Solving the Kodaira-Spencer Problem using Harmonic Analysis on Torus Bundles over S^1

by

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Contents

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Last but certainly not least, I thank my friends for the all the good times and all the good board games. They helped me keep my sanity during lockdown.

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. I declare that the work contained within this thesis is my own original work, except where otherwise indicated, cited, or commonly known.

Chapter 1 introduces the reader to the field of almost complex geometry and to the Kodaira-Spencer problem. I claim no originality for the content found here, except in the presentation and choice of results.

Chapters 2 and 4, as well as Sections 3.2 and 5.4, are based on work which has been submitted for publication [15].

Section 3.1 is based on collaborative work with Weiyi Zhang which has been submitted for publication [13, 14].

The remaining sections of Chapter 5 consist of new results not yet submitted for publication anywhere.

Abstract

In this thesis we will consider the spaces of $\bar{\partial}$ and Bott-Chern harmonic differential forms $\mathcal{H}^{p,q}_{\bar{\partial}}$ & $\mathcal{H}^{p,q}_{BC}$, defined on an almost complex manifold equipped with a metric compatible with the almost complex structure. In 1954, Kodaira and Spencer asked whether the Hodge numbers $h^{p,q}_{\bar{\partial}} := \dim \mathcal{H}^{p,q}_{\bar{\partial}}$ are all invariant of the choice of metric. We will answer this question in the negative. Furthermore, in the case of compact almost complex 4-manifolds we will give a full account of the values of p and q for which both $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC} := \dim \mathcal{H}^{p,q}_{BC}$ are or are not independent of the metric.

Specifically, we find examples of compact 4-manifolds where $h^{0,1}_{\bar{\partial}}, h^{2,1}_{\bar{\partial}}, h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ all change depending on the metric, even if we restrict ourselves to the special class of almost Kähler metrics. We also show that the only possible values for $h^{1,1}_{\bar{\partial}}$ are b_- and $b_- + 1$, while the value of $h^{1,1}_{BC}$ is always $b_- + 1$. Here b_- denotes an invariant given by the number of d-harmonic anti-self-dual 2-forms.

In order to obtain these results, we are required to solve a system of partial differential equations. We therefore introduce a decomposition of L^2 functions on torus bundles over S^1 which allows us to rewrite this system into a family of ordinary differential equations, which we can solve by describing the Stokes phenomenon, and a family of algebraic equations, which are equivalent to the Gauss circle problem.

Chapter 0

Introduction

An almost complex structure is defined to be a linear map $J : TM \to TM$ acting on the tangent bundle of a manifold, such that $J^2 = -id$. These structures were first introduced in the 1940s by Ehresmann and Hopf and later popularised by Gromov's introduction of pseudoholomorphic curves [11]. Today their study encompasses topics from the famous Hopf problem, which asks whether S^6 admits a complex structure [16], to Goldberg's conjecture that any compact almost Kähler-Einstein manifold is Kähler [9].

Almost complex geometry also has strong ties to Hodge theory, first developed by Hodge in the 1930s, building on work by de Rham (see [10, 18] for a good introduction to the theory). An almost complex structure allows us to decompose the space of complex valued k-forms as the sum of spaces of (p, q)-forms

$$\mathcal{A}^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M).$$

From this decomposition, we can define the Dolbeault cohomology on any compact, complex manifold by

$$H^{p,q}_{\bar{\partial}}(M) := \frac{\ker \bar{\partial}|_{\mathcal{A}^{p,q}}}{\operatorname{im} \bar{\partial}|_{\mathcal{A}^{p,q-1}}}.$$

The dimension of these spaces $h^{p,q}_{\bar{\partial}} := \dim H^{p,q}_{\bar{\partial}}$ provides a collection of important invariants of the complex manifold, known as the *Hodge numbers*. The Hodge numbers can in fact be defined on any almost complex manifold (see Chapter 1), however their definition in general requires the introduction of a non-unique metric, known as an *almost Hermitian* metric. In Hirzebruch's 1954 paper [17] Kodaira and Spencer ask:

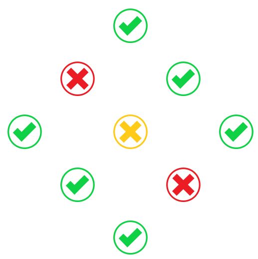
Question 1.21 (Kodaira-Spencer)**.** *Given a compact almost complex manifold, are the numbers $h^{p,q}_{\bar{\partial}}$ independent of the choice of almost Hermitian metric?*

According to an update of Hirzebruch's problem list [20], until very recently, next to no progress had been made towards answering this question, with the exception of attempts to develop a harmonic theory for almost Kähler manifolds by Donaldson [5] and a version of Hodge theory for strictly nearly Kähler 6-manifolds developed by Verbitsky [31].

The main result of this thesis is to finally give an answer to the Kodaira-Spencer question. In fact, in Chapter 3 we prove a number of results which together give a full description, for compact almost complex 4-manifolds, of when $h^{p,q}_{\bar{\partial}}$ does or does not depend on the almost Hermitian metric. By laying out the non-trivial values of $h^{p,q}_{\bar{\partial}}$ in the following way

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

known as a Hodge diamond, we can summarise many of the results of Chapter 3.



: Invariant of almost Hermitian metrics,

: Not invariant of almost Hermitian metrics, but invariant of almost Kähler metrics,

: Not invariant of almost Kähler metrics.

In particular, notice the unruly behaviour of $h^{0,1}_{\bar\partial}$ and $h^{2,1}_{\bar\partial}$ (These two are always equal by Serre duality, see Chapter 1). By considering the example of the Kodaira-Thurston manifold, a 3-torus bundle over S^1, we prove

Theorem 3.12. *On compact almost complex 4-manifolds, $h^{0,1}_{\bar\partial}$ is not in general invariant of the choice of almost Hermitian metric. In particular, on the Kodaira-Thurston manifold we can find a family of almost Hermitian metrics, compatible with a fixed almost complex structure, over which $h^{0,1}_{\bar\partial}$ takes multiple different values.*

We therefore answer the Kodaira-Spencer question in the negative. In fact, this example gives us an even stronger result! It turns out that $h^{0,1}_{\bar\partial}$ can vary with our choice of metric even when our choice is restricted to a special class of almost Hermitian metrics known as almost Kähler metrics.

Furthermore, by considering a family of almost complex structures $J_{a,b}$ along with a compatible almost Kähler metric $g_{a,b}$, we also prove

Theorem 3.9. *$h^{0,1}_{\bar\partial}$ can be computed for all members of the continuous family of non-integrable almost Kähler structures given by $(J_{a,b}, g_{a,b})$, $a, b \in \mathbb{R}$, $b \neq 0$, on the Kodaira-Thurston manifold. Furthermore, for any $n \in \mathbb{Z}^+$ such that $8 \nmid n$, there is a b such that $h^{0,1}_{\bar\partial} = n$.*

So, by varying the almost Kähler structure continuously it is possible to make the value of $h^{0,1}_{\bar\partial}$ arbitrarily large.

When compared with $h^{0,1}_{\bar\partial}$, $h^{1,1}_{\bar\partial}$ is much more well-behaved. In fact we will prove that on any almost complex manifold there are only two values it can take.

Theorem 3.18. *On a compact almost Hermitian 4-manifold we have either $h^{1,1}_{\bar\partial} = b_-$ or $b_- + 1$.*

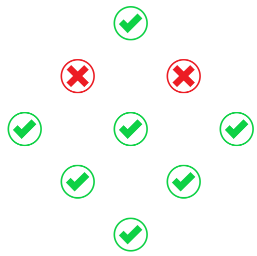
Here b_- denotes the dimension of the space of anti-self-dual, d-harmonic forms, *i.e.* the forms α for which $*\alpha = -\alpha$, $d\alpha = 0$ and $d^*\alpha = 0$. Importantly, this number is a topological invariant.

In Chapter 4, we turn our attention to the Bott-Chern cohomology groups, defined on complex manifolds by

$$H^{p,q}_{BC}(M) := \frac{\ker \partial|_{\mathcal{A}^{p,q}} \cap \ker \bar\partial|_{\mathcal{A}^{p,q}}}{\operatorname{im} \partial\bar\partial|_{\mathcal{A}^{p-1,q-1}}},$$

with $h^{p,q}_{BC} := \dim H^{p,q}_{BC}$. These groups were first introduced by Kodaira and Spencer [19], in order to prove the invariance of the Kähler condition under sufficiently small deformations of the almost complex structure. As with the Dolbeault cohomology, we can extend the definition of $h^{p,q}_{BC}$ to almost complex manifolds by introducing an almost Hermitian metric.

In Chapter 4 we will prove another set of results which together give a full description, for compact almost complex 4-manifolds, of when $h^{p,q}_{BC}$ does or does not depend on the choice of almost Hermitian metric. These results can be summarised in a Hodge diamond as follows:



: Invariant of almost Hermitian metrics,

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As with $h^{0,1}_{\bar\partial}$, we show that $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ may both take a range of values on a single almost complex manifold by performing calculations on the Kodaira-Thurston manifold. Specifically, we are able to show that, for the family of almost Kähler structures considered in the above theorems, the values of $h^{0,1}_{\bar\partial}, h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ are all equal. We can therefore deduce the following results:

Theorem 4.4. *$h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ can both be computed for all members of the continuous family of non-integrable almost Kähler structures given by $(J_{a,b}, g_{a,b})$, $a, b \in \mathbb{R}$, $b \neq 0$, on the Kodaira-Thurston manifold. Furthermore, for any $n \in \mathbb{Z}^+$ such that $8 \nmid n$, there is a b such that $h^{2,1}_{BC} = h^{1,2}_{BC} = n$.*

Theorem 4.5. *On compact almost complex 4-manifolds, $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ are not in general invariant of the choice of almost Kähler metric. In particular, on the Kodaira-Thurston manifold we can find a family of almost Hermitian metrics, compatible with a fixed almost complex structure, over which $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ take multiple different values.*

We are also able to determine the exact value of $h^{1,1}_{BC}$ on any compact almost Hermitian 4-manifold.

Theorem 4.10. *Given any compact almost Hermitian 4-manifold, we always have* $h^{1,1}_{BC} = b_- + 1$.

The calculations of $h^{0,1}_{\bar\partial}$ in Chapter 3 and of $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ in Chapter 4, described above, both require the solution of a system of linear partial differential equations (PDEs) on the Kodaira-Thurston manifold. This is made possible by the introduction of a harmonic analytic technique in Chapter 2. The main result of this technique is a decomposition of L^2 functions on any torus bundle over S^1.

Theorem 2.8. *Let M be a mapping torus of an n-dimensional torus, given by the matrix $A \in GL_n(\mathbb{Z})$. The space of L^2 functions on M decomposes in the following way.*

$$L^2(M) = \left(\widehat{\bigoplus_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}| = \infty}}} \mathcal{H}_{\mathbf{y}} \right) \oplus \left(\widehat{\bigoplus_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}| = N < \infty}}} \widehat{\bigoplus_{t_0 \in \mathbb{Z}}} \mathcal{H}_{t_0,\mathbf{y}} \right),$$

where

$$\mathcal{H}_{\mathbf{y}} = \left\{ \sum_{\xi \in \mathbb{Z}} f(t+\xi) e^{2\pi i \mathbf{y} \cdot A^{\xi} \mathbf{x}} \;\middle|\; f \in L^2(\mathbb{R}) \right\}$$

and

$$\mathcal{H}_{t_0,\mathbf{y}} = \left\{ C e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y} \cdot A^{\xi} \mathbf{x} \right)} \;\middle|\; C \in \mathbb{C} \right\}.$$

Here $\hat\oplus$ denotes the direct sum followed by the closure with respect to the L^2 norm.

Given $\mathbf{y} \in \mathbb{Z}^n$ we use $\mathrm{Orb}_{\mathbf{y}}$ here to denote the orbit of the group generated by the transpose A^T acting on $\mathbf{y}$, while $\mathcal{O}$ denotes the set of all such orbits.

We find that the PDEs involved in calculating $h^{0,1}_{\bar\partial}$, $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ consist of differential operators which generally preserve the spaces $\mathcal{H}_{\mathbf{y}} \cap C^\infty$ and $\mathcal{H}_{t_0,\mathbf{y}} \cap C^\infty$. It therefore suffices to find just the solutions contained within $\mathcal{H}_{\mathbf{y}} \cap C^\infty$ and $\mathcal{H}_{t_0,\mathbf{y}} \cap C^\infty$ as any smooth solution can then be obtained as a linear combination of these.

This technique is not, however, limited to the Kodaira-Thurston manifold; the same ideas can be applied to solve PDEs on other torus bundle over S^1. In Chapter 5 we demonstrate how one might go about calculating $h^{p,q}_{\bar\partial}$ on a number of different such torus bundles. We will consider examples with $\mathbb{E}^4$, Nil^4 and $Sol^3 \times \mathbb{E}$ geometry, in addition to the Kodaira-Thurston manifold, which we will show has $Nil^3 \times \mathbb{E}$ geometry.

Chapter 1

Almost Complex Geometry and the Kodaira-Spencer problem

We begin with an introduction to the field of almost complex geometry, giving preliminary results and definitions which we will be using throughout this thesis. The contents of this chapter may be found in any good textbook on complex geometry and algebraic geometry (see *e.g.* [10, 18]).

1.1 Integrable Almost Complex Structures

Definition 1.1. An *almost complex structure* on a smooth manifold M is a tangent bundle automorphism $J : TM \to TM$ which satisfies $J^2 = -id$. We call the pair (M, J) an *almost complex manifold*.

An almost complex structure induces a decomposition of the complexified tangent bundle $T_{\mathbb{C}}M := TM \otimes \mathbb{C}$ into a sum of its $+i$ and $-i$ eigenspaces

$$T_{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$$

with projection into the first and second components given by

$$\mathrm{proj}_1 : \mathbf{v} \mapsto \frac{1}{2}(\mathbf{v} - iJ\mathbf{v}) \qquad \mathrm{proj}_2 : \mathbf{v} \mapsto \frac{1}{2}(\mathbf{v} + iJ\mathbf{v}).$$

This gives rise to a similar decomposition on the dual tangent bundle

$$T^*_{\mathbb{C}}M = T^*_{1,0}M \oplus T^*_{0,1}M$$

which ultimately leads to the following result.

Proposition 1.2. *On an almost complex manifold the graded algebra of complex-valued k-forms $\mathcal{A}^k_{\mathbb{C}}(M) := \Gamma(\bigwedge^k_{\mathbb{C}} M)$ decomposes into a bigraded algebra*

$$\mathcal{A}^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(M)$$

where we define

$$\mathcal{A}^{p,q}(M) := \Gamma\left(\bigwedge\nolimits^{p,q} M\right)$$
$$\bigwedge\nolimits^{p,q} M := \bigwedge\nolimits^{p} T^*_{1,0}M \wedge \bigwedge\nolimits^{q} T^*_{0,1}M.$$

Differential forms contained in $\mathcal{A}^{p,q}(M)$ are said to have bidegree (p,q).

The exterior derivative $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$ can be written as a sum of four components

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}$$

which change the bidegree of the form they act upon by $(+2,-1), (+1,0), (0,+1)$ and $(-1,+2)$, respectively. The fact that μ and $\bar{\mu}$ as well as ∂ and $\bar{\partial}$ are complex conjugates of each other follows simply from the fact that the exterior derivative d is itself invariant under conjugation.

Other important properties of $\mu, \partial, \bar{\partial}$ and $\bar{\mu}$ also arise from the properties of d. For instance, d is a graded derivation, a property which descends directly to its components, giving us the following.

Proposition 1.3. *Every component of the exterior derivative is a graded derivation, that is to say for any $\alpha \in \mathcal{A}^k(M)$, $\beta \in \mathcal{A}^l(M)$ we have*

$$\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^k \alpha \wedge \delta\beta$$

for all $\delta \in \{\mu, \partial, \bar{\partial}, \bar{\mu}\}$.

Another property of the exterior derivative is that it squares to zero. Although this fact does not descend to the components of d, we can consider the components of the expression $d^2 = 0$ separately, to obtain a family of seven identities which hold for $\mu, \partial, \bar{\partial}$ and $\bar{\mu}$.

Proposition 1.4. *On any almost complex manifold, the following relations hold for the components of the exterior derivative.*

$$\begin{cases} \mu^2 = 0, \\ \mu\partial + \partial\mu = 0, \\ \mu\bar{\partial} + \partial^2 + \bar{\partial}\mu = 0, \\ \mu\bar{\mu} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\mu}\mu = 0, \\ \partial\bar{\mu} + \bar{\partial}^2 + \bar{\mu}\partial = 0, \\ \bar{\partial}\bar{\mu} + \bar{\mu}\bar{\partial} = 0, \\ \bar{\mu}^2 = 0. \end{cases}$$

At this point it is worth explaining in what way an almost complex structure really is "almost complex". To this end we will consider how a complex structure on a smooth manifold gives rise to an almost complex structure.

Example 1.5. Let X denote a complex manifold with complex dimension n. Around any point $p \in X$ we have an open neighbourhood U which is identified with some $V \subset \mathbb{C}^n$ by a smooth co-ordinate chart $\psi : U \to V$. If we write a point in V using the co-ordinates

$$(x_1 + iy_1, \ldots, x_n + iy_n)$$

then we can locally define an almost complex structure J by the maps

$$\begin{aligned} J : \frac{\partial}{\partial x_i} &\mapsto \frac{\partial}{\partial y_i} \\ \frac{\partial}{\partial y_i} &\mapsto -\frac{\partial}{\partial x_i}. \end{aligned}$$

Let J_1 and J_2 be two locally defined almost complex structures corresponding to different co-ordinate charts ψ_1 and ψ_2. We can use the holomorphic transition map $\psi_2 \circ \psi_1^{-1}$ to show that J_1 and J_2 in fact describe the same almost complex structure. First, by writing $\psi_2 \circ \psi_1^{-1}$ using local co-ordinates as

$$(\tilde{x}_1 + i\tilde{y}_1, \ldots, \tilde{x}_n + i\tilde{y}_n) = (\psi_2 \circ \psi_1^{-1})(x_1 + iy_1, \ldots, x_n + iy_n),$$

the Cauchy-Riemann equations tell us that

$$\frac{\partial \tilde{x}_i}{\partial x_j} = \frac{\partial \tilde{y}_i}{\partial y_j} \qquad\qquad \frac{\partial \tilde{x}_i}{\partial y_j} = -\frac{\partial \tilde{y}_i}{\partial x_j}.$$

Then we can show the following:

$$
\begin{aligned}
J_2\,(\psi_2\circ\psi_1^{-1})_*\left(\frac{\partial}{\partial x_i}\right) &= J_2\sum_{j=1}^{n}\left(\frac{\partial\tilde{x}_j}{\partial x_i}\frac{\partial}{\partial\tilde{x}_j}+\frac{\partial\tilde{y}_j}{\partial x_i}\frac{\partial}{\partial\tilde{y}_j}\right)\\
&= \sum_{j=1}^{n}\left(\frac{\partial\tilde{x}_j}{\partial x_i}\frac{\partial}{\partial\tilde{y}_j}-\frac{\partial\tilde{y}_j}{\partial x_i}\frac{\partial}{\partial\tilde{x}_j}\right)\\
&= \sum_{j=1}^{n}\left(\frac{\partial\tilde{y}_j}{\partial y_i}\frac{\partial}{\partial\tilde{y}_j}+\frac{\partial\tilde{x}_j}{\partial y_i}\frac{\partial}{\partial\tilde{x}_j}\right)\\
&= (\psi_2\circ\psi_1^{-1})_*\left(\frac{\partial}{\partial y_i}\right) = (\psi_2\circ\psi_1^{-1})_*\left(J_1\frac{\partial}{\partial x_i}\right).
\end{aligned}
$$

This demonstrates that our local definition of J does not depend on the choice of coordinate chart. The local descriptions can therefore be patched together forming a globally well-defined almost complex structure.

The above example shows how a complex structure will always give rise to an almost complex structure. The reverse, however, is not always the case.

Definition 1.6. Let (M, J) be an almost complex manifold. We say that J is *integrable* if it arises from a description of M as a complex manifold.

The Newlander-Nirenberg theorem, originally proven in [23] (see also [22]), gives us a way to identify which almost complex structures are integrable.

Theorem 1.7 (Newlander-Nirenberg)**.** *An almost complex structure J is integrable if and only if the Nijenhuis tensor, acting on vector fields by*

$$N_J(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY],$$

is identically zero.

Corollary 1.8. *An almost complex structure J is integrable if and only if the exterior derivative can be given by $d = \partial + \bar{\partial}$, i.e. $\mu = 0$.*

Using the above corollary, in the case of integrable almost complex structures we can simplify the seven identities in Proposition 1.4 to just three.

Proposition 1.9. *On any almost complex manifold (M, J), if J is integrable the fol-*

lowing relations hold for the components of the exterior derivative:

$$\begin{cases} \partial^2 = 0, \\ \partial\bar{\partial} = -\bar{\partial}\partial, \\ \bar{\partial}^2 = 0. \end{cases}$$

We define the Dolbeault cohomology and the Bott-Chern cohomology as follows.

Definition 1.10. On an integrable almost complex manifold (M, J), we can define the Dolbeault cohomology groups by

$$H^{p,q}_{\bar{\partial}}(M) := \frac{\ker \bar{\partial}|_{\mathcal{A}^{p,q}}}{\operatorname{im} \bar{\partial}|_{\mathcal{A}^{p,q-1}}}.$$

The dimension of these groups is given by the *Hodge numbers* $h^{p,q}_{\bar{\partial}} := \dim H^{p,q}_{\bar{\partial}}$.

Definition 1.11. On an integrable almost complex manifold (M, J), we can define the Bott-Chern cohomology groups by

$$H^{p,q}_{BC}(M) := \frac{\ker \partial|_{\mathcal{A}^{p,q}} \cap \ker \bar{\partial}|_{\mathcal{A}^{p,q}}}{\operatorname{im} \partial\bar{\partial}|_{\mathcal{A}^{p-1,q-1}}}.$$

We will use $h^{p,q}_{BC} := \dim H^{p,q}_{BC}$ to denote the dimension of these groups.

The definitions of $H^{p,q}_{\bar{\partial}}$ and $H^{p,q}_{BC}$ only make sense because the identities in Proposition 1.9 hold. For instance, for the Dolbeault cohomology to be well-defined requires that $\bar{\partial}^2 = 0$. Unfortunately, this is not the case for non-integrable almost complex structures. In the next section we will introduce different definitions for $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$, which are equivalent to the definitions given above when J is integrable, but which remain well-defined when J is non-integrable.

1.2 The Almost Complex Hodge Numbers

In order to define $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ on all almost complex manifolds we must first define some new structures.

Definition 1.12.

i) A Riemannian metric g on an (almost) complex manifold (M, J) is called an *(almost) Hermitian metric* if it is invariant under J, *i.e.*

$$g(\cdot, \cdot) = g(J\cdot, J\cdot).$$

In this case we say that g and J are compatible and call (M, J, g) an *(almost) Hermitian manifold.*

ii) Given an (almost) Hermitian manifold (M, J, g), the *fundamental form* ω is a real, alternating, bilinear form defined by

$$\omega(\cdot, \cdot) := g(J\cdot, \cdot).$$

Definition 1.13. We say that an (almost) Hermitian metric is *(almost) Kähler* if the corresponding fundamental form ω is symplectic *i.e.* if $d\omega = 0$.

Proposition 1.14. *On an almost Hermitian manifold (M, J, g) the almost complex structure, almost Hermitian metric and fundamental form exist as a compatible triple (J, g, ω). Given any two structures the third can be constructed, as follows:*

i)

$$\omega(\cdot, \cdot) = g(J\cdot, \cdot),$$

ii)

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot),$$

iii)

$$J = \psi_g^{-1} \circ \psi_\omega.$$

Here we define $\psi_g : \mathbf{v} \mapsto g(\mathbf{v}, \cdot)$ and $\psi_\omega : \mathbf{v} \mapsto \omega(\mathbf{v}, \cdot)$.

Given an almost complex manifold, Proposition 1.14 tells us that specifying an almost Hermitian metric is equivalent to specifying the corresponding fundamental form. As a consequence some authors, myself included, will sometimes engage in a slight abuse of notation by referring to the fundamental form as if it were the almost Hermitian metric, even though it is technically not a metric.

Remark 1.15. *Given an almost Hermitian manifold (M, J, g), the almost Hermitian metric induces an inner product on the dual tangent space. If we let $v_1, \dots v_{2n}$ denote an orthonormal basis at some point in the dual tangent space, we can extend the inner product to $\bigwedge^k M$. This is accomplished by taking all wedge products $v_{i_1} \wedge \dots \wedge v_{i_k} \in \bigwedge^k M$ to be orthonormal. The extension of g to differential forms will also be denoted by g.*

The existence of an almost complex structure implies a natural orientation. Taking this together with an almost Hermitian metric, we obtain a volume form dV with which we can then define the Hodge star operator.

Definition 1.16. Let (M, J, g) be an oriented almost Hermitian manifold with real dimension $2n$. We define the *Hodge star operator* to be the map $* : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{n-q,n-p}(M)$ given by the relation

$$\alpha \wedge *\bar{\beta} = g(\alpha, \beta) dV.$$

We can then define the operators

$$\mu^* = - * \bar{\mu} *, \quad \partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial *, \quad \bar{\mu}^* = - * \mu * .$$

Remark 1.17. *If the almost Hermitian manifold (M, J, g) is compact, then $\mu^*, \partial^*, \bar{\partial}^*$ and $\bar{\mu}^*$ are all adjoint operators with respect to the inner product on $\mathcal{A}^k_{\mathbb{C}}(M)$ given by*

$$(\cdot, \cdot) := \int_M g(\cdot, \cdot) \, dV,$$

as the notation would suggest.

We list a few properties of the Hodge star here.

Proposition 1.18. *Let (M, J, g) be a compact almost Hermitian manifold, with real dimension $2n$.*

i) Let $v_1, \dots, v_{2n}$ be an orthonormal basis of T^*M such that $v_1 \wedge v_2 \wedge \dots \wedge v_{2n}$ is the volume form. If $\{v_{i_1}, \dots, v_{i_{2n}}\} = \{v_1, \dots, v_{2n}\}$, then we have

$$*v_{i_1} \wedge \dots v_{i_k} = s \, v_{i_{k+1}} \wedge \dots \wedge v_{i_{2n}}$$

where $s = \operatorname{sign}(i_1, \dots, i_{2n})$.

ii) The square of the Hodge star is the identity up to a sign. That is to say, given $\alpha \in \bigwedge^k M$, we have

$$* * \alpha = (-1)^k \alpha.$$

iii) The Hodge star is self-adjoint up to a sign. That is to say, given $\alpha \in \bigwedge^k M$ and $\beta \in \bigwedge^{2n-k} M$, we have

$$g(*\alpha, \beta) = (-1)^k g(\alpha, *\beta).$$

Now that we have introduced all of the necessary prerequisites, we can give new definitions of $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ which extend the cohomological definitions to compact almost complex manifolds.

Definition 1.19. Given an almost Hermitian manifold, we can define the elliptic operators

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

$$\Delta_{BC} = \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\partial + \bar{\partial}^*\bar{\partial}$$

known respectively as the *$\bar{\partial}$-Laplacian* and the *Bott-Chern Laplacian.*

The spaces of *$\bar{\partial}$-harmonic* and *Bott-Chern harmonic* (p, q)-forms corresponding to these Laplacians are then given by

$$\mathcal{H}^{p,q}_{\bar{\partial}} := \ker(\Delta_{\bar{\partial}} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}),$$
$$\mathcal{H}^{p,q}_{BC} := \ker(\Delta_{BC} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}),$$

and we define $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ to be the dimensions of these spaces.

When restricted to compact, complex manifolds the above definitions of $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ are equivalent to the definitions involving cohomology groups. This fact is a consequence of the following result:

Proposition 1.20. *On any compact complex manifold, every cohomology class in $H^{p,q}_{\bar{\partial}}$ or $H^{p,q}_{BC}$ contains a unique (up to scaling) $\bar{\partial}$-harmonic or Bott-Chern harmonic form, respectively. That is to say, we have*

$$H^{p,q}_{\bar{\partial}} \cong \mathcal{H}^{p,q}_{\bar{\partial}}, \qquad H^{p,q}_{BC} \cong \mathcal{H}^{p,q}_{BC}.$$

The $\bar{\partial}$-Laplacian is clearly elliptic. In [24] Piovani and Tomassini prove that the Bott-Chern Laplacian is also elliptic. By a well-known property of elliptic operators (see *e.g.* Chapter 3 of [32]), $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ are therefore always finite when defined on compact manifolds.

Since the definitions of $H^{p,q}_{\bar{\partial}}$ and $H^{p,q}_{BC}$ do not depend on the almost Hermitian metric, we know that $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ are also independent of the metric. It is this property which prompted Kodaira and Spencer to ask the following question, which appeared as question 20 in Hirzebruch's 1954 paper [17].

Question 1.21. *[Kodaira-Spencer] Given a compact almost complex manifold, are the numbers $h^{p,q}_{\bar{\partial}}$ independent of the choice of almost Hermitian metric?*

We could equally well ask this question of $h^{p,q}_{BC}$. Over the course of this thesis we will answer this Kodaira-Spencer problem, as well as the Bott-Chern version of the problem, in the negative.

1.3 Properties of $\mathcal{H}_{\bar{\partial}}^{p,q}$ and $\mathcal{H}_{BC}^{p,q}$

We conclude this chapter by describing some of the properties of $\mathcal{H}_{\bar{\partial}}^{p,q}$ and $\mathcal{H}_{BC}^{p,q}$ which will prove useful in later Chapters. To start with we show that the definitions of these spaces as the kernels of $\Delta_{\bar{\partial}}$ and Δ_{BC} can be rewritten as a collection of simpler conditions.

Proposition 1.22. *For any compact almost Hermitian manifold and a general (p, q)-form $s \in \mathcal{A}^{p,q}$, we have*

$$s \in \mathcal{H}_{\bar{\partial}}^{p,q} \iff \Delta_{\bar{\partial}} s = 0 \iff \begin{cases} \bar{\partial} s = 0 \\ \partial * s = 0 \end{cases},$$

$$s \in \mathcal{H}_{BC}^{p,q} \iff \Delta_{BC} s = 0 \iff \begin{cases} \partial s = 0 \\ \bar{\partial} s = 0 \\ \partial\bar{\partial} * s = 0 \end{cases}.$$

Proof. First note that $\partial * s = 0$ is equivalent to $\bar{\partial}^* s = 0$, as $\bar{\partial}^* = - * \partial *$ and the Hodge star is an invertible linear map. In the case of $\mathcal{H}_{\bar{\partial}}^{p,q}$ it therefore suffices to prove that $\Delta_{\bar{\partial}} s = 0$ if and only if $\bar{\partial} s = 0$ and $\bar{\partial}^* s = 0$. Clearly, if $\bar{\partial} s = 0$ and $\bar{\partial}^* s = 0$ then $\Delta_{\bar{\partial}} s = 0$, the opposite direction however is a bit trickier. By assuming that $\Delta_{\bar{\partial}} s = 0$ and taking the inner product (as defined in Remark 1.17) with s we see that

$$(\Delta_{\bar{\partial}} s, s) = (\bar{\partial}\bar{\partial}^* s, s) + (\bar{\partial}^* \bar{\partial} s, s) = 0.$$

But $(\bar{\partial}^* \bar{\partial} s, s) = (\bar{\partial} s, \bar{\partial} s) = \left\|\bar{\partial} s\right\|^2$ and similarly $(\bar{\partial}\bar{\partial}^* s, s) = \left\|\bar{\partial}^* s\right\|^2$, so we therefore require $\bar{\partial} s = 0$ and $\bar{\partial}^* s = 0$.

The proof in the case of $\mathcal{H}_{BC}^{p,q}$ follows the exact same reasoning.

□

Now we introduce a symmetry on the spaces $\mathcal{H}_{\bar{\partial}}^{p,q}$ known as Serre duality.

Proposition 1.23 (Serre duality)**.** *On any compact almost Hermitian manifold with real dimension $2n$, the space $\mathcal{H}_{\bar{\partial}}^{p,q}$ satisfies*

$$*\overline{\mathcal{H}_{\bar{\partial}}^{p,q}} = \mathcal{H}_{\bar{\partial}}^{n-p,n-q}$$

and thus $h_{\bar{\partial}}^{p,q} = h_{\bar{\partial}}^{n-p,n-q}$.

Proof. From the previous proposition we know that $s \in \mathcal{H}^{p,q}_{\bar{\partial}}$ is equivalent to

$$\bar{\partial} s = 0 \qquad \partial * s = 0.$$

We then take the conjugate of these two conditions and make use of the Hodge star property $*^2 = (-1)^{p+q}$ when acting on a form of bidegree (p,q). From this we see that these conditions are equivalent, respectively, to

$$\partial * (*\bar{s}) = 0 \qquad \bar{\partial}(*\bar{s}) = 0.$$

Thus we have $s \in \mathcal{H}^{p,q}_{\bar{\partial}}$ if and only if $*\bar{s} \in \mathcal{H}^{n-p,n-q}_{\bar{\partial}}$. $\square$

There is unfortunately no such symmetry on the spaces of Bott-Chern harmonic forms. However, we can use a similar argument to equate them to the spaces of Aeppli harmonic forms.

Definition 1.24. On a compact almost Hermitian manifold, we define the space of *Aeppli harmonic* (p,q)-forms to be

$$\mathcal{H}^{p,q}_A := \ker(\Delta : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}),$$

where Δ_A denotes the *Aeppli Laplacian*

$$\Delta_A = \bar{\partial}\partial\partial^*\bar{\partial}^* + \partial^*\bar{\partial}^*\bar{\partial}\partial + \bar{\partial}\partial^*\partial\bar{\partial}^* + \partial\bar{\partial}^*\bar{\partial}\partial^* + \bar{\partial}\bar{\partial}^* + \partial\partial^*.$$

We can then define $h^{p,q}_A := \dim \mathcal{H}^{p,q}_A$. As in Prop. 1.22 we can show that this space can equally be defined by

$$s \in \mathcal{H}^{p,q}_A \iff \Delta_A s = 0 \iff \begin{cases} \partial * s = 0 \\ \bar{\partial} * s = 0 \\ \bar{\partial}\partial s = 0 \end{cases}.$$

In the case of an integrable almost complex structure, we can also define the *Aeppli cohomology*

$$H^{p,q}_A := \frac{\ker \bar{\partial}\partial}{\operatorname{im}\partial + \operatorname{im}\bar{\partial}}$$

which is isomorphic to $\mathcal{H}^{p,q}_A$ whenever both are defined.

Proposition 1.25. *On any compact almost Hermitian manifold with real dimension* $2n$*,* *the spaces* $\mathcal{H}^{p,q}_{BC}$ *and* $\mathcal{H}^{p,q}_{A}$ *relate to each other as follows:*

$$*\overline{\mathcal{H}^{p,q}_{BC}} = \mathcal{H}^{n-p,n-q}_{A}$$

and therefore we have $h^{p,q}_{BC} = h^{n-p,n-q}_{A}$.

Proof. Taking the complex conjugate of the conditions

$$\partial * s = 0 \qquad \bar{\partial} * s = 0 \qquad \bar{\partial}\partial s = 0$$

gives us the conditions

$$\bar{\partial}(*\bar{s}) = 0 \qquad \partial(*\bar{s}) = 0 \qquad \partial\bar{\partial} * (*\bar{s}) = 0,$$

respectively. Thus $s \in \mathcal{H}^{p,q}_{A}$ if and only if $*\bar{s} \in \mathcal{H}^{p,q}_{BC}$. □

In Chapter 4 we focus on proving a number of results for the space of Bott-Chern harmonic forms. These results may also be applied to the space of Aeppli harmonic forms through the use of the above proposition.

Chapter 2

Harmonic Analysis on Torus Bundles over S^1

In this chapter we make use of classical Fourier theory to derive a decomposition of the space of L^2 functions on torus bundles over S^1. These results were originally detailed in [15] and can be used to simplify or solve certain linear PDEs. In fact, in Chapters 3 and 4 these results will form the backbone of a technique used to calculate $h^{p,q}_{\bar{\partial}}$ and $h^{p,q}_{BC}$ on the Kodaira-Thurston manifold.

Often a torus bundle over S^1 can be viewed as a group G modulo a discrete subgroup Γ acting by left-multiplication. In this case, the decomposition derived in this chapter can also be viewed as a decomposition of the right regular representation

$$
\begin{aligned}
&R : G \to Aut(L^2(\Gamma\backslash G)) \\
&R(g)f(\Gamma h) = f(\Gamma hg) \qquad\qquad g, h \in G, \quad f \in L^2(\Gamma\backslash G)
\end{aligned}
$$

(see Remarks 2.9 and 5.4 for more details). The reason for deriving these results via classical Fourier theory instead is that it allows us to give a more explicit description of the decomposition, which is needed in Chapters 3 and 4.

Going forwards we will assume the reader is familiar with some of the basic results of classical Fourier analysis, for an introduction to this field (see *e.g* [28]).

2.1 Decomposition of functions

Let M be any n-torus bundle over S^1. This can be described as the mapping torus of an n-torus determined by a matrix $A \in GL_n(\mathbb{Z})$. In other words, M is given by $\mathbb{R}^{n+1}$

with points $(t, \mathbf{x})$ identified by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x} + \eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t + \xi \\ A^{\xi}\mathbf{x} \end{pmatrix} \tag{2.1}$$

for all $\xi \in \mathbb{Z}$, $\eta \in \mathbb{Z}^n$.

When t is fixed, $\mathbf{x}$ describes a point on a torus. This means any smooth function $f \in C^{\infty}(M)$, when viewed as a function on $\mathbb{R}^{n+1}$ satisfying

$$f(t, \mathbf{x}) = f(t, \mathbf{x} + \eta) \quad \text{and} \quad f(t, \mathbf{x}) = f(t + \xi, A^{\xi}\mathbf{x}) \tag{2.2}$$

can be decomposed into the Fourier series

$$f(t, \mathbf{x}) = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}$$

where we define

$$\mathcal{F}_{\mathbf{x}_0}(f)(t) = \int_{[0,1]^n} f(t, \mathbf{x}) e^{-2\pi i \mathbf{x}_0 \cdot \mathbf{x}}\, d\mathbf{x}.$$

Here we have to be careful: notice that we have no guarantee that the summands $\mathcal{F}_{\mathbf{x}_0}(f) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}$ will satisfy the same condition (2.2) as f, and so the summands are not themselves smooth functions on M. In particular, it is the second condition of (2.2) that may fail. We do however have the following result.

Proposition 2.1. *A function $f \in C^{\infty}(\mathbb{R}^{n+1})$ satisfies* (2.2) *if and only if it can be written as the Fourier series*

$$f(t, \mathbf{x}) = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}}$$

such that

$$\mathcal{F}_{(A^T)^{\xi}\mathbf{x}_0}(t) = \mathcal{F}_{\mathbf{x}_0}(f)(t + \xi)$$

for all $\xi \in \mathbb{Z}$.

Proof. It is clear that f has a Fourier expansion if and only if it satisfies the first condition of (2.2). Taking the expansion of the second condition we see that

$$\sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}} = \sum_{\mathbf{x}_0 \in \mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t + \xi) e^{2\pi i \mathbf{x}_0 \cdot A^{\xi}\mathbf{x}}$$

or equivalently

$$\sum_{\mathbf{x}_0\in\mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t)e^{2\pi i\mathbf{x}_0\cdot\mathbf{x}} = \sum_{\mathbf{x}_0\in\mathbb{Z}^n} \mathcal{F}_{(A^T)^{-\xi}\mathbf{x}_0}(f)(t+\xi)e^{2\pi i\mathbf{x}_0\cdot\mathbf{x}}.$$

By the uniqueness of Fourier coefficients, this is identical to requiring

$$\mathcal{F}_{(A^T)^{\xi}\mathbf{x}_0}(t) = \mathcal{F}_{\mathbf{x}_0}(f)(t+\xi).$$

□

This proposition suggests that by grouping together terms in the expansion, we can obtain a decomposition of f into smooth functions on M.

Definition 2.2. Let $\mathrm{Orb}_{\mathbf{y}}$ denote the orbit of the point $\mathbf{y}\in\mathbb{Z}^n$ being acted on by the group generated by the transpose matrix A^T. That is to say we have

$$\mathrm{Orb}_{\mathbf{y}} = \{(A^T)^{\xi}\mathbf{y} \mid \xi\in\mathbb{Z}\}.$$

We use these orbits to partition $\mathbb{Z}^n$ and define $\mathcal{O}$ to be the set of all such orbits.

Proposition 2.3. *Any $f\in C^\infty(M)$ can be written as the series*

$$\sum_{\substack{\mathrm{Orb}_{\mathbf{y}}\in\mathcal{O}\\|\mathrm{Orb}_{\mathbf{y}}|=\infty}} \left(\sum_{\xi\in\mathbb{Z}} \mathcal{F}_{\mathbf{y}}(f)(t+\xi)e^{2\pi i\mathbf{y}\cdot A^{\xi}\mathbf{x}}\right) + \sum_{\substack{\mathrm{Orb}_{\mathbf{y}}\in\mathcal{O}\\|\mathrm{Orb}_{\mathbf{y}}|<\infty}} \left(\sum_{\xi=0}^{|\mathrm{Orb}_{\mathbf{y}}|-1} \mathcal{F}_{\mathbf{y}}(f)(t+\xi)e^{2\pi i\mathbf{y}\cdot A^{\xi}\mathbf{x}}\right)$$

and we have

$$\left(\sum_{\xi\in\mathbb{Z}} \mathcal{F}_{\mathbf{y}}(f)(t+\xi)e^{2\pi i\mathbf{y}\cdot A^{\xi}\mathbf{x}}\right) \in C^\infty(M)$$

$$\left(\sum_{\xi=0}^{|\mathrm{Orb}_{\mathbf{y}}|-1} \mathcal{F}_{\mathbf{y}}(f)(t+\xi)e^{2\pi i\mathbf{y}\cdot A^{\xi}\mathbf{x}}\right) \in C^\infty(M)$$

in the cases where $\mathbf{y}\in\mathbb{Z}^n$ satisfies $|\mathrm{Orb}_{\mathbf{y}}|=\infty$, respectively $|\mathrm{Orb}_{\mathbf{y}}|<\infty$.

Proof. By partitioning $\mathbb{Z}^n$ into the orbits $\mathrm{Orb}_{\mathbf{y}}$ we can write

$$\sum_{\mathbf{x}_0\in\mathbb{Z}^n} \mathcal{F}_{\mathbf{x}_0}(f)(t)e^{2\pi i\mathbf{x}_0\cdot\mathbf{x}} = \sum_{\mathrm{Orb}_{\mathbf{y}}\in\mathcal{O}} \sum_{\mathbf{x}_0\in\mathrm{Orb}_{\mathbf{y}}} \mathcal{F}_{\mathbf{x}_0}(f)(t)e^{2\pi i\mathbf{x}_0\cdot\mathbf{x}}.$$

Then by Proposition 2.1, if we have $\mathbf{x}_0 = (A^T)^\xi \mathbf{y}$ for some $\xi \in \mathbb{Z}$, then we can write

$$\mathcal{F}_{\mathbf{x}_0}(f)(t) = \mathcal{F}_{\mathbf{y}}(f)(t+\xi)$$

and thus

$$\sum_{\mathbf{x}_0 \in \mathrm{Orb}_{\mathbf{y}}} \mathcal{F}_{\mathbf{x}_0}(f)(t) e^{2\pi i \mathbf{x}_0 \cdot \mathbf{x}} = \sum_{\xi} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}}$$

with ξ ranging over different values depending on the size of $\mathrm{Orb}_{\mathbf{y}}$. □

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = N$ for some $N < \infty$ the function $\mathcal{F}_{\mathbf{y}}(f)$ is periodic with period N, and so we can further decompose it as follows

Proposition 2.4. *Given $f \in C^\infty(M)$ and any $\mathbf{y} \in \mathbb{Z}^n$ such that $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$, we can write*

$$\mathcal{F}_{\mathbf{y}}(f)(t) = \sum_{t_0 \in \mathbb{Z}} \mathcal{G}_{t_0,\mathbf{y}}(f) e^{\frac{2\pi i t_0 t}{N}}$$

where $\mathcal{G}_{t_0,\mathbf{y}} \in \mathbb{C}$ is defined by

$$\mathcal{G}_{t_0,\mathbf{y}}(f) = \frac{1}{N} \int_0^N \mathcal{F}_{\mathbf{y}}(f)(t) e^{-\frac{2\pi i t_0 t}{N}}\, dt.$$

Proof. This is simply the Fourier expansion of the periodic function $\mathcal{F}_{\mathbf{y}}(f)(t)$. □

Corollary 2.5. *In the decomposition of f in Proposition 2.3, the summand*

$$\left(\sum_{\xi=0}^{N-1} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right) \in C^\infty(M)$$

can be further decomposed into

$$\sum_{t_0 \in \mathbb{Z}} \left(\mathcal{G}_{t_0,\mathbf{y}}(f) e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y} \cdot A^\xi \mathbf{x} \right)} \right)$$

such that each term

$$\left(\mathcal{G}_{t_0,\mathbf{y}}(f) e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y} \cdot A^\xi \mathbf{x} \right)} \right)$$

is itself a smooth function on M.

Proof. This result is achieved by substituting the expression for $\mathcal{F}_{\mathbf{y}}(f)$ in Proposition

2.4 into the summand. That the terms of the decomposition are themselves smooth functions on M can be verified through the use of Proposition 2.1. $\square$

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = \infty$ there does not seem to be any further useful decomposition of $\mathcal{F}$, however there are additional properties which $\mathcal{F}$ must satisfy.

Proposition 2.6. *For any $f \in C^\infty(M)$ and any $\mathbf{y} \in \mathbb{Z}^n$ such that $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, we require that all derivatives of $\mathcal{F}_{\mathbf{y}}(f)(t)$ tend to zero as $t \to \pm\infty$ faster than any power of $\left|(A^T)^\xi \mathbf{y}\right|$ grows as $\xi \to \pm\infty$. Specifically, for any compact set $K \subset \mathbb{R}$ we require*

$$\sup_{\substack{t\in K\\ \xi\in\mathbb{Z}}} \left| \left\|(A^T)^\xi \mathbf{y}\right\|^p \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) \right| < \infty$$

for all $p, q \in \mathbb{N}$.

Proof. First, note that given any smooth function $f \in C^\infty(\mathbb{R}^{n+1})$ satisfying (2.2), all its derivatives must be bounded over any compact $\tilde{K} \subset \mathbb{R}^{n+1}$. If we take $\tilde{K} = [0,1]^n \times K$, we see that the Fourier coefficients $\mathcal{F}_{\mathbf{x}_0}$ of all the derivatives of f must be bounded for t ranging over K. Importantly, this bound is independent of $\mathbf{x}_0 \in \mathbb{Z}^n$.

The Fourier coefficients of the derivatives of f can take the form of $M(\mathbf{x}_0) \frac{d^q}{dt^q} \left(\mathcal{F}_{\mathbf{x}_0}(f)(t)\right)$ for any monomial M and any $q \in \mathbb{N}$. This means for all monomials M and all $q \in \mathbb{N}$ we require

$$\sup_{\substack{t\in K\\ \mathbf{x}_0\in\mathbb{Z}^n}} \left| M(\mathbf{x}_0) \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{x}_0}(f)(t) \right| < \infty$$

and thus if we restrict our attention to $\mathbf{x}_0 \in \mathrm{Orb}_{\mathbf{y}}$ we require

$$\sup_{\substack{t\in K\\ \mathbf{x}_0\in\mathrm{Orb}_{\mathbf{y}}}} \left| M(\mathbf{x}_0) \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{x}_0}(f)(t) \right| = \sup_{\substack{t\in K\\ \xi\in\mathbb{Z}}} \left| M((A^T)^\xi \mathbf{y}) \frac{d^q}{dt^q} \mathcal{F}_{\mathbf{y}}(f)(t+\xi) \right| < \infty.$$

$M(\mathbf{x}_0)$ can then be chosen to be $\|\mathbf{x}_0\|^p$ for arbitrarily large $p \in \mathbb{N}$, giving us the desired result. $\square$

Corollary 2.7. *For any $f \in C^\infty(M)$ and any $\mathbf{y} \in \mathbb{Z}^n$ such that $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, we require that $\mathcal{F}_{\mathbf{y}}(f)(t) \in \mathcal{S}(\mathbb{R})$. Here $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions*

$$\mathcal{S}(\mathbb{R}) = \left\{ h(t) \in C^\infty(\mathbb{R}) \,\middle|\, \sup_{t\in\mathbb{R}} \left| t^p \frac{d^q}{dt^q} h(t) \right| < \infty, \text{ for all } p, q \in \mathbb{N} \right\}.$$

Proof. If $|\mathrm{Orb}_{\mathbf{y}}| = \infty$ then $\left\|(A^T)^{\xi}\mathbf{y}\right\|$ must blow up as $\xi \to \pm\infty$, since an infinite orbit cannot repeat the same point twice. Furthermore, since the number of lattice points within a bounded region of $\mathbb{Z}^n$ grows like R^n with the radius R of the region, it must be the case that $\left\|(A^T)^{\xi}\mathbf{y}\right\|$ blows up at least as fast as $|\xi|^{\frac{1}{n}}$. Substituting this speed of growth into the above proposition gives the definition of $\mathcal{S}(\mathbb{R})$.

Note that if $\left\|(A^T)^{\xi}\mathbf{y}\right\|$ blows up faster than polynomially, then the Proposition yields an even stricter condition on $\mathcal{F}_{\mathbf{y}}$ than Schwartz. □

The decomposition of smooth functions described above can be extended to a deeper result, describing a decomposition of L^2 functions.

Theorem 2.8. *Let M be a mapping torus of an n-dimensional torus, given by the matrix $A \in GL_n(\mathbb{Z})$. The space of L^2 functions on M decomposes in the following way.*

$$L^2(M) = \left(\widehat{\bigoplus_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}| = \infty}}} \mathcal{H}_{\mathbf{y}} \right) \oplus \left(\widehat{\bigoplus_{\substack{\mathrm{Orb}_{\mathbf{y}} \in \mathcal{O} \\ |\mathrm{Orb}_{\mathbf{y}}| = N < \infty}}} \widehat{\bigoplus_{t_0 \in \mathbb{Z}}} \mathcal{H}_{t_0,\mathbf{y}} \right),$$

where

$$\mathcal{H}_{\mathbf{y}} = \left\{ \sum_{\xi \in \mathbb{Z}} f(t+\xi) e^{2\pi i \mathbf{y}\cdot A^{\xi}\mathbf{x}} \;\middle|\; f \in L^2(\mathbb{R}) \right\}$$

and

$$\mathcal{H}_{t_0,\mathbf{y}} = \left\{ C e^{2\pi i \frac{t_0 t}{N}} \sum_{\xi=0}^{N-1} e^{2\pi i \left(\frac{t_0 \xi}{N} + \mathbf{y}\cdot A^{\xi}\mathbf{x} \right)} \;\middle|\; C \in \mathbb{C} \right\}.$$

Here $\hat{\bigoplus}$ denotes the direct sum followed by the closure with respect to the L^2 norm.

Proof. From Propositions 2.3 and 2.4 we see that any smooth function can be decomposed in the way described above. We then obtain the desired result by taking the closure of $C^{\infty}(M)$ with respect to the L^2 norm. □

Remark 2.9. *If a manifold M can be written as a group G modulo a discrete subgroup Γ acting by left-multiplication, then the right regular representation*

$$R : G \to Aut(L^2(\Gamma\backslash G))$$
$$R(g)f(\Gamma h) = f(\Gamma h g) \qquad g, h \in G, \quad f \in L^2(\Gamma\backslash G)$$

gives rise to a decomposition of $L^2(\Gamma\backslash G)$ into irreducible components, i.e. a decomposition into the subspaces preserved by $R(g)$ for all $g \in G$.

This often coincides with the decomposition described in Theorem 2.8. For example, let $A \in GL_n(\mathbb{Z})$ be chosen such that A^t is real valued for all $t \in \mathbb{R}$, and define a manifold M by the identification (2.1). *This is equivalent to the manifold given by $M := \Gamma\backslash G$, where we define $G := \mathbb{R}^{n+1}$ with the group operation*

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \circ \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} t + t' \\ \mathbf{x} + A^t\mathbf{x}' \end{pmatrix}$$

and $\Gamma := \mathbb{Z}^{n+1} \subset G$.

It is then a simple matter to check that the space $\mathcal{H}_{\mathbf{y}}$ is preserved by the regular representation. For any $(t', \mathbf{x}') \in G$ we have

$$\begin{aligned} R(t', \mathbf{x}') & \left(\sum_{\xi \in \mathbb{Z}} f(t+\xi) e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \right) \\ &= \sum_{\xi \in \mathbb{Z}} f(t + t' + \xi) e^{2\pi i \mathbf{y} \cdot A^\xi (\mathbf{x} + A^t \mathbf{x}')} \\ &= \sum_{\xi \in \mathbb{Z}} f(t + t' + \xi) e^{2\pi i \mathbf{y} \cdot A^{t+\xi} \mathbf{x}'} e^{2\pi i \mathbf{y} \cdot A^\xi \mathbf{x}} \in \mathcal{H}_{\mathbf{y}}, \end{aligned}$$

since $f(t+t')e^{2\pi i \mathbf{y}\cdot A^t \mathbf{x}'} \in L^2(\mathbb{R})$. The same is not always true for $\mathcal{H}_{t_0,\mathbf{y}}$ in general, except when $|\mathrm{Orb}_{\mathbf{y}}| = 1$. In which case

$$\mathcal{H}_{t_0,\mathbf{y}} = \left\{ C e^{2\pi i (t_0 t + \mathbf{y}\cdot\mathbf{x})} \,\middle|\, C \in \mathbb{C} \right\}$$

and for any $(t', \mathbf{x}') \in G$ we have

$$\begin{aligned} R(t', \mathbf{x}') & \left(C e^{2\pi i (t_0 t + \mathbf{y}\cdot\mathbf{x})} \right) \\ &= C e^{2\pi i (t_0 (t+t') + \mathbf{y}\cdot(\mathbf{x} + A^t \mathbf{x}'))} \\ &= C e^{2\pi i (t_0 t' + \mathbf{y}\cdot\mathbf{x}')} e^{2\pi i (t_0 t + \mathbf{y}\cdot\mathbf{x})} \in \mathcal{H}_{t_0,\mathbf{y}}, \end{aligned}$$

since $Ce^{2\pi i (t_0 t' + \mathbf{y}\cdot\mathbf{x}')} \in \mathbb{C}$. Here we used the fact that A^T acts on $\mathbf{y}$ as the identity and therefore $\mathbf{y} \cdot A^t\mathbf{x}' = \mathbf{y} \cdot \mathbf{x}'$.

2.2 What do the orbits look like?

It will be useful now to consider what the orbits of $\mathbf{y} \in \mathbb{Z}^n$ actually look like. In particular, when exactly is $|\mathrm{Orb}_{\mathbf{y}}| < \infty$. First, we define the generalised eigenvectors of A.

Definition 2.10. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ be the eigenvalues of $A \in GL_n(\mathbb{Z})$ with values repeated for geometric multiplicity. Then any n linearly independent vectors $\mathbf{v}_{i,j} \in \mathbb{C}^n$ with $i = 1, \ldots, k$ and $j = 1, \ldots, m_i$ such that

$$(A - \lambda_i)^j \mathbf{v}_{i,j} = 0 \quad \text{but} \quad (A - \lambda_i)^{j-1} \mathbf{v}_{i,j} \neq 0$$

are called generalised eigenvectors of A. Note that when $j = 1$ we just have the standard eigenvectors of A. Furthermore, we can make a choice of $\mathbf{v}_{i,j}$ so that when i is fixed, the sequence $\mathbf{v}_{i,1}, \mathbf{v}_{i,2}, \ldots, \mathbf{v}_{i,m_i}$ forms a Jordan chain of length m_i. This means for all $j \neq 1$ we have

$$(A - \lambda_i)\mathbf{v}_{i,j} = \mathbf{v}_{i,j-1} \tag{2.3}$$

and for $j = 1$ we have

$$(A - \lambda_i)\mathbf{v}_{i,1} = 0. \tag{2.4}$$

These $\mathbf{v}_{i,j}$ can be used to describe when the orbit of the group generated by A^T acting on $\mathbf{y} \in \mathbb{Z}^n$ is finite.

Proposition 2.11. *Let $\mathbf{v}_{i,j}$ be the generalised eigenvectors of $A \in GL_n(\mathbb{Z})$ as defined above, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$. Given $\mathbf{y} \in \mathbb{Z}^n$, if $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$ it must be the case that $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ except for when i and j are chosen such that $\lambda_i^N = 1$ and $j = m_i$*

Proof. If $\mathrm{Orb}_{\mathbf{y}}$ is a finite subset of $\mathbb{Z}^n$, then $(A^T)^\xi \mathbf{y}$ must be bounded over $\xi \in \mathbb{Z}$. This means $\mathbf{v}_{i,j} \cdot ((A^T)^\xi \mathbf{y}) = (A^\xi \mathbf{v}_{i,j}) \cdot \mathbf{y}$ must be bounded over $\xi \in \mathbb{Z}$ for all $\mathbf{v}_{i,j}$.

From (2.4) we know that $A\mathbf{v}_{i,1} = \lambda \mathbf{v}_{i,1}$ and thus

$$A^\xi \mathbf{v}_{i,1} \cdot \mathbf{y} = \lambda^\xi \mathbf{v}_{i,1} \cdot \mathbf{y}.$$

But if $|\lambda_i| > 1$ then λ_i^ξ will blow up as $\xi \to \infty$ and if $|\lambda_i| < 1$ then it will blow up as $\xi \to -\infty$. From this we conclude that $|\mathrm{Orb}_{\mathbf{y}}| < \infty$ only if $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$ for all i such that $|\lambda_i| \neq 1$. Rewriting (2.3) as $A\mathbf{v}_{i,j} = \lambda \mathbf{v}_{i,j} + \mathbf{v}_{i,j-1}$ and using $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$ we can apply the above argument again to prove the same result for $\mathbf{v}_{i,2}$. In fact, continuing by induction, we see that $|\mathrm{Orb}_{\mathbf{y}}|$ is finite only if $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ for all i and j such that $|\lambda_i| \neq 0$.

Now, consider the case when $|\lambda_i| = 1$. From (2.3) we can see that when $m_i \geq 2$ then

$$A^{\xi}\mathbf{v}_{i,2} = \lambda^{\xi}\mathbf{v}_{i,2} + \xi\lambda^{\xi-1}\mathbf{v}_{i,1}.$$

This means $A^{\xi}\mathbf{v}_{i,2} \cdot \mathbf{y}$ will blow up as $\xi \to \pm\infty$ unless $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$. Similarly, if $\mathbf{v}_{i,1} \cdot \mathbf{y} = 0$ then the same argument works to show $A^{\xi}\mathbf{v}_{i,3} \cdot \mathbf{y}$ will blow up unless $\mathbf{v}_{i,2} \cdot \mathbf{y} = 0$, provided $m_i \geq 3$. Repeating this procedure, we find that $|\mathrm{Orb}_{\mathbf{y}}| < \infty$ implies that $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ for all i and j such that $|\lambda_i| = 1$ and $j < m_i$

Finally, it remains to consider the case of $\mathbf{v}_{i,m_i}$. If $|\mathrm{Orb}_{\mathbf{y}}| = N$ then we know that $(A^T)^N\mathbf{y} = \mathbf{y}$, and also we have shown that $\mathbf{v}_{i,j} \cdot \mathbf{y} = 0$ for all $j \neq m_i$. The following must therefore hold.

$$\begin{aligned}
\mathbf{v}_{i,m_i} \cdot \mathbf{y} &= \mathbf{v}_{i,m_i} \cdot (A^T)^N\mathbf{y} \\
&= A^N\mathbf{v}_{i,m_i} \cdot \mathbf{y} \\
&= \lambda_i^N\mathbf{v}_{i,m_i} \cdot \mathbf{y}
\end{aligned}$$

Thus $|\mathrm{Orb}_{\mathbf{y}}| = N$ requires that for all i, either $\mathbf{v}_{i,m_i} \cdot \mathbf{y} = 0$ or $\lambda_i^N = 1$ $\square$

Corollary 2.12. *Whenever $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$, it holds that*

$$A\mathbf{v}_{i,j} \cdot \mathbf{y} = \begin{cases} e^{2\pi i\theta_i}\mathbf{v}_{i,j} \cdot \mathbf{y} & \textit{if } \lambda_i^N = 1 \textit{ and } j = m_i \\ 0 & \textit{otherwise} \end{cases}$$

where $\theta_i \in \mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2}]$ is some rational number depending on i satisfying $N\theta_i \in \mathbb{Z}$.

2.3 Properties of the decomposition

We would now like to consider some of the properties of this decomposition, which will be useful when performing calculations in the following Chapters. But in order to do this we must first construct a special frame on M.

Definition 2.13. Given any invertible matrix $A \in GL_n(\mathbb{Z})$, and for some choice of matrix logarithm $\log A$ we can define the power $A^t := e^{t\log A}$ for all $t \in \mathbb{R}$. Note that such a logarithm always exists, but it may not be unique and it may be complex valued.

Throughout this paper, the choice of $\ln A$ will always be made such that

$$A^t\mathbf{v}_{i,j} \cdot \mathbf{y} = \begin{cases} e^{2\pi i\theta_i t}\mathbf{v}_{i,j} \cdot \mathbf{y} & \text{if } \lambda_i^N = 1 \text{ and } j = m_i \\ 0 & \text{otherwise} \end{cases}$$

for $\theta_i \in \mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2}]$

Using the generalised eigenvectors of A given by $\mathbf{v}_{i,j}$, a smooth frame for the complexified tangent bundle of M can be given by

$$\epsilon_0 = \frac{\partial}{\partial t} \qquad \epsilon_{i,j} = A^t \mathbf{v}_{i,j} \cdot \nabla_{\mathbf{x}}.$$

Here we are using $\nabla_{\mathbf{x}} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$ to denote the gradient excluding the variable t. We verify that this is indeed a well-defined frame on M in the following proposition:

Proposition 2.14. *Viewing M as a torus bundle over S^1, any smooth frame of the complexified tangent bundle on a single fibre may be extended to a smooth frame on all of M.*

Proof. We can assume, without loss of generality, that we are starting with a frame on the $t = 0$ fibre, where t is parametrising the base space S^1, as in the definition of M (2.1).

Let $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{C}^n$ be smooth maps sending each point $\mathbf{x} \in \mathbb{T}^n$ to n linearly independent vectors. Then the collection $\{\mathbf{a}_i \cdot \nabla_{\mathbf{x}}\}_{i=1,\ldots,n}$ defines a general frame on the $t = 0$ fibre. A frame for $T_{\mathbb{C}}M$ is then given by $u_0 = \frac{\partial}{\partial t}$ and $u_i = A^t \mathbf{a}_i \cdot \nabla_{\mathbf{x}}$ with $i = 1, \ldots, n$

These are indeed all well-defined vector fields on M, in particular they do not conflict with the identification of points given in (2.1). To check the first identification, simply note that the maps $\mathbf{a}_i(\mathbf{x})$ are defined on the torus. For the second we consider the map

$$\phi_\xi : \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \mapsto \begin{pmatrix} t + \xi \\ A^\xi \mathbf{x} \end{pmatrix}$$

with $\xi \in \mathbb{Z}$ and try to show that u_i are invariant under the pushforward. Certainly this is true of $\frac{\partial}{\partial t}$, and we also know that, for $i = 1, \ldots, n$, we have

$$\begin{aligned}(\phi_\xi)_*(\mathbf{e}_i \cdot \nabla_{\mathbf{x}}) &= (\phi_\xi)_* \frac{\partial}{\partial x_i} \\ &= A^\xi \mathbf{e}_i \cdot \nabla_{\mathbf{x}}\end{aligned}$$

with $\mathbf{e}_i$ signifying the standard basis vector $(0, \ldots, 1, \ldots, 0)$ with a 1 in the i^{th} position.

Therefore

$$\begin{aligned}(\phi_\xi)_* u_i(t,\mathbf{x}) &= (\phi_\xi)_*(A^t\mathbf{a}_i \cdot \nabla_{\mathbf{x}}) \\ &= A^{t+\xi}\mathbf{a}_i \cdot \nabla_{\mathbf{x}} \\ &= u_i(t+\xi, A^\xi \mathbf{x}).\end{aligned}$$

□

It should be noted that if A has a real-valued logarithm and we choose $\mathbf{a}_i$ to be maps into $\mathbb{R}^n$, then the construction in the above proof will give us a smooth frame on the standard, non-complexified tangent bundle. Furthermore, in this case we can define a group operation on $\mathbb{R}^{n+1}$ given by

$$\begin{pmatrix} t_1 \\ \mathbf{x}_1 \end{pmatrix} \circ \begin{pmatrix} t_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 \\ \mathbf{x}_1 + A^{t_1}\mathbf{x}_2 \end{pmatrix}.$$

The smooth frame given above by ϵ_0 and $\epsilon_{i,j}$ is left-invariant with respect to this group operation. We therefore call this frame the *special left-invariant frame* on M.

Proposition 2.15. *Given any $\mathbf{y} \in \mathbb{Z}^n$ and any $f \in C^\infty(M)$, $\mathcal{F}_{\mathbf{y}}$ has the properties*

i)

$$\mathcal{F}_{\mathbf{y}}(\epsilon_0 f)(t) = \epsilon_0\, \mathcal{F}_{\mathbf{y}}(f)(t),$$

ii)

$$\mathcal{F}_{\mathbf{y}}(\epsilon_{i,j} f)(t) = 2\pi i A^t \mathbf{v}_{i,j} \cdot \mathbf{y}\, \mathcal{F}_{\mathbf{y}}(f).$$

Proof. Since $\mathcal{F}_{\mathbf{y}}(f)$ is just one of the Fourier coefficients of f in the standard expansion

$$f(t,\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^n} \mathcal{F}_{\mathbf{y}}(f)(t) e^{2\pi i \mathbf{y}\cdot\mathbf{x}}$$

this proposition is simply restating results from classical Fourier analysis, □

Proposition 2.16. *Given any $\mathbf{y} \in \mathbb{Z}^n$ such that $|\mathrm{Orb}_{\mathbf{y}}| = N < \infty$ and any $f \in C^\infty(M), \mathcal{G}_{t_0,\mathbf{y}}$ has the properties*

i)

$$\mathcal{G}_{t_0,\mathbf{y}}(\epsilon_0 f)(t) = 2\pi i \frac{t_0}{N}\, \mathcal{G}_{t_0,\mathbf{y}}(f),$$

ii)

$$\mathcal{G}_{t_0,\mathbf{y}}(\epsilon_{i,j} f) = \begin{cases} 2\pi i \mathbf{v}_{i,j} \cdot \mathbf{y}\, \mathcal{G}_{t_0+N\theta_i,\mathbf{y}}(f) & \text{if } \lambda_i^N = 1 \text{ and } j = m_i \\ 0 & \text{otherwise} \end{cases}.$$

With θ_i defined as in Corollary 2.12.

Proof. For part *i*), we make use of the result *i*) in the previous proposition along with the definition of $\mathcal{G}_{t_0,\mathbf{y}}$ to write

$$\begin{aligned} \mathcal{G}_{t_0,\mathbf{y}}(\epsilon_0 f)(t) &= \frac{1}{N}\int_0^N \mathcal{F}_{\mathbf{y}}(\epsilon_0 f) e^{-\frac{2\pi i t_0 t}{N}}\, dt \\ &= \frac{1}{N}\int_0^N \left(\epsilon_0 \mathcal{F}_{\mathbf{y}}(f)\right) e^{-\frac{2\pi i t_0 t}{N}}\, dt. \end{aligned}$$

Then, since $\mathcal{F}_{\mathbf{y}}(f)(t)$ is periodic with period N, we can make use of integration by parts to get

$$\begin{aligned} \frac{1}{N}\int_0^N \left(\epsilon_0 \mathcal{F}_{\mathbf{y}}(f)\right) e^{-\frac{2\pi i t_0 t}{N}}\, dt &= -\frac{1}{N}\int_0^N \mathcal{F}_{\mathbf{y}}(f) \left(\epsilon_0 e^{-\frac{2\pi i t_0 t}{N}}\right) dt \\ &= 2\pi i \frac{t_0}{N}\frac{1}{N}\int_0^N \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i t_0 t}{N}}\, dt \\ &= 2\pi i \frac{t_0}{N} \mathcal{G}_{t_0,\mathbf{y}}(f). \end{aligned}$$

For part *ii*), we make use of the result *ii*) in the previous proposition to write

$$\begin{aligned} \mathcal{G}_{t_0,\mathbf{y}}(\epsilon_{i,j} f) &= \frac{1}{N}\int_0^N \mathcal{F}_{\mathbf{y}}(\epsilon_{i,j} f) e^{-\frac{2\pi i t_0 t}{N}}\, dt \\ &= \frac{1}{N}\int_0^N 2\pi i A^t \mathbf{v}_{i,j} \cdot \mathbf{y} \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i t_0 t}{N}}\, dt. \end{aligned}$$

Then, because of the way A^t was defined in Definition 2.13, we get

$$\frac{1}{N}\int_0^N 2\pi i A^t \mathbf{v}_{i,j} \cdot \mathbf{y} \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i t_0 t}{N}}\, dt = 0$$

unless $\lambda_i^N = 1$ and $j = m_i$, in which case

$$\begin{aligned} \frac{1}{N}\int_0^N 2\pi i A^t \mathbf{v}_{i,j} \cdot \mathbf{y} \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i t_0 t}{N}}\, dt &= 2\pi i \mathbf{v}_{i,j} \cdot \mathbf{y} \frac{1}{N}\int_0^N \mathcal{F}_{\mathbf{y}}(f) e^{-\frac{2\pi i (t_0+N\theta_i) t}{N}}\, dt \\ &= 2\pi i \mathbf{v}_{i,j} \cdot \mathbf{y}\, \mathcal{G}_{t_0+N\theta_i,\mathbf{y}}(f). \end{aligned}$$

□

Chapter 3

Solving the Kodaira-Spencer Problem

In this chapter we will focus on compact almost complex 4-manifolds, giving a full description of the bidegrees for which $h^{p,q}_{\bar\partial}$ does or does not depend on the choice of almost Hermitian metric.

In particular, we will give a negative answer to the Kodaira-Spencer problem, by explicitly calculating $h^{0,1}_{\bar\partial}$ on the Kodaira-Thurston manifold for a family of Kähler metrics. Since the Kodaira-Thurston manifold is a 3-torus bundle over S^1, these calculations will make use of the results from the previous chapter to simplify a PDE system to a countably infinite collection of ordinary differential equations (ODEs) and algebraic equations. We will then find the ODEs can be reduced to a Stokes phenomenon problem, while the algebraic equations are equivalent to the Gauss circle problem. These calculations were first presented in [13, 14].

We will also make use of a characterisation of $\mathcal{H}^{1,1}_{\bar\partial}$ given in [29] to show that on a compact 4-manifold, $h^{1,1}_{\bar\partial}$ may only take two values: b_- or b_-+1; a result first presented in [15]. Which of the two values is taken by $h^{1,1}_{\bar\partial}$ will depend on the almost Hermitian metric chosen, thus we will demonstrate that the behaviour of $h^{1,1}_{\bar\partial}$ also gives a negative answer to the Kodaira-Spencer question.

But to start with, let us consider the simplest case: bidegrees of the form $(p,0)$ and (p,n). In [3], Haojie Chen and Weiyi Zhang determined the following:

Proposition 3.1. *If (M,J) describes an almost complex manifold with real dimension $2n$, then the values $h^{p,0}_{\bar\partial}$ and $h^{p,n}_{\bar\partial}$ are invariant of any almost Hermitian metric for all $p\in\mathbb{Z}$.*

Proof. Given an almost Hermitian metric g compatible with J, we can characterise a $\bar{\partial}$-harmonic $(p,0)$-form $s \in \mathcal{H}^{p,0}_{\bar{\partial}}$ by the equations $\bar{\partial}s = 0$ and $\bar{\partial}^* s = 0$. But if s has bidegree $(p,0)$ then $\bar{\partial}^* s$ has bidegree $(p,-1)$ and so must be zero. We therefore only need to consider the first equation $\bar{\partial}s = 0$, which has no dependence on the metric, and so neither does $h^{p,0}_{\bar{\partial}}$.

By Serre duality, we know that $h^{p,0}_{\bar{\partial}} = h^{n-p,n}_{\bar{\partial}}$. Thus $h^{p,n}_{\bar{\partial}}$ does not depend on the chosen metric. It should be noted however that, unlike $\mathcal{H}^{p,0}_{\bar{\partial}}$, the space $\mathcal{H}^{p,n}_{\bar{\partial}}$ *may* depend on the metric. $\square$

On 4-manifolds we now have only two cases left to be considered: $h^{0,1}_{\bar{\partial}}$ and $h^{1,1}_{\bar{\partial}}$. The case of $h^{2,1}_{\bar{\partial}}$ being equivalent to that of $h^{0,1}_{\bar{\partial}}$ by Serre duality. Giving a description for both of these cases is quite a bit trickier, in fact it will take up the rest of this chapter.

3.1 $h^{0,1}_{\bar{\partial}}$ on the Kodaira-Thurston manifold

If we consider the specific example of the Kodaira-Thurston manifold, we can show that $h^{0,1}_{\bar{\partial}}$ depends on the metric by direct calculation. But first we must define the Kodaira-Thurston manifold.

Definition 3.2. The Kodaira-Thurston manifold, denoted by KT^4, is a compact 4-manifold defined by taking $\mathbb{R}^4$ and identifying points by

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t + t_0 \\ x + x_0 \\ y + y_0 \\ z + z_0 + t_0 y \end{pmatrix}$$

for all $t_0, x_0, y_0, z_0 \in \mathbb{Z}$. This description characterises the Kodaira-Thurston manifold as a torus bundle over S^1, as given in the previous chapter by (2.1) with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The special left-invariant global frame on the tangent space of KT^4 is given by

$$\epsilon_0 = \frac{\partial}{\partial x}, \quad \epsilon_1 = \frac{\partial}{\partial x}, \quad \epsilon_2 = \frac{\partial}{\partial y} + t\frac{\partial}{\partial z}, \quad \epsilon_3 = \frac{\partial}{\partial z}$$

and the accompanying dual frame is given by

$$\epsilon^0 = dt, \quad \epsilon^1 = dx, \quad \epsilon^2 = dy, \quad \epsilon^3 = dz - tdy.$$

Note that we cannot use the vector field $\frac{\partial}{\partial y}$ when defining a global frame as it is not well-defined on KT^4.

We define a family of almost complex structures $J_{a,b}$ acting on the special global frame by the matrix

$$J_{a,b} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{pmatrix}$$

with $a, b \in \mathbb{R}$, $b \neq 0$ and $c := -\frac{a^2+1}{b}$. For this family of almost complex structures, we have a global frame on $T_{1,0}KT^4$ given by the vector fields

$$V_1 = \frac{1}{2}(\epsilon_0 - i\epsilon_1) \quad \& \quad V_2 = \frac{1}{2}\left(\epsilon_2 - \frac{a-i}{b}\epsilon_3\right)$$

along with a dual frame on $T^*_{1,0}KT^4$

$$\phi^1 = \epsilon^0 + i\epsilon^1 \quad \& \quad \phi^2 = (1 - ai)\epsilon^2 - ib\epsilon^3.$$

This dual frame satisfies the structure equations

$$d\phi^1 = 0$$

$$d\phi^2 = \frac{b}{4}\left(\phi^{12} + \phi^{1\bar{2}} + \phi^{2\bar{1}} - \phi^{\bar{1}\bar{2}}\right).$$

Here, we use $\phi^{i\bar{j}}$ as shorthand for $\phi^i \wedge \bar{\phi^j}$. Let us also define an almost Hermitian metric along with the corresponding fundamental form by

$$g_{a,b} = \phi^1 \otimes \bar{\phi^1} + \phi^2 \otimes \bar{\phi^2} + \bar{\phi^1} \otimes \phi^1 + \bar{\phi^2} \otimes \phi^2$$

$$\omega_{a,b} = i\left(\phi^{1\bar{1}} + \phi^{2\bar{2}}\right).$$

This is the metric for which V_1 and V_2 are orthonormal. The Hodge star defined by this metric acts on ϕ^1 and ϕ^2 as follows:

$$\begin{aligned} * : \phi^1 &\mapsto \phi^{12\bar{2}}, \\ \phi^2 &\mapsto -\phi^{12\bar{1}}. \end{aligned}$$

Example 3.3. Given the almost Hermitian structure defined above on KT^4 we can calculate $h^{0,1}_{\bar{\partial}}$. Let a general smooth $(0,1)$-form $s \in \mathcal{A}^{p,q}(KT^4)$ be given by $s = f\bar{\phi}^1 + g\bar{\phi}^2$ for some pair of smooth functions $f, g \in C^\infty(KT^4)$. Since KT^4 is compact, we know that s is $\bar{\partial}$-harmonic if and only if the two conditions $\bar{\partial}s = 0$ and $\bar{\partial}^* s = - * \partial * s = 0$. From these conditions we obtain a pair of PDEs

$$\begin{cases} -\bar{V}_2(f) + \bar{V}_1(g) + \frac{b}{4}g = 0, \\ V_1(f) + V_2(g) = 0. \end{cases} \tag{3.1}$$

It is at this point that the machinery developed in the previous chapter comes into play. By applying the maps $\mathcal{F}$ and $\mathcal{G}$ to the above PDEs we will be able to determine conditions on the components of f and g in the decomposition of Theorem 2.8 and thereby find solutions to the two PDEs. Looking at the orbits of points $\mathbf{y} \in \mathbb{Z}^3$ under the action of A^T, we see that there are two cases we must consider:

1) If $\mathbf{y} = (l, m, n)$ with $n \neq 0$ then $(A^T)^\xi \mathbf{y} = (l, m + \xi n, n)$ for all $\xi \in \mathbb{Z}$. In which case the orbit $\mathrm{Orb}_{\mathbf{y}}$ is infinite and blows up with polynomial speed as $\xi \to \pm\infty$. Since $\mathbf{y}$ could be replaced with any other element of $\mathrm{Orb}_{\mathbf{y}}$, we can assume without losing generality that we have $0 \leq m < |n|$.
2) If $\mathbf{y} = (l, m, 0)$ then $A^T \mathbf{y} = \mathbf{y}$. In which case the orbit $\mathrm{Orb}_{\mathbf{y}}$ has size 1.

3.1.1 Case 1: $|\mathrm{Orb}_{\mathbf{y}}| = \infty$

Let $\mathbf{y} = (l, m, n)$ for some $l, m, n \in \mathbb{Z}$. If we take our PDEs (3.1) and look at the Fourier coefficients given by $\mathcal{F}_{\mathbf{y}}$, then using Prop. 2.15 we obtain a system of ODEs, which can be written as

$$\frac{d}{dt}\begin{pmatrix} \mathcal{F}_{l,m,n}(f) \\ F_{l,m,n}(g) \end{pmatrix} = (A_n t + B_{l,m,n}) \begin{pmatrix} F_{l,m,n}(f) \\ F_{l,m,n}(g) \end{pmatrix} \tag{3.2}$$

with

$$A_n = 2\pi n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B_{l,m,n} = 2\pi \begin{pmatrix} l & m - n\frac{a-i}{b} \\ m - n\frac{a+i}{b} & \frac{b}{4\pi}i - l \end{pmatrix}.$$

If we now assume that $n \neq 0$ and $0 \leq m < |n|$, we know from Cor. 2.7 that $\mathcal{F}_{\mathbf{y}}(f)$ and $\mathcal{F}_{\mathbf{y}}(g)$ must be Schwartz functions. Conversely, any pair of Schwartz functions $\alpha, \beta \in \mathcal{S}(\mathbb{R})$ solving our ODE system for some fixed $\mathbf{y} = (l, m, n)$, gives rise to a solution

$$f = \sum_{\xi \in \mathbb{Z}} \alpha(t+\xi) e^{2\pi i (lx + (m+n\xi)y + nz)}$$

$$g = \sum_{\xi \in \mathbb{Z}} \beta(t+\xi) e^{2\pi i (lx + (m+n\xi)y + nz)}.$$

That is to say, the solution given by choosing functions f and g so that the Fourier coefficient $\mathcal{F}_{\mathbf{y}}$ is equal to α and β respectively, and all other Fourier coefficients $\mathcal{F}_{\mathbf{x_0}}$, with $\mathbf{x_0} \notin \mathrm{Orb}_{\mathbf{y}}$, are equal to zero.

The coefficients of the ODE system (3.2) are analytic in $\mathbb{R}$ (in fact, in $\mathbb{C}$), and it has an irregular singularity (*i.e.* essential singularity) of order two at infinity. By standard ODE theory (*e.g.* Chapters 3 and 5 of [4]), there are two linearly independent analytic solutions of (3.2). If we consider the fundamental matrices of the ODE systems at both positive and negative infinities, they are of the form $e^{Q_0 t^2 + Q_1 t} t^a P(t^{-1})$, where $P(t^{-1})$ is a formal power series in t^{-1} and Q_0 is the diagonal matrix $diag(\pi n, -\pi n)$.

Hence, as $t \to +\infty$ in (3.2) we have two independent local solutions, one that grows like $e^{|n|\pi t^2}$ and one that decays like $e^{-|n|\pi t^2}$, and likewise for $t \to -\infty$. If we have a single solution that decays in both directions then it must be Schwartzian, though we may instead have two independent solutions that both blow up at one end while decaying at the other. Clearly, we have this situation if $B_{l,m,n} = 0$, although it is never zero when $n \neq 0$. However, we cannot always have this situation, as otherwise (3.2) would have a Schwartzian solution for every $\mathbf{y}$ and thus the elliptic system (3.1) would have infinitely many solutions which is absurd.

What we have is in essence a real version of the Stokes phenomenon problem, which asks how the asymptotic behaviour of a solution in one direction corresponds to its asymptotic behaviour in other directions. To solve this problem we are required to introduce the following theorem.

Theorem 3.4. *Let $A, B \in M_2(\mathbb{C})$ be matrices and let A have two distinct, real eigenvalues λ_1, λ_2 with $\lambda_1 > 0 > \lambda_2$ then the equation*

$$\frac{d}{dt}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (At + B)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \tag{3.3}$$

has a pair of solutions $\alpha, \beta \in \mathcal{S}(\mathbb{R})$ if and only if the following holds: Given $P \in GL(2, \mathbb{C})$

such that PAP^{-1} is diagonal and writing PBP^{-1} as $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ we have $b_2 b_3 \in (\lambda_1 - \lambda_2) \cdot \mathbb{Z}^-$.

Clearly if λ_1 and λ_2 are both positive then all pairs of solutions α, β will blow up in both the positive and negative directions, while if they are both negative all pairs α, β will decay in both directions. Note also that the arguments we use below should still apply when A has two complex eigenvalues λ_1, λ_2 with $\operatorname{Re}\lambda_1 > 0 > \operatorname{Re}\lambda_2$. Here however we will restrict our attention to the real situation to simplify the notation and also because it is sufficient for all our applications.

Proof. If we write down the second order ODE satisfied by α or β, the coefficients will involve a third order polynomial of t, and there is no efficient method known to study these types of equations. The trick is here is to simplify the above equation (3.3) slightly by left-multiplying the solution by P and adding an $e^{-\frac{1}{2}\lambda_2 t^2}$ term inside the derivative. This replaces A with a matrix with only one non-zero entry, such that our equation becomes

$$\frac{d}{dt}\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \left(\begin{pmatrix} \lambda_1 - \lambda_2 & 0 \\ 0 & 0 \end{pmatrix} t + PBP^{-1}\right)\begin{pmatrix} \psi \\ \phi \end{pmatrix}, \tag{3.4}$$

where

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = e^{-\frac{1}{2}\lambda_2 t^2} P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Since $\mathcal{S}(\mathbb{R})$ is closed under addition, and the matrix P is invertible, we can say that $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a pair of Schwartzian functions if and only if $e^{\frac{1}{2}\lambda_2 t^2}\begin{pmatrix} \psi \\ \phi \end{pmatrix}$ is a pair of Schwartzian functions. In order to complete the proof, it therefore suffices to describe when it is we have solutions ψ and ϕ, such that $e^{\frac{1}{2}\lambda_2 t^2}\psi, e^{\frac{1}{2}\lambda_2 t^2}\phi \in \mathcal{S}(\mathbb{R})$.

We can show both ϕ and ψ must satisfy a second order ODE, both of which can be solved using a Laplace integral transform:

$$\psi'' - ((\lambda_1 - \lambda_2)t + b_1 + b_4)\psi' + ((\lambda_1 - \lambda_2)b_4 t + b_1 b_4 - b_2 b_3 - (\lambda_1 - \lambda_2))\psi = 0, \tag{3.5}$$
$$\phi'' - ((\lambda_1 - \lambda_2)t + b_1 + b_4)\phi' + ((\lambda_1 - \lambda_2)b_4 t + b_1 b_4 - b_2 b_3)\phi = 0. \tag{3.6}$$

As detailed in [4], in order to find a function h that satisfies

$$(p_2 t + q_2)h'' + (p_1 t + q_1)h' + (p_0 t + q_0)h = 0,$$

we can write h as

$$h(t) = \int_C \varphi(s) e^{st} ds$$

where C is some contour in the complex plane $\mathbb{C}$.

Then, defining

$$P(s) = p_2 s^2 + p_1 s + p_0$$
$$Q(s) = q_2 s^2 + q_1 s + q_0$$

and choosing C so that

$$V(s) = \exp\left(\int^s \frac{Q(\sigma)}{P(\sigma)} d\sigma\right) e^{st}$$

takes the same value at both (possibly infinite) endpoints for all t when s parameterises the contour C, we can find a solution

$$\varphi(s) = \frac{1}{P(s)} \exp\left(\int^s \frac{Q(\sigma)}{P(\sigma)} d\sigma\right) \quad \& \quad h(t) = \int_C \frac{V(s)}{P(s)} ds. \tag{3.7}$$

In principle, there are no conditions on the choice of contour, except that $V(s)$ must take the same value at both endpoints. We could, for instance, choose C to be a loop. However, away from the zeros of $P(s)$, the function $\frac{V(s)}{P(s)}$ is holomorphic, and thus its integral around a loop is zero. This still gives us a solution to the ODE, but it is the trivial solution $h(t) = 0$. Similarly, if two contours differ only up to a deformation that avoids the zeros of $P(s)$, then we obtain the same solution from both of them. We must therefore be careful to ensure that the contours we choose lead to two distinct, non-trivial solutions.

In our specific case, first solving (3.6) for ϕ we find that

$$P_\phi(s) = (\lambda_1 - \lambda_2)(b_4 - s), \qquad Q_\phi(s) = s^2 - (b_1 + b_4)s + (b_1 b_4 - b_2 b_3),$$

which gives us the solution

$$\phi(t) = \frac{1}{\lambda_2 - \lambda_1} \int_C (s - b_4)^{\frac{b_2 b_3}{\lambda_1 - \lambda_2} - 1} \exp\left(-\frac{1}{\lambda_1 - \lambda_2}\left(\frac{s^2}{2} - b_1 s\right) + ts\right) ds \tag{3.8}$$

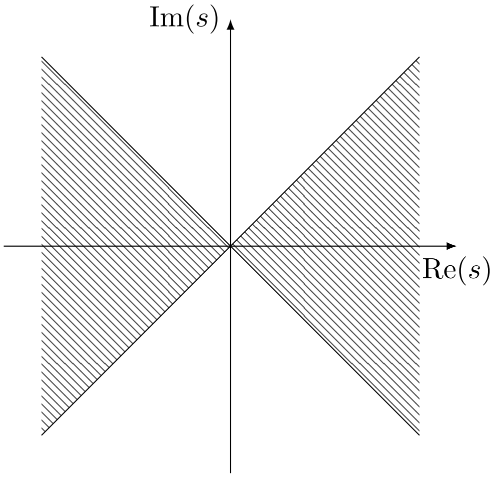
with

$$V_\phi(s) = (s - b_4)^{\frac{b_2 b_3}{\lambda_1 - \lambda_2}} \exp\left(-\frac{1}{\lambda_1 - \lambda_2}\left(\frac{s^2}{2} - b_1 s\right) + ts\right).$$

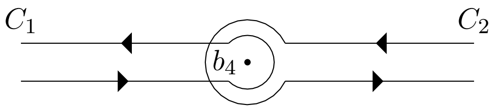
Let us first consider the case when $\frac{b_2 b_3}{\lambda_1 - \lambda_2} \notin \mathbb{Z}$. Notice that the definition of V_ϕ

involves a non-integer power, and is therefore only well-defined after we make a choice of which branch to work on. The exact choice is inconsequential, but the fact that a choice has been made is not. Since $P(b_4) = 0$, we might like to choose the contour C to be a loop around the point $s = b_4$, but such a contour would have its endpoints lying in two different branches of $V(s)$. This means $V(s)$ does not take the same value at both endpoints.

Instead, we will make use of the fact that the function V_ϕ tends to zero as s grows large within the shaded regions, regardless of branch.



This means that we can define a contour C_1, starting and ending at negative real infinity, encircling the point $s = b_4$, for which $V_\phi = 0$ at both endpoints. Likewise, we can define a contour C_2, starting and ending at positive real infinity, encircling the point $s = b_4$. Let ϕ_1 and ϕ_2 denote the solutions to (3.4) arising from the these contours.



This is in essence the same as in the treatment for the physicists' Hermite equation from the mathematical appendices of [21], and as in Appendix §a there, we will use a substitution to explore the behaviour of solutions as $t \to \pm\infty$.

Let us define

$$u := \frac{s - b_1}{\lambda_1 - \lambda_2} - t.$$

Substituting u into (3.8) in place of s, our expression for ϕ becomes

$$\phi(t) = -\exp\left(\frac{(b_1 + (\lambda_1 - \lambda_2)t)^2}{2(\lambda_1 - \lambda_2)}\right) \int_{\tilde{C}} ((\lambda_1 - \lambda_2)(u + t) + b_1 - b_4)^{\frac{b_2 b_3}{\lambda_1 - \lambda_2} - 1} e^{-\frac{1}{2}(\lambda_1 - \lambda_2)u^2} du.$$

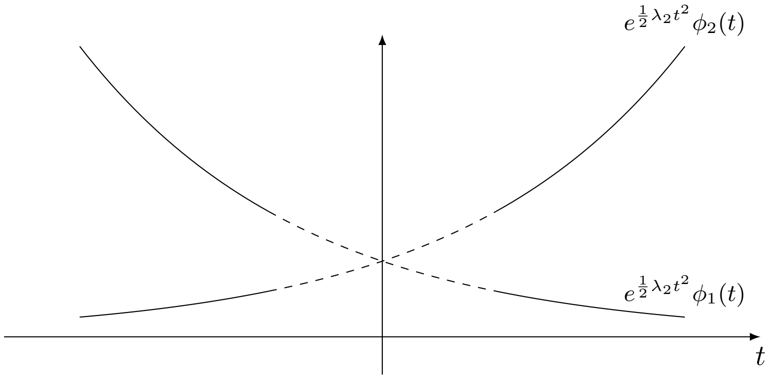
Use $\tilde{C}_1$ and $\tilde{C}_2$ to denote the new contours transformed from C_1 and C_2 after substitution of u. We can see that as $t \to +\infty$, $\tilde{C}_1$ will shift to the left causing the integral along $\tilde{C}_1$

$$\int_{\tilde{C}} ((\lambda_1 - \lambda_2)(u+t) + b_1 - b_4)^{\frac{b_2 b_3}{\lambda_1 - \lambda_2} - 1} e^{-\frac{1}{2}(\lambda_1 - \lambda_2)u^2} du.$$

to decay like $e^{-\frac{1}{2}(\lambda_1-\lambda_2)t^2}$ and hence $e^{\frac{1}{2}\lambda_2 t^2}\phi_1$ will decay at the rate of $e^{\frac{1}{2}\lambda_2 t^2}$.

$\tilde{C}_2$, on the other hand, will tend towards a contour along the whole horizontal direction. The integral along $\tilde{C}_2$ only changes with polynomial speed, meaning $e^{\frac{1}{2}\lambda_2 t^2}\phi_2$ grows like $e^{\frac{1}{2}\lambda_1 t^2}$.

As $t \to -\infty$ the contours shift to the right instead. This results in $e^{\frac{1}{2}\lambda_2 t^2}\phi_1$ now being the one to grow like $e^{\frac{1}{2}\lambda_1 t^2}$, while $e^{\frac{1}{2}\lambda_2 t^2}\phi_2$ is the one decaying like $e^{\frac{1}{2}\lambda_2 t^2}$. Clearly this means any linear combination of these two functions will blow up at either $\infty, -\infty$ or both.



If $\frac{b_2 b_3}{\lambda_1 - \lambda_2} \in \mathbb{Z}^- \cup \{0\}$, then the value of V_ϕ no longer depends upon a choice of branch. The integrals along the horizontal directions of the paths of integration cancel, and the two integrals along $\tilde{C}_1$ and $\tilde{C}_2$ reduce to an integral along a loop around $s = b_4$, or equivalently $u = \frac{b_4 - b_1}{\lambda_1 - \lambda_2}' - t$. We will call this loop $\tilde{C}_3$. If we choose our contour to be $\tilde{C}_3$ we find that the integral part of our expression for ϕ decays like $e^{-\frac{1}{2}(\lambda_1-\lambda_2)t^2}$ in both directions. The solution $\phi(t)$ therefore grows at most as e^{Kt} (when $b_4 = 0$, it is essentially an Hermite polynomial), and $e^{\frac{1}{2}\lambda_2 t^2}\phi$ decays as $e^{\frac{1}{2}\lambda_2 t^2}$ at both ends. We have found a Schwartzian function!

The same argument applies to (3.5) for ψ when $\frac{b_2 b_3}{\lambda_1 - \lambda_2} + 1 \in \mathbb{Z}^- \cup \{0\}$, *i.e.* $\frac{b_2 b_3}{\lambda_1 - \lambda_2} \in \mathbb{Z}^-$. In this case, $e^{\frac{1}{2}\lambda_2 t^2}\psi$ also decays as $e^{\frac{1}{2}\lambda_2 t^2}$ at both ends. Furthermore, we can take the function ψ obtained by integrating around $\tilde{C}_3$ and plug it into the first relation of (3.4) to find the corresponding function ϕ. It turns out this ϕ solves the second relation of (3.4) and also (3.6), in fact it is equal to the ϕ in the previous paragraph, up to multiplication by a constant. The pair (ϕ, ψ) therefore gives rise to a pair of Schwartzian functions (α, β) solving (3.3), whenever $\frac{b_2 b_3}{\lambda_1 - \lambda_2} \in \mathbb{Z}^-$. We remark that this is the only $\mathcal{S}(\mathbb{R})$ solution of (3.3) as by ODE theory there is only one solution which decays as $e^{\lambda_2 t^2}$ at $+\infty$ (or $-\infty$).

Lastly, we study the solution of (3.5) when $\frac{b_2 b_3}{\lambda_1 - \lambda_2} + 1 \in \mathbb{Z}^+$. Contours C_1, C_2 and C_3 all give the trivial solution. Instead we define the contours C_4 and C_5 to be the lines parallel to the real axis, running from b_4 to $-\infty$ and $+\infty$ respectively. These satisfy the condition that $V_\psi(s) = (s - b_4)^{\frac{b_2 b_3}{\lambda_1 - \lambda_2} + 1} \exp\Big(-\frac{1}{\lambda_1 - \lambda_2}\Big(\frac{s^2}{2} - b_1 s\Big) + ts\Big) = 0$ at both endpoints and so give rise to two independent solutions ψ_4 and ψ_5 of (3.5). We have (as in (3.8) for ϕ)

$$\psi_{4,5}(t) = \frac{1}{\lambda_2 - \lambda_1} \int_{C_{4,5}} (s - b_4)^{\frac{b_2 b_3}{\lambda_1 - \lambda_2}} \exp\left(-\frac{1}{\lambda_1 - \lambda_2}\left(\frac{s^2}{2} - b_1 s\right) + ts\right) ds.$$

Then we can use exactly the same argument as we did for the case when $\frac{b_2 b_3}{\lambda_1 - \lambda_2} \notin \mathbb{Z}$ to see that $e^{\frac{1}{2}\lambda_2 t^2}\psi_4$ decays like $e^{\frac{1}{2}\lambda_2 t^2}$ as $t \to +\infty$ and grows like $e^{\frac{1}{2}\lambda_1 t^2}$ as $t \to -\infty$ whilst the opposite is true of $e^{\frac{1}{2}\lambda_2 t^2}\psi_5$. This means when $\frac{b_2 b_3}{\lambda_1 - \lambda_2} + 1 \in \mathbb{Z}^+$ all solutions blow up in either the positive or negative directions. Thus, there are no $\mathcal{S}(\mathbb{R})$ solutions. $\square$

It should be noted that in this last case we could write our solutions explicitly in terms of the error function

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

First by directly calculating ψ_4 and ψ_5 when $\frac{b_2 b_3}{\lambda_1 - \lambda_2} = 0$ and 1. Then, using integration by parts, we could find a recurrence relation allowing us to write solutions for all other positive integer values of $\frac{b_2 b_3}{\lambda_1 - \lambda_2}$ in terms of these first two.

Remark 3.5. *In the preceding theorem we considered solutions as functions of a real variable t. If instead we allow t to take complex values we see that the study of asymptotic behaviour as $t \to \pm\infty$ is really just a restriction of the Stokes phenomenon to $\mathbb{R}$. It would be interesting to see if this method could be used to describe the complete picture of the Stokes phenomenon for this linear system.*

Applying Theorem 3.4 to equation (3.2), when $n > 0$, we have $\lambda_1 = 2\pi n, \lambda_2 = -2\pi n$ and

$$PB_{l,m,n}P^{-1} = 2\pi \begin{pmatrix} m - \frac{na}{b} + \frac{b}{8\pi}i & l - \frac{n}{b}i - \frac{b}{8\pi}i \\ l + \frac{n}{b}i - \frac{b}{8\pi}i & -m + \frac{na}{b} + \frac{b}{8\pi}i \end{pmatrix},$$

where $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. In order for us to have a pair of solutions $\alpha, \beta \in \mathcal{S}(\mathbb{R})$ we must have

$$\pi\left(l - \frac{n}{b}i - \frac{b}{8\pi}i\right)\left(l + \frac{n}{b}i - \frac{b}{8\pi}i\right) \in n\mathbb{Z}^-.$$

The imaginary part of the left hand side is $-\frac{b}{4}li$, so we can only have solutions when l is zero. Then looking at the real part and setting $l = 0$, we also need b to satisfy

$$\pi\left(\left(\frac{n}{b}\right)^2 - \left(\frac{b}{8\pi}\right)^2\right) \in n\mathbb{Z}^-.$$

That is to say the only time we can have Schwartzian solutions to (3.2) is when there is some $u \in \mathbb{Z}^-$ such that b is a solution to

$$b^4 + 64\pi nub^2 - 64\pi^2 n^2 = 0, \tag{3.9}$$

or in terms of $d = \frac{b}{8\pi}$,

$$\left(8\pi d^2\right)^2 + 8nu\left(8\pi d^2\right) - n^2 = 0. \tag{3.10}$$

For (3.10) to hold requires $8\pi d^2 \in \mathbb{Z}[\sqrt{D}]$ for some integer $D > 0$. For example, since π is a transcendental number, no rational number $d = \frac{p}{q}$ can satisfy equation (3.10) for any choice of $u, n \in \mathbb{Z}$.

When $n < 0$, we have essentially the same discussion, with n replaced with $|n|$ in all the above relations. Note that whether or not a solution exists is independent of the value of m.

3.1.2 Case 2: $|\mathrm{Orb}_{\mathbf{y}}| = 1$

Now let $\mathbf{y} = (l, m, 0)$ for some $l, m \in \mathbb{Z}$. For these values of $\mathbf{y}$ the functions $\mathcal{F}_{\mathbf{y}}(f)$ and $\mathcal{F}_{\mathbf{y}}(g)$ are periodic, and so we can apply an additional Fourier expansion to the ODE system (3.2). This is equivalent to applying $\mathcal{G}_{k,\mathbf{y}}$ for all $k \in \mathbb{Z}$ to the PDEs (3.1), which Prop. 2.16 tells us will yield the following system of algebraic equations

$$\begin{pmatrix} -m & l + ik - \frac{b}{4\pi}i \\ l - ik & m \end{pmatrix}\begin{pmatrix} \mathcal{G}_{k,\mathbf{y}}(f) \\ \mathcal{G}_{k,\mathbf{y}}(g) \end{pmatrix} = 0.$$

which the coefficients $\mathcal{G}_{k,l,m,0}(f)$ and $\mathcal{G}_{k,l,m,0}(g)$ must satisfy if f and g are to be a pair of solutions. In fact, any pair of complex numbers $\alpha, \beta \in \mathbb{C}$ satisfying the above will produce a pair of solutions

$$f = \alpha\, e^{2\pi i(kt+lx+my)}$$

$$g = \beta\, e^{2\pi i(kt+lx+my)}.$$

It is only possible for non-trivial solutions to exist when

$$\det \begin{pmatrix} -m & l + ik - \frac{b}{4\pi} i \\ l - ik & m \end{pmatrix} = 0 \tag{3.11}$$

The rank of this matrix is always at least 1, so whenever it has zero determinant we obtain a single independent solution.

Looking first at the imaginary part of (3.11), we see that $l = 0$. Applying this to the real part gives us

$$m^2 + k^2 + \frac{b}{4\pi} k = 0.$$

Setting $d = \frac{b}{8\pi}$, we can rearrange this to get the condition

$$m^2 + (k - d)^2 = d^2. \tag{3.12}$$

Each $k, m \in \mathbb{Z}$ satisfying the above corresponds to a single independent solution to (3.1). These solutions are given by

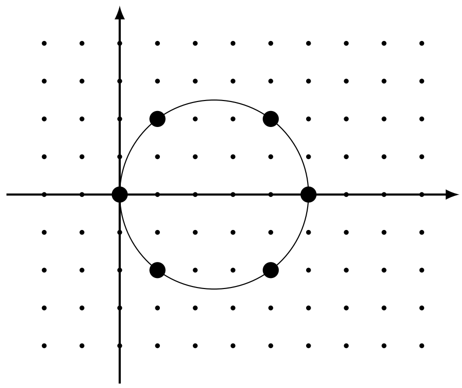
$$f = mC_1 e^{2\pi i(kt+my)}, \quad g = ikC_1 e^{2\pi i(kt+my)},$$

for any $C_1 \in \mathbb{C}$, except in the case when $m = k = 0$ for which the above pair is just the trivial solution $f = g = 0$. The non-trivial pair of solutions corresponding to this case is instead given by

$$f = C_2, \quad g = 0$$

for any $C_2 \in \mathbb{C}$.

So, how many solutions $k, m \in \mathbb{Z}$ does (3.12) actually have, for any given $d = \frac{b}{8\pi} \in \mathbb{R}\backslash\{0\}$? This is equivalent to asking how many lattice points lie on a circle with centre $(d, 0)$ and radius d. For instance, when $d = \frac{5}{2}$ we have 6 solutions as shown below.



When d is an integer this problem is very well understood and the number of such integer pairs is denoted $r_2(d^2)$, see for instance [12]. First we write d^2 as a unique product of prime numbers

$$d^2 = 2^{\alpha_0} p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t},$$

where $p_i \equiv 3 \mod 4$ for all i and $q_j \equiv 1 \mod 4$ for all j. The number of solutions is then given by

$$h^{0,1}_{\bar{\partial}} = 4(\beta_1 + 1)(\beta_2 + 1) \dots (\beta_t + 1).$$

This reveals the interesting fact that by changing our choice of b we can make $h^{0,1}$ become arbitrarily large. It should be noted that if any of the powers of the p_i's were odd then we would not have any solutions, but since we are looking at a square number the powers are guaranteed to be even.

Moreover, when $d = \frac{p}{q}$ with $\gcd(p, q) = 1$ and q is small, we can also compute the number of solutions.

Theorem 3.6. *For our family of almost complex structures $J_{a,b}$ and the almost Hermitian metrics $g_{a,b}$ on KT^4, whenever $d = \frac{p}{q} \in \mathbb{Q}$, the Hodge number $h^{0,1}_{\bar{\partial}}$ is equal to the number of the integer pairs (k, m) satisfying* (3.12)*. Furthermore, if $\gcd(p, q) = 1$ and $q \leq 5$, we have*

$$h^{0,1}_{\bar{\partial}} = \begin{cases} 4(\beta_1 + 1)(\beta_2 + 1) \dots (\beta_t + 1) & \textit{if } q = 1, \\ 2(\beta_1 + 1)(\beta_2 + 1) \dots (\beta_t + 1) & \textit{if } q = 2, \\ (\beta_1 + 1)(\beta_2 + 1) \dots (\beta_t + 1) & \textit{if } q = 3, \\ (\beta_1 + 1)(\beta_2 + 1) \dots (\beta_t + 1) & \textit{if } q = 4, \\ (\beta_1 + 1)(\beta_2 + 1) \dots (\beta_t + 1) & \textit{if } q = 5. \end{cases}$$

where $p^2 = 2^{\alpha_0} p_1^{\alpha_1} \dots p_s^{\alpha_s} q_1^{\beta_1} \dots q_t^{\beta_t}$ *with* $p_i \equiv 3 \mod 4$ *for all* i *and* $q_j \equiv 1 \mod 4$ *for all* j.

Proof. First, any rational number $d = \frac{p}{q}$ cannot solve (3.10). Hence, all solutions to the PDE system (3.1) are provided by linear combinations of the solutions derived from the case when $|\mathrm{Orb}_{\mathbf{y}}| = 1$, *i.e.* the number of lattice points satisfying (3.12). In the following, we compute this number.

In the proof, we always write $q(k-d) = qk' - d'$ where $k' = k - \lfloor \frac{p}{q} \rfloor$ and $d' = q\{\frac{p}{q}\}$. By abusing notation, we usually write k for k' in the following.

The case of $q = 1$ is solved above.

When $q = 2$, then p is odd. We can rewrite (3.12) as $(2k-1)^2 + (2m)^2 = p^2$. For any integer solution (x, y) of $x^2 + y^2 = p^2$, one and only one from (x, y) and (y, x) is of the type $(2k-1, 2m)$. Thus, $h_{\bar{\partial}}^{0,1}$ is half of the number of lattice points on $x^2 + y^2 = p^2$.

When $q = 3$, then p is not divisible by 3. Rewrite (3.12) as $(3k-d')^2 + (3m)^2 = p^2$, where d' is 1 or 2. For any integer solution (x, y) of $x^2 + y^2 = p^2$, one and only one among (x, y), $(x, -y)$, (y, x) and $(-y, x)$ is of the type $(3k - d', 3m)$ for a given d'. Thus $h_{\bar{\partial}}^{0,1}$ is a quarter of the number of lattice points on $x^2 + y^2 = p^2$.

When $q = 4$, then $p \equiv 1$ or $3 \pmod 4$. Rewrite (3.12) as $(4k - d')^2 + (4m)^2 = p^2$, where d' is 1 or 3. We look at the equation $x^2 + y^2 = p^2$ modulo 8, then the even term has to be a multiple of 4. Hence, for any integer solution (x, y) of $x^2 + y^2 = p^2$, one and only one among (x, y), $(x, -y)$, (y, x) and $(-y, x)$ is of the type $(4k - d', 4m)$ for a given $d' = 1$ or 3. Thus $h_{\bar{\partial}}^{0,1}$ is a quarter of the number of lattice points on $x^2 + y^2 = p^2$.

When $q = 5$, then p is not divisible by 5. Rewrite (3.12) as $(5k-d')^2 + (5m)^2 = p^2$, where d' is $1, 4$ or $2, 3$. We look at the equation $x^2 + y^2 = p^2$ modulo 5, the left hand side is 1 mod 5 if $d' = 1, 4$, or is 4 mod 5 if $d' = 2, 3$. In both cases, for any integer solution (x, y) of $x^2 + y^2 = p^2$, one and only one among (x, y), $(x, -y)$, (y, x) and $(-y, x)$ is of the type $(5k - d', 5m)$ for a given d'. Thus $h_{\bar{\partial}}^{0,1}$ is a quarter of the number of lattice points on $x^2 + y^2 = p^2$. □

The above argument cannot continue for $q \geq 6$ as we have $4^2 + 3^2 = 5^2 + 0^2 = 5^2$. It would be interesting to know in general how many integer solutions of (3.12) there are.

Corollary 3.7. *For any nonnegative integer* $n = 4K, 2K$ *or* K *where* K *is odd, there is an almost complex structure and compatible almost Hermitian metric on* KT^4 *whose* $h_{\bar{\partial}}^{0,1} = n$.

Proof. When $K = 1$, we take $b = 8\pi, 4\pi, 2\pi$ respectively.

When $K > 1$, we take $b = \frac{8\pi \cdot 5^{\frac{K-1}{2}}}{q}$ where $q = 1, 2, 3$ respectively. These are Schinzel circles [26]. $\square$

We notice that for the vast majority of members of the family of almost complex structures $J_{a,b}$, we have $h^{0,1}_{\bar\partial} = 1$ as this holds for any irrational $d = \frac{b}{8\pi}$ which does not solve (3.10). On the other hand, we can compute $h^{0,1}_{\bar\partial}$ for those d that do solve (3.10).

Proposition 3.8. *If some d (with $8\pi d^2 \in \mathbb{Z}[\sqrt{D}]$ for some $D \in \mathbb{Z}^+$) solves* (3.10) *for a given $n \in \mathbb{Z} \setminus \{0\}$ and a certain $u \in \mathbb{Z}^-$, then $h^{0,1}_{\bar\partial} = 2|n| + 1$ for the almost complex structure $J_{a,8\pi d}$, $\forall a \in \mathbb{R}$, with its standard orthonormal metric on KT^4.*

Proof. Notice for $n \neq 0$, $\pm n$ gives the same equation (3.10) to solve where n is replaced by $|n|$. Hence, without loss, we can assume $n > 0$.

For any d, there is only one $n > 0$ that could solve (3.10). If there is another $N > 0$ and $U \in \mathbb{Z}^-$ solving (3.10) for d, then

$$n(2u + \sqrt{4u^2 + 1}) = N(2U + \sqrt{4U^2 + 1}).$$

This holds only when $n = N$ and $u = U$.

Hence, by Theorem 3.4, for each integer value of m between 0 and $n - 1$, there will be a Schwartzian solution to (3.2). Similarly, there will be n Schwartzian solutions when we start with $-n$.

Moreover, we have one and only one solution contributed by $(l, m) = (0, 0)$ in the $|\mathrm{Orb}_{\mathbf{y}}| = 1$ case as d is irrational.

In total, we have $2|n| + 1$ dimensions of solutions to (3.1). This implies $h^{0,1}_{\bar\partial} = 2|n| + 1$. $\square$

In particular, Corollary 3.7 and Proposition 3.8 together imply the following theorem

Theorem 3.9. *$h^{0,1}_{\bar\partial}$ can be computed for all members of the continuous family of non-integrable almost Kähler structures given by $(J_{a,b}, g_{a,b})$, $a, b \in \mathbb{R}$, $b \neq 0$, on the Kodaira-Thurston manifold. Furthermore, for any $n \in \mathbb{Z}^+$ such that $8 \nmid n$, there is a b such that $h^{0,1}_{\bar\partial} = n$.*

3.1.3 Varying the almost Hermitian metric

Now that we have demonstrated a method for calculating $h^{0,1}_{\bar\partial}$ on the Kodaira-Thurston manifold, let us consider a broad family of almost Hermitian metrics g_λ compatible with the almost structure $J_{a,b}$, for some $a, b \in \mathbb{R}$, $b \neq 0$.

Definition 3.10. Define a family of metrics by

$$g_\lambda = \phi^1 \otimes \bar{\phi}^1 + \bar{\phi}^1 \otimes \phi^1 + \lambda(\phi^2 \otimes \bar{\phi}^2 + \bar{\phi}^2 \otimes \phi^2),$$

or equivalently the fundamental form

$$\begin{aligned} \omega_\lambda &= i(\phi^1 \wedge \bar{\phi}^1 + \lambda \phi^2 \wedge \bar{\phi}^2) \\ &= 2(dt \wedge dx - \lambda dy \wedge dz). \end{aligned}$$

with $\lambda \in \mathbb{R}$, $\lambda > 0$. The dependence of g_λ on a and b is omitted here for the sake of notational simplicity.

Clearly, we have $d\omega_\lambda = 0$ for all λ, thus the almost Hermitian metrics g_λ describe a family of almost Kähler structures. These are the metrics for which V_1 and $\frac{1}{\sqrt{\lambda}} V_2$ are orthonormal, and the resulting Hodge star is described by its action on ϕ^1 and ϕ^2 as follows:

$$\begin{aligned} * : \phi^1 &\mapsto \lambda \phi^{12\bar{2}} \\ \phi^2 &\mapsto -\phi^{12\bar{1}} \end{aligned}$$

Example 3.11. As before, we write a smooth $(0,1)$-form $s \in \mathcal{A}^{0,1}(KT^4)$ as $f\bar{\phi}^1 + g\bar{\phi}^2$ for some pair of smooth functions $f, g \in C^\infty(KT^4)$. Then $s \in \mathcal{H}^{0,1}_{\bar{\partial}}$ if and only if $\bar{\partial} s = 0$ and $\bar{\partial}^* s = 0$. From these conditions we obtain the PDEs

$$\begin{cases} -\bar{V}_2(f) + \bar{V}_1(g) + g\frac{b}{4} &= 0, \\ \rho V_1(f) + V_2(g) &= 0. \end{cases} \tag{3.13}$$

Again we will split our calculations up into the two cases $|\mathrm{Orb}_{\mathbf{y}}| = \infty$ and $|\mathrm{Orb}_{\mathbf{y}}| = 1$.

In case 1, we have $\mathbf{y} = (l, m, n) \in \mathbb{Z}^3$, with $n \neq 0$ and $0 \leq m < |n|$. Applying $\mathcal{F}_{\mathbf{y}}$ to (3.13) we conclude that the Fourier coefficients of f and g must satisfy

$$\frac{d}{dt} \begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix} = (A_n t + B_{l,m,n}) \begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix} \tag{3.14}$$

where

$$A_n = 2\pi n \begin{pmatrix} 0 & \frac{1}{\lambda} \\ 1 & 0 \end{pmatrix}, \qquad B_{l,m,n} = 2\pi \begin{pmatrix} l & \frac{1}{\lambda}\left(m - \frac{a-i}{b} n\right) \\ m - \frac{a+i}{b} n & i\frac{b}{4\pi} - l \end{pmatrix}.$$

By choosing a matrix

$$P = \frac{\sqrt{2}}{2}\begin{pmatrix} \sqrt{\lambda} & 1 \\ \sqrt{\lambda} & -1 \end{pmatrix},$$

which diagonalises A_n, we can calculate

$$PA_nP^{-1} = \frac{2\pi n}{\sqrt{\lambda}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$PB_{l,m,n}P^{-1} = 2\pi\begin{pmatrix} \frac{1}{\sqrt{\lambda}}\left(m - \frac{an}{b}\right) + \frac{b}{8\pi}i & l - \frac{n}{b\sqrt{\lambda}}i - \frac{b}{8\pi}i \\ l + \frac{n}{b\sqrt{\lambda}}i - \frac{b}{8\pi}i & -\frac{1}{\sqrt{\lambda}}\left(m - \frac{an}{b}\right) + \frac{b}{8\pi}i \end{pmatrix}.$$

Theorem 3.4 then tells us we have a Schwartzian solution to (3.14) if and only if

$$4\pi^2\left(l - \frac{n}{b\sqrt{\lambda}}i - \frac{b}{8\pi}i\right)\left(l + \frac{n}{b\sqrt{\lambda}}i - \frac{b}{8\pi}i\right) \in \frac{4\pi n}{\sqrt{\lambda}}\mathbb{Z}^-.$$

The imaginary part of the left hand side is $-lb\pi$ so we are forced to set $l = 0$, this leaves us with the condition that for some $u \in \mathbb{Z}^+$

$$b^4\lambda - 64\pi nub^2\sqrt{\lambda} - 64n^2\pi^2 = 0,$$

or alternatively if we set $d = \frac{b}{8\pi}$

$$\left(8\pi\sqrt{\lambda}d^2\right)^2 - 8nu\left(8\pi\sqrt{\lambda}d^2\right) - n^2 = 0. \tag{3.15}$$

This can be solved to show that there are no solutions unless $8\pi\sqrt{\lambda}d^2$ can be written as quadratic integer in $\mathbb{Z}[\sqrt{D}]$ for some $D \in \mathbb{Z}$.

Now considering case 2, we have $\mathbf{y} = (l, m, 0)$, with $l, m \in \mathbb{Z}$. Applying $\mathcal{G}_{k,\mathbf{y}}$ to (3.13), for $k \in \mathbb{Z}$, we see that f and g must satisfy the algebraic equation

$$\begin{pmatrix} -m & l - ik - \frac{b}{4\pi}i \\ \lambda(l - ik) & m \end{pmatrix}\begin{pmatrix} \mathcal{G}_{k,\mathbf{y}}(f) \\ \mathcal{G}_{k,\mathbf{y}}(g) \end{pmatrix} = 0$$

This gives us an independent solution whenever

$$m^2 + \lambda\left(l^2 + k^2 - \frac{b}{4\pi}k - \frac{b}{4\pi}li\right) = 0$$

The imaginary part tells us $l = 0$. Applying this to the real part, and setting $d = \frac{b}{8\pi}$ we see that we have a solution for every $k, m \in \mathbb{Z}$ satisfying

$$(k-d)^2 + \left(\frac{m}{\sqrt{\lambda}}\right)^2 = d^2. \tag{3.16}$$

The corresponding solutions being given by

$$f = mC_1 e^{2\pi i(kt+my)}, \; g = \rho ik C_1 e^{2\pi i(kt+my)}$$

for $C_1 \in \mathbb{C}$, except when $k = m = 0$, in which case we obtain the solution

$$f = C_2 \quad g = 0$$

for $C_2 \in \mathbb{C}$

Bringing together the two cases described above we obtain the following result, thereby answering the Kodaira-Spencer question in the negative.

Theorem 3.12. *On compact almost complex 4-manifolds, $h^{0,1}_{\bar{\partial}}$ is not in general invariant of the choice of almost Hermitian metric. In particular, on the Kodaira-Thurston manifold we can find a family of almost Hermitian metrics, compatible with a fixed almost complex structure, over which $h^{0,1}_{\bar{\partial}}$ takes multiple different values.*

Proof. Consider the example of the Kodaira-Thurston manifold equipped with the almost Hermitian structure $(KT^4, J_{a,b}, g_\lambda)$ calculated above. We can fix the almost complex structure by setting $d = \frac{b}{8\pi} = 1$ and choosing some $a \in \mathbb{R}$.

First, we will consider the value of $h^{0,1}$ for the metric given by $\lambda = 1$. In case 1, where $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, we get no solutions, since $8\pi\sqrt{\lambda}d^2 = 8\pi$ is not a quadratic integer. In case 2, where $|\mathrm{Orb}_{\mathbf{y}}| = 1$, the equation

$$(k-1)^2 + m^2 = 1$$

has four integer solutions: $(k, m) = (0, 0), (1, 1), (1, -1), (2, 0)$. Therefore, when $\lambda = 1$, the space $\mathcal{H}^{0,1}_{\bar{\partial}}$ is generated by the four elements

$$\bar{\phi}^1, \quad e^{2\pi i(t+y)}(\bar{\phi}^1 + i\bar{\phi}^2), \quad e^{2\pi i(t-y)}(-\bar{\phi}^1 + i\bar{\phi}^2), \quad e^{4\pi iy}\bar{\phi}^2$$

and thus $h^{0,1} = 4$

Now, we consider the the value of $h^{0,1}$ when $\lambda = \frac{1}{4}$. Again, in case 1, we get no solutions as $8\pi\sqrt{\lambda}d^2 = 4\pi$ is not a quadratic integer. In case 2 however, the equation

$$(k-1)^2 + 4m^2 = 1$$

only has two integer solutions: $(k,m) = (0,0),(2,0)$. Therefore, when $\lambda = \frac{1}{4}$, the space $\mathcal{H}^{0,1}_{\bar{\partial}}$ is generated by

$$\bar{\phi}^1, \quad e^{4\pi i y}\bar{\phi}^2$$

and thus $h^{0,1} = 2 \neq 4$. □

Furthermore, since g_λ describes a family of almost Kähler metrics, the above proof shows that $h^{0,1}_{\bar{\partial}}$ may take different values even when we restrict our attention to almost Kähler metrics. Thus yielding the corollary:

Corollary 3.13. *On compact almost complex 4-manifolds, $h^{0,1}_{\bar{\partial}}$ is not invariant of the choice of almost Kähler metric.*

3.2 $h^{1,1}_{\bar{\partial}}$ on compact 4-manifolds

Now, if we are aiming for a complete description of the metric invariance of the Hodge numbers $h^{p,q}_{\bar{\partial}}$ on almost complex 4-manifolds, it only remains to consider the behaviour of $h^{1,1}_{\bar{\partial}}$. To that end in this section we will need to make use of the concept of a Gauduchon metric.

Definition 3.14. Given an almost complex manifold with real dimension $2n$, we say that an Hermitian metric is *Gauduchon* when its corresponding fundamental form ω satisfies the condition

$$\bar{\partial}\partial(\omega^{n-1}) = 0.$$

In particular, on almost complex 4-manifolds, a Gauduchon metric satisfies $\bar{\partial}\partial\omega = 0$. A classical result of Gauduchon (see [8]) tells us that

Proposition 3.15. *On a compact almost complex manifold, any conformal class of metrics contains a unique (up to uniform scaling) Gauduchon metric.*

To see why Gauduchon metrics are useful for determining $h^{1,1}_{\bar{\partial}}$, consider the following proposition, proven in [29].

Proposition 3.16. *On any almost complex manifold with real dimension $2n$, $\mathcal{H}^{p,q}_{\bar{\partial}}$ is a conformal invariant whenever $p+q=n$.*

Proof. Recall that a (p,q)-form s is in $\mathcal{H}^{p,q}_{\bar\partial}$ if and only if $\bar\partial s = 0$ and $\partial * s = 0$. The only metric dependence contained within these two conditions comes from the Hodge star in the second condition, it therefore suffices to show that the Hodge stars arising from two conformal metrics have the same effect when acting on a (p,q)-form with $p+q=n$.

Let g and $\tilde g$ be conformal metrics. We can find an everywhere positive function Φ such that $\tilde g = \Phi g$. We can show that the two Hodge star operators resulting from g and $\tilde g$ differ from each other in the following way:

$$*_{\tilde g} = \Phi^{p+q-n} *_g$$

when acting on a (p,q)-form. In particular, $*_{\tilde g} = *_g$ when $p+q=n$. $\square$

In the case of almost complex 4-manifolds, we can see that $\mathcal{H}^{1,1}_{\bar\partial}$ is a conformal invariant. When determining the value of $h^{1,1}_{\bar\partial}$, we may therefore assume we are working with a Gauduchon metric without loosing any generality.

In [29], Tardini and Tomassini prove the following characterisation of $\mathcal{H}^{1,1}_{\bar\partial}$:

Theorem 3.17. *On any compact almost complex 4-manifold, equipped with a compatible Gauduchon metric, we can write*

$$\mathcal{H}^{1,1}_{\bar\partial} = \{a\omega + \gamma \,|\, a \in \mathbb{C}, *\gamma = -\gamma, id^c\gamma = ad\omega\} \tag{3.17}$$

where ω is the corresponding fundamental form.

Here we define $d^c := J^{-1}dJ$ with J acting on a (p,q)-form as multiplication by i^{p-q}.

Using this characterisation it is possible to put bounds on the possible values of $h^{1,1}_{\bar\partial}$.

Theorem 3.18. *On a compact almost Hermitian 4-manifold we have either $h^{1,1}_{\bar\partial} = b_-$ or $b_- + 1$.*

Proof. Since $h^{1,1}_{\bar\partial}$ is a conformal invariant we assume without loss of generality that the metric is Gauduchon. Denote by ω the fundamental form.

If s denotes a d-harmonic anti-self-dual $(1,1)$-form, *i.e.* a $(1,1)$-form for which $(dd^* + d^*d)s = 0$ and $*s = -s$, then by the same arguments as in Prop. 1.22 we know that $ds = 0$ and $d^*s = 0$. In particular $\bar\partial^* s = 0$ and $\bar\partial s = 0$. We therefore obtain the inclusion

$$\mathcal{H}^-_g \otimes \mathbb{C} \subseteq \mathcal{H}^{1,1}_{\bar\partial}$$

where $\mathcal{H}^-_g$ denotes the space of real-valued d-harmonic anti-self-dual forms. When this inclusion is an equality then clearly we have $h^{1,1}_{\bar\partial} = b_-$. Suppose instead that $\mathcal{H}^{1,1}_{\bar\partial}$ has some element $a_0\omega + \gamma_0$ which is not in $\mathcal{H}_g^- \otimes \mathbb{C}$. Here a_0 is a constant and γ_0 is an anti-self-dual form satisfying $id^c\gamma_0 = a_0 d\omega$. Note that a_0 cannot be zero, as that would leave us with a d-harmonic anti-self-dual form. A general element of $\mathcal{H}_{\bar\partial}^{1,1}$ given by $a\omega+\gamma$ can then be rewritten as an element of $\mathcal{H}_g^- \otimes \mathbb{C}$ plus a multiple of the single additional element $a_0\omega + \gamma_0$

$$a\omega + \gamma = \frac{a}{a_0}\left(a_0\omega + \gamma_0\right) + \frac{1}{a_0}\left(a_0\gamma - a\gamma_0\right),$$

thus giving us $h_{\bar\partial}^{1,1} = b_- + 1$.

To see that the anti-self-dual form $a_0\gamma - a\gamma_0$ is indeed d-harmonic, first note that $d^c(a_0\gamma - a\gamma_0) = a_0 d^c\gamma - a d^c\gamma_0 = 0$. Then, since $d^c = J^{-1}dJ$ and J is the identity when acting on $(1,1)$-forms, it follows that $d(a_0\gamma - a\gamma_0) = 0$. As our form is anti-self-dual we therefore also have $d * (a_0\gamma - a\gamma_0) = 0$. □

Corollary 3.19. *On a compact almost complex 4-manifold endowed with a fundamental form ω corresponding to a Gauduchon metric, we have $h_{\bar\partial}^{1,1} = b_- + 1$ if and only if there exists an anti-self-dual $(1,1)$-form γ satisfying the equation*

$$id^c\gamma = d\omega. \tag{3.18}$$

Proof. If such a γ exists then $\omega + \gamma$ is $\bar\partial$-harmonic, along with b_- many linearly independent elements of $\mathcal{H}_g^- \otimes \mathbb{C}$, therefore $h_{\bar\partial}^{1,1} = b_- + 1$.

Conversely, if $h_{\bar\partial}^{1,1} = b_- + 1$, then there must be some form in $\mathcal{H}_{\bar\partial}^{1,1}$ other than those contained in $\mathcal{H}_g^- \otimes \mathbb{C}$, *i.e.* a form which can be written as $a_0\omega + \gamma_0$ with $a_0 \neq 0$ such that $id^c\gamma_0 = a_0 d\omega$. Thus $\gamma = \frac{1}{a_0}\gamma_0$ gives us the desired solution. □

In [6], Draghici, Li and Zhang prove that, for integrable almost complex manifolds (M,J), $h_{\bar\partial}^{1,1}$ takes the value b_-+1 when (M,J) is Kähler and otherwise takes the value b_-. Partially extending this result to non-integrable manifolds, in [29] Tardini and Tomassini prove that if a compact almost Hermitian 4-manifold (M,J,g) is *globally conformally almost Kähler* then $h_{\bar\partial}^{1,1} = b_- + 1$, whereas if (M,J,g) is *strictly locally conformally almost Kähler* then $h_{\bar\partial}^{1,1} = b_-$. The following result is therefore obtained:

Corollary 3.20. *Given a compact almost complex manifold (M,J), $h_{\bar\partial}^{1,1}$ is not in general invariant of the choice of almost Hermitian metric. It is, however, invariant of almost Kähler metrics.*

Remark 3.21. *Given an almost Hermitian manifold X with fundamental form ω, we can write*

$$d\omega = \alpha \wedge \omega$$

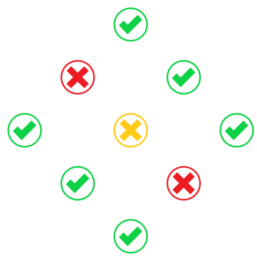
for some $\alpha \in \mathcal{A}^1$. X is said to be globally conformally almost Kähler whenever α is exact, whereas it is locally conformally almost Kähler if α is closed.

We verify in the example at the end of this chapter that both globally and strictly locally conformally almost Kähler metrics can indeed exist on the same almost complex manifold, thereby giving different values of $h^{1,1}_{\bar\partial}$ for different almost Hermitian metrics.

Combining the results of Prop. 3.1, Thm. 3.12, Cor. 3.13 and Cor. 3.20 we now have a full description of when $h^{p,q}_{\bar\partial}$ is or is not almost Hermitian metric independent. By laying out the values of $h^{p,q}_{\bar\partial}$ in a Hodge diamond

$$
\begin{array}{ccccc}
 & & h^{2,2}_{\bar\partial} & & \\
 & h^{2,1}_{\bar\partial} & & h^{1,2}_{\bar\partial} & \\
h^{2,0}_{\bar\partial} & & h^{1,1}_{\bar\partial} & & h^{0,2}_{\bar\partial} \\
 & h^{1,0}_{\bar\partial} & & h^{0,1}_{\bar\partial} & \\
 & & h^{0,0}_{\bar\partial} & &
\end{array}
$$

we can summarize these results as follows:



: Invariant of almost Hermitian metrics,

: Not invariant of almost Hermitian metrics, but invariant of almost Kähler metrics,

: Not invariant of almost Kähler metrics.

Returning to the case of $h^{1,1}_{\bar\partial}$, in order to have a complete description it remains to consider what happens when the metric is neither globally, nor strictly locally, almost Kähler. We therefore ask

Question 3.22. *On a compact almost Hermitian 4-manifold, does the value of $h^{1,1}_{\bar\partial}$ give a full description of whether an almost Hermitian metric is conformally almost Kähler? Specifically, in the case when the metric is not locally conformally almost Kähler (and thus also not globally conformally almost Kähler) do we have $h^{1,1}_{\bar\partial} = b_-$?*[1]

Although the answer to this is not known, we can prove a similar result for the dimension of the space of d-harmonic $(1,1)$-forms, which we will denote by $h^{1,1}_d$.

Theorem 3.23. *Let (M, J, g) be a compact almost Hermitian 4-manifold with fundamental form ω. We have $h^{1,1}_d = b_- + 1$ if ω is in the conformal class of an almost Kähler metric, otherwise $h^{1,1}_d = b_-$.*

Proof. As in the proof of the previous theorem, we use the fact that $h^{1,1}_d$ is a conformal invariant and thereby assume ω is a Gauduchon metric. Furthermore, all almost Kähler metrics are Gauduchon, so the conformal class of ω contains an almost Kähler metric if and only if ω is almost Kähler itself.

On compact manifolds we know a differential form s is d-harmonic if and only if

$$ds = 0 \qquad d * s = 0.$$

From this we can see that the Hodge star maps d-harmonic forms to d-harmonic forms, meaning that if some $(1,1)$-form s is in $\mathcal{H}^{1,1}_d$ so too are its self-dual and anti-self-dual components, $\frac{1}{2}(s + *s)$ and $\frac{1}{2}(s - *s)$. Furthermore, we have the inclusion

$$\mathcal{H}^{1,1}_d \subseteq \mathcal{H}^{1,1}_{\bar\partial}$$

and so from (3.17) we know we can write any d-harmonic $(1,1)$-form as $a\omega + \gamma$ with $a \in \mathbb{C}$ a constant and γ an anti-self-dual form. But the self-dual component of this is only harmonic if $d\omega = 0$ or $a = 0$ and so either ω is almost Kähler and we have $h^{1,1}_d = b_- + 1$ or all d-harmonic $(1,1)$-forms are anti-self-dual and we have $h^{1,1}_d = b_-$. $\square$

From this result we see that the above question is equivalent to asking whether $h^{1,1}_d$ and $h^{1,1}_{\bar\partial}$ are always equal on compact Hermitian 4-manifolds.

We conclude this section with a calculation of $h^{1,1}_{\bar\partial}$ for a family of almost complex structures and compatible metrics. In doing so we will see that, at least for this family of almost Hermitian structures, Question 3.22 has a positive answer.

[1] Since this thesis was written, Piovani and Tomassini have found examples of non locally conformally almost Kähler structures on compact 4-manifolds, for which $h^{1,1}_{\bar\partial} = b_- + 1$ [25], thereby answering this question in the negative.

Example 3.24. Let M be a compact manifold, given by identifying the points in $\mathbb{R}^4$ by the equivalence relations

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t \\ x + x_0 \\ y + y_0 \\ z + z_0 \end{pmatrix} \qquad \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} t + t_0 \\ x + t_0 y + \frac{1}{2}t_0^2 z \\ y + t_0 z \\ z \end{pmatrix}$$

for all $x_0, y_0, z_0 \in \mathbb{Z}$, $t_0 \in 2\mathbb{Z}$. This is equivalent to the group Nil^4 modulo the discrete subgroup $\mathbb{Z} \ltimes \mathbb{Z}^3$ (see Section 5.2).

M has a smooth global frame given by

$$\epsilon_1 = \frac{\partial}{\partial t}, \qquad \epsilon_2 = \frac{\partial}{\partial x}, \qquad \epsilon_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad \epsilon_4 = \frac{1}{2}t^2\frac{\partial}{\partial x} + t\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

along with the dual frame

$$\epsilon^1 = dt, \qquad \epsilon^2 = dx - tdy + \frac{1}{2}t^2 dz, \qquad \epsilon^3 = dy - tdz, \qquad \epsilon^4 = dz.$$

We can then define an almost complex structure J, such that $J\,\epsilon_1 = \epsilon_2$ and $J\,\epsilon_3 = \epsilon_4$. A pair of vector fields, spanning $T_p^{1,0}M$ at every point $p \in M$, can then be defined by

$$V_1 = \frac{1}{2}(\epsilon_0 - i\epsilon_1) \qquad \& \qquad V_2 = \frac{1}{2}(\epsilon_2 - i\epsilon_4),$$

along with their dual $(1,0)$-forms, given by

$$\phi^1 = \epsilon^1 + i\epsilon^2 \qquad \& \qquad \phi^2 = \epsilon^3 + i\epsilon^4.$$

These forms satisfy the structure equations

$$\begin{aligned} d\phi^1 &= -i\,\epsilon^1 \wedge \epsilon^3 \\ &= -\frac{i}{4}\left(\phi^{12} + \phi^{1\bar{2}} - \phi^{2\bar{1}} + \phi^{\bar{1}\bar{2}}\right), \end{aligned}$$

$$\begin{aligned} d\phi^2 &= -\epsilon^1 \wedge \epsilon^4 \\ &= \frac{i}{4}\left(\phi^{12} - \phi^{1\bar{2}} - \phi^{2\bar{1}} - \phi^{\bar{1}\bar{2}}\right), \end{aligned}$$

with $\phi^{i\bar{j}}$ used here as shorthand for $\phi^1 \wedge \bar{\phi}^2$. From this we can see that J is a non-integrable almost complex structure, namely we have $\bar{\mu}\phi^1 = \bar{\mu}\phi^2 = -\frac{i}{4}\phi^{\bar{1}\bar{2}} \neq 0$.

Now it only remains to choose a family of almost Hermitian metrics

$$\omega_\lambda = i\left((1+\lambda^2)\phi^{1\bar{1}} - \lambda\phi^{1\bar{2}} - \lambda\phi^{2\bar{1}} + \phi^{2\bar{2}}\right)$$

varying over some $\lambda \in \mathbb{R}$, defined such that $V_1 + \lambda V_2$ and V_2 form a unitary basis on $T_p^{1,0}M$. Using the structure equations we can calculate

$$d\omega_\lambda = \frac{\lambda}{2}\left(\phi^{12\bar{1}} + \phi^{1\bar{1}\bar{2}}\right),$$

from which we can see firstly that ω_λ is an almost Kähler metric if and only if $\lambda = 0$ and secondly that $\partial\bar{\partial}\omega_\lambda = 0$, and thus ω_λ is Gauduchon for all λ.

Furthermore, we can write

$$d\omega_\lambda = \alpha_\lambda \wedge \omega_\lambda$$

with

$$\begin{aligned} \alpha_\lambda &= -\frac{\lambda}{2}i\left(\lambda\phi^1 - \phi^2 - \lambda\bar{\phi}^1 + \bar{\phi}^2\right) \\ d\alpha_\lambda &= -\frac{\lambda^2}{8}\left(\phi^{12} + \phi^{1\bar{2}} - \phi^{2\bar{1}} + \phi^{\bar{1}\bar{2}}\right). \end{aligned}$$

ω_λ is therefore neither *globally* nor *locally conformally almost Kähler* except when $\lambda = 0$.

Finding $h^{1,1}_{\bar{\partial}}$ then amounts to asking whether there exists an anti-self-dual γ solving

$$id^c\gamma = d\omega_\lambda.$$

Since J is the identity on $(1,1)$-forms, this is equivalent to

$$iJ^{-1}d\gamma = d\omega_\lambda.$$

If such a γ exists that would mean

$$\begin{aligned} Jd\omega_\lambda &= \frac{i\lambda}{2}\left(\phi^{1\bar{1}\bar{2}} - \phi^{12\bar{1}}\right) \\ &= 2\lambda\epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 \\ &= 2\lambda dt \wedge (dx \wedge dy - t dx \wedge dz + \frac{1}{2}t^2 dy \wedge dz) \end{aligned}$$

is an exact 3-form. But consider the closed submanifold given by $z = 0$. The pullback onto this submanifold is $2\lambda dt \wedge dx \wedge dy$, which by Stokes' theorem cannot be exact, since its integral over the submanifold is non-zero, the only exception to this being when $\lambda = 0$. Thus, in all the cases when ω_λ is not *globally almost Kähler*, there is no solution to (3.18) and so $h^{1,1}_{\bar{\partial}} = b_-$.

Chapter 4

Bott-Chern Harmonic Forms

In this chapter, we will present a number of results concerning the values taken by $h^{p,q}_{BC}$. These results parallel those found for $h^{p,q}_{\bar\partial}$ in Chapter 3, namely we will give a full account of when $h^{p,q}_{BC}$ is or is not metric independent for compact almost complex 4-manifolds, in the process answering a Bott-Chern version of the Kodaira-Spencer problem. Furthermore, we will show that in the case of bidegrees $(2,1)$ and $(1,2)$ the value of $h^{p,q}_{BC}$ can be made arbitrarily large by varying the almost complex structure, while for bidegree $(1,1)$ we always have $h^{1,1}_{BC} = b_- + 1$. Most of the results of this chapter can be found in [15].

For many values of (p,q), proving the metric invariance of $h^{p,q}_{BC}$ is a relatively trivial affair and so we will not spend too long on these cases.

Proposition 4.1. *On any compact almost Hermitian 4-manifold $h^{p,q}_{BC}$ is metric independent when (p,q) is equal to $(2,0),(0,2),(1,0),(0,1),(0,0)$ or $(2,2)$.*

Proof. Bott-Chern harmonic $(0,0)$-forms are given by constant functions, since Δ_{BC} is elliptic. Similarly Bott-Chern harmonic $(2,2)$-forms are just constant multiples of the volume form, so although $\mathcal{H}^{2,2}_{BC}$ might change with the metric, $h^{2,2}_{BC}$ does not.

For the remaining cases recall that a (p,q)-form s is Bott-Chern harmonic if and only if it satisfies the three conditions

$$\partial s = 0 \qquad \bar\partial s = 0 \qquad \partial\bar\partial * s = 0.$$

When $(p,q) = (2,0),(0,2),(1,0)$ or $(0,1)$ the third condition is always true leaving behind the first two conditions which do not depend on the metric. $\square$

The more interesting cases are those when $(p,q) = (1,1),(2,1)$ and $(1,2)$. These will require a more detailed consideration.

4.1 $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ on the Kodaira-Thurston manifold

We start with the cases of $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$, both of which can be shown to vary in response to a changing almost Kähler metric. To demonstrate this, we will again consider the Kodaira-Thurston manifold, with the same almost complex structure as in Definition 3.2. The metric will again be given by

$$g_\lambda = \phi^1 \otimes \bar{\phi}^1 + \bar{\phi}^1 \otimes \phi^1 + \lambda(\phi^2 \otimes \bar{\phi}^2 + \bar{\phi}^2 \otimes \phi^2),$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$. What follows is essentially a more general, completed version of a calculation in [24].

Example 4.2. Let a general $(2,1)$-form be given by $f\phi^{12\bar{1}} + g\phi^{12\bar{2}}$. Then from the conditions $\bar{\partial}s = 0$ and $\partial\bar{\partial} * s = 0$ we see that $s \in \mathcal{H}^{2,1}_{BC}$ if and only if the following PDEs hold.

$$\begin{cases} \lambda V_1\bar{V}_1(f) + V_2\bar{V}_1(g) - \frac{b}{4}\lambda V_1(f) + \frac{b}{4}\lambda\bar{V}_1(f) - \frac{b}{4}\bar{V}_2(g) - \frac{b^2}{8}\lambda f = 0 \\ \lambda V_1\bar{V}_2(f) + V_2\bar{V}_2(g) + \frac{b}{4}\lambda V_2(f) = 0 \\ \bar{V}_1(g) - \bar{V}_2(f) = 0 \end{cases} \tag{4.1}$$

Performing a Fourier expansion on the second and third PDEs with respect to x, y and z, we find that the Fourier coefficients $\mathcal{F}_{\mathbf{y}}(f)$ and $\mathcal{F}_{\mathbf{y}}(g)$ must satisfy the ODE system

$$\frac{d}{dt}\begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix} = (A_n t + B_{l,m,n})\begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix} \tag{4.2}$$

for all $\mathbf{y} = (l, m, n) \in \mathbb{Z}^3$, where

$$A_n = 2\pi n \begin{pmatrix} 0 & \frac{1}{\lambda} \\ 1 & 0 \end{pmatrix}, \quad B_{l,m,n} = \begin{pmatrix} l - \frac{b}{4\pi}i & \frac{1}{\lambda}\left(m - n\frac{a-i}{b}\right) \\ m - n\frac{a+i}{b} & -l \end{pmatrix}.$$

The ODE obtained from the first PDE can be derived from the above system and therefore adds no new information.

The above system is very similar to the system (3.2) derived in Example 3.3 and, as in that Example, we can split our calculation into the cases where $|\mathrm{Orb}_{\mathbf{y}}| = \infty$ and $|\mathrm{Orb}_{\mathbf{y}}| = 1$.

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, we require $\mathcal{F}_{\mathbf{y}}(f), \mathcal{F}_{\mathbf{y}}(g) \in \mathcal{S}(\mathbb{R})$. Applying Theorem 3.4 we see in this case we have an independent solution given by the pair of functions

$$f = \sum_{\xi\in\mathbb{Z}} \mathcal{F}_{l,m,n}(f)(t+\xi)e^{2\pi i(lx+(m+n\xi)y+nz)}$$

$$g = \sum_{\xi\in\mathbb{Z}} \mathcal{F}_{l,m,n}(g)(t+\xi)e^{2\pi i(lx+(m+n\xi)y+nz)}$$

to (4.1) for all $0 \leq m < |n|$, whenever $l = 0$ and n satisfies

$$64\pi^2n^2 - 64\pi nub^2\sqrt{\lambda} - b^4\lambda = 0$$

for some negative integer u. Or equivalently, if we set $d = \frac{b}{8\pi}$,

$$n^2 - 64\pi nud^2\sqrt{\lambda} - 64\pi^2d^4\lambda = 0. \tag{4.3}$$

Note that if d and λ are both rational this case gives us no solutions as π is transcendental.

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = 1$, we require $\mathcal{F}_{\mathbf{y}}(f), \mathcal{F}_{\mathbf{y}}(g)$ to be periodic in t. So taking another Fourier expansion we see that the coefficients $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$ must satisfy

$$\lambda\left(l - ik - \frac{b}{4\pi}i\right)\mathcal{G}_{k,l,m,0}(f) + m\mathcal{G}_{k,l,m,0}(g) = 0$$
$$m\mathcal{G}_{k,l,m,0}(f) = (l+ik)\mathcal{G}_{k,l,m,0}(g).$$

This can be solved directly to find the solution

$$s = \phi^{12\bar{2}}$$

when $k = 0$, and the solution

$$s = ike^{2\pi i(kt+my)}\phi^{12\bar{1}} + me^{2\pi i(kt+my)}\phi^{12\bar{2}}$$

when $k \neq 0$ and $k, m \in \mathbb{Z}$ satisfy

$$\frac{m^2}{\sqrt{\lambda}} + (k+d)^2 = d^2. \tag{4.4}$$

Here we again set $d = \frac{b}{8\pi}$.

The above equations (4.3) and (4.4) are nearly identical to (3.15) and (3.16) from

Example 3.11, in fact they have the same number of solutions. We can therefore conclude that for the family of almost Hermitian structures $(KT^4, J_{a,b}, g_\lambda)$ considered here, we always have $h^{2,1}_{BC} = h^{0,1}_{\bar\partial}$.

Example 4.3. Now let us denote a general $(1,2)$-form by $f\phi^{1\bar1\bar2} + g\phi^{2\bar1\bar2}$, for some pair of smooth functions $f, g \in C^\infty(KT^4)$. Then from the conditions $\partial s = 0$ and $\partial\bar\partial * s = 0$ we see that $s \in \mathcal{H}^{1,2}_{BC}$ if and only if the following PDEs hold.

$$\begin{cases} \lambda V_1\bar V_1(f) + V_1\bar V_2(g) + \frac{b}{4}\lambda V_1(f) - \frac{b}{4}\lambda\bar V_1(f) - \frac{b}{4}\bar V_2(g) - \frac{b^2}{16}\lambda f = 0 \\ \lambda V_2\bar V_1(f) + V_2\bar V_2(g) + \frac{b}{4}\lambda V_2(f) = 0 \\ V_1(g) - V_2(f) = 0 \end{cases} \tag{4.5}$$

Applying the same Fourier expansion as before, the second and third equations give us a similar ODE system

$$\frac{d}{dt}\begin{pmatrix} \mathcal{F}_{l,m,n}(f) \\ \mathcal{F}_{l,m,n}(g) \end{pmatrix} = 2\pi\left[\begin{pmatrix} 0 & \frac{n}{\lambda} \\ n & 0 \end{pmatrix} t + \begin{pmatrix} -l + \frac{b}{4\pi}i & -\frac{1}{\lambda}\left(m - n\frac{a+i}{b}\right) \\ -m + n\frac{a-i}{b} & l \end{pmatrix}\right]\begin{pmatrix} \mathcal{F}_{l,m,n}(f) \\ \mathcal{F}_{l,m,n}(g) \end{pmatrix}.$$

Again splitting the calculations into two cases, we find in the first case that we have an independent solution for all $0 \le m < |n|$, whenever $n \ne 0$ satisfies

$$n^2 - 64\pi nud^2\sqrt{\lambda} - 64\pi^2 d^4\lambda = 0$$

for some negative integer u. In the second case we have solutions

$$s = \phi^{2\bar1\bar2}$$

and

$$s = ike^{2\pi i(kt+my)}\phi^{1\bar1\bar2} - me^{2\pi i(kt+my)}\phi^{2\bar1\bar2}$$

for all $k, m \in \mathbb{Z}$, with $k \ne 0$, satisfying

$$\frac{m^2}{\sqrt{\lambda}} + (k-d)^2 = d^2.$$

From the above we see that for this family of almost Hermitian structures we have $h^{1,2}_{BC} = h^{2,1}_{BC} = h^{0,1}_{\bar\partial}$ (although this need not be the case in general).

From these two examples, we see that the results of Theorem 3.9 and Theorem 3.12, which concern the values taken by $h^{0,1}_{\bar\partial}$, may be extended to apply also to $h^{2,1}_{BC}$ and

$h^{1,2}_{BC}$.

Theorem 4.4. *$h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ can both be computed for all members of the continuous family of non-integrable almost Kähler structures given by $(J_{a,b}, g_{a,b})$, $a, b \in \mathbb{R}$, $b \neq 0$, on the Kodaira-Thurston manifold. Furthermore, for any $n \in \mathbb{Z}^+$ such that $8 \not| n$, there is a b such that $h^{2,1}_{BC} = h^{1,2}_{BC} = n$.*

Theorem 4.5. *On compact almost complex 4-manifolds, $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ are not in general invariant of the choice of almost Kähler metric. In particular, on the Kodaira-Thurston manifold we can find a family of almost Hermitian metrics, compatible with a fixed almost complex structure, over which $h^{2,1}_{BC}$ and $h^{1,2}_{BC}$ take multiple different values.*

We can therefore answer a Bott-Chern version of the Kodaira-Spencer in the negative: the values of $h^{p,q}_{BC}$ on a compact almost complex manifold are not in general independent of the choice of almost Hermitian metric.

4.2 $h^{1,1}_{BC}$ on compact 4-manifolds

We will now consider the case of $\mathcal{H}^{1,1}_{BC}$. Firstly, we shall prove a Bott-Chern version of Proposition 3.16.

Proposition 4.6. *On any almost complex manifold with real dimension $2n$, $\mathcal{H}^{p,q}_{BC}$ is a conformal invariant whenever $p + q = n$.*

Proof. A (p, q)-form s is in $\mathcal{H}_{BC}$ if and only if it satisfies $\partial s = 0$, $\bar{\partial} s = 0$ and $\partial\bar{\partial} * s = 0$. The only metric dependence in these three conditions comes from the Hodge star operator. In the proof of Proposition 3.16 we saw that, when acting on (p, q)-forms with $p + q = n$, the Hodge star is a conformal invariant, thus $\mathcal{H}^{p,q}_{BC}$ is a conformal invariant whenever $p + q = n$. □

In the case of 4-manifolds, we therefore find that $\mathcal{H}^{1,1}_{BC}$ is a conformal invariant. So, in our consideration of $\mathcal{H}^{1,1}_{BC}$ in this section, we may always assume that our metric is Gauduchon.

In [24] Piovani and Tomassini prove the following characterisation of $\mathcal{H}^{1,1}_{BC}$:

Proposition 4.7. *On any compact almost complex 4-manifold, equipped with a compatible Gauduchon metric, we can write*

$$\mathcal{H}^{1,1}_{BC} = \{a\omega - \gamma \,|\, a \in \mathbb{C}, *\gamma = -\gamma, d\gamma = ad\omega\}, \tag{4.6}$$

where ω is the corresponding fundamental form.

From this characterisation of $\mathcal{H}^{1,1}_{BC}$ we can find the following bounds on the values that $h^{1,1}_{BC}$ can take:

Proposition 4.8. *On a compact almost Hermitian 4-manifold we have either* $h^{1,1}_{BC} = b_-$ *or* $b_- + 1$.

Proof. Without loss of generality, we assume the metric is Gauduchon and denote by ω the corresponding fundamental form.

Writing the elements of $\mathcal{H}^{1,1}_{BC}$ as $a\omega - \gamma$, as in (4.6), we see that when $a = 0$ we have all the $(1,1)$-forms γ for which $*\gamma = -\gamma$ and $d\gamma = 0$. Any such $(1,1)$-form also satisfies $d * \gamma = 0$, so these are the anti-self-dual, d-harmonic forms. This gives us the inclusion

$$\mathcal{H}^-_g \otimes \mathbb{C} \subseteq \mathcal{H}^{1,1}_{BC}.$$

When this inclusion is an equality then clearly we have $h^{1,1}_{BC} = b_-$.

Suppose instead there exists a $(1,1)$-form $a_0\omega - \gamma_0$ that is Bott-Chern harmonic but not contained in $\mathcal{H}^-_g \otimes \mathbb{C}$, with a_0 a non-zero constant and γ_0 an anti-self-dual form satisfying $d\gamma_0 = a_0 d\omega$. Then any other element $a\omega - \gamma$ of $\mathcal{H}^{1,1}_{BC}$ can be written as a multiple of $a_0\omega - \gamma_0$ plus an element of $\mathcal{H}^-_g$ as follows:

$$a\omega - \gamma = \frac{a}{a_0}(a_0\omega - \gamma_0) + \frac{1}{a_0}(a\gamma_0 - a_0\gamma).$$

In this case we therefore have $h^{1,1}_{BC} = b_- + 1$.

$\square$

Corollary 4.9. *On a compact almost complex 4-manifold endowed with a fundamental form ω corresponding to a Gauduchon metric, we have $h^{1,1}_{BC} = b_- + 1$ if and only if there exists an anti-self-dual $(1,1)$-form γ satisfying the equation*

$$d\gamma = d\omega. \tag{4.7}$$

Proof. Clearly if such a γ exists then we have the Bott-Chern harmonic form $\omega - \gamma \notin \mathcal{H}^-_g \otimes \mathbb{C}$. Conversely, if there exists a Bott-Chern harmonic form $a_0\omega - \gamma_0 \notin \mathcal{H}^-_g \otimes \mathbb{C}$ then γ is given by $\frac{1}{a_0}\gamma_0$. $\square$

It turns out that solutions to the above equation (4.7) can be found by making use of the Hodge decomposition which allows us to write k-forms as the following sum:

$$\mathcal{A}^k = d\mathcal{A}^{k-1} \oplus \mathcal{H}^k_d \oplus d^*\mathcal{A}^{k+1}.$$

Theorem 4.10. *Given any compact almost Hermitian 4-manifold, we always have* $h^{1,1}_{BC} = b_- + 1$.

Proof. From the conformal invariance of $h^{1,1}_{BC}$ we may assume without losing generality that the fundamental form ω is Gauduchon. Then taking the Hodge decomposition we can write

$$\omega = d\alpha + h + d^*\beta$$

for some $\alpha \in \mathcal{A}^1, h \in \mathcal{H}^2_d$ and $\beta \in \mathcal{A}^3$. Then defining a 2-form

$$\gamma := d * \beta + d^*\beta$$

we have

$$d\omega = dd^*\beta = d\gamma$$

and thus γ is a solution to (4.7).

It only remains to show that γ is anti-self-dual. Using the definition of d^* along with the fact that the square of the Hodge star when applied to a k-form is given by $** = (-1)^k$, we can see that

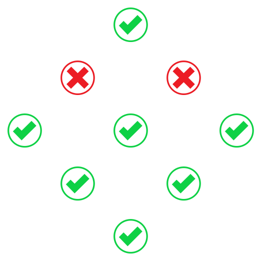
$$\begin{aligned} *\gamma &= *d * \beta - * * d * \beta \\ &= -d^*\beta - d * \beta \\ &= -\gamma. \end{aligned}$$

We therefore find that $\mathcal{H}^{1,1}_{BC}$ is generated by b_- many elements of $\mathcal{H}^-_g \otimes \mathbb{C}$, together with $\omega - \gamma$. Thus $h^{1,1}_{BC}$ is always $b_- + 1$. $\square$

Combining Proposition 4.1, Theorem 4.5 and Theorem 4.10 we now have a full description for compact 4-manifolds of when the values of $h^{p,q}_{BC}$ are or are not independent of the choice Hermitian metric. If we lay out these values in a Hodge diamond

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

we can summarize these results as follows:



: Invariant of almost Hermitian metrics,

: Not invariant of almost Kähler metrics.

4.3 Birational invariance of $h_{BC}^{p,0}$

In Theorem 5.5 of [3] Haojie Chen and Weiyi Zhang prove that $h_{\bar{\partial}}^{p,0}$ is birationally invariant on compact 4-manifolds for any $p \in \{0, 1, 2\}$. This means that if two compact almost complex 4-manifolds X and Y are separated by a sequence of almost complex manifolds $X = X_0, X_1, X_2 \ldots, X_{k+1} = Y$ along with a sequence of degree one pseudoholomorphic maps $u_0, \ldots u_k$ such that $u_{2i-1} : X_{2i-1} \to X_{2i}$ and $u_{2i} : X_{2i+1} \to X_{2i}$ then $h_{\bar{\partial}}^{p,0}(X) = h_{\bar{\partial}}^{p,0}(Y)$. Here we define a pseudoholomorphic map $u : (M_1, J_1) \to (M_2, J_2)$ between almost complex manifolds to be one whose pushforward f_* satisfies

$$J_2 \circ f_* = f_* \circ J_1.$$

It turns out this result can be extended to show that the numbers $h_{BC}^{p,0}$ are also birational invariants.

Theorem 4.11. *Let $u : X \to Y$ be a degree one pseudoholomorphic map between compact almost complex 4-manifolds. Then $h_{BC}^{p,0}(X) = h_{BC}^{p,0}(Y)$ for any $p \in \{0, 1, 2\}$.*

Proof. From [3] we know that the pullback with respect to u describes a bijection

$$u^* : \mathcal{H}_{\bar{\partial}}^{p,0}(Y) \to \mathcal{H}_{\bar{\partial}}^{p,0}(X).$$

Restricting this to the forms $s \in \mathcal{H}_{\bar{\partial}}^{p,0}(Y)$ which satisfy $\partial s = 0$ gives us

$$u^* : \mathcal{H}_{BC}^{p,0}(Y) \to \mathcal{H}_{BC}^{p,0}(X).$$

The injectivity of this map follows directly from the injectivity of u^* acting on $\mathcal{H}^{p,0}_{\bar{\partial}}(Y)$, so it only remains to prove surjectivity.

Since u^* is invertible when acting on $\mathcal{H}^{p,0}_{\bar{\partial}}(Y)$ we know that for any $s \in \mathcal{H}^{p,0}_{BC}(X)$ there is some $t \in \mathcal{H}^{p,0}_{\bar{\partial}}(Y)$ such that $u^*t = s$. By Theorem 1.5 in [33] we know there is a finite set $Y_1 \subset Y$ such that the restriction

$$u : X \backslash u^{-1}(Y_1) \to Y \backslash Y_1$$

is a diffeomorphism. This means we have

$$t\big|_{X\backslash u^{-1}(Y_1)} = (u^{-1})^* s\big|_{Y\backslash Y_1}$$

and so $\partial t = 0$ on $Y\backslash Y_1$. But since t is smooth and $\overline{Y\backslash Y_1} = Y$, we must have $\partial t = 0$ on all of Y, thus $t \in \mathcal{H}^{p,0}_{BC}(Y)$ and $u^*\big|_{\mathcal{H}^{p,0}_{BC}(Y)}$ is surjective. $\square$

Corollary 4.12. *$h^{0,p}_{BC}$ is a birational invariant on compact almost complex 4-manifolds for any $p = 0, 1$ or 2.*

Proof. Recall that $s \in \mathcal{H}^{p,q}_{BC}$ if and only if the following conditions hold

$$\bar{\partial} s = 0 \qquad \partial s = 0 \qquad \partial\bar{\partial} * s = 0.$$

If s is either a $(p, 0)$-form or a $(0, p)$-form for any $p = 0, 1$ or 2 then the third condition is always true for reasons of bidegree. The remaining two conditions, when taken together, are unchanged by a conjugation of s. The corollary therefore follows simply from the fact that $\mathcal{H}^{0,p}_{BC} = \overline{\mathcal{H}^{p,0}_{BC}}$. $\square$

Chapter 5

Calculations of $h_{\bar{\partial}}^{p,q}$ on other Torus Bundles

Throughout this thesis we have exclusively focused on the example of the Kodaira-Thurston manifold. Let us now broaden our attention to other torus bundles. In this chapter, we will consider how the decomposition of functions described in Theorem 2.8 behaves for other torus bundles and how this might be applied to solving PDEs on these manifolds.

Specifically, we will attempt to calculate the values of $h_{\bar{\partial}}^{0,1}$ for a range of manifolds. As was the case for the Kodaira-Thurston in Section 3.1, we will often find that this amounts to solving countably many ODE systems and countably many algebraic equations. Many of these ODE systems cannot yet be solved, however by solving the algebraic equations a partial solution can always be found.

5.1 Geometries in 4-dimensions

We would like to examine the effect of changing the behaviour of the orbits

$$\mathrm{Orb}_{\mathbf{y}} := \{(A^T)^{\xi}\mathbf{y} \mid \xi \in \mathbb{Z}\}$$

for $\mathbf{y} \in \mathbb{Z}^n, A \in GL_n(\mathbb{Z})$, where the choice of A determines a manifold as in (2.1). The behaviour of these orbits turns out to be closely linked to the geometry of the manifold and so the examples considered in this chapter will be split up accordingly.

We have already considered the case of the Kodaira-Thurston manifold, for which we have two types of orbit, characterised by $|\mathrm{Orb}_{\mathbf{y}}| = 1$ and $|\mathrm{Orb}_{\mathbf{y}}| = \infty$. We will see that the Kodaira-Thurston manifold has the geometric structure of $Nil^3 \times \mathbb{E}$.

Additionally, in this chapter we will consider an example of a manifold with the geometric structure of Nil^4, which has the same two types of orbit as the Kodaira-Thurston manifold. We will consider a family of manifolds with the geometric structure of $Sol^3 \times \mathbb{E}$, which also have orbits characterised by $|\mathrm{Orb}_{\mathbf{y}}| = 1$ and $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, except in this case the elements of the infinite orbits grow with exponential speed. Finally, we will consider a manifold with $\mathbb{E}^4$, *i.e.* Euclidean, geometry which has been specifically chosen so as to produce orbits with finite size greater than 1.

First we shall introduce the concept of a geometry as defined by Thurston [27, 30].

Definition 5.1. A *geometry* is given by a pair (X, G), consisting of a simply connected manifold X along with a Lie group of diffeomorphisms G acting transitively on X with compact stabilizers at every point. We say that a compact manifold M has the *geometric structure* of (X, G) if M is diffeomorphic to X/Γ for some discrete subgroup $\Gamma \subset G$ acting freely on X.

We now define the geometries which we shall be considering in this chapter.

Definition 5.2. Euclidean geometry consists of $X = \mathbb{E}^4$ together with the isometries of translation and rotation. In other words $G = SO_4(\mathbb{R}) \ltimes \mathbb{R}^4$.

The group $Nil^3 \times \mathbb{E}$ is given by $\mathbb{R}^4$ together with the group operation

$$\begin{pmatrix} t_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} t_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 \\ x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 + t_1 y_2 \end{pmatrix}$$

i.e. it is the semi-direct product of $\mathbb{R}$ acting on $\mathbb{R}^3$ by the matrix

$$C_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

The group Nil^4 is given similarly by $\mathbb{R}^4$ with the group operation

$$\begin{pmatrix} t_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} t_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 \\ x_1 + x_2 + t_1 y_2 + \frac{1}{2} t_1^2 z_2 \\ y_1 + y_2 + t_1 z_2 \\ z_1 + z_2 \end{pmatrix}$$

i.e. the semi-direct product of $\mathbb{R}$ acting on $\mathbb{R}^3$ by the matrix

$$C_2(t) = \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, the group $Sol^3 \times \mathbb{E}$ can be viewed as $\mathbb{R}^4$ with the group operation

$$\begin{pmatrix} t_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} t_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 \\ x_1 + x_2 \\ y_1 + e^{t_1} y_2 \\ z_1 + e^{-t_1} z_2 \end{pmatrix},$$

i.e. a semi-direct product of $\mathbb{R}$ acting on $\mathbb{R}^3$ by the matrix

$$C_3(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}.$$

If we set $X = Nil^3 \times \mathbb{E}, Nil^4$ or $Sol^3 \times \mathbb{E}$ we obtain a geometry by choosing G to be the diffeomorphisms given by X acting on itself by left multiplication.

Proposition 5.3. *The Kodaira-Thurston manifold KT^4 has the geometric structure of $Nil^3 \times \mathbb{R}$*

Proof. Recall the definition of the Kodaira-Thurston manifold as $\mathbb{R}^4$ with points identified by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x} + \eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t + \xi \\ A_1^{\xi} \mathbf{x} \end{pmatrix}$$

for all $\xi \in \mathbb{Z}$ and $\eta \in \mathbb{Z}^3$, where $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. This is equivalent to defining KT^4 as X/Γ, with $X = Nil^3 \times \mathbb{E}$ and $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^3 \subset Nil^3 \times \mathbb{E}$, where Γ is acting on X by left multiplication. The first of the above identifications is given by the $\mathbb{Z}^3$ part of Γ, while the second is given by the $\mathbb{Z}$ part.

The equivalence of these definitions follows from that fact that we can write

$$A_1^t = \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} = C_1(t)$$

for all $t \in \mathbb{R}$. $\square$

Remark 5.4. *As we noted in Remark 2.9, the decomposition of L^2 functions described in Theorem 2.8 can often be understood from a representation theory viewpoint. For instance, in the case of the Kodaira-Thurston manifold $KT^4 := \mathbb{Z} \ltimes \mathbb{Z}^3 \backslash Nil^3 \times \mathbb{E}^1$, it can be understood as the decomposition into irreducible components of the right regular representation of $Nil^3 \times \mathbb{E}^1$ acting on $L^2(KT^4)$.*

In fact, the irreducible representations of Nil^3 (also called the Heisenberg group) are classified by the Stone-von Neumann theorem. The resulting decomposition of $L^2(Nil^3)$ with respect to the regular representation is well-understood and descends to a decomposition of $L^2(\mathbb{Z} \ltimes \mathbb{Z}^2 \backslash Nil^3)$ (see [1, 7]). The exact same decomposition is derived when Theorem 2.8 is applied to the manifold $\mathbb{Z} \ltimes \mathbb{Z}^2 \backslash Nil^3$ (also called the Heisenberg manifold).

5.2 $h_{\bar\partial}^{0,1}$ on a manifold with Nil^4 geometry

Definition 5.5. We define a manifold M_2 by identifying points in $\mathbb{R}^4$ by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x} + \eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t + \xi \\ A_2^{\xi} \mathbf{x} \end{pmatrix}$$

for all $\xi \in \mathbb{Z}$ and $\eta \in \mathbb{Z}^3$, where we set $A_2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. This is exactly the manifold we considered in Example 3.24 up to a rescaling of t by a factor of 2.

Proposition 5.6. *The manifold M_2 defined above has the geometric structure of Nil^4.*

Proof. Similar to the proof that the Kodaira-Thurston manifold has $Nil^3 \times \mathbb{E}$ geometry, we can give an equivalent definition of M_2 as X/Γ, with $X = Nil^4$ and $\Gamma = 2\mathbb{Z} \ltimes \mathbb{Z}^3 \subset Nil^4$, where Γ is acting on X by left multiplication.

Their equivalence follows from

$$A_2^t = \exp \begin{pmatrix} 0 & 2t & 0 \\ 0 & 0 & 2t \\ 0 & 0 & 0 \end{pmatrix} = C_2(2t)$$

for all $t \in \mathbb{R}$. $\square$

Using the generalised eigenvectors of the matrix A_2 we can define a left-invariant frame for M_2

$$\epsilon_0 = \partial_t, \qquad \epsilon_1 = \partial_x, \qquad \epsilon_2 = 2t\partial_x + \partial_y, \qquad \epsilon_3 = 2t^2\partial_x + 2t\partial_y + \partial_z$$

along with the dual frame

$$\epsilon^0 = dt, \qquad \epsilon^1 = dx - 2tdy + 2t^2dz, \qquad \epsilon^2 = dy - 2tdz, \quad e^3 = dz.$$

We will now demonstrate how the techniques of Chapter 2 can be applied to solving PDEs on M_2. Specifically, we shall consider the equations involved in calculating $h^{0,1}_{\bar{\partial}}$

Example 5.7. We consider the almost complex structure on M_2 given by

$$J : \epsilon_0 \mapsto \epsilon_1, \quad \epsilon_2 \mapsto \epsilon_3$$

for which a frame on the space $T_{1,0}M_2$ along with a dual frame on $T^*_{1,0}M_2$ can be given by

$$V_1 = \frac{1}{2}(\epsilon_0 - i\epsilon_1), \quad V_2 = \frac{1}{2}(\epsilon_2 - i\epsilon_3),$$
$$\phi^1 = \epsilon^0 + i\epsilon^1, \quad \phi^2 = \epsilon^2 + i\epsilon^3.$$

For this frame we have the structure equations

$$\begin{aligned} d\phi^1 &= -2i\epsilon^0 \wedge \epsilon^2 \\ &= -\frac{i}{2}\left(\phi^{12} + \phi^{1\bar{2}} - \phi^{2\bar{1}} + \phi^{\bar{1}\bar{2}}\right), \end{aligned}$$

$$\begin{aligned} d\phi^2 &= -2\epsilon^0 \wedge \epsilon^3 \\ &= -\frac{i}{2}\left(\phi^{12} - \phi^{1\bar{2}} - \phi^{2\bar{1}} - \phi^{\bar{1}\bar{2}}\right). \end{aligned}$$

We also define the Hermitian metric such that V_1 and V_2 are orthonormal. This corresponds to the fundamental form

$$\omega = 2\left(\epsilon^0 \wedge \epsilon^1 + \epsilon^2 \wedge \epsilon^3\right).$$

(Note that $d\omega$ is zero, so this defines an almost Kähler structure)

A general $(0,1)$-form $s \in \mathcal{A}^{0,1}$ can be written as $f\bar{\phi^1} + g\bar{\phi^2}$ for $f, g \in C^\infty(M_2)$. A differential form is $\bar{\partial}$-harmonic if and only if it satisfies $\bar{\partial}s = 0$ and $\partial * s = 0$. From these two conditions we obtain a pair of PDEs involving the functions f and g.

$$\begin{cases} -\bar{V_2}(f) + \bar{V_1}(g) - \frac{i}{2}f + \frac{i}{2}g = 0, \\ V_1(f) + V_2(g) = 0. \end{cases} \tag{5.1}$$

As in the Kodaira-Thurston case, we can now simplify these equations by applying the maps $\mathcal{F}$ and $\mathcal{G}$, while making use of the results of Proposition 2.15 and 2.16. Looking at the orbits of $\mathbf{y} \in \mathbb{Z}^3$ under the action of the transpose matrix A_2^T, we see that there are two cases to consider:

1) If $\mathbf{y} = (l, m, n)$ with $(l, m) \neq (0, 0)$ then $(A^T)^\xi \mathbf{y} = (l, 2l\xi + m, 2l\xi^2 + 2m\xi + n)$ for all $\xi \in \mathbb{Z}$. In which case the orbit $\mathrm{Orb}_{\mathbf{y}}$ is infinite and blows up with polynomial speed as $\xi \to \pm\infty$.
2) If $\mathbf{y} = (0, 0, n)$ then $A^T \mathbf{y} = \mathbf{y}$. In which case the orbit $\mathrm{Orb}_{\mathbf{y}}$ has size 1.

Case 1: $|\mathrm{Orb}_{\mathbf{y}}| = \infty$

We can simplify the equations (5.1) by taking a Fourier expansion with respect to x, y and z. For any $\mathbf{y} = (l, m, n) \in \mathbb{Z}^3$ we can use Prop. 2.15 to show that the Fourier coefficients $\mathcal{F}_{\mathbf{y}}(f)$ and $\mathcal{F}_{\mathbf{y}}(g)$ must satisfy the ODE system

$$\frac{d}{dt}\begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix} = \left(A_l t^2 + B_{l,m} t + C_{l,m,n}\right)\begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix}$$

where

$$A_l = 2\pi\begin{pmatrix} 0 & -2l \\ -2l & 0 \end{pmatrix}, \qquad B_{l,m} = 2\pi\begin{pmatrix} 0 & -2(m + li) \\ -2(m - li) & 0 \end{pmatrix}$$

$$C_{l,m,n} = 2\pi\begin{pmatrix} -l & -n - mi \\ -n + mi + \frac{1}{2\pi}i & l - \frac{1}{2\pi}i \end{pmatrix}$$

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, *i.e.* either $l \neq 0$ or $m \neq 0$, Cor. 2.7 tells us that $\mathcal{F}_{\mathbf{y}}(f)$ and $\mathcal{F}_{\mathbf{y}}(g)$ must be Schwartz. In fact, any pair of functions $\alpha, \beta \in \mathcal{S}(\mathbb{R})$ solving the above ODE system, for some such choice of $\mathbf{y}$, gives rise to a pair of solutions to (5.1) given by

$$f = \sum_{\xi \in \mathbb{Z}} \alpha(t+\xi) e^{2\pi i (lx + (2\xi l + m)y + (2\xi^2 l + 2\xi m + n)z)}$$

$$g = \sum_{\xi \in \mathbb{Z}} \beta(t+\xi) e^{2\pi i (lx + (2\xi l + m)y + (2\xi^2 l + 2\xi m + n)z)}.$$

Unfortunately, there does not seem to be any known solution to this ODE system. However, below we *will* be able to give a full description of the solutions that arise from Case 2.

Remark 5.8. *To see why this ODE system is trickier to solve than the system* (3.2) *on the Kodaira-Thurston manifold, we could try to solve the system using the same Laplace transform method from the proof of Theorem 3.4. First by defining*

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \frac{1}{\sqrt{2}} e^{2\pi\left(\frac{2}{3} l t^3 + m t^2 + \left(n - \frac{i}{2\pi}\right) t\right)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_{\mathbf{y}}(f) \\ \mathcal{F}_{\mathbf{y}}(g) \end{pmatrix}$$

and thereby rewriting the ODE as

$$\frac{d}{dt} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \left(\tilde{A}_l t^2 + \tilde{B}_{l,m} t + \tilde{C}_{l,m,n} \right) \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

where

$$\tilde{A}_l = 2\pi \begin{pmatrix} -4l & 0 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{B}_{l,m} = 2\pi \begin{pmatrix} -4m & 2li \\ 2li & 0 \end{pmatrix},$$

and

$$\tilde{C}_{l,m,n} = 2\pi \begin{pmatrix} -2n + \frac{i}{2\pi} & -l + mi - \frac{i}{2\pi} \\ -l - mi & 0 \end{pmatrix}.$$

From this we can derive a second order ODE in ϕ which takes the form of

$$p(t)\phi'' + q(t)\phi' + r(t)\phi = 0 \tag{5.2}$$

for some polynomials $p(t), q(t), r(t)$ of degree three. We could try to solve this using the Laplace transform, substituting

$$\phi = \int_C \varphi(s) e^{st} ds.$$

In the case of the Kodaira-Thurston manifold this transforms the second order differential *equations* (3.5) *and* (3.6) *into first order differential equations with respect to* φ*, which can be solved to obtain solutions as described in* (3.7)*. In the above case, however, we instead get a third order differential equation with respect to* φ*, which is no easier to solve than* (5.2) *itself.*

Case 2: $|\mathrm{Orb}_{\mathbf{y}}| = 1$

In the case when $\mathbf{y} = (0, 0, n) \in \mathbb{Z}^3$, the functions $\mathcal{F}_{\mathbf{y}}(f)$ and $\mathcal{F}_{\mathbf{y}}(g)$ are periodic. We are therefore able to apply another Fourier expansion, to give us the coefficients $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$ for all $k \in \mathbb{Z}$. Using Prop. 2.16 this turns our ODE from Case 1 into an algebraic equation

$$\begin{pmatrix} ki & n \\ n - \frac{1}{2\pi}i & ki + \frac{1}{2\pi}i \end{pmatrix} \begin{pmatrix} \mathcal{G}_{k,\mathbf{y}}(f) \\ \mathcal{G}_{k,\mathbf{y}}(g) \end{pmatrix} = 0$$

which $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$ must satisfy. In fact, for any $\alpha, \beta \in \mathbb{C}$ satisfying this equation for some $k, n \in \mathbb{Z}$ we have a solution to (5.1) given by

$$f = \alpha\, e^{2\pi i(kt+nx)}$$

$$g = \beta\, e^{2\pi i(kt+nx)}.$$

Clearly, we only have a solution when k and l satisfy

$$k^2 + n^2 + \frac{1}{2\pi}(k - ni) = 0.$$

From the imaginary part of the above condition, we require that $n = 0$. The real part then tells us that the only integer value of k that gives rise to a solution is $k = 0$. This corresponds to the family of solutions

$$f = g = C$$

with $C \in \mathbb{C}$.

5.3 $h^{0,1}_{\bar{\partial}}$ on a manifold with $Sol^3 \times \mathbb{E}^1$ geometry

Definition 5.9. Let $A_3 \in GL_3(\mathbb{Z})$ be chosen such that it can be diagonalised to give $\mathrm{diag}(1, e^{\kappa}, e^{-\kappa}) = PA_3P^{-1}$ by some $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & s \end{pmatrix} \in GL_3(\mathbb{R})$, where $\kappa \in \mathbb{R}$ is some positive number. We then define M_3 to be the manifold given by identifying points in

$\mathbb{R}^4$ by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x}+\eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t+\xi \\ A_3^{\xi}\mathbf{x} \end{pmatrix}$$

for all $\xi \in \mathbb{Z}$ and $\eta \in \mathbb{Z}^3$.

Note that we cannot simply choose $A_3 = \mathrm{diag}(1, e^{\kappa}, e^{-\kappa})$ as this would not be an invertible integer valued matrix unless $\kappa = 0$.

Proposition 5.10. *The manifold M_3 as defined above has the geometric structure of $Sol^3 \times \mathbb{E}$.*

Proof. Let the group $\tilde{G}$ be given by the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^3$ with $\mathbb{R}$ acting on $\mathbb{R}^3$ with the matrix

$$P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\kappa t} & 0 \\ 0 & 0 & e^{-\kappa t} \end{pmatrix} P.$$

Then M_3 can be viewed as the quotient $\tilde{G}/\Gamma$ of $\tilde{G}$ by the discrete subgroup $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^3$ acting by left multiplication. To see that M_3 has $Sol^3 \times \mathbb{E}$ geometry, we simply note that we have an isomorphism $\tilde{G} \to Sol^3 \times \mathbb{E}$ given by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \mapsto \begin{pmatrix} \kappa t \\ P\mathbf{x} \end{pmatrix}.$$

□

Using the eigenvectors of the matrix A, $P^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $P^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $P^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, corresponding to eigenvalues $1, e^{\kappa}$ and $e^{-\kappa}$, we can construct the special left-invariant frame on M_2:

$$\epsilon_0 = \frac{\partial}{\partial t}, \quad \epsilon_1 = \frac{\partial}{\partial x}, \quad \epsilon_2 = \frac{e^{\kappa t}}{ps - qr}\left(s\frac{\partial}{\partial y} - r\frac{\partial}{\partial z}\right) \quad \epsilon_3 = \frac{e^{-\kappa t}}{ps - qr}\left(-q\frac{\partial}{\partial y} + p\frac{\partial}{\partial z}\right)$$

along with its dual frame

$$\epsilon^0 = dt, \qquad \epsilon^1 = dx, \qquad \epsilon^2 = e^{-\kappa t}\left(pdy + qdz\right) \qquad \epsilon^3 = e^{\kappa t}\left(rdy + sdz\right)$$

Example 5.11. As in the previous section we can define an almost complex structure by the maps

$$J : \epsilon_0 \mapsto \epsilon_1, \quad \epsilon_2 \mapsto \epsilon_3.$$

We then have a pair of frames on the spaces $T_{1,0}M_2$ and $T^*_{1,0}M_2$ given by

$$V_1 = \frac{1}{2}(\epsilon_0 - i\epsilon_1), \quad V_2 = \frac{1}{2}(\epsilon_2 - i\epsilon_3),$$

$$\phi^1 = \epsilon^0 + i\epsilon^1, \quad \phi^2 = \epsilon^2 + i\epsilon^3.$$

The structure equations for this frame is $d\phi^1 = 0$ and

$$\begin{aligned} d\phi^2 &= -\kappa\epsilon^0 \wedge (\epsilon^2 - i\epsilon^3) \\ &= -\frac{\kappa}{2}(\phi^{1\bar{2}} + \phi^{\bar{1}\bar{2}}). \end{aligned}$$

Given an Hermitian metric such that V_1 and V_2 are orthonormal, the fundamental form is

$$\omega = 2\left(e^0 \wedge e^1 + e^2 \wedge e^3\right).$$

Unlike in the previous section, this metric is not almost Kähler, instead we have $d\omega = (ps - qr)dy \wedge dz \neq 0$.

In order to calculate $h^{0,1}_{\bar{\partial}}$, we write a general $(0,1)$-form as $s = f\bar{\phi}^1 + g\bar{\phi}^2$, with $f, g \in C^\infty(M_3)$. Then the conditions $\bar{\partial}s = 0$ and $\partial * s = 0$ tell us that s is $\bar{\partial}$-harmonic if and only if f and g satisfy

$$\begin{cases} -\bar{V_2}(f) + \bar{V_1}(g) = 0 \\ V_1(f) + V_2(g) = 0 \end{cases} \tag{5.3}$$

Now we wish to apply the maps $\mathcal{F}$ and $\mathcal{G}$ to simplify these PDEs. First we divide the orbits of $\mathbf{y} \in \mathbb{Z}^3$ under A_3^T into two cases. Note that A_3^T has three eigenvalues of $1, e^\kappa$ and $e^{-\kappa}$.

1) If $\mathbf{y} = (l, m, n)$, with $(m, n) \neq (0, 0)$, then it has a component in both of the eigenspaces corresponding to the eigenvalues e^κ and $e^{-\kappa}$. The orbit $\mathrm{Orb}_\mathbf{y}$ is therefore infinite with $(A_3^T)^\xi\mathbf{y}$ blowing up exponentially fast as $\xi \to \pm\infty$.
2) If $\mathbf{y} = (l, 0, 0)$, then then $A^T\mathbf{y} = \mathbf{y}$. In which case the orbit $\mathrm{Orb}_\mathbf{y}$ has size 1.

It is not possible for any $\mathbf{y} \in \mathbb{Z}^3$ to be contained entirely within, say, the eigenspace corresponding to e^κ. If it were, then as $\xi \to -\infty$ we would find that $(A_3^T)^\xi\mathbf{y}$ would become arbitrarily small. But this is not possible if we require $(A_3^T)^\xi\mathbf{y} \in \mathbb{Z}^3$ for all ξ.

Case 1: $|\mathrm{Orb}_\mathbf{y}| = \infty$

For any $\mathbf{y} = (l, m, n) \in \mathbb{Z}^3$ we can find conditions on the Fourier coefficients of f and g by applying $\mathcal{F}_\mathbf{y}$ to the equations in (5.3). Using Prop. 2.15 we see that these coefficients must satisfy the ODE system

$$\frac{d}{dt}\begin{pmatrix}\mathcal{F}_{\mathbf{y}}(f)\\ \mathcal{F}_{\mathbf{y}}(g)\end{pmatrix} = \left(A_{m,n}e^{\kappa t} + B_l + C_{m,n}e^{-\kappa t}\right)\begin{pmatrix}\mathcal{F}_{\mathbf{y}}(f)\\ \mathcal{F}_{\mathbf{y}}(g)\end{pmatrix}$$

where

$$A_{m,n} = -2\pi\frac{sm - rn}{ps - qr}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \qquad B_l = -2\pi l\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$C_{m,n} = -2\pi\frac{-qm + pn}{ps - qr}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

In the case when $|\mathrm{Orb}_{\mathbf{y}}| = \infty$, Prop. 2.6 gives us a condition on the functions $\mathcal{F}_{\mathbf{y}}(f)(t)$ and $\mathcal{F}_{\mathbf{y}}(g)(t)$ that is even stronger than $\mathcal{S}(\mathbb{R})$. Namely all of their derivatives must tend to zero as $t \to \pm\infty$ faster than any function of the form $e^{-\tau t}$ for any $\tau \in \mathbb{R}$.

Any smooth functions $\alpha(t), \beta(t)$ solving the ODE system and decaying sufficiently fast as $t \to \pm\infty$, give rise to a solution to (5.3)

$$f = \sum_{\xi\in\mathbb{Z}} \alpha(t+\xi)e^{2\pi i\mathbf{y}\cdot A_3^{\xi}\mathbf{x}}$$

$$g = \sum_{\xi\in\mathbb{Z}} \beta(t+\xi)e^{2\pi i\mathbf{y}\cdot A_3^{\xi}\mathbf{x}}$$

As in the previous section, giving a complete account of the solutions in the infinite orbit case proves tricky, whereas the finite orbit case below is relatively simple.

Case 2: $|\mathrm{Orb}_{\mathbf{y}}| = 1$

In the case when $\mathbf{y} = (l, 0, 0) \in \mathbb{Z}^3$, we can apply an additional Fourier expansion, yielding a condition on the coefficients $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$ for all $k \in \mathbb{Z}$

$$\begin{pmatrix}0 & k+il\\ k-il & 0\end{pmatrix}\begin{pmatrix}\mathcal{G}_{k,\mathbf{y}}(f)\\ \mathcal{G}_{k,\mathbf{y}}(g)\end{pmatrix} = 0$$

There are only non-trivial solutions to this when we have $k = l = 0$. This corresponds to two families of solutions

$$f = const. \qquad g = 0$$

and

$$f = 0 \qquad g = const.$$

5.4 $h^{0,1}_{\bar\partial}$ and $h^{0,2}_{\bar\partial}$ on a manifold with Euclidean geometry

Definition 5.12. We define the manifold M_4 to be given by setting

$$A_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and identifying points in $\mathbb{R}^4$ by

$$\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t \\ \mathbf{x}+\eta \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \sim \begin{pmatrix} t+\xi \\ A_4^{\xi}\mathbf{x} \end{pmatrix}$$

for all $\eta \in \mathbb{Z}^3$ and all $\xi \in \mathbb{Z}$.

Proposition 5.13. *The manifold M_4 as defined above has Euclidean geometry, i.e. it has the geometric structure of $\mathbb{E}^4$.*

Proof. Since we have $A_4 \in SO_4(\mathbb{R})$, we can define M_4 to be X/Γ, where $X = \mathbb{E}^4$ and Γ is the discrete subgroup of $SO_4(\mathbb{R}) \ltimes \mathbb{R}^4$ given by

$$\left\{ \left(A_4^{\xi}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) \,\middle|\, \xi \in \mathbb{Z}, \eta \in \mathbb{Z}^3 \right\}.$$

$\square$

The matrix A_4 has eigenvalues of 1, $e^{-\frac{2}{3}\pi i}$ and $e^{\frac{2}{3}\pi i}$ corresponding to eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} e^{\frac{2}{3}\pi i} \\ e^{-\frac{2}{3}\pi i} \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} e^{\frac{2}{3}\pi i} \\ e^{-\frac{2}{3}\pi i} \\ 1 \end{pmatrix}.$$

We therefore define a smooth frame on the complexified tangent bundle by

$$\epsilon_0 = \frac{\partial}{\partial t} \qquad \epsilon_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \nabla_{\mathbf{x}} \qquad \epsilon_2 = e^{-\frac{2}{3}\pi i t} \begin{pmatrix} e^{\frac{2}{3}\pi i} \\ e^{-\frac{2}{3}\pi i} \\ 1 \end{pmatrix} \cdot \nabla_{\mathbf{x}} \qquad \epsilon_3 = e^{\frac{2}{3}\pi i t} \begin{pmatrix} e^{-\frac{2}{3}\pi i} \\ e^{\frac{2}{3}\pi i} \\ 1 \end{pmatrix} \cdot \nabla_{\mathbf{x}}$$

where we define $\nabla_{\mathbf{x}} := \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$. The dual frame is given by

$$\epsilon^0 = dt, \qquad \epsilon^1 = \frac{1}{3}\left(dx + dy + dz\right),$$

$$\epsilon^2 = \frac{e^{\frac{2}{3}\pi i t}}{3}\left(e^{-\frac{2}{3}\pi i}dx + e^{\frac{2}{3}\pi i}dy + dz\right), \qquad \epsilon^3 = \frac{e^{-\frac{2}{3}\pi i t}}{3}\left(e^{\frac{2}{3}\pi i}dx + e^{-\frac{2}{3}\pi i}dy + dz\right).$$

5.4.1 Calculating $h^{0,1}_{\bar{\partial}}$

Example 5.14. Let an almost complex structure J be defined by the mapping

$$\epsilon_0 \mapsto \frac{1}{2}(\epsilon_2 + \epsilon_3) \qquad \text{and} \qquad \epsilon_1 \mapsto -\frac{i}{2}(\epsilon_2 - \epsilon_3).$$

We can then find smooth frames for $T_{1,0}M_4$ and $T^*_{1,0}M_4$

$$V_1 = \frac{1}{2}\left(\epsilon_0 - \frac{i}{2}(\epsilon_2 + \epsilon_3)\right) \qquad V_2 = \frac{1}{2}\left(\epsilon_1 + \frac{1}{2}(\epsilon_2 - \epsilon_3)\right)$$

$$\phi^1 = \epsilon^0 + i(\epsilon^2 + \epsilon^3) \qquad \phi^2 = \epsilon^1 - \epsilon^2 + \epsilon^3.$$

The structure equations for this choice of frame is

$$d\phi^1 = \frac{\pi}{6}\left(\phi^{12} - \phi^{1\bar{2}} - \phi^{2\bar{1}} - \phi^{\bar{1}\bar{2}}\right)$$

$$d\phi^2 = \frac{\pi}{3}\phi^{1\bar{1}}.$$

We choose the metric so that V_1 and V_2 are orthonormal. This corresponds to the fundamental form

$$\omega = 2\epsilon^0 \wedge (\epsilon^2 + \epsilon^3) + 2i\epsilon^1 \wedge (\epsilon^2 - \epsilon^3).$$

In this case

$$d\omega = \frac{4}{3}\pi\epsilon^0 \wedge \epsilon^1 \wedge (\epsilon^2 + \epsilon^3) \neq 0,$$

thus ω does not define an almost Kähler structure.

In order to calculate $h^{0,1}_{\bar{\partial}}$, we let a general $(0,1)$-form be written as $s = f\bar{\phi}^1 + g\bar{\phi}^2$.

The two requirements $\bar{\partial}s = 0$ and $\partial * s = 0$ give rise to the two PDEs

$$\begin{cases} -(\bar{V}_2 - \frac{\pi}{6})f + \bar{V}_1 g = 0, \\ V_1 f + V_2 g = 0. \end{cases} \tag{5.4}$$

As in our previous calculations, in order to simplify the above PDEs, we first must describe the behaviour of the orbits of $\mathbf{y} \in \mathbb{Z}^3$ under the action of A_4^T.

1) If $\mathbf{y} = (n, n, n)$ for some $n \in \mathbb{Z}$, the $A_4^T \mathbf{y} = \mathbf{y}$. In which case the orbit $\mathrm{Orb}_{\mathbf{y}}$ is finite and has size 1.
2) If $\mathbf{y} = (l, m, n)$, such that l, m and n are not all equal, then A^T permutes the entries of $\mathbf{y}$. In particular, $(A_4^T)^3 \mathbf{y} = \mathbf{y}$ and so in this case the orbit $\mathrm{Orb}_{\mathbf{y}}$ is still finite, but now has size 3.

Case 1: $|\mathrm{Orb}_{\mathbf{y}}| = 1$

For any $\mathbf{y} = (n, n, n) \in \mathbb{Z}^3$, after expanding with respect to x, y and z we see that the Fourier coefficients given by $\mathcal{F}_{\mathbf{y}}$ are periodic, with a period of 1. We can therefore take another expansion with respect to the remaining variable t. This is to say, we can write

$$\mathcal{F}_{\mathbf{y}}(f) = \sum_{k \in \mathbb{Z}} \mathcal{G}_{k,\mathbf{y}}(f) e^{2\pi i k t},$$

$$\mathcal{F}_{\mathbf{y}}(g) = \sum_{k \in \mathbb{Z}} \mathcal{G}_{k,\mathbf{y}}(g) e^{2\pi i k t}.$$

Applying the map $\mathcal{G}_{k,\mathbf{y}}$ to the equations in (5.4), for each $k \in \mathbb{Z}$, we obtain the condition

$$\begin{pmatrix} k & 3n \\ -3n - \frac{i}{6} & k \end{pmatrix} \begin{pmatrix} \mathcal{G}_{k,\mathbf{y}}(f) \\ \mathcal{G}_{k,\mathbf{y}}(g) \end{pmatrix} = 0,$$

on $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$. This has non-trivial solutions if and only if k and n satisfy

$$k^2 + 9n^2 + 3n\frac{i}{6} = 0$$

which is only possible when $k = n = 0$. Corresponding to this case, we have the solution

$$f = 0 \qquad g = const.$$

Case 2: $|\mathrm{Orb}_{\mathbf{y}}| = 3$

Now let $\mathbf{y} = (l, m, n)$ with l, m and n not all equal. In this case, $\mathcal{F}_{\mathbf{y}}$ still gives us periodic functions, but now with period 3 and so our expansion looks like

$$\mathcal{F}_{\mathbf{y}}(f)(t) = \sum_{k\in\mathbb{Z}} \mathcal{G}_{k,\mathbf{y}}(f) e^{\frac{2\pi i k t}{3}},$$

$$\mathcal{F}_{\mathbf{y}}(g)(t) = \sum_{k\in\mathbb{Z}} \mathcal{G}_{k,\mathbf{y}}(g) e^{\frac{2\pi i k t}{3}}.$$

For the sake of notational simplicity we define

$$\alpha_1 = l + m + n \qquad \alpha_2 = e^{\frac{2}{3}\pi i} l + e^{-\frac{2}{3}\pi i} m + n \qquad \alpha_3 = e^{-\frac{2}{3}\pi i} l + e^{\frac{2}{3}\pi i} m + n.$$

Then by Proposition 2.16 we can say that $\mathcal{G}_k(f)$ satisfies

$$\begin{aligned}
\mathcal{G}_{k,\mathbf{y}}(\epsilon_0 f) &= 2\pi i k \mathcal{G}_{k,\mathbf{y}}(f),\\
\mathcal{G}_{k,\mathbf{y}}(\epsilon_1 f) &= 2\pi i \alpha_1 \mathcal{G}_{k,\mathbf{y}}(f),\\
\mathcal{G}_{k,\mathbf{y}}(\epsilon_2 f) &= 2\pi i \alpha_2 \mathcal{G}_{k-1,\mathbf{y}}(f)\\
\mathcal{G}_{k,\mathbf{y}}(\epsilon_3 f) &= 2\pi i \alpha_3 \mathcal{G}_{k+1,\mathbf{y}}(f),
\end{aligned}$$

and likewise for $\mathcal{G}_k(g)$.

Applying $\mathcal{G}_{k,\mathbf{y}}$ to (5.4) we therefore obtain the following pair of equations:

$$\begin{aligned}
&\frac{\alpha_3}{2}\mathcal{G}_{k-1,\mathbf{y}}(f) - (\alpha_1 + \frac{i}{6})\mathcal{G}_{k,\mathbf{y}}(f) - \frac{\alpha_2}{2}\mathcal{G}_{k+1,\mathbf{y}}(f)\\
&\quad + \frac{i\alpha_3}{2}\mathcal{G}_{k-1,\mathbf{y}}(g) + \frac{k}{3}\mathcal{G}_{k,\mathbf{y}}(g) + \frac{i\alpha_2}{2}\mathcal{G}_{k+1,\mathbf{y}}(g) = 0
\end{aligned}$$

$$\begin{aligned}
&-\frac{i\alpha_3}{2}\mathcal{G}_{k-1,\mathbf{y}}(f) + \frac{k}{3}\mathcal{G}_{k,\mathbf{y}}(f) - \frac{i\alpha_2}{2}\mathcal{G}_{k+1,\mathbf{y}}(f)\\
&\quad + \frac{\alpha_3}{2}\mathcal{G}_{k-1,\mathbf{y}}(g) + \alpha_1\mathcal{G}_{k,\mathbf{y}}(g) - \frac{\alpha_2}{2}\mathcal{G}_{k+1,\mathbf{y}}(g) = 0.
\end{aligned}$$

By choosing to cancel either the terms $\mathcal{G}_{k-1,\mathbf{y}}(f)$ & $\mathcal{G}_{k-1,\mathbf{y}}(g)$ or the terms $\mathcal{G}_{k+1,\mathbf{y}}(f)$ & $\mathcal{G}_{k+1,\mathbf{y}}(g)$ we can simplify to the pair of equations

$$\left(\frac{k}{3} + \frac{1}{6} - i\alpha_1\right)\mathcal{G}_{k,\mathbf{y}}(f) - i\alpha_2\mathcal{G}_{k+1,\mathbf{y}}(f) + i\left(\frac{k}{3} - i\alpha_1\right)\mathcal{G}_{k,\mathbf{y}}(g) + \alpha_2\mathcal{G}_{k+1,\mathbf{y}}(g) = 0$$

$$-i\alpha_3\mathcal{G}_{k-1,\mathbf{y}}(f) + \left(\frac{k}{3} - \frac{1}{6} + i\alpha_1\right)\mathcal{G}_{k,\mathbf{y}}(f) + \alpha_3\mathcal{G}_{k-1,\mathbf{y}}(g) - i\left(\frac{k}{3} + i\alpha_1\right)\mathcal{G}_{k,\mathbf{y}}(g) = 0.$$

Evaluating the second of these at $k+1$ instead of k we can cancel either the $\mathcal{G}_{k+1,\mathbf{y}}(f)$ term or the $\mathcal{G}_{k+1,\mathbf{y}}(g)$ term. In this way we can write our equations as the recurrence relation

$$\begin{pmatrix} \mathcal{G}_{k+1,\mathbf{y}}(f) \\ \mathcal{G}_{k+1,\mathbf{y}}(g) \end{pmatrix} = \frac{6}{(4k+3+12i\alpha_1)} B_k \begin{pmatrix} \mathcal{G}_{k,\mathbf{y}}(f) \\ \mathcal{G}_{k,\mathbf{y}}(g) \end{pmatrix}$$

where

$$B_k = \begin{pmatrix} -i\left[\left(\frac{k}{3}+\frac{1}{6}\right)\left(\frac{k}{3}+\frac{1}{3}\right)+\alpha_1^2-\frac{1}{6}i\alpha_1-\alpha_2\alpha_3\right] & \left[\frac{k}{3}\left(\frac{k}{3}+\frac{1}{3}\right)+\alpha_1^2-\frac{1}{3}i\alpha_1-\alpha_2\alpha_3\right] \\ -\left[\left(\frac{k}{3}+\frac{1}{6}\right)^2+\alpha_1^2+\alpha_2\alpha_3\right] & -i\left[\frac{k}{3}\left(\frac{k}{3}+\frac{1}{6}\right)+\alpha_1^2-\frac{1}{6}i\alpha_1+\alpha_2\alpha_3\right] \end{pmatrix},$$

and so the values of $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$ for all $k \in \mathbb{Z}$ are determined by a choice for $\mathcal{G}_{0,\mathbf{y}}(f)$ and $\mathcal{G}_{0,\mathbf{y}}(g)$. Since we are looking for smooth solutions f and g, we require that $\mathcal{F}_{\mathbf{y}}(f)(t) = \sum_{k\in\mathbb{Z}} \mathcal{G}_{k,\mathbf{y}}(f) e^{\frac{2\pi i k t}{3}}$ be smooth, and likewise for $\mathcal{F}_{\mathbf{y}}(g)(t)$. By the properties of the Fourier series of smooth functions, this is equivalent to asking that the sequences $\mathcal{G}_{k,\mathbf{y}}(f)$ and $\mathcal{G}_{k,\mathbf{y}}(g)$ are Schwartz, *i.e.* they are contained in

$$\mathcal{S}(\mathbb{Z}) = \left\{ (a_k)_{k\in\mathbb{Z}} \,\middle|\, \sup_{k\in\mathbb{Z}} |k^p a_k| < \infty \ \textit{for all}\ p \in \mathbb{N} \right\}.$$

5.4.2 Calculating $h_{\bar{\partial}}^{0,2}$

In all the examples considered in this section so far we have only partial solutions. In this last example, we use the same methods to calculate $h_{\bar{\partial}}^{0,2}$ on the manifold M_4 equipped with the almost Hermitian structure defined at the start of Example 5.14. For this example we are able to obtain a full solution.

Example 5.15. In order to calculate $h_{\bar{\partial}}^{0,2}$, we let a general $(0,1)$-form be written as $s = f\bar{\phi}^1 + g\bar{\phi}^2$, for some smooth function $f \in C^\infty(M_4)$. For s to be $\bar{\partial}$-harmonic we require $\bar{\partial}s = 0$, which is trivially true and $\partial * s = 0$, which is equivalent to the PDE system

$$\begin{cases} V_1(f) = 0, \\ V_2(f) + \frac{\pi}{6} f = 0. \end{cases} \tag{5.5}$$

We again split our calculations into the two cases where $|\mathrm{Orb}_{\mathbf{y}}| = 1$ and where $|\mathrm{Orb}_{\mathbf{y}}| = 3$.

<u>*Case 1:* $|\mathrm{Orb}_{\mathbf{y}}| = 1$</u>

For any $\mathbf{y} = (n,n,n) \in \mathbb{Z}^3$, we can apply the map $\mathcal{G}_{k,\mathbf{y}}$ to (5.5), for all $k \in \mathbb{Z}$, to obtain the condition

$$\begin{cases} k\mathcal{G}_{k,\mathbf{y}}(f) = 0, \\ \left(3n - \frac{i}{6}\right)\mathcal{G}_{k,\mathbf{y}}(f) = 0. \end{cases}$$

This is only possible if we have $\mathcal{G}_{k,\mathbf{y}}(f) = 0$, corresponding to the trivial solution $f = 0$.

<u>*Case 2:* $|\mathrm{Orb}_{\mathbf{y}}| = 3$</u>

Now let $\mathbf{y} = (l, m, n)$ with l, m and n not all equal. Again defining

$$\alpha_1 = l + m + n \qquad \alpha_2 = e^{\frac{2}{3}\pi i}l + e^{-\frac{2}{3}\pi i}m + n \qquad \alpha_3 = e^{-\frac{2}{3}\pi i}l + e^{\frac{2}{3}\pi i}m + n,$$

we see that applying $\mathcal{G}_{k,\mathbf{y}}$ to (5.5) and using Prop. 2.16 we obtain the pair of equations:

$$\begin{cases} \alpha_3\mathcal{G}_{k+1,\mathbf{y}}(f) + 2ik\mathcal{G}_{k,\mathbf{y}}(f) + \alpha_2\mathcal{G}_{k-1,\mathbf{y}}(f) = 0, \\ -\alpha_3\mathcal{G}_{k+1,\mathbf{y}}(f) + \left(2ik - \frac{i}{3}\right)\mathcal{G}_{k,\mathbf{y}}(f) + \alpha_2\mathcal{G}_{k-1,\mathbf{y}}(f) = 0. \end{cases}$$

Cancelling out either the $\mathcal{G}_{k-1,\mathbf{y}}(f)$ or $\mathcal{G}_{k+1,\mathbf{y}}(f)$ terms, we can combine the above equations to obtain the recurrence relations

$$\begin{cases} \mathcal{G}_{k+1,\mathbf{y}}(f) = -\frac{6\alpha_1 + i(6k+1)}{6\alpha_3}\mathcal{G}_k(f), \\ \mathcal{G}_{k,\mathbf{y}}(f) = -\frac{6\alpha_2}{6\alpha_1 + i(6k-1)}\mathcal{G}_{k-1}(f). \end{cases}$$

These only describe the same recurrence relation when $\mathbf{y}$ is chosen such that

$$\alpha_1^2 + i\alpha_1(2k+1) = \alpha_2\alpha_3 + \left(k - \frac{1}{6}\right)\left(k + \frac{5}{6}\right)$$

for all $k \in \mathbb{Z}$. No such $\mathbf{y}$ exists, and therefore there can be no sequence $(\mathbf{G}_{k,\mathbf{y}}(f))_{k\in\mathbb{Z}}$ satisfying both of them.

Bringing the two cases together, we conclude that (5.5) has no non-trivial solutions: $h^{0,2}_{\bar{\partial}} = 0$.

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