checked in each case whether there were any such examples without regular orbits. We carried out these computations in Magma. The results of these computations are tabulated in the next section.

We know from Theorem 2.1 that G has a normal subgroup F, which is a central product of a subgroup $U = Z(F)$ and an extraspecial group E of order r^{2e+1}, where $|U|$ divides $p^a - 1$ and U acts irreducibly on a subspace W of V of dimension a. Since G is quasi-primitive, U acts homogeneously on V, and by [7, Lemma 1.10] (applied with M, K and F equal to U, $\mathbb{F}_p$ and $\mathbb{F}_{p^a}$), we can regard V as a vector space over the field $\mathbb{F}_{p^a}$ of order p^a, and we have $A = C_G(U) \leq C_{\mathrm{GL}(d,p)}(U) \cong \mathrm{GL}(d/a, p^a)$.

This last isomorphism follows from [4, Chapter 3, Theorem 5.4 (iii)] (although the statement assumes that U acts absolutely irreducibly on U, the proof of this ismorphism does not). Alternatively, note that the containment $C_{\mathrm{GL}(d,p)}(U) \leq \mathrm{GL}(d/a, p^a)$ is proved in [7, Lemma 1.10], and $\mathrm{GL}(d/a, p^a) \leq C_{\mathrm{GL}(d,p)}(U)$ holds because U homogeneous and abelian implies that the elements of U are scalar matrices as elements of $\mathrm{GL}(d/a, p^a)$.

Furthermore, G is isomorphic to a subgroup of the normalizer of $C_{\mathrm{GL}(d,p)}(U)$ in $\mathrm{GL}(d, p)$. We claim that this normalizer is isomorphic to the group $\Gamma L(d/a, p^a)$ of semilinear maps $\mathbb{F}_{p^a}^{d/a} \to \mathbb{F}_{p^a}^{d/a}$, which has $\mathrm{GL}(d/a, p^a)$ as a normal subgroup with

$$\Gamma L(d/a, p^a)/\mathrm{GL}(d/a, p^a) \cong \mathrm{Gal}(\mathbb{F}_{p^a}, \mathbb{F}_p),$$

which is cyclic of order a. To see this, note first that the elements of $\Gamma L(d/a, p^a)$ induce linear maps $\mathbb{F}_p^d \to \mathbb{F}_p^d$, and the subgroup $\mathrm{GL}(d/a, p^a)$ corresponds to $C_{\mathrm{GL}(d,p)}(U)$, so the normalizer of $C_{\mathrm{GL}(d,p)}(U)$ contains (a group isomorphic to) $\Gamma L(d/a, p^a)$. To see that this is the full normalizer, consider the conjugation action of an element of the normalizer on $Z \cup \{0\} \cong \mathbb{F}_{p^a}$, where $Z := Z(C_{\mathrm{GL}(d,p)}(U))$. Since this action preserves both the additive and multiplicative structures of $Z \cup \{0\}$, it induces a field automorphism of $\mathbb{F}_{p^a}$, and the claim follows. Then, since $A = C_G(U)$, the quotient group G/A can be identified with a subgroup of $\mathrm{Gal}(\mathbb{F}_{p^a}, \mathbb{F}_p)$.

We shall now summarize some properties of extraspecial and symplectic-type groups and their representations. Convenient background references for much of this material are [5, Section 4.6] or [4, Section 5.5].

For a prime r and integer $e \geq 1$ there are two isomorphism types of extraspecial r-groups E of order r^{2e+1}, and they both arise as central products of e extraspecial groups of order r^3 ([4, Chapter 5, Theorem 5.2] or [5, Proposition 4.6.1]). There are $r - 1$ equivalence classes of faithful absolutely irreducible representations in characteristics other than r, and they all have dimension e and are distinguished by their actions on $Z(E)$ ([4, Chapter 5, Theorem 5.4] or [5, Proposition 4.6.3]). Since $\mathrm{Aut}(E)$ acts transitively on the $r - 1$ nontrivial elements of $Z(E)$, these $r-1$ representations are quasi-equivalent to each other; that is, they are equivalent under the action of $\mathrm{Aut}(E)$.

Assume first that $ea = d$ (or, equivalently, that $b = 1$). Then, since e is the dimension of all non-linear absolutely irreducible representations of E in characteristic $p \neq r$, the group E must be absolutely irreducible as a subgroup of $\mathrm{GL}(e, p^a)$ and, for a given isomorphism type of E, since its faithful absolutely irreducible representations are quasi-equivalent, there is a unique conjugacy class of subgroups of $\mathrm{GL}(e, p^a)$ isomorphic to E.

Our methods for the cases when r is odd and even are slightly different, so we consider them separately. Suppose first that r is odd. We claim that E must have exponent r. The other isomorphism type of extraspecial group has exponent r^2, and its elements of order r form a characteristic subgroup E_r of index r in E with non-cyclic center of order r^2 [4, Chapter 5, Theorem 5.2 (ii)]. So E_r has no faithful irreducible representations, but it acts faithfully on V, so it cannot be acting homogeneously, contradicting the quasi-primitivity of G.

There is existing functionality in MAGMA for constructing E as a subgroup of $\mathrm{GL}(e, p^a)$ and its normalizer N_A in $\mathrm{GL}(e, p^a)$ (which is not usually a solvable group). The group N_A has the structure $Z_0 r^{1+2e}.\mathrm{Sp}(2e, r)$ (with $|E| = r^{1+2e}$), where $Z_0 := Z(\mathrm{GL}(d/a, p^a))$ is the group of scalar matrices. (This result is stated in various places, such as [5, Table 4.6.B], but it seems hard to find a proof in the literature. To see why it is true, note that any automorphism of E must preserve the associated bilinear form defined by commutators in E modulo scalars, and an automorphism induced by an element of $\mathrm{GL}(e, p^a)$ must centralize scalar matrices, so it preserves the form absolutely, and corresponds to an element of $\mathrm{Sp}(2e, r)$. Conversely, any element of $\mathrm{Sp}(2e, r)$ induces an automorphism of E that centralizes $Z(E)$, so it preserves the equivalence class of the representation defined by E, and is hence induced by an element of $\mathrm{GL}(e, p^a)$.) The group E consists of the elements of order dividing r in $O_r(N_A)$, and so it is characteristic in N_A.

After constructing N_A, we embed it in $\mathrm{GL}(d, p)$ using the natural embedding $\mathrm{GL}(e, p^a) \to \mathrm{GL}(d, p)$. Then, as a subgroup of $\mathrm{GL}(d, p)$, $Z_0 E$ acts irreducibly with centralizing field $\mathbb{F}_{p^a}$, so $C_{\mathrm{GL}(d,p)}(Z_0 E) = Z_0$. Note that the normal subgroup A of the group G that we are attempting to construct is the intersection of G with N_A. (This follows from the fact that A is the intersection of G with $\mathrm{GL}(e, p^a)$ under our identification of $\mathrm{GL}(e, p^a)$ with $C_{\mathrm{GL}(d,p)}(U)$.) So G is a subgroup of $N := N_{\mathrm{GL}(d,p)}(N_A)$, and the method that we chose to find G involves computing this group N. To do that, we compute $\mathrm{Aut}(N_A)$, and then check which outer automorphisms of N_A can be induced by conjugation in $\mathrm{GL}(d, p)$. (This uses the fact that $C_N(N_A) \leq N_A$, which follows from $C_{\mathrm{GL}(d,p)}(Z_0 E) = Z_0 \leq N_A$.) This automorphism group computation was one of the slowest parts of the complete process, and it is possible that there are faster ways of computing N from N_A, but it eventually completed successfully in all of the examples.

After computing N, we compute its subgroups of increasingly large index, by repeated application of the MaximalSubgroups command in MAGMA, using conjugacy testing to ensure that we only consider one representative of each N-conjugacy class of subgroups. For each such subgroup, we test whether it is solvable and quasi-primitive. If so, then we test whether it has regular orbits. If so then we do not need to consider any of its proper subgroups, because they would also have regular orbits. If not, then we have identified an example without regular orbits.

The situation is more complicated when $r = 2$. In that case, by ([4, Chapter 5, Theorem 5.2 (iii)] or [5, Proposition 4.6.1 (iii)]), the extraspecial groups of order 2^3 are Q_8 and D_8, the dihedral and quaternion groups of order 8, and those of order 2^{2e+1} are: E^+, the central product of e copies of D_8; and E^-, the central product of $e-1$ copies of D_8 and one of Q_8. (Note that $D_8 * D_8 \cong Q_8 * Q_8$.)

It is straightforward to check that the central products $D_8 * C_4$ and $Q_8 * C_4$ of D_8 and Q_8 with a cyclic group of order 4 are isomorphic groups of order 16, and so the central products $E^+ * C_4$ and $E^- * C_4$ are isomorphic groups S of order 2^{2e+2}, which are known as 2-groups of *symplectic-type*. By [5, Proposition 4.6.3], the faithful absolutely irreducible representations of E^+, E^-, and S in characteristic $p \neq 2$ are quasi-equivalent and have dimension e. Those of E^+ and E^- can be written over $\mathrm{GF}(p^a)$ for any odd prime p and any $a \geq 1$, whereas those of S can be written over $\mathrm{GF}(p^a)$ if and only if $4 \mid p^a - 1$; i.e. if and only if either $p \equiv 1 \bmod 4$, or $p \equiv 3 \bmod 4$ and a is even.

Suppose first that $4 \mid p^a - 1$. Since the elements of the C_4 central factor of S are represented by scalar matrices, the normalizer $N(S)_A$ of S in $\mathrm{GL}(e, p^a)$ contains the normalizers of E^+ and of E^-, and so we can deal with both of these cases together by computing it. The group $N(S)_A$ has the structure $Z_0 2^{1+2e}.\mathrm{Sp}(2e, 2)$, and S consists of the elements of $O_2(N(S)_A)$ of order dividing 4, so S is characteristic in $N(S)_A$. We use the same process as for the case with r odd but with S in place of E.

When 4 does not divide $p^a - 1$, we have $E = O_2(G) \lhd G$, where E can be isomorphic to either E^+ or E^-, and we must carry out the computations for these two cases separately. By [5, Table 4.6.B], we have $N_A = N_{\mathrm{GL}(e,p^a)}(E) \cong Z_0 2^{1+2e}.\mathrm{GO}^+(2e, 2)$ and $N_0 = Z_0 2^{1+2e}.\mathrm{GO}^-(2e, 2)$ in the two cases, and we proceed as in the case r odd in both cases.

It remains to consider the case $d/(ea) = b > 1$. Then, by quasi-primitivity, the group E acts homogeneously as a subgroup of $\mathrm{GL}(d, p)$ and, since it is centralized by U, which acts as scalar multiplication as a subgroup of $\mathrm{GL}(d/a, p^a)$, the group E also acts homogeneously as a subgroup of $\mathrm{GL}(d/a, p^a)$. So it has b isomorphic absolutely irreducible constituents, each of dimension e over $\mathbb{F}_{p^a}$. Now, by [4, Theorem 3.5.4], we have $C_A := C_{\mathrm{GL}(d/a,p^a)}(E) \cong \mathrm{GL}(b, p^a)$, and hence also $\mathrm{CC}_A := C_{\mathrm{GL}(d/a,p^a)}(C_A) \cong \mathrm{GL}(e, p^a)$. Now the normalizer in $\mathrm{GL}(d, p)$ of E also normalizes $N_{\mathrm{CC}_A}(E) \cong N_{\mathrm{GL}(e,p^a)}(E)$, and hence it also normalizes the subgroup $N_A := \langle C_A, N_{\mathrm{CC}_A}(E) \rangle$ (or $N(S)_A := \langle C_A, N_{\mathrm{CC}_A}(S) \rangle$ when $4 \mid p^a - 1$).

We can compute $N_{\mathrm{CC}_A}(E)$ (or $N_{\mathrm{CC}_A}(S)$) as in the case $b = 1$, and C_A is also straightforward to compute, so we can compute the group N_A (or $N(S)_A$), and we use this to construct the normalizer in $\mathrm{GL}(d, p)$ of E and its subgroups in the same way as in the case $b = 1$.

4. Table of results

Here is a list of those entries in Table *2.2* for which there is at least one example of a group with no regular orbit. The leftmost column lists the number of the corresponding row in Table 2.2 or 2.3. In each case we give the number "num gps" of such examples (up to conjugacy in $\mathrm{GL}(d, p)$), and the order "max $|G|$" of the largest example. In cases where $r = 2$ and 4 does not divide $2^a - 1$, we have handled the E^+ and E^- cases separately.

The whole data package about these groups' structure is long and cannot be written explicitly in a paper, so it is provided in separate files. All groups are constructed as matrix groups in suitable fields, and each file corresponds to all the groups in a particular row in Table *4.1*.

Table 4.1. Parameters of quasi-primitive solvable groups that do not have a regular orbit.

No.	e	p	d	a	b	num gps	max $\lvert G\rvert$	Note
1	16	3	16	1	1	12	15925248	E^-
3	9	2	18	2	1	40	559872	
9	8	3	8	1	1	27	18432	E^+
9	8	3	8	1	1	71	165888	E^-
10	8	5	8	1	1	22	331776	
19	4	3	4	1	1	14	2304	E^+
19	4	3	4	1	1	9	640	E^-
20	4	5	4	1	1	24	4608	
21	4	7	4	1	1	17	6912	E^+
22	4	3	8	2	1	72	18432	
23	4	11	4	1	1	4	11520	E^+
24	4	13	4	1	1	5	13824	
25	4	17	4	1	1	4	18432	
28	4	5	8	2	1	3	55296	
48	3	2	6	2	1	7	1296	
49	3	7	3	1	1	4	1296	
50	3	13	3	1	1	2	2592	
51	3	2	12	4	1	8	12960	
52	3	19	3	1	1	1	3888	
53	3	5	6	2	1	10	10368	
62	2	3	2	1	1	2	48	
63	2	5	2	1	1	2	96	
64	2	7	2	1	1	2	144	
65	2	3	4	2	1	13	384	
66	2	11	2	1	1	2	240	
67	2	13	2	1	1	2	288	
68	2	17	2	1	1	3	384	
69	2	19	2	1	1	2	432	
71	2	5	4	2	1	16	1152	
72	2	3	6	3	1	2	1872	
74	2	7	4	2	1	7	2304	
75	2	3	8	4	1	10	7680	
117	4	3	8	1	2	9	2304	

5. Acknowledgement

Yang was partially supported by a grant from the Simons Foundation (No 499532, YY). The authors are grateful to the referees for the valuable suggestions which greatly improved the manuscript.

References

[1] Wieb Bosma, John Cannon, Catherine Playoust, The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4) (1997), 235–265. Computational algebra and number theory (London, 1993).
[2] S. Dolfi and E. Pacifici, 'Zeros of Brauer characters and linear actions of finite groups', *J. Algebra* 340 (2011), 104–113.
[3] S. Dolfi and E. Pacifici, 'Zeros of Brauer characters and linear actions of finite groups: small primes', *J. Algebra* 399 (2014), 343–357.
[4] D. Gorenstein, *Finite Groups*, Harper and Row, 1968.
[5] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, Cambridge University Press, Cambridge, 1990.
[6] M. Lewis, G. Navarro, T. R. Wolf, 'p-parts of character degrees and the index of the Fitting subgroup', *J. Algebra* 411 (2014), 182–190.
[7] A. Lucchini, F. Menegazzo, M. Morigi, 'On the number of generators and composition length of finite linear groups', *J. Algebra* 243 (2001), 427–447.
[8] I. N. Ponomarenko, 'Bases of Schurian antisymmetric coherent configurations and isomorphism test for Schurian tournaments', *J. Math. Sci. (N.Y.)* 192 (2013), 316–338.
[9] Y. Yang, 'Regular orbits of finite primitive solvable groups', *J. Algebra* 323 (2010), 2735–2755.
[10] Y. Yang, 'Regular orbits of finite primitive solvable groups, II', *J. Algebra* 341 (2011), 23–34.
[11] Y. Yang, 'Regular orbits of nilpotent subgroups of solvable linear groups', *J. Algebra* 325 (2011), 56–69.
[12] Y. Yang, 'Large character degrees of solvable $3'$-groups', *Proc. Amer. Math. Soc.* 139 (2011), 3171–3173.
[13] Y. Yang, 'Blocks of small defect', *J. Algebra* 429 (2015), 192–212.
[14] Y. Yang, A. Vasil'ev, E. Vdovin, 'Regular orbits of finite primitive solvable groups, III', *J. Algebra* 590 (2022), 139–154.

Mathematics Institute, University of Warwick, Coventry CV47AL, UK.
Email address: DerekHo1127@gmail.com

Three Gorges Mathematical Research Center, China Three Gorges University, Yichang, Hubei 443002, China and Department of Mathematics, Texas State University, 601 University Drive, San Marcos, TX 78666, USA.
Email address: yang@txstate.edu