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The random cluster model on the complete graph via large deviations

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Abstract

We study the emergence of the giant component in the random cluster model on the complete graph, which was first studied by Bollobás et al. (1996). We give an alternative analysis using a thermodynamic/large deviations approach introduced by Biskup et al. (2007) for the case of percolation. In particular, we compute the rate function for large deviations of the size of the largest connected component of the random graph for $q \geq 1$.

1. Introduction

The random cluster model was introduced by Fortuin and Kasteleyn in [7] as a generalisation of several existing models in statistical physics. For a finite, simple graph $G = (V, E)$, the random cluster model with *edge weight* $p \in [0, 1]$ and *cluster weight* $q > 0$ is the probability measure $\phi_{G,p,q}$ on the state space $\Omega_G = \{0, 1\}^E$ defined by

$$\phi_{G,p,q}[\omega] := \frac{\{\prod_{e\in E} p^{\omega_e}(1-p)^{1-\omega_e}\}q^{k(\omega)}}{Z_{G,p,q}}, \quad \omega = (\omega_e)_{e\in E} \in \Omega_G, \tag{1}$$

where

$$Z_{G,p,q} := \sum_{\omega\in\Omega_G} \{\prod_{e\in E} p^{\omega_e}(1-p)^{1-\omega_e}\}q^{k(\omega)} \tag{2}$$

is the normalising partition function, and $k(\omega)$ is the number of connected components in the graph $G(\omega) = (V, E(\omega))$ with $E(\omega) = \{e \in E : \ \omega_e = 1\}$. For an element $\omega \in \Omega_G$, we say that

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the edge e is *open* if $e \in E(\omega)$, otherwise, it is *closed*. The measure $\phi_{G,p,q}$ may be equivalently interpreted as a probability distribution on the set of random edge subgraphs $G(\omega) = (V, E(\omega))$ of G. For $q = 1$, we recover the percolation model, for which we use the standard shorthand $\phi_{G,p}$.

We will be interested in the particular case of the complete graph $K_n = (V_n, E_n)$ when we choose $\lambda \in [0, \infty)$ and rescale the edge weight as $p = \lambda/n$. For the corresponding random cluster measure, we use the shorthand $\phi_{n,\lambda,q} := \phi_{K_n,\lambda/n,q}$. Similarly, we denote the partition function as $Z_{n,\lambda,q}$, and use the shorthand $\phi_{n,\lambda}$ in the case of percolation. For fixed $q \geq 1$, the random cluster model is stochastically ordered in λ, and we study how the random graph obtained under the measure $\phi_{n,\lambda,q}$ varies with respect to this parameter.

One particular quantity of interest is the size of the largest connected component in the random graph. In their groundbreaking paper [6], Erdős and Rényi studied the measure $\phi_{n,\lambda}$ and showed the existence of a critical parameter $\lambda_c = 1$ marking the emergence of a *giant* component containing a positive proportion of the total number of vertices asymptotically almost surely. More specifically, they showed that, with probability tending to one as $n \to \infty$ under the percolation measure $\phi_{n,\lambda}$, the largest component of the graph $K_n(\omega) = (V_n, E_n(\omega))$ is of order $\log n$ for $\lambda < 1$ and of order n for $\lambda > 1$. This abrupt change in the size of the largest component (in the limit as $n \to \infty$) is known as an *asymptotic phase transition*.

The main idea in the proof of the asymptotic phase transition of [6] is an exploration process whereby one chooses a vertex and sequentially inspects which vertices are connected to it in the random graph. Provided one has not yet explored a large fraction of the vertices, this exploration can be approximated by a Poisson branching process with mean λ in the limit as $n \to \infty$. The emergence of a giant component then corresponds to the *survival* of the corresponding Poisson branching process. This method is not crucially dependent on the structure of the complete graph, and has been successfully applied to establish similar asymptotic phase transitions on a variety of families of graphs, such as the hypercube in [1], and expander graphs in [2].

For $q \neq 1$, the presence of the factor $q^{k(\omega)}$ in the random cluster measure $\phi_{n,\lambda,q}$ creates an additional complexity — the states of the edges in the random graph are no longer independent and it is no longer clear that the exploration process may be used. Nevertheless, it has been established by Bollobás, Grimmett, and Janson in [4] that the largest component of the complete graph undergoes an asymptotic phase transition at a particular value $\lambda_c(q)$. The claim is proven by randomly colouring the vertices using two colours (say red and green) in a particular way so that the distribution of edges on the *red vertices* is given by a percolation measure, to which we may apply the exploration process used in [6]. Note that the colouring argument depends crucially on the fact that for a fixed set R of red vertices, the conditional distribution of edges yields a random percolation on the complete graph induced on the set R. This is a particular fact that does not generalise to more structured families of graphs.

The purpose of the present paper is to provide a new analysis of the random cluster model $\phi_{n,\lambda,q}$ on the complete graph, with the hope that it may be extended to wider families of graphs. Our analysis employs the methods introduced by Biskup, Chayes and Smith in [3], who analysed the largest component for percolation on the complete graph using a large deviations approach instead of branching processes. We extend this approach to the random cluster model on the complete graph for $q \geq 1$. In particular, we compute the rate function for large deviations of the size of the largest connected component, thereby recovering the asymptotic phase transition proven in [4]. In addition, we obtain a limit for the *free energy* of the random cluster model. This was also computed in [4] via the colouring argument, but used to study the large deviations of the *number* of connected components, rather than their size.

As a byproduct of our analysis, we also obtain the exponential decay rate for the events that the random graph is connected and acyclic, respectively.

2. Statement of results

In this section, we state the main results of the paper. Fix $q > 0$ and $\lambda > 0$. We use $\phi_{n,\lambda,q}$ to denote the random cluster probability measure on the complete graph K_n with edge weight $p = \lambda/n$ and cluster weight q. Similarly, we denote the corresponding partition function by $Z_{n,\lambda,q}$. In some of our results, we will analyse the measure of an event $A \subset \Omega_{K_n}$ before normalisation, in which case we write $Z_{n,\lambda,q}[A] := Z_{n,\lambda,q}\phi_{n,\lambda,q}[A]$. When $q = 1$, we drop the subscript q entirely and use the standard percolation notation $\phi_{n,\lambda}$.

Let $\mathcal{N}_r$ be the number of connected components in $K_n(\omega)$ of size larger than r, and let $\mathcal{V}_r$ be the set of vertices in $K_n(\omega)$ belonging to components of size larger than r. In addition, recall the *entropy function*, defined for $\theta \in (0, 1)$ by

$$S(\theta) = -\theta \log \theta - (1-\theta)\log(1-\theta). \tag{3}$$

Following [3], we define two functions on $[0, \infty)$ by

$$\pi_1(x) = 1 - e^{-x}, \quad \Psi(x) = \left(\log x - \frac{1}{2}\left[x - \frac{1}{x}\right]\right) \wedge 0. \tag{4}$$

Note that $\Psi(x) < 0$ if and only if $x > 1$. Finally, we define the function

$$\begin{aligned}\Phi(\theta, \lambda, q) = S(\theta) &+ (1-\theta)\log[1 - \pi_1(\lambda\theta)] + \theta \log \pi_1(\lambda\theta)\\ &+ (1-\theta)\{\Psi(\tfrac{\lambda(1-\theta)}{q}) - (\tfrac{q-1}{2q})\lambda(1-\theta) + \log q\}.\end{aligned} \tag{5}$$

Our main result may then be stated as follows:

Theorem 2.1. *Fix $q > 0$ and $\lambda > 0$. Then, for every $\theta \in [0, 1]$,*

$$\lim_{\epsilon \downarrow 0} \lim_{n\to\infty} \frac{1}{n} \log \phi_{n,\lambda,q}\big[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\big] = \Phi(\theta, \lambda, q) - \sup_{\theta\in[0,1]} \Phi(\theta, \lambda, q). \tag{6}$$

In the language of large deviations, Theorem 2.1 says that in the limit $\epsilon \downarrow 0$, the sequence of random variables $|\mathcal{V}_{\epsilon n}|/n$ satisfies the large deviations principle with rate function $\Phi(\theta, \lambda, q)$. Moreover, if $\theta^* \in [0, 1]$ maximises $\Phi(\theta, \lambda, q)$, then Theorem 2.1 implies that approximately $\theta^* n$ vertices of $K_n(\omega)$ belong to components of order n. The following lemma tells us that in fact, these vertices all belong to a single component of size $\theta^* n$:

Lemma 2.2. *Fix $q > 0$ and $\lambda > 0$. Then, for every $\lambda > 0$ and $\epsilon > 0$ there exists $c = c(\lambda, \epsilon) > 0$ such that for every $\theta > \epsilon > 0$,*

$$\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{\epsilon n} = 1] \geq (1 - e^{-cn})\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor]. \tag{7}$$

It remains to specify the maximiser θ^*. We will see in Section 5 that the value θ^* maximising $\Phi(\theta, \lambda, q)$ is given by

$$\theta(\lambda, q) = \begin{cases} 0 & \text{if } \lambda < \lambda_c(q) \\ \theta_{\max} & \text{if } \lambda \geq \lambda_c(q) \end{cases} \tag{8}$$

where the *critical* value λ_c is defined as

$$\lambda_c(q) = \begin{cases} q & \text{if } q \leq 2 \\ 2\left(\frac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2 \end{cases} \tag{9}$$

and $\theta_{\max}$ is the largest solution of the *mean field equation*

$$e^{-\lambda\theta} = \frac{1-\theta}{1+(q-1)\theta}. \tag{10}$$

The solutions to (10) are discussed in detail in ([4], Lemma 2.5). In particular, it is shown there that $\theta_{\max} > 0$ for $\lambda > \lambda_c$. Consequently, Theorem 2.1 implies that the largest component of the graph $K_n(\omega)$ is of order n asymptotically almost surely when $\lambda > \lambda_c$. In this way, we recover the asymptotic phase transition for the size of the largest connected component established in [4]. The behaviour for $\lambda = \lambda_c$ is more complicated, and will not be discussed here.

Theorem 2.1 was proven in [3] for the special case of percolation by conditioning on the set A of vertices contained in the largest component and computing the exponential rates of the events that A is connected and that A^c does not contain any large components. We will extend this approach to the random cluster model. To this end, let K be the event that $K_n(\omega)$ is connected, let B_r be the event that $K_n(\omega)$ contains no components of size larger than r and let L be the event that $K_n(\omega)$ is acyclic. The following three theorems, which generalise ([3], Theorems 2.3, 2.4 and 2.5), yield the exponential rates of these three events, beginning with the event K:

Theorem 2.3. *Fix $q > 0$ and $\lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}\big[K\big] = \log \pi_1(\lambda). \tag{11}$$

Moreover, convergence is uniform for λ belonging to compact subsets of $[0,\infty)$.

Observe that the limit obtained in (11) is independent of q. This is no coincidence; on the event K, $K_n(\omega)$ is connected and so $k(\omega) = 1$. It follows that $Z_{n,\lambda,q}[K] = q\phi_{n,\lambda}[K]$, and so the proof given in ([3], Theorem 2.3) for percolation trivially generalises to the random cluster model.

Next, we compute the exponential rate of the event L. On this event, we have the correspondence $k(\omega) = n - |E_n(\omega)|$ between the number of components and edges of the graph $K_n(\omega)$. In particular, it is possible to absorb the cluster weight q into the edge weight and extend ([3], Theorem 2.4) in the following form:

Theorem 2.4. *Fix $q > 0$ and $\lambda > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}\big[L\big] = \lim_{r\to\infty}\lim_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}\big[B_r \cap L\big] = \Psi(\tfrac{\lambda}{q}) - (\tfrac{q-1}{2q})\lambda + \log q. \tag{12}$$

Moreover, convergence is uniform for λ belonging to compact subsets of $(0,\infty)\setminus\{q\}$.

As in [3], one may prove that the exponential rate of the event that $K_n(\omega)$ contains only small components coincides with the exponential rate of the event that $K_n(\omega)$ is acyclic. This is not surprising, as in [4] it was shown that almost all vertices outside of the giant component belong to trees. This argument is summarised in the following analogue of ([3], Theorem 2.5):

Theorem 2.5. *Fix $q > 0$ and $\lambda > 0$. Then*

$$\lim_{r\to\infty}\liminf_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}\big[B_r\big] = \lim_{r\to\infty}\limsup_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}\big[B_r\big] = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}\big[L\big] \tag{13}$$

and

$$\lim_{\epsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[B_{\epsilon n}\big]=\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[B_{\epsilon n}\big]=\lim_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[L\big]. \tag{14}$$

Moreover, convergence is uniform for λ *belonging to compact subsets of* $(0,\infty)\setminus\{q\}$.

Observe that the preceding three theorems compute the exponential rates of the events K, L and B_r for the random cluster measure *before* normalisation. Indeed, we will compute the rate function for the size of the largest connected component in the following form:

Theorem 2.6. *Fix* $q>0$ *and* $\lambda>0$. *Then, for every* $\theta\in[0,1]$,

$$\lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[|\mathcal{V}_{\epsilon n}|=\lfloor\theta n\rfloor\big]=\Phi(\theta,\lambda,q). \tag{15}$$

Moreover, convergence is uniform for λ *belonging to compact subsets of* $(0,\infty)\setminus\{q\}$.

In order to turn these theorems into statements about probabilities in the random cluster model, we will reintroduce the partition function using the following result:

Theorem 2.7. *Fix* $q>0$ *and* $\lambda>0$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}=\sup_{\theta\in[0,1]}\Phi(\theta,\lambda,q). \tag{16}$$

The limit of Eq. (16) is known as the *free energy* of the random cluster model, and is not a new quantity of interest; it was computed in ([4], Theorem 2.6). In Lemma 5.2, we show that our computation agrees with theirs.

The structure of the free energy provides some hint as to its derivation. Indeed, for fixed $\epsilon>0$, one may decompose the partition function $Z_{n,\lambda,q}$ according to the number of vertices in components of size at least ϵn to obtain

$$Z_{n,\lambda,q}=\sum_{k=0}^{n}Z_{n,\lambda,q}\big[|\mathcal{V}_{\epsilon n}|=(\tfrac{k}{n})n\big]. \tag{17}$$

We seek to apply Theorem 2.6 to each of the terms on the right hand side of (17). We may then dominate the sum in (17) in terms of its largest summand, and apply the *Laplace Principle* when taking the appropriate limit. However, these steps involve some technicalities, which will be discussed in detail in Section 5. In particular, uniform convergence of the rate function is required in order to pass to a supremum in the limit. This is the reason why it is specified in our theorems, while it was not required for the random graphs in [3].

3. Acyclic graphs

In this section, we prove Theorems 2.4 and 2.5 by analysing the quantity $Z_{n,\lambda,q}[L\cap B_r]$, as was done for percolation in [3]. Let m_l be the number of connected components of size l in $K_n(\omega)$. Then, the weight of a configuration $\omega\in L$ is given by

$$Z_{n,\lambda,q}[\omega]=q^{\sum m_l}(1-\tfrac{\lambda}{n})^{\binom{n}{2}-n+\sum m_l}\prod_{l\geq 1}(\tfrac{\lambda}{n})^{(l-1)m_l}. \tag{18}$$

If $\omega \in L \cap B_r$, then we must also have $m_l = 0$ when $l > r$. Observe that, conditionally on the numbers $m_1, \ldots, m_n$, there are $\frac{n!}{\prod_l m_l!(l!)^{m_l}}$ ways of choosing appropriately sized components in $K_n(\omega)$. In addition, each component of size l has $a_l = l^{l-2}$ possible spanning trees. Thus

$$Z_{n,\lambda,q}[L \cap B_r] = \sum_{\substack{\sum l m_l = n \\ m_l = 0 \forall l > r}} \frac{n!}{\prod_l [m_l!(l!)^{m_l}]} q^{\sum m_l} (1 - \tfrac{\lambda}{n})^{\binom{n}{2} - n + \sum m_l} \prod_{l \geq 1} \left[a_l (\tfrac{\lambda}{n})^{l-1} \right]^{m_l}. \tag{19}$$

Introducing the sum

$$Q_{n,k,r} = \sum_{\substack{\sum l m_l = n \\ \sum m_l = k \\ m_l = 0 \forall l > r}} \prod_{l \geq 1} \left(\frac{a_l}{l!} \right)^{m_l} \frac{1}{m_l!} \tag{20}$$

allows us to rewrite (19) as

$$Z_{n,\lambda,q}[L \cap B_r] = n! (\tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{\binom{n}{2} - n} \sum_{k=1}^{n} (\tfrac{\lambda}{n})^{-k} [q(1 - \tfrac{\lambda}{n})]^k Q_{n,k,r}. \tag{21}$$

The Eq. (21) employs the same re-arrangement as that done for percolation in [3], with the random cluster model introducing an extra factor of q^k in the summand. We cite the following proposition regarding the asymptotic behaviour of the quantity $Q_{n,k,r}$:

Proposition 3.1 (*[3], Proposition 4.1*). *Consider the polynomial*

$$F_r(s) = \sum_{l=1}^{r} \frac{s^l a_l}{l!}. \tag{22}$$

Then for all $n, k, r \geq 1$

$$Q_{n,k,r} \leq \frac{1}{k!} \inf_{s>0} \frac{F_r(s)^k}{s^n}. \tag{23}$$

Moreover, for each $\eta > 0$ there is an $n_0 < \infty$ and a sequence $(c_r)_{r \geq 1}$ of positive numbers such that for all $n \geq n_0$, $k \geq 1$ and $r \geq 2$ such that $k < (1-\eta)n$ and $rk > n(1+\eta)$, we have

$$Q_{n,k,r} \geq \frac{c_r}{\sqrt{n}} \frac{1}{k!} \inf_{s>0} \frac{F_r(s)^k}{s^n}. \tag{24}$$

Applying Stirling's Formula to the factorial, we may use Proposition 3.1 to obtain an upper bound on the summand in (21) of

$$(\tfrac{\lambda}{n})^{-k} [q(1 - \tfrac{\lambda}{n})]^k Q_{n,k,r} \leq e^{o(n)} \inf_{s>0} \exp\{n \Theta_r(s, k/n, \lambda/q)\}, \tag{25}$$

where the error term is bounded uniformly for λ belonging to compact subsets of $(0, \infty)$, and the function $\Theta_r(s, \theta, \alpha)$ is given by

$$\Theta_r(s, \theta, \alpha) = -\theta \log \alpha - \theta \log \theta + \theta + \theta \log F_r(s) - \log s. \tag{26}$$

The notation $\alpha = \lambda/q$ is chosen to agree with [3]. Notice that a sum of n non-negative terms is sandwiched between its maximal term and n times its maximal term. This implies an upper bound on Eq. (21) of

$$Z_{n,\lambda,q}[L \cap B_r] \leq n! n (\tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{\binom{n}{2} - n} e^{o(n)} \sup_k \inf_{s>0} \exp\{n \Theta_r(s, k/n, \lambda/q)\} \tag{27}$$

and a lower bound (for fixed $\eta > 0$) on Eq. (21) of

$$Z_{n,\lambda,q}[L \cap B_r] \geq n!(\tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{\binom{n}{2}-n} e^{o(n)} \sup_k \inf_{s>0} \exp\{n \Theta_r(s, k/n, \lambda/q)\}, \tag{28}$$

where the supremum is now taken over k satisfying $\frac{1}{r}(1+\eta)n < k < (1-\eta)n$. If we can show that the supremum of Θ is contained in this interval (for sufficiently small η and sufficiently large r) then the bounds (27) and (28) will coincide. This supremum is evaluated in ([3], Lemma 4.2). We reproduce their lemma here, as it will be important to check that any convergence is uniform:

Lemma 3.2 *([3], Lemma 4.2). Let $\alpha > 0$ and $r \geq 2$. Then there is a unique $(s_r, \theta_r) \in [0, \infty] \times [1/r, 1]$ for which*

$$\Theta_r(s_r, \theta_r, \alpha) = \sup_{1/r \leq \theta \leq 1} \inf_{s>0} \Theta_r(s, \theta, \alpha). \tag{29}$$

Moreover, $s_r \in (0, \infty)$ satisfies the limit

$$\lim_{r\to\infty} s_r = \begin{cases} \alpha e^{-\alpha} & \text{if } \alpha < 1 \\ \frac{1}{e} & \text{if } \alpha > 1 \end{cases} \tag{30}$$

and $\theta_r \in (1/r, 1)$ satisfies the limit

$$\lim_{r\to\infty} \theta_r = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } \alpha < 1 \\ \frac{1}{2\alpha} & \text{if } \alpha > 1 \end{cases} \tag{31}$$

Combining (30) and (31) yields the limit

$$\lim_{r\to\infty} \Theta_r(s_r, \theta_r, \alpha) = \begin{cases} 1 + \frac{\alpha}{2} - \log\alpha & \text{if } \alpha < 1 \\ 1 + \frac{1}{2\alpha} & \text{if } \alpha > 1 \end{cases} \tag{32}$$

Moreover, convergence is uniform for α belonging to compact subsets of $(0, \infty) \setminus \{1\}$.

Note that we may rewrite the limit (32) as

$$\lim_{r\to\infty} \Theta_r(s_r, \theta_r, \alpha) = 1 + \tfrac{\alpha}{2} - \log\alpha + \Psi(\alpha). \tag{33}$$

As $\alpha = \lambda/q$, the convergence (in r) will be uniform for λ belonging to compact subsets of $(0, \infty) \setminus \{q\}$, which is precisely the claim of Theorem 2.4.

Proof. We follow the proof of [3, Lemma 4.2]. By setting the partial derivatives of Θ_r equal to 0, we see that the maximising pair (s_r, θ_r) is a solution of the equations

$$sF_r'(s) = \alpha, \quad F_r(s) = \alpha\theta. \tag{34}$$

We will obtain the limits (30) and (31) by analysing the above equations, beginning with the equation $sF_r'(s) = \alpha$. By definition, $sF_r'(s)$ is equal to the sum

$$\sum_{l=1}^{r} a_l \frac{s^l}{(l-1)!} = \sum_{l=1}^{r} \frac{1}{l} \frac{l^l}{l!} s^l. \tag{35}$$

When $s > \frac{1}{e}$, we may apply Stirling's Formula to the factorial in (35) to see that the sum diverges. When $s \leq \frac{1}{e}$, the sum instead converges to the *Lambert* function $W(s)$ satisfying $We^{-W} = s$ (a fact which we cite from [5]). We now consider two cases, depending on the value of α:

1. Suppose first that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_1 < 1$. Define

$$r_1 = \inf\{r : \ (1/e)F_r'(1/e) > \alpha_1\} \tag{36}$$

and observe that for every $r \geq r_1$ and $\alpha \leq \alpha_1$, we must have $s_r < 1/e$. In particular, the error

$$\Delta_r(s) := W(s) - sF_r'(s) \tag{37}$$

may be uniformly bounded by $\Delta_r := \Delta_r(1/e)$, which converges to 0 as $r \to \infty$. As $s_r F_r'(s_r) = \alpha$, we see that $|W(s_r) - \alpha| \leq \Delta_r$. Moreover, by the Mean Value Theorem, we have

$$|s_r - \alpha e^{-\alpha}| = |W(s_r)e^{-W(s_r)} - \alpha e^{-\alpha}| \leq ce^{-c}|W(s_r) - \alpha| \tag{38}$$

for some $c \in (\alpha - \Delta_r, \alpha + \Delta_r)$. As $ce^{-c} \leq 1$, we deduce that $|s_r - \alpha e^{-\alpha}| \leq \Delta_r$, which converges to 0 uniformly.

2. Next, suppose that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_0 > 1$. As $s_r F_r'(s_r) < W(s_r)$ for $s_r < \frac{1}{e}$ and $W(\frac{1}{e}) = 1$, it follows that $s_r \geq \frac{1}{e}$. Conversely, we may discard lower order terms in the sum (35) to obtain the bound

$$sF_r'(s) \geq r^{-3/2}(es)^r. \tag{39}$$

If we write $s = \gamma/e$, then (39) exceeds α_1 if

$$\gamma \geq \exp(\tfrac{1}{r}\log\alpha_1 + \tfrac{3}{2r}\log r) := \gamma_r. \tag{40}$$

Thus $|s_r - \frac{1}{e}| \leq (\gamma_r - 1)/e$, which converges to 0 uniformly.

We have proven (30). Next, we analyse the equation $F_r(s) = \alpha\theta$ in order to prove (31). As before, we consider two cases, depending on the value of α:

1. Suppose first that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_1 < 1$. To find a limit for θ_r, we find a limit for $F_r(s_r)$ by integrating the equation

$$sF_r'(s) = W(s) - \Delta_r(s). \tag{41}$$

Using the identity $We^{-W} = s$ and its derivative $W'(1 - W)e^{-W} = 1$ with respect to s, we obtain the differential equation

$$F_r'(s) = W'(1 - W) - \frac{\Delta_r(s)}{s}, \tag{42}$$

which has the solution

$$F_r(s) = W - \tfrac{1}{2}W^2 - \int_0^s \tfrac{\Delta_r(t)}{t}dt. \tag{43}$$

Plugging in the equations $F_r(s_r) = \alpha\theta_r$ and $W(s_r) = \alpha + \Delta_r(s_r)$ yields

$$\theta_r = 1 - \tfrac{1}{2}\alpha + \tfrac{1}{\alpha}\left(\Delta_r(s_r) - \alpha\Delta_r(s_r) - \tfrac{1}{2}\Delta_r(s_r)^2 - \int_0^{s_r} \tfrac{\Delta_r(t)}{t}dt\right). \tag{44}$$

As α is bounded away from 0, s_r is bounded below $1/e$ for r sufficiently large, and $\Delta_r(t)/t$ converges to 0 as $r \to \infty$, the error term in (44) converges to 0 uniformly.

2. Next, suppose that α belongs to a bounded interval $[\alpha_0, \alpha_1]$ with $\alpha_0 > 1$, and recall that in this case, s_r converges to e^{-1} uniformly. By the triangle inequality, we have

$$|F_r(s_r) - \tfrac{1}{2}| \leq |F_r(s_r) - F_r(e^{-1})| + |F_r(e^{-1}) - \tfrac{1}{2}|. \tag{45}$$

The second term converges to 0 independently of α. For the first, we apply the Mean Value Theorem to obtain

$$|F_r(s_r) - F_r(e^{-1})| \leq F_r'(c)|s_r - e^{-1}| \tag{46}$$

for some $c \in (e^{-1}, s_r)$. Noting that $F_r'(s)$ is increasing and $F_r'(s_r) = \alpha/s_r$ is bounded, we deduce that the first term must also converge to 0 uniformly. The result follows after substituting $F_r(s_r) = \alpha\theta_r$.

Thus the limit (31) holds. Finally, we observe that

$$\Theta_r(s_r, \theta_r, \alpha) = \theta_r - \log s_r. \tag{47}$$

The limit (32) then follows from the limits (30) and (31). □

Note that the supremum in Lemma 3.2 is taken over the interval $\theta \in [1/r, 1]$, whereas the sums in the bounds of Eqs. (27) and (28) are over discrete subsets of the interval $[1/r, 1]$. We claim that the supremums over these two sets coincide in the limit as $n \to \infty$. Let $\theta_{r,n} = \frac{1}{n}\lfloor \theta_r n \rfloor$ and define $s_{r,n}$ to be the number s satisfying

$$\Theta_r(s_{r,n}, \theta_{r,n}, \alpha) = \inf_{s>0} \Theta_r(s, \theta_{r,n}, \alpha). \tag{48}$$

It is sufficient to prove that the pair $(s_{r,n}, \theta_{r,n})$ converges to the pair (θ_r, s_r) as $n \to \infty$:

Lemma 3.3. *Fix $\alpha > 0$ and $r \geq 2$. Then*

$$\lim_{n\to\infty} \Theta_r(s_{r,n}, \theta_{r,n}, \alpha) = \Theta_r(s_r, \theta_r, \alpha). \tag{49}$$

Moreover, convergence is uniform for α in compact subsets of $(0, \infty) \setminus \{1\}$.

To prove Lemma 3.3, we will need the following proposition:

Proposition 3.4. *Let $P(x) = \sum_{r=0}^{n} a_r x^r$ be a polynomial with non-negative co-efficients, and consider the function*

$$Q(x) = \frac{P(x)}{xP'(x)}. \tag{50}$$

Then for $x > 0$, $Q(x)$ is decreasing.

Proof. Differentiation yields

$$Q'(x) = \frac{1}{x^2 P'(x)^2}\Big(xP'(x)^2 - P(x)P'(x) - xP(x)P''(x)\Big). \tag{51}$$

It will be sufficient to show that

$$x^3 P'(x)^2 Q'(x) = \sum_{s=0}^{n}\sum_{t=0}^{n}[sta_s a_t - ta_s a_t - t(t-1)a_s a_t]x^{s+t}, \tag{52}$$

is negative. Writing $m = s + t$ (and noting that the sum is the same when interchanging the order of summation of s and t) this may be rewritten as

$$x^3 P'(x)^2 Q(x) = \frac{1}{2} \sum_{m=0}^{2n} \sum_{r=0}^{n} [2r(m-r) - m - r(r-1) - (m-r)(m-r-1)] a_r a_{m-r} x^m$$
$$= -\frac{1}{2} \sum_{m=0}^{2n} \sum_{r=0}^{n} (m-2r)^2 a_r a_{m-r} x^m \tag{53}$$

where we have set $a_l = 0$ for any $l < 0$. This is negative, as required. □

Proof of Lemma 3.3. Recall that

$$\Theta_r(s, \theta, \alpha) = -\theta \log \alpha - \theta \log \theta + \theta + \theta \log F_r(s) - \log s. \tag{54}$$

Using the triangle inequality, we may bound the difference between $\Theta_r(s_{r,n}, \theta_{r,n}, \alpha)$ and $\Theta_r(s_r, \theta_r, \alpha)$ by the sum of the differences of the terms in (54). If we can show that each of these differences converges to 0, we will be done:

1. First, consider the term $|1 - \log \alpha||\theta_r - \theta_{r,n}|$, and note that $|\theta_r - \theta_{r,n}| \le 1/n$ by definition. As the function $1 - \log \alpha$ is uniformly bounded on compact subsets of $\alpha \in (0, \infty) \setminus \{1\}$, this term converges uniformly to 0 as $n \to \infty$.
2. Next, consider the term $|\theta_r \log \theta_r - \theta_{r,n} \log \theta_{r,n}|$. By the triangle inequality, we have
$$|\theta_r \log \theta_r - \theta_{r,n} \log \theta_{r,n}| \le |\theta_r (\log \theta_r - \log \theta_{r,n})| + |(\theta_r - \theta_{r,n}) \log \theta_{r,n}|, \tag{55}$$
where both of the terms in the right hand side converge uniformly to 0 by uniform continuity of the logarithm away from 0.
3. Next, consider the term $|\log s_r - \log s_{r,n}|$. Recall that, at a stationary point, θ is given by
$$\theta = \frac{F_r(s)}{s F_r'(s)}. \tag{56}$$
By Proposition 3.4, θ is decreasing when viewed as a function of s. Fix $a < b$ such that, on our chosen compact subset of $\alpha \in (0, \infty) \setminus \{1\}$,
$$\theta(b) < \theta_{r,n} < \theta_r < \theta(a). \tag{57}$$
On $[a, b]$, $\theta(s)$ is continuous and injective. In particular, it has a uniformly continuous inverse $s = h(\theta)$ on the compact set $[\theta(b), \theta(a)]$, and so $s_r - s_{r,n}$ converges uniformly to 0. As the logarithm is uniformly continuous on intervals bounded away from 0, the same holds for $\log s_r - \log s_{r,n}$.
4. Finally, consider the term $|\theta_r \log F_r(s_r) - \theta_{r,n} \log F_r(s_{r,n})|$. By the triangle inequality, we have
$$|\theta_r \log F_r(s_r) - \theta_{r,n} \log F_r(s_{r,n})| \le |\theta_r \log F_r(s_r) - \theta_r \log F_r(s_{r,n})| + |\theta_r \log F_r(s_{r,n}) - \theta_{r,n} \log F_r(s_{r,n})|. \tag{58}$$
The second term converges to 0 as $n \to \infty$ by continuity of F_r. For the first term, we observe that
$$\log F_r(s_{r,n}) - \log F_r(s_r) = \int_{s_r}^{s_{r,n}} \frac{F_r'(t)}{F_r(t)} dt \le r \int_{s_r}^{s_{r,n}} t \, dt \le r s_{r,n}(s_{r,n} - s_r). \tag{59}$$
For n sufficiently large, $s_{r,n}$ is uniformly bounded and so this converges to uniformly. □

We have established that θ_r converges to a limit uniformly for λ belonging to compact subsets of $(0,\infty)\setminus\{q\}$. Moreover, this limit belongs to a compact subset of $(0,1)$. In particular, one may choose $\eta > 0$ sufficiently small and $r > 0$, $n > 0$ sufficiently large to ensure that $\theta_{r,n}n$ belongs to the interval $[\frac{1}{r}(1+\eta)n, (1-\eta)n]$. Then, we may apply Proposition 3.1 to the bounds of Eqs. (27) and (28) to obtain

$$Z_{n,\lambda,q}[L\cap B_r] = e^{o(n)}\exp\{n(-1-\tfrac{1}{2}\lambda+\log\lambda+\Theta_r(s_{r,n},\theta_{r,n},\lambda/q))\}, \tag{60}$$

where the $o(n)$ term is bounded uniformly for λ belonging to compact subsets of $(0,\infty)\setminus\{q\}$.

Proof of Theorem 2.4. From (60), we have that

$$\begin{aligned}\frac{1}{n}\log Z_{n,\lambda,q}[L\cap B_r] = {} & \Theta_r(s_{r,n},\theta_{r,n},\lambda/q) - \Theta_r(s_r,\theta_r,\lambda/q)\\ &+ \Theta_r(s_r,\theta_r,\lambda/q) - 1 - \tfrac{1}{2}\lambda + \log\lambda\\ &+ \tfrac{o(n)}{n}.\end{aligned} \tag{61}$$

As the event B_n holds for every possible graph, one may set $r = n$, take the limit as $n\to\infty$, and apply Lemmas 3.2 and 3.3 to obtain an upper bound of

$$\limsup_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[L\big] \le \Psi(\tfrac{\lambda}{q}) - (\tfrac{q-1}{2q})\lambda + \log q. \tag{62}$$

Conversely, for fixed $r\ge 2$, we may apply the inclusion $L\supset L\cap B_r$ and take the limit as $n\to\infty$ to obtain the lower bound

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[L\big] &\ge \liminf_{n\to\infty}\frac{1}{n}\log Z_{n,\lambda,q}\big[L\cap B_r\big]\\ &= \Theta_r(s_r,\theta_r,\lambda/q) - 1 - \tfrac{1}{2}\lambda + \log\lambda,\end{aligned}$$

which converges to $\Psi(\frac{\lambda}{q}) - (\frac{q-1}{2q})\lambda + \log q$ in the limit as $r\to\infty$ by Lemma 3.2. □

Next, we prove Theorem 2.5. We need one preliminary lemma, corresponding to ([3], Lemma 5.1):

Lemma 3.5. *Fix $q > 0$ and $\lambda > 0$. Then*

$$Z_{n,\lambda,q}[B_r] \le Z_{n,\lambda,q}[L](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn}. \tag{63}$$

Proof. We follow the proof of ([3], Lemma 5.1), showing that

$$\phi_{n,\lambda,q}[B_r] \le \phi_{n,\lambda,q}[L](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn}. \tag{64}$$

Given a set of vertices $S\subset\{1,\dots,n\}$, let C_S denote the restriction of the graph $K_n(\omega)$ to S, and let T be a tree on S. Conditionally on the event $C_S\supset T$, all vertices of S belong to the same component of $K_n(\omega)$. In particular, any edge in $E(S)\setminus T$ is open independently with probability $1-\lambda/n$, and so

$$\frac{\phi_{n,\lambda,q}[C_S = T]}{\phi_{n,\lambda,q}[C_S\supset T]} = (1-\tfrac{\lambda}{n})^{\binom{|S|}{2}-|S|+1} \ge (1-\tfrac{\lambda}{n})^{\frac{1}{2}|S|^2}. \tag{65}$$

Let K_S be the event that C_S is connected. Then

$$\phi_{n,\lambda,q}[K_S] \le \sum_T \phi_{n,\lambda,q}[C_S\supset T] \le (1-\tfrac{\lambda}{n})^{-\frac{1}{2}|S|^2}\phi_{n,\lambda,q}[C_S \text{ is a tree}]. \tag{66}$$

Now, let L_r denote the event that each component of $K_n(\omega)$ is either acyclic or has size at most r, and note that $B_r \subset L_r$. Let $\{S_j\}$ be a partition of $\{1,\dots,n\}$ and let $\phi_{n,\lambda,q}[\{S_j\}]$ denote the probability that $\{S_j\}$ are the connected components of $K_n(\omega)$. Conditioning the event L_r on the partition $\{S_j\}$ of connected components, we have

$$\phi_{n,\lambda,q}[L_r] = \sum_{\{S_j\}} \phi_{n,\lambda,q}[\{S_j\}]\phi_{n,\lambda,q}[L_r \mid \{S_j\}]. \tag{67}$$

Moreover, conditionally on the partition $\{S_j\}$ of connected components, the states of edges in different components are independent. In particular, we may write

$$\begin{aligned}\phi_{n,\lambda,q}[L_r|\{S_j\}] &= \prod_{j:|S_j|>r} \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}] \\ &= \prod_{j} \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}] \prod_{j:|S_j|\le r} \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}]^{-1} \\ &\le \prod_{j} \phi_{n,\lambda,q}[C_{S_j} \text{ is a tree} \mid K_{S_j}] \prod_{j:|S_j|\le r} (1-\tfrac{\lambda}{n})^{-\frac{1}{2}|S_j|^2}\end{aligned}$$

where the inequality in the third line is a consequence of (65). As $|S_j| \le r$ for every factor in the second product and the sum over $|S_j|$ is at most n, we obtain

$$\phi_{n,\lambda,q}[L_r \mid \{S_j\}] \le \phi_{n,\lambda,q}[L \mid \{S_j\}](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn}. \tag{68}$$

Finally, we see that

$$\begin{aligned}\phi_{n,\lambda,q}[B_r] &\le \phi_{n,\lambda,q}[L_r] \\ &= \sum_{\{S_j\}} \phi_{n,\lambda,q}[\{S_j\}]\phi_{n,\lambda,q}[L_r \mid \{S_j\}] \\ &\le \sum_{\{S_j\}} \phi_{n,\lambda,q}[\{S_j\}]\phi_{n,\lambda,q}[L \mid \{S_j\}](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn} \\ &= \phi_{n,\lambda,q}[L](1-\tfrac{\lambda}{n})^{-\frac{1}{2}rn}\end{aligned}$$

which establishes (64). We obtain (63) by multiplying both sides of (64) by the partition function. □

Proof of Theorem 2.5. The theorem is a consequence of the following inequalities:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[L] &\le \lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_r], \\ \lim_{r\to\infty} \liminf_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_r] &\le \lim_{\epsilon\downarrow 0} \liminf_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_{\epsilon n}], \\ \lim_{r\to\infty} \limsup_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_r] &\le \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_{\epsilon n}], \\ \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_{\epsilon n}] &\le \lim_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[L].\end{aligned} \tag{69}$$

To prove the first inequality, we apply the inclusion $B_r \supset B_r \cap L$ and Theorem 2.4 to see that

$$\liminf_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[B_r] \ge \liminf_{n\to\infty} \frac{1}{n} Z_{n,\lambda,q}[B_r \cap L] \to \lim_{n\to\infty} \frac{1}{n}\log Z_{n,\lambda,q}[L] \tag{70}$$

as $r \to \infty$. To prove the second inequality, fix $r \geq 2$, $\epsilon > 0$, and let $N = \lceil r/\epsilon \rceil$. Then, for every $n \geq N$, we have $\epsilon n \geq \epsilon \lceil r/\epsilon \rceil \geq r$. As a result, $B_{\epsilon n} \supset B_r$, and

$$\frac{1}{n} \log Z_{n,\lambda,q}[B_r] \leq \frac{1}{n} \log Z_{n,\lambda,q}[B_{\epsilon n}]. \tag{71}$$

One may take the infimum over $m \geq n$, followed by the limit as $n \to \infty$, on both sides to obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}[B_r] \leq \liminf_{n\to\infty} \frac{1}{n} \log Z_{n,\lambda,q}[B_{\epsilon n}]. \tag{72}$$

As r and ϵ were arbitrary, taking the limits $r \to \infty$ and $\epsilon \downarrow 0$ yields the second inequality. The proof of the third inequality is similar. To prove the fourth inequality, we apply Lemma 3.5 in conjunction with the inequality $1 - x \geq e^{-2x}$ (valid for $x \leq \frac{1}{2}\log 2$) to obtain

$$\frac{1}{n} \log Z_{n,\lambda,q}[B_{\epsilon n}] \leq \frac{1}{n} \log Z_{n,\lambda,q}[L] + \lambda\epsilon, \tag{73}$$

from which the inequality follows after taking the limit superior as $n \to \infty$ and the limit as $\epsilon \downarrow 0$. $\square$

4. Uniqueness and the largest component

In this section, we prove Theorem 2.6 and Lemma 2.2. The proof of the latter is identical to that of ([3], Lemma 6.2), and requires only the following analogue of ([3], Lemma 6.1), which estimates the probability of the event $K_{\epsilon,2}$ that $K_n(\omega)$ is connected or has exactly two connected components each of size at least ϵn:

Lemma 4.1 (*[3], Lemma 6.1*). *Fix $q > 0$. Then for all $\lambda_0 > 0$ and $\epsilon_0 > 0$ there exists $c_1 = c_1(\lambda_0, \epsilon_0) > 0$ such that for all $\epsilon \geq \epsilon_0$ and $\lambda \leq \lambda_0$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \phi_{n,\lambda,q}[K^c | K_{\epsilon,2}] < -c_1. \tag{74}$$

Proof. Observe that

$$\phi_{n,\lambda,q}[K^c | K_{\epsilon,2}] = \frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K] + \phi_{n,\lambda,q}[K]} \leq \frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]}. \tag{75}$$

It suffices to show that the ratio on the right hand side of (75) decays to zero exponentially in n, with a rate that is uniformly bounded in $\epsilon \geq \epsilon_0$ and $\lambda \leq \lambda_0$. Observe that $\omega \in K_{\epsilon,2} \setminus K$ if and only if we may find a set $A \subset K_n$ (where A depends on ω) of vertices of size between ϵn and $n - \epsilon n$ such that A, A^c are connected components of $K_n(\omega)$ and there are no open edges between them. We count the configurations satisfying these conditions. Let $E(A, A^c)$ be the set of edges between A and its complement in $K_n(\omega)$, and suppose that $|A| = k$. It can be seen that A is disconnected from A^c in $K_n(\omega)$ with probability

$$\phi_{n,\lambda,q}[E(A, A^c) = \emptyset] = \frac{Z_{k,\lambda k/n,q} Z_{n-k,\lambda(1-k/n),q}}{Z_{n,\lambda,q}} \left(1 - \tfrac{\lambda}{n}\right)^{k(n-k)}. \tag{76}$$

Conditionally on the above event, A is connected in $K_n(\omega)$ with probability

$$\phi_{k,\lambda k/n,q}[K] = Z_{k,\lambda k/n,q}[K] / Z_{k,\lambda k/n,q}, \tag{77}$$

and A^c is connected in $K_n(\omega)$ with probability

$$\phi_{n-k,\lambda(1-k/n),q}[K] = Z_{n-k,\lambda(1-k/n),q}[K] / Z_{n-k,\lambda(1-k/n),q}. \tag{78}$$

Note that there are $\binom{n}{k}$ choices for A, and that we have counted any pair (A, A^c) twice. Thus, we have the equation

$$Z_{n,\lambda,q}[K_{\epsilon,2} \setminus K] = \frac{1}{2} \sum_{\epsilon n \le k \le n-\epsilon n} \binom{n}{k} \left(1 - \frac{\lambda}{n}\right)^{k(n-k)} Z_{k,\lambda k/n,q}[K] Z_{n-k,\lambda(1-k/n),q}[K].$$

Applying Theorem 2.3, we obtain

$$\frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]} = e^{o(n)} \sum_{\epsilon n \le k \le n-\epsilon n} \binom{n}{k} \frac{\pi_1(\lambda \frac{k}{n})^k \pi_1(\lambda(1-\frac{k}{n}))^{n-k}}{\pi_1(\lambda)^n} \left(1 - \frac{\lambda}{n}\right)^{k(n-k)}, \tag{79}$$

where $\pi_1(\lambda) = 1 - e^{-\lambda}$, and the $e^{o(n)}$ term is the error term found in Theorem 2.3. The lemma now concludes identically to ([3], Lemma 6.1), rewriting (79) as

$$\frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]} = e^{o(n)} \sum_{\epsilon n \le k \le n-\epsilon n} e^{n[\Xi(k/n) - \Xi(0)]}, \tag{80}$$

where the function Ξ is defined by

$$\Xi(\theta) = S(\theta) + \theta \log \pi_1(\lambda\theta) + (1-\theta) \log \pi_1(\lambda(1-\theta)) - \lambda\theta(1-\theta). \tag{81}$$

We now bound the sum in (80) by n times its maximal summand. As Ξ is convex and symmetric around the point $1/2$, the summand is maximised at the endpoints. In particular, we have the bound

$$\frac{\phi_{n,\lambda,q}[K_{\epsilon,2} \setminus K]}{\phi_{n,\lambda,q}[K]} \le e^{o(n)} e^{n[\Xi(\epsilon) - \Xi(0)]}. \tag{82}$$

More explicitly, we may take any value $c_1 < \Xi(0) - \Xi(\epsilon)$ provided that n is sufficiently large. $\square$

Proof of Lemma 2.2. The proof is identical to that of ([3], Lemma 6.2). In particular, it will suffice to prove that

$$\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor,\ \mathcal{N}_{\epsilon n} > 1] \le e^{-cn} \phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor]. \tag{83}$$

For a given vertex x, let $\mathcal{C}_x$ denote the component of $K_n(\omega)$ containing x. On the event $\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{\epsilon n} > 1\}$, we may find a pair of vertices $x, y \in [n]$ in $K_n(\omega)$ such that $|\mathcal{C}_x| \ge \epsilon n$, $|\mathcal{C}_y| \ge \epsilon n$ and $x \nleftrightarrow y$. Define the following two events for the random graph:

$$A_1 = \{|\mathcal{C}_x| \ge \epsilon n\} \cap \{|\mathcal{C}_y| \ge \epsilon n\} \cap \{x \nleftrightarrow y\},$$
$$A_2 = \{|\mathcal{C}_x| \ge \epsilon n\} \cap \{|\mathcal{C}_y| \ge \epsilon n\}.$$

As the complete graph is transitive, the probabilities of the events A_1 and A_2 do not depend on the particular choices of x and y. By the union bound, it follows that

$$\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{\epsilon n} > 1] \le n^2 \phi_{n,\lambda,q}[\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_1]. \tag{84}$$

We now condition further on the set $\mathcal{C}_x \cup \mathcal{C}_y$. For a given set $\mathcal{C} \subset [n]$, let D be the event that $\mathcal{C}$ is disjoint from $\mathcal{C}^c$ in $K_n(\omega)$ and that its complement contains $\lfloor \theta n \rfloor - |\mathcal{C}|$ vertices in components of size at least ϵn. Then

$$\phi_{n,\lambda,q}[\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_1] = \sum_{\mathcal{C} \subset [n]} \phi_{n,\lambda,q}[A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \cap D],$$
$$= \sum_{\mathcal{C} \subset [n]} \phi_{n,\lambda,q}[A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \mid D] \phi_{n,\lambda,q}[D].$$

Write $m = |\mathcal{C}|$, $\tilde{\lambda} = \lambda\theta$, and $\tilde{\epsilon} = \epsilon/\theta$. On the event D, the random cluster measure restricts to the random cluster measure $\phi_{\theta n,\tilde{\lambda},q}$ on $\mathcal{C}$. Moreover, we have the following correspondences between events:

$$A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} = K^c \cap K_{\tilde{\epsilon},2},$$
$$A_2 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} = K_{\tilde{\epsilon},2}.$$

As $\tilde{\epsilon} \geq \epsilon$ for every $\theta > 0$, we may apply Lemma 4.1 to deduce that

$$\phi_{n,\lambda,q}[A_1 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \mid D] \leq e^{-c_1 m} \phi_{n,\lambda,q}[A_2 \cap \{\mathcal{C}_x \cup \mathcal{C}_y = \mathcal{C}\} \mid D], \tag{85}$$

which allows us to rewrite (84) as

$$\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{\epsilon n} > 1] \leq n^2 e^{-c_1 m} \phi_{n,\lambda,q}[\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_2]. \tag{86}$$

The result follows as $\{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\} \cap A_2 \subset \{|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor\}$ and $m \geq \epsilon n$. □

Proof of Theorem 2.6. By Lemma 2.2, it is sufficient to prove that

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \lfloor \theta n \rfloor, \mathcal{N}_{\epsilon n} = 1] = \Phi(\theta, \lambda, q). \tag{87}$$

The case $\theta = 1$ reduces to Theorem 2.3 and the case $\theta = 0$ reduces to Theorems 2.4 and 2.5. Let $\theta \in (0, 1)$, $\epsilon \in (0, \theta)$ and assume that θn is an integer. Given a configuration ω, observe that $\omega \in \{|\mathcal{V}_{\epsilon n}| = \theta n, N_{\epsilon n} = 1\}$ if and only if we may find a subset $A \subset K_n$ of vertices (where A depends on ω) of size θn such that A is a connected component of $K_n(\omega)$, A^c contains no connected components of $K_n(\omega)$ of size exceeding ϵn, and $E(A, A^c) = 0$. We count the possible configurations which satisfy these conditions. Note that there are $\binom{n}{\theta n}$ possible choices for A, and A is disconnected from A^c in $K_n(\omega)$ with probability

$$\phi_{n,\lambda,q}[E(A, A^c) = \emptyset] = \frac{Z_{\theta n,\lambda\theta,q} Z_{(1-\theta)n,\lambda(1-\theta),q}}{Z_{n,\lambda,q}} \left(1 - \tfrac{\lambda}{n}\right)^{\theta(1-\theta)n^2}. \tag{88}$$

Conditionally on this event, A is connected in $K_n(\omega)$ with probability

$$\phi_{\theta n,\lambda\theta,q}[K] = Z_{\theta n,\lambda\theta,q}[K]/Z_{\theta n,\lambda\theta,q}, \tag{89}$$

and A^c does not contain any components of size exceeding ϵn in $K_n(\omega)$ with probability

$$\phi_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}] = Z_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}]/Z_{(1-\theta)n,\lambda(1-\theta),q}. \tag{90}$$

Thus, we have the equation

$$Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = \theta n, N_{\epsilon n} = 1] = \binom{n}{\theta n} (1 - \tfrac{\lambda}{n})^{\theta n(1-\theta)n} Z_{\theta n,\lambda\theta,q}[K] Z_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}]. \tag{91}$$

We now take logarithms, divide by n and take the limit as $n \to \infty$. Note first that

$$\lim_{n \to \infty} \frac{1}{n} \log\left(\binom{n}{\theta n} (1 - \lambda/n)^{\theta n(1-\theta)n}\right) = S(\theta) + (1 - \theta) \log[1 - \pi_1(\lambda\theta)]. \tag{92}$$

Applying Theorem 2.3, we also have

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{\theta n,\lambda\theta,q}[K] = \theta \log \pi_1(\lambda\theta). \tag{93}$$

Finally, we apply Theorems 2.4 and 2.5 to obtain

$$\lim_{\epsilon\downarrow 0} \lim_{n\to\infty} \frac{1}{n} \log Z_{(1-\theta)n,\lambda(1-\theta),q}[B_{\epsilon n}] =(1-\theta)\times \left\{ \Psi(\tfrac{\lambda(1-\theta)}{q}) - (\tfrac{q-1}{2q})\lambda(1-\theta) + \log q \right\}. \tag{94}$$

Summing these limits yields the result. Convergence is uniform as each of the individual limits converge uniformly. □

5. Free energy

In this section, we prove Theorem 2.7. To begin, we let $\epsilon > 0$. Then, we may decompose the partition function as

$$Z_{n,\lambda,q} = Z_{n,\lambda,q}[B_{\epsilon n}] + \sum_{k>\epsilon n} Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = k]. \tag{95}$$

By Lemma 2.2, we may write

$$Z_{n,\lambda,q} = Z_{n,\lambda,q}[B_{\epsilon n}] + (1-o(1)) \sum_{k>\epsilon n} Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = k,\ \mathcal{N}_{\epsilon n} = 1]. \tag{96}$$

We aim to apply Theorem 2.6 to each summand in (96). Recall that

$$Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = k, N_{\epsilon n} = 1] = \binom{n}{k} (1-\tfrac{\lambda}{n})^{k(n-k)} Z_{k,\lambda k/n,q}[K] Z_{n-k,\lambda(1-k/n),q}[B_{\epsilon n}]. \tag{97}$$

In order to apply Theorem 2.6 to all of the summands in (96) simultaneously, we require that the term $\lambda(1-k/n)$ belongs to a compact subset of $(0,\infty) \setminus \{q\}$. It will suffice to prove that the terms for which k/n is close to 1 or $1-q/\lambda$ have negligible probability, which we do via the following two tail inequalities for sufficiently small ϵ:

$$\begin{aligned} Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{\epsilon n} = 1] &\leq o(1) Z_{n,\lambda,q}, \\ Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| \leq (1-q/\lambda+\epsilon)n,\ \mathcal{N}_{\epsilon n} = 1] &\leq o(1) Z_{n,\lambda,q}. \end{aligned} \tag{98}$$

Equivalently, we show that

$$\begin{aligned} \phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{\epsilon n} = 1] &\leq o(1), \\ \phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| \leq (1-q/\lambda+\epsilon)n,\ \mathcal{N}_{\epsilon n} = 1] &\leq o(1). \end{aligned} \tag{99}$$

Both inequalities in (99) may be proven via direct comparisons with percolation, using the following law of large numbers for the size of the largest component of $K_n(\omega)$ under the percolation measure $\phi_{n,\lambda}$, cited from [9]:

Theorem 5.1 (*[9], Theorem 4.8*)**.** *Fix $\lambda > 1$ and let $p = \lambda/n$. Then, for every $\nu \in (\frac{1}{2}, 1)$ there exists $\delta = \delta(\lambda,\nu) > 0$ such that*

$$\phi_{n,\lambda}[||\mathcal{C}_1| - \theta(\lambda,1)n| \geq n^{\nu}] = O(n^{-\delta}). \tag{100}$$

We begin with the first inequality of (99). Note that the random cluster measure $\phi_{n,\lambda,q}$ is stochastically dominated by the percolation measure $\phi_{n,\lambda}$ for $q \geq 1$ (see e.g. ([8], Theorem 3.21)), yielding the upper bound

$$\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{\epsilon n} = 1] \leq \phi_{n,\lambda}[|\mathcal{V}_{\epsilon n}| \geq (1-\epsilon)n,\ \mathcal{N}_{\epsilon n} = 1]. \tag{101}$$

As the percolation measure is stochastically ordered in the edge weight p, we may assume that $\lambda > 1$, in which case we are done by Theorem 5.1 provided we take $\epsilon < 1 - \theta(\lambda, 1)$.

We now turn to the second inequality of (99), noting first that it is only relevant if $\lambda > q$. If this is the case, then the random cluster measure $\phi_{n,\lambda,q}$ stochastically dominates the supercritical percolation measure $\phi_{n,\lambda/q}$, again by ([8], Theorem 3.21). As a result, Theorem 5.1 may be applied to show that

$$\phi_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| \leq (\theta(\lambda/q, 1) - \epsilon)n,\ \mathcal{N}_{\epsilon n} = 1] = O(n^{-\delta}). \tag{102}$$

Write $\alpha = \lambda/q$, as before. We claim that $\theta(\alpha, 1) > 1 - \alpha^{-1}$. As $\theta(\alpha, 1)$ solves the equation

$$\alpha = -\frac{1}{\theta(\alpha, 1)} \log(1 - \theta(\alpha, 1)) \tag{103}$$

it will be sufficient to prove that

$$-\frac{1}{\theta(\alpha, 1)} \log(1 - \theta(\alpha, 1)) < (1 - \theta(\alpha, 1))^{-1}, \tag{104}$$

which is a consequence of the inequality $(1-x)\log(1-x)+x > 0$ for $x \in (0, 1)$. In particular, we are done provided we take $\epsilon < \theta(\lambda/q, 1) - 1 + q/\lambda$.

Applying both inequalities of (98) to (96), we see that

$$(1 - o(1))Z_{n,\lambda,q} = Z_{n,\lambda,q}[B_{\epsilon n}] + \sum_{k > \max\{\epsilon, \theta(\lambda/q,1)-\epsilon\}}^{(1-\epsilon)n} Z_{n,\lambda,q}[|\mathcal{V}_{\epsilon n}| = k,\ \mathcal{N}_{\epsilon n} = 1]. \tag{105}$$

We are finally in a position to prove Theorem 2.7:

Proof of Theorem 2.7. Let s_n be the maximal summand of Eq. (105). As the summands converge uniformly, so too does the maximal summand, with limit

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log s_n = \sup_{\theta > \theta(\lambda/q,1)} \Phi(\theta, \lambda, q). \tag{106}$$

Moreover, as the sum is bounded between s_n and ns_n, we have

$$\frac{1}{n} \log s_n \leq \frac{1}{n} \log((1 - o(1))Z_{n,\lambda,q}) \leq \frac{1}{n} \log s_n + \frac{1}{n} \log n. \tag{107}$$

Taking the limits as $n \to \infty$ and $\epsilon \downarrow 0$ in that order yields the result. □

We have proven that the free energy of the random cluster model converges to the supremum of the function $\Phi(\theta, \lambda, q)$. Finally, we evaluate this supremum. In particular, the following lemma shows that our computation of the free energy agrees with the one found in ([4], Theorem 2.6):

Lemma 5.2. *Let $q > 0$ and $\lambda > 0$. Then*

$$\sup_{\theta \in [0,1]} \Phi(\theta, \lambda, q) = \sup_{\theta > \theta(\lambda/q,1)} \Phi(\theta, \lambda, q) = \frac{g(\theta(\lambda, q))}{2q} - \left(\frac{q-1}{2q}\right)\lambda + \log q \tag{108}$$

where the function $g : (0, 1) \to \mathbb{R}$ *is defined by*

$$g(\theta) = -(q-1)(2-\theta)\log(1-\theta) - [2+(q-1)\theta]\log[1+(q-1)\theta]. \tag{109}$$

Proof. We separate the argument into the cases $\lambda(1-\theta) > q$ and $\lambda(1-\theta) < q$, corresponding to the regions in which the Ψ function is defined. We use the shorthand notation

$$a = \frac{1-\theta}{q\theta}, \ b = \frac{e^{-\lambda\theta}}{1-e^{-\lambda\theta}}, \ k = \lambda\theta. \tag{110}$$

When $\lambda(1-\theta) > q$, the derivative of Φ with respect to θ is given by

$$\frac{\partial}{\partial\theta}\Phi(\theta, \lambda, q) = (kb) - \log(kb) - 1. \tag{111}$$

As $x - \log x - 1 \geq 0$ with equality if and only if $x = 1$, the derivative in (111) is equal to zero only if $kb = 1$. This is equivalent to the equation $1 + \lambda\theta = e^{\lambda\theta}$, for which the only solution is $\theta = 0$. When $\lambda(1-\theta) < q$, we obtain the derivative

$$\frac{\partial}{\partial\theta}\Phi(\theta, \lambda, q) = (\log a - ka) - (\log b - kb). \tag{112}$$

The function $\log x - kx$ is convex, with a maximum at $x = \frac{1}{k}$. We know that $a \leq \frac{1}{k}$ by assumption, and $b \leq \frac{1}{k}$ is a consequence of the inequality $1 + \lambda\theta \leq e^{\lambda\theta}$. As a result, the derivative in (112) is equal to zero only if $a = b$, which may be rearranged to see that the maximising value θ^* satisfies the mean field Eq. (10). Conversely, any solution θ to the mean-field equation satisfies the assumption (and so is a stationary point), as

$$\frac{\lambda(1-\theta)}{q} = ka = kb \leq 1. \tag{113}$$

We may now assume θ^* satisfies the mean-field equation. Under this assumption, one may rewrite $\Phi(\theta^*, \lambda, q)$ in the form

$$\Phi(\theta^*, \lambda, q) = \frac{1}{2q}g(\theta^*) - \frac{q-1}{2q}\lambda + \log q. \tag{114}$$

This is the form taken in ([4], Theorem 2.6). It remains to show that (114) is maximised when we take the solution $\theta(\lambda, q)$ to the mean field equation. We quote the following properties of the function g from [4]:

$$\begin{aligned} g(0) &= g'(0) = 0, \\ g''(\theta) &= -\frac{q(q-1)[q-2-2(q-1)\theta]\theta}{(1-\theta)^2[1+(q-1)\theta]^2}. \end{aligned} \tag{115}$$

For $q \leq 2$, $g(\theta)$ is a convex, increasing function and the result is clear. For $q > 2$, $g(\theta)$ is initially decreasing. Moreover, $g''(\theta)$ has a zero at $\theta = \frac{q-2}{2(q-1)}$, and is increasing thereafter. In particular, $g(\theta)$ is convex for $\theta > \frac{q-2}{2(q-1)}$ and has only one zero in this region, which we may compute as $\theta_c = \frac{q-2}{q-1}$. Note that θ_c is the largest solution to the mean-field equation for $\lambda = \lambda_c$.

We claim that $\theta_{\max}$ is increasing as a function of λ. If $\theta_{\max}(\lambda) = 0$ then this is obvious, so we may assume that $\theta_{\max}(\lambda) > 0$. Let $\epsilon > 0$, and define the function

$$h(\theta) := e^{-(\lambda+\epsilon)\theta} - \frac{1-\theta}{1+(q-1)\theta}. \tag{116}$$

Noting that $h(\theta_{\max}(\lambda)) < 0$ and $h(1) > 0$, it follows that h has a zero in the interval $(\theta_{\max}(\lambda), 1)$ i.e. $\theta_{\max}(\lambda + \epsilon) > \theta_{\max}(\lambda)$.

We may now conclude. If $\lambda < \lambda_c$, then $\theta_{\max}(\lambda) < \theta_{\max}(\lambda_c)$ and so $g(\theta_{\max}(\lambda)) < 0$. In particular, $\theta^* = 0$ maximises the free energy. Conversely, if $\lambda > \lambda_c$ then it follows that $g(\theta_{\max}(\lambda)) > 0$. As $g(\theta)$ is convex for $\theta > \frac{1}{2}\theta_c$, it follows that $\theta_{\max}$ is the solution maximising the function $g(\theta)$, and so $\theta^* = \theta_{\max}$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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