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$\sqrt{\log t}$ -SUPERDIFFUSIVITY FOR A BROWNIAN PARTICLE IN THE CURL OF THE 2D GFF

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The present work is devoted to the study of the large time behaviour of a critical Brownian diffusion in two dimensions, whose drift is divergence-free, ergodic and given by the curl of the 2-dimensional Gaussian free field. We prove the conjecture, made in (*J. Stat. Phys.* **147** (2012) 113–131), according to which the diffusion coefficient $D(t)$ diverges as $\sqrt{\log t}$ for $t \rightarrow \infty$. Starting from the fundamental work by Alder and Wainwright (*Phys. Rev. Lett.* **18** (1967) 988–990), logarithmically superdiffusive behaviour has been predicted to occur for a wide variety of out-of-equilibrium systems in the critical spatial dimension $d = 2$. Examples include the diffusion of a tracer particle in a fluid, self-repelling polymers and random walks, Brownian particles in divergence-free random environments and, more recently, the 2-dimensional critical Anisotropic KPZ equation. Even if in all of these cases it is expected that $D(t) \sim \sqrt{\log t}$, to the best of the authors' knowledge, this is the first instance in which such precise asymptotics is rigorously established.

1. Introduction. In the present work, we study the motion of a Brownian particle in \mathbb{R}^2 , subject to a random, time-independent drift ω given by the curl of the two-dimensional Gaussian Free Field (2d GFF). Namely, we look at the SDE, which is (formally) given by

$$(1.1) \quad dX(t) = \omega(X(t)) dt + dB(t), \quad X(0) = 0,$$

where $B(t)$ is a standard two-dimensional Brownian motion and

$$x \mapsto \omega(x) = (\omega_1(x), \omega_2(x))$$

is defined as

$$(1.2) \quad x = (x_1, x_2) \mapsto \omega(x) = (\partial_{x_2} \xi(x), -\partial_{x_1} \xi(x)),$$

with ξ the 2d GFF. As written, (1.1) is ill-posed due to the singularity of the drift ω . In fact, not only classical stochastic analytical tools would fail in characterising (even) its law but it would also be *critical* for the recent techniques established in [4, 7] as its spatial regularity is way below the threshold identified therein.¹ Nevertheless, we are interested in its large time behaviour, and hence we regularise ξ by convolving it with a C^∞ bump function (see Section 2 for details), so that ω is well-defined pointwise and smooth. Note that the vector field ω is everywhere orthogonal to the gradient of the field ξ and, therefore, parallel to its level lines. As a consequence, the particle is subject to two very different mechanisms: the

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¹Formally, the 2d GFF is in C^α , $\alpha < 0$, the latter being the space of Hölder distributions with regularity α (see [4] for the definition), so that $\omega \in C^{\alpha-1}$. In the aforementioned works, the threshold regularity is $-2/3$ so that (1.1) falls indeed out of their scope.

drift tends to push the motion *along the level lines of the GFF*, while the Brownian noise tends to make it diffuse *isotropically*. Our main theorem is a *sharp* superdiffusivity result: the mean square displacement $\mathbf{E}[|X(t)|^2]$ (under the joint law of the Brownian noise and of the random drift) is of order $t\sqrt{\log t}$ for $t \rightarrow \infty$, up to multiplicative loglog corrections. This proves a conjecture of B. Tóth and B. Valkó [22] and, in a broader perspective, it is the first proof of the $\sqrt{\log t}$ -*superdiffusivity phenomenon* conjectured to occur in a large class of (self-)interacting diffusive systems in dimension $d = 2$ (see the discussion below).

To put the model and the result into context, let us observe first that the vector field ω is divergence-free and that its law is translation-invariant and ergodic. Brownian diffusions in ergodic, divergence-free vector fields have been introduced in the physics and mathematics literature as a (toy) model for a tracer particle evolving in an incompressible turbulent flow. If the energy spectrum of the vector field (i.e., the Fourier transform $e(p)$ of the trace of the covariance matrix $R(x - y) = \{\mathbf{E}(\omega_a(x)\omega_b(y))\}_{a,b \leq d}$, with d the space dimension) satisfies the integrability condition²

$$(1.3) \qquad \int_{\mathbb{R}^d} \frac{e(p)}{|p|^2} dp < \infty,$$

the behaviour of the particle is known to be diffusive on large scales [11, 12] (see also [14, 21] for analogous results obtained, via different methods, in the discrete setting of random walks in divergence-free random environments). In the robustly superdiffusive case, where the integral in (1.3) has a power-law divergence for small p , it turns out that $\mathbf{E}[|X(t)|^2]$ grows like t^ν for some $\nu > 1$ [13]. The case under consideration in this work instead, where $d = 2$ and ω is the curl of the GFF, is precisely at the boundary between the diffusive and the super-diffusive case: $e(p)$ is essentially constant for p small, the integral (1.3) diverges logarithmically at small momenta and logarithmic corrections to diffusivity are expected.

Logarithmic corrections to diffusivity in two-dimensional out-of-equilibrium systems have a long history. The seminal works [1, 23] of Alder and Wainwright lead the way, in that they predicted that the velocity auto-correlation of a tracer particle diffusing in a fluid behaves like $t^{-d/2}$ in dimension $d \geq 3$ and like $1/(t\sqrt{\log t})$ in the critical dimension $d = 2$. This translates into the fact that, in two dimensions, the mean square displacement $\mathbf{E}[|X(t)|^2]$ of the particle should grow like $tD(t)$ with

$$(1.4) \qquad D(t) \approx \sqrt{\log t} \quad \text{as } t \rightarrow \infty.$$

The quantity $D(t)$ takes the name of (bulk) diffusion coefficient. The same prediction was obtained by Forster, Nelson and Stephen [8] via renormalization group methods. Subsequently, anomalous logarithmic corrections as in (1.4) were conjectured to occur for several other two-dimensional (self-)interacting diffusions, including self-repelling random walks and Brownian polymers³ [2, 18, 19, 22], lattice gas models [16], the diffusion (1.1) in the curl of the $2d$ GFF [22] and, more recently, the two-dimensional Anisotropic KPZ equation ($2d$ AKPZ) [6]. We emphasize that in all of these cases, it is known or conjectured that the analogous models behave diffusively ($D(t) \sim 1$) in dimension $d \geq 3$ (see, for instance, [9] for the self-interacting random walks and Brownian polymers).

From a rigorous viewpoint, results available so far fall short of the conjecture (1.4). Until recently, the best estimates obtained can be summarised into bounds of the form

$$(1.5) \qquad \log \log t \lesssim D(t) \lesssim \log t$$

²The integral in the left-hand side is known as the “Péclet number” [11].

³There has been some controversy in the physics literature as to the value of the exponent ζ of the logarithm in (1.4) for self-repelling random walks. The values $\zeta = 0.4$ and $\zeta = 1$ have been proposed [2, 18, 19], in addition to the $\zeta = 1/2$ prediction [22] based on the Alder–Wainwright argument.

(see [22] for $2d$ self-repelling Brownian polymers and for the SDE (1.1), and [16] for two-dimensional lattice fluids). More recently, two of the authors together with D. Erhard proved in [6] that, for the $2d$ AKPZ equation, one has

$$(1.6) \quad (\log t)^a \lesssim D(t) \lesssim (\log t)^{1-a}$$

for some sufficiently small $a > 0$; after the present work was completed, in a second version of [6] the result for the $2d$ AKPZ equation has been also improved to $a = 1/2$. (All the above cited results have been shown in the sense of Laplace transform.)

For the SDE (1.1) under consideration in the present work, we establish for the first time the conjectured behaviour (1.4), up to corrections that are polynomial in $\log \log t$ (see Theorem 2.2 below). The result holds again in the sense of Laplace transform; see, however, Remark 2.3 for its implications in real time.

Our argument is based on an iterative analysis of the resolvent of the generator of the Markov process given by the environment seen from the particle (see (2.3) below). This is inspired by the method employed by H.-T. Yau [24] to prove $(\log t)^{2/3}$ corrections to the diffusivity of the two-dimensional Asymmetric Simple Exclusion Process ($2d$ ASEP) and, more closely, by the techniques developed in [6] to determine (1.6) for the $2d$ AKPZ equation. Note that the exponent $2/3$ of the logarithmic corrections of $2d$ ASEP is different from the exponent $1/2$ in (1.4), reflecting the fact that the two models belong to two different universality classes, as emphasized already in [16, 22]. From a technical point of view, a crucial difference between the two models is that for $2d$ ASEP the iterative method in [24] provides, at each step k of the recursion, upper/lower bounds for $D(t)$ of the form $(\log t)^{v_k}$, with v_k converging exponentially fast to $2/3$ as $k \rightarrow \infty$. In our case, on the other hand, at step k the method naturally provides lower (resp., upper) bounds of order $(\log \log t)^k/k!$ (resp., $k! \log t / (\log \log t)^k$) and we have to run the iteration for a number of steps of order $k = k(t) \approx \log \log t$ (instead of $k(t) \approx \log \log \log t$ as in [24]) to reach the final result. As a consequence, in contrast with [24], we cannot afford to lose a multiplicative constant at each step of the iteration (such multiplicative constants are responsible for the sub-optimal result (1.6) in the first version of [6]), and a much finer analysis of the resolvent is needed. Further, we get a significantly sharper control of sub-leading corrections to $D(t)$ with respect to $2d$ ASEP, namely, a multiplicative correction that is polynomial in $\log \log t$ (see Remark 2.3), to be compared with the corrections of order $\exp((\log \log \log t)^2)$ for $2d$ ASEP [24].

Organization of the article. The rest of this work is organized as follows. In Section 2, we rigorously define the model (1.1) and state the main result. In Section 3, we introduce the main tools: we recall the generator of the environment seen from the particle process and we describe the space on which it acts. Section 4 is devoted to the analysis of the generator and the derivation of the crucial recursive bounds, while in Section 5, the proof of the main result is given. At last, in the Appendix, we collect some technical estimates needed in Sections 4 and 5.

2. The model and main result. The Brownian diffusion in the curl of the 2-dimensional Gaussian free field is the stochastic process $t \mapsto X(t) \in \mathbb{R}^2$ given by the solution of the SDE (1.1) where $B(t)$ is a standard two-dimensional Brownian motion and

$$x \mapsto \omega(x) = (\omega_1(x), \omega_2(x))$$

is a smooth, divergence-free, random vector field on \mathbb{R}^2 , given by the curl of (a smoothed version of) the two-dimensional Gaussian free field. To be more precise, let us introduce the following assumption, which will be in place throughout the paper.

ASSUMPTION 2.1. Let $V: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a radially symmetric bump function, that is, a function such that there exists $U: \mathbb{R}^2 \rightarrow \mathbb{R}$, which is a smooth function in $C^\infty(\mathbb{R}^2)$, radially symmetric, decaying sufficiently (say, exponentially) fast at infinity and such that $\int_{\mathbb{R}^2} U(x) \, dx = 1$, for which

(2.1)

$$V = U * U.$$

Let V satisfy Assumption 2.1 and U be such that (2.1) holds. Then ω is a centred Gaussian field whose law \mathbb{P} (and corresponding expectation \mathbb{E}) is defined as follows. Let $\xi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the two-dimensional Gaussian free field convoluted with U , that is, the centred Gaussian field with covariance

$$\mathbb{E}(\xi(x)\xi(y)) = V * g(x - y) \quad \text{for all } x, y \in \mathbb{R}^2,$$

where $g(x) = -\log |x|$. Then, we define ω as the curl of the scalar field ξ , that is, as in (1.2), which clearly satisfies for all $x, y \in \mathbb{R}^2$,

(2.2)

$$\mathbb{E}(\omega_\ell(x)) = 0, \quad \mathbb{E}(\omega_k(x)\omega_\ell(y)) = -\tilde{\partial}_k \tilde{\partial}_\ell V * g(x - y), \quad k, \ell = 1, 2,$$

for $\tilde{\partial}_1 := \partial_{x_2}$ and $\tilde{\partial}_2 := -\partial_{x_1}$. Note that, while convolving the full-plane Gaussian free field with U is a somewhat formal operation (because the field is only defined up to a constant), the derivatives of the convolved field and, therefore, ω are (pointwise) defined without any ambiguity and are smooth with respect to x .

It is well known [11], Chapter 11, that, since ω sampled from \mathbb{P} is divergence-free, translation invariant and ergodic, the law \mathbb{P} is stationary for the Markov process of the environment seen from the particle, that is, the time-evolving field $t \mapsto \omega(t, \cdot)$ given by

(2.3)

$$\omega(t, x) \stackrel{\text{def}}{=} \omega(X(t) + x), \quad x \in \mathbb{R}^2.$$

2.1. *Main result.* Our main result is a sharp estimate on the super-diffusivity of the process X . For $t > 0$, let $\mathbf{E}(|X(t)|^2)$ denote the mean square displacement of X at time t —the expectation being taken with respect to the joint randomness of the vector field ω and of the Brownian noise B in (1.1). Let us remark that $\mathbf{E}(X(t)) = 0$ because the law of the environment is symmetric and $X(0) = 0$.

Throughout the present article, we will be working with its Laplace transform, given by

(2.4)

$$D(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t} \mathbf{E}(|X(t)|^2) \, dt, \quad \lambda > 0.$$

Note that if in (1.1) there were no drift, one would trivially have $X(t) = B(t)$ so that $\mathbf{E}(|X(t)|^2) = 2t$ and $D(\lambda) = 2\lambda^{-2}$. In [22], it was conjectured that

(2.5)

$$D(\lambda) \stackrel{\lambda \rightarrow 0}{\approx} \lambda^{-2} \sqrt{|\log \lambda|},$$

corresponding in real time to

$$\mathbf{E}(|X(t)|^2) \stackrel{t \rightarrow \infty}{\approx} t \sqrt{\log t}$$

(the diffusion coefficient mentioned in the abstract is $D(t) = t^{-1} \mathbf{E}(|X(t)|^2)$. From a rigorous point of view, in the aforementioned work it was proved that

$$C_1 \lambda^{-2} \log |\log \lambda| \leq D(\lambda) \leq C_2 \lambda^{-2} |\log \lambda|$$

for some positive constants C_1, C_2 , for sufficiently small λ . In this work, we establish the conjecture (2.5) in full.

THEOREM 2.2. *For every $\varepsilon > 0$, there exists constants $C_{\pm}(\varepsilon)$ such that, for every $0 < \lambda < 1$,*

$$(2.6) \quad C_{-}(\varepsilon)(\log |\log \lambda|)^{-1-\varepsilon} \leq \lambda^2 \frac{D(\lambda)}{\sqrt{|\log \lambda|}} \leq C_{+}(\varepsilon)(\log |\log \lambda|)^{1+\varepsilon},$$

where D is defined according to (2.4).

The exponent $1 + \varepsilon$ in the sub-dominant corrections can presumably be improved by some additional technical work, but we do not pursue this here. The constants $C_{\pm}(\varepsilon)$ implicitly depend also on the choice of bump function V .

REMARK 2.3. By a well-established argument (see [20]) the upper bound in (2.6) implies an upper bound for the diffusivity in real time of the form

$$\mathbf{E}(|X(t)|^2) \leq O(t\sqrt{\log t}(\log \log t)^{1+\varepsilon}).$$

Deducing a pointwise (in time) lower bound on $\mathbf{E}(|X(t)|^2)$ from the behaviour for $\lambda \rightarrow 0$ of the Laplace transform is much more delicate. That said, one can easily get (applying for instance [3], Theorem 1.7.1) the following:

$$\limsup_{t \rightarrow \infty} \frac{\mathbf{E}(|X(t)|^2)}{t\sqrt{\log t}(\log \log t)^{-1-\varepsilon}} > 0.$$

3. Preliminaries. By rotation invariance, one has $\mathbf{E}(|X(t)|^2) = \mathbf{E}(X_1(t)^2 + X_2(t)^2) = 2\mathbf{E}(X_1(t)^2)$, and we write

$$(3.1) \quad X_1(t) = B_1(t) + \int_0^t \phi(\omega_s) ds =: B_1(t) + F_1(t),$$

where $t \mapsto \omega_t$ is the environment seen from the particle (recall (2.3)), and

$$(3.2) \quad \phi(\omega) \stackrel{\text{def}}{=} \omega_1(0).$$

Recall that $\mathbf{E}X_1(t) = 0$. The first term on the right-hand side of (3.1) has variance t , so to prove (2.6), it is sufficient to show

$$(3.3) \quad \frac{C_{-}(\varepsilon)}{(\log |\log \lambda|)^{1+\varepsilon}} \leq \frac{\lambda^2}{\sqrt{|\log \lambda|}} \int_0^\infty e^{-\lambda t} \mathbf{E}[F_1(t)^2] dt \leq C_{+}(\varepsilon)(\log |\log \lambda|)^{1+\varepsilon}.$$

The starting point in the study of $\mathbf{E}[F_1(t)^2]$ is the understanding of the environment process $t \mapsto \omega_t$. As argued in [22], this is a Markov process, whose generator will be denoted by \mathcal{G} , on the Fréchet space of C^∞ , divergence-free two-dimensional vector fields with derivatives growing slower than any power at infinity. As the field is stationary, ergodic and divergence-free, the probability measure \mathbb{P} is stationary for the environment process [11], Chapter 11. This ensures that, as in [5], Lemma 5.1, we have

$$(3.4) \quad \tilde{D}(\lambda) \stackrel{\text{def}}{=} \int_0^\infty e^{-\lambda t} \mathbf{E}[F_1(t)^2] dt = \frac{2}{\lambda^2} \mathbb{E}[\phi(\omega)(\lambda - \mathcal{G})^{-1}\phi(\omega)],$$

with ϕ defined in (3.2). Hence, our analysis will focus on the resolvent $(\lambda - \mathcal{G})^{-1}$. Recall that \mathbb{E} is the expectation with respect to the stationary law of the environment.

A first necessary step is to describe how \mathcal{G} acts on elements in $L^2(\mathbb{P})$, for which we need a more accurate description of the latter space. Since \mathbb{P} is Gaussian (and given by the law of the curl of the smoothed Gaussian free field ω), $L^2(\mathbb{P})$ admits a Wiener chaos decomposition, which we now briefly describe.

Let H_0 be the set containing constant random variables and H_n be the closure of the span of

$$(3.5) \qquad \left\{ \psi = \sum_{j_1, \dots, j_n=1}^2 \int_{\mathbb{R}^{2n}} f_{\mathbf{j}}(x_{1:n}) : \prod_{l=1}^n \omega_{j_l}(x_l) : dx_{1:n} \right\},$$

where $x_{1:n}$ is a shorthand notation for (x_1, \dots, x_n) , $\mathbf{j} \stackrel{\text{def}}{=} (j_1, \dots, j_n)$, $: \dots :$ denotes the Wick product associated to the measure \mathbb{P} and the symmetric functions $f_{\mathbf{j}}$'s are such that

$$(3.6) \qquad \hat{\psi}(p_{1:n}) \stackrel{\text{def}}{=} \sum_{j_1, \dots, j_n=1}^2 \prod_{k=1}^n \tilde{p}_{k,j_k} \hat{f}_{\mathbf{j}}(p_{1:n})$$

satisfies

$$(3.7) \qquad \int_{\mathbb{R}^{2n}} \prod_{i=1}^n \frac{\widehat{V}(p_i)}{|p_i|^2} |\hat{\psi}(p_{1:n})|^2 dp_{1:n} < \infty.$$

Above, $\hat{f}_{\mathbf{j}}$ is the Fourier transform of $f_{\mathbf{j}}$, $\tilde{p}_{k,1} = (p_k)_2$ and $\tilde{p}_{k,2} = -(p_k)_1$, with $(p_k)_\ell$ the ℓ th component, $\ell = 1, 2$, of p_k . Also, \widehat{V} is the Fourier transform of the bump function V .

REMARK 3.1. The intuition behind (3.7) becomes clear upon noting that the components of ω are nothing but the derivatives of the smoothed Gaussian field ξ (see (1.2)). Indeed, by performing an n -fold integration by parts in (3.5), using the fact that, in Fourier space, $\partial_{(x_j)_\ell}$ corresponds to multiplication by $\iota(p_j)_\ell$ (with $\iota = \sqrt{-1}$) and writing the covariance (2.2) of the field ξ in Fourier variables, one sees that (3.7) is just the $L^2(\mathbb{P})$ norm squared of ψ in (3.5).

REMARK 3.2. It is easy to see that the random variable $\phi(\omega) = \omega_1(0)$ belongs to H_1 and has kernel $\hat{\phi}(p) = p_2$.

Then, by [17], Theorem 1.1.1, $L^2(\mathbb{P})$ can be orthogonally decomposed as

$$(3.8) \qquad L^2(\mathbb{P}) = \bigoplus_{n=0}^\infty H_n$$

and the expectation of the scalar product of $F, G \in L^2(\mathbb{P})$ satisfies

$$\mathbb{E}[FG] = \sum_{n=1}^\infty \langle \psi_n, \phi_n \rangle.$$

Above, ψ_n and ϕ_n are the kernels of the projections of F and G onto H_n and the scalar product appearing at the right-hand side is given by

$$(3.9) \qquad \langle \psi_n, \phi_n \rangle \stackrel{\text{def}}{=} n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \frac{\widehat{V}(p_j)}{|p_j|^2} \overline{\hat{\psi}_n(p_{1:n})} \hat{\phi}_n(p_{1:n}) dp_{1:n}.$$

REMARK 3.3. In what follows, we will implicitly identify a random variable F in H_n of the form (3.5) with its kernel $\hat{\psi}_n$ in Fourier space, since this mapping is an isometry from H_n to the set $L^2_{\text{sym}}(\mathbb{R}^{2n})$ of symmetric functions on \mathbb{R}^{2n} endowed with the scalar product $\langle \cdot, \cdot \rangle$ in (3.9). In the same spirit, we will identify linear operators acting on $L^2(\mathbb{P})$ with the corresponding linear operators acting on $\bigoplus_n L^2_{\text{sym}}(\mathbb{R}^{2n})$, and with a slight abuse of notation, we will denote them using the same symbol.

REMARK 3.4. With respect to [9, 22], we are using different normalization conventions in (3.5) and in the scalar product in (3.9). More specifically, in the conventions of [9, 22] there would be a factor $1/\sqrt{n!}$ in front of the integral in (3.5) and no factor $n!$ in (3.9). In other words, our kernels ψ_n equal those of [9, 22] times $1/\sqrt{n!}$. Our conventions are consistent with those of [6] and of [10, 17]; we refer to these latter references for more details on Wiener chaos analysis.

We are now ready to move back to the analysis of the generator \mathcal{G} of the environment process. As noted in [22], \mathcal{G} can be written as

$$\mathcal{G} = -\Delta + \mathcal{A}_+ - \mathcal{A}_+^*,$$

where $-\Delta$ and $\mathcal{A} \stackrel{\text{def}}{=} \mathcal{A}_+ - \mathcal{A}_+^*$, respectively, denote the symmetric and anti-symmetric part of \mathcal{G} with respect to \mathbb{P} , and \mathcal{A}_+^* is the adjoint of \mathcal{A}_+ in $L^2(\mathbb{P})$. The action of $-\Delta$ and \mathcal{A} in Fock space is explicit. First of all, Δ maps the n th chaos H_n into itself while \mathcal{A}_+ (resp., \mathcal{A}_+^*) maps H_n into H_{n+1} (resp., H_{n-1}) and can therefore be interpreted as a “creation” (resp., annihilation) operator. Moreover, Δ is diagonal in Fourier space as it acts as a Fourier multiplier on the kernels, while \mathcal{A}_+ is not. Adopting the convention in Remark 3.3, one has (see also [22], Section 2.1)⁴

$$(3.10) \quad \begin{aligned} \widehat{(-\Delta)\psi_n}(p_{1:n}) &= \left| \sum_{i=1}^n p_i \right|^2 \widehat{\psi_n}(p_{1:n}), \\ \widehat{\mathcal{A}_+\psi_n}(p_{1:n+1}) &= \iota \frac{1}{n+1} \sum_{i=1}^{n+1} \left(p_i \times \sum_{j=1}^{n+1} p_j \right) \widehat{\psi_n}(p_{1:n+1 \setminus i}), \end{aligned}$$

$\iota = \sqrt{-1}$, for $\psi_n \in H_n$. Above and throughout, we denote by $p_{1:n+1 \setminus i}$ the collection $p_{1:n} = (p_1, \dots, p_n)$ with p_i removed. Also, for a, b two vectors in \mathbb{R}^2 , by $a \times b$ we mean the scalar given by the vertical component of the usual cross product $a \times b$, with a, b viewed as vectors in \mathbb{R}^3 . Explicitly, $a \times b = |a||b| \sin \theta$, where θ is the angle between a and b .

REMARK 3.5. Observe that the “Laplacian” $-\Delta$ is different from the one appearing in [6], which acts instead as multiplication by $\sum_{i=1}^n |p_i|^2$. This represents a major technical difference, which forces us to significantly modify the arguments therein.

At last, in light of the notation and conventions introduced above (see, in particular, Remark 3.3) we rewrite (3.4) in Fock space as

$$(3.11) \quad \int_0^\infty e^{-\lambda t} \mathbf{E}[F_1(t)^2] dt = \frac{2}{\lambda^2} \langle \phi, (\lambda - \mathcal{G})^{-1} \phi \rangle,$$

where ϕ is the random variable $\phi(\omega)$ in (3.2), which lives in H_1 (see Remark 3.2).

4. The generator equation and the diffusivity. In order to obtain suitable bounds on the right-hand side of (3.11), one should in principle solve the generator equation $(\lambda - \mathcal{G})\psi = \phi$ and then try to evaluate $\langle \phi, \psi \rangle$. While ϕ belongs to the first chaos, the operator \mathcal{G} is not diagonal in the chaos decomposition and finding ψ explicitly is a rather challenging task. A way out was first devised in [15]. The idea is to truncate the generator \mathcal{G} by defining

⁴If we had adopted the normalization conventions analogous to those of [9], the factor $1/(n+1)$ in (3.10) would be replaced by $1/\sqrt{n+1}$. This is due to the different definition of the kernels; see Remark 3.4 above.

$\mathcal{G}_n \stackrel{\text{def}}{=} I_{\leq n} \mathcal{G} I_{\leq n}$, with $I_{\leq n}$ the orthogonal projection onto $H_{\leq n} \stackrel{\text{def}}{=} \bigoplus_{k \leq n} H_k$ (the chaoses up to order n), and then consider the solution $\psi^{(n)} \in H_{\leq n}$ of the truncated generator equation

(4.1)
$$(\lambda - \mathcal{G}_n) \psi^{(n)} = \phi.$$

The advantage of this procedure is that it provides upper and lower bounds (depending on the parity of n) on (3.11). Indeed, the following lemma, which was first proven in [15], Lemma 2.1, (and whose proof straightforwardly carries out in the present case) holds.

LEMMA 4.1. *For every $n \geq 1$, one has*

(4.2)
$$\langle \phi, \psi^{(2n)} \rangle \leq \langle \phi, (\lambda - \mathcal{G})^{-1} \phi \rangle = \langle \phi, \psi \rangle \leq \langle \phi, \psi^{(2n+1)} \rangle.$$

Equation (4.1) coincides with the following hierarchical system of n equations, one for each component $\psi_k^{(n)}$ of $\psi^{(n)}$,

(4.3)
$$\begin{cases} (\lambda - \Delta) \psi_n^{(n)} - \mathcal{A}_+ \psi_{n-1}^{(n)} = 0, \\ (\lambda - \Delta) \psi_{n-1}^{(n)} - \mathcal{A}_+ \psi_{n-2}^{(n)} + \mathcal{A}_+^* \psi_n^{(n)} = 0, \\ \dots, \\ (\lambda - \Delta) \psi_1^{(n)} + \mathcal{A}_+^* \psi_2^{(n)} = 0. \end{cases}$$

Since ϕ belongs to the first chaos and different chaoses are orthogonal, in order to estimate the terms at the left- and right-hand side of (4.2) we only need to know $\psi_1^{(n)}$. The latter in turn can be obtained by solving the system (4.3) iteratively starting from $k = n$ so that we get

(4.4)
$$\langle \phi, \psi^{(n)} \rangle = \langle \phi, \psi_1^{(n)} \rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_n)^{-1} \phi \rangle,$$

where the self-adjoint operators \mathcal{H}_j are recursively defined as

(4.5)
$$\begin{aligned} \mathcal{H}_1 &\stackrel{\text{def}}{=} 0, \\ \mathcal{H}_{j+1} &= \mathcal{A}_+^* (\lambda - \Delta + \mathcal{H}_j)^{-1} \mathcal{A}_+ \quad \text{for } j \geq 1. \end{aligned}$$

We remark that these operators are positive and leave each chaos invariant, that is, $\mathcal{H}_j H_n \subset H_n$, for all $j, n \in \mathbb{N}$.

4.1. *Operator recursive estimates.* In view of (4.2) and (4.4), the proof of Theorem 2.2 must entail a good understanding of the operators \mathcal{H}_j 's in (4.5). In particular, we need to derive suitable (upper and lower) bounds on them and this is the content of the main result of this section, Theorem 4.2. To state it, we need a few preliminary definitions.

For $k \in \mathbb{N}$, $x > 0$ and $z \geq 0$, we define L , LB_k and UB_k as follows:

(4.6)
$$L(x, z) = z + \log(1 + x^{-1}),$$

(4.7)
$$LB_k(x, z) = \sum_{0 \leq j \leq k} \frac{(\frac{1}{2} \log L(x, z))^j}{j!} \quad \text{and} \quad UB_k(x, z) = \frac{L(x, z)}{LB_k(x, z)}$$

and for $k \geq 1$, σ_k , as

$$\sigma_k(x, z) = \begin{cases} UB_{\frac{k-2}{2}}(x, z) & \text{if } k \text{ is even,} \\ LB_{\frac{k-1}{2}}(x, z) & \text{if } k \text{ is odd.} \end{cases}$$

Note that $\sigma_1 \equiv 1$. All the properties we need on the functions UB_k , LB_k are summarized in Lemma A.1. Further, let

(4.8)
$$z_k(n) = K_1(n + k)^{2+2\varepsilon} \quad \text{and} \quad f_k(n) = K_2 \sqrt{z_k(n)},$$

where K_1, K_2 are absolute constants (chosen sufficiently large, so that (4.20), (4.24) and (4.25) below, hold) and ε is the small positive constant that appears in the statement of Theorem 2.2.

Finally, for $k \geq 1$ let S_k be the operator whose multiplier is σ_k , that is,

$$S_k = \begin{cases} f_k(\mathcal{N})\sigma_k(\lambda - \Delta, z_k(\mathcal{N})) & \text{if } k \text{ is even,} \\ \frac{1}{f_k(\mathcal{N})}(\sigma_k(\lambda - \Delta, z_k(\mathcal{N})) - f_k(\mathcal{N})) & \text{if } k \text{ is odd,} \end{cases}$$

where \mathcal{N} is the number operator acting on the n th chaos as multiplication by n , that is, $(\mathcal{N}\phi_n) = n\phi_n$ for $\phi_n \in H_n$. We are now ready to state the following theorem.

THEOREM 4.2. *For any $\varepsilon > 0$, the constants K_1, K_2 in (4.8) can be chosen in such a way that the following holds. For $0 < \lambda \leq 1$ and $k \geq 1$, one has the operator bounds*

$$(4.9) \quad \mathcal{H}_{2k-1} \geq c_{2k-1}(-\Delta)S_{2k-1}$$

and

$$(4.10) \quad \mathcal{H}_{2k} \leq c_{2k}(-\Delta)S_{2k},$$

where $c_1 = 1$ and

$$(4.11) \quad c_{2k} = \frac{\pi}{c_{2k-1}} \left(1 + \frac{1}{k^{1+\varepsilon}} \right), \quad c_{2k+1} = \frac{\pi}{c_{2k}} \left(1 - \frac{1}{(k+1)^{1+\varepsilon}} \right).$$

REMARK 4.3. A crucial aspect we need to stress is that, as $j \rightarrow \infty$, c_{2j} tends to a finite constants larger than 1, while c_{2j+1} tends to a strictly positive constant smaller than 1.

4.2. Generalities about the operators. In this section, we collect some preliminary facts and bounds concerning operators in Fock space. In all the statements herein, S will be a *diagonal operator*, meaning that S commutes with \mathcal{N} (i.e., it maps the n th chaos H_n into itself) and is diagonal in the Fourier basis, that is, it acts in Fourier space as multiplication by a function of the momenta. The Fourier multiplier of S will be denoted \mathfrak{s} , and actually \mathfrak{s} is the collection $(\mathfrak{s}_n)_{n \geq 1}$, with \mathfrak{s}_n the Fourier multiplier on H_n . It is understood that \mathfrak{s}_n is a symmetric function of its n arguments.

LEMMA 4.4. *Let S be a positive diagonal operator and let \mathfrak{s} be its Fourier multiplier. For any element ψ of H_n , we can write*

$$\langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle = \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Diag}} + \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Off}},$$

where the “diagonal part” is defined as

$$(4.12) \quad \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Diag}} \stackrel{\text{def}}{=} n! \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 \mathfrak{s}_{n+1}(p_{1:n+1}) \left(p_{n+1} \times \sum_{j=1}^n p_j \right)^2 dp_{1:n+1}$$

while the “off-diagonal part” is

$$(4.13) \quad \begin{aligned} & \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Off}} \\ & \stackrel{\text{def}}{=} n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \overline{\hat{\psi}(p_{1:n})} \hat{\psi}(p_{1:n+1 \setminus n}) \mathfrak{s}_{n+1}(p_{1:n+1}) \\ & \quad \times \left(p_{n+1} \times \sum_{i=1}^{n+1} p_i \right) \left(p_n \times \sum_{i=1}^{n+1} p_i \right) dp_{1:n+1}. \end{aligned}$$

PROOF. Expanding the inner product using (3.10), we obtain

$$\begin{aligned} & \langle \mathcal{A}_+ \psi_n, S \mathcal{A}_+ \psi_n \rangle \\ &= \frac{(n+1)!}{(n+1)^2} \int_{\mathbb{R}^{2(n+1)}} \prod_{i=1}^{n+1} \frac{\widehat{V}(p_i)}{|p_i|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \left| \sum_{i=1}^{n+1} \widehat{\psi}(p_{1:n+1 \setminus i}) \left(p_i \times \sum_{j=1}^{n+1} p_j \right) \right|^2 dp_{1:n+1}. \end{aligned}$$

The “diagonal” and “off-diagonal” refer to the squared sum. The former is the contribution of the squared summands while the latter comes from all the cross terms. Hence, the diagonal part is

$$\begin{aligned} & \frac{n!}{n+1} \int_{\mathbb{R}^{2(n+1)}} \prod_{i=1}^{n+1} \frac{\widehat{V}(p_i)}{|p_i|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \sum_{i=1}^{n+1} |\widehat{\psi}(p_{1:n+1 \setminus i})|^2 \left(p_i \times \sum_{j=1}^{n+1} p_j \right)^2 dp_{1:n+1} \\ &= n! \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} |\widehat{\psi}(p_{1:n})|^2 \mathfrak{s}_{n+1}(p_{1:n+1}) \left(p_{n+1} \times \sum_{j=1}^n p_j \right)^2 dp_{1:n+1}, \end{aligned}$$

where we pulled out the sum and used that $\widehat{\psi}$ is symmetric in its arguments. For the off-diagonal part, one follows the same procedure. Since there are in total $n(n+1)$ summands, a factor n is left in front of the integral. \square

The next two results will be used in the bounds on the diagonal and off-diagonal parts, respectively. In order to appreciate them, note that at the right-hand side of both (4.12) and (4.13), there appears the vector product.

LEMMA 4.5. *Let S be a positive diagonal operator, and let \mathfrak{s} be the associated Fourier multiplier. If for every integer n and for every $p_{1:n} \in \mathbb{R}^{2n}$ with $\sum_{k=1}^n p_k \neq 0$,*

$$(4.14) \quad \int_{\mathbb{R}^2} \widehat{V}(q) (\sin \theta)^2 \mathfrak{s}_{n+1}(p_{1:n}, q) dq \leq \tilde{\mathfrak{s}}_n(p_{1:n})$$

with θ the angle between q and $\sum_{k=1}^n p_k$, then for every ψ ,

$$(4.15) \quad \langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Diag}} \leq \langle \psi, (-\Delta) \tilde{S} \psi \rangle,$$

where \tilde{S} is the diagonal operator whose Fourier multiplier is $\tilde{\mathfrak{s}}$.

If the inequality in (4.14) is reversed, then (4.15) holds with the reversed inequality.

PROOF. Starting from (4.12) and denoting $q = p_{n+1}$ we get that the left-hand side equals

$$\begin{aligned} & n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \frac{\widehat{V}(p_j)}{|p_j|^2} |\widehat{\psi}(p_{1:n})|^2 \left| \sum_{k=1}^n p_k \right|^2 \int_{\mathbb{R}^2} \mathfrak{s}_{n+1}(p_{1:n}, q) \widehat{V}(q) (\sin \theta)^2 dq dp_{1:n} \\ & \leq n! \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \frac{\widehat{V}(p_j)}{|p_j|^2} |\widehat{\psi}(p_{1:n})|^2 \left| \sum_{k=1}^n p_k \right|^2 \tilde{\mathfrak{s}}_n(p_{1:n}) = \langle \psi, (-\Delta) \tilde{S} \psi \rangle, \end{aligned}$$

where we used that $a \times b = |a||b| \sin \theta$, θ being the angle between a and b , and (4.14). Since every step except the assumption is an equality, the other direction also holds. \square

LEMMA 4.6. *Let S be a diagonal, positive operator with Fourier multiplier \mathfrak{s} . If for every integer n and every $p_{1:n} \in \mathbb{R}^{2n}$, one has*

$$\left| \sum_{i=1}^n p_i \right| \int_{\mathbb{R}^2} \widehat{V}(q) \frac{(\sin \theta)^2 \mathfrak{s}_{n+1}(p_{1:n}, q)}{|q + \sum_{i=1}^{n-1} p_i|} dq \leq \tilde{\mathfrak{s}}_n(p_{1:n})$$

with θ the angle between q and $\sum_{i=1}^n p_i$, then for every $n \in \mathbb{N}$, $\psi \in H_n$ one has

$$|\langle \psi, \mathcal{A}_+^* S \mathcal{A}_+ \psi \rangle_{\text{Off}}| \leq n \langle \psi, (-\Delta) \tilde{S} \psi \rangle,$$

with \tilde{S} the diagonal operator of Fourier multiplier \tilde{s} .

PROOF. We start by bounding the left-hand side of (4.13) as

$$\begin{aligned} & n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})| |\hat{\psi}(p_{1:n+1 \setminus n})| \mathfrak{s}_{n+1}(p_{1:n+1}) \\ & \times \left| p_{n+1} \times \sum_{k=1}^n p_k \right| \left| p_n \times \left(\sum_{k=1}^{n-1} p_k + p_{n+1} \right) \right| dp_{1:n+1} \\ (4.16) \quad & = n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \Phi(p_{1:n+1}) \Phi(p_{1:n-1}, p_{n+1}, p_n) \\ & \times \left| \sum_{k=1}^{n-1} p_k + p_{n+1} \right| \left| \sum_{k=1}^n p_k \right| dp_{1:n+1}, \end{aligned}$$

where

$$\Phi(p_{1:n+1}) = \frac{|\hat{\psi}(p_{1:n})| |p_{n+1} \times \sum_{k=1}^n p_k|}{|\sum_{k=1}^{n-1} p_k + p_{n+1}|}.$$

We apply Cauchy–Schwarz and exploit symmetry of ψ to bound (4.16) from above by

$$n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \Phi(p_{1:n+1})^2 \left| \sum_{k=1}^{n-1} p_k + p_{n+1} \right| \left| \sum_{k=1}^n p_k \right| dp_{1:n+1}.$$

Now set $s_1 = \sum_{k=1}^n p_k$, $s_2 = \sum_{k=1}^{n-1} p_k$ and $q = p_{n+1}$, which gives

$$\begin{aligned} & n!n \int_{\mathbb{R}^{2(n+1)}} \prod_{j=1}^{n+1} \frac{\widehat{V}(p_j)}{|p_j|^2} \mathfrak{s}_{n+1}(p_{1:n+1}) \frac{|\hat{\psi}(p_{1:n})|^2 |q \times s_1|^2}{|q + s_2|} |s_1| dp_{1:n+1} \\ & = n!n \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \left(\frac{\widehat{V}(p_j)}{|p_j|^2} dp_j \right) |\hat{\psi}(p_{1:n})|^2 |s_1|^3 \int_{\mathbb{R}^2} \mathfrak{s}_{n+1}(p_{1:n}, q) \frac{\widehat{V}(q)(\sin \theta)^2}{|q + s_2|} dq \\ & \leq n!n \int_{\mathbb{R}^{2n}} \prod_{j=1}^n \frac{\widehat{V}(p_j)}{|p_j|^2} |\hat{\psi}(p_{1:n})|^2 |s_1|^2 \tilde{s}_n(p_{1:n}) dp_{1:n} = n \langle \psi, (-\Delta) \tilde{S} \psi \rangle, \end{aligned}$$

which concludes the proof. \square

4.3. Proof of Theorem 4.2. This section is devoted to Theorem 4.2. We will first show the lower bound and then the upper bound, both by induction on k . The induction switches from lower to upper bounds and vice versa, as follows: For $k = 1$, the bound (4.9) will be trivial; given (4.9) for $k = 1$, we will deduce (4.10) with $k = 1$, then (4.9) with $k = 2$ and so on.

PROOF OF THE LOWER BOUND (4.9). For $k = 1$, (4.9) trivially holds as \mathcal{H}_1 is by definition zero while \mathcal{S}_1 is nonpositive if the constant K_2 in the definition (4.8) is large enough.

We need then to prove (4.9) with $k \geq 1$ and $2k - 1$ replaced by $2k + 1$. Assume by induction that (4.10) holds. Then we have

$$(4.17) \quad \mathcal{H}_{2k+1} = \mathcal{A}_+^*(\lambda - \Delta + \mathcal{H}_{2k})^{-1} \mathcal{A}_+ \geq \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k} \mathcal{S}_{2k}))^{-1} \mathcal{A}_+.$$

For $\psi \in H_n$, we apply Lemma 4.4 with $S = (\lambda - \Delta(1 + c_{2k} \mathcal{S}_{2k}))^{-1}$ and we split

$$(4.18) \quad \langle \psi, \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k} \mathcal{S}_{2k}))^{-1} \mathcal{A}_+ \psi \rangle$$

into diagonal and off-diagonal part. In order to control the former from below, we exploit Lemma 4.5 according to which it suffices to consider

$$(4.19) \quad \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2}{\lambda + |p + q|^2(1 + c_{2k} f_{2k}(n+1) \text{UB}_{k-1}(\lambda + |p + q|^2, z_{2k}(n+1)))} dq,$$

where $p = \sum_{i=1}^n p_i \neq 0$ and θ is the angle between p and q . Note that the functions f_{2k}, z_{2k} have argument $n+1$ because $\mathcal{A}_+ \psi \in H_{n+1}$ but, by (4.8), $f_{2k}(n+1) = f_{2k+1}(n)$, $z_{2k}(n+1) = z_{2k+1}(n)$. To lighten the notation, throughout the proof we will omit the argument n and write z_{2k+1}, f_{2k+1} instead of $z_{2k+1}(n), f_{2k+1}(n)$.

The denominator in (4.19) is upper bounded by

$$c_{2k} f_{2k+1} \left(1 + \frac{1}{f_{2k+1}}\right) (\lambda + |p + q|^2 \text{UB}_{k-1}(\lambda + |p + q|^2, z_{2k+1})),$$

as c_{2k}, f_{2k+1} and UB_{k-1} are all larger than one. Thus we can concentrate on

$$\int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2}{\lambda + |p + q|^2 \text{UB}_{k-1}(\lambda + |p + q|^2, z_{2k+1})} dq.$$

For this, we first apply Lemmas A.2 and A.4 to obtain the lower bound

$$\begin{aligned} & \frac{\pi}{2} \int_{\lambda+|p|^2}^1 \frac{d\rho}{\rho \text{UB}_{k-1}(\rho, z_{2k+1})} - C_{\text{Diag}} \frac{\text{LB}_{k-1}(\lambda + |p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}} \\ & \geq \frac{\pi}{2} \int_{\lambda+|p|^2}^1 \frac{d\rho}{(\rho + \rho^2) \text{UB}_{k-1}(\rho, z_{2k+1})} - C_{\text{Diag}} \frac{\text{LB}_k(\lambda + |p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}}. \end{aligned}$$

From Lemma A.1, we have that the primitive of the integrand is $-2 \text{LB}_k(\rho, z_{2k+1})$, so that the last expression equals

$$\begin{aligned} & \pi \text{LB}_k(\lambda + |p|^2, z_{2k+1}) - \pi \text{LB}_k(1, z_{2k+1}) - C_{\text{Diag}} \frac{\text{LB}_k(\lambda + |p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}} \\ & \geq \pi \text{LB}_k(\lambda + |p|^2, z_{2k+1}) - \pi \frac{f_{2k+1}}{2} - C_{\text{Diag}} \frac{\text{LB}_k(\lambda + |p|^2, z_{2k+1})}{\sqrt{z_{2k+1}}}, \end{aligned}$$

where in the first inequality we need to choose K_2 large enough in (4.8) so that for all k and n ,

$$(4.20) \quad \text{LB}_k(1, z_{2k+1}) \leq \sqrt{L(1, z_{2k+1})} = \sqrt{\log(2) + z_{2k+1}} \leq \frac{1}{2} f_{2k+1}$$

(see also (A.1)). Altogether, the diagonal part of (4.18) is lower bounded as $\langle \psi, (-\Delta) \tilde{S} \psi \rangle$, with

$$(4.21) \quad \tilde{S} = \left(1 + \frac{1}{f_{2k+1}(1)}\right)^{-1} \frac{\pi}{c_{2k}} \left[\frac{\text{LB}_k(\lambda - \Delta, z_{2k+1}(\mathcal{N}))}{f_{2k+1}(\mathcal{N})} \left(1 - \frac{C_{\text{Diag}}}{\pi \sqrt{z_{2k+1}(1)}}\right) - \frac{1}{2} \right]$$

(in two instances, we have lower bounded $z_{2k+1} = z_{2k+1}(n)$ and $f_{2k+1} = f_{2k+1}(n)$ with the same quantities for $n = 1$).

For the off-diagonal terms in (4.18), we use Lemma 4.6 so that, calling $p := \sum_{i=1}^n p_i$ and $p' := \sum_{i=1}^{n-1} p_i$, we have to upper bound

$$(4.22) \quad n|p| \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2}{(\lambda + |p+q|^2(1 + c_{2k} f_{2k+1} \text{UB}_{k-1}(\lambda + |p+q|^2, z_{2k+1})))|p' + q|} dq.$$

Thanks to Lemmas A.3 and A.4, applied with $f(x, z) = c_{2k} f_{2k+1} \text{UB}_{k-1}(x, z)$ and $g(x, z) = \frac{1}{c_{2k} f_{2k+1}} \text{LB}_{k-1}(x, z)$, this expression is upper bounded by

$$(4.23) \quad \begin{aligned} & \frac{n C_{\text{off}}}{c_{2k} f_{2k+1} z_{2k+1}} \text{LB}_{k-1}(\lambda + |p|^2, z_{2k+1}) \\ & \leq \frac{n C_{\text{off}}}{c_{2k} f_{2k+1} z_{2k+1}} \text{LB}_k(\lambda + |p|^2, z_{2k+1}) \\ & \leq \frac{C_{\text{off}}}{c_{2k} f_{2k+1}} \frac{1}{K_1(2k+1)^{1+\varepsilon}} \text{LB}_k(\lambda + |p|^2, z_{2k+1}), \end{aligned}$$

where we used monotonicity properties of LB_k , the definition of $z_{2k+1} = z_{2k+1}(n)$ in (4.8) and in particular the fact that

$$\frac{n}{z_{2k+1}(n)} = \frac{n}{K_1(2k+1+n)^{2+2\varepsilon}} \leq \frac{1}{K_1(2k+1+n)^{1+\varepsilon}}.$$

Combining (4.21) and (4.23), together with Lemmas 4.5 and 4.6, we conclude that $\mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k} \mathcal{S}_{2k}))^{-1} \mathcal{A}_+$ is lower bounded by

$$(-\Delta) \frac{\pi}{c_{2k}} \left[\frac{\text{LB}_k(\lambda - \Delta, z_{2k+1}(\mathcal{N}))}{f_{2k+1}(\mathcal{N})} A - B \right],$$

where A and B are given by

$$\begin{aligned} A &= \left(1 - \frac{C_{\text{Diag}}}{\pi \sqrt{z_{2k+1}(1)}} \right) \left(1 + \frac{1}{f_{2k+1}(1)} \right)^{-1} - \frac{C_{\text{off}}}{\pi K_1(2k+1)^{1+\varepsilon}}, \\ B &= \frac{1}{2} \left(1 + \frac{1}{f_{2k+1}(1)} \right)^{-1} \end{aligned}$$

and thanks to (4.17) the same lower bound holds for \mathcal{H}_{2k+1} . Note that, provided the constants K_1, K_2 in (4.8) are large, one has

$$(4.24) \quad A \geq 1 - \frac{1}{(k+1)^{1+\varepsilon}}, \quad B \leq 1 - \frac{1}{(k+1)^{1+\varepsilon}}.$$

Therefore, we have proven (4.9) (with $2k+1$ instead of $2k-1$) with c_{2k+1} given by (4.11). \square

PROOF OF THE UPPER BOUND (4.10). For $k \geq 1$, again by the induction hypothesis we have

$$\mathcal{H}_{2k} = \mathcal{A}_+^*(\lambda - \Delta + \mathcal{H}_{2k-1})^{-1} \mathcal{A}_+ \leq \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k-1} \mathcal{S}_{2k-1}))^{-1} \mathcal{A}_+.$$

We split $\langle \psi, \mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k-1} \mathcal{S}_{2k-1}))^{-1} \mathcal{A}_+ \psi \rangle$ into diagonal and off-diagonal parts as in Lemma 4.4. By Lemma 4.5 for the diagonal part, we need to upper bound the integral

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2}{\lambda + |p+q|^2(1 + \frac{c_{2k-1}}{f_{2k}} (\text{LB}_{k-1}(\lambda + |p+q|^2, z_{2k}) - f_{2k}))} dq \\ & \leq \frac{f_{2k}}{c_{2k-1}} \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2}{\lambda + |p+q|^2 \text{LB}_{k-1}(\lambda + |p+q|^2, z_{2k})} dq, \end{aligned}$$

where we used $f_{2k-1}(n+1) = f_{2k}(n)$, the same for z (and we suppressed the argument of both) and, in the second step, exploited the fact that $c_{2k-1} < 1$ and $f_{2k} > 1$. By Lemmas A.2 and A.4, the latter is bounded above by

$$\frac{f_{2k}\pi}{2c_{2k-1}} \left(\int_{\lambda+|p|^2}^1 \frac{d\rho}{\rho \text{LB}_{k-1}(\rho, z_{2k})} + \frac{C_{\text{diag}} \text{UB}_{k-1}(\lambda + |p|^2, z_{2k})}{\sqrt{z_{2k}}} \right).$$

The integral can be controlled via Lemma A.5, so that

$$\begin{aligned} \int_{\lambda+|p|^2}^1 \frac{d\rho}{\rho \text{LB}_{k-1}(\rho, z_{2k})} &\leq \int_{\lambda+|p|^2}^1 \frac{d\rho}{(\rho + \rho^2) \text{LB}_{k-1}(\rho, z_{2k})} + C \frac{\text{UB}_{k-1}(\lambda + |p|^2, z_{2k})}{z_{2k}} \\ &\leq 2 \text{UB}_{k-1}(\lambda + |p|^2, z_{2k}) + C \frac{\text{UB}_{k-1}(\lambda + |p|^2, z_{2k})}{z_{2k}}, \end{aligned}$$

the last passage being a consequence of Lemma A.1.

For the off-diagonal terms, we argue as in the analysis of (4.22), so that we need to control

$$n|p| \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2}{(\lambda + |p+q|^2)(1 + \frac{c_{2k-1}}{f_{2k}}(\text{LB}_{k-1}(\lambda + |p+q|^2, z_{2k}) - f_{2k}))|p'+q|} dq.$$

Once again, we can pull out the factor $\frac{f_{2k}}{c_{2k-1}}$ and apply once more Lemmas A.3 and A.4, this time with $f(x, z) = \text{LB}_{k-1}(x, z)$ and $g(x, z) = \text{UB}_{k-1}(x, z)$. Hence, we obtain

$$\frac{f_{2k}}{c_{2k-1}} \frac{n C_{\text{off}} \text{UB}_{k-1}(\lambda + |p|^2, z_{2k})}{z_{2k}} \leq \frac{f_{2k}}{c_{2k-1}} \frac{C_{\text{off}} \text{UB}_{k-1}(\lambda + |p|^2, z_{2k})}{K_1(n+2k)^{1+2\varepsilon}}.$$

Collecting these upper bounds and using the fact that $z_{2k}(n) > z_{2k}(1)$, we conclude that $\mathcal{A}_+^*(\lambda - \Delta(1 + c_{2k-1}S_{2k-1}))^{-1}\mathcal{A}_+$ is upper bounded by

$$\frac{\pi}{c_{2k-1}} A'(-\Delta) S_{2k},$$

where this time, upon choosing K_1 big enough, we have

$$(4.25) \quad A' = 1 + \frac{C_{\text{diag}}}{\pi \sqrt{K_1}(2k)^{1+\varepsilon}} + \frac{C}{\pi K_1(2k)^{2+2\varepsilon}} + \frac{C_{\text{off}}}{\pi K_1(2k)^{1+2\varepsilon}} \leq 1 + \frac{1}{k^{1+\varepsilon}}.$$

It follows that (4.10) holds with c_{2k} satisfying (4.11).

Let us remark that constants K_1 and K_2 such that (4.20), (4.24) and (4.25) hold for all $k, n \in \mathbb{N}$ clearly exist, so that the proof of Theorem 4.2 is concluded. \square

5. Proof of Theorem 2.2. This section is devoted to the proof of Theorem 2.2 and shows how to exploit the iterative bounds derived in the previous section. Recall from Section 3 that it suffices to prove (2.6) with $D(\lambda)$ replaced by $\tilde{D}(\lambda)$ defined in (3.4).

PROOF OF THEOREM 2.2. Let us begin with the upper bound. By Lemma 4.1 and (3.11), we have

$$\frac{\lambda^2}{2} \tilde{D}(\lambda) \leq \langle \phi, \psi^{(2k+1)} \rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_{2k+1})^{-1} \phi \rangle,$$

for ϕ such that $\hat{\phi}(q) = q_2$ (see Remark 3.2), which in turn, by Theorem 4.2, is bounded above by

$$\begin{aligned}
 & \langle \phi, (\lambda - \Delta(1 + c_{2k+1}\mathcal{S}_{2k+1}))^{-1}\phi \rangle \\
 (5.1) \quad &= \int_{\mathbb{R}^2} \frac{\widehat{V}(q)}{|q|^2} \frac{|\hat{\phi}(q)|^2 dq}{\lambda + |q|^2(1 + \frac{c_{2k+1}}{f_{2k+1}}(\text{LB}_k(\lambda + |q|^2, z_{2k+1}) - f_{2k+1}))} \\
 &\leq \frac{f_{2k+1}}{c_{2k+1}} \int_{\mathbb{R}^2} \widehat{V}(q) \frac{dq}{\lambda + |q|^2 \text{LB}_k(\lambda + |q|^2, z_{2k+1})}.
 \end{aligned}$$

Note that, as $\phi \in H_1$, f_{2k+1} and z_{2k+1} are $f_{2k+1}(1)$ and $z_{2k+1}(1)$, that is, are constants depending only on k .

In view of (4.11), we can replace c_{2k+1} with its $k \rightarrow \infty$ limit. By equation (A.9) in Lemma A.2, (5.1) is controlled, up to a multiplicative absolute constant, by

$$\begin{aligned}
 & f_{2k+1} \left[\int_{\lambda}^1 \frac{d\rho}{\rho \text{LB}_k(\rho, z_{2k+1})} + \frac{\text{UB}_k(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\
 (5.2) \quad &\lesssim f_{2k+1} \left[\int_{\lambda}^1 \frac{d\rho}{(\rho + \rho^2) \text{LB}_k(\rho, z_{2k+1})} + \frac{\text{UB}_k(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\
 &\lesssim f_{2k+1} \text{UB}_k(\lambda, z_{2k+1}) \lesssim f_{2k+1} \frac{\text{L}(\lambda, 0) + z_{2k+1}}{\text{LB}_k(\lambda, 0)},
 \end{aligned}$$

where in the first inequality we applied Lemma A.5, in the second Lemma A.1 and in the last the monotonicity of $\text{LB}_k(\cdot, z)$ with respect to z . We now recall that the central limit theorem, applied to Poisson random variables of rate one, gives that

$$(5.3) \quad \lim_{k \rightarrow \infty} \sum_{j=0}^k \frac{k^j}{j!} e^{-k} = \frac{1}{2}.$$

Hence, by choosing

$$(5.4) \quad k = k(\lambda) = \left\lfloor \frac{\log \text{L}(\lambda, 0)}{2} \right\rfloor,$$

in (5.3) and recalling the definition of LB_k in (4.7), we have that for λ sufficiently small

$$(5.5) \quad \frac{e^{-k}}{\text{LB}_k(\lambda, 0)e^{-k}} \lesssim \frac{1}{\sqrt{\text{L}(\lambda, 0)}}.$$

Plugging this into (5.2) and using the definition of $z_{2k+1} = z_{2k+1}(1)$ and f_{2k+1} in (4.8), we ultimately get the upper bound

$$\lambda^2 \tilde{D}(\lambda) \lesssim (\log \text{L}(\lambda, 0))^{1+\varepsilon} \sqrt{\text{L}(\lambda, 0)}$$

which is the desired one, since

$$\text{L}(\lambda, 0) = \log \left(1 + \frac{1}{\lambda} \right)^{\lambda \rightarrow 0} |\log \lambda|.$$

For the lower bound, we argue similarly. Again by Lemma 4.1, we have

$$\frac{\lambda^2}{2} \tilde{D}(\lambda) \geq \langle \phi, \psi^{(2k)} \rangle = \langle \phi, (\lambda - \Delta + \mathcal{H}_{2k})^{-1}\phi \rangle,$$

for ϕ such that $\hat{\phi}(q) = q_2$, which in turn, by Theorem 4.2, is bounded below by

$$(5.6) \quad \begin{aligned} \langle \phi, (\lambda - \Delta(1 + c_{2k}\mathcal{S}_{2k}))^{-1}\phi \rangle &\geq \int_{\mathbb{R}^2} \frac{\widehat{V}(q)}{|q|^2} \frac{|\hat{\phi}(q)|^2 dq}{\lambda + |q|^2(1 + c_{2k}f_{2k}\text{UB}_{k-1}(\lambda + |q|^2, z_{2k}))} \\ &\gtrsim \frac{1}{f_{2k}} \int_{\mathbb{R}^2} \frac{\widehat{V}(q)}{|q|^2} \frac{|\hat{\phi}(q)|^2 dq}{\lambda + |q|^2\text{UB}_{k-1}(\lambda + |q|^2, z_{2k})}. \end{aligned}$$

We restrict the integral to the cone where $|q_2|^2 \geq (1/2)|q|^2$ and we get that (5.6) is lower bounded by

$$\frac{C}{f_{2k}} \int_{\mathbb{R}^2} \widehat{V}(q) \frac{dq}{\lambda + |q|^2\text{UB}_{k-1}(\lambda + |q|^2, z_{2k})},$$

where now the integral is unrestricted because the integrand depends only on $|q|$. We can now apply again equation (A.9) in Lemma A.2, so that overall (5.6) is lower bounded, up to a multiplicative absolute constant, by

$$(5.7) \quad \begin{aligned} &\frac{1}{f_{2k+1}} \left[\int_{\lambda}^1 \frac{d\rho}{\rho\text{UB}_{k-1}(\rho, z_{2k+1})} - \frac{\text{LB}_{k-1}(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\ &\geq \frac{1}{f_{2k+1}} \left[\int_{\lambda}^1 \frac{d\rho}{(\rho + \rho^2)\text{UB}_{k-1}(\rho, z_{2k+1})} - \frac{\text{LB}_{k-1}(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\ &\gtrsim \frac{1}{f_{2k+1}} \left[\text{LB}_k(\lambda, z_{2k+1}) - \text{LB}_k(1, z_{2k+1}) - \frac{\text{LB}_k(\lambda, z_{2k+1})}{\sqrt{z_{2k+1}}} \right] \\ &\gtrsim \frac{1}{f_{2k+1}} [\text{LB}_k(\lambda, z_{2k+1}) - f_{2k+1}], \end{aligned}$$

where in the second inequality we used Lemma A.1, and $\text{LB}_{k-1} \leq \text{LB}_k$, while in the last (4.20) and that, for k large enough, $1 - 1/\sqrt{z_{2k+1}}$ is bounded below by a strictly positive constant. Now, the $-f_{2k+1}$ just gives a constant contribution, which can be absorbed by decreasing the value of C if λ is small enough. Using the inequality in (5.5) for k as in (5.4), we see that

$$\text{LB}_k(\lambda, 0) \gtrsim \sqrt{L(\lambda, 0)},$$

which, together with the definition of f_{2k+1} in (4.8), gives

$$\lambda^2 \tilde{D}(\lambda) \gtrsim (\log L(\lambda, 0))^{-1-\varepsilon} \sqrt{L(\lambda, 0)}.$$

Hence, (3.3) follows at once and, by (3.1) and the discussion thereafter, so does Theorem 2.2. \square

APPENDIX

Here, we collect some the technical estimates about the integrals involved in the proofs. We also include some of the properties of the functions LB_k and UB_k from [6], Lemma C.3.

LEMMA A.1. *For $k \in \mathbb{N}$ let L , LB_k and UB_k be the functions defined in (4.6) and (4.7). Then L , LB_k and UB_k are decreasing in the first variable and increasing in the second. For any $x > 0$ and $z \geq 1$, the following inequalities hold:*

$$(A.1) \quad \begin{aligned} 1 &\leq \text{LB}_k(x, z) \leq \sqrt{L(x, z)}, \\ 1 &\leq \sqrt{z} \leq \sqrt{L(x, z)} \leq \text{UB}_k(x, z) \leq L(x, z). \end{aligned}$$

Moreover, for any $0 < a < b$, we have

$$(A.2) \quad \int_a^b \frac{dx}{(x^2 + x) \text{UB}_k(x, z)} = 2(\text{LB}_{k+1}(a, z) - \text{LB}_{k+1}(b, z)),$$

$$(A.3) \quad \int_a^b \frac{dx}{(x^2 + x) \text{LB}_k(x, z)} \leq 2(\text{UB}_k(a, z) - \text{UB}_k(b, z)).$$

At last, we also have

$$(A.4) \quad \begin{aligned} \partial_x \text{L}(x, z) &= -\frac{1}{x^2 + x}, & \partial_x \text{LB}_k(x, z) &= -\frac{1}{2(x^2 + x) \text{UB}_{k-1}(x, z)}, \\ \partial_x \text{UB}_k(x, z) &= -\frac{1}{2(x^2 + x) \text{LB}_k(x, z)} \left(1 + \frac{(\frac{1}{2} \log \text{L}(x, z))^k}{k! \text{LB}_k(x, z)} \right). \end{aligned}$$

PROOF. All of these properties were shown in [6], Lemma C.3. For completeness, we add here the proof.

The two chains of inequalities in (A.1) are a direct consequence of the respective definitions. A computation of the partial derivative with respect to the second variable yields the desired monotonicity. Furthermore, we have that

$$(A.5) \quad \partial_x \text{L}(x, z) = -\frac{1}{x^2 + x}, \quad \partial_x \text{LB}_k(x, z) = -\frac{1}{2} \frac{\text{LB}_{k-1}(x, z)}{(x^2 + x) \text{L}(x, z)}$$

and

$$(A.6) \quad \begin{aligned} \partial_x \text{UB}_k(x, z) &= -\frac{\text{LB}_k(x, z) - \frac{1}{2} \text{LB}_{k-1}(x, z)}{(x^2 + x) (\text{LB}_k(x, z))^2} \\ &= -\frac{1}{2(x^2 + x) \text{LB}_k(x, z)} \left[1 + \frac{(\frac{1}{2} \log \text{L}(x, z))^k}{k! \text{LB}_k(x, z)} \right], \end{aligned}$$

which are all strictly negative for any $x > 0$ and $z \geq 1$. The above computation of the partial derivatives moreover reveals that

$$\int_a^b \frac{dx}{(x^2 + x) \text{UB}_k(x, z)} = 2 \int_b^a \partial_x \text{LB}_{k+1}(x, z) dx = 2[\text{LB}_{k+1}(a, z) - \text{LB}_{k+1}(b, z)],$$

which is (A.2). For (A.3), notice that

$$\begin{aligned} \int_a^b \frac{dx}{(x^2 + x) \text{LB}_k(x, z)} &= \int_b^a \partial_x \text{UB}_k(x, z) dx + \frac{1}{2} \int_a^b \frac{\text{LB}_{k-1}(x, z)}{(x^2 + x) \text{LB}_k(x, z)^2} dx \\ &\leq \int_b^a \partial_x \text{UB}_k(x, z) dx + \frac{1}{2} \int_a^b \frac{1}{(x^2 + x) \text{LB}_k(x, z)} dx, \end{aligned}$$

where the last inequality follows from the fact that all the terms are positive and for all x we have $\text{LB}_{k-1}(x, z) \leq \text{LB}_k(x, z)$. Bringing the last term to the left-hand side gives the required estimate. \square

LEMMA A.2. Let V be a bump function satisfying Assumption 2.1. Let $z > 1$, $f(\cdot, z) : [0, \infty) \mapsto [1, \infty)$ be a strictly decreasing, differentiable function such that

$$(A.7) \quad -\frac{f(x)}{x} \leq f'(x) < 0 \quad \text{for all } x \in \mathbb{R}$$

and $g(\cdot, z) : [0, \infty) \mapsto [1, \infty)$ a strictly decreasing function such that $g(x, z)f(x, z) \geq z$. Then there exists a constant $C_{\text{Diag}} > 0$ such that for all $z > 1$, the following bound holds:

$$(A.8) \quad \left| \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{\lambda + |p + q|^2 f(\lambda + |p + q|^2, z)} - \frac{\pi}{2} \int_{\lambda + |p|^2}^1 \frac{d\rho}{\rho f(\rho, z)} \right| \leq C_{\text{Diag}} \frac{g(\lambda + |p|^2, z)}{\sqrt{z}},$$

where $0 \neq p \in \mathbb{R}^2$, θ is the angle between p and q and it is understood that the second integral is zero if $\lambda + |p|^2 \geq 1$.

Moreover, for $\lambda \leq 1$,

$$(A.9) \quad \left| \frac{1}{2} \int_{\mathbb{R}^2} \frac{\widehat{V}(q) dq}{\lambda + |q|^2 f(\lambda + |q|^2, z)} - \frac{\pi}{2} \int_{\lambda}^1 \frac{d\rho}{\rho f(\rho, z)} \right| \leq C_{\text{Diag}} \frac{g(\lambda, z)}{\sqrt{z}}.$$

PROOF. As z is fixed throughout, we suppress the dependence of f and g on it. At first, we use the triangle inequality to split the left-hand side of (A.8) into

$$(A.10) \quad \left| \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{\lambda + |p + q|^2 f(\lambda + |p + q|^2)} - \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{(\lambda + |p + q|^2)f(\lambda + |p + q|^2)} \right|$$

$$(A.11) \quad + \left| \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{(\lambda + |p + q|^2)f(\lambda + |p + q|^2)} - \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{(\lambda + |p|^2 + |q|^2)f(\lambda + |p|^2 + |q|^2)} \right|$$

$$(A.12) \quad + \left| \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{(\lambda + |p|^2 + |q|^2)f(\lambda + |p|^2 + |q|^2)} - \frac{\pi}{2} \int_{\lambda + |p|^2}^1 \frac{d\rho}{\rho f(\rho)} \right|.$$

We will bound these three terms separately. For the first, we re-write it as

$$\begin{aligned} & \left| \lambda \int_{\mathbb{R}^2} \frac{\widehat{V}(q)(\sin \theta)^2 (f(\lambda + |p + q|^2) - 1) dq}{(\lambda + |p + q|^2 f(\lambda + |p + q|^2))(\lambda + |p + q|^2) f(\lambda + |p + q|^2)} \right| \\ & \leq \lambda \int_{\mathbb{R}^2} \frac{dq}{(\lambda + |p + q|^2 f(\lambda + |p + q|^2))(\lambda + |p + q|^2)}. \end{aligned}$$

The latter can be further split into two parts, corresponding to $|p + q| \leq |p|$ and $|p + q| > |p|$. In the first case, $f(\lambda + |p + q|^2) \geq f(\lambda + |p|^2)$, and thus we obtain the upper bound

$$\begin{aligned} & \frac{\lambda}{f(\lambda + |p|^2)} \int_{\mathbb{R}^2} \frac{dq}{\left(\frac{\lambda}{f(\lambda + |p|^2)} + |p + q|^2\right)(\lambda + |p + q|^2)} \\ & \leq \frac{\lambda}{f(\lambda + |p|^2)} \left(\int_{\mathbb{R}^2} \frac{dq}{\left(\frac{\lambda}{f(\lambda + |p|^2)} + |p + q|^2\right)^2} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \frac{dq}{(\lambda + |p + q|^2)^2} \right)^{\frac{1}{2}} \\ & = \frac{C\lambda}{f(\lambda + |p|^2)} \frac{f(\lambda + |p|^2)^{\frac{1}{2}}}{\lambda} = \frac{C}{\sqrt{f(\lambda + |p|^2)}} \leq C \frac{g(\lambda + |p|^2)}{\sqrt{z}}, \end{aligned}$$

for some positive constant C . For the other case, we use that $f(\lambda + |p + q|^2) \geq \frac{z}{g(\lambda + |p|^2)}$. Applying the same steps as above, we get an upper bound of the form

$$\frac{C\sqrt{g(\lambda + |p|^2)}}{\sqrt{z}} \leq \frac{Cg(\lambda + |p|^2)}{\sqrt{z}},$$

which holds as $g \geq 1$.

Now we look at (A.11). First, note that the restriction of each integral to the region $|q + p| < |p|$ has an upper bound of the desired form. Indeed for the first integral we can use

$(\sin \theta)^2 \leq \frac{|p+q|^2}{|p|^2}$ (which holds for any q_1 and q_3 by elementary Euclidean geometry) to obtain

$$\begin{aligned} & \int_{|p+q| < |p|} \frac{\widehat{V}(q)(\sin \theta)^2 dq}{(\lambda + |p+q|^2)f(\lambda + |p+q|^2)} \\ & \leq |p|^{-2} \int_{|p+q| < |p|} \frac{dq}{f(\lambda + |p+q|^2)} \leq \frac{C}{f(\lambda + |p|^2)} \leq \frac{Cg(\lambda + |p|^2)}{\sqrt{z}}. \end{aligned}$$

For the second integral in (A.11), we can bound from above $|\sin \theta| \leq 1$, the denominator from below by $|p|^2 f(\lambda + 5|p|^2)$ and notice that the area of integration is of order $|p|^2$.

As for the region $|q+p| \geq |p|$, define $h(x) = xf(x)$. By (A.7), $|h'(x)| \leq 2|f(x)|$, therefore,

$$\begin{aligned} & |h(\lambda + |p+q|^2) - h(\lambda + |p|^2 + |q|^2)| \\ & \leq 2|p+q|^2 - |p|^2 - |q|^2 |f(\min(\lambda + |p+q|^2, \lambda + |p|^2 + |q|^2))|, \end{aligned}$$

since f is positive and decreasing. Therefore, we get

$$\begin{aligned} & \int_{|p+q| \geq |p|} \frac{\widehat{V}(q)(\sin \theta)^2 |h(\lambda + |p+q|^2) - h(\lambda + |p|^2 + |q|^2)| dq}{(\lambda + |p+q|^2)(\lambda + |p|^2 + |q|^2)f(\lambda + |p+q|^2)f(\lambda + |p|^2 + |q|^2)} \\ & \lesssim \int_{|p+q| \geq |p|} \frac{\widehat{V}(q)(\sin \theta)^2 |p||q| \cos \theta dq}{(\lambda + |p+q|^2)(\lambda + |p|^2 + |q|^2)f(\max(\lambda + |p+q|^2, \lambda + |p|^2 + |q|^2))} \\ & \lesssim |p| \int_{|p+q| \geq |p|} \frac{|q| dq}{(\lambda + |p+q|^2)(\lambda + |p|^2 + |q|^2)f(\lambda + 2|p|^2 + 2|q|^2)} \\ & \lesssim \frac{g(\lambda + |p|^2)}{z} |p| \int_{|p+q| \geq |p|} \frac{|q| dq}{(\lambda + |p|^2 + |q|^2)(\lambda + |p+q|^2)} \leq C \frac{g(\lambda + |p|^2)}{\sqrt{z}} \end{aligned}$$

as can be seen by further splitting the last integral into the region where $|q| \geq 2|p|$ and the complementary one, and using $z > 1$. This concludes the estimate of the second term.

For (A.12), we split the first integral into two regions, one such that $|q|^2 \geq 1 - (\lambda + |p|^2)$ and the other given by its (possibly empty) complement. Note that on the first $\lambda + |p|^2 + |q|^2 \geq 1$. Therefore, the integral can be bounded above by

$$\frac{1}{z} \int_{|q|^2 \geq 1 - (\lambda + |p|^2)} \widehat{V}(q)g(\lambda + |p|^2 + |q|^2) dq \leq \frac{g(\lambda + |p|^2)}{z} \int_{\mathbb{R}^2} \widehat{V}(q) dq = C \frac{g(\lambda + |p|^2)}{z}.$$

To treat the second, since $\widehat{V}(\cdot)$ is smooth and rotationally invariant, there is a constant C such that $|\widehat{V}(q) - \widehat{V}(0)| < C|q|^2$ for $|q| \leq 1$. We can now write the remaining integral as

$$\begin{aligned} & \int_{|q|^2 < 1 - (\lambda + |p|^2)} \frac{(\sin \theta)^2 dq}{(\lambda + |p|^2 + |q|^2)f(\lambda + |p|^2 + |q|^2)} \\ & + \int_{|q|^2 < 1 - (\lambda + |p|^2)} \frac{(\widehat{V}(0) - \widehat{V}(q))(\sin \theta)^2 dq}{(\lambda + |p|^2 + |q|^2)f(\lambda + |p|^2 + |q|^2)}. \end{aligned}$$

By passing to polar coordinates and setting $\rho = \lambda + |p|^2 + |q|^2$, the first summand can be immediately seen to equal the second integral in (A.12). The second summand instead can be controlled via

$$\begin{aligned} & \int_{|q|^2 < 1 - (\lambda + |p|^2)} \frac{|\widehat{V}(0) - \widehat{V}(q)|(\sin \theta)^2 dq}{(\lambda + |p|^2 + |q|^2)f(\lambda + |p|^2 + |q|^2)} \\ & \leq \frac{Cg(\lambda + |p|^2)}{z} \int_{|q|^2 < 1 - (\lambda + |p|^2)} \frac{|q|^2 dq}{\lambda + |p|^2 + |q|^2} \leq C \frac{g(\lambda + |p|^2)}{z}. \end{aligned}$$

Thus, collecting all the estimates obtained so far, (A.8) follows at once.

Finally, to see (A.14), we recall that (A.8) holds uniformly for $p \neq 0$. Letting $p \rightarrow 0$, the second integral and the right-hand side of (A.8) tend to the analogous quantities in (A.14). As for the first integral in (A.8), for $p \rightarrow 0$ the integral over $|q|$ and θ factorizes, and we get the first integral in (A.14) times $1/2$ (coming from the average of $(\sin \theta)^2$). \square

LEMMA A.3. *Let the assumptions of Lemma A.2 be in place. Then there exists a constant $C_{\text{off}} > 0$ such that, for every q_1, q_2 ,*

$$|q_1| \int_{\mathbb{R}^2} \frac{\widehat{V}(q_3)(\sin \theta)^2 dq_3}{(\lambda + |q_1 + q_3|^2 f(\lambda + |q_1 + q_3|^2))|q_2 + q_3|} \leq C_{\text{off}} \frac{g(\lambda + |q_1|^2)}{z}$$

with θ the angle between q_1 and q_3 .

PROOF. Throughout the proof, the constant C appearing in the bounds is independent of q_1, q_2 and q_3 and might change from line to line.

We split \mathbb{R}^2 into three regions, $\Omega_1 = \{q_3 : |q_1 + q_3| < \frac{|q_1|}{2}\}$, $\Omega_2 = \{q_3 : |q_2 + q_3| < \frac{|q_1|}{2}\}$ and $\Omega_3 = \mathbb{R}^2 \setminus (\Omega_1 \cup \Omega_2)$. Note that Ω_1 and Ω_2 might not be disjoint, but this is no issue as we are proving an upper bound.

In Ω_1 , we exploit the monotonicity of f to bound $f(\lambda + |q_1 + q_3|^2) \geq f(\lambda + \frac{1}{4}|q_1|^2)$. Moreover, we estimate $(\sin \theta)^2 \leq \frac{|q_3 + q_1|^2}{|q_1|^2}$ and \widehat{V} by a constant to get

$$\begin{aligned} & |q_1| \int_{\Omega_1} \frac{\widehat{V}(q_3)(\sin \theta)^2 dq_3}{(\lambda + |q_1 + q_3|^2 f(\lambda + |q_1 + q_3|^2))|q_2 + q_3|} \\ & \leq C |q_1|^{-1} \int_{\Omega_1} \frac{|q_1 + q_3|^2 dq_3}{(\lambda + |q_1 + q_3|^2 f(\lambda + \frac{1}{4}|q_1|^2))|q_2 + q_3|} \\ & \leq C \frac{|q_1|^{-1}}{f(\lambda + \frac{1}{4}|q_1|^2)} \int_{\Omega_1} \frac{dq_3}{|q_2 + q_3|} \\ & \leq \frac{C}{f(\lambda + \frac{1}{4}|q_1|^2)} \leq \frac{C}{f(\lambda + |q_1|^2)} \leq C \frac{g(\lambda + |q_1|^2)}{z}, \end{aligned}$$

the last step from the third to the fourth line being a consequence of the fact that, on Ω_1 , $\frac{|q_1|}{2} < |q_3| < \frac{3}{2}|q_1|$.

For Ω_2 , we estimate the sine differently, that is,

$$(A.13) \quad (\sin \theta)^2 \leq \frac{4|q_3 + q_1|^2}{|q_1|^2 \vee (\frac{1}{4}|q_2|^2)},$$

which holds as, for $|q_2| \leq 2|q_1|$ this is just a weaker estimate than the previous one, while for $|q_2| \geq 2|q_1|$ we claim that, in the region Ω_2 the right-hand side is always greater or equal to 1 (and thus the inequality holds as well). Indeed, notice that since $|q_2| \geq 2|q_1|$, we have

$$\frac{4|q_3 + q_1|^2}{|q_1|^2 \vee (\frac{1}{4}|q_2|^2)} = 16 \frac{|q_3 + q_1|^2}{|q_2|^2}.$$

Assume by contradiction that $|q_3 + q_1| < \frac{1}{4}|q_2|$. Then

$$|q_3 + q_2| \geq |q_2 - q_1| - |q_1 + q_3| > |q_2| - |q_1| - \frac{1}{4}|q_2| \geq \frac{1}{4}|q_2|,$$

where in the last step we used once again that $|q_2| \geq 2|q_1|$. Now, on Ω_2 , $|q_2 + q_3| < \frac{1}{2}|q_1|$, so that, in conclusion,

$$\frac{1}{4}|q_2| < |q_3 + q_2| < \frac{1}{2}|q_1| \leq \frac{1}{4}|q_2|$$

which is a contradiction. Hence, $|q_3 + q_1| \geq \frac{1}{4}|q_2|$, from which (A.13) follows.

Plugging (A.13) into the integral, we get

$$\begin{aligned} & |q_1| \int_{\Omega_2} \frac{\widehat{V}(q_3)(\sin \theta)^2 dq_3}{(\lambda + |q_1 + q_3|^2 f(\lambda + |q_1 + q_3|^2))|q_2 + q_3|} \\ & \leq C \frac{|q_1|}{|q_1|^2 \vee (\frac{1}{4}|q_2|^2)} \int_{\Omega_2} \frac{dq_3}{|q_2 + q_3| f(\lambda + |q_1 + q_3|^2)}. \end{aligned}$$

Now we use the monotonicity of f to bound the previous integral from above by

$$\begin{aligned} & C \frac{|q_1|}{(|q_1|^2 \vee (\frac{1}{4}|q_2|^2)) f(\lambda + (\frac{3}{2}|q_1| + |q_2|)^2)} \int_{\Omega_2} \frac{dq_3}{|q_2 + q_3|} \\ (A.14) \quad & = C \frac{|q_1|^2}{(|q_1|^2 \vee (\frac{1}{4}|q_2|^2)) f(\lambda + (\frac{3}{2}|q_1| + |q_2|)^2)}. \end{aligned}$$

We now bound this term by maximizing over $|q_2|$. It is easy to see that it is monotonically increasing for $|q_2| < 2|q_1|$. For $|q_2| \geq 2|q_1|$, we will prove that it is monotonically decreasing. Since f satisfies assumption (A.7), for any $a, b \geq 0$ we have

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{r^2 f(a + (b+r)^2)} \right) &= - \frac{2rf + 2r^2(b+r)f'}{r^4 f^2} = - \frac{2}{r^3 f^2} (f + r(b+r)f') \\ &\leq - \frac{2}{r^3 f} \left(1 - \frac{r(b+r)}{a + (b+r)^2} \right) < 0, \end{aligned}$$

where we suppressed the argument of f and f' because it does not change. Thus, the maximum over q_2 of the right-hand side of (A.14) is achieved at $|q_2| = 2|q_1|$ and reads

$$\frac{C}{f(\lambda + (\frac{7}{2}|q_1|)^2)} \leq \frac{Cg(\lambda + (\frac{7}{2}|q_1|)^2)}{z} \leq \frac{Cg(\lambda + |q_1|^2)}{z}.$$

We are left to consider the integral over Ω_3 . In this case, we first bound the $(\sin \theta)^2 \leq 1$ and then apply the Hölder inequality with exponents $\frac{3}{2}$ and 3, to the two functions $((\lambda + |q_1 + q_3|^2) f(\lambda + |q_1 + q_3|^2))^{-1}$ and $|q_2 + q_3|^{-1}$ with respect to the measure $\widehat{V}(q_3) dq_3$, so that we obtain

$$\begin{aligned} & |q_1| \int_{\Omega_3} \frac{\widehat{V}(q_3)(\sin \theta)^2 dq_3}{(\lambda + |q_1 + q_3|^2 f(\lambda + |q_1 + q_3|^2))|q_2 + q_3|} \\ (A.15) \quad & \leq |q_1| \left(\int_{\Omega_3} \frac{\widehat{V}(q_3) dq_3}{(\lambda + |q_1 + q_3|^2 f(\lambda + |q_1 + q_3|^2))^{\frac{3}{2}}} \right)^{\frac{2}{3}} \left(\int_{\Omega_3} \frac{\widehat{V}(q_3) dq_3}{|q_2 + q_3|^3} \right)^{\frac{1}{3}}. \end{aligned}$$

The second integral is upper bounded by a constant factor times $|q_1|^{-1}$.

In the first integral of (A.15), we make the change of variables $q = q_1 + q_3$, bound the bump function \widehat{V} by a constant and then pass to polar coordinates, hence we get

$$(A.16) \quad C \int_{\frac{|q_1|}{2}}^{\infty} \frac{\varrho d\varrho}{(\lambda + \varrho^2 f(\lambda + \varrho^2))^{\frac{3}{2}}}.$$

We split the domain of integration into two parts, $\rho^2 > \lambda$ and $\rho^2 \leq \lambda$ (the second one might be empty). In the first, we note that

$$\lambda + \varrho^2 f(\lambda + \varrho^2) \geq \frac{1}{2}(\lambda + \varrho^2)f(\lambda + \varrho^2).$$

Using $f(x) \geq \frac{z}{g(x)}$, we obtain that this part of (A.16) is upper bounded by

$$\begin{aligned} C \int_{\frac{\sqrt{\lambda}}{2}}^{\infty} \frac{\varrho \, d\varrho}{((\lambda + \varrho^2)f(\lambda + \varrho^2))^{\frac{3}{2}}} &\leq C \int_{\frac{|q_1|}{2}}^{\infty} \frac{\varrho \, d\varrho}{((\lambda + \varrho^2)f(\lambda + 4\varrho^2))^{\frac{3}{2}}} \\ (A.17) \qquad &\leq C \left(\frac{g(\lambda + |q_1|^2)}{z} \right)^{\frac{3}{2}} \int_{\frac{|q_1|}{2}}^{\infty} \frac{\varrho \, d\varrho}{(\lambda + \varrho^2)^{\frac{3}{2}}} \\ &\leq C |q_1|^{-1} \left(\frac{g(\lambda + |q_1|^2)}{z} \right)^{\frac{3}{2}}, \end{aligned}$$

where in the last step we estimated the integral by dropping λ from the denominator.

We now turn to the second part of the integral, where $\rho^2 \leq \lambda$. We use the following:

$$\int_{\frac{|q_1|}{2}}^{\sqrt{\lambda}} \frac{\varrho \, d\varrho}{(\lambda + \varrho^2 f(\lambda + \varrho^2))^{\frac{3}{2}}} \leq \frac{1}{f(2(\lambda + |q_1|^2))^{\frac{3}{2}}} \int_{\frac{|q_1|}{2}}^{2\sqrt{\lambda}} \frac{d\varrho}{\varrho^2} \leq C |q_1|^{-1} \frac{g(\lambda + |q_1|^2)^{\frac{3}{2}}}{z^{\frac{3}{2}}}.$$

In conclusion, plugging these estimates into (A.15), we get that the integral over Ω_3 is upper bounded by

$$C \frac{g(\lambda + |q_1|^2)}{z},$$

and, collecting all the bounds derived so far, the statement follows at once. \square

LEMMA A.4. *The functions $UB_k(\cdot, z)$ and $LB_k(\cdot, z)$ satisfy the conditions of the previous lemmas.*

PROOF. By definition $UB_k(\cdot, z) LB_k(\cdot, z) = L(\cdot, z) \geq z$. From Lemma A.1, we get that $LB_k(x, z) > 1$ and $UB_k(x, z) > 1$ for all x and that their derivatives are both negative. Equation (A.4) implies that

$$\partial_x LB_k(x, z) = -\frac{LB_{k-1}(x, z)}{2(x^2 + x)L(x, z)} \geq -\frac{LB_k(x, z)}{x}$$

and

$$\partial_x UB_k(x, z) \geq -\frac{UB_k(x, z)}{(x^2 + x)L(x, z)} \geq -\frac{UB_k(x, z)}{x}$$

which gives (A.7). \square

LEMMA A.5. *For any $z \geq 1$, $\lambda \in \mathbb{R}_+$ and $p \in \mathbb{R}^2$ such that $\lambda + |p|^2 \leq 1$, we have*

$$\left| \int_{\lambda+|p|^2}^1 \frac{d\rho}{\rho LB_k(\rho, z)} - \int_{\lambda+|p|^2}^1 \frac{d\rho}{(\rho + \rho^2) LB_k(\rho, z)} \right| \leq \frac{UB_k(\lambda + |p|^2, z)}{z}.$$

PROOF. Note that the difference of integrals equals

$$0 \leq \int_{\lambda+|p|^2}^1 \frac{d\rho}{(1 + \rho) LB_k(\rho, z)} = \int_{\lambda+|p|^2}^1 \frac{UB_k(\rho, z)}{(1 + \rho)L(\rho, z)} d\rho \leq \frac{UB_k(\lambda + |p|^2, z)}{z}$$

because $UB_k(\cdot, z)$ is decreasing and $L(\cdot, z) \geq z$. \square

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